Feedback Stabilization on Linearly Perturbed Solid Body Rotational Flow

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Abstract

The method of feedback control on the stability of vortices is studied for an inviscid incompressible base flow subjected to an axisymmetric disturbance in a circular pipe with non-periodic boundary conditions. The investigation first focuses on the linear asymptotic equation with the control parameter applied. This is done to investigate the dynamics of the first growth rate branch curve because of the ease to apply constraints and the reduced complexity (allowing for analytical solutions) of the linear asymptotic equation. Numerical analysis indicates that the asymptotic equation is controllable, prompting the investigation of the same control mechanism applied to the linear WR equation. Since the asymptotic equation is only accurate to swirls up to and near the first growth rate branch, investigation focuses on whether the global equation is controllable and the effectiveness of said control mechanism. While this technique is only based on the specific solid body rotation flow (for its weakly non-linearity), investigation will pave way for improving the control mechanism and better understanding the dynamics of vortex stability of non-linear vortex flows.

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Contents

Chapter 1

Introduction

The field of hydrodynamic stability is a classical area of research under the wider genre of fluid mechanics. In particular, the phenomenon of vortex breakdown (one aspect of hydrodynamic stability) has gained steady interest in the past 50 years with the advancement of modern technologies, producing varied application of swirling flows in both experimental and theoretical areas. Vortex breakdown refers to the phenomenon in which the creation of stagnation point within the flow produces a sudden change in the topology of the streamlines. This phenomenon is widespread and involved in a variety of applications involving swirling flow of fluids such as the delta wing of a combat aircraft; the onset of instabilities of the vortex (often referred to as 'burst' or 'breakdown') generated at the wings are of particular concern, due to the direct impact it has to the stability, controllability and ultimately, aircraft performance. Many other technological applications with similar concerns exist, such as hydro-cyclone separators, combustion chambers, and meteorological studies, all of which involves in one form or another, the swirling flow of fluids in general.

Swirling flows (or 'vortices') generally refers to spinning currents of liquid or gas (fluids), which are in spiral motion of a center with closed streamlines. The spinning nature of fluids allows for the use of cylindrical coordinates (r, θ, x) to better express the dynamics of flows. The two main types of vortices are free (irrotational) vortex, and forced (rotational) vortex, the former describes a flow where the azimuthal velocity v of the flow changes as a function of the radius r, and the azimuthal velocities of each contour are different. The latter vortex refers to an ideal flow named solid body rotation, unlike the free vortex, the lack of shearing force allows the vortex to move as a whole, all contours are locked in with one another, and the motion looks akin to a solid cylinder rotating. This ideal flow with the lack of shear force allows for simplification of many complicated situations involved with the vortex breakdown phenomenon, and will be the focus of this paper while investigating the principles of different aspects involved in the stability of the vortex.

Classical stability by Lord Rayleigh [11] (Rayleigh 1916) details the study of vortex stability in an infinitely long pipe by setting periodic inlet and outlet conditions (for other papers on classical results refer to [7] (Leibovich 1984), [1] (Ash & Khorrami 1995), and [14] (Synge 1933)). This allowed the establishment of the circulation criterion for a swirling flow in an infinitely long, straight, circular pipe, which stated that flows are linearly neutrally stable subjected to axisymmetric perturbations if and only if the absolute value of the circulation function increases monotonically with the radius r from the center [16] (Wang & Rusak 2011). One particular flow type that always satisfy this criterion is the solid body rotation flow. When subjected to perturbations, solid body rotational flow will always remain stable because the criterion is always satisfied provided that the perturbation was axisymmetrical, and inlet and outlet conditions remain periodic. Howard & Gupta later extended this criterion to accommodate flows with an axial velocity $W(r)$, to show that the flow is linearly stable when subjected to axisymmetrical perturbations provided that the condition $\Phi > (dW/dr)^2/4$ was met [6] (Howard & Gupta 1962), further strengthening of the criterion can be seen in [8] (Leibovich & Stewartson 1983), [9] (Lessen, Singh & Paillet 1974) etc. However, the strengthening and confirmation of classical theorization failed to address an important issue that often renders experimental approaches to the classical theory to be inaccurate, for example, instability may actually occur for a solid body rotational flow whereas classical theory predicts stability under the required conditions. This is due to the fact that classical theorization limits the evolution of axisymmetrical perturbations to a periodic scheme of the inlet and outlet conditions of the pipe. The result is that periodic conditions often cannot accurately reflect realistic flow physics of the vortex, and situations that do present periodic conditions are often few and far between. Non-periodic conditions may actually generate different perturbation modes from periodic conditions.

Studies in [17] (Wang & Rusak 1996a) showed that non-periodic inlet and outlet conditions do in fact interfere with the stability of the perturbation, producing drastically different results from the classical theorization. With the placement of these non-periodic boundary conditions, the pipe dimension now more accurately reflects an industrial schematic of a swirling flow in a pipe. This allowed for a more realistic simulation of the flow dynamics that addresses the problem of the onset of instability unaddressed by classical theory. Under this scheme, solid body rotational flow can produce positive growth of the perturbations which may eventually lead to vortex instability even though classical theory predicts stability, and each growth rate branch associated with a range of swirl increases as the current swirl departs further from Benjamin's critical swirl [2] (Benjamin 1962). For detailed analysis and stability of other vortex type subjected to axisymmetrical perturbations refer to [17] (Wang & Rusak 1996a) and [18] (Wang & Rusak 1996b).

Due to the complex nature of the stability equation for general base flow subjected to non-periodic boundary conditions, it may be difficult to manipulate the equation itself or change the dynamics by tweaking with some constraints or conditions. Therefore it seems logical to derive some reduced form of the stability equation that governs this problem. Essentially the focus of the reduced form of the stability equation will be on the first few growth rate branches of the full stability equation. By utilizing this approach, a reduced form (or the asymptotic) stability equation that is much simplified than its full equation counterpart can be produced. The asymptotic equation is simpler to change and place constraints, however its downfall is that the equation is only accurate for the first few growth rate branches, i.e., the swirl cannot increase that far away from Benjamin's critical swirl. Comparison will show that subsequent growth rate branches produces a significant amount of difference which cannot be ignored, but for the purpose of the asymptotic equation accurately predicting the first branch the accuracy is high enough. Since the subsequent branches all behave similarly to the branch before, one would hope that any changes made to the first branch will produce the same effect on the later branches.

Since positive growth of the perturbation is produced by adding non-periodic boundary conditions (for solid body rotational flow), a method for quenching the growing perturbation or to control it was investigated in hoping that the perturbation can somehow be contained to allow the vortex to remain stable. A way to achieve this is by placing the asymptotic equation under feedback stabilization control. Essentially the feedback control places a control parameter $u(t)$ within the asymptotic evolution equation for perturbation, and the control parameter can be adjusted according to the significance of the growth of the perturbation. This approach was developed by noting instability involves the transfer of kinetic energy of the perturbation, and that certain conditions will sometimes cause the rate of transfer or production of kinetic energy to drop, delaying the onset of instability, while some conditions speeds up the process (for details see [19] (Wang, Taylor & Ku Akil 2010) and [16] (Wang & Rusak 2011), for alternative control method see [5] (Gallaire, Chomaz & Huerre 2004)). However, the current control method involves a control parameter within the stream, which makes the parameter difficult to adjust under realistic circumstances.

In this research paper, an alternative representation of the control parameter will be investigated. We will attempt to transfer the control parameter from within the stream to an equivalent condition at the inlet of the pipe. This adjustment allows for the ease of change for the control parameter experimentally. The development of the control parameter for the asymptotic equation guarantees stability for a solid body rotation flow subjected to axisymmetrical disturbances provided that the control meets certain requirements. However the asymptotic equation only accurately represents the first growth rate branch (and the nearby swirl range), and the effect of the control on later branches is unknown. Therefore we will attempt to develop a new method of control for the linear Wang & Rusak equation which enforces the effect of control by changing the circulation condition at the inlet of the pipe. But to derive a condition for which complete control can be achieved for this new control method, it is necessary to rescale and reduce the linear WR equation to arrive at a third order asymptotic form of the WR equation. Doing so allows us to investigate what the inlet condition for circulation represents in the asymptotic equation scheme, and a requirement for the inlet circulation can be found. Since both equations have near identical first growth rate branch, one would hope that for the full equation, the first growth rate branch at the very least can be completely controlled like the asymptotic equation; investigation will then be focused on the later branches, which departs from the prediction of the asymptotic equation. Currently it is predicted that the eliminated terms from the reduction towards the asymptotic equation may have minor effect close to the first branch, but as the swirl increases further from the critical swirl, the eliminated terms may gain importance, which can possibly eventually render the control parameter unable to stabilize the perturbation, leading to positive growth and vortex instability. However, the new approach to control the linear WR stability problem was found to still be able to successfully suppress the growing perturbation contrary to what was predicted. This signifies that feedback stabilization is a robust form of stability control for solid body rotational stability problem. Numerical analysis gave evidence of reduced effectiveness of the control parameter, however, the control parameter after reaching the minimum requirement is still able to control the linear WR equation, and both the linear asymptotic equation and the linear WR equation can be controlled by the feedback control mechanism with solid body rotational flow subjected to axisymmetrical disturbance. Further investigation is needed to verify the effectiveness of feedback control on other flow types which involves a higher level of non-linearity effect. Since solid body rotation has a fairly weak non-linear effect that only manifest itself for sufficiently large perturbation modes, the linear WR and weakly non-linear asymptotic equation will no longer be able to accurately describe the perturbation dynamics.

Chapter 2

Linear Global Stability Analysis

An inviscid, incompressible, axisymmetric swirling flow subjected to infinitesimal axisymmetric disturbances will be examined using the Euler equations of motion. The chapter will attempt to clarify the difference between classical stability theorization proposed by Lord Rayleigh [11] (Rayleigh 1916), and a new stability theorization by Wang & Rusak [17] (Wang & Rusak 1996a). The lack of boundary conditions that reflects physical operations in the classical theorization means that classical stability problem often fails to predict instability for experimental flows that may have been stable when subjected to periodic boundary conditions seen in the classical theory. This problem was addressed by Wang & Rusak, by placing non-periodic boundary conditions instead of periodic boundary conditions that produce Fourier mode solutions. This research paper will focus mainly on solid body rotation flow which is able to provide analytical solutions for most of the situations, and the dynamics of the perturbations will be studied using the linear WR stability equation proposed by Wang & Rusak. For stability analysis of alternative flow profiles, some involving viscosity, see excellent research papers [20] (Wang & Rusak 1997), [4] (Galleire & Chomaz 2004) which discuss said flows leading to vortex breakdown.

2.1 Classical (Local) stability problem

For the purpose of this paper, the phenomenon of vortex breakdown will be examined under a few restrictions placed on the base flow and its perturbations, the necessity of these restrictions shall be explained in later section where they serve the purpose of simplification. We consider a flow model of an inviscid, incompressible, and axisymmetric flow field in a straight with no bends cylindrical pipe. To examine this flow field we will need to utilize the Euler equations of motion (for hydrodynamics), and for the convenience of calculation the equations will be represented in cylindrical coordinates (r, θ, x), instead of the usual formulation in cartesian. The velocity component of each coordinate will be represented by (u, v, w), the radial, azimuthal, and axial velocity respectively.

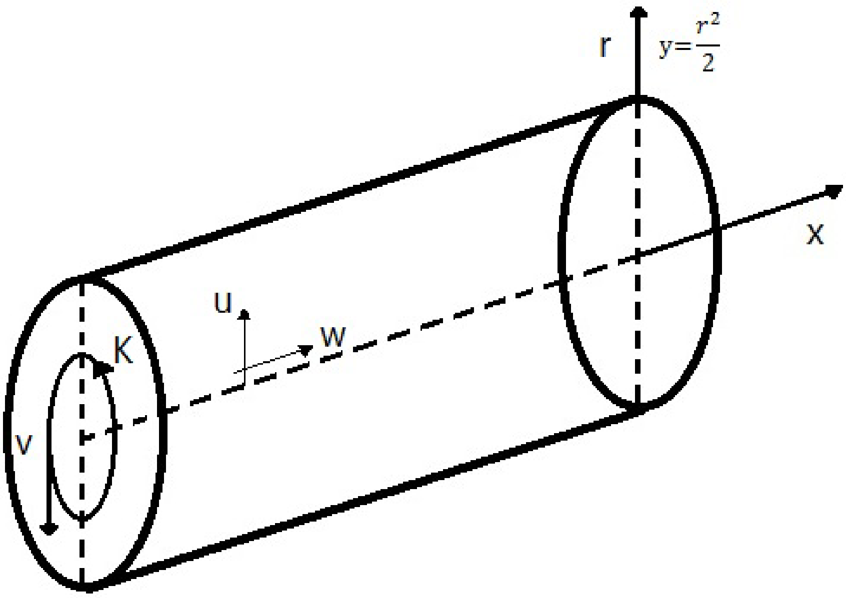

Figure 2.1: Flow configuration and coordinate display.

Therefore the full Euler equations of motion in cylindrical coordinates are:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial x} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r},$$
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial x} - \frac{uv}{r} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta},$$
$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{2.1}$$

along with the equation of mass conservation:

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial x} = 0, \tag{2.2}$$

where the variable p represents the pressure in the system.

To analyze the stability of flow problems with the use of the cylindrical equations of motion it is desirable to first look into the classical studies conducted by Lord Rayleigh. The classical problem describes the motion and stability/instability for two concentric rotating fluid cylinders, this result translates particularly well to the study of vortex stability since the relation between the two concentric cylinders reflects the condition of free or forced vortex formation. Rayleigh considered a swirling flow of an inviscid fluid with angular velocity $v = V(r)$ (velocity profile from the axis of rotation with radius r). By simple reasoning using physical arguments, Rayleigh derived his circulation criterion [11] (Rayleigh 1916) which stated that a necessary condition for stability of a swirling flow subjected to axisymmetrical disturbances is that the square of the circulation function does not decrease anywhere in the flow, i.e.,

$$\Gamma = \frac{1}{r^3}\frac{d}{dr}(rV)^2 > 0,$$

where $K = rV$ is the circulation function, a function that is dependent upon the swirl of the base flow. Note that this describes a pure vortex flow that only depends on the radius r, there exists no axial or radial velocity, the lack of viscosity (stemmed from inviscid flow) and shear allowed for a forced movement causing the swirling flow to move as if it were a solid cylinder. This criterion shows that for the ideal flow of a forced vortex such as solid body rotation subjected to infinitesimal disturbances, the vortex is linearly stable for any swirl level. The criterion is later strengthened by Synge who showed that by placing a strict inequality the criterion gains the property to be also a sufficient condition for ensuring linear stability [14] (Synge 1933). [6] (Howard & Gupta 1962) generalized this result by expanding to flows that have both axial and radial velocity components present, which allowed for analysis on other flow conditions such as Q-vortex model given by [7] (Leibovich 1984).

Now consider a base flow; $u = w = 0$, $v = V(r)$, and $p = P(r)$, this flow is in fact a steady basic solution of (2.1)-(2.2), we will perturb this flow by an infinitesimal amount to obtain the perturbed base flow;

$$(u, v, w) = (\tilde{u}, V(r) + \tilde{v}, \tilde{w}), \qquad \text{and} \qquad p = P(r) + \tilde{p},$$

the substitution of this perturbed flow into (2.1)-(2.2) and eliminating high order infinitesimal terms result in the equations of motion taking the following form;

$$\begin{aligned} &\frac{\partial \tilde{u}}{\partial t} + \Omega \frac{\partial \tilde{u}}{\partial \theta} - 2\Omega \tilde{v} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial r}, \\ &\frac{\partial \tilde{v}}{\partial t} + \Omega \frac{\partial \tilde{v}}{\partial \theta} + \left(\frac{dV}{dr} + \frac{V}{r}\right) = -\frac{1}{\rho}\frac{1}{r}\frac{\partial \tilde{p}}{\partial \theta}, \\ &\frac{\partial \tilde{w}}{\partial t} + \Omega \frac{\partial \tilde{w}}{\partial \theta} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x}, \end{aligned} \tag{2.3}$$

with the mass conservation equation;

$$\frac{\partial \tilde{u}}{\partial r} + \frac{\tilde{u}}{r} + \frac{1}{r}\frac{\partial \tilde{v}}{\partial \theta} + \frac{\partial \tilde{w}}{\partial x} = 0. \tag{2.4}$$

(Notation used here follows classical theorisation, $V(r) = r\Omega(r)$ where the swirl $\Omega(r)$ is an arbitrary function of r.)

