The Maxim Criterion and Randomised Behaviour Reconsidered

Matthew Ryan
University of Auckland, m.ryan@auckland.ac.nz

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Matthew Ryan[†]

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Abstract

Mixed strategy equilibria are often regarded as unconvincing behavioral predictions (eg. Stahl (1988)). Furthermore, while many games possess mixed equilibria, explicit randomization is rare in practice. We argue that both problems arise because conventional game theory excludes *maximin* behavior. By considering a generalization of Nash equilibrium, we prove that there exist plausible equilibria for 2×2 games which involve randomized choice. Interestingly, the randomizing player always adopts a maximin strategy. Maximin behavior is therefore crucial to explaining randomization. We also prove that games with randomized equilibria are *non-generic*, hence unlikely to be observed in practice.

Keywords: Maximin criterion, randomization, 2×2 games, maxmin expected utility.

JEL classification: C72, D81

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[†]Department of Economics, University of Auckland, Private Bag 92019, Auckland, New Zealand. Phone +64-9-373 7999. Facsimile +64-9-373 7427. E-mail: m.ryan@auckland.ac.nz

1 Introduction

Conventional game theory does not provide a convincing treatment of randomized behavior. Many games have mixed equilibria, but none of these equilibria offer a compelling motivation for randomization[1].

To illustrate, consider the familiar game of Matching Pennies, whose payoff bi-matrix is depicted in Figure 1.

		2	
		α	β
1	A	1,-1	-1,1
	B	-1,1	1,-1

Figure 1

As is well known, this game has a unique Nash equilibrium in which each player chooses each of his or her pure strategies with probability $\frac{1}{2}$. However, since the players are indifferent amongst *all* of their strategies, it is difficult to see why one should expect them to behave precisely as the equilibrium dictates.

One response to this problem is to reinterpret the equilibrium. Harsanyi's (1973) payoff perturbations, or Brown's (1951) "fictitious play", characterize equilibria as de-

[1] Recent discussions of the point may be found in Stahl (1988) and Cheng and Zhu (1995).

scribing (limiting) distributions of *pure* strategy choices over time, or across a population of player types. Under these interpretations, the Matching Pennies equilibrium does not imply that players actually randomize. However, such descriptive devices leave us none the wiser about how players *do* in fact behave in a one-off Matching Pennies encounter.

The early literature on zero-sum games arguably offers a more satisfying analysis of Matching Pennies. This literature asserts that (a) players in zero-sum games face considerable uncertainty about their rival's choice of strategy; and (b) *maximin* behavior is a natural response to such uncertainty. The famous *Minmax Theorem* proves that maximin always selects an equilibrium strategy in a zero-sum game[2].

In Matching Pennies, for example, the equilibrium strategies are also the players' unique maximin strategies: the *worst* that can happen to a player when playing the equilibrium strategy, is *strictly better* than the worst-case scenario associated with *any other strategy*. This provides a positive incentive to conform to the equilibrium. However, modern game theory excludes this important motive for randomization, because the maximin rule is incompatible with expected utility (EU) preferences. Indeed, a *strict* preference for *any* randomized strategy is incompatible with EU.

Historically, maximin vanished from game theory because of problems in the analysis of non-zero-sum games. When analysing Battle of the Sexes, for example, Luce and Raiffa (1957, pp.93-4) observe that the maximin and Nash predictions diverge, resulting in a theoretical dilemma. Game theorists abandoned maximin in favor of Nash.

However, while the EU behavior implicit in Nash equilibrium is incompatible with maximin, the basic logic of equilibrium is not. Lo (1996) has recently shown how

[2]See Luce and Raiffa (1957, pp.71-2).

to generalize this logic to encompass non-EU preferences. Lo's analysis is based on Gilboa and Schmeidler's (1989) *maxmin expected utility (MMEU)*, which includes both (subjective) expected utility and maximin as special cases, along with a range of other behaviors.

In the present paper, we use Lo's framework to study 2×2 games. We establish conditions under which randomized choice is *strictly* optimal in equilibrium. It is shown that these conditions *always* imply maximin behavior, *whether or not the game is zero-sum*. Thus, within the broad MMEU class, maximin behavior is fundamental to all randomized choice. Furthermore, we show that randomization is *non-generic*. That is, the observation of randomized behavior is a "zero probability" event. Since explicit randomization is rare in practice, this is an appealing result.

The paper is organized as follows. The next section reviews MMEU and Lo's equilibrium concepts. The main results are presented in section 3, and discussed in section 4, along with some related literature. Section 5 concludes.

2 MMEU preferences and normal form games

2.1 SEU versus the Ellsberg "paradox"

Each player in a normal form game faces a decision problem under uncertainty. Since players may randomize their strategy choice, an element of risk is also potentially present[3]. The decision-making framework of Anscombe and Aumann (1963) is therefore appropriate. In particular, we shall let Ω denote a finite set of states, and X a

[3] We are here maintaining Knight's (1921) well-known distinction between risk and uncertainty.

mixture set (Herstein and Milnor (1953)) of "roulette lotteries". The objects of choice are *acts*, being mappings which associate an outcome in X to each state in Ω. The set of acts may therefore be denoted X^Ω. In what follows we shall (ab)use the notation x to denote both the outcome in X, and also the *constant act* which delivers outcome x contingent on each $\omega \in \Omega$. The intended meaning should be clear from the context.

If a preference ordering $\succeq$ on X^Ω satisfies the axioms of subjective expected utility (SEU), then there will exist an affine utility function $u : X \to \mathbb{R}$, and a probability

$$p \in \Delta(\Omega) \;=\; \left\{ q \in [0,1]^\Omega \,\middle|\, \sum_{\omega \in \Omega} q(\omega) \;=\; 1 \right\}$$

such that

$$f \succeq g \quad \Leftrightarrow \quad \sum_{\omega \in \Omega} p(\omega)(u \circ f)(\omega) \;\geq\; \sum_{\omega \in \Omega} p(\omega)(u \circ g)(\omega).$$

One of the key SEU axioms is *independence*. It may be stated as follows: for all $f, g, h \in X^\Omega$ and all $\alpha \in (0, 1)$,

$$f \succ g \quad \Leftrightarrow \quad \alpha f + (1 - \alpha)h \succ \alpha g + (1 - \alpha)h \tag{1}$$

An implication of this axiom is that randomization can never make an SEU-maximizer better off. Suppose, for example, that $f \sim g = h$. The act $\alpha f + (1 - \alpha)g$ is equivalent to choosing act f with probability α, and choosing g with probability $1 - \alpha$. Condition (1) implies the following indifference relations:

$$\alpha f + (1 - \alpha)g \sim g \sim f.$$

From this one may deduce the familiar facts that an SEU-maximizer will only randomize when indifferent amongst all acts assigned non-zero probability; and will further be indifferent between choosing any of these acts directly, and randomizing over them in any fashion whatsoever. This phenomenon is clearly at the heart of the our difficulties with the Matching Pennies equilibrium.

