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# Patent Licensing with Spillovers

Reiko Aoki[*] Yair Tauman[†]

[*]University of Auckland, r.aoki@auckland.ac.nz

[†]Tel Aviv University, tauman@post.tau.ac.il

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# Patent Licensing with Spillovers

Reiko Aoki* Yair Tauman†

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**Abstract**

The purpose of this paper is to study the effect of spillover on extent of licensing when cost reducing innovation is introduced and licensed to a number of oligopolistic firms. We characterize the equilibrium number of licenses that are sold through an auction. An increase in the number of licenses has two effects. First, it increases the competition between the licensees. Second, due to spillover, the non-licensees become more efficient contributing to even more competition. We find that despite these effects, a patentee of a significant innovation will sell more licenses when there is spillover than without spillover thereby inducing even more competition. In this case, consumer surplus will be greater with spillover. However, if the innovation is less significant, then the patentee will sell less licenses with spillover thereby restrict competition. In this case the market price will be higher and the consumer surplus will be smaller.

*University of Auckland, r.aoki@auckland.ac.nz
†SUNY Stony Brook and Tel-Aviv University

# 1 Introduction

The paper deals with the optimal licensing of a cost reducing patented innovation when there is spillover. The patent holder is concerned with licensing the optimal number of licenses when licenses are sold through a first price auction. The spillover will reduce the non-licensee cost of production, although not as much as with the patented technology itself. We consider a case where the amount of non-licensee cost reduction is proportional to the number of licensees. Thus when there is spillover, more licenses makes every firm (licensee or not) more efficient.

One expects that if the patent owner has the monopoly power over the innovation (which he does with a patent), it is in his best interest to limit competition by selling to fewer number of licensees when there is spillover as compared to the no spillover case and by that maintaining the non-licensees less efficient. Quite surprisingly, we find that if the innovation is relatively significant, there will actually be more licenses sold when there is spillover compared to the no spillover case and thereby the market will be more competitive and consumers are better off. This result is consistent with the argument that spillover encourages licensing which Ordover (1991) has made informally. Our result also provides another efficiency theory of spillover. In the traditional efficiency theory (Spence (1986)), society benefits from greater "bang per buck" of R&D investment when there is spillover. In our framework, consumers benefit from the increased competition in the market from more efficient firms as result of spillover.

Levin, Klevorick, Nelson and Winter (1987) have observed a positive correlation between patenting and licensing. They have suggested that either a patent served as an "announcement" that a technology was available or it provided "complementary information" to a license. However a patent is also an important source of spillover, as documented by Jaffe, Trajtenberg and Henderson (1993) and others (Bernstein (1989), Suzuki (1993)). A similar relationship between patents, spillover, and licensing have also been reflected in the preliminary analysis of a survey of Japanese firms concerning appropriability and licensing (Institute of Intellectual Property 1994)). We provide another explanation of this phenomenon: patents increase licensing because of the spillovers they provide.

We employ a very simple oligopoly model with $n$ firms and a patentee, following the formulation of Kamien and Tauman (1986) of licensing without spillover. All firms produce a homogeneous product in a Cournot (quantity setting) market. The equilibrium winning bid size is the difference between profits with and without the license. The profit of the patent owner will be the total rent extracted from the licensees.

When there is spillover, increasing the number of licenses has two effects on revenue: one due to increase in internal competition and the other due to increase in external competition. Internal competition (competition induced by licensees) increases since there are more efficient firms. With spillover, external competition (the competition induced by the non-licensees) may increase since although there are fewer non-licensees, each firm becomes more efficient because of spillover. External competition always decreases with number of licenses when there is no spillover.

Kamien and Tauman (1986) have shown that when there is no spillover and the innovation is significant (relative to the demand intensity and the market size), the optimal number of licenses sold is the minimum required for the non-licensee firms to exit. In this case the market price is driven down to the pre-innovation marginal cost. The same phenomenon occurs when spillover exists, except that now the non-licensee firms are more efficient (due to spillover) and therefore more licenses are needed to drive them out of the market. When this happens the market price falls below the pre-innovation marginal cost and consumers are the ones who benefit from spillover.

If the innovation is relatively less significant, then either there are not sufficient number of firms in the market to force non-licensees to exit, or the patent holder has to sell licenses to so many firms that the incremental loss in rents due to internal competition dominates the gains from eliminating external competition. With spillover, the bid size decreases very quickly and the benefit of an extra license is exhausted at a smaller number of licenses compared to the no spillover case. Thus when innovation is not significant, there are less licenses with spillover. More inefficient firms implies a higher price and the consumers are worse off. Regardless of innovation magnitude, spillover hurts the patentee for any number of licenses. Thus even when fewer licenses are sold, patentee never does better with spillover.

Previous works on optimal licensing have not considered spillover.When there is no spillover and if the market is oligopolistic, first price auction has been found to be best for the patentee compared to royalties and fixed fees when demand is linear (Kamien and Tauman (1986)) and optimal among a wide class of licensing strategies under a more general demand structure (Kamien, Oren and Tauman (1992)). Katz and Shapiro (1986) have also characterized optimal licensing, in a competitive market. Ferchtman and Kamien (1992) has analyzed optimal cross licensing of complementary technologies. But again, neither of the approaches took into account spillover. Muto (1987) has analyzed the effect of licensing on information dissemination when there is spillover. By using cooperative game theory, he focuses on the equilibrium distribution of technology in the market. Our focus is the strategic behavior of the patentee.

We present a general framework, applicable to both licensing with and without spillover in the following section. The framework is general enough for any form of spillover that reduces marginal cost and highlights the two effects of spillover. The model is analyzed with spillover in section 3. We then compare the results to the case of no spillover, including welfare implications, in section 4. Robustness of our results is discussed in section 5.

## 2 General Framework

There are $n$ identical firms with constant marginal cost $c$. All sell a homogeneous product with linear demand $p = a - q$. A new process innovation will reduce the constant marginal cost of production from $c$ to $c - \epsilon$. Following Arrow (1963), a process innovation is drastic if and only if the post-innovation monopolist price does not exceed the pre-innovation marginal cost. In the linear case, we obtain that an innovation is drastic if $(a - c)/\epsilon \leq 1$. Let $\alpha = (a - c)/\epsilon$. The parameter $\alpha$ measures the magnitude of innovation (smaller the $\alpha$ is, the larger is the magnitude of the innovation.) If an innovation is drastic, it is always optimal to have a monopoly and there is no spillover. Thus, we assume for our analysis that $\alpha > 1$. We assume that patent owner is unable to produce the product himself and thus must license. [1]

We first present a general framework, applicable to licensing with and without spillover, and with any type of spillover. When licenses are sold to the $x$ proportion of firms, there will be $nx$ licensee firms with lower cost $c_L$ and $n(1-x)$ non-licensee firms with higher marginal cost $c_N$. If the innovation lowers cost from $c$ to $c-\epsilon$, then $c_L = c-\epsilon$. If there is no spillover, then $c_N = c$. If there is spillover, then $c-\epsilon < c_N < c$. If spillover depends on number of licenses then $c_N$ will be a function of $x$.