These are the linearized equations of motion that describes the linearized problem for an inviscid perturbed base flow. The inviscid nature of the base flow signifies the lack of viscosity, which allows for the elimination of pressure relating terms, and the additional condition of axisymmetricity of the disturbed base flow allowed further simplification through the elimination of θ relating terms as well as partials and differentials of θ. Using equations (2.3)-(2.4) we are able to obtain an equation that dictates an initial value problem;

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}+\frac{\partial^2}{\partial x^2}\right)\tilde{u}+\Gamma\frac{\partial^2\tilde{u}}{\partial x^2}=0. \tag{2.5}$$

Although lacking the full picture from simplification through various conditions and linearization, stability analysis on the equations do provide a starting point for investigation on said equations. A simplistic analysis of the stability of the perturbed flow can be achieved by utilizing normal modes which separates the dependence of t and x with r through exponentials. Let k be the wave number in the axial direction, it is then possible to write $\tilde{u}=\hat{u}(r)\exp(st+ikx)$, substituting the normal mode analysis into equation (2.5) we can obtain;

$$\frac{d}{dr}\left[\frac{d\hat{u}}{dr}+\frac{\hat{u}}{r}\right]-k^2\hat{u}=\frac{k^2}{s^2}\Gamma\hat{u}. \tag{2.6}$$

In [14] (Synge 1933) it was noted that (2.6) along with boundary conditions $\hat{u}=0$, at $r=R_1=R_2$ (radius of the two concentric cylinders) is of the classic Sturm-Liouville problem type, with eigenvalues k^2/s^2 (s is an arbitrary constant). Synge therefore concluded using the Sturm-Liouville theorem that the eigenvalues should all be negative if the circulation criterion $\Gamma>0$ throughout the radial direction. The wave number k must be a real and positive quantity, which in term suggests that the variable s must be a complex quantity, producing a purely imaginary exponent. Since the analysis of this equation was done regarding the base flow of solid body rotation, the conclusion suggested that the flow will be **always** stable due to the fact that with solid body rotation flow the circulation criterion Γ will always be positive, suggesting the flow to always be stable (for other instability phenomenon relating to classical theorization, refer to [3] (Drazin & Reid 1981)). However, the stability assured in classical studies is rarely the case in reality with flows often become unstable even with solid body rotation. Many research papers published later have tried to rectify this problem by suggesting many other physical mechanics which may have influenced the flow that leads it towards instability, but the core misunderstanding of the classical stability problem remains the same; Rayleigh stability suggested long pipe approach with periodic boundary conditions given in the axial direction, the Fourier mode approach allows for the separation of the axial and radial component of any flow. This presents no problem in theorization, but in reality this is rarely the case, often pipes are not sufficiently long enough to apply the long pipe approach and set periodic boundary conditions for the middle section of the pipe, which may actually be experiencing periodic boundary conditions. Wang and Rusak suggested the placement of non-periodic boundary conditions that mimics industrial operations, which allowed for predictions of instability seen in reality [17] (Wang & Rusak 1996a). The following chapters will be investigating the Wang & Rusak stability problem to clarify the difference with classical stability, and develop further understanding of the workings of the instability mechanism, hopefully this will lead to improved prediction of the onset of instability that may help with the development of some control parameter which can quench the growing perturbations.

2.2 Circulation, stream, and vorticity function

Since this research focuses on the analysis of Wang & Rusak stability problem subjected to disturbed base flow of a solid body rotation flow, the approach here will be different to that of the classical stability problem theorized by Rayleigh. While the starting point is the same (using the Euler equations of motion), analysis of stability (hence instability) will be investigated using two approaches, one being the WR stability analysis detailed in this chapter, and the long wave asymptotic approach which shall be explained in the following chapter.

Following the same premise set in section 2.1, we consider an inviscid, incompressible axisymmetric flow in a cylindrical domain. The coordinate system remains in the cylindrical domain (r, θ, x), with the same velocity (u, v, w) respectively for each coordinate. With the inviscid and axisymmetric nature of the flow we can eliminate terms relating pressure and θ component from (2.1)-(2.2). To investigate the stability of the vortex we would like to see the motion of the flow subjected to perturbation, and the easiest way to do this is to plot the stream lines of particles in the flow, to do so we would like to utilize variables such as the circulation function K, azimuthal vorticity χ (essentially the tendency of the fluid to 'spin'), and the stream function ψ. Doing this avoids complicated manipulation involving elementary variables such as velocity, and a compact form of the Euler equations of motion can also be obtained (see [15] (Szeri & Holmes 1988)). Let $K = rv$, and the stream function ψ is

$$(u, w) = \frac{1}{r}\left(-\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial r}\right),$$

also, the azimuthal vorticity χ is defined as

$$\chi = \frac{1}{r}\left(\frac{\partial u}{\partial x} - \frac{\partial w}{\partial r}\right) = -\frac{1}{r^2}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r}\right),$$

utilizing the three substitutions, the newly formed evolution equations written in K, ψ, and χ is thus

$$\frac{\partial K}{\partial t} + \{\psi, K\} = 0, \tag{2.7}$$

$$\frac{\partial \chi}{\partial t} + \{\psi, \chi\} = \frac{1}{4y^2}\frac{\partial (K^2)}{\partial x}, \tag{2.8}$$

where $y = r^2/2$ is the new radial coordinate, and $\{f, g\}$ represents the Poisson bracket:

$$\{f, g\} = \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial y}.$$

(2.7) describes the transport of circulation along the path line of a particular chosen flow, and (2.8) accounts for the interaction of the azimuthal vorticity χ along said flow path line and vorticity which has been rescaled by gradient of the circulation function. By placing the flow in the pipe under certain boundary conditions that reflects realistic operation which involves a vortex generator at one end of the pipe, working steadily and continuously, it is now possible to investigate the instability caused by the disturbance which is different to that of classical theorization. We assume that for all time $t \geq 0$ and for $0 \leq y \leq 1/2$, the initial stream function is $\psi(0, y, t) = \psi_0(y)$ and circulation function $K(0, y, t) = K_0(y)\omega\tilde{K}_0(y)$. The inlet conditions must satisfy axisymmetricity, namely that they are all reduced to zero at $y = 0$. $\psi_0(y)$ is the inlet volumetric flux profile, the new parameter ω is the base swirl which is strictly positive, and the inlet circulation profile $\tilde{K}_0(y)$ is rescaled with ω. Notice the definition of the azimuthal vorticity means that $\chi(0, x, t)$ can be fixed by letting $\psi_{xx}(0, y, t) = 0$, since ψ_{0y} is a preset quantity at the inlet, $\chi(0, y, t) = -\psi_{0yy}$, and the axial component of the inlet vorticity is removed, leaving radial components only. The inlet conditions are set to be purely in radial direction, this is done so axial interference at the inlet with the flow can be avoided (i.e., the flow heading in from the inlet will remain so).

The conditions at the outlet of the pipe will also be fixed, there are two possibilities that can be explored in this research paper, the first condition refers to a sufficiently long pipe, where the discharged flow exhibits no radial velocity, essentially the flow being discharged is free flowing with no alteration at the outlet, so for all time t, $\psi_x(L, y, t) = 0$ for $0 \leq y \leq 1/2$, L is the length of pipe which should be $>> 1$ to be sufficiently long, all conditions are the same for y negative due to axisymmetricity. This condition was used in the analysis of Wang & Rusak stability (WR) [17] (Wang & Rusak 1996a), [18] (Wang & Rusak 1996b). Another outlet condition refers to when the pipe has a discharge device placed at the outlet; $\psi(L, y, t) = \psi_0(y)$ which fix the flux profile at the outlet for all time t and $0 \leq y \leq 1/2$. As mentioned before, symmetry condition is placed at the center line of the pipe, i.e. $\psi(x, 0, t) = 0$ for $0 \leq x \leq L$ and all time t. Boundary condition will also be given for the wall of the pipe so at $y = 1/2$ the stream function will be fixed at $\psi(x, 1/2, t) = \psi_0(1/2)$ for $0 \leq x \leq L$ and all time t.

With the placement of the boundary conditions, the equations of motion departs from the infinite pipe, periodic boundary problem stated in the classical analysis and becomes a non-periodic boundary value problem, and the boundary conditions produce restriction on the equation which complicates the analysis. For example, in Rayleigh stability a normal mode analysis is able to be used since the classical approach allows the separation of the axial component with the radial component, although this was only demonstrated on solid body rotation base flow, separation of variables is actually achievable for every flow under the classical scheme, but the introduction of non-periodic boundary conditions produces two dimensional PDE which for most base flows, will not allow the separation of variables except for solid body rotation flow seen in [17] (Wang & Rusak 1996a). The research paper will utilize this convenience allowed by solid body rotation flow as a basis for analyzing the new stability problem posed by non-periodic boundary conditions of the inlet and outlet of the pipe.

2.3 Wang & Rusak stability problem

The new stability problem devised by Wang & Rusak addresses the problem in Rayleigh stability, namely the lack of realistic boundary conditions which allowed for stability of solid body rotational flow subjected to infinitesimal disturbances, the non-periodic boundary conditions of the Wang & Rusak stability problem produces instability of the disturbed flow when classical theory predicts stability. To get to the WR stability problem, we start off with defining a steady base flow much like in section 2.2;

$$\psi = \psi_0(y), \qquad K = K_0(y) = \sqrt{\Omega}\tilde{K}_0(y), \tag{2.9}$$

the variable Ω is the square of the swirl ratio (ω) of the flow. The base flow (2.9) is a steady solution of the equations (2.7) and (2.8) and the boundary conditions posed in the previous section, it is also a solution to the Squire-Long equation (also known as the Bragg-Hawthorne equation, for full derivations see [13] (Squire 1956) and [10] (Long 1953));

$$\psi_{yy} + \frac{\psi_{xx}}{2y} = H'(\psi;\Omega) - \frac{I'(\psi;\Omega)}{2y},$$

H is the total head function of the flow, $H = p/\rho + (u^2 + v^2 + w^2)/2$ where p is the static pressure, ρ is the density of the flow, and $I = K^2/2$.

Following the procedure in classical theorization, we will disturb the base flow by an infinitesimal amount to analyze the dynamics of the perturbation:

$$\psi(x,y,t) = \psi_0(y) + \epsilon\psi_1(x,y,t) + ..., \quad (2.10)$$

$$K(x,y,t) = K_0(y) + \epsilon K_1(x,y,t)..., \quad (2.11)$$

the coefficient ϵ is scaling factor where $\epsilon << 1$, ψ_1 and K_1 are the arbitrary stream function disturbance and the circulation disturbance respectively. Substituting (2.10) and (2.11) into the governing equations (2.7) and (2.8), plus the elimination of second order ϵ terms, we will arrive at the linearized equations of motion which describes the dynamics of the perturbation of the swirling flow;

$$K_{1t} + \psi_{0y}K_{1x} - K_{0y}\psi_{1x} = 0, \quad (2.12)$$

$$-\frac{K_0}{2y^2}K_{1x} + \chi_{1t} + \psi_{0y}\chi_{1x} - \chi_{0y}\psi_{1x} = 0, \quad (2.13)$$

χ_1 is the azimuthal vroticity disturbance from substituting the stream function with disturbance term; $\chi_1 = -(\psi_{1yy} + \psi_{1xx}/2y)$. The new governing equations (2.12) and (2.13) must now satisfy a new set of boundary conditions for the perturbations:

$$\psi_1(x,0,t) = 0, \quad \psi_1(x,\frac{1}{2},t) = 0 \quad \text{for} \quad 0 \le x \le L,$$

$$\psi_1(0,y,t) = 0, \quad \psi_{1xx}(0,y,t) = 0, \quad K_1(0,y,t) = 0 \quad \text{for} \quad 0 \le y \le \frac{1}{2},$$

$$\psi_{1x}(L,y,t) = 0, \quad \textit{or} \quad \psi_1(L,y,t) = 0 \quad \text{for} \quad 0 \le y \le \frac{1}{2}, \quad (2.14)$$

from the boundary conditions above one can deduce that $\chi_1(0,y,t) = 0$, this simply signifies that we do not generate any azimuthal vorticity disturbance at the inlet of the pipe.

Combining the definition of χ_1 and equations (2.12) and (2.13) will lead to a partial differential equation which the stream function disturbance ψ_1 is able to satisfy;

$$\left(\psi_{1yy} + \frac{\psi_{1xx}}{2y}\right)_{xx} + \frac{2}{\psi_{0y}}\left(\psi_{1yy} + \frac{\psi_{1xx}}{2y}\right)_{xt} + \frac{1}{\psi_{0y}^2}\left(\psi_{1yy} + \frac{\psi_{1xx}}{2y}\right)_{tt} + \frac{\chi_{0y}}{\psi_{0y}^2}\psi_{1xt} - \left(H'(\psi;\Omega) - \frac{I'(\psi;\Omega)}{2y}\right)\psi_{1xx} = 0, \tag{2.15}$$

this is the equation of motion in ψ_1, the solution of which can be used to plot the stream line of the perturbation. Now rearrange (2.13) into a function of K_{1x} and substitute it into equation (1.12) to obtain an equation in K_{1t}:

$$K_{1t} = -\psi_{0y}\frac{2y^2}{K_0(y)}(\chi_{1t} + \psi_{0y}\chi_{1x} - \chi_{0y}\psi_{1x}) + K_{0y}\psi_{1x}, \tag{2.16}$$

if we integrate (2.16) in time, we can obtain K_1. The solutions of (2.15) and (2.16) should satisfy the boundary conditions for the perturbation in (2.14).

To study the linearized stability problem for the perturbed swirling flow, a suitable eigenmode analysis of $\psi_1 = \phi(x,y)e^{\sigma t}$ and $K_1 = k(x,y)e^{\sigma t}$ will be used (k here has been reassigned as the mode analysis form of the circulation disturbance, instead of the wave number). In general, the variables σ, ϕ, and k are complex functions. By substituting mode analysis into the above equations (2.15) and (2.16), we can obtain a partial differential equation for solving the stream function perturbation mode ϕ:

$$\left(\phi_{yy} + \frac{\phi_{xx}}{2y} - \left(H'(\varphi;\Omega) - \frac{I'(\varphi;\Omega)}{2y}\right)\phi\right)_{xx} + \frac{\sigma\chi_{0y}}{\varphi_{0y}^2}\phi_x + \frac{2\sigma}{\varphi_{0y}}\left(\phi_{yy} + \frac{\phi_{xx}}{2y}\right)_x + \frac{\sigma^2}{\varphi_{0y}}\left(\phi_{yy} + \frac{\phi_{xx}}{2y}\right) = 0. \tag{2.17}$$

(2.17) along with the eigenmode boundary conditions are the Wang & Rusak stability problem for general base flow, while this equation is very useful for determining stability of flows numerically, it is impossible to discern the exact nature due to the involvement of the spatial variables x and y that does not usually allow for separation of variables. The equation can also be simplified based on the base flow that is chosen to be applied to the equation, by substituting for the same disturbed base flow (2.10), (2.11), and the definition of χ, we are able to transform the Squire-Long equation to:

$$-H''(\psi_0;\Omega) + \frac{\Omega\tilde{I}''(\psi_0)}{2y} = \frac{\chi_{0y}}{\psi_{0y}} + \frac{K_0 K_{0y}}{2y^2\psi_{0y}^2},$$

applying this to WR stability equation will reduce the complexity of the equation greatly. The functions ϕ and k must now satisfy a new set of boundary conditions:

$$\begin{aligned}
&\phi(x,0)=0, \quad \phi(x,\frac{1}{2})=0, \quad \text{for every } \ 0\le x\le L,\\
&\phi(0,y)=0, \quad \phi_{xx}(0,y)=0, \quad k(0,y)=0, \quad \text{for every } \ 0\le y\le \frac{1}{2},\\
&\phi_x(L,y)=0, \quad \text{for every } \ 0\le y\le \frac{1}{2},
\end{aligned} \tag{2.18}$$

To find $k(x,y)$, substitute the mode analysis into equation (2.16):

$$\sigma k = -\psi_{0y}\frac{2y^2}{K_0(y)}(\sigma\chi^* + \psi_{0y}\chi_x^* - \chi_{0y}\phi_x) + K_{0y}\phi_x,$$

where $\chi^* = -(\phi_{yy} + \phi_{xx}/2y)$. From the above equation and the new set of boundary conditions, the boundary condition $k(0,y)=0$ can be replaced by a new condition which involves ϕ only terms,

$$\begin{aligned}
\sigma k(0,y) &= -\psi_{0y}\frac{2y^2}{K_0(y)}(\sigma\chi^*(0,y) + \psi_{0y}\chi_x^*(0,y) - \chi_{0y}\phi_x(0,y)) + K_{0y}\phi_x(0,y)\\
&= \psi_{0y}^2\frac{2y^2}{K_0}\left[\phi_{yyx}(0,y) + \frac{\phi_{xxx}(0,y)}{2y} + \frac{\chi_{0y}}{\psi_{0y}}\phi_x(0,y)\right] + K_{0y}\phi_x(0,y)\\
&= \phi_{yyx}(0,y) + \frac{\phi_{xxx}(0,y)}{2y} + \left(\frac{\chi_{0y}}{\psi_{0y}} + \frac{K_0K_{0y}}{2y^2\psi_{0y}^2}\right)\phi_x(0,y) = 0,
\end{aligned} \tag{2.19}$$

so equation (2.17) and its boundary conditions will then be purely dependent on ϕ;

$$\phi_{yyx}(0,y) + \frac{\phi_{xxx}(0,y)}{2y} - \left(H''(\psi_0;\Omega) - \frac{\Omega\tilde{I}''(\psi_0)}{2y}\right)\phi_x(0,y) = 0,$$
$$\text{for every} \quad 0 \leq y \leq \frac{1}{2}. \tag{2.20}$$

2.4 Analysis on Solid Body Rotation flow

The WR stability problem introduced in the previous section involves both axial and vertical component, both components are intertwined in a way that does not allow for separation of variables from conventional methods and for most base flows selected, this renders analytical solutions impossible, since the stability equation cannot be solved. However there exists one base flow which can be used to produce analytical solution much like the one obtained in classical analysis in section 2.1, that is to utilize solid body rotation base flow, exactly as classical analysis. **S**olid **B**ody **R**otation (**SBR**) flow is the only base flow for which separation of variable can be achieved for the stability equation with non-periodic boundary conditions.