However, the independence axiom is regularly violated in experimental studies of decision-making. As an example, we shall present a simple variation on Ellsberg's (1961) classic "two-urn experiment" (*ibid.*, pp.650-3).

An urn contains 100 balls, each of which is known to be either black or red in color, but the proportion of each color in the urn is unknown. The experimenter draws a ball from the urn completely at random, concealing its color from the subject. The subject is then offered the choice between a bet in which she wins \$1 if the ball is black, nothing otherwise; and a bet in which she wins \$1 if the ball is red, nothing otherwise. Subjects are typically indifferent between these two bets. If instead, a choice is offered between either of these bets, and a bet in which a fair coin is tossed, \$1 being won if the coin comes up heads, nothing if tails, subjects generally express a strict preference for the latter bet.

To see that these reasonable-sounding preferences violate (1), let $\Omega = \{B, R\}$ and $X = \Delta(\{0,1\})$. The state B (respectively, R) corresponds to the ball drawn randomly from the urn being black (respectively, red) in color. The set X is the set of all lotteries whose prizes are either \$0 or \$1. Let us describe a typical element of X by the probability of winning \$1. That is, the lottery in X which gives the prize \$0 with probability π, and \$1 with probability $1-\pi$, will be identified with the number $1-\pi$. The three acts in the experiment may therefore be denoted f, g and h, where:

$$f(B) = 1, \quad f(R) = 0;$$

$$g(B) = 0, \quad g(R) = 1;$$

$$h(B) = 0.5, \quad h(R) = 0.5.$$

Observe that $h = 0.5f + 0.5g$, so the preferences $h \succ f \sim g$ contradict (1).

Does this violation of independence indicate irrationality, or is the independence axiom theoretically flawed as a necessary condition for rational choice? Almost certainly the latter. Indeed, subjects in experiments such as the one just described are usually not persuaded to abandon their preferences by having the independence axiom explained to them (Ellsberg (1961, pp.655-6), Camerer and Weber (1992, section 3.1.2)). Furthermore, the preference ordering $h \succ f \sim g$ is consistent with the maximin decision criterion. This criterion has long been proposed in the theoretical literature as a sound basis for choice in situations (such as the present example) of complete uncertainty[4].

In the experiment above, suppose that one suggested to the subjects that they consider choosing between acts f and g by tossing a coin. This randomized choice is formally equivalent to h, in the sense that there is probability 0.5 of winning \$1, irrespective of the color of the ball. Randomization therefore raises the subject's *security level* (Luce and Raiffa (1957, p.61)) from $0 \in X$ (the security level of f and g), to $0.5 \in X$. Thus, just as h is strictly preferred to either f or g, so would one expect this randomized decision scheme to be preferred to either f or g. Raiffa (1961) reports (informal) experimental evidence which lends support to this conclusion.

The theoretical appeal of randomization in general, and maximin in particular, is this potential to raise one's security level by hedging against uncertainty, "just as one may raise one's 'security level' in betting on the horses by not placing all of one's money on a single horse," (Luce and Raiffa (1957, p.70)).

[4]See, for example, Wald (1950, p.18).

2.2 Gilboa and Schmeidler's MMEU theory

Gilboa and Schmeidler (1989) show that the independence axiom may be relaxed to admit such hedging behavior, without sacrificing a tractable mathematical representation of preferences. They replace independence with the following two axioms, which are jointly weaker (and remain so in the presence of the other SEU axioms). First is the axiom of *certainty-independence*: for all $f, g \in X^\Omega$, all $x \in X$, and all $\alpha \in (0, 1)$,

$$f \succ g \quad \Leftrightarrow \quad \alpha f + (1-\alpha)x \succ \alpha g + (1-\alpha)x.$$

This restriction of independence is motivated by the fact that constant acts have no hedging value. Second is the axiom of *uncertainty aversion*, which captures the potential benefits of hedging: for all $f, g \in X^\Omega$ and all $\alpha \in (0, 1)$,

$$f \sim g \quad \Rightarrow \quad \alpha f + (1-\alpha)g \succeq f \tag{2}$$

If the preference ordering $\succeq$ satisfies Gilboa and Schmeidler's MMEU axioms, then there will exist an affine utility function $u : X \to \mathbb{R}$, and a closed, convex $C \subseteq \Delta(\Omega)$, such that for any $f, g \in X^\Omega$,

$$f \succeq g \quad \Leftrightarrow \quad \min_{p \in C} \sum_{\omega \in \Omega} p(\omega)(u \circ f)(\omega) \geq \min_{p \in C} \sum_{\omega \in \Omega} p(\omega)(u \circ g)(\omega) \tag{3}$$

The ordering $\succeq$ satisfies the SEU axioms if and only if C is a singleton.

One may observe that $\succeq$ coincides with the maximin rule (i.e. the ranking of acts by their security levels) when $C = \Delta(\Omega)$. MMEU is also compatible with Ellsberg's proposed resolution of his own paradox (Ellsberg (1961, pp.661-5)). He suggested that decision-makers, when faced with uncertainty, might reasonably employ a convex combination of some SEU decision criterion, and maximin. Such hybrid decision rules correspond to cases in which the set C is a "contaminated prior", familiar from the theory of robust Bayesian inference (Berger (1980, Section 4.7)). That is (in standard notation):

$$C = (1 - \varepsilon)\{p\} + \varepsilon\Delta(\Omega),$$

where $p \in \Delta(\Omega)$ and $\varepsilon \in (0, 1)$.

2.3 Solutions for normal form games

Lo (1996) undertakes the task of integrating MMEU decision theory into the analysis of normal form games[5]. This re-introduces the potential for maximin behavior in such games.

Lo takes a subjective approach to equilibrium, in the spirit of Aumann (1987). That is, equilibria specify players' *beliefs*, which in the MMEU case, take the form of sets of probabilities. The equilibrium conditions require these beliefs to be "reasonable". Players ought to believe that each rival chooses his or her strategy so as to maximize the rival's own preferences.