The equilibrium outputs, $q_L(\cdot)$ of licensees, and $q_N(\cdot)$ of non-licensees, will be the equilibrium outputs of a Cournot game with $nx$ low cost $(c_L)$ firms and $n(1-x)$ high cost $(c_N)$ firms.

$$q_L(c_L, c_N, x) = (a - (n(1-x)+1)c_L + n(1-x)c_N)/(n+1),$$
$$q_N(c_L, c_N, x) = (a - (nx+1)c_N + nxc_L)/(n+1). \quad (1)$$

Note that for large values of $x$, non-licensee output maybe zero. There are too many efficient firms for the inefficient firm to produce and there will be no external competition as result. The equilibrium bid $(b(c_L, c_N, x))$ will be the licensee profit (Cournot equilibrium profit with $nx$ firms with

[1] The complete game theoretic formulation is as follows (Kamien (1992)). There are three stages. In the first stage, patent owner will choose $x$, the proportion of firms to sell the license to. This then will be made common knowledge. In the second stage, firms bid for the license. Winners will be the $nx$ firms with the highest bids. Each firm knows if it is a winner (licensee) or not (non-licensee). In the third stage, each firm chooses output. The strategy of the patent owner will be $x \in [0,1]$. A firm's strategy will be a pair of functions, $b(x)$ and $q(x,s)$, where $b(x) : [0,1] \to [0,\infty)$ and $q(x,s) : [0,1] \times \{licensee, non-licensee\} \to [0,\infty)$. By requiring optimal behavior at each stage, we are characterizing the subgame perfect Nash equilibrium of this game.

marginal cost $c_L$) in this case. Otherwise, since the non-licensees are producing, the equilibrium bid will be exactly the difference between profits with and without license. We can summarize the equilibrium bid as follows:

$$b(c_L, c_N, x) = \begin{cases} \left(\frac{a-c_L}{nx+1}\right)^2, & q_N(c_L, c_N, x) = 0, \\ q_L(c_L, c_N, x)^2 - q_N(c_L, c_N, x)^2, & q_N(c_L, c_N, x) > 0. \end{cases} \tag{2}$$

Let $x_1$ be the smallest $x \geq 0$ such that $q_N(c_L, c_N, x) = 0$. This is the smallest number of licenses at which external competition will be eliminated. We make the following two observations from (1). First, that absent of spillover, $q_N(\cdot)$ is monotonically decreasing in $x$. Thus without spillover, $q_L(c_L, c_N, x) = 0$ for all $x > x_1$. Secondly, $x_1$ is decreasing in $c_N - c_L$. It takes more efficient firms to drive the inefficient firms out of the market when the magnitude of innovation is small.

The patentee's profit ($\pi_P(\cdot)$) is,

$$\pi_P(c_L, c_N, x) = \begin{cases} n\pi_P^0(c_L, c_N, x), & q_N(c_L, c_N, x) = 0, \\ n\pi_P^+(c_L, c_H, x), & q_N(c_L, c_N, x) > 0. \end{cases}$$

where,

$$\pi_P^0(c_L, c_N, x) = xb(c_L, c_N, x), \quad q_N(c_L, c_N, x) = 0,$$
$$\pi_P^+(c_L, c_H, x) = xb(c_L, c_N, x), \quad q_N(c_L, c_N, x) > 0.$$

The bid $(b(c_L, c_H, c))$ varies according to (2). The patentee chooses $x$ to maximize this profit. Let $x_0$ denote the maximizer of $\pi_P^+(\cdot)$ on [0,1].

Now we are able to characterize the "basic structure" of optimal strategy of the patentee (optimal choice of $x$). There are two possibilities: (i) $x_1 \leq x_0$, and (ii) $x_1 > x_0$. (i) occurs when the magnitude of innovation is large so that inefficient firms exit the market with small $x_1$. Once $x_1$ proportion of firms have licenses, increasing licenses further simply increases internal competition. Thus just enough licenses to eliminate external competition are sold and the optimal proportion is $x_1$. Case (ii) occurs when magnitude of innovation is small. Since the cost differential is small, it is not possible to eliminate external competition. It is optimal to license to $x_0$ proportion of firms in this case. There will be $n(1 - x_0)$ firms with the higher marginal cost $c_N$ producing positive outputs. [2]

So far we have not considered spillover. Let us now differentiate the two scenarios. We denote the non-licensee marginal cost with spillover by $c_N^S$ and the marginal cost without spillover by $c_N^{NS}$. It is $c_N^{NS} = c$ by definition. We assume spillover is such that the non-licensee marginal cost is reduced more as number of licenses increase, i.e., $\frac{dc_N^S}{dx} < 0$ and $c_L < c_N^S < c$.

The immediate effect of spillover is that it reduces the "effective" magnitude of innovation, i.e., $c_N^S - c_L < c_N^{NS} - c_L = \epsilon$. Accordingly, the critical value $x_1$ with spillover $(x_1^S)$ should be greater than without spillover $(x_1^{NS})$. Because the effective cost differential is smaller with spillover, it takes more efficient firms to eliminate external competition. Thus we expect more licensing with spillover when innovation magnitude is large enough for case

[2] There may be $x > x_1$ for which $q_N(c_L, c_N, x) > 0$ if $q_N(\cdot)$ is not monotonic, as is the case when there is spillover. However, it can be shown that $\pi_P^0(c_L, c_N, x) > \pi_P^+(c_L, c_H, x)$ for such $x$.

(i) to occur. [3]

Now we consider case (ii). The maximizer $x_0$ ($x_0^S$ with spillover and $x_0^{NS}$ without) can be characterized from the first order condition,

$$\frac{d\pi_P^+(c_N, c_L, x)}{dx} = b(c_N, c_L, x) + x\frac{db(c_N, c_L, x)}{dx}. \tag{3}$$

This reflects the typical monopolist's trade-off when selling one more unit. The first term is the marginal gain of the extra sale and the second term is the marginal cost from declining bid (price) of all other sales. Both terms are influenced by spillover. Since $b(\cdot)$ is increasing in $c_N$, [4]

$$b(c_N^S, c_L, x) < b(c_N^{NS}, c_L, x). \tag{4}$$

The second term of (3) has two parts,

$$\frac{db(c_N, c_L, x)}{dx} = \frac{\partial b(c_N, c_L, x)}{\partial x} + \frac{\partial b(c_N, c_L, x)}{\partial c_N}\frac{dc_N}{dx}. \tag{5}$$

From $\frac{\partial b(c_N, c_L, x)}{\partial x} = \frac{-2(c_N - c_L)^2}{n+1}$, we have

$$0 > \frac{\partial b(c_N^{NS}, c_L, x)}{\partial x} > \frac{\partial b(c_N^S, c_L, x)}{\partial x}. \tag{6}$$

The two inequalities (4) and (6) follow from the fact that spillover reduces the effective magnitude of innovation. The situation with spillover can be similar to that of a smaller innovation without spillover. Let us call this the

[3] Of course the parameter values for which case (i) occurs differ between when there is spillover and when there is no spillover. This is discussed in detail in section 4.

[4] This follows immediately from the fact that licensee output is increasing and non-licensee output is decreasing in $c_N$ and (2).

efficiency effect of spillover.

The second term in (5) captures the other effect of spillover. This term disappears when there is no spillover since $\frac{dc_N^{NS}}{dx} = 0$. With spillover, the second term will be negative since $\frac{dc_N^S}{dx} < 0$ and $b(\cdot)$ is increasing in $c_N$. Spillover aggravates the negative externality a monopolist suffers from selling more and thus increases the marginal cost of selling one more unit. By selling one more unit, a monopolist not only moves down along the demand curve but also experiences spillover shifting the demand curve itself down. We will refer to this as the externality effect.

The efficiency and externality effects influence the optimal $x_0$ in opposite ways. If the efficiency effect dominates, patentee behaves as if innovation magnitude were smaller and we expect $x_0^S > x_0^{NS}$. If the externality effect is dominant, number of licenses are restricted to reduce the marginal cost from erosion of bids and thus $x_0^S < x_0^{NS}$. In order to determine under which conditions each effect dominates, we must be more specific about the form of spillover.

# 3 Analysis with Spillover

Now we assume that spillover is proportional to the number of licensees. That is, $c_N^S = c - x\epsilon$. $c_N^S$ can be thought of as the expected marginal cost when a firm can learn and reduce cost by interacting with a licensee. A non-licensee interacts with a licensee with probability $x$ and with a non-licensee with probability $1 - x$. Its cost will be reduced to $c - \epsilon$ if it interacts with a licensee but there is no reduction from interacting with a non-licensee. Thus the expected marginal cost will be $x(c-\epsilon)+(1-x)c = c - x\epsilon$.