So instead of the general base flow considered in the derivation of section 2.3, we now consider a columnar base flow with uniform axial speed which describes the flow of solid body rotation:

$$u = 0, \qquad v = \omega r, \qquad w = w_0,$$

so now we have $\psi_0 = w_0 y$, $K_0 = 2\omega y$ (since $K_0 = \omega\tilde{K}_0$, where $\tilde{K}_0 = 2y$.), and $\chi_0 = 0$, therefore $\psi_{0y} = w_0$, $\chi_{0y} = 0$ and now $-H''(\psi_0;\Omega) + \Omega\tilde{I}''(\psi_0)/2y = 4\omega^2/(2yw_0^2)$. The quantity w_0 is chosen to be one for the convention of numerical analysis and to simplify the equation. Equation (2.17) will now take the form:

$$\left(\phi_{yy}+\frac{\phi_{xx}}{2y}+\frac{\Omega}{2y}\phi\right)_{xx}+2\sigma\left(\phi_{1yy}+\frac{\phi_{xx}}{2y}\right)_{x} + \sigma^2\left(\phi_{yy}+\frac{\phi_{xx}}{2y}\right)=0, \tag{2.21}$$

where $\Omega = 4\omega^2$. In the case of solid body rotational flow, an analytical solution is found by noting that equation (2.21) is separable in the radial and axial component, so a separation of variables solution can be found:

$$\phi(x,y)=\Phi(y)\varphi(x), \tag{2.22}$$

where the function $\Phi(y)$ is a solution of

$$\Phi_{yy}+\frac{\Omega_B}{2y}\Phi=0,$$
$$\Phi(0)=\Phi(\frac{1}{2})=0. \tag{2.23}$$

Solution of (2.23) along with its boundary conditions is $\Phi_B = \sqrt{y}J_1(\sqrt{2\Omega_B}\sqrt{y})$, with $\sqrt{\Omega_B} = 2\omega_B = 3.83171$ and is the Benjamin's critical swirl of a solid body rotation flow [2] (Benjamin 1962).

By substituting the separation of variables solution into (2.21), one can obtain an ordinary differential equation in terms of $\varphi(x)$:

$$\varphi_{xxxx}+2\sigma\varphi_{xxx}+\left(\Omega-\Omega_B+\sigma^2\right) - 2\sigma\Omega_B\varphi_x-\sigma^2\Omega_B\varphi=0. \tag{2.24}$$

do the same with the boundary condition (2.20):

$$\Phi_{yy}(y)\varphi_{xx}(0)+\frac{\Phi(y)\varphi_{xxx}(0)}{2y}+\frac{4\omega^2}{2y}\Phi(y)\varphi_x(0)=0,$$
$$\Rightarrow \quad \frac{\Phi(y)}{2y}(-\Omega_B\varphi_x(0)+\varphi_{xxx}(0)+\Omega\varphi_x(0))=0,$$
$$\Rightarrow \quad (\Omega_B-\Omega)\varphi_x(0)-\varphi_{xxx}(0)=0. \tag{2.25}$$

From previous analysis up to the Wang & Rusak stability equation the solution of (2.24) must then satisfy the following conditions;

$$\begin{aligned} &\varphi(0) = 0, \qquad \varphi_{xx}(0) = 0, \qquad (\Omega_B - \Omega)\varphi_x(0) - \varphi_{xxx} = 0, \\ &\varphi_x(x_0) = 0 \qquad \text{for} \qquad 0 \leq x \leq x_0. \end{aligned} \tag{2.26}$$

To solve (2.24) let $\varphi(x) = e^{ax}$:

$$a^4 + 2\frac{\sigma}{w_0}a^3 + \left(\Omega - \Omega_B + \frac{\sigma^2}{w_0^2}\right)a^2 - 2\frac{\sigma}{w_0}\Omega_B a - \frac{\sigma^2}{w_0^2}\Omega_B = 0,$$

$$\Rightarrow (a^2 - \Omega_B)(a + \sigma)^2 + \Omega a^2 = 0. \tag{2.27}$$

By using the exponential to solve (2.24) the resulting equation (2.27) is now a polynomial in terms of a, an equation which has four roots, the four roots are the linearly independent modes of φ, together they form a linear combination of the eigenmodes

$$\varphi(x) = C_1e^{a_1x} + C_2e^{a_2x} + C_3e^{a_3x} + C_4e^{a_4x},$$

where only the real portion of these terms are taken into consideration, as the coefficients C_i are generally complex. The coefficients can be found by applying the boundary conditions to the above equation:

Condition (1) $\quad \varphi(0) = 0 \quad \Rightarrow \quad C_1 + C_2 + C_3 + C_4 = 0,$

Condition (2) $\quad \varphi_{xx}(0) = 0 \quad \Rightarrow \quad a_1^2C_1 + a_2^2C_2 + a_3^2C_3 + a_4^2C_4 = 0,$

Condition (3) $\quad (\Omega_B - \Omega)\varphi_x(0) - \varphi_{xxx}(0) = 0 \quad \Rightarrow$

$$\begin{aligned} &(a_1(\Omega_B - \Omega) - a_1^3)C_1 + (a_2(\Omega_B - \Omega) - a_2^3)C_2 \\ &+ (a_3(\Omega_B - \Omega) - a_3^3)C_3 + (a_4(\Omega_B - \Omega) - a_4^3)C_4 = 0, \end{aligned}$$

Condition (4) $\varphi_x(x_0) = 0 \quad \Rightarrow \quad a_1 C_1 e^{a_1 x_0} + a_2 C_2 e^{a_2 x_0} + a_3 C_3 e^{a_3 x_0} + a_4 C_4 e^{a_4 x_0} = 0.$

We can form a matrix equation with the resulting system of equations from applying the boundary conditions:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 \\ a_1(\Omega_B - \Omega) - a_1^3 & a_2(\Omega_B - \Omega) - a_2^3 & a_3(\Omega_B - \Omega) - a_3^3 & a_4(\Omega_B - \Omega) - a_4^3 \\ a_1 e^{a_1 x_0} & a_2 e^{a_2 x_0} & a_3 e^{a_3 x_0} & a_4 e^{a_4 x_0} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = 0,$$

a nontrivial solution of the matrix equation exists if and only if

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 \\ a_1(\Omega_B - \Omega) - a_1^3 & a_2(\Omega_B - \Omega) - a_2^3 & a_3(\Omega_B - \Omega) - a_3^3 & a_4(\Omega_B - \Omega) - a_4^3 \\ a_1 e^{a_1 x_0} & a_2 e^{a_2 x_0} & a_3 e^{a_3 x_0} & a_4 e^{a_4 x_0} \end{vmatrix} = 0,$$

by setting a particular value for the swirl, we are able to match the coefficients a_i and C_i according to the two matrix equations, this will then allow us to determine the growth rate σ of the stability equation.

2.5 Analysis of the perturbation growth rate

Using Matlab, the solution to this problem can be easily found. Firstly the polynomial (2.27) is solved by matching coefficients a_i to the determinant with a preset swirl. The coefficients a_i along with the preset swirl will then be used to match for the coefficients C_i, and finally all parameters will be used to solve for the growth rate σ according to a particular swirl. For convenience w_0 shall be set to equal to one, the length of pipe will be set as 6, and the vertical component $0 \geq y \geq 1/2$. The numerical values obtain for the adjusted swirl and growth rate is then converted to true swirl ω for better comparison with existing results:

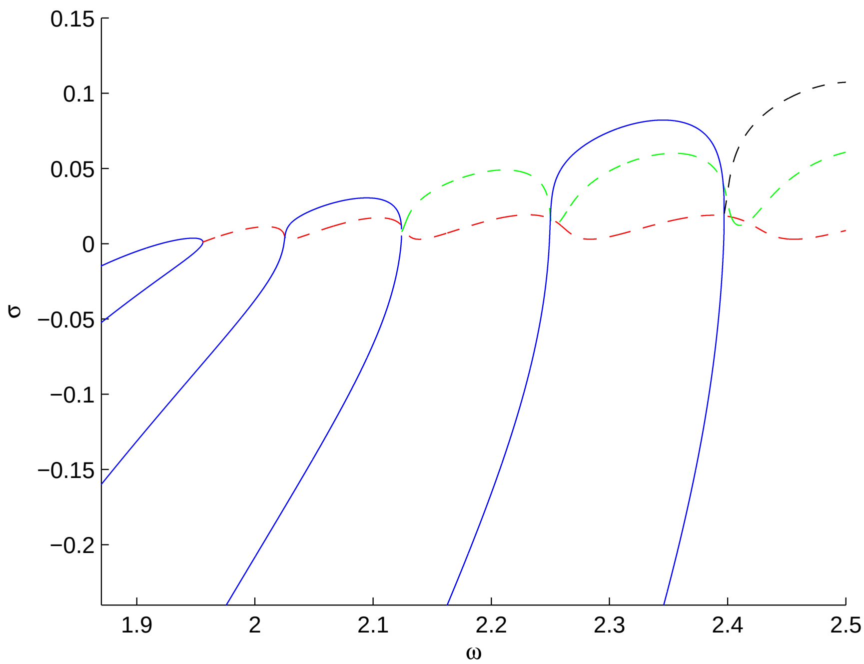

Figure 2.2: Growth rate curve of solid body rotational flow.

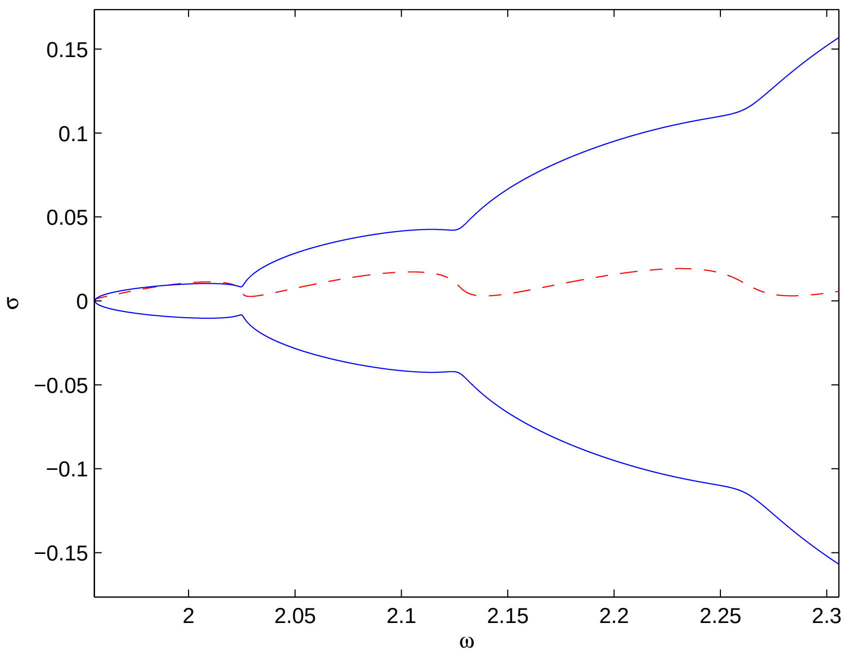

Figure 2.3: Complex modes of solid body rotational flow (first complex branch).

The resulting plot shows three growth rate branches of the perturbations (but not limited to), each of the branches generate two σ zero points, referred to as critical swirls, note that there exists other branches further along the ω axis. Figure 2.2 has solid lines representing purely real solutions, while the dashed lines represent the real part of the complex solutions generated at the maximum swirl points of each branch. Figure 2.3 shows the growth rate curve of complex modes, with the dashed line representing the real part of the complex solution, and the solid lines representing the imaginary portion of the complex conjugate.

The relation may seem counter intuitive, as one would expect the perturbation growth to increase for all increasing swirls (i.e., the flow spinning faster and faster producing growing perturbation). The discrepancy can be explained by the added boundary conditions applied onto the flow at the inlet and outlet unlike that of the infinite pipe approach (periodic boundary conditions) examined in classical results. As the swirl increases, the perturbation travels upstream towards the inlet, but inlet boundary conditions will now force the perturbation back down the stream, it is the application of the inlet and outlet boundary condition that can be seen on the plot generating many branches. The growth branches will increase as swirl becomes larger, this means that eventually for a particular swirl range some sets of σ will be large enough to cause the perturbation to grow to a point that leads to breakdown of the vortex that was formed. This is quiet the departure from classical results, which suggests that solid body rotation flow subjected to infinitesimal disturbances will be stable for all time with periodic boundary conditions. The instability of WR stability problem came from large positive growth of the perturbation which can eventually no longer be contained causing the vortex structure to disintegrate as a result (for details see [17] (Wang & Rusak 1996a)).

While the WR stability problem produces entirely accurate prediction of the growth rate curve for linear perturbations of any flow, it is still a simplified form of equations (2.7) and (2.8), since the substitution of linear perturbation eliminates the non-linearity motion of the perturbation, but the focus of this research paper is on solid body rotational flow which is weakly non-linear, the difference between linear and non-linear behavior for SBR flow is only distinguishable for very large perturbation, essentially making the weakly non-linear motion of SBR equivalent to linear motion for a sufficiently large range of swirl values, and for such large perturbations and large swirl range both the linear and weakly non-linear equations will be inaccurate in predicting the large, non-linear perturbation dynamics, which is outside the scope of this research. (2.17) is still highly complex despite the simplification from linearization, however, it is noticeable from the graph produced by numerical interpretation that the growth rate branches follow a certain trend, and that each branch that comes after is a larger version of the former branch. This property can be exploited, since branches behave similarly, it makes sense to just deal with one branch, simplify the mathematics to allow for change of conditions or adding control terms to change the dynamics. An asymptotic approach to the stability problem will be taken, since the branches behave similarly, it follows that changes applied to the first branch should have the same effect to the subsequent branches.

Chapter 3

Long wave (Asymptotic) Analysis

This chapter will investigate an alternative method to study the dynamics of the perturbed flow. The growth rate branches of the linear WR equation signified that all branches behave in a similar manner, but with increased values for growth rate σ as swirl increases further away from Benjamin's critical swirl. Noting this, it stands to reason that the complexity of the linear WR equation can be reduced by focusing the dynamics onto the first growth rate branch. This approach produces the asymptotic equation that accurately predicts the dynamics of the first growth rate branch and nearby swirl range, but the accuracy is lowered as the swirl increases. However for the purpose of reducing the complexity of the full equation, the asymptotic equation is able to produce the first branch accurately enough, therefore changing the pipe dynamics or setting extra constraints should be accurately reflected back to the full equation (at least for small swirl). For further discussion of the non-linear asymptotic equation (non-solid body rotational base flow), refer to [12] (Rusak, Wang, Xu & Taylor 2011).

3.1 Weakly non-linear asymptotic analysis

Equations (2.17) and (2.21) in chapter 2 gives detailed stability relation between the swirl and the growth rate of the perturbation for particular base flow, and (2.21) in particular gives the relationship of solid body rotational flow. While the equation gives accurately the eigenmodes of the growth rate curve for linear perturbation, the equation itself is difficult to analyze and even more so to tweak parameters or placing constraints that can eventually change the dynamics of flow in the pipe. Therefore it seemed logical to produce an equation that is essentially a reduced form of the original equations. Back to the full equations of motion (2.7) and (2.8), most of the parameters are the same as the previous chapter, except now $L >> 1$, where $L = x_0$. The analysis will take on a rescaled pipe approach essentially focusing the dynamics onto the first growth rate branch and nearby swirl levels (at near ϵ swirl levels), with $\epsilon = 1/L^2$ as the rescaling factor so the rescaled pipe length $X = \sqrt{\epsilon}x$ and the rescaled time parameter $t^* = t/\epsilon^{\frac{3}{2}}$, the asymptotic equation will focus on near Benjamin's critical swirl levels so that $\omega = \omega_B + \Delta\omega$, we assume an asymptotic expansion for the flow dynamics problem:

$$\begin{aligned}
\psi(X, y, t^*) &= \psi_0(y) + \epsilon\psi_1(X, y, t^*) + \epsilon_2\psi_2(X, y, t^*) + O(\epsilon_3),\\
\chi(X, y, t^*) &= -\psi_{0yy}(y) + \epsilon\chi_1(X, y, t^*) + \epsilon_2\chi_2(X, y, t^*) + O(\epsilon_3),\\
K(X, y, t^*) &= \omega\tilde{K}_0(y) + \epsilon K_1(X, y, t^*) + \epsilon_2 K_2(X, y, t^*) + O(\epsilon_3),
\end{aligned} \tag{3.1}$$

we assume that $\Delta\omega$ is of the order of ϵ, ϵ_2 is of the order of ϵ^2 and $\epsilon_3 \sim \epsilon^3$. The various perturbation functions must satisfy the boundary conditions;

$$\begin{aligned}
&\psi_1(0, y, t^*) = \psi_{1XX}(0, y, t^*) = K_1(0, y, t^*) = 0 \quad \text{for} \quad 0 \le y \le \frac{1}{2},\\
&\psi_2(0, y, t^*) = \psi_{2XX}(0, y, t^*) = K_2(0, y, t^*) = 0 \quad \text{for} \quad 0 \le y \le \frac{1}{2},\\
&\psi_{1X}(1, y, t^*) = \psi_{2X}(1, y, t^*) = 0 \quad \text{for} \quad 0 \le y \le \frac{1}{2},\\
&\psi_1(X, 0, t^*) = K_1(X, 0, t^*) = 0 \quad \text{for} \quad 0 \le X \le 1,
\end{aligned}$$

$$\psi_2(X,0,t^*) = K_2(X,0,t^*) = 0 \quad \text{for} \quad 0 \leq X \leq 1,$$
$$\psi_1(X,\frac{1}{2},t^*) = K_1(X,\frac{1}{2},t^*) = 0 \quad \text{for} \quad 0 \leq X \leq 1,$$
$$\psi_2(X,\frac{1}{2},t^*) = K_2(X,\frac{1}{2},t^*) = 0 \quad \text{for} \quad 0 \leq X \leq 1, \tag{3.2}$$

substituting (3.1) into equations (2.7) and (2.8) along with the set of boundary conditions (3.2), and balance for the order of perturbation terms, i.e., expansion for χ gives

$$\chi = -\psi_{0yy}(y) + \epsilon\left(-\psi_{1yy} - \epsilon\frac{\psi_{1XX}}{2y}\right) + \epsilon_2\left(-\psi_{2yy} - \epsilon\frac{\psi_{2XX}}{2y}\right) + O(\epsilon_3),$$

which we can balance the orders to obtain:

$$\chi_0 = -\psi_{0yy}, \qquad \chi_1 = -\psi_{1yy}, \qquad \epsilon_2\chi_2 = -\epsilon_2\psi_{2yy} - \epsilon^2\frac{\psi_{1XX}}{2y}.$$