Consider the case of a finite, two-player normal form game

$$G = \langle S_1, S_2; u_1, u_2 \rangle,$$

where S_i is player i's finite set of pure strategies, and $u_i(s_1, s_2)$ is player i's von Neumann–Morgenstern (vNM) utility from the outcome associated with pure strat-

[5]Klibanoff (1993) attempts a similar integration. However, Klibanoff's normal form solution concepts involve larger departures from the spirit of Nash equilibrium, so we focus on Lo's analysis here. The relationship between Lo's and Klibanoff's solutions is discussed in Lo (1996, section 8.1).

egy profile (s_1, s_2). We let $M_i = \Delta(S_i)$ denote player i's set of mixed strategies for G. Let us also define $S = S_1 \times S_2$ and $M = M_1 \times M_2$.

A Nash equilibrium of G specifies a pair (m_1, m_2), with $m_1 \in M_1$ and $m_2 \in M_2$. Component m_i of this pair is interpreted *either* as the mixed strategy employed by player i, *or* as the probabilistic belief of player $j \neq i$ about how i might play. SEU decision-making is clearly an implicit assumption under the latter interpretation.

Lo (1996) proposes two generalizations of standard normal form solution concepts to encompass MMEU decision-making. The first – Beliefs Equilibrium (BE) – is a direct analogue of Nash equilibrium. The second – Strict Beliefs Equilibrium (SBE) – generalizes the notion of a strict Nash equilibrium (Fudenberg and Tirole (1991, pp.11-12)[6]. Although strict Nash equilibria can only ever involve *pure* strategies, an SBE may, as we shall see, entail randomization by one of the players. We therefore pursue the implications of both solution concepts.

Specialized to the case of finite, two-player games, the definitions of BE and SBE may be rendered as follows:

Definition 1 *A pair (C_1, C_2) of non-empty, closed, convex subsets of M_1 and M_2 respectively, form a* (Strict) Beliefs Equilibrium *of G if*

$$C_1 \subseteq (=) BR_1(C_2) \text{ and } C_2 \subseteq (=) BR_2(C_1),$$

where

$$BR_i(C_j) = \arg\max_{m_i \in M_i} \left[\min_{m_j \in C_j} \sum_{(s_1,s_2)\in S} m_1(s_1) m_2(s_2) u_i(s_1, s_2) \right].$$

[6] Lo's analogy is somewhat strained in the SBE case, which is arguably closer in spirit to the notion of a *quasi-strict* Nash equilibrium (Fudenberg and Tirole (1991, p.12)).

The set $BR_i(C_j)$ is the set of best responses (in M_i) of player i given belief set

$$C_j \subseteq M_j = \Delta(S_j).$$

A Strict Beliefs Equilibrium requires that each player's belief set coincide with his or her rival's set of best responses[7]. A Beliefs Equilibrium, by contrast, requires only that each player's belief set be *contained* in the set of best responses for the player's rival. Therefore, every SBE is a BE, but not conversely.

Lo does not exclude the possibility that players may wish to employ randomized strategies. It may therefore seem odd that player beliefs are defined as sets of probabilities over rival *pure* strategy sets. Would it not be more appropriate for C_i to be a set of probabilities over M_i? The reasons for beliefs being formed over pure strategy sets are threefold. First, the mathematical convenience of a finite state space. Second, for ease of comparison with the familiar Nash equilibrium notion. In particular, $(m_1, m_2) \in M$ is a Nash equilibrium of G if and only if $(\{m_1\}, \{m_2\})$ is a BE.[8] The third, and most substantive reason, is that it makes essentially no difference whether one defines player i's beliefs on S_j or M_j. A closed[9], convex set of probabilities on M_j induces, by the usual reduction of compound lotteries, a closed, convex set of probabilities on S_j. Moreover,

[7]The set $BR_i(C_j)$ is closed and convex by standard arguments.

[8]Thus, every two-player game possesses at least one BE. However, not all two-player games possess an SBE – see Lo (1996, Example 3).

[9]Lo (1996) endows M_j with the usual Borel σ-algebra, and adopts the weak* topology for finitely additive probabilities on this measurable space.

the preferences of an MMEU-maximizer with either belief set will be identical. A more detailed discussion of this point may be found on pp.448-9 of Lo (1996).

3 Maximin and randomized equilibria

Let us call the (S)BE (C_1, C_2) of

$$G = \langle S_1, S_2;\ u_1, u_2 \rangle$$

randomized if there exists an $i \in \{1, 2\}$ such that $BR_i(C_j)$ contains *no pure strategies.* A randomized (S)BE is one in which some player must randomize in order to be playing a best response. By this definition, the usual Nash equilibrium of Matching Pennies, which is equivalent to the BE

$$\left(\left\{\left(\frac{1}{2}, \frac{1}{2}\right)\right\}, \left\{\left(\frac{1}{2}, \frac{1}{2}\right)\right\}\right) \tag{4}$$

is *not* randomized.

In this section, we shall identify those 2×2 games which possess randomized (S)BE's. It will be shown that such games are "rare", and that the randomizing player will always use his or her maximin strategy. Moreover, we shall also demonstate that there are *no* randomized (S)BE's in which *both* players must randomize. These results are all corollaries of the following theorem:

Theorem 1 *Let*

$$G = \langle S_1 = \{A, B\},\ S_2 = \{\alpha, \beta\};\ u_1,\ u_2 \rangle$$

be a 2×2 *game, and* C_2 *a closed, convex subset of* M_2. *If* $BR_1(C_2)$ *contains no pure* *strategies, then player 1 has a unique maximin strategy in* G, *and* $BR_1(C_2)$ *is a singleton containing this maximin strategy.*

The theorem says that if player 1 must randomize to play a best response, then player 1 must choose a maximin strategy, and there is only one such strategy. Obviously, an analogous result to Theorem 1 holds for player 2.

We shall break the proof of Theorem 1 into three steps.

STEP 1: The game G may be represented by the following bi-matrix:

		2	
		α	β
1	A	$u_1(A,\alpha)$, $u_2(A,\alpha)$	$u_1(A,\beta)$, $u_2(A,\beta)$
	B	$u_1(B,\alpha)$, $u_2(B,\alpha)$	$u_1(B,\beta)$, $u_2(B,\beta)$

Figure 2

In what follows we shall identify a typical element of M_1 with the probability q which it places on $A \in S_1$; and a typical element of M_2 with the probability p which it places on $\alpha \in S_2$.