The Cournot equilibrium outputs for the licensee and the non-licensee, $q_L^S(x)$ and $q_N^S(x)$ respectively, are,

$$q_L^S(x) = \begin{cases} q_L(c-\epsilon, c-x\epsilon, x) = \frac{\epsilon}{nx+1}(\alpha+1), & q_N^S(x)=0, \\ q_L(c-\epsilon, c-x\epsilon, x) = \frac{\epsilon}{n+1}\left\{\alpha+1+n(x-1)^2\right\}, & q_N^S(x)>0, \end{cases}$$

$$q_N^S(x) = q_N(c-\epsilon, c-x\epsilon, x) = \max\left\{0, \frac{\epsilon}{n+1}\{\alpha + x(-1+n(x-1))\}\right\}.$$

The specific form of spillover makes $q_N^S(x)$ non-monotonic in $x$. However, it is easy to verify that,

**Lemma 1.** *The Cournot equilibrium output for a non-licensee is,*

$$\text{If } \alpha \geq \frac{(n+1)^2}{4n}, \quad \text{then } q_N^S(x) > 0 \quad \forall x.$$

$$\text{If } \alpha < \frac{(n+1)^2}{4n}, \quad \text{then } \begin{cases} q_N^S(x) > 0, & x \notin [x_1^S, x_2^S], \\ q_N^S(x) = 0, & x \in [x_1^S, x_2^S]. \end{cases}$$

*where, $x_1^S, x_2^S$ satisfy $q_N^S(x) = 0$ and $x_1^S < x_2^S$.*

Non-licensees produce nothing if their marginal cost is too high and there are many efficient firms ($x > x_1^S$). This is true independent of spillover. But because of the spillover, if there are enough licensees ($x > x_2^S$), the non-licensees become efficient enough to produce.

$\pi_P^0(x)$, patentee profit when non-licensees exit and $\pi_P^{S+}(x)$, patentee profit when non-licensees produce positive outputs, can be found using $\pi_P^0(c_L, c_N, x)$ and $\pi_P^+(c_L, c_N, x)$. The maximizer $x_0^S$ is found from $\pi_P^{S+}(x)$.

We find that the optimal proportion $x^{*S}$, follows the "basic structure" described in the previous section. We summarize it in the following proposition. Proof is in the Appendix.

**Proposition 1.** *The optimal proportion of licensees with spillover is,*

$$x^{*S} = \begin{cases} x_1^S, & \alpha \leq \frac{(3n-5)(n+1)}{16n}, \\ x_0^S, & \alpha > \frac{(3n-5)(n+1)}{16n}, \end{cases}$$

*where $x_0^S$ maximizes $\pi_P^{S+}(x)$ on $[0, 1]$.*

If the innovation is significant ($\alpha$ is small, Figure 1, region (I)), non-licensees are at a significant disadvantage, even with the spillover. Thus optimal strategy is to eliminate external competition, $x^{*S} = x_1^S$. Licensee must pay a fee equal to the difference in profit of having and not having a license. Thus in this case, the patent owner is able to extract all the rents.

If the magnitude of innovation is small ($\alpha$ is large, Figure 1, regions (II) to (IV)), patentee is unable to eliminate external competition. The optimal proportion is chosen to balance the marginal benefit and cost described in equation (3) and thus $x^{*S} = x_0^S$.

## 4 Comparison with No Spillover Case

Kamien and Tauman (1986) have characterized optimal licensing policy when there is no spillover. The outputs of the licensee and the non-licensee when there is no spillover, $q_L^{NS}(x)$ and $q_N^{NS}(x)$, are,

$$q_N^{NS}(x) = q_N(c-\epsilon, c, x) = \max\left\{0, \frac{\epsilon}{n+1}(\alpha - nx)\right\},$$

$$q_L^{NS}(x) = q_L(c-\epsilon, c, x) = \begin{cases} \frac{\epsilon}{nx+1}(\alpha+1), & q_N^{NS}(x) = 0, \\ \frac{\epsilon}{n+1}(\alpha+1+n(1-x)), & q_N^{NS}(x) > 0. \end{cases}$$

It is easy to see that $x_1^{NS} = \frac{\alpha}{n}$. The profit of the patent owner in the no spillover case is,

$$\pi_P^{NS}(x) = \begin{cases} \pi_P^{NS0}(x) = \frac{\epsilon^2 n}{(xn+1)^2} x(\alpha+1)^2, & x \geq x_1^{NS}, \\ \pi_P^{NS+}(x) = \frac{\epsilon^2 n}{n+1} x\{2\alpha+1+n-2nx\}, & x < x_1^{NS}. \end{cases}$$

$\pi_P^{N0}(x)$ is monotonically decreasing for $x > x_1^{NS}$ and $x_0^{NS} = \frac{2\alpha+n+1}{4n}$ is the unique global maximum of $\pi_P^{N+}(x)$.

The optimal proportion of firms $x^{*NS}$ is charaterized below and follows the aforementioned "basic structure".

**Proposition 2 (Kamien and Tauman (1986)).** *The optimal proportion of licensees without spillover is,*

$$x^{*NS} = \begin{cases} x_1^{NS}, & \alpha \leq \frac{n+1}{2}, \\ x_0^{NS}, & \frac{n+1}{2} < \alpha \leq \frac{n-1+\sqrt{2n(n+1)}}{2}, \\ 1, & \frac{n-1+\sqrt{2n(n+1)}}{2} < \alpha. \end{cases}$$

If the innovation is significant (Figure 1, regions (I) and (II)), it is optimal to eliminate external competition and thus $x^{*NS} = x_1^{NS}$. If the innovation is small (Figure 1, regions (III) and (IV)), then the optimal proportion is determined by (3) and $x^{*NS} = x_0^{NS}$.

We now compare the optimal proportions of licensees with and without spillover. The effect of spillover depends on the innovation magnitude, $\alpha$.

**Proposition 3.** *The optimal proportion of licensees with and without spillover have the following relationships.*

$$\text{(I)}\quad \textit{If}\ \alpha \leq \frac{(3n-5)(n+1)}{16n},\ \textit{then}\ (1)\begin{cases} x^{*NS} = x_1^{NS} \\ x^{*S} = x_1 \end{cases},(2)\ x^{*NS} < x^{*S},$$

$$\text{(II)}\quad \textit{If}\ \frac{(3n-5)(n+1)}{16n} < \alpha \leq \frac{n+1}{2},\ \textit{then}\ (1)\begin{cases} x^{*NS} = x_1^{NS} \\ x^{*S} = x_0 \end{cases},(2)\ x^{*NS} \lesseqqgtr x^{*S} \Leftrightarrow \alpha \lesseqqgtr \alpha^0(n)$$

$$\text{(III)}\quad \textit{If}\ \frac{n+1}{2} < \alpha \leq \frac{n-1+\sqrt{2n(n+1)}}{2},\ \textit{then}\ (1)\begin{cases} x^{*NS} = x_0^{NS} \\ x^{*S} = x_0 \end{cases},(2)\ x^{*NS} > x^{*S},$$

$$\text{(IV)}\quad \textit{If}\ \frac{n-1+\sqrt{2n(n+1)}}{2} < \alpha,\ \textit{then}\ (1)\begin{cases} x^{*NS} = 1 \\ x^{*S} = x_0 \end{cases},(2)\ x^{*NS} > x^{*S},$$

*where, $\alpha^0(n)$ is a value of $\alpha$ in region (II) defined in the proof.*

The proposition is illustrated in Figure 1.

For very large magnitude of innovation (region (I)), external competition is eliminated independent of existence of spillover. There will be more licenses with spillover because only the efficiency effect matters and there is more licenses with spiilover as a direct consequence of $x_1^{NS} < x_1^S$. As magnitude of innovation decreases ($\alpha$ increases, region (II)), the magnitude of innovation is small enough so that non-licensees produce a positive amount when there is spillover ($x^{*S} = x_0^S$) but large enough so that non-licensees will produce zero when there is no spillover ($x^{*NS} = x_1^{NS}$).

As innovation magnitude decreases further, (regions (III)), optimal proportion is determined by equation (3) for both scenerios. Within this region, if the innovation size is large, then the efficiency effect dominates and $x^{*NS} < x^{*S}$. For smaller magnitudes of innovation, externality effect dominates and it becomes $x^{*NS} > x^{*S}$. When innovation size is even smaller (region (IV)), then $x^{NS} = 1 > x_0$. When there is spillover, it is never optimal for the licenses to be sold for every firm in the market ($x_0^S < 1$). Given the type of spillover considered, the spillover will be perfect for the last firm. So the willingness to pay of the last firm is 0.