Note that asymptotic method involving order balancing does not always guarantee the asymptotic method will work, as sometimes the variables being asymptotically expanded end up not being able to balance out. The details of the derivation can be seen in [12] (Rusak, Wang, Xu, & Taylor 2011). Eventually we will arrive at a differential equation for the function ψ_1:

$$\psi_{1yy} = \left(\frac{\chi_{0y}}{\psi_{0y}} + \omega_B^2\frac{\tilde{K}_0\tilde{K}_{0y}}{2y^2\psi_{0y}^2}\right)\psi_1 = 0, \tag{3.3}$$

(3.3) is a differential equation involving y only, therefore we can look for a separation of variables solution $\psi(X,y,t^*) = \phi_1(y)A(X,t^*)$, where ϕ_1 is a function that satisfies

$$\phi_{1yy} = \left(\frac{\chi_{0y}}{\psi_{0y}} + \omega_B^2\frac{\tilde{K}_0\tilde{K}_{0y}}{2y^2\psi_{0y}^2}\right)\phi_1 = 0,$$
$$\phi_1(0) = \phi_1(\frac{1}{2}) = 0, \tag{3.4}$$

(3.4) is the Benjamin's eigenvalue problem where the first eigenvalue of the problem is assigned as Benjamin's critical swirl ω_B, and the corresponding eigenfunction is designated as ϕ_B. Note that $\epsilon_2\psi_2(X, y, t^*)$ can also be expanded in a similar manner, so now we can write

$$\psi(X, y, t^*) = \psi_0(y) + \epsilon\phi_B(y)A(X, t^*) + \epsilon_2\phi_2(y)B(X, t^*) + O(\epsilon_3), \tag{3.5}$$

and the corresponding boundary conditions for A, B, ϕ_2, K_1, K_2 for all $t^* \geq 0$ are:

$$\begin{aligned} &\phi_2(0) = 0, \quad \phi_2\left(\frac{1}{2}\right) = 0, \\ &A(0, t^*) = 0, \quad A_{XX}(0, t^*) = 0, \quad A_{XX}(1, t^*) = 0, \\ &B(0, t^*) = 0, \quad B_{XX}(0, t^*) = 0, \quad B_X(1, t^*) = 0, \\ &K_1(0, y, t^*) = 0, \quad K_2(0, y, t^*) = 0. \end{aligned} \tag{3.6}$$

We are now able to expand $K(X, yt^*)$ and $\chi(X, y, t^*)$ in $A(X, t*)$ and $B(X, t^*)$, following the order balancing and derivation procedures in [12] (Rusak, Wang, Xu & Taylor 2011), we would eventually arrive at the equation in $A(X, t^*)$ and $B(X, t^*)$:

$$\begin{aligned} &\left(\phi_{2yy} + \left(\omega_B^2 \frac{\tilde{K}_0\tilde{K}_{0y}}{2y^2\psi_{0y}^2} - \frac{\chi_{0y}}{\psi_{0y}}\right)\phi_2\right)B_X \\ &- A_{t^*}\left(\omega_B^2\phi_B\frac{\tilde{K}_0\tilde{K}_{0y}}{2y^2\psi_{0y}^3} - \frac{\phi_{Byy}}{\psi_{0y}}\right) \\ &+ A_{XXX}\frac{\phi_B}{2y} \\ &+ (A^2)_X\frac{1}{2}\left(\frac{\omega_B^2}{y\psi_{0y}^{3/2}}\left(\frac{\tilde{K}_0\tilde{K}_{0y}}{y\psi_{0y}^{3/2}}\right)_y + \frac{1}{\psi_{0y}}\left(\frac{\psi_{0yyy}}{\psi_{0y}}\right)_y\right)\phi_B^2 \\ &+ k_\omega A_X\frac{\tilde{K}_0\tilde{K}_{0y}}{2y^2\psi_{0y}^2}\phi_B = 0, \end{aligned} \tag{3.7}$$

multiplying the above equation by ϕ_B, and integrated over $0 \leq y \leq 1/2$. Using integration by parts on the ϕ_2 terms along with the boundary conditions for ϕ_2, resulting in ϕ_2 terms vanishing which leads to the resulting equation for $A(X, t^*)$:

$$N_s A_{t^*} - \delta A_{XXX} + N_1 (A^2)_X - k_\omega N_2 A_X = 0, \tag{3.8}$$

with the variables defined as:

$$N_s = \int_0^{1/2} \left(\omega_B^2 \frac{\tilde{K}_0 \tilde{K}_{0y}}{y^2 \psi_{0y}^3} + \frac{\chi_{0y}}{\psi_{0y}^2} \right) \phi_B^2 dy,$$
$$\delta = \int_0^{1/2} \frac{\phi_B^2}{2y} dy,$$
$$N_1 = -\frac{1}{2} \int_0^{1/2} \left(\frac{\omega_B^2}{y \psi_{0y}^{3/2}} \left(\frac{\tilde{K}_0 \tilde{K}_{0y}}{y \psi_{0y}^{3/2}} \right)_y + \frac{1}{\psi_{0y}} \left(\frac{\chi_{0y}}{\psi_{0y}} \right)_y \right) \phi_B^3 dy,$$
$$N_2 = \int_0^{1/2} \frac{\tilde{K}_0 \tilde{K}_{0y}}{2y^2 \psi_{0y}^2} \phi_B^2 dy.$$

Finally, let $\tau = N_S/\delta$, $\alpha = N_1/\delta$ and $\beta = N_2/\delta$, which results in the weakly non-linear model asymptotic equation for the evolution of $A(X, t^*)$ in the range $0 \leq X \leq 1$:

$$\tau A_{t^*} = A_{XXX} - \alpha (A^2)_X + k_\omega \beta A_X. \tag{3.9}$$

The evolution of the mode axial shape function $A(X, t^*)$ can be viewed in two distinct areas in which one deals with the approximated linear relationship due to the minute step size in space and time (described by [17] (Wang and Rusak 1996a), and the inclusion of a non-linear term into the original linear relationship to approximately simulate the dynamics under non-linear perturbation conditions, the weakly non-linearity stemmed from the simplification during derivation, where higher order ϵ terms were dropped due to the small effect it has on the dynamics for near Benjamin's critical swirl levels. Detailed analysis of the non-linear perturbation equation can be found in [12] (Rusak, Wang, Xu and Taylor 2011).

3.2 Analysis on linearized asymptotic equation

Since this paper deals mostly with linear perturbations for a solid body rotation flow, the non-linear element of the reduced equation can be taken out by noting that the variable $\alpha = 0$ which is produced by solid body rotation flow, this result came from substituting the solid body rotational base flow into the variables, if we let $w_0 = 1$ as a convention for numerical analysis and to simplify the equation, the SBR base flow variables are: $\psi_0 = y, \chi_0 = 0, K_0 = \omega\tilde{K}_0, \tilde{K}_0 = 2y$, then α, β and τ becomes:

$$\alpha = \frac{N_1}{\delta} = -\frac{1}{2}\int_0^{\frac{1}{2}} \left(\frac{\omega_B^2}{y}\left(\frac{4y}{y}\right)_y\right)\phi_B^3 dy \div \int_0^{1/2} \frac{\phi_B^2}{2y} dy = 0,$$
$$\beta = \frac{N_2}{\delta} = \int_0^{\frac{1}{2}} \frac{4y}{2y^2}\phi_B^2 dy \div \int_0^{1/2} \frac{\phi_B^2}{2y} dy = 4,$$
$$\tau = \frac{N_S}{\delta} = \int_0^{\frac{1}{2}} \left(\omega_B^2 \frac{4y}{y^2}\right)\phi_B^2 dy \div \int_0^{1/2} \frac{\phi_B^2}{2y} dy = 8\omega_B^2 = 2\Omega_B.$$

$\alpha = 0$ is not to say that SBR flow with higher order perturbations behaves linearly, but that for the non-linear effect to occur, the asymptotic equation would have to include higher order ϵ terms to depict this effect (the linear asymptotic equation for SBR can accurately predict non-linear SBR behavior since SBR has very weak non-linear effect when subjected to perturbations). This is essentially the same as if the asymptotic equation has been derived using linear perturbations, details of which can be seen in [16] (Wang & Rusak 2011). Since non-linearity for SBR only manifest itself for large perturbation, the dynamics for linear and weakly on-linear equation generates the same growth rate curve. Therefore it is more convenient and accurately enough to work with the linearized equation instead of that of the non-linear for SBR flow.

The similarity of the growth rate for both equations would suggest that the two equations should behave in the same manner for all growth rate branches, due to the fact that SBR has very weak non-linear behavior, which is caused by small perturbation producing an α of zero that linearizes the non-linear asymptotic equation. So instead of adding further constraints or changing certain conditions on an equation which cannot be analytically solved for, we will use the simplified linear equation instead, since the focus of the asymptotic equation of this paper is on solid body rotation flow to begin with. In the next chapter, a control parameter will be added to the equation, to see if the growing perturbation is able to be quenched.

The linearized equation is :

$$\tau A_{t^*} = A_{XXX} + k_\omega \beta A_X, \tag{3.10}$$

by using a suitable engenmode solution $A(X, t^*) = \bar{A}(X) \exp(\sigma^* t^*)$, the asymptotic motion equation can be transformed into the asymptotic stability equation:

$$\sigma^* \tau \bar{A} = \bar{A}_{XXX} + k_\omega \beta \bar{A}_X, \tag{3.11}$$

where $\beta = 4$, and $\tau = 8\omega_B^2$ when substituted for solid body rotational scheme. This stability equation is analyzed along with the non-linear asymptotic stability equation.

To ensure the accuracy of linear asymptotic equation with the linear WR equation, both are calculated numerically and plotted under the same solid body rotation base flow (parameters set the same as section 2.5) and compared:

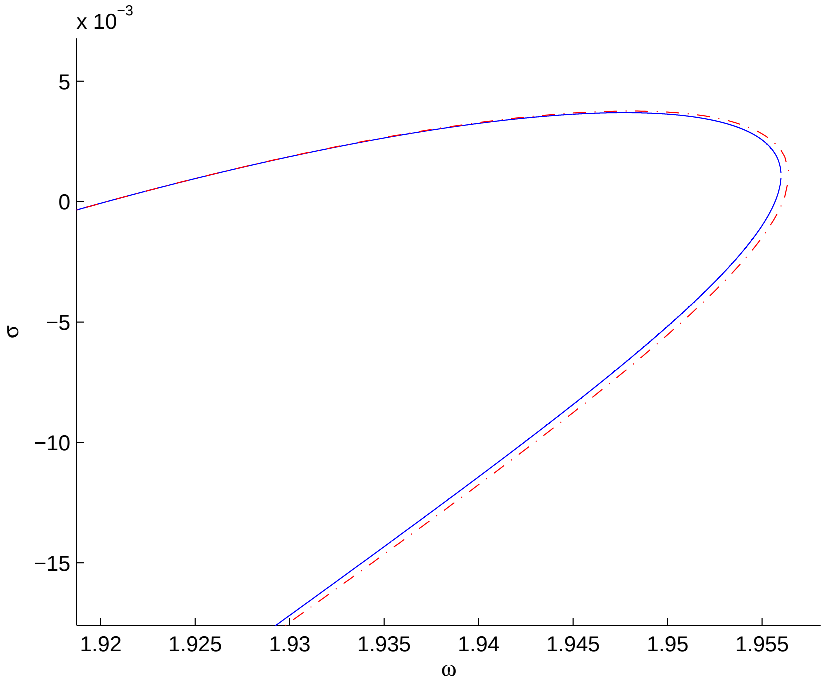

Figure 3.1: Growth rate σ comparison (first branch).

The blue solid line represents the linear WR equation, and the red represents the linear asymptotic equation. Although the growth rate of the first branch of both the asymptotic and WR equation exhibits very little difference, subsequent branches display a much more significant difference, this lack of accuracy is contributed to applying the asymptotic approach to the stability equation i.e. the smaller terms of the equation were dropped after rescaling using ϵ. Since ϵ is of the order $\omega - \omega_B$, the difference in subsequent branches becomes much larger as ω moves further away from ω_B .

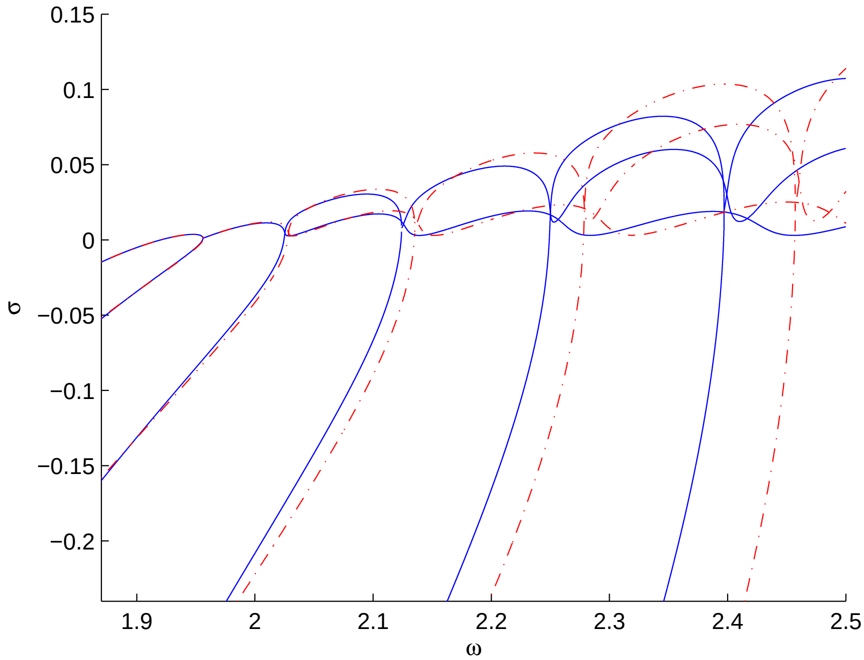

Figure 3.2: Growth rate σ comparison (large swirl).

This lack of accuracy for the subsequent branches is what prompted the main focus of this research paper, that although the linear asymptotic equation's perturbation can be totally controlled by the feedback stabilization method, the status of the same control onto the linear WR equation with the same base flow is currently unknown, therefore we would like to investigate if the control method introduced in the next chapter on the asymptotic equation can be translated onto the linear WR equation, and whether or not the method would yield similar results for both the asymptotic and the WR equation. In [12] (Rusak, Wang, Xu & Taylor 2011) it was shown that non-linear asymptotic equation demonstrated uncontrollable perturbations, the controllability depended on two parameters, α and δ, where α relates to the degree of non-linearity and δ is the amount of perturbation present. The combination of the two together ultimately dictates if instability can be delayed or sped up, but the presence of non-linearity signified that eventually the stability always cannot be controlled by feedback stabilization (detailed in next chapter). However, without the non-linearity by using SBR flow (for sufficiently small perturbation), it can be found that the perturbation is controllable by meeting a certain requirement. Since linear asymptotic equation is a reduction of the full equation, it was generally predicted that reduction of full equation may have the same effect as linearization, which may produce uncontrollable instability, when the same control is applied to the linear WR equation. Fortunately, the same control is able to stabilize the vortex even for the linear WR equation, which shall be discussed in the next chapter.

Chapter 4

Feedback Control Analysis

The theorization of the asymptotic equation greatly reduces the complexity of the linear WR stability problem, this allowed for easier application of constraints on to the stability equation. Therefore we seek some control variable to quench the growing perturbation, since the use of non-periodic boundary conditions produces instability for solid body rotational flow. Theorization will start by noticing the onset of instability relates directly to the production of kinetic energy of the flow (for details on transfer of kinetic energy see [19] (Wang, Taylor & Ku Akil 2009) and [16] (Wang & Rusak 2011)), therefore a control parameter that reduces the production of kinetic energy was introduced [12] (Rusak, Wang, Xu & Taylor 2011). Analysis showed that control parameter can successfully control the perturbation for solid body rotational flow for the linear asymptotic equation (provided that the control parameter meets certain requirements), however the asymptotic equation is only accurate for low swirl levels. In this paper the effect of the control parameter on the linear WR stability equation will be investigated to see if the control can still be effective as the swirl increases to the second and third branch and so forth, this is done by developing a new method which the effect of control is enforced by adjusting the circulation at the inlet, this novel approach gives physical meaning to the control parameter, and feasibility of application.