Let us define the functions $h^A : [0,1] \to \mathbb{R}$ and $h^B : [0,1] \to \mathbb{R}$ as follows:

$$h^A(p) \;=\; pu_1(A,\alpha) \,+\, (1-p)\,u_1(A,\beta);$$

$$h^B(p) = pu_1(B,\alpha) + (1-p)\,u_1(B,\beta).$$

For each $p \in M_2$, the function h^A gives player 1's expected return from playing the pure strategy $A \in S_1$ in response. Function h^B has a similar interpretation.

Since $BR_1(C_2)$ contains no pure strategies, it must be the case that *neither*

$$(u_1(A,\alpha),\, u_1(A,\beta)) \geq (u_1(B,\alpha),\, u_1(B,\beta))\,,$$

nor

$$(u_1(A,\alpha),\, u_1(A,\beta)) \leq (u_1(B,\alpha),\, u_1(B,\beta))\,.$$

It follows that

$$\left[h^A(1) - h^B(1)\right]\,\left[h^A(0) - h^B(0)\right] \,<\, 0.$$

The equation $h^A(p) - h^B(p) = 0$ must therefore have a *unique* solution $\hat{p}$ in $[0,1]$, and furthermore $\hat{p} \in (0,1)$. In particular, the function $h^A - h^B$ is strictly monotone, and we shall assume (without loss of generality)[10] that it is *strictly increasing.*

STEP 2: We next show that h^A is strictly increasing, and h^B strictly decreasing. That is:

$$u_1(A,\alpha) > u_1(A,\beta) \tag{5}$$

and

$$u_1(B,\beta) > u_1(B,\alpha) \tag{6}$$

In the terminology of Schmeidler (1989), this implies that player 1's pure strategies are not *comonotone.* Intuitively, (5) and (6) suggest that there may some hedging value to player 1 from mixing the two pure strategies. We shall now prove (5). The proof of (6) is omitted, as it is virtually identical.

[10]If $h^A - h^B$ is strictly *decreasing,* simply switch the labels on player 1's pure strategies.

The closed, convex set $C_2 \subseteq M_2$, must take the form $[\underline{p}, \overline{p}]$, for some $\underline{p}$ and some $\overline{p}$ satisfying $0 \leq \underline{p} \leq \overline{p} \leq 1$. Suppose, contrary to (5), that $u_1(A, \alpha) \leq u_1(A, \beta)$, and hence that h^A is decreasing. Since $h^A - h^B$ is strictly increasing (Step 1), h^B must be strictly decreasing. Thence, for any $q \in [0, 1]$:

$$\begin{aligned}
\min_{p \in [\underline{p}, \overline{p}]} \; qh^A(p) + (1-q)h^B(p) &= qh^A(\overline{p}) + (1-q)h^B(\overline{p}) \\
&\leq \max\left\{h^A(\overline{p}),\ h^B(\overline{p})\right\} \\
&= \max\left\{\min_{p \in [\underline{p}, \overline{p}]} h^A(p),\ \min_{p \in [\underline{p}, \overline{p}]} h^B(p)\right\}
\end{aligned}$$

But this contradicts the fact that $BR_1(C_2)$ contains no pure strategies, proving (5).

STEP 3: We may now conclude the argument with the aid of a simple diagram (cf. Luca and Raiffa (1957, Appendix 3)). Figure 3 plots the functions h^A and h^B. Steps 1 and 2 establish the qualitative features of Figure 3: h^A is strictly increasing; h^B is strictly decreasing; and the two lines intersect at some $\hat{p} \in (0, 1)$.

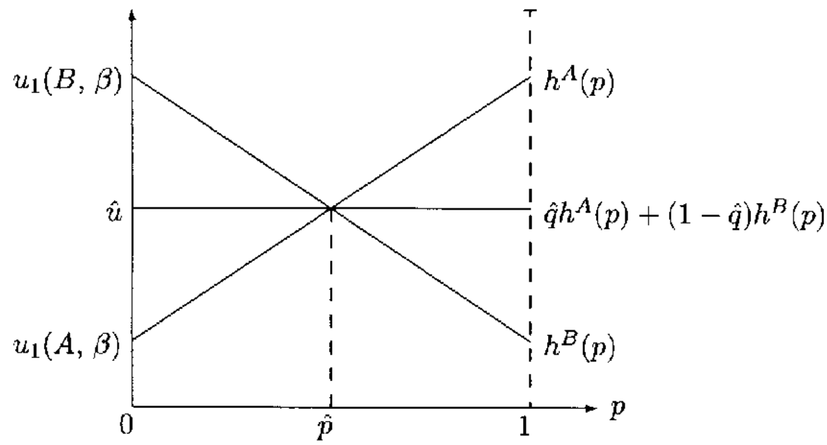

Figure 3

Suppose player 1 employs the mixed strategy $q \in M_1$. The associated payoff function $qh^A + (1-q)h^B$, being a weighted average of h^A and h^B, will lie "between" the latter two lines, passing through their intersection. Indeed, the payoff function associated with any of player 1's mixed strategies may be obtained by pivoting h^A clockwise about its intersection with h^B. One such payoff function, $\hat{q}h^A + (1-\hat{q})h^B$, is indicated in Figure 3. This particular function has zero slope and corresponds to the unique maximin strategy for player 1. That is, for $q \neq \hat{q}$, we observe from Figure 3 that

$$\min_{p\in[0,1]} \; qh^A(p) + (1-q)h^B(p) \; < \; \min_{p\in[0,1]} \; \hat{q}h^A(p) + (1-\hat{q})h^B(p) \; = \; \hat{u}.$$

Given C_2, player 1 chooses $q \in [0,1]$ to maximize

$$\min_{p\in[\underline{p},\overline{p}]} \; qh^A(p) + (1-q)h^B(p).$$

That is, they choose a payoff function whose *lowest* point (in Figure 3) over the range $[\underline{p}, \overline{p}]$ is *at least as high* as the lowest point for any other payoff function. If $\overline{p} \leq \hat{p}$, then it is geometrically obvious that $q = 0$ satisfies this condition. Similarly, if $\underline{p} \geq \hat{p}$, then $q = 1$ will clearly be optimal. Since $BR_1(C_2)$ contains no pure strategies, it follows that $\hat{p} \in (\underline{p}, \overline{p})$. In particular, C_2 must be a non-degenerate interval. The situation is depicted in Figure 4.