Without spillover, the optimal proportion decreases with innovation size. The extreme case is the drastic innovation when only one firm will produce. It is easy to verify that $x_1^S$ is increasing in $\alpha$. Unfortunately, we are unable verify that $x_0^S$ is monotonic in $\alpha$. It could be that the externality effect makes the relationship non-monotonic. We also observe that independent of existence of spillover, market power from cost advantage is less in less concentrated markets. The critical values of $\alpha$ between regions decreases with $n$. In fact for smaller values of $n$, region (I), where non-licensees are driven out of the market doest not exist (Figure 1).

We now look at the effect of spillover on patentee profit and consumer surplus.

**Proposition 4.** *Patent owner's profit will always be greater when there is no spillover for every $\alpha$.*

*Proof.* At every $x$, bid without spillover is no less than with spillover. Thus at every $x$, patent owner's profit is greater when there is no spillover. In particular, this is true at $x^{*S}$, i.e., $\pi_P^{NS}(x^{*S}) \geq \pi_P(x^{*S})$. The equality will hold only when bids are equal which occurs when non-licensees are producing zero both with and without spillover. By definition, $\pi_P^{NS}(x^{*NS}) \geq \pi_P^{NS}(x^{*S})$. This inequality is actually strict whenever $x^{*NS} \neq x^{*S}$ since the maximum is unique. When $x^{*NS} = x^{*S}$, non-licensee is producing a positive amount with spillover. Thus the first inequality is strict. □

Let the equilibrium aggregate outputs with and without spillover be $q^S$ and $q^{NS}$. Then $q^S = x^{*S} n q_L(x^{*S}) + (1 - x^{*S}) n q_N(x^{*S})$ and $q^{NS} = x^{*NS} n q_L^{NS}(x^{*NS}) + (1 - x^{*NS}) n q_N^{NS}(x^{*NS})$. The relative size of the outputs will depend on the size of innovation.

**Proposition 5.** *The aggregate output with optimal licensing rule with and without spillover have the following relationship.*

$$q^S \gtreqqless q^{NS} \quad \Leftrightarrow \quad \alpha \lesseqqgtr \alpha^*(n),$$

*where $\alpha^*(n)$ is a value of $\alpha$ greater than $\alpha^0(n)$ defined in the proof.*

The proposition is illustrated in Figure 2. Existence of spillover may be good or bad for consumers, depending on the magnitude of innovation. If innovation is significant ($\alpha$ is small), then consumers are better off with spillover. When innovaion magnitude is large, with and without spillover, non-licensees do not produce. Under both scenerios, all firms that are producing are licensees and have the same marginal cost, $c - \epsilon$. Aggregate output increases with the number of producers with Cournot oligopoly. Thus $q^S > q^{NS}$ follows directly from $x^{*S} > x^{*NS}$. Consumers benefit from the efficiency effect of spillover.

If innovation is small ($\alpha$ is large), then consumers are worse off with spillover. The relationship $x^{*S} < x^{*NS}$ implies that there are more non-licensees and each non-licensee produces more when there is spillover. However the output of each licensee is always greater than that of a non-licensee and the negative effect of less aggregate licensee output from spillover dominates.

## 5 Concluding Remarks

A patentee must consider two forms of competition: internal and external. Internal competition refers to competition among licensees (efficient firms) and external competition refers to the competition from non-licensees (inefficient firms).

When maginitude of innovation is significant, it is optimal to license just enough licenses to eliminate external competition. The efficiency effect of spillover increases external competition, requiring more efficient firms to eliminate it. Thus spillover increases number of licenses and consumers are better off because of lower prices.

When maginitude of innovation is small, optimal number of licenses is chosen to balance the marginal benefit and cost of one extra sale. Externality effect of spillover increases the extra cost. Thus spillover requires the patentee to reduce the number of licenses. There will be less licenses with spillover and consumers are worse off.

The externality effect is very sensitive to the form of spillover considered. In our case, the *marginal* spillover, $\frac{c_N^S}{dx} = -\epsilon$ is independent of number of licenses. It is also independent of demand size and cost, measured by $a - c$. The reason why the externality effect dominates is becuase the size of spillover is relatively large compared to profit (which depends on $a - c$) when innovation is small. Thus we conjecture that there may be forms of spillover where the relationship between maginitude of innovation and dominace of externatliy effect is different, possibly opposite.

## Appendix

**Proof of Proposition 1:** Patent owner's profit, $\pi_P^S(x)$ will be as follows:

$$\pi_P^S(x) = \begin{cases} \pi_p^{S0}(x) = \frac{\epsilon^2 n}{(xn+1)^2} x(\alpha+1)^2, & x \in [x_1, x_2] \text{ and } \alpha < \frac{(n+1)^2}{4n}. \\ \pi_p^{S+}(x) = \frac{\epsilon^2 n}{n+1} x(1-x)\{(1-2x)(1-x)n + 2\alpha + 1 + x\}, & x \notin [x_1, x_2] \text{ or } \alpha \geq \frac{(n+1)^2}{4n}. \end{cases}$$

The "basic structure" of the optimal proportion, $x^{*S}$ is as described in section 2. The proof is complicated by the fact that spillover makes $q_N^S(x)$ and $q_L^S(x)$ not monotonic in $x$. One consequence is that $q_L^S(x)$ maybe positive for very large values (Lemma 1, $x > x_2^S$). Because of the non-monotonicity of outputs, $\pi_p^{S+}$ may have two local maxima. It becomes necessary to determine which optima is the global maximum and how the maximizer $x_0^S$ is relates to $x_1^S$ and $x_2^S$. The proof does the following: We first characterize $x_0^S$ in step (1). Then we establish conditions determining $x_1^S \gtreqless x_0^S$ in step (2). In steps (3) and (4) we show that there is never an incentive to license to more than $x_2^S$.

**(1)** There are at most two local maxima of $\pi_P$ on $[0, 1]$. Assume they occur at $\underline{x}_0$ and $\bar{x}_0$. Then the following are true,

**(i)** $\underline{x}_0 < \frac{1}{2}$ and it always exists and is always the global maximum.

**(ii)** $\frac{1}{2} < \bar{x}_0$ exists when $n^2 - 30n + 3 - 32\alpha n > 0$.

**Proof of (i)** We first check the second order derivative,

$$\frac{\partial^2 \pi_p^{S+}}{\partial x^2} = -24nx^2 + (30n - 6)x - 8n - 4\alpha.$$

$$\frac{\partial^2 \pi_p^+}{\partial x^2}|_{x=0} = -8n - 4\alpha < 0, \quad \frac{\partial^2 \pi_p^+}{\partial x^2}|_{x=1} = -2n - 6 - 4\alpha < 0. \quad (7)$$

We check the discriminant of $\frac{\partial^2 \pi_p^+}{\partial x^2}$,

$$\begin{aligned} D &= (30n-6)^2 - 4(24n)(8n+4\alpha) \\ &= 6^2(25n^2 - 10n + 1) - 4 \times 3(8^2 n^2 + 8 \times 4\alpha) \\ &= 12(11n^2 - 30n + 3 - 32\alpha n) \end{aligned}$$

$D < 0 \Longrightarrow \frac{\partial^2 \pi_p^+}{\partial x^2} < 0 \ \ \forall x \in [0,1]$ from (7). Also,

$$\frac{\partial \pi_p^+}{\partial x}|_{x=0} = n + 2\alpha + 1 > 0, \quad \frac{\partial \pi^+}{\partial \pi}|_{x=1} = -2(2\alpha + 2) < 0.$$

Thus there is a unique $\underline{x}_0$ such that $\frac{\partial \pi_p^+}{\partial x}|_{x=\underline{x}_0} = 0$ and it is the global maximum. In addition,

$$\frac{\partial \pi_p^+}{\partial x}|_{x=\frac{1}{2}} = \frac{1}{4}(1-n) < 0 \Longrightarrow \ \underline{x}_0 < \frac{1}{2}. \ \square$$