4.1 Control implementation

Now that instability of solid body rotation has been established using boundary conditions that reflects realistic pipe conditions, it is desirable to find ways that are able to diminish this growing effect experienced by perturbations that eventually leads to vortex breakdown. Studies in [16] (Wang & Rusak 2011) on the energy transfer mechanism of the perturbed flow on the linear WR stability equation indicated that the transfer of kinetic energy played an important role in the dynamics of the vortex. At near critical swirl level, the production and loss of kinetic energy of the perturbed flow may ultimately determine whether the onset of vortex breakdown can be delayed. Studies suggested that for general flows such as Lamb-Oseen or Q vortex, even though the flow is at near critical level, changes such as pipe contraction and compressibility increases the loss of kinetic energy which leads to a delay of vortex breakdown, while mechanisms such as slight viscosity, inlet vorticity disturbances add to production of kinetic energy, which ultimately promotes the onset of vortex instability before swirl reaches the critical level. The importance of this mechanism leads to further investigation in [12] (Rusak, Wang, Xu, and Taylor 2011) and [16] (Wang & Rusak 2011), this time conducted on the asymptotic equation. It stands to reason that an effective way to delay the onset of instability will need to be investigated in an energy point of view, which led to the implementation of feedback stabilization, involving a control parameter that can be tweaked in order to produce a net loss of kinetic energy eventually leading to stable flow.

Consider equation (3.9), we will now introduce a control term $u(t^*)$ into the equation:

$$\underbrace{\tau A_{t^*}}_{[A]} = \underbrace{A_{XXX}}_{[B]} \underbrace{-\alpha(A^2)_X}_{[C]} \underbrace{+k_\omega \beta A_X}_{[D]} \underbrace{+u(t^*)}_{[E]}, \tag{4.1}$$

and its boundary conditions:

$$
\begin{aligned}
&A(0,t^*) = A_{XX}(0,t^*) = A_X(1,t^*) = 0, \quad t^* \geq 0, \\
&A(X,0) = f(X), \quad \text{for} \quad 0 < X < 1.
\end{aligned}
$$

It is desirable to determine a function $u(t^*)$ so that the energy will decay to zero or stay bounded in some sense (preferably in an exponential relationship). To calculate the energy, we multiply equation (4.1) with the multiplier A_{XX} and integrate over the rectangle R defined by $0 < X < 1$ and $0 < t < T$:

$$
\begin{aligned}
\underbrace{\int\int_R \tau A_{XX} A_{t^*} dX dt^*}_{[\text{A}]} &= \tau \int_0^T \left([A_X A_{t^*}]_{X=0}^{X=1} - \int_0^1 A_X A_{Xt^*} dx \right) dt \\
&= -\tau \int\int_R A_X A_{Xt^*} dX dt^* \\
&= -\tau \int\int_R \frac{1}{2} \frac{\partial}{\partial t^*} (A_X)^2 dX dt^* \\
&= E(0) - E(T),
\end{aligned}
$$

where we define the associated kinetic energy to be

$$
E(t^*) = \frac{\tau}{2} \int_0^1 (A_X)^2 dX, \tag{4.2}
$$

also,

$$
\begin{aligned}
\underbrace{\int\int_R A_{XX} A_{XXX} dX dt^*}_{[\text{B}]} &= \int_0^T \frac{1}{2} [(A_{XX})^2]_{X=0}^{X=1} dt^* \\
&= \frac{1}{2} \int_0^T (A_{XX}(1,T))^2 dt^*,
\end{aligned}
$$

$$\underbrace{-\alpha\int\int_R (A)^2_X A_{XX}dXdt^*}_{[C]} = -2\alpha\int\int_R AA_XA_{XX}dXdt^*$$
$$= -\alpha\int_0^T \left([A(A_X)^2]_{X=0}^{X=1} - \int_0^1 (A_X)^3 dX\right)dt^*$$
$$= \alpha\int\int_R (A_X)^3 dXdt^*,$$

$$\underbrace{k_\omega\beta\int\int_R A_{XX}A_XdXdt^*}_{[D]} = \frac{1}{2}k_\omega\beta\int_0^T [(A_X)^2]_{X=0}^{X=1}dt^*$$
$$= -\frac{1}{2}k_\omega\beta\int_0^T (A_X(0,t^*))^2dt^*,$$

$$\underbrace{\int\int_R A_{XX}u(t^*)dXdt^*}_{[E]} = -\int_0^T A_X(0,t^*)u(t^*)dt^*,$$

leads to the following relationship in energy:

$$E(T) - E(0) = -\frac{1}{2}\int_0^T (A_{XX}(1,t^*))^2dt^* - \alpha\int\int_R (A_X)^3dXdt^* + \frac{1}{2}k_\omega\beta\int_0^T (A_X(0,t^*))^2dt^* + \int_0^T A_X(0,t^*)u(t^*)dt^*. \quad (4.3)$$

The first term of the right hand side of equation (4.3) can be seen to always have a stabilizing effect (always produce negative change in energy). The second term however can have both stabilizing and destabilizing effect. When current swirl ω_1 is smaller than critical swirl ω, A_X is positive which will create a stabilizing effect, but at current swirl larger than critical swirl, A_X is negative along most of the pipe, the term transforms from loss of energy to production of energy. The third term depends on the difference between swirl and Benjamin's critical swirl. Notice that there is competition between the second and the third term, since stabilizing of one term would mean destabilizing of the other term due to k_ω involved in the third term. Lastly the fourth term is the one with the control parameter implemented; it must be sufficiently large to overcome the total destabilizing effect produced by the second and third term.

To investigate the control parameter, we will look at the case of $\alpha = 0$ which is directly applicable to solid body rotation flow, and linearize the equation;

$$\begin{aligned} E(T) - E(0) = &- \frac{1}{2}\int_0^T (A_{XX}(1,t^*))^2 dt^* \\ &+ \frac{1}{2}k_\omega \beta \int_0^T (A_X(0,t^*))^2 dt^* + \int_0^T A_X(0,t^*)u(t^*)dt^* \\ &\leq \frac{1}{2}k_\omega \beta \int_0^T (A_X(0,t^*))^2 dt^* + \int_0^T A_X(0,t^*)u(t^*)dt^*. \end{aligned} \tag{4.4}$$

We will introduce a feedback law here for the control parameter:

$$u(t^*) = -\frac{1}{2}\gamma A_X(0,t^*), \tag{4.5}$$

where $\gamma > 0$ is some constant representing control gain, so now the relation (4.4) is:

$$E(T) \leq E(0) + \frac{1}{2}(k_\omega \beta - \gamma)\int_0^T (A_X(0,t^*))^2 dt^* \leq 0, \tag{4.6}$$

provided that γ is sufficiently large, and $\gamma \geq \ k_\omega\beta$, we can achieve $E(0) > E(T)$, and the overall energy will be non-increasing. This feedback stabilization method shows that provided the correct control parameter is chosen, the perturbation can be effectively controlled for all swirl levels as long as the dynamics are under linear regime. As previously mentioned, by adding a control function into the evolution equation, one would hope that the kinetic energy calculated would decrease to zero as time goes to infinity: $E(T) \to 0, t^* \to \infty$. The best case scenario would be that the energy decays uniformly in an exponential relationship, in which case there are positive constants C and μ such that:

$$E(t^*) \leq CE(0)e^{-\mu t^*}$$

for all situations with finite energy.

4.2 Alternative control parameter representation

In the previous section the control parameter was introduced, and the derivation was based on feedback stabilization introduced within the equation. Although the parameter itself is essentially a constant (subjecting to initial condition $A_X(0)$), it is very difficult to implement physically while the parameter is inside the equation as it is the same as adjusting the flow inside of the pipe. Therefore we seek an alternative representation of the same control parameter, turning it from a control parameter inside the pipe to a boundary condition at the inlet of the pipe. By doing so the control parameter gains physical meaning, the flow can now be changed at the inlet of the pipe to adapt to the situation that is needed to stabilize the perturbation.

Let the pure time component be A^0, the equation is then divided into two sections where A^0 denotes time portion of the pipe and A^L denotes the spatial portion of the pipe, and $A = A^0 + A^L$ is then the superposition of the time and space of the pipe dynamics. Since A_{XXX} and A_X does not involve time variation terms the pure time portion of the equation is then:

$$\begin{aligned} \tau A^0_{t^*} &= u(t^*) = -\frac{1}{2}\gamma A_X(0,t^*) \\ \Rightarrow \int_0^t \tau A^0_{t^*} dt &= -\int_0^t \frac{\gamma}{2} A_X(0,t^*)dt \\ \Rightarrow A^0 &= -\int_0^t \frac{\gamma}{2\tau} A_X(0,t^*)dt, \end{aligned} \tag{4.7}$$

by setting $\alpha = 0$ in equation (4.1), the linearized controlled asymptotic equation is:

$$\tau A_{t^*} = A_{XXX} + k_\omega \beta A_X + u(t^*), \tag{4.8}$$

substitute in $A = A^0 + A^L$ into equation (4.8):

$$\begin{aligned} \tau(A^0 + A^L)_{t^*} &= (A^0 + A^L)_{XXX} + k_\omega \beta (A^0 + A^L)_X + u(t^*) \\ &\Rightarrow \tau A^L_{t^*} + \tau A^0_{t^*} = A^L_{XXX} + k_\omega \beta A^L_X + u(t^*). \end{aligned} \tag{4.9}$$

Since $\tau A^0_{t^*} = u(t^*)$, we can cancel the two terms in (4.9), yielding:

$$\tau A^L_{t^*} = A^L_{XXX} + k_\omega \beta A^L_X. \tag{4.10}$$

Now to redefine the boundary condition $A(0, t^*) = 0$:

$$A^0(t^*) + A^L(0, t^*) = -\int_0^t \frac{\gamma}{2\tau} A_X(0, t^*)dt + A^L(0, t^*) = 0$$

$$\Rightarrow A^L(0, t^*) = \int_0^t \frac{\gamma}{2\tau} A_X(0, t^*)dt$$

$$\Rightarrow A^L_{t^*}(0, t^*) = \frac{\gamma}{2\tau} A^L_X(0, t^*), \tag{4.11}$$

other boundary conditions attached with the linearized asymptotic equation remains the same. To analyze this new relation that transformed the feedback stabilization into an initial control problem we utilize normal mode analysis to seek solution(s), let $A^L = a(X)e^{-i\sigma^* t^*}$, this is used to decouple the time and spatial component of the equation. Substitute this into equation (4.10):

$$-i\tau\sigma^* a(X)e^{-i\sigma^* t^*} = a_{XXX}(X)e^{-i\sigma^* t^*} + k_\omega \beta a_X(X)e^{-i\sigma^* t^*}$$
$$\Rightarrow -i\tau\sigma^* a(X) = a_{XXX}(X) + k_\omega \beta a_X(X), \tag{4.12}$$

we can therefore obtain the stability equation with new control parameter.

Now an alternative approach has been obtained, the control parameter has been shifted from being inside of the pipe to the beginning of the pipe. If we continued with derivation such as what has been demonstrated in section 1.4, a similar matrix can obtained, due to the third order nature of the asymptotic stability equation, the size of the matrix will be 3×3 instead of 4×4. However the matrix analysis fails due to a significant error produced in the formulation, once the matrix is formed, the some of the elements involved in the matrix are divided by σ^*, i.e., the matrix analysis approach fails at critical swirls. This is a significant hurdle that needs to be overcome since with the control parameter added, we would like to analyze when the growth rate branches will be completely controlled, meaning that the maximum point of a growth rate branch will be at exactly zero, so as the swirl gets closer to the critical swirl, the matrix approach produces inaccuracies due to σ^* tending towards zero.

To overcome this division by zero problem with σ^*, we will now use shooting method matching the boundary conditions with the differential equation directly to calculate the solution instead of the determinant method; first convert the differential equation into a system of equations, let $Z_1 = a$, $Z_2 = a_X$, and $Z_3 = a_{XX}$:

$$\begin{aligned} \frac{dZ_1}{dX} &= Z_2, \\ \frac{dZ_2}{dX} &= Z_3, \\ \frac{dZ_3}{dX} &= \sigma^* \tau Z_1 - k_\omega \beta Z_2, \end{aligned} \tag{4.13}$$

with the initial conditions:

$$a(0) = \frac{\gamma}{2\tau\sigma^*} a_X(0), \qquad a_X(0) = 1, \qquad a_{XX}(0) = 0, \tag{4.14}$$

At the beginning of the shooting method, an arbitrary σ^* will be chosen (should still be sufficiently close to actual value) to shoot for the outlet condition with a range of swirl values. The system of equations will be solved according to this preset σ^* and the control parameter γ, then the system of equations will be solved again, but this time with a new σ^* stepped by an infinitesimal amount $\Delta\sigma^*$. This is done so that the Newton-Raphson method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

can be utilized to obtain an accurate value for the true σ^* for the swirl chosen shooting for the outlet condition. Numerically represented, $f(x_n)$ is equivalent to the solution calculated using the preset σ^*, and the difference between the second solution for the system of equations and the first solution divided by the step in σ^*, which is $\Delta\sigma^*$, represent the finite difference approximation of $f'(x_n)$. x_n represents the previous σ^* (or the preset σ^* for the first step), take away the fraction to obtain the new (more accurate) σ^* which is x_{n+1}. This procedure is set within a loop to improve the accuracy of the growth rate calculated until the accuracy has surpassed a certain expectation, then the entire process is executed again for the next swirl inside the range of swirls to produce an accurate (to a certain degree) growth rate branch. With this method not only can we solve the problem but we can also calculate the progression of the eigenmodes much easier with better accuracy.

The control parameter $u(t^*)$ stated in equation (4.5) along with the condition that $\gamma \geq k_\omega \beta$ suggests that when $\gamma = k_\omega \beta$, the perturbation can be just totally controlled, and the energy of the perturbation will be bounded. If we place 1/2 together with $k_\omega \beta$ (essentially $\gamma/2$) to be one variable as per the derivation of this section, we can notice that the roles of constants and variables can be exchanged (for the purpose of analysis). $\gamma = k_\omega \beta$ can be kept as the constant depending on the swirl given, and the control gain can now be the original constant 1/2. So for the case when the equation is just totally controllable, there must exist some control gain that has to equal to 1/2, and for anything larger can be viewed as $\gamma > k_\omega \beta$. So instead of changing γ, one can change the constant attached to it instead.

4.3 Numeric prediction theory

To have a sense of whether the numerical methods are calculating correct and accurate results, the critical swirls which generates zero point σ^* can be determined theoretically, therefore to have an indication of where and how the eigenmodes will proceed with control parameter, we can try to find when the control parameter will cause the critical swirls to be exactly the maximum value of each branch, indicating exact control, and subsequently higher values of the control parameter will also completely control the growth rate. To find the control parameter value that first completely controls the growth rate branch in the asymptotic equation, we look at the original asymptotic control problem, instead of the converted boundary condition control:

$$A_{XXX} + k_\omega \beta A_X - \sigma^* \tau A - \frac{\gamma}{2} A_X(0) = 0$$
$$A_{XX}(0) = 0 \qquad A_X(1) = 0 \qquad A(0) = 0$$

Another way to view this equation is to see this as an inhomogeneous differential equation, since $A_X(0)\gamma/2$ is a constant, and $A_X(0) = 1$. To solve this equation, we will take the approach of forming a characteristic polynomial and a particular integral to obtain a full solution using superposition. First the characteristic polynomial, e^{aX}:

$$a^3 e^{aX} + k_\omega \beta a e^{aX} - \sigma^* \tau e^{aX} = 0$$

$$\Rightarrow a^3 + k_\omega \beta a - \sigma^* \tau = 0$$

since we want to find the critical swirls for which the growth rate $\sigma^* = 0$:

$$a^3 + k_\omega \beta a = 0 \quad \Rightarrow \quad a(a^2 + k_\omega \beta) = 0$$

so $a = 0$, and $a = \pm\sqrt{k_\omega \beta} i$, now form the characteristic equation:

$$A(X) = C_1 \sin \sqrt{k_\omega \beta} X + C_2 \cos \sqrt{k_\omega \beta} X + C_3$$

by fitting initial conditions $A_{XX}(0) = 0$ and $A(0) = 0$, C_2 and C_3 can be eliminated, let $\alpha = \frac{\gamma}{2}$ and combining with particular integral to obtain:

$$A(X) = C_1 \sin \sqrt{k_\omega \beta} X + \frac{\alpha}{k_\omega \beta} X$$

In the derivation it was assumed that $A_X(0) = 1$ and $A_X(1) = 0$, we will now fit the solution with the two equation:

$$\text{For} \quad A_X(0) = 1 \qquad \Rightarrow C_1 \sqrt{k_\omega \beta} \cos(0) + \frac{\alpha}{k_\omega \beta} = 1$$

$$\Rightarrow C_1 = \left(1 - \frac{\alpha}{k_\omega \beta}\right) \frac{1}{\sqrt{k_\omega \beta}}$$

$$\text{For} \quad A_X(1) = 0 \qquad \Rightarrow C_1\sqrt{k_\omega\beta}\cos(\sqrt{k_\omega\beta}) + \frac{\alpha}{k_\omega\beta} = 0$$

$$\Rightarrow \cos(\sqrt{k_\omega\beta}) = -\frac{1}{C_1}\frac{\alpha}{(k_\omega\beta)^{\frac{3}{2}}}$$

combining the two equations together to obtain:

$$\cos(\sqrt{k_\omega\beta}) = -\frac{\alpha}{(k_\omega\beta)^{\frac{3}{2}}}\frac{1}{\frac{1}{\sqrt{k_\omega\beta}} - \alpha(k_\omega\beta)^{-\frac{3}{2}}}$$

$$= \frac{-\alpha}{k_\omega\beta - \alpha}$$

$$\Rightarrow \cos(\sqrt{k_\omega\beta}) = \frac{\alpha}{\alpha - k_\omega\beta}$$

since $\alpha = \gamma/2$ we let $\delta = 1/2$ so that $\gamma/2 = \delta k_\omega\beta$, substituting into the above equation:

$$\cos(\sqrt{k_\omega\beta}) = \frac{\delta k_\omega\beta}{\delta k_\omega\beta - k_\omega\beta} = \frac{\delta}{\delta - 1} \tag{4.15}$$

The left hand side of (4.15) is a cosine function with a range limit of -1 or 1, to find the case when the eigenvalues are just totally controlled (max $\sigma^* = 0$), equate $\delta/(\delta - 1) = -1$, to find $\delta = 1/2$. The control parameter is now changed to a constant, at exactly half, the eigenvalues are all exactly controlled for the third order asymptotic equation, for cases smaller than half, the eigenvalues will be unable to be controlled since $\delta/(\delta - 1) < -1$ producing more zero points indicating the existence of positive eigenvalues. If the control parameter is larger than half, $\delta/(\delta - 1) > -1$, the invalid equation indicates that there exists no zero σ^* points, and the eigenvalues are all controlled. Counter intuitively, excess amount of control actually reduces the effectiveness of feedback stabilization, with the best control scenario at $\delta \approx 1$, the closer it is to one, the more effective the control parameter is. So to ensure the accuracy of the growth rate branches using numerical methods we will now calculate the critical swirls:

$$\cos(\sqrt{k_\omega \beta}) = -1$$

this indicates that zero points will occur at odd number of π:

$$k_\omega = \frac{2\omega_B(\omega - \omega_B)}{\epsilon} = \frac{(n\pi)^2}{4}$$

$$\Rightarrow \omega = \frac{\epsilon(n\pi)^2}{8\omega_B} + \omega_B$$

where $n = 1, 3, 5, 7.....$, the position number of each element is representative of the branch number. Since $\omega_B = 1.915855$ according to the above formula, the first zero point should occur at $\omega_1 = 1.933743$, the zero point of the second branch at $\omega_2 = 2.076841$ and the third zero point will be at $\omega_3 = 2.363037$.