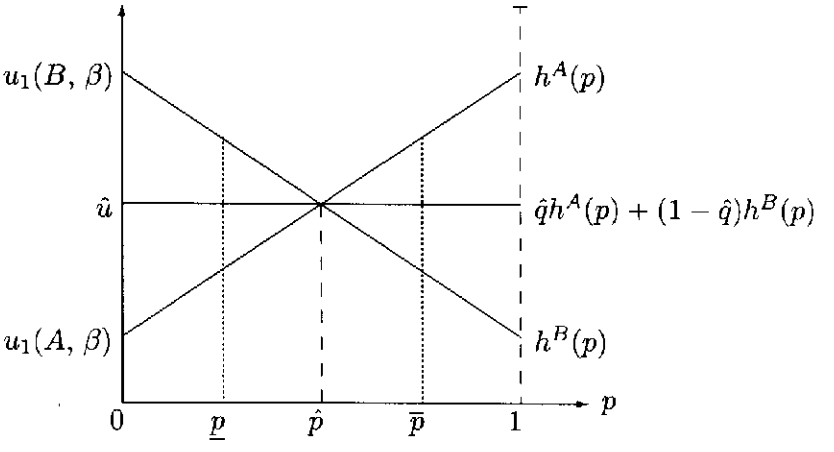

Figure 4

If $q \neq \hat{q}$, then it is obvious from Figure 4 that

$$\min_{p \in [\underline{p}, \overline{p}]} qh^A(p) + (1-q)h^B(p) < \hat{u} = \min_{p \in [\underline{p}, \overline{p}]} \hat{q}h^A(p) + (1-\hat{q})h^B(p).$$

That is, $BR_1(C_2)$ is a singleton containing player 1's (unique) maximin strategy. This completes the proof of Theorem 1.

We immediately obtain the following:

Corollary 1 *If (C_1, C_2) is a (S)BE of the 2×2 game G, and $BR_i(C_j)$ contains no pure strategies, then player i has a unique maximin strategy, and plays this strategy in the (S)BE.*

Only slightly more effort yields:

Corollary 2 *If (C_1, C_2) is a randomized (S)BE of the 2×2 game G, then exactly one player must randomize in order to play a best response.*

If $BR_i(C_j)$ contains no pure strategies, then it is a singleton by Theorem 1. Hence, player j has SEU preferences, and cannot *strictly* prefer any randomized strategy over *both* pure strategies.

Finally, one may use Theorem 1 to prove that randomized equilibria are "rare", in the following sense. Any 2×2 game may be identified with a point in $\mathbb{R}^8$; i.e. a vector with eight elements, corresponding to the eight payoffs in Figure 2, suitably ordered. If a game is chosen "at random" from this set, its having a randomized BE is a "probability zero" event.

Corollary 3 Generically *(Kreps and Wilson (1982, p.877)), 2×2 games have no randomized BE's. That is, the closure in $\mathbb{R}^{S \times S}$ of the set of 2×2 games which possess at least one randomized BE has Lebesgue measure zero.*

The basic idea is simple. Suppose (C_1, C_2) is a randomized BE of the 2×2 game G such that $BR_1(C_2)$ contains no pure strategies. Then, by Corollary 1 and the proof of Theorem 1, C_2 is a non-degenerate interval, while $C_1 \subseteq BR_1(C_2)$ is a singleton.

This means that player 2 has SEU preferences, and in particular, that all of M_2 is a best response to player 1's unique maximin strategy. Games for which player 1 has a unique maximin strategy that makes player 2 indifferent amongst all strategies in M_2 are non-generic.

To formalize the argument, let

$$x_1 = u_1(A,\alpha) - u_1(A,\beta),$$

$$x_2 = u_1(B,\beta) - u_1(B,\alpha),$$

$$y_1 = u_2(A,\alpha) - u_2(A,\beta),$$

and

$$y_2 = u_2(B,\beta) - u_2(B,\alpha).$$

From the proof of Theorem 1, if $BR_1(C_2)$ contains no pure strategies, there exists a unique $\hat{q} \in (0,1)$ such that

$$\begin{aligned} \hat{q}u_1(A,\alpha) + (1-\hat{q})u_1(B,\alpha) &= \hat{q}u_1(A,\beta) + (1-\hat{q})u_1(B,\beta) \\ \Rightarrow \quad \hat{q} &= \frac{x_2}{x_1+x_2}. \end{aligned}$$

Furthermore, $BR_1(C_2) = \{\hat{q}\}$, and hence $C_1 = \{\hat{q}\}$. Since the set $C_2 \subseteq BR_2(C_1)$ must be a non-degenerate interval, it is necessary that

$$\begin{aligned} \hat{q}u_2(A,\alpha) + (1-\hat{q})u_2(B,\alpha) &= \hat{q}u_2(A,\beta) + (1-\hat{q})u_2(B,\beta) \\ \Rightarrow \quad \frac{x_2}{x_1+x_2} &= \frac{y_2}{y_1+y_2} \\ \Rightarrow \quad x_1y_2 &= x_2y_1 \end{aligned} \tag{7}$$

Therefore, the set of 2×2 games with BE in which player 1 must randomize is contained in the subset of $\mathbb{R}^8$ satisfying (7). This (closed) subset clearly has Lebesgue measure zero. The set of 2×2 games with a BE in which player 2 must randomize is contained in a similarly defined subset of $\mathbb{R}^8$. Since the union of two sets of Lebesgue measure zero also has Lebesgue measure zero, Corollary 3 is proved.

4 Discussion

4.1 Interpretation of Corollaries 1–3

Corollaries 1 and 3 contribute renewed support for traditional views about randomized play in normal form games. On the one hand, casual empiricism suggests that explicit randomization is uncommon, and one might expect game theory to reflect this. Corollary 3 confirms such expectations. On the other hand, the potential for randomization to raise an uncertain player's security level, especially in the context of zero-sum games, has traditionally been used to argue that the maximin choice criterion may plausibly explain randomized choice in a limited range of circumstances. Corollary 1 shows that maximin behavior motivates *all* randomized equilibria of 2×2 games, whether zero-sum or not.