**Proof of (ii)**

$$\begin{aligned} & D \geq 0, \ \frac{\partial^3 \pi_p^+}{\partial x^2} = 30n - 48xn - 6 \text{ and } \frac{\partial^3 \pi_p^+}{\partial x^2} = 0 \Leftrightarrow x = \frac{30n-6}{48n} = \frac{5}{8} - \frac{1}{8n} \in (0,1), \\ \Longrightarrow \quad & \exists x', x'' \text{ s.t. } \begin{cases} \frac{\partial^2 \pi_p^+}{\partial x^2} & < 0 \text{ for } x < x', x > x'' \\ \frac{\partial^2 \pi_p^+}{\partial x^2} & > 0 \text{ otherwise }. \end{cases} \\ \Longrightarrow \quad & \exists \underline{x}_0 < x', \exists \bar{x}^0 > x'' \text{ s.t. } \frac{\partial \pi_p^+}{\partial x} = 0, \text{ i.e., local maxima }. \end{aligned}$$

$\pi_p$ consists of 2 parts. $x(1-x)$ and $[(1-2x)(1-x)n + 2\alpha + 1 + x]$. The first term is symmetric around $x = \frac{1}{2}$, the second term is symmetric around $x = \frac{3}{4} - \frac{1}{4}n \in (\frac{1}{2}, 1)$. Thus, for $\forall x \in (\frac{1}{2}, 1), \exists x' \in (0, \frac{1}{2})$ s.t. $\pi_p^+(x') > \pi_p^+(x) \Rightarrow \underline{x}_0$ is the global maximum. □

**(2)** $\frac{\partial \pi_p^+}{\partial x}|_{x=x_1} \gtreqless 0 \Leftrightarrow \alpha \gtreqless \bar{\alpha}$, where $\bar{\alpha}$ is defined below.

**Proof:** From the definitions,

$$\begin{aligned}
\pi_p^+ &= nx\{(q_L)^2 - (q_N)^2\}, \\
\frac{\partial \pi_p^+}{\partial x} &= n\{(q_L)^2 - (q_N)^2\} + nx\{2q_L\frac{\partial q_L}{\partial x} - 2q_N\frac{\partial q_N}{\partial x}\}, \\
\frac{\partial \pi_p^+}{\partial x}|_{x=x_1} &= n(q_L)^2 + nx(2q_L\frac{\partial q_L}{\partial x}) = nq_L(q_L + 2x\frac{\partial q_L}{\partial x}), \\
q_L &= \frac{\epsilon}{n+1}(\alpha + 1 + n(1-x)^2), \\
\frac{\partial q_L}{\partial x} &= \frac{\epsilon}{n+1}(-2n)(1-x), \\
\Longrightarrow \frac{\partial \pi_p^+}{\partial x}|_{x=x_1} &= nq_L\frac{\epsilon}{n+1}(\alpha + 1 + n(1-x_1)(1-5x_1)).
\end{aligned}$$

We check the sign of $\alpha + 1 + n(1-x_1)(1-5x_1)$. First note that,

$$q_N = \frac{\epsilon}{n+1}\{\alpha + x(1-n) + x^2 n\} \quad \text{and} \quad q_N|_{x=x_1} = 0.$$

It follows that, $\alpha + x_1(1-n) + x_1^2 n \implies (x_1)^2 n = x_1(n-1) - \alpha$. This implies

$$
\begin{aligned}
&\alpha+1+n(1-x_1)(1-5x_1)=\alpha+1+n-6nx_1+5n(x_1)^2\\
&=\alpha+1+n-6nx_1+5(x_1(n-1)-\alpha)\\
&=(-5-n)x_1+n+1-4\alpha\\
&=-(n+5)\cdot\frac{n-1-\sqrt{(n-1)^2-4n\alpha}}{2n}+n+1-4\alpha\\
&=\frac{1}{2n}\{-(n+5)(n-1)+2n(n+1-4\alpha)+(n+5)\sqrt{(n-1)^2-4n\alpha}\}\\
&=\frac{1}{2n}\{n^2-2n+5-8n\alpha+(n+5)\sqrt{(n-1)^2-4n\alpha}\}.
\end{aligned}
$$

If $\alpha < \frac{n^2-2n+5}{8n}$, then $\frac{\partial\pi_p}{\partial x}|_{x=x_1} > 0$. If $\alpha > \frac{n^2-2n+5}{8n}$, then we need to compare the absolute values of the two terms. Note that,

$$
(n+5)^2(\sqrt{(n-1)^2-4n\alpha})^2-(n^2-2n+5-8n\alpha)^2 \tag{8}
$$

$$
=4n(-16n\alpha^2+(3n^2-(8n-5)\alpha+(3n-5)(n+1)). \tag{9}
$$

Let $\alpha_1 \equiv \frac{3n^2-18n-5+\sqrt{(3n^2-18n-5)^2+64n^3(3n-5)(n+1)}}{32n^2} = \frac{(3n-5)(n+1)}{16n}$. Then if $\alpha > \frac{n^2-2n+5}{8n}$, then $\frac{\partial\pi_p}{\partial x}|_{x=x_1} \gtreqless 0 \Leftrightarrow \alpha \gtreqless \alpha_1$. Together, we have,

$$
\frac{\partial\pi_p^+}{\partial x}|_{x=x_1} \gtreqless 0 \Leftrightarrow \alpha \gtreqless \bar{\alpha} \equiv \max\{\alpha_1, \frac{n^2-2n+5}{8n}\} = \alpha_1. \ \square
$$

**(3)** $\frac{\partial\pi_p^0}{\partial x} < 0 \forall x \in [x_1, x_2]$.

**Proof:** From the definition and taking the derivative,

$$\frac{\partial \pi_p^0}{\partial x} = \frac{(\alpha+1)^2(1-nx)}{(nx+1)^3} \gtreqqless 0 \Leftrightarrow x \lesseqqgtr \frac{1}{n}.$$

$$x_1 - \frac{1}{n} = \frac{n-1-\sqrt{(n-1)^2-4n\alpha}}{2n} - \frac{1}{n} = \frac{1}{2n}(n-3-\sqrt{(n-1)^2-4n\alpha}) > 0.$$

$$\Longrightarrow \frac{\partial \pi_p^0}{\partial x} < 0 \text{ for } x > x_1. \ \square$$

**(4)** $\pi_p^+(x_1) > \pi_p^+(\bar{x}_0)$ for $n \geq 6$.

**Proof:** Recall that $\bar{x}_0 > \frac{1}{2}$. We first find an upper bound, $\bar{\pi}$, of $\pi_p^+(\bar{x}_0)$. Then we show that $\pi_p^+(\frac{1}{2}) > \bar{\pi}$.

**(i)** $\bar{x}_0 > \frac{3}{4} - \frac{1}{4}n$ (we already showed $\bar{x}_0 > \frac{1}{2}$).

**Proof:** $\pi_p(x) = f(x) \cdot g(x)$ where

$$f(x) = x(1-x), \quad g(x) = (27-1)(2-1)n + 2\alpha + 1 + x.$$

$f(x)$ is monotonically decreasing on $x > \frac{1}{2}$. $g(x)$ attains minimum at $x_b \equiv \frac{3}{4} - \frac{1}{4n}$. We check the discriminant of $g(x)$,

$$D_g = (1-3n)^2 - 4(2n)(n+2\alpha+1) = n^2 + 1 - 16n\alpha - 14n.$$

If $D_g < 0$, then $g(x)$ is positive for $\frac{1}{2} < x < 1$, and decreasing on $\frac{1}{2} < x < x_b \Longrightarrow \bar{x}_0 > x_b$. If $D_g > 0$, then $\bar{x}_0 > x_b$ if $\frac{\partial \pi_p}{\partial x}|_{x=x_b} > 0$. In fact, $\frac{\partial \pi_p}{\partial x}|_{x=x_b} = \frac{(n^2+1-16n\alpha-14n)(n-1)}{n^2} > 0$. $\square$

**(ii)** We divide $[\frac{3}{4} - \frac{1}{4}n, 1]$ into 3 intervals. We show that 3 upper bounds (for each interval) are all less than $\pi_p(x_1)$.