4.4 Effectiveness of control (Asymptotic)

With the shooting method, and the alternative representation of the control parameter, we are able to accurately produce the growth rate branches by finding a particular growth σ^* and perform shooting for the outlet condition with the matching swirl. Numerical analysis will start from control parameter of 0.5, since in the last section it was noted that at 0.5 the linear asymptotic equation should be able to be just completely controlled (since $k_\omega \beta/2$ is the minimum requirement for control), therefore we should expect to see all growth rate branches falling below the zero mark. For the case of control $= 0.5$:

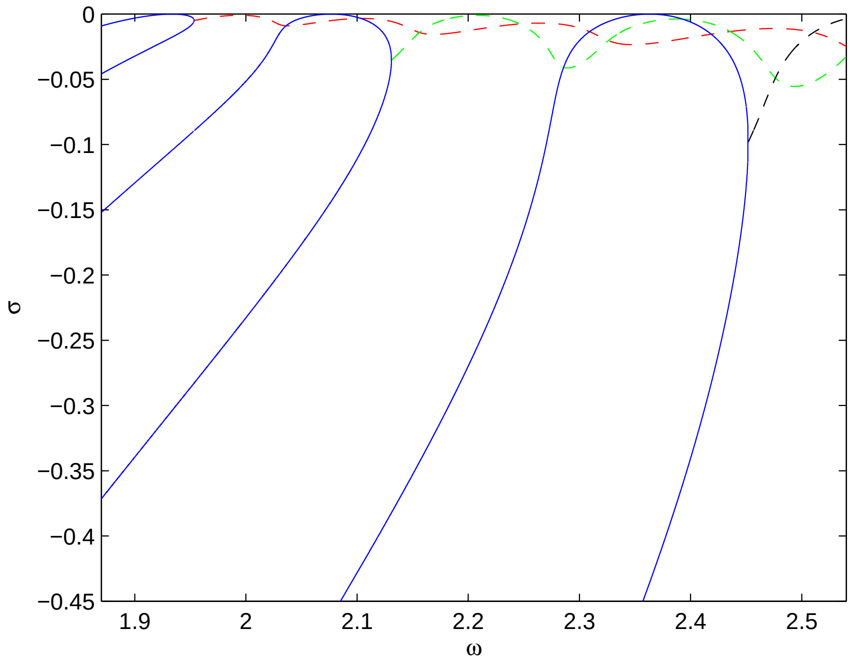

Figure 4.1: Swirl ω $(= \sqrt{\Omega}/2)$ against growth rate σ comparison (Control=0.5)

This is the case as stated in the original theory of feedback control, that in order to fully control the growth rate, the control parameter must be some constant that is $\geq k_\omega \beta/2$. So for control parameter set at exactly $k_\omega \beta/2$, we should be able to see the growth rates being just completely controlled as can be seen in the above figure. Although this case has been proven in the derivation of the control parameter, it is reassuring to see numerical interpretation matching the theory. It can also be noted that the complex branches developed from the respective real branches are totally controlled as well, and the complex branches exhibit slight damping behavior.

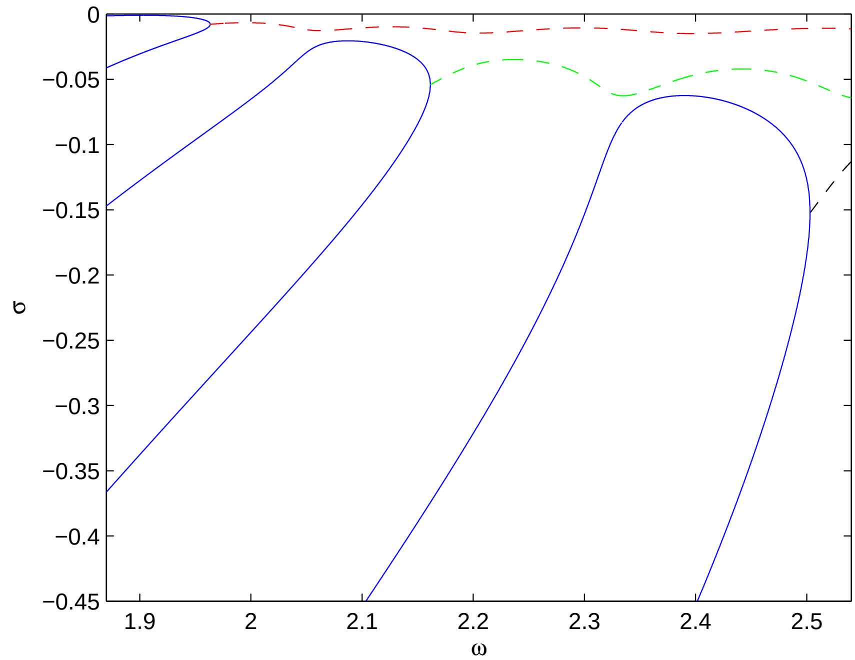

Figure 4.2: Swirl ω ($= \sqrt{\Omega}/2$) against growth rate σ comparison (Control=0.95)

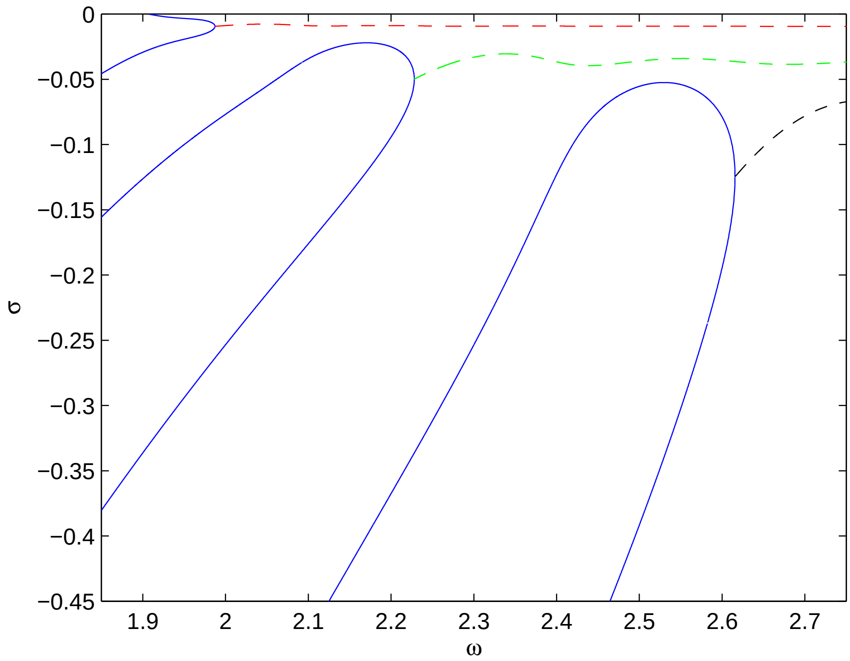

Figure 4.3: Swirl ω $(= \sqrt{\Omega}/2)$ against growth rate σ comparison (Control=1.3)

Figures 4.2 and 4.3 show the behavior of the growth rate branches near the control parameter of one (or $\gamma = 2k_\omega\beta$) for the case of solid body rotation. The figures show the first three branches of the growth rate, notice that as the control parameter is increased from 0.5; the branches are being stretched and transported down the swirl axis. Figure 4.2 showed a significant lowering of the growth rate branches, signifying the effectiveness of the control parameter increases as the control parameter itself increase. Although figure 4.3 seems to suggest that with a control parameter of 1.3, the overall effect of the control seems to be weaker than figure 4.2 with a control parameter of 0.95, the two figures does not exhibit a significant amount of difference, both control gains can still effectively control the perturbation growth rate. The two figures indicate that there is some optimal control gain that exists between 0.95 and 1.3, which is calculated to be one in the previous section.

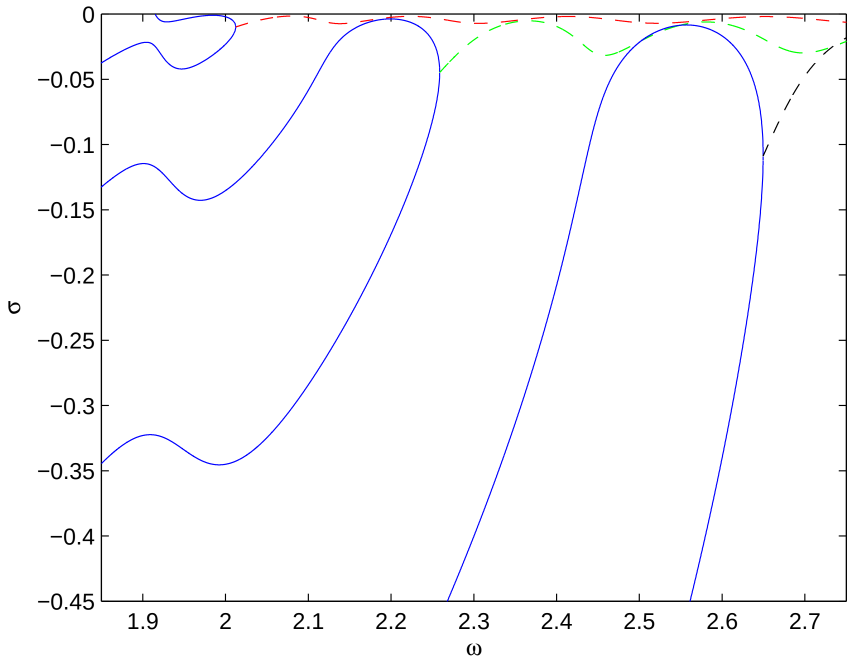

Figure 4.4: Swirl ω ($= \sqrt{\Omega}/2$) against Rescaled growth σ comparison (Control=5)

Figure 4.4 showed the dynamics of the growth rate branches with a control parameter of 5. We can see that this figure exhibits significantly less branch lowering than the three previous scenarios, the effectiveness of control is approaching the case of control $= 0.5$, this showed that the effectiveness of control is significantly reduced as it passes a particular value, but as the control increase, the branches are being transferred further down the swirl axis, this suggests that the larger the control parameter, the larger the swirl is needed to move from the first branch to later branches. This is important particularly since we know that the first growth rate branch is accurate enough to represent the linear WR stability equation, so the dynamics of the controlled asymptotic equation should be the exact same (or similar) for the controlled linear WR equation. Complete control of the first branch would suggest complete control for the WR equation as well, even if we don't know the dynamics of later branches for the WR equation, increasing the control parameter transfers the first branch along the swirl axis, and larger swirls are needed to reach later branches that may still have positive growth rate, hence delaying the onset of instability. However, control gain of 5 still produced a growth rate branch plot that indicates significant loss in the effectiveness of control, signifying that excessive control is actually unnecessary and may actually reduce the effect.

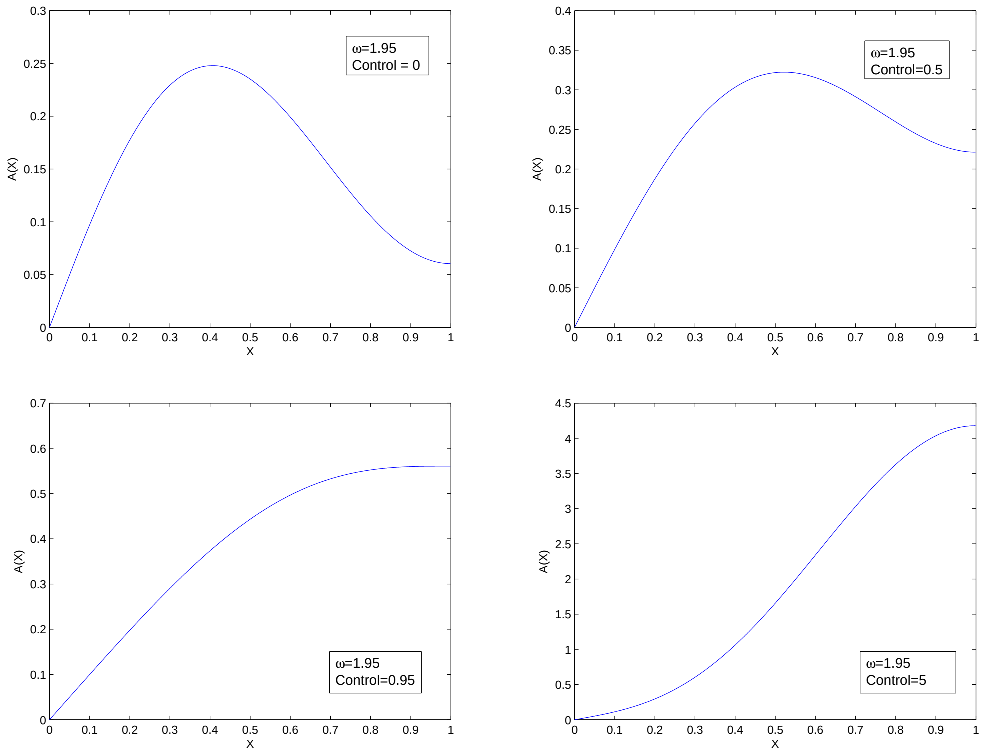

Figure 4.5: Eigenmode A(X) for specific swirl ω (for control $= 0, 0.5, 0.95, 5$ left to right).

The four plots show the evolution of the eigenfunction with a specific swirl value (hence a specific growth rate), the figure shows normal evolution with no control, as the swirl increases, the mode shape starts to change, the increase in swirl causes the mode shape to create more bends as can be seen in plot with a control of zero. By placing a control parameter, the mode shape behavior clearly exhibit the transportation effect. As the control parameter is increased, the mode shape creates less bends and starts to resemble the shape of modes with smaller swirl when there is no control, although the negative growth rate cannot be easily seen from the mode shape plots, the transportation of the growth rate branches down the swirl axis can be seen clearly.

