Simple Games: Weightedness and Generalizations

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Preface

This thesis contributes to the program of numerical characterisation and classification of simple games outlined in the classical monograph of von Neumann and Morgenstern. One of the most fundamental questions of this program is what makes a simple game a weighted majority game. The necessary and sufficient conditions that guarantee weightedness were obtained by Elgot and refined by Taylor and Zwicker. If a simple game does