By Corollary 1, it should be noted, a strict preference to randomize entails the use of a maximin *strategy*, but may not involve maximin *preferences*. A player who strictly prefers to randomize in some BE, may nevertheless have a preference ranking over M_1 which differs from the maximin ranking. As an example, consider Matching Pennies (Figure 1). Let C_1 be a singleton containing player 1's maximin strategy (i.e. $C_1 = \{\frac{1}{2}\}$); and let $C_2 = [0, \frac{3}{4}]$. Then (C_1, C_2) is a randomized BE of Matching Pennies, since $BR_1(C_2) = C_1$ and $BR_2(C_1) = M_2$. However, player 1 strictly prefers

$q = \frac{1}{4}$ to $q = \frac{3}{4}$, even though each of these strategies has the same security level.

It should be clear from the proof of Corollary 3, however, that in any randomized *Strict* Beliefs Equilibrium, the randomizing player must have maximin preferences. Therefore, when randomized behavior is strictly optimal in an equilibrium of a 2×2 game, it is necessarily maximin behavior, and for the case of randomized SBE's, it is explicitly motivated by the maximin decision criterion.

The arguments of the maximin school are further confirmed by noting that every zero-sum 2×2 game satisfies (7). If G is zero-sum, then $x_1 = -y_1$ and $x_2 = -y_2$, so

$$x_1 y_2 \;=\; -y_1 y_2 \;=\; x_2 y_1.$$

In fact, one may show[11] that for any zero-sum 2×2 game in which player i has no *pure* maximin strategy, player i's maximin strategy is unique, and furthermore, there will exist a randomized SBE for this game in which player i has maximin preferences. Therefore, whenever maximin implies randomization in a 2×2 zero-sum game, one may support this prediction by way of an SBE.[12]

In relation to Corollary 3, it is interesting to note that the well-known game of Battle of the Sexes (BoS), a version of which appears in Figure 5, does *not* possess a randomized BE.

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[11] See, for example, Figure 4 in Appendix 3 of Luce and Raiffa (1957).

[12] It is straightforward to verify that all of the claims in this paragraph generalize to the class of *strictly competitive* 2×2 games (Osborne and Rubinstein (1994, p.21)).

		2	
		α	β
1	A	2,1	-1,-1
	B	-1,-1	1,2

Figure 5

Suppose that (C_1, C_2) is a BE of BoS, and $BR_1(C_2)$ contains no pure strategies. Then by Theorem 1, $C_1 = \{\frac{2}{5}\}$, since player 1's unique maximin strategy is to choose A with probability $\frac{2}{5}$. But $BR_2(\{\frac{2}{5}\}) = \{0\}$; that is, player 2's best response to player 1's maximin strategy is β. Thus, we must have $C_2 = \{0\}$. As $C_1 \cap BR_1(\{0\}) = \emptyset$, there is no BE of BoS in which player 1 must randomize. A similar argument rules out BE's in which player 2 must randomize[13].

[13] Luce and Raiffa (1957, Section 5.3) encounter similar problems in reasoning about BoS. The fact that the mixed Nash equilibrium strategies (i.e. $\frac{3}{5} \in M_1$ and $\frac{2}{5} \in M_2$) are not maximin, makes this equilibrium, in their view, highly implausible. In particular, if the players anticipate this mixed equilibrium, each player would seem to have good reason to deviate to his or her maximin strategy. Given this tension between the mixed Nash and maximin strategies, it is interesting to observe that the following is a symmetric BE of BoS:

$$\left(C_1 = \left[\frac{2}{5}, \frac{3}{5}\right], C_2 = \left[\frac{2}{5}, \frac{3}{5}\right]\right).$$

In this BE, each player's beliefs form an interval, with the end points equal to the mixed Nash and maximin strategies of the player's rival. This BE is not randomized, since each player has a pure strategy best response. However, the equilibrium beliefs reflect exactly the ambiguity which Luce and Raiffa express regarding the relative merits of the maximin and mixed Nash strategies.

Nevertheless, there do exists non-zero-sum games which possess randomized (S)BE's. For example, the belief profile

$$\left(C_1 = \left\{\frac{1}{2}\right\}, C_2 = M_2\right)$$

constitutes a randomized SBE of the "coordination game" in Figure 6.

		2	
		α	β
1	A	1,1	-1,-1
	B	-1,-1	1,1

Figure 6

Observe that player 1 employs the maximin rule, and strictly prefers to randomize.

While Corollaries 1 and 3 tend to confirm established intuition, Corollary 2 seems almost counter-intuitive. In fact, the curious asymmetry implied by Corollary 2 is quite natural, and reveals a flaw in the "old-fashioned" analysis of zero-sum games. Recall that this analysis predicts maximin behavior by *both* players in Matching Pennies, due to the uncertainty they face. But if maximin is *uniquely* distinguished as each player's optimal response, then where is the uncertainty which motivates such behavior? Thus, on closer inspection, the "old-fashioned" logic is not so compelling after all, and is in fact contrary to the spirit of equilibrium.[14]

[14]This statement in no way contradicts the Minmax Theorem. In the case of Matching Pennies,

Suppose, however, that only one player in Matching Pennies is uncertain. This player will surely employ her maximin strategy, thereby justifying the absence of uncertainty in the mind of her rival. Moreover, the rival player will thus be indifferent amongst all available strategic responses, and this indifference will sustain the original player's uncertainty. This is exactly the logic behind the randomized BE's of Matching Pennies, and indeed of any other 2×2 game.

It is easy to identify the BE's of Matching Pennies. Ironically, all but the usual Nash equilibrium are randomized. That is, (4) is the only non-randomized BE. The remainder come in two varieties, depending on which player randomizes. The BE's in which player 1 necessarily randomizes take the form

$$\left(C_1 = \left\{\frac{1}{2}\right\},\ C_2 = [\underline{p}, \overline{p}]\right),$$

for any $\underline{p}, \overline{p} \in [0,1]$ such that $0.5 \in (\underline{p}, \overline{p})$. Conversely, any pair of this form is a BE of Matching Pennies. The remaining BE's – in which player 2 necessarily adopts a randomized strategy – are all pairs

$$\left(C_1 = [\underline{q}, \overline{q}],\ C_2 = \left\{\frac{1}{2}\right\}\right)$$

with $0.5 \in (\underline{q}, \overline{q})$. The SBE's are $\left(C_1 = \left\{\frac{1}{2}\right\}, C_2 = M_2\right)$ and $\left(C_1 = M_1, C_2 = \left\{\frac{1}{2}\right\}\right)$.

