Economics Department

Economics Working Papers

The University of Auckland *Year* 1999

What Do Uncertainty-Averse Decision-Makers Believe? A Note

Matthew Ryan
University of Auckland, m.ryan@auckland.ac.nz

What Do Uncertainty-Averse Decision-Makers Believe?
A Note

Matthew J. Ryan*

February, 1999

Abstract

The aim of this note is to plug an important gap in our understanding of the epistemic foundations of uncertainty-averse behavior. For *Choquet expected utility* maximizers (Schmeidler (1989)), the beliefs which motivate uncertainty-averse choice are frequently identified using Dow and Werlang's (1994) notion of support for *convex capacities*. Building on the work of Morris (1997), we present a new, *preference-based* belief operator which is is shown to characterize such epistemic inferences. This makes their behavioral foundations transparent, and enables readier comparison with alternative epistemic models for such behavior.

Economists, and especially game theorists, are frequently interested in the beliefs which motivate choice behavior. Such beliefs are often imputed from some representation of the decision-maker's preferences. The probability in a representation of *subjective expected utility (SEU)* preferences, for example, is commonly

*Department of Economics, University of Auckland, Private Bag 92019, Auckland, New Zealand. Ph. +64-9-373 7999. Fax +64-9-373 7427. Email : `m.ryan@auckland.ac.nz`

taken to encapsulate the decision-maker's various epistemic attitudes. More controversially, the *convex capacity* in a *Choquet expected utility (CEU)* representation of uncertainty-averse preferences is often given a similar interpretation[1].

It is an important task to connect such epistemic models to preferences, so that their behavioral foundations are made explicit. Morris (1997) makes a useful contribution to this endeavor. The purpose of the present note is to extend the work of Morris, and in particular to clarify the basis for epistemic inferences based on uncertainty-averse CEU preferences (Schmeidler (1989)). These are a proper subset of the *maxmin expected utility (MMEU)* preferences of Gilboa and Schmeidler (1989), and are particularly prominent in economic applications of CEU[2].

We discuss the relationship between Morris' preference-based belief operators, and extant notions of support for convex capacities. It is shown that a significant gap exists in our understanding of the behavioral foundations of Dow and Werlang's (1994) support concept (DW-support), which is commonly adopted in applications of CEU[3]. In section 3 we present two results which help to establish these foundations. First, it is shown that Morris' *weak belief* operator characterizes the *intersection* of the DW-supports. Second, a new belief operator – *firm belief* – is defined, and is shown to provide a similar characterization for the *union* of DW-supports. *Firm belief* is thus an important concept for understanding uncertainty-averse decision-making.

The note concludes with some general thoughts on the potential pitfalls in trying to identify epistemic motivations for uncertainty-averse choice behavior.

[1] See Dow and Werlang (1994), Eichberger and Kelsey (1994, 1995), Marinacci (1996) and Mukerji (1995, 1997), among others.

[2] See, for example, the references in note 1.

[3] See, for example, Eichberger and Kelsey (1994, 1995), Marinacci (1996) and Mukerji (1995).

1. Uncertainty-averse preferences

In Morris (1997), *acts* are elements of $\mathbb{R}^\Omega$, being mappings from a finite state space Ω, to monetary outcomes. However, since MMEU and Schmeidler's (1989) version of CEU are both derived within the "two-stage" framework of Anscombe and Aumann (1963), we shall adapt Morris' analysis to that framework. In particular, we take outcomes to be elements of the (mixture) set X of all (monetary) lotteries on $\mathbb{R}$ having finite support.

We consider the uncertainty-averse CEU preference orderings on X^Ω; that is to say, preference orderings which satisfy axioms (i), (ii), (iv), (v) and (vii) of Schmeidler (1989), plus "uncertainty aversion" (Schmeidler (1989, p.582)).