**Proof:** Define the intervals by,

$$I_1 = [\frac{3}{4} - \frac{1}{4}n, m], \quad I_2 = [m, 1 - \frac{2}{2n}], \quad I_3 = [1 - \frac{1}{2n}, 1],$$

where $m = \frac{7n-3}{8n} = \frac{1}{2}(\frac{3}{4} - \frac{1}{4n} + 1 - \frac{1}{2n})$. Let $\pi_i$ denote upper bound of interval $I_i$, i.e. $\pi_p(x) < \bar{\pi}_i \ \forall x \in I_i$. We construct $\bar{\pi}_i$ below:

$$\bar{\pi}_1 = f(\frac{3}{4} - \frac{1}{4n})g(m), \quad \bar{\pi}_2 = f(m)g(1 - \frac{1}{2n}), \quad \bar{\pi}_3 = f(1 - \frac{1}{2n})g(1).$$

By choice of $1 - \frac{1}{2n}$, $g(1 - \frac{1}{2n}) = 2\alpha + \frac{3}{2}, \Longrightarrow \bar{\pi}_2 > 0$. $\bar{\pi}_1$ may be negative.

$$\begin{aligned}
\bar{\pi}_1 &= \frac{(3n-1)(n+1)}{16n^2} \times (-\frac{3n^2 - 54n + 3 - 64n\alpha}{32n}) \\
&= \frac{-(3n-1)(n+1)13n^2 - 54n + 3 - 64n\alpha)}{512n^3} \\
\bar{\pi}_2 &= \frac{(12n-3)(n+3)(4\alpha+3)}{n^2} \\
\bar{\pi}_3 &= \frac{(2n-1)(\alpha+1)}{2n^2}.
\end{aligned}$$

We simplify $\pi_p(x_1)$ using the fact that $q_N(x_1) = 0$, i.e,

$$q_N(x_1) = 0 \Longrightarrow x_1^2 n = x_1(n-1) - \alpha.$$

By substituting this repeatedly into $\pi_p(x_1)$, we have,

$$\pi_p^+(x_1) = \frac{n+1}{n}(\alpha n + \alpha + x_1 - x_1 n\alpha).$$

$$\pi_p^+(x_1) - \bar{\pi}_1 = \frac{n+1}{512n^3}(-512x_1n^2\alpha + 512x_1n + 320\alpha n^2 + 576n\alpha + 9n^3 - 165n^2 + 6n - 3)$$

$$\begin{aligned}
(\quad) &> -512x_1n^2\alpha + 320\alpha n^2 + 576n\alpha + 9n^3 - 165n^2 \text{ ( from } 6n > 3 \text{ since } n \geq 6) \\
&> -256n^2\alpha + 320\alpha n^2 + 576n\alpha + 9n^3 - 165n^2 \text{ ( from } x_1 < \frac{1}{2}) \\
&= 64\alpha n^2 + 576n\alpha + 9n^3 - 165n^2 \\
&= n(9n^2 + 64\alpha n + 576\alpha - 165n) \\
&> n(9n^2 + 64n + 576 - 165n)2 > 1 \\
&= n(9n^2 - 101n + 576)
\end{aligned}$$

Let $h(n) = 9n^2 - 101n + 576$. Then,

$h'(n) = 18n - 101, h'(6) = 18\times6 - 101 > 0, h(6) = 9\times36 - 101\times6 + 576 > 0.$

This implies, $h(n) > 0$ for $n \geq 6$. That is, for $n \geq 6$, $\pi_p^+(x_1) - \bar{\pi}_1 > 0$.

$$\pi_p^+(x_1) - \bar{\pi}_2 = \frac{1}{128n^2}(-128x_1n\alpha + 128x_1 + 184n\alpha + 164\alpha - 128x_1n^2d + 128x_1n + 100\alpha n^2 - 21n^2 - 54n + 27)$$

$$
\begin{aligned}
(\quad) &> -64n\alpha + 128x_1 + 184n\alpha + 164\alpha - 64n^2\alpha + 128x_1 n + 100n^2\alpha \\
&- 21n^2 - 54n + 27 \quad x_1 < \frac{1}{2} \\
&= 120n\alpha + 128x_1 + 164\alpha + 36n^2\alpha + 128x_1 n - 21n^2 - 54n + 27 \\
&> (120\alpha - 54)n + (36\alpha - 21)n^2 > 0 \ \text{( since } \alpha > 1).
\end{aligned}
$$

Thus we have, $\pi_p(x_1) > \bar{\pi}_2$.

$$
\pi_p^+(x_1) - \bar{\pi}_3 = \frac{2(n+1)n(\alpha n + \alpha + x_1 - x_1 n\alpha)}{2n^2} - \frac{(2n-1)(\alpha+1)}{2n^2}.
$$

Since $n \geq 6$,

$$
= \begin{cases} (n+1)n > 2n - 1 \\ \alpha n + \alpha + x_1 - x_1 n\alpha - (\alpha+1) = (\alpha n - 1)(1 - x_1) > 0 \end{cases}
$$

It follows that,

$$
\pi_p^+(x_1) > \bar{\pi}_3.
$$

From (i), we know that $\bar{x}_0 \in I_i \exists i \in \{1, 2, 3\}$. Thus

$$
\pi_p^+(x_1) > \bar{\pi}_i > \pi_p^+(\bar{x}_0). \ \square
$$

**Remark** For $n \leq 5, 11n^2 - 30n - 32n\alpha + 3 < 0$ and $(n-1)^2 - 4n\alpha < 0$ since $\alpha > 1$. From (1), we have $x^{*S} = x_0$ for $\alpha \geq \frac{(n-1)^2}{4n}$. From (2),(3),(4), $\frac{(n-1)^2}{4n} > \frac{(3n-5)(n+1)}{16n}$, and $\frac{(n-1)^2}{4n} < 1$ for $n \leq 5$, we have the proposition. $\square$

**Proof of Proposition 3**

(I) $\alpha \leq \dfrac{(3n-5)(n+1)}{16n} \quad \Rightarrow \quad \begin{cases} x^{*NS} = x_1^{NS} = \frac{\alpha}{n} \\ x^{*S} = x_1 = \frac{(n-1)-\sqrt{(n7)^2-4n\alpha}}{2n} \end{cases}$.

$$x_1^{NS} - x_1^S = \frac{2\alpha - n + 1 + \sqrt{(n7)^2 - 4n\alpha}}{2n},$$

$$(n7)^2 - 4n\alpha - (-2\alpha + n - 1)^2 = -4\alpha(\alpha + 1) < 0.$$

$$\Longrightarrow x_1^{NS} < x_1^S$$

(II) $\dfrac{(3n-5)(n+1)}{16n} < \alpha \leq \dfrac{n+1}{2} \Rightarrow \begin{cases} x^{*NS} = x_1^{NS} = \frac{\alpha}{n} \\ x^{*S} = x_0, where \frac{\partial \pi_p^+}{\partial x}(x_0^S) = 0. \end{cases}$.