Before concluding this part of the discussion, we should point out that the issue of equilibrium randomization is considerably more complex outside the 2×2 case.

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this theorem merely notes that each player's maximin strategy is a best response to the other player's maximin strategy, in the sense of *maximizing expected utility*. But since both players are assumed to employ *non-EU* behavior, this would seem to be the wrong equilibrium test to apply. A more appropriate test is that proposed by Lo, and this test eliminates the outcome in which both players adopt the maximin rule (Corollary 2).

Theorem 1, for example, fails for larger games. To illustrate, consider the 2×3 game in Figure 7.

		2		
		α	β	γ
1	A	-1,0	2,0	1,0
	B	1,0	-2,0	1,0

Figure 7

It is evident that player 1's maximin strategy for this game is $\frac{1}{2} \in M_1$, yielding a security level of 0. Any other strategy will provide a security level strictly less than 0.

Suppose that a typical element of M_2 is denoted $(p_\alpha, p_\beta, p_\gamma)$, where p_α is the probability of α, p_β is the probability of β, and $p_\gamma = 1 - p_\alpha - p_\beta$. Let

$$C_2 = \left\{ (p_\alpha, p_\beta, p_\gamma) \in M_2 \,\middle|\, p_\alpha \leq \frac{1}{4} \text{ and } p_\beta \leq \frac{1}{4} \right\}.$$

This is clearly a closed and convex set. In fact, for the purposes of the calculations to follow, it is useful to note that C_2 is the convex hull of the set

$$\left\{ (0,0,1), \left(0, \frac{1}{4}, \frac{3}{4}\right), \left(\frac{1}{4}, 0, \frac{3}{4}\right), \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right) \right\} \subseteq M_2.$$

Since

$$\min_{p \in C_2} p_\alpha u_1(A,\alpha) + p_\beta u_1(A,\beta) + p_\gamma u_1(A,\gamma) = \frac{1}{4} u_1(A,\alpha) + \frac{3}{4} u_1(A,\gamma) = \frac{1}{2}$$

and

$$\min_{p \in C_2} p_\alpha u_1(B,\alpha) + p_\beta u_1(B,\beta) + p_\gamma u_1(B,\gamma) = \frac{1}{4}u_1(B,\beta) + \frac{3}{4}u_1(B,\gamma) = \frac{1}{4},$$

$A \in S_1$ is player 1's best pure strategy response given C_2. Furthermore, player 1 is *indifferent* between pure strategy A and his maximin strategy, as a straightforward calculation will confirm. However, if $\hat{m}_1 \in M_1$ denotes the mixed strategy in which A is chosen with probability $\frac{2}{3}$, and

$$u_1(\hat{m}_1, s_2) = \frac{2}{3}u_1(A, s_2) + \frac{1}{3}u_1(B, s_2)$$

for each $s_2 \in S_2$, then

$$\min_{p \in C_2} p_\alpha u_1(\hat{m}_1,\alpha) + p_\beta u_1(\hat{m}_1,\beta) + p_\gamma u_1(\hat{m}_1,\gamma) =$$
$$\frac{1}{4}u_1(\hat{m}_1,\alpha) + \frac{1}{4}u_1(\hat{m}_1,\beta) + \frac{1}{2}u_1(\hat{m}_1,\gamma) = \frac{7}{12} > \frac{1}{2}.$$

Therefore, $BR_1(C_2)$ contains no pure strategies, but nor does it contain player 1's maximin strategy.

4.2 Related literature

Holler (1990) compares maximin and Nash equilibrium strategies in the context of 2×2 games. He presents a result which bears some similarities to Theorem 1. In particular, for 2×2 games in which neither player has a pure maximin strategy (i.e. maximin implies randomization), Holler shows that any Nash equilibrium in non-degenerately mixed strategies will deliver to players their maximin payoffs. Such equilibria are termed "unprofitable".

To understand the logic of Holler's result, suppose that player 1 has no pure maximin strategy in G. Then, with only minor modifications, the arguments in Steps 1-2 of the proof of Theorem 1 show that the situation must be (qualitatively) as depicted in Figure 3. If player 1 employs a non-pure strategy in a Nash equilibrium of G, it must be the case that player 2's equilibrium strategy is $\hat{p} \in M_2$. Player 1's equilibrium payoff will thus be $\hat{u}$, the security level associated with player 1's maximin strategy, $\hat{q} \in M_1$.

As Holler notes, although these mixed strategy equilibria generate maximin *payoffs*, they need not involve the use of maximin *strategies*. The game of BoS is a classic example. The equilibrium strategies may therefore needlessly lower players' security levels, so Holler (following Harsanyi) suggests that maximin provides a better prediction of behavior than Nash equilibrium.

Thus, like Luce and Raiffa (1957), Holler perceives a conflict between maximin behavior and Nash equilibrium. However, unlike the majority of modern game theorists, Holler favors maximin over Nash. The main point of the present paper is that maximin is *not* incompatible with the logic of equilibrium, and *both* are fundamental to a proper understanding of randomized behavior in 2×2 games.

In fact, Lo's BE and SBE notions are not the only ways to generalize equilibrium logic. Other generalizations exist, some providing alternative motivatations for randomized behavior which do not involve maximin choice. However, these typically treat games as decision problems under *risk* rather than uncertainty.[15]

[15] There also exist equilibrium notions based on the *Choquet expected utility (CEU)* theory of choice under uncertainty. Examples include Dow and Werlang (1994), Eichberger and Kelsey (1994), Mukerji (1995) and Marinacci (1996). In the CEU framework, subjective uncertainty is described by a single "non-additive probability" – or *capacity* – rather than a set of probabilities (as in MMEU). However, the two decision theories are closely related. In particular, every CEU preference ordering which satisfies the uncertainty aversion axiom (2) is in fact an MMEU preference ordering (Schmeidler (1989)). Since

Cheng and Zhu (1995), for example, employ the *quadratic utility* model of Chew, Epstein and Segal (1991) to study 2×2 games[16]. They show that every game possessing a unique Nash equilibrium in non-pure strategies, will also possess a *randomized* equilibrium under quadratic utility maximization. Cheng and Zhu's theory therefore predicts at least as much randomized behavior as conventional Nash equilibrium. In particular, randomization is generic in Cheng and Zhu's framework.