Such preferences may be represented by an affine utility function, $u : X \to \mathbb{R}$, and a convex capacity on Ω. The latter is a mapping[4] $v : \mathcal{P}(\Omega) \to [0,1]$ which satisfies the following conditions: $v(\emptyset) = 0$, $v(\Omega) = 1$, $v(E) \leq v(F)$ whenever $E \subseteq F$, and (convexity) $v(E \cup F) + v(E \cap F) \geq v(E) + v(F)$ for any $E, F \in \mathcal{P}(\Omega)$. The components u and v combine to represent preferences $\succeq$ as follows: given any two acts f and g,

$$f \succeq g \quad \textit{iff} \quad \int_\Omega (u \circ f)\ dv \geq \int_\Omega (u \circ g)\ dv \tag{1.1}$$

The integral in (1.1) is the *Choquet integral* (see Choquet (1953-4)).

Such preferences also have an equivalent MMEU representation, involving the same utility function, and a closed, convex set of probabilities on Ω obtained as the *core* of v (Schmeidler (1989, Proposition)):

$$f \succeq g \quad \textit{iff} \quad \min_{p \in \text{core}(v)} \int_\Omega (u \circ f)\ dp \geq \min_{p \in \text{core}(v)} \int_\Omega (u \circ g)\ dp$$

where[5]

$$\text{core}(v) = \{p \in \Delta(\Omega) |\ p(E) \geq v(E) \quad \forall E \in \mathcal{P}(\Omega)\}.$$

[4]The notation $\mathcal{P}(\Omega)$ indicates the set of all subsets of Ω.

[5]We use $\Delta(\Omega)$ to denote the set of probabilities on Ω.

Consider uncertainty-averse preferences $\succeq$, with CEU representation (u, v). For ease of comparison with Morris (1997), let us define the induced ordering[6] $\succeq^u$ on $\mathbb{R}^\Omega$ by the condition: $x \succeq^u y$ if $f \succeq g$ for some $f, g \in X^\Omega$ such that $u(f) = x$ and $u(g) = y$. This ordering is well-defined, and may be taken to be complete without loss of generality (see Gilboa and Schmeidler (1989, Lemmas 3.2 and 3.3)).

We adopt a couple of common notational conventions. If $x \in \mathbb{R}$, we take the liberty of interpreting x as a constant vector in $\mathbb{R}^\Omega$ as convenient. When $\{A_1, A_2, ..., A_n\}$ is a partition of Ω, and $x^i \in \mathbb{R}^\Omega$ for each $i \in \{1, 2, ..., n\}$, then $(x^1_{A_1}, x^2_{A_2}, ..., x^n_{A_n})$ denotes the vector with ω component equal to the x^i_ω such that $\omega \in A_i$.

We shall express Morris' belief operators in terms of $\succeq^u$. The formalities will therefore be identical, but the interpretation slightly different. While his preferences are defined on vectors of *monetary* outcomes, ours are defined on vectors of *von Neumann-Morgenstern utility* outcomes. Provided one assumes, within Morris' framework, his *monotonicity* postulate [P5*], nothing of conceptual substance will be lost in this translation. That is, all the belief operators discussed here and in Morris (1997) are defined entirely in terms of the ordinal ranking of outcomes in each state. In particular, none of the belief operators depends on the particular utility function u employed in the CEU representation of preferences.

For the remainder of the paper, we shall suppose that at each state $\omega \in \Omega$, the decision-maker has *uncertainty-averse CEU preferences* $\succeq_\omega$ over X^Ω (and an associated ordering $\succeq^u_\omega$ over $\mathbb{R}^\Omega$). We shall use v_ω to denote the convex capacity in any CEU representation of $\succeq_\omega$.

[6]The superscript in $\succeq^u$ reflects the fact that u is defined only up to positive linear transformations (cf. v, which is uniquely determined) in the CEU representation of $\succeq$ (Schmeidler (1989, Theorem)).

2. Possibility correspondences and capacity supports

A *belief operator* $B : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ specifies the decision-maker's beliefs at each state. Event $E \subseteq \Omega$ is believed at $\omega \in \Omega$ if $\omega \in B(E)$. Any belief operator is required to satisfy the following four intuitive restrictions[7]:

B1: $B(\Omega) = \Omega$.