We check the sign of

$$\frac{\partial \pi_p^*}{\partial x}(\frac{\alpha}{n}) = \frac{\epsilon^2}{n(n+1)}(-8\alpha^3 + 11\alpha^2 n - 6\alpha n^2 - 3\alpha^2 + n^3 + n^2)$$

Let $f(\alpha) = -8\alpha^3 + 11\alpha^2 n - 6\alpha n^2 - 3\alpha^2 + n^3 + n^2$. Then,

$$f\left(\frac{(3n-5)(n+1)}{16n}\right) = \frac{(107n4 - 10n^3 + 84n^2 - 30n + 25)(n+1)(n+5)}{512n^3} > 0,$$

$$f\left(\frac{n+1}{2}\right) = -\frac{(n^2+7)}{4n} < 0.$$

We also have,

$$\frac{df}{d\alpha} = -24\alpha^2 + 22n\alpha - 6n^2 - 6\alpha < 0 \;\; \forall \alpha \text{ when } n \geq 2.$$

From the sign of the discriminant,

$$D_f = (22n-6)^2 - 4(24)(6n^2) = 4n^2(2n^2+8n-3) < 0 \ \ n \geq 2,$$

we have $\exists \alpha^0(n) \in [\frac{(3n-5)(n+1)}{16n}, \frac{n+1}{2}]$ such that

$$f(\alpha) \gtreqless 0 \Leftrightarrow \ \alpha \gtreqless \alpha_0 \Leftrightarrow \frac{\partial \pi_p^+}{\partial x}(\frac{\alpha}{n}) \gtreqless 0.$$

Since $f(\frac{n}{2}) = \frac{-n(n+1)}{4(n+1)} < 0$ it must be that $\alpha_0 < \frac{n}{2}$. Thus $\frac{\partial \pi_p^+}{\partial x}(\frac{\alpha}{n}) > 0$ occurs when $\alpha < \frac{n}{2}$ or $\frac{\alpha}{n} < \frac{1}{2}$, implying $\frac{\alpha}{n} < x_0$. $\frac{\partial \pi_p^+}{\partial x}(\frac{\alpha}{n}) < 0$ implies $\frac{\alpha}{n} > x_0$. That is,

$$\alpha \gtreqless^{\exists} \alpha^0(n) \Leftrightarrow x_1^{NS} \gtreqless x_0.$$

$$\text{(III)} \quad \frac{n+1}{2} < \alpha \leq \frac{n-1+\sqrt{2n(n+1)}}{2} \Rightarrow \begin{cases} x^{*NS} = x_0^{NS} \\ x^{*S} = x_0 = \frac{2\alpha+n+1}{4n} \end{cases}.$$

We check the value of

$$\frac{\partial \pi_p^+}{\partial x}(x_0^{NS}) = \frac{1}{16}(2\alpha+n+1)(-3n^2+8n+6n\alpha-8\alpha^2-14\alpha-5)$$

Let $f(x) = -3n^2+8n+6n\alpha-8\alpha^2-14\alpha-5$. Then,

$$\frac{df}{d\alpha} = 6n - 16\alpha - 14, \ \ \frac{df}{d\alpha}|_{\alpha=\frac{n+1}{2}} = -2n-22 < 0.$$

This implies that, since $\frac{df}{d\alpha}$ is decreasing in $\alpha$, $\frac{df}{d\alpha} < 0 \ \ \forall \alpha \geq \frac{n+1}{2}$. Together with, $f(\frac{n+1}{2}) = -2n^2 - 14 < 0$, that implies $f$ is negative $\forall \alpha \geq \frac{n+1}{2}$. That is, we have

$$\frac{\partial \pi_p^+}{\partial x}\left(\frac{2\alpha+n+'}{4n}\right)<0 \ \forall \alpha \geq \frac{n+1}{2}.$$

That is,

$$x_0^{NS}=\frac{2\alpha+n+1}{4n}>x_0, \quad \text{i.e. } x^{*NS}>x^{*NS}.$$

When $\alpha=\frac{n-1+\sqrt{2n(n+1)}}{2}$, $x^{*NS}$ takes two values, $x_0^{NS}$ and 1.

(IV) $$\frac{n-1+\sqrt{2n(n+1)}}{2}<\alpha \Rightarrow \begin{cases} x^{*NS}=1 \\ x^{*S}=x_0 \end{cases}.$$

Obviously $x_0<1$.

**Proof of Proposition 5** Using the definitions for $q_L, q_N$ etc., in $x$ we get

$$\begin{aligned} q^S &= \frac{n(x+\alpha)+nx(1-x)}{n+1}=\frac{n(2x+\alpha-x^2)}{n+1} \\ q^{NS} &= \frac{n(x+\alpha)}{n+1} \end{aligned}$$

The formula is valid only when $q_N \geq 0$. Fortunately this is the case in equilibrium, i.e. at $x^{*S}$ and $x^{*NS}$.

Immediately, we know that $q^{NS}<q^S$ when $x^{*NS}\leq x^{*S}$. This occurs in region I and part of region II ($\alpha \leq \alpha_0$). In fact, for region I, we know

$$\begin{aligned} x^{*NS} &= x_1^{NS}=\frac{\alpha}{n}, \\ x^{*S} &= x_1^S=\frac{n-1-\sqrt{(n-1)^2-4n\alpha}}{2n}. \end{aligned}$$

Then we have

$$\begin{cases} q^S = -\frac{\sqrt{(n-1)^2-4n\alpha}-2n\alpha+1-n}{2n} \\ \quad = \alpha + \frac{n-1-\sqrt{(n-1)^2-4n\alpha}}{2n}, \\ q^{NS} = \alpha. \end{cases}$$

In region IV, $x^{*S} = x_0^S < 1 = x^{*NS}$. Thus $q^S < q^{NS}$.

To see what happens in part of region II ($\alpha > \alpha^0(n)$) and region III, we verify how $x^{*S}$ and $x^{*NS}$ changes with $\alpha$.

$$x^{*NS} = \begin{cases} x_1^{NS} & = \frac{\alpha}{n} & \alpha < \frac{n+1}{2} & \text{(region II)} \\ x_0^{NS} & = \frac{2\alpha+n+1}{4n} & \alpha \geq \frac{n+1}{2} & \text{(region III)} \end{cases}$$

We see that $x^{*NS}$ is increasing in $\alpha$. On the other hand, $x^{*S} = x_0$ satisfies

$$\frac{\partial \pi_p^+}{\partial x} = 0.$$

Thus we have

$$\frac{dx_0^S}{d\alpha} = -\frac{\frac{\partial^2 \pi_p^+}{\partial \alpha \partial x}}{\frac{\partial^2 \pi_p^+}{\partial x^2}}.$$

Since $x_0^S$ is a maximum, $\frac{\partial^2 \pi_p^+}{\partial x^2} < 0$. Thus sign of $\frac{dx_0^S}{d\alpha}$ = sign of $\frac{\partial^2 \pi_p^+}{\partial \alpha \partial x}$. Straightforward calculation yields $\frac{\partial^2 \pi_p^+}{\partial \alpha \partial x} = -\frac{n(4x-2)}{n+1} < 0$. That is, $x^{*S}$ is decreasing in $\alpha$.

From continuity of $x_0, x_1^{NS}, x_0^{NS}$ in $\alpha$ and the continuity of $q^S$ and $q^{NS}$ in $x$, we see that there is $^\exists \alpha^*(n) > \alpha^0(n)$, such that at $\alpha^*(n), q^S = q^{NS}$. Furthermore,