In general, approaches to randomized choice based on non-EU risk preferences suffer two fundamental drawbacks. First, they cannot encompass maximin behavior, and therefore rule out a classic motive for randomization. Second, the assumed absence of (subjective) uncertainty strikes one as artificial. It seems, for example, that intractable uncertainty must lie at the heart of any satisfying analysis of games such as Matching Pennies. As Luce and Raiffa (1957, p.14) ruefully observe:

> "Intuitively, the problem of conflict of interest is, for each participant, a problem of decision making under a mixture of risk and uncertainty, the uncertainty arising from his ignorance as to what the others will do. In game theory one attempts to idealize this problem in such a way as to transform it into interacting problems of individual decision making under risk. In point of fact, assumptions about the motivations of players are not sufficient to eliminate completely the uncertainty aspects of the problem, as

uncertainty aversion captures precisely the hedging value of randomization, there is no loss in essential generality (and some gains in terms of convenience) from confining attention to the MMEU class for the purposes of our analysis.

[16]Crawford's (1990) notion of equilibrium, encompasses an even wider class of non-EU risk preferences.

we shall become acutely aware."

Gilboa and Schmeidler's MMEU theory allows us to model equilibrium without recourse to this restrictive "idealization". One may directly examine the contribution of equilibrium uncertainty in generating randomized behavior.

5 Conclusion

The SEU paradigm precludes a proper understanding of randomized choice. In the present paper, we have therefore chosen to view matters from an MMEU perspective. The advantages of such a perspective are threefold. First, MMEU preferences are compatible with the *strict* optimality of randomized choice. Second, the MMEU framework allows "subjective uncertainty" to persist in equilibrium. When players have multiple best responses, such uncertainty is natural. Finally, maximin behavior is covered by MMEU. This is important, given the historical significance of the maximin criterion, and its potential to explain randomization as a useful hedge against uncertainty.

Our analysis demonstrates that equilibrium randomization necessarily involves the choice of a maximin strategy. For Strict Beliefs Equilibria, the randomizing player is motivated by the maximin choice criterion. These results confirm and extend traditional explanations for randomization based on the security levels of different strategies.

Randomized behavior, it transpires, is non-generic. While there exist 2×2 games in which some player expresses a strict preference to randomize; the existence of randomized equilibria is *not* robust to small perturbations of the game's payoffs.

Finally, we have shown that when uncertainty leads some player to choose randomly, this player's behavior is completely determinate, and hence the other player faces no uncertainty. Randomization is therefore an "asymmetric" phenomenon, contrary to conventional analysis.

References

Anscombe, F.J. and R.J. Aumann, 1963, A definition of subjective probability, *Annals of Mathematical Statistics* 34, 199–205.

Aumann, R.J., 1987, Correlated equilibrium as an expression of Bayesian rationality, *Econometrica* 55, 1–18.

Berger, J.O., 1980, *Statistical Decision Theory and Bayesian Analysis*, Second edition (Springer-Verlag, New York).

Brown, G., 1951, Iterative solutions of games by fictitious play, in T.C. Koopmans, ed., *Activity Analysis of Production and Allocation* (Wiley and Sons, New York) 374–376.

Camerer, C. and M. Weber, 1992, Recent developments in modeling preferences: Uncertainty and ambiguity, *Journal of Risk and Uncertainty* 5, 325–370.

Cheng, L.K. and M. Zhu, 1995, Mixed-strategy Nash equilibrium based upon expected utility and quadratic utility, *Games and Economic Behavior* 9, 139–150.

Chew, S.H., L.G. Epstein and U. Segal, 1991, Mixture symmetry and quadratic utility, *Econometrica* 59, 139–163.

Crawford, V., 1990, Equilibrium without independence, *Journal of Economic Theory* 50, 127–154.

Dow, J. and S.R.Da C. Werlang, 1994, Nash equilibrium under Knightian uncertainty: Breaking down backward induction, *Journal of Economic Theory* 64, 305–324.

Eichberger, J. and D. Kelsey, 1994, Non-additive beliefs and game theory, University of Melbourne Research Paper No. 398.

Ellsberg, D., 1961, Risk, ambiguity and the Savage axioms, *Quarterly Journal of Economics* 75, 643–669.

Fudenberg, D. and J. Tirole, 1991, *Game Theory* (MIT Press, Cambridge, MA).

Gilboa, I. and D. Schmeidler, 1989, Maxmin expected utility with a non-unique prior, *Journal of Mathematical Economics* 18, 141–153.

Harsanyi, J., 1973, Games with randomly disturbed payoffs: A new rationale for mixed-strategy equilibrium points, *International Journal of Game Theory* 2, 1–23.

Herstein, I.N. and J. Milnor, 1953, An axiomatic approach to measurable utility, *Econometrica* 23, 291–297.

Holler, M.J., 1990, The unprofitability of mixed-strategy equilibria in two-person games: A second Folk Theorem, *Economics Letters* 32, 319 323.

Klibanoff, P., 1993, Uncertainty, decision, and normal-form games, Mimeo, MIT.

Knight, F., 1921, *Risk, Uncertainty and Profit* (Houghton Mifflin, Cambridge, MA).

Kreps, D. and R. Wilson, 1982, Sequential equilibria, *Econometrica* 50, 863–894.

Lo, K.C., 1996, Equilibrium in beliefs under uncertainty, *Journal of Economic Theory* 71, 443–484.

Luce, R.D. and H. Raiffa, 1957, *Games and Decisions* (John Wiley and Sons, New York).

Machina, M.J. and D. Schmeidler, 1992, A more robust definition of subjective probability, *Econometrica* 60, 745–780.

Marinacci, M., 1996, Ambiguous games, Mimeo, University of Toronto.

Mukerji, S., 1995, A theory of play for games in strategic form when rationality is not common knowledge, Mimeo, University of Southampton.

Osborne, M.J. and A. Rubinstein, 1994, *A Course in Game Theory* (MIT Press, Cambridge, MA).

Raiffa, H., 1961, Risk, ambiguity and the Savage axioms: Comment, *Quarterly Journal of Economics* 75, 690–694.

Schmeidler, D., 1989, Subjective probability and expected utility without additivity, *Econometrica* 57, 571–587.

Stahl, D.O., 1988, On the instability of mixed-strategy Nash equilibria, *Journal of Economic Behavior and Organization* 9, 59–69.

Wald, A., 1950, *Statistical Decision Functions* (John Wiley and Sons, New York).