B2: $B(\emptyset) = \emptyset$.

B3: $B(E) \cap B(F) \subseteq B(E \cap F)$.

B4: $E \subseteq F \;\Rightarrow\; B(E) \subseteq B(F)$.

Given a belief operator B, we may define the associated *possibility correspondence* $P : \Omega \to \mathcal{P}^*(\Omega)$, where $\mathcal{P}^*(\Omega)$ denotes the set of all *non-empty* subsets of Ω, as follows:

$$P(\omega) = \bigcap \{E \mid \omega \in B(E)\}.$$

If B satisfies [B1]–[B4], we may recover B from P using

$$B(E) = \{\omega \in \Omega \mid P(\omega) \subseteq E\}.$$

Morris (1997) defines (*inter alia*) the following two belief operators based on preference information[8].

Definition 2.1. *Given* $\{(\succeq_\omega, \succeq_\omega^u)\}_{\omega \in \Omega}$, *the* Savage belief *operator is defined by*

$$B^S(E) = \left\{\omega \in \Omega \,\middle|\, (x_E, y_{E^c}) \succeq_\omega^u (x_E, z_{E^c}) \;\; \textit{for all } x, y, z \in \mathbb{R}^\Omega \right\}.$$

[7]Observe that [B3] and [B4] together imply $B(E) \cap B(F) = B(E \cap F)$.

[8]Definitions 2.1 and 2.2 below adapt Morris' Definitions 1 and 3 respectively to our preference framework. We have also denoted the two operators somewhat differently.

Definition 2.2. *Given* $\{(\succeq_\omega, \succeq^u_\omega)\}_{\omega\in\Omega}$, *let*

$$P^W(\omega) \;=\; \left\{\omega' \in \Omega \,\middle|\, (\forall x \gg y)\,(\exists z \ll y) \;\; \textit{such that} \;\; \left(x_{\{\omega'\}},\, z_{\{\omega'\}^c}\right) \succ^u_\omega y\right\}.$$

The weak belief *operator is defined by*

$$B^W(E) \;=\; \{\omega \in \Omega |\, P^W(\omega) \subseteq E\}\,.$$

The Savage belief operator satisfies [B1]–[B4] for the class of CEU preferences (Morris (1997, Theorem 2). However, weak belief may violate [B2], even if we restrict attention to *uncertainty-averse* CEU preferences (cf. Morris (1997, Example 3)). For instance, if $|\Omega| > 1$, and $v_\omega(E) = 0$ for every $E \neq \Omega$, then $P^W(\omega) = \emptyset$ and hence $\omega \in B(\emptyset)$.

If $\succeq_\omega$ is an SEU ordering, then v_ω is a *probability*, and $P^W(\omega) = P^S(\omega)$ is its *support* (Morris (1997, Example 1)).

For general uncertainty-averse CEU orderings, $P^S(\omega)$ may differ from $P^W(\omega)$. The former is the *outer support* (Ryan (1998, Definition 8 and Lemma 1)) of v_ω. This notion of support is employed in Groes *et al.*(1998) and Lo (1995, Chapter 3), the latter in an MMEU context, so these authors adopt (at least implicitly) the Savage belief operator in order to impute epistemic motivations for choice behavior.

The weak belief correspondence $P^W(\omega)$ is equivalent to the *LM-support* of v_ω (Morris (1997, Examples 3 and 4)). The terminology "LM-support" is due to Ryan (1998), but the concept originates in Lo (1995, Chapter 3) and Marinacci (1996), and refers to the set

$$\bigcap_{p\in \mathrm{core}(v_\omega)} \mathrm{supp}(p) \;=\; \left\{\, \omega' \in \Omega \mid v_\omega(\{\omega'\}) > 0 \,\right\} \tag{2.1}$$

In particular, convexity of v_ω implies

$$\left\{\, \omega' \in \Omega \mid v_\omega(\{\omega'\}) > 0 \,\right\} \;=\; \left\{\, \omega' \in \Omega \mid v_\omega(E \cup \{\omega'\}) > v_\omega(E) \;\; \forall E \subseteq \{\omega'\}^c \,\right\}$$

(cf. Morris (1997, Example 4)).

3. Characterizing Dow and Werlang's (1994) supports.

Unfortunately, neither $P^S(\omega)$ nor $P^W(\omega)$ are able to characterize one of the most commonly encountered support concepts for convex capacities – that defined by Dow and Werlang (1994), and dubbed *DW-support* in Ryan (1998). The set $A \subseteq \Omega$ is a DW-support of the convex capacity v if A is $\subseteq$-minimal with respect to the property that $v(A^c) = 0$. That is, $v(A^c) = 0$, and $v(B^c) > 0$ whenever B is a proper subset of A.

The DW-support does not lead directly to a possibility correspondence, since it is well-known (Dow and Werlang (1994, p.311)) that a convex capacity may have more than one DW-support[9]. However, the union and intersection of the DW-supports determine two natural bounds on any possibility correspondence derived from the DW-support concept. We shall provide a preference-based characterization for each of these.

Given uncertainty-averse CEU preferences $\succeq$, $DW(\succeq)$ will denote the set of DW-supports of the convex capacity in any CEU representation of $\succeq$. Let

$$P^{\vee}(\omega) = \bigcup \{A \,|A \in DW(\succeq_\omega)\}$$

and

$$P^{\wedge}(\omega) = \bigcap \{A \,|A \in DW(\succeq_\omega)\}.$$

Proposition 3.1. $P^{\wedge}(\omega) = P^W(\omega)$.

Proof: We must show that $P^{\wedge}(\omega)$ is the LM-support of v_ω. That $P^{\wedge}(\omega)$ contains the LM-support of v_ω is obvious from (2.1) and the monotonicity of v_ω. Let us

[9] In particular, the *negative belief* operator (Morris (1997, p.230))

$$B^-(E) = \{\omega \in \Omega |\; v_\omega(E^c) = 0\}$$

does not satisfy [B3] (although [B1], [B2] and [B4] are easily verified). As an illustration, take $E = \{1, 2\}$ and $F = \{1, 3\}$ in Dow and Werlang's (1994, p.311) example of a convex capacity with multiple DW-supports.

prove the reverse inclusion. Suppose that $v_\omega(\{\omega'\}) = 0$, so that ω' is *not* an element of the LM-support of v_ω. We must show that $\omega' \notin P^\wedge(\omega)$. To do so, we shall indicate how to construct a DW-support for v_ω which excludes ω'. Begin with the set $E^{(0)} = \Omega\setminus\{\omega'\}$ and observe that $v_\omega\left(\Omega\setminus E^{(0)}\right) = 0$. If $v_\omega(B^c) > 0$ for every proper subset B of $E^{(0)}$ we are done. If not, let $E^{(1)}$ be any proper subset of $E^{(0)}$ which satisfies $v_\omega\left(\Omega\setminus E^{(1)}\right) = 0$. Continuing in this fashion, and observing that $v_\omega(\emptyset^c) = v_\omega(\Omega) = 1$, we shall obtain some $E^{(n)} \neq \emptyset$ which is a DW-support of v_ω and excludes ω'. □

Therefore, the intersection of DW-supports induces an epistemic model based on Morris' *weak belief.* Definition 2.2 provides a preference-based characterization of such beliefs.

By contrast, the possibility correspondence $P^\vee(\omega)$ is obtained from a belief operator which is distinct from both Savage and weak belief, and is in fact intermediate between these two. In particular,

$$P^W(\omega) = P^\wedge(\omega) \subseteq P^\vee(\omega) \subseteq P^S(\omega),$$

and both containments may be strict.

That $P^\wedge(\omega) \subseteq P^\vee(\omega)$ is obvious, and the possibility of strict inclusion follows from the existence of convex capacities with non-unique DW-supports.

The inclusion $P^\vee(\omega) \subseteq P^S(\omega)$ is easily confirmed by noting that every DW-support is contained in the outer support[10]. To see that the inclusion may be strict, suppose that $A \subseteq \Omega$, with $1 < |A| < |\Omega|$, and consider the convex capacity v_ω defined as follows: $v_\omega(\Omega) = 1$, $v_\omega(E) = 0.5$ if $A \subseteq E \neq \Omega$, and $v_\omega(E) = 0$ otherwise. One may easily determine that Ω is the outer support of v_ω, while A is the union of its DW-supports (which consist of the singleton subsets of A).

[10] If $\omega' \in \Omega\setminus P^S(\omega)$, then (Morris (1997, Example 4)) $v_\omega\left(\{\omega'\} \cup E\right) = v_\omega(E)$ for any E. Thus, if A is a DW-support of v_ω, we must have $A \subseteq P^S(\omega)$, since $v_\omega\left(\{\omega'\} \cup A^c\right) = v_\omega(A^c) = 0$.

It therefore remains to provide a preference characterization for the belief operator associated with $P^{\vee}(\omega)$.

Definition 3.2. *Given* $\{(\succeq_\omega, \succeq_\omega^u)\}_{\omega\in\Omega}$, *the* firm belief *operator is defined as*

$$B^{\vee}(E) = \{\omega \in \Omega \,|\, (\forall\omega' \in E^c)\, (\forall F \subseteq \Omega)\, (\exists x, y, z \in \mathbb{R} \text{ with } x > y)$$

$$\text{such that } \left(z_{F\backslash\{\omega'\}}, y_{F^c\cup\{\omega'\}}\right) \succeq_\omega^u (x_F, y_{F^c})\} .$$

If E is *firmly believed*, then it is always possible to compensate for a reduction in the utility obtained some previously "good" state $\omega' \in E^c$ by suitably increasing the utility obtained in the remaining "good" states.

Proposition 3.3. *The operator $B^{\vee}$ satisfies [B1]–[B4].*

Proof: [B1] and [B4] are trivially satisfied. To verify [B2], note that for each $\omega \in \Omega$, it is always possible to find ω' and F such that $v_\omega(F\backslash\{\omega'\}) = 0$ and $v_\omega(F) > 0$ (since v_ω is monotone, $v_\omega(\emptyset) = 0$ and $v_\omega(\Omega) = 1$). Finally, [B3] must obtain, since $(E \cap F)^c = E^c \cup F^c$. □

Proposition 3.4. $B^{\vee}(E) = \{\omega \in \Omega |\, P^{\vee}(\omega) \subseteq E\}$.

Proof: Let us first observe that $\omega' \in P^{\vee}(\omega)$ if and only if there exists a DW-support of v_ω containing ω'. This is the case *if and only if* there exists some $E \subseteq \{\omega'\}^c$ such that $v_\omega(E^c\diagdown\{\omega'\}) = 0$ and $v_\omega(E^c) > 0$. The "only if" part is obvious (take E to be the DW-support which contains ω', with ω' removed). For the converse, we may construct a DW-support for v_ω containing ω' as follows. Begin with $E \cup \{\omega'\}$. Note that $v_\omega(E^c\diagdown\{\omega'\}) = v_\omega((E \cup \{\omega'\})^c) = 0$. If $v_\omega((F \cup \{\omega'\})^c) = 0$ for some proper subset F of E, replace $E\cup\{\omega'\}$ with $F\cup\{\omega'\}$, and so on (cf. the proof of Proposition 3.1). This process will terminate at some (possibly empty) $B \subseteq E$ such that $v_\omega((B \cup \{\omega'\})^c) = 0$ and $v_\omega((F \cup \{\omega'\})^c) > 0$

for every proper subset F of E. The set $B \cup \{\omega'\}$ will be a DW-support for v_ω since $v_\omega(A^c) \geq v_\omega(E^c) > 0$ for any $A \subseteq B$.

We may now apply Ryan (1998, Proposition 1(iii)) to conclude that $\omega' \in P^\vee(\omega)$ if and only if there exists some $E \subseteq \{\omega'\}^c$ such that

$$(x_{E^c}, y_E) \succ^u_\omega \left(z_{E^c \setminus \{\omega'\}}, y_{E \cup \{\omega'\}}\right)$$

for all $x, y, z \in \mathbb{R}$ with $x > y$.

Since $P^\vee(\omega) \subseteq E$ if and only if $\omega' \notin P^\vee(\omega)$ for every $\omega' \in E^c$, the result follows. □

4. Discussion

Epistemic inferences about uncertainty-averse CEU maximizers are typically made on the basis of some support concept for convex capacities, used in the manner of a possibility correspondence. This note has identified the preference-based notions of belief which are implicit in these epistemic inferences.

It is clearly valuable to have the behavioral foundations for epistemic inferences thereby made explicit. However, there remains the important question of how the modeller is to choose a suitable (preference-based) notion of belief.

Divergence of the various belief operators creates a modelling quandary, and one should proceed with due caution. For example, I have argued elsewhere[11] that when weak and firm belief diverge (i.e. when $P^\wedge(\omega) \neq P^\vee(\omega)$), a misspecification of the state space may be indicated, thus rendering any epistemic inferences highly questionable.

Indeed, not all rational choice behavior need be motivated by beliefs. It is well-known, for example, that the maximin decision criterion is compatible with

[11] See especially the discussion in section 5 of Ryan (1998).

uncertainty-averse CEU preferences. Maximin would seem to be a "beliefs-free" decision rule; or at least, its application keeps the decision-maker's beliefs hidden from view. One should therefore not be over-zealous in the search for an epistemic motivation for all choice behavior.

References

[1] Anscombe, F.J., and R.J. Aumann: 1963, "A Definition of Subjective Probability," *Annals of Mathematical Statistics* **34**, 199–205.

[2] Choquet, G.: 1953-4, "Theory of Capacities," *Annales de L'Institut Fourier* **V**, 131–295.

[3] Dow, J. and S.R.Da C. Werlang: 1994, "Nash Equilibrium Under Knightian Uncertainty: Breaking Down Backward Induction," *Journal of Economic Theory* **64**, 305–324.

[4] Eichberger, J. and D. Kelsey: 1994, "Non-Additive Beliefs and Game Theory," University of Melbourne, Department of Economics Research Paper No. 398.

[5] Eichberger, J. and D. Kelsey: 1995, "Signalling Games With Uncertainty," *mimeo*, University of Birmingham.

[6] Gilboa, I. and D. Schmeidler: 1989, "Maxmin Expected Utility With a Non-Unique Prior," *Journal of Mathematical Economics* **18**, 141–153.

[7] Groes, E., H.J. Jacobsen, B. Sloth and T. Tranaes: 1998, "Nash Equilibrium with Lower Probabilities," *Theory and Decision* **44**(1), 37–66.

[8] Lo, K.C.: 1995, "Games With Uncertainty Averse Players," *unpublished Ph.D. dissertation*, University of Toronto.

[9] Marinacci, M.: 1996, "Ambiguous Games," *mimeo*, University of Toronto.

[10] Morris, S.: 1997, "Alternative Definitions of Knowledge," in M.O.L. Bacharach *et al.* (eds.), *Epistemic Logic and the Theory of Games and Decisions*, Kluwer Academic, Dordrecht, The Netherlands.

[11] Mukerji, S.: 1995, "A Theory of Play for Games in Strategic Form When Rationality is not Common Knowledge," *mimeo*, University of Southampton.

[12] Mukerji, S.: 1997, "Understanding the Nonadditive Probability Decision Model," *Economic Theory* **9**, 23–46.

[13] Ryan, M.J.: 1998, "Rational Belief Revision by Uncertainty-Averse Decision-Makers," *mimeo*, University of Auckland.

[14] Schmeidler, D.: 1989, "Subjective Probability and Expected Utility Without Additivity," *Econometrica* **57**(3), 571–587.