$$\alpha \gtreqless \alpha^* \Leftrightarrow q^S \gtreqless q^{NS}.$$

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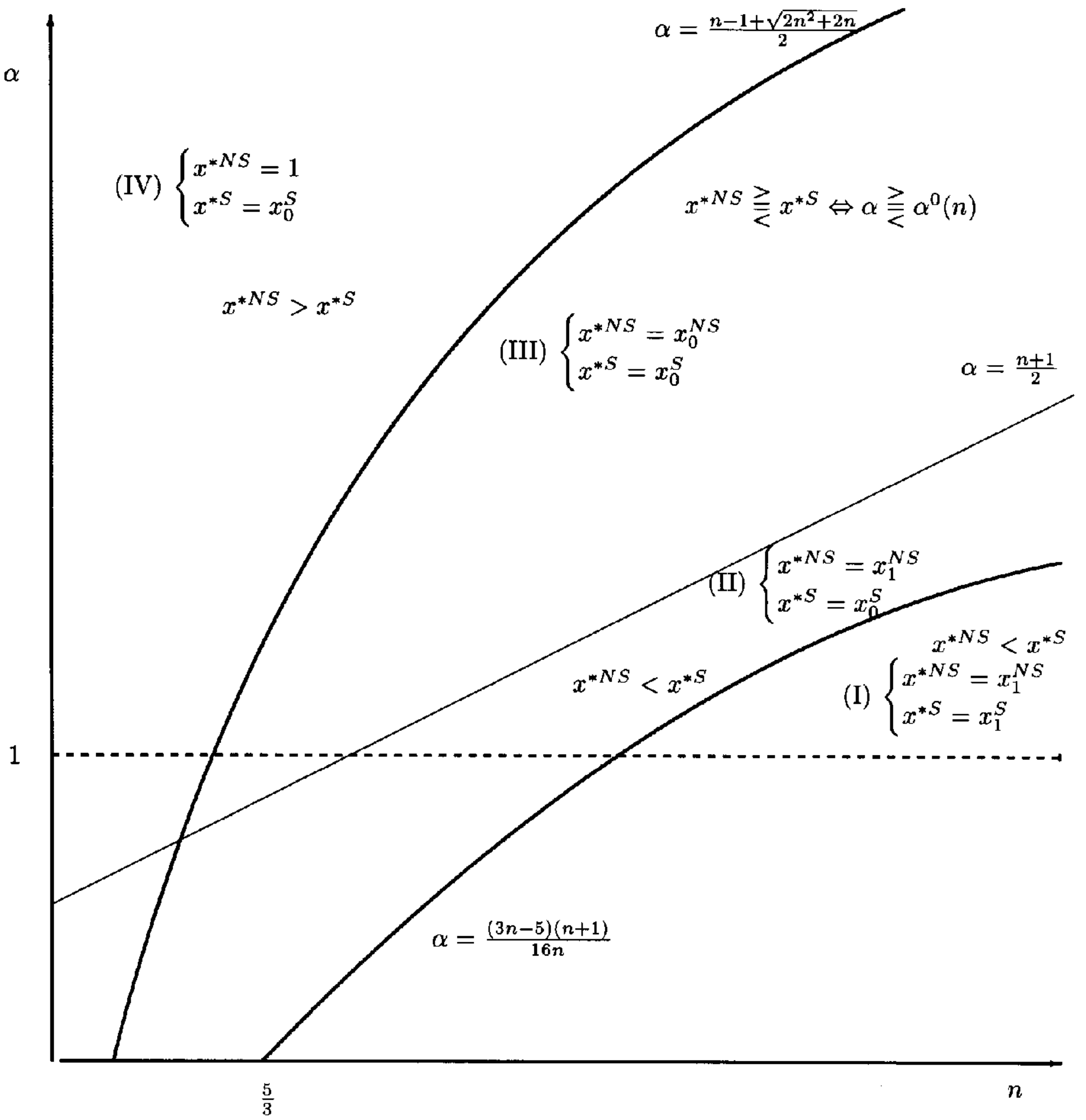
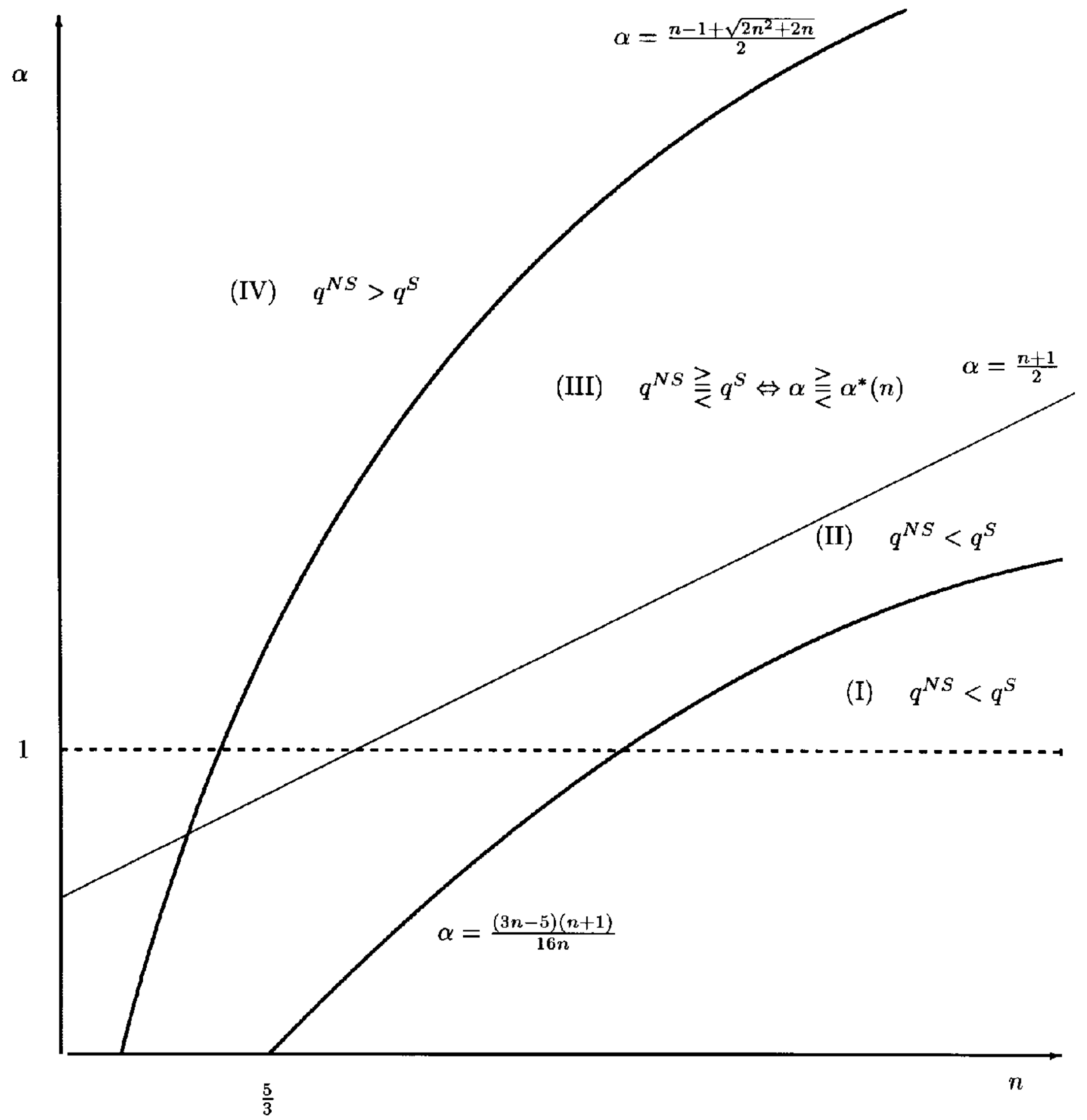


$\alpha = \frac{n-1+\sqrt{2n^2+2n}}{2}$
$\alpha$
(IV) $\begin{cases} x^{*NS} = 1 \\ x^{*S} = x_0^S \end{cases}$
$x^{*NS} \gtreqless x^{*S} \Leftrightarrow \alpha \gtreqless \alpha^0(n)$
$x^{*NS} > x^{*S}$
(III) $\begin{cases} x^{*NS} = x_0^{NS} \\ x^{*S} = x_0^S \end{cases}$
$\alpha = \frac{n+1}{2}$
(II) $\begin{cases} x^{*NS} = x_1^{NS} \\ x^{*S} = x_0^S \end{cases}$
$x^{*NS} < x^{*S}$
$x^{*NS} < x^{*S}$
(I) $\begin{cases} x^{*NS} = x_1^{NS} \\ x^{*S} = x_1^S \end{cases}$
1
$\alpha = \frac{(3n-5)(n+1)}{16n}$
$\frac{5}{3}$
$n$


**Figure 1**



$\alpha$
$\alpha = \frac{n-1+\sqrt{2n^2+2n}}{2}$
(IV) $q^{NS} > q^S$
(III) $q^{NS} \gtreqless q^S \Leftrightarrow \alpha \gtreqless \alpha^*(n)$
$\alpha = \frac{n+1}{2}$
(II) $q^{NS} < q^S$
(I) $q^{NS} < q^S$
1
$\alpha = \frac{(3n-5)(n+1)}{16n}$
$\frac{5}{3}$
$n$


**Figure 2**