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# **Automatic Structures**

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## Abstract

This thesis investigates structures that are presentable by finite automata working synchronously on tuples of finite words. The emphasis is on understanding the expressiveness and limitations of automata in this setting. In particular, the thesis studies the classification of classes of automatic structures, the complexity of the isomorphism problem, and the relationship between definability and recognisability.

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