4.5 Linear WR motion equation with control

In section 3.2 it was shown that there exist differences between the growth rate branches of the asymptotic equation and the linear WR stability equation. The level of difference for each branch increases in magnitude as the current swirl moves further away from Benjamin's critical swirl (ω_B), however the first branch was deemed accurate enough due to the difference between the two equations to be insignificant when compared to branches further along the swirl axis. This allowed the asymptotic equation to provide an accurate depiction of the dynamics of the linear WR stability equation for at most the first growth rate branch and nearby swirl levels i.e., $\omega - \omega_B$ is small enough to be in the order of ϵ. Section 4.1 saw the introduction of a control parameter in hoping that the growth rate σ can in some way be suppressed, delaying the onset of vortex instability. Theoretical analysis has proven that for the linear asymptotic equation ($\alpha = 0$ equivalent to solid body rotational flow) the first growth rate branch can in fact be totally controlled, providing that the control parameter meets the condition $\gamma \geq k_\omega \beta$. This is particularly interesting since the first growth rate branch of the asymptotic equation is near identical to that of the linear WR equation, therefore similar dynamics should govern both equations for the first growth rate branch. We will look for a way to control the linear WR problem by altering the inlet condition, since this is more practical to implement than involving the motion equation itself with the control parameter, and through the inlet condition that involves the control we are able to investigate whether or not this method allows for stability of the linear WR problem. We will attempt to control the perturbation by changing the circulation of the inlet flow, the adjustment will be according to the dynamics of the flow itself, exhibiting feedback effect, to deduce the requirement for complete control to occur for the perturbation and what to set for the inlet circulation for stability, the motion equation for solid body rotation flow and its boundary conditions needs to be determined first. We start by revisiting (2.15):

$$\left(\psi_{1yy} + \frac{\psi_{1xx}}{2y}\right)_{xx} + \frac{2}{\psi_{0y}}\left(\psi_{1yy} + \frac{\psi_{1xx}}{2y}\right)_{xt} + \frac{1}{\psi_{0y}^2}\left(\psi_{1yy} + \frac{\psi_{1xx}}{2y}\right)_{tt} + \frac{\chi_{0y}}{\psi_{0y}^2}\psi_{1xt} - \left(H'(\psi;\Omega) - \frac{I'(\psi;\Omega)}{2y}\right)\psi_{1xx} = 0,$$

the focus of this research is on the stability of the solid body rotational flow subjected to disturbances, therefore we shall simplify the above equation through substituting parameters for solid body rotation base flow. By substituting $u = 0$, $v = \omega r$, and $w = w_0$ (hence $\psi_0 = y$, $K_0 = 2\omega y$, and $\chi_0 = 0$), (2.15) is simplified and separation of variables is now possible. w_0 is chosen to be one as a convention and to further simplify the equation, after substitution we obtain:

$$\left(\psi_{1yy} + \frac{\psi_{1xx}}{2y}\right)_{xx} + 2\left(\psi_{1yy} + \frac{\psi_{1xx}}{2y}\right)_{xt} + \left(\psi_{1yy} + \frac{\psi_{1xx}}{2y}\right)_{tt} + \frac{4\omega^2}{2y}\psi_{1xx} = 0. \tag{4.16}$$

Now we look for a separation of variables solution $\psi(x, y, t) = \Phi(y)\varphi(x, t)$, where Φ is a function of Benjamin's eigenvalue problem (2.13). Substitute the separation of variables solution into (4.17);

$$\left(\Phi_{yy}\varphi + \frac{\Phi\varphi_{xx}}{2y}\right)_{xx} + 2\left(\Phi_{yy}\varphi + \frac{\Phi\varphi_{xx}}{2y}\right)_{xt} + \left(\Phi_{yy}\varphi + \frac{\Phi\varphi_{xx}}{2y}\right)_{tt} + \frac{4\omega^2}{2y}\Phi\varphi_{xx} = 0, \tag{4.17}$$

notice that the same substitution of $\Phi_{yy} = -\Omega_B/2y\Phi$ can be used following the derivation of section 2.4, apply the substitution to obtain an equation with the elimination of y component;

$$
\begin{aligned}
&-\Omega_B\varphi_{xx}+\varphi_{xxxx}-2\Omega_B\varphi_{xt}+2\varphi_{xxxt}-\Omega_B\varphi_{tt}+\varphi_{xxtt}+\Omega\varphi_{xx}=0\\
&\Rightarrow\quad \varphi_{xxxx}+(\Omega-\Omega_B)\varphi_{xx}+2\varphi_{xxxt}+\varphi_{xxtt}-2\Omega_B\varphi_{xt}-\Omega_B\varphi_{tt}=0. \qquad (4.18)
\end{aligned}
$$

The exact same procedure can be performed on equation (2.16), substitute for solid body rotational flow, and then utilize a separation of variables solution $\psi_1(x,y,t)=\Phi(y)\varphi(x,t)$, $K_1=\Phi(y)k(x,t)$, and Φ_{yy}:

$$
\begin{aligned}
&\Phi k_t=-\frac{2y^2}{2\omega y}\left[-\left(-\frac{\Omega_B}{2y}\Phi\varphi+\frac{\Phi\varphi_{xx}}{2y}\right)_t+\left(-\frac{\Omega_B}{2y}\Phi\varphi+\frac{\Phi\varphi_{xx}}{2y}\right)_x\right]+2\omega\Phi\varphi_x\\
&\Rightarrow\quad k_t=\frac{1}{2\omega}\varphi_{xxx}+\left(2\omega-\frac{\Omega_B}{2\omega}\right)\varphi_x+\frac{1}{2\omega}\varphi_{xxt}-\frac{\Omega_B}{2\omega}\varphi_t. \qquad (4.19)
\end{aligned}
$$

Eventually we would arrive at the linear WR motion equation with its boundary conditions for solid body rotational flow:

$$
\begin{aligned}
&\varphi_{xxxx}+(\Omega-\Omega_B)\varphi_{xx}+2\varphi_{xxxt}+\varphi_{xxtt}-2\Omega_B\varphi_{xt}-\Omega_B\varphi_{tt}=0\\
&\varphi(0,t)=0,\qquad \varphi_{xx}(0,t)=0,\qquad \varphi_x(x_0,t)=0,\\
&k_t(0,t)=\frac{1}{2\omega}\varphi_{xxx}(0,t)+\left(2\omega-\frac{\Omega_B}{2\omega}\right)\varphi_x(0,t), \qquad (4.20)
\end{aligned}
$$

the three inlet boundary conditions basically dictates that the flow coming in from the inlet is not changed, the flux, vorticity and circulation are all what has been preset for the flow coming into the pipe, and the outlet boundary condition states that the flow remains what has reached the outlet if the flow is not subjected to control, in the approach to control the perturbation for the linear WR equation, we choose to alter the inlet circulation $k_t(0,t)$, originally the inlet circulation is set as zero for the non-control case, now we can alter the inlet circulation to try to stabilize the unstable flow caused by growing perturbation according to the dynamics of the flow. To determine what is required of the inlet circulation to completely control the positive growth rates of the perturbation, we can look to the derivation of the reduction of linear WR equation to the third order asymptotic form. The reason for choosing the inlet circulation is so that a feasible control involving changing the circulation only can be achieved, whereas altering other inlet conditions involve changing the flux and motion of the incoming flow that may be complicated in realizing.

To arrive at some control parameter that the inlet circulation should be set to in order to achieve control, equations (4.18) and (4.19) shall be rescaled according to section 3.1, using the rescaling parameter $\epsilon = 1/L^2$, the rescaled variables are:

$$\begin{aligned}
&x \Rightarrow X, \qquad X = \sqrt{\epsilon}x \\
&t \Rightarrow t^*, \qquad t^* = \epsilon^{\frac{3}{2}}t \\
&\sigma \Rightarrow \sigma^*, \qquad \sigma^* = \frac{\sigma}{\epsilon^{\frac{3}{2}}} \\
&\Omega = 4\omega^2, \qquad \Omega_B = 4\omega_B^2, \qquad k_\omega = 2\omega_B\frac{\omega - \omega_B}{\epsilon},
\end{aligned} \tag{4.21}$$

we can expand and rescale $(\Omega - \Omega_B)$ using (4.21):

$$\begin{aligned}
(\Omega - \Omega_B) &= 4(\omega^2 - \omega_B^2) \\
&= 4[(\omega - \omega_B)^2 + 2\omega\omega_B - 2\omega_B^2] \\
&= 4\left[\frac{\epsilon}{\epsilon}(\omega - \omega_B)^2 + \frac{\epsilon}{\epsilon}2\omega_B(\omega - \omega_B)\right] \\
&= 4[(\omega - \omega_B)^2 + k_\omega\epsilon].
\end{aligned} \tag{4.22}$$

Notice that from (4.21), $\varphi(x,t) = \Psi(X,t^*)$, and the derivative becomes $\varphi_x(x,t) = \Psi_X(X,t^*)\sqrt{\epsilon}$, $\varphi_t(x,t) = \Psi_{t^*}(X,t^*)\epsilon^{3/2}$. Utilize (4.21) to rescale (4.18):

$$\begin{aligned}
\epsilon^2\Psi_{XXXX} &+ \epsilon\left[4\left((\omega - \omega_B)^2 + k_\omega\epsilon\right)\right]\Psi_{XX} \\
&+ \epsilon^3 2\Psi_{XXXt^*} + \epsilon^4\Psi_{XXt^*t^*} - \epsilon^2 2\Omega_B\Psi_{Xt^*} - \epsilon^3\Omega_B\Psi_{t^*t^*} = 0,
\end{aligned} \tag{4.23}$$

divide (4.23) by ϵ^2 to obtain the rescaled version of linear WR motion equation;

$$\begin{aligned}
&\Psi_{XXXX} + \frac{4(\omega-\omega_B)^2}{\epsilon}\Psi_{XX} + 4k_\omega\Psi_{XX} \\
&\qquad + \epsilon 2\Psi_{XXXt^*} + \epsilon^2\Psi_{XXt^*t^*} - 2\Omega_B\Psi_{Xt^*} - \epsilon\Omega_B\Psi_{t^*t^*} = 0. \qquad (4.24)
\end{aligned}$$

To produce an asymptotic form of the linear WR motion equation, elimination of the higher order terms $O(\epsilon)$ on equation (4.24) will be done, note that for the asymptotic equation the range of swirl shall be confined to within the first growth rate branch and nearby swirl levels, this confinement keeps $\omega - \omega_B$ within the order of ϵ, and the equation (4.24) can be simplified through $4(\omega-\omega_B)^2/\epsilon \sim \epsilon$, therefore we can obtain the fourth order asymptotic form of the linear WR motion equation;

$$\begin{aligned}
&\Psi_{XXXX} + 4k_\omega\Psi_{XX} - 2\Omega_B\Psi_{Xt^*} = 0 \\
\Rightarrow \quad & 2\Omega_B\Psi_{Xt^*} = \Psi_{XXXX} + 4k_\omega\Psi_{XX}. \qquad (4.25)
\end{aligned}$$

The same procedure is applied to (4.19) as well;

$$\begin{aligned}
\epsilon^{\frac{3}{2}}\kappa_{t^*} &= \epsilon^{\frac{3}{2}}\frac{1}{2\omega}\Psi_{XXX} + \epsilon^{\frac{1}{2}}\left(2\omega - \frac{\Omega_B}{2\omega}\right)\Psi_X + \epsilon^{\frac{5}{2}}\frac{1}{2\omega}\Psi_{XXt^*} - \epsilon^{\frac{3}{2}}\frac{\Omega_B}{2\omega}\Psi_{t^*} \\
\Rightarrow \quad \kappa_{t^*} &= \frac{1}{2\omega}\Psi_{XXX} + \frac{4k_\omega}{2\omega}\Psi_X - \frac{\Omega_B}{2\omega}\Psi_{t^*}, \qquad (4.26)
\end{aligned}$$

set $\kappa_{t*}(0,t^*)$ to produce a boundary condition at the intlet for the motion equation:

$$\kappa_{t^*}(0,t^*) = \frac{1}{2\omega}\Psi_{XXX}(0,t^*) + \frac{4k_\omega}{2\omega}\Psi_X(0,t^*), \qquad (4.27)$$

other boundary conditions include

$$\Psi(0,t^*) = 0, \qquad \Psi_{XX}(0,t^*) = 0, \qquad \Psi_X(1,t^*) = 0, \qquad (4.28)$$

in keeping the convention of numerical analysis in previous chapters, the length of the pipe is set to be $L = 6$, hence at the end of the pipe $X = 1$. To obtain a motion equation analogous to the asymptotic equation of (3.10), equation (4.25) will be integrated in X:

$$\int_0^X (\Psi_{XXXX} + 4k_\omega \Psi_{XX})dX = \int_0^X 2\Omega_B \Psi_{Xt^*} dX$$

$$\begin{aligned}\Psi_{XXX} + 4k_\omega \Psi_X - 2\Omega_B \Psi_{t^*} \\ - [\Psi_{XXX}(0,t^*) + 4k_\omega \Psi_X(0,t^*)] = 0,\end{aligned} \tag{4.29}$$

$2\Omega_B \Psi_{t^*}(0,t^*)$ is eliminated by using the boundary condition $\Psi(0,t^*) = 0$. To eliminate the inlet conditions in (4.29), we notice that the condition for circulation at the inlet has a similar form to the inlet conditions that we are trying to eliminate, by setting the inlet circulation to be equal to zero ($2\omega\kappa_{t^*}(0,t^*) = 0$), we can completely eliminate the extra terms in (4.29), this presents us with the third order asymptotic form of the linear WR equation of motion with no control:

$$\begin{aligned}&\Psi_{XXX} + 4k_\omega \Psi_X - 2\Omega_B \Psi_{t^*} = 0, \\ &\Psi(0,t^*) = 0, \qquad \Psi_{XX}(0,t^*) = 0, \qquad \Psi_X(1,t^*) = 0,\end{aligned}$$

which is in the same form as the linear third order asymptotic motion equation. However, when (4.29) is compared to the linear third order asymptotic equation that involves a control term (4.8), we are able to see that instead of eliminating the extra terms, they can be set to be equal to the control parameter $u(t^*)$. By doing so the definition of the inlet circulation now changes, and $2\omega\kappa_{t^*}(0,t^*)$ can now be altered as a control condition to stabilize the perturbation. Following the formulation for the control parameter in section 4.1, we can arrive at the conclusion that the inlet circulation condition $2\omega\kappa_{t^*}(0,t^*)$ must be equal to the feedback control parameter $u(t^*)$, therefore a new condition at the inlet where the circulation of the flow at the inlet can be changed to stabilize the perturbation can be formulated:

$$2\omega\kappa_{t^*}(0,t^*) = u(t^*) = \frac{1}{2}\gamma\Psi_X(0,t^*) = \Psi_{XXX}(0,t^*) + 4k_\omega \Psi_X(0,t^*), \tag{4.30}$$

the linear WR motion equation with control is therefore:

$$\begin{aligned}
&\varphi_{xxxx} + (\Omega - \Omega_B)\varphi_{xx} + 2\varphi_{xxxt} + \varphi_{xxtt} - 2\Omega_B\varphi_{xt} - \Omega_B\varphi_{tt} = 0, \\
&\varphi(0,t) = 0, \qquad \varphi_{xx}(0,t) = 0, \qquad \varphi_x(x_0,t) = 0, \\
&\varphi_{xxx}(0,t) = \frac{1}{2}\gamma\varphi_x(0,t) - (\Omega - \Omega_B)\,\varphi_x(0,t),
\end{aligned} \tag{4.31}$$

changing the inlet condition of $\varphi_{xxx}(0,t)$ is equivalent to changing the inlet circulation $k_t(0,t)$, rescaling can also be performed to produce the control problem for the linear WR motion equation that involves rescaled time and pipe length for better comparison with previous numerical results. The control problem above is still a form of feedback stabilization, changing the inlet circulation is dependent on the dynamics of the perturbation.

In deducing what the inlet circulation should be in order to control the third order asymptotic form of the linear WR motion equation, the relation $2\omega k_{t^*}(0,t^*) = u(t^*)$ is the new boundary control method for the reduced form of the inlet circulation $\kappa_{t^*}(0,t^*)$, by letting this equal to the control parameter $u(t^*)/2\omega$, it is possible to completely control the growth rate of the perturbation for swirl near the Benjamin's critical swirl. However when the reduced terms of $\kappa_{t^*}(0,t^*)$ is added to produce the control problem for the linear WR motion equation, the additional terms may change the growth rate of the perturbations, since only the growth rate near ω_B can be accurately determined by the reduced equation. For $\gamma = k_\omega\beta$ the first growth rate branch can still be completely controlled, since swirl range is still near the critical swirl, however, for swirl range further away from the critical swirl, the terms eliminated after reduction from both the linear WR motion equation and boundary condition $\kappa_{t^*}(0,t^*)$ may play a larger role in reducing the effect of the control parameter, which may lead to instability as the swirl increases.

4.6 Effectiveness of control

To analyze the control problem of the linear WR motion equation, we use suitable eigenmode analysis $\varphi(x,t) = \tilde{\varphi}(x)e^{\sigma t}$ and $k(0,t) = \tilde{k}(0)e^{\sigma t}$ on the control problem (4.31):

$$\begin{aligned}
&\tilde{\varphi}_{xxxx} + (\Omega - \Omega_B)\tilde{\varphi}_{xx} + 2\sigma\tilde{\varphi}_{xxxt} + \sigma^2\tilde{\varphi}_{xx} - 2\Omega_B\sigma\tilde{\varphi}_x - \Omega_B\sigma^2\tilde{\varphi} = 0,\\
&\tilde{\varphi}(0,t) = 0, \qquad \tilde{\varphi}_{xx}(0,t) = 0, \qquad \tilde{\varphi}_x(x_0,t) = 0,\\
&\tilde{\varphi}_{xxx}(0,t) = \frac{1}{2}\gamma\tilde{\varphi}_x(0,t) - (\Omega - \Omega_B)\,\tilde{\varphi}_x(0,t), \qquad (4.32)
\end{aligned}$$

this is the control problem for the linear WR stability equation, which is used to analyze the relationship between σ and ω under feedback stabilization control method, the control parameter here can still be considered as $k_\omega\beta/2$ instead of $k_\omega\beta$, which signifies that changing the constant is the same as changing $k_\omega\beta$, and we shall call the constant the control gain. In section 3.2, a noticeable difference between the asymptotic equation and the linear WR equation with no control can be seen, particularly in later branches where the linear WR equation produces smaller positive growth rate branches. This difference suggests that the eliminated terms when reducing to the asymptotic equation plays a larger part in later branches, causing the growth rate to decrease, a preliminary deduction can be seen from the linear WR stability equation, since all the eliminated terms involves the growth rate σ, and their collective effect decreases the growth rate branch while σ remains positive, the collective effect should be to increase the growth rate branches when σ is negative. Section 4.4 showed total controllability of all branches for the asymptotic equation, this suggests that total control is achieved by reducing the equation, therefore a counter argument can be formulated, that with the unreduced linear WR equation, the terms that were reduced may play an important role on later branches which concerns whether or not these later branches can be controlled. Section 3.2 suggests these reduced terms have a largely negative effect while σ is positive, which would lead one to deduce that at the control gain of 0.5, the later branches may not be able to be controlled due to the reduced terms (since σ being negative should create a largely positive effect on the growth rate branches). To investigate this, numerical analysis will be done on the control problem for the linear WR equation, at control gain of = 0.5:

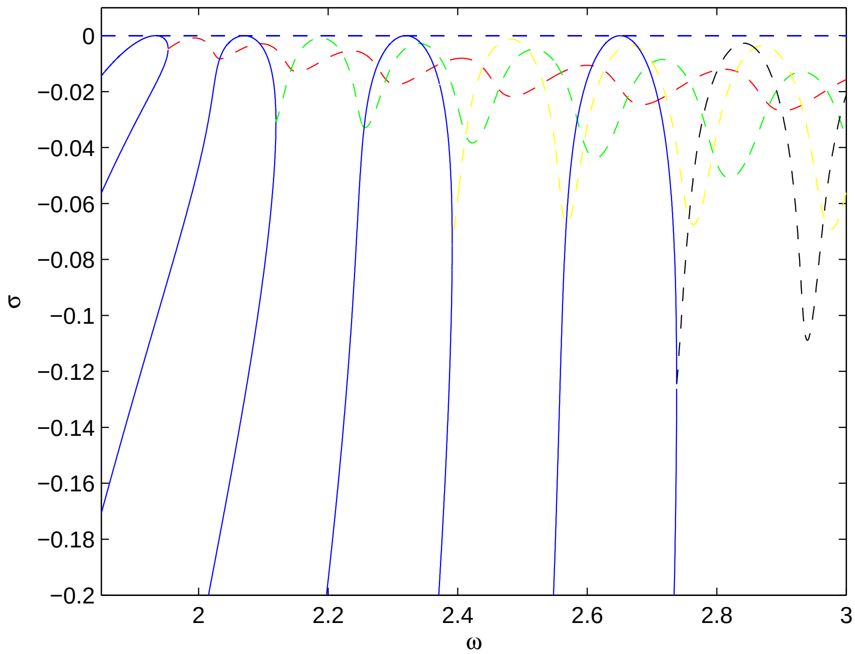

Figure 4.6: Perturbation growth of linear WR equation (Control gain = 0.5).

numerical analysis is done on the rescaled parameters of the pipe, where $X = 1$ is the rescaled full length of the pipe, this is done as a convention for numerical analysis. Fortunately at the same requirement for the control parameter linear WR equation showed similar behavior to the asymptotic equation. At the control gain of 0.5, both equations can totally control all the growth rate branches, with the complex branches also exhibiting dampening effect. Numerical analysis suggests that even with the reduced terms present, the positive effect they have on the growth rate branches can still be controlled by the controlled term, reinforcing the robustness of feedback control on both equations with a solid body rotational base flow subjected to axisymmetric disturbances. At control = 0.95:

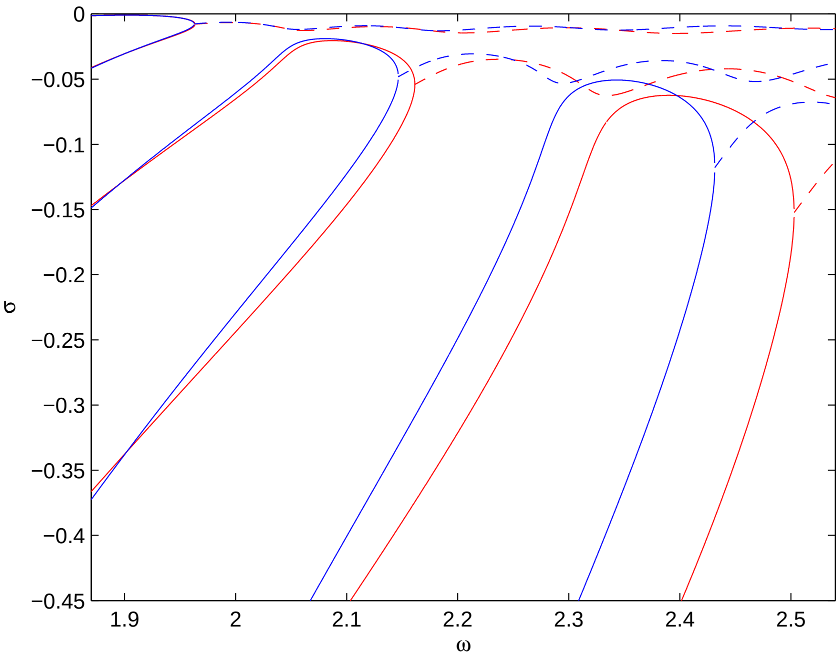

Figure 4.7: Asymptotic and WR at control of 0.95.

figure 4.7 shows the growth rate curves of both the asymptotic equation and the linear WR equation with control, blue line represents the linear WR equation while the red represents asymptotic motion. From this figure we can see clearly that the effectiveness of the feedback stabilization method is less effective against the linear WR equation, the larger the swirl the less the growth rate branch lowering, also the transportation of the growth rate branches down the swirl axis is also less prominent than the asymptotic equation, although complex modes exhibit less dampening, the difference is not very discernable, the main difference is the maximum swirl points for each branches where the respective complex modes start.

At control = 1.3:

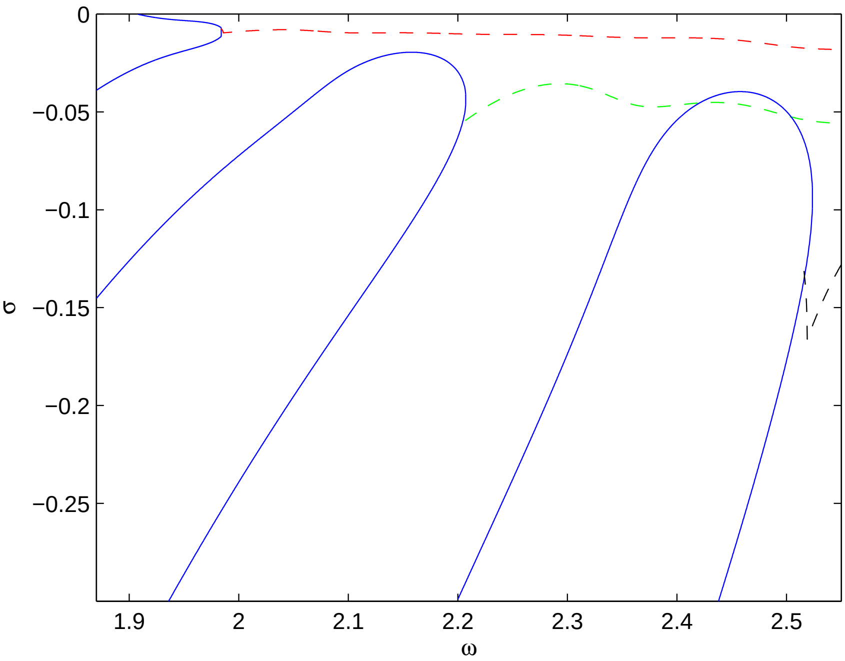

Figure 4.8: Linear WR equation (Control gain = 1.3).

Increasing the control parameter follows the same trend as the asymptotic equation with control, with the optimal control somewhere between 0.95 and 1.3, however, the numerical prediction theory only works with the asymptotic equation, this means that the optimal control for the linear WR equation is not necessary at control gain = 1. In fact, at control gain of 1.3, the reduced effectiveness of the control parameter seems to be stronger than the asymptotic equation, which suggests the optimal control gain to be smaller, but very close to 1. The growth rate branches still exhibit translation across the swirl axis, shifting to the right as the swirl is increasing.

At control = 5:

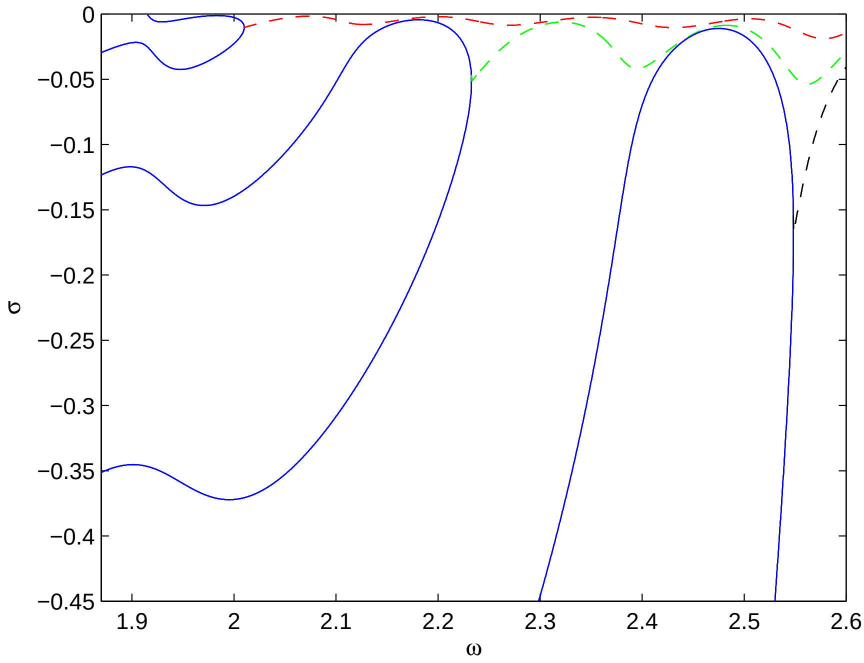

Figure 4.9: Linear WR equation (Control gain = 5).

From figures 4.7, 4.8, and 4.9 it is possible to discern that the effectiveness of the control term is reduced. When compared to the asymptotic equation, the lowering effect of the growth rate branches is reduced, and the transporting effect is also reduced, this phenomenon can be contributed to the existence of the reduced terms in the linear global equation, since they were shown to have an overall positive effect on the growth rate branches (for σ negative) when there is no control, it is possible they exhibit the same behavior when there is control, which is proven by the numerical analysis.

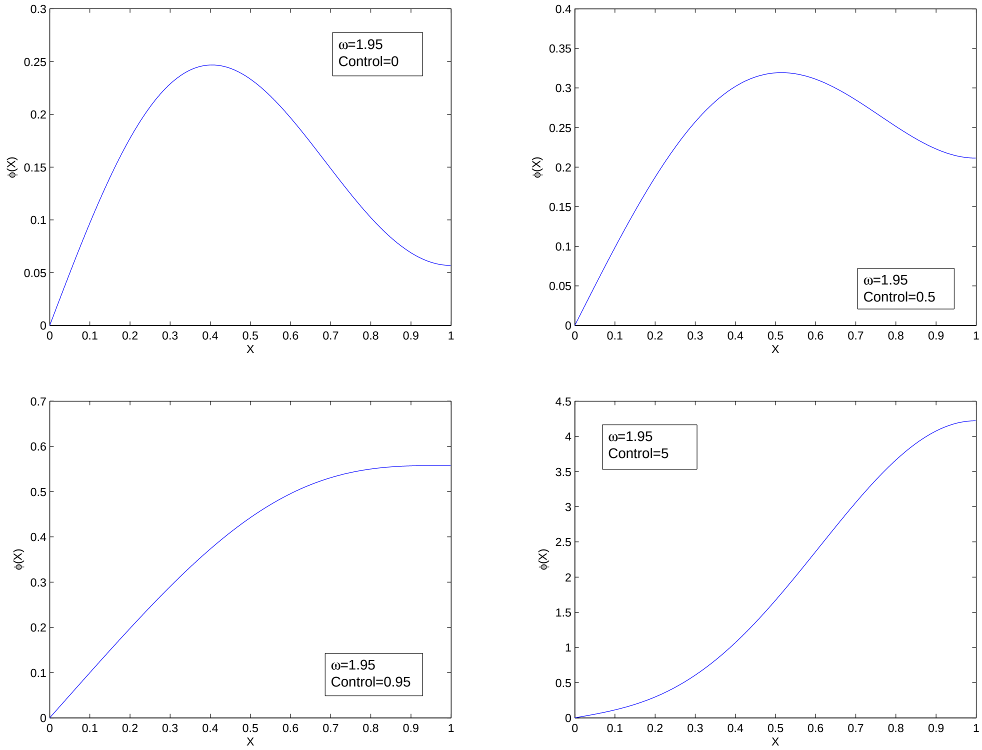

Figure 4.10: Eigenmode $\Psi(X)$ for specific swirl ω (for control $= 0, 0.5, 0.95, 5$ left to right).

Figure 4.10 showed a similar behavior of the eigenmodes to the asymptotic equation eigenmodes in section 4.4. The transporting effect can be easily discerned in the figure; however, it is noticeable that the transporting effect was reduced, since a control parameter of 0.5 showed less flattening of the curve near the end of the pipe (resembles the eigenmode with less swirl when there is no control), whilst the same control for the asymptotic equation has the eigenmode show more flattening of the curve (resembling eigenmode with larger swirl when there is no control), another indication that the effect of the control parameter is reduced when applied to the linear WR equation.

Concluding Remarks

The placement of non-periodic boundary conditions allowed for prediction of instability of seemingly stable flow under classical theorization, now the stability equation can reflect realistic operations of the swirling flows. The ability to build models that accurately interprets the dynamics of perturbation enabled further research to improve the understanding of the vortex stability problem. By adding a control mechanism to the linear asymptotic equation, we gain the ability to control the growing perturbation by altering the generation of kinetic energy of the flow. However, adding control to the asymptotic equation only allows for accurate depiction of the dynamics for swirls near Benjamin's critical swirl, i.e., $\omega - \omega_B$ must be in the order of ϵ, the limitation of the asymptotic equation only allows for accurate prediction of the first growth rate branch and nearby swirls, also the control parameter being inside the flow makes physical alteration of the control parameter difficult. Therefore an alternative representation of the control parameter was developed; the new control parameter switches the control from inside the flow to a condition at the inlet of the pipe.

Numerical analysis in section 4.3 showed that for the control parameter to be equal to the requirement for γ, the growth rate branches can be just completely controlled. By increasing the control gain one can notice that the effectiveness of control increases, but peaks at control gain = 1, which is $\gamma = 2k_\omega\beta$, any control gain value larger than this reduces the effectiveness of the control parameter, hence reducing the lowering of the growth rate branches. But an interesting effect of increasing the control parameter is that the growth rate branches are transported down the swirl axis, now a larger swirl is needed to depart from the first growth rate branch. Since the asymptotic equation accurately predicts the dynamics of the first growth rate branch, the larger the control gain, the larger the swirl is needed to reach the second growth rate branch, and since the first branch can always be controlled (provided that γ meets the requirement), the onset of instability is delayed in the sense that a larger than before swirl is needed to reach the later branches that may cause instability (having positive growth rate).

To investigate whether the actual growth rate branches can be completely controlled, a method was developed for the linear WR equation, where the effect of control is applied by changing the inlet condition for the circulation of the flow. This is a new control method since the linear WR equation is different to the asymptotic equation, the two control problems do not share the same control method, however, it is still possible to derive a condition for the inlet circulation to control the first growth rate branch, this is done by rescaling and reducing the linear WR equation to the same form as the linear asymptotic equation, and matching the inlet circulation condition with the control parameter that was set for the linear asymptotic equation, this allows for the development of a condition that needs to be fulfilled for the inlet circulation in order for the first branch to be totally controlled. We are able to investigate the unsolved problems with the asymptotic equation, since the asymptotic equation is a reduced form of the linear WR equation, it was generally predicted that the terms eliminated due to reduction should have a stronger effect as the swirl increases, and that for the control gain of 0.5 (when the first branch can be just completely controlled) the later branches along the swirl axis may actually have positive growth rate, since the effect of the reduced terms are more prominent in later branches (shown in section 3.2, there exists significant difference in later growth rate branches between the asymptotic and WR equations). Numerical analysis showed that linear WR equation follows a similar pattern to the asymptotic equation, and the results were different from what was expected. At control gain of 0.5, the linear WR stability equation produced a growth rate branch plot that shows the growth rate branches can be totally controlled as well, contrary to what was expected. Increasing the control gain also follows the same pattern seen in the asymptotic equation, with branch lowering peaking at control gain of close to one, similar to the asymptotic equation. However, the main difference between the two equations is that the transportation effect of the growth rate branches down the swirl axis with increasing control gain is lower in the linear WR case than the asymptotic case. This seems to suggest that with the reduced terms intact, the growth rate branches now require a higher control gain to transport the branches as far as the same case seen in asymptotic equation. So even though the linear WR equation with control can still be completely controlled for all growth rate branches of the solid body rotation flow, the reduced terms reduces the effect of the control parameter on the perturbation, a swirl level less than what was require to reach the second growth rate branch in the asymptotic equation can then reach the second branch for the linear WR equation.

This signifies that for flows that may produce non-linear effects on the stability equations, the control term may not be as effective as the linear case suggests, and investigation on the effectiveness of the control parameter on the non-linear asymptotic equation needs to be done to clarify the usefulness and the ability to control the growing perturbation for feedback stabilization. A better understanding in the mechanism of perturbation with some control method may ultimately pave way to a successful control method that allows for absolute control even with the non-linear effect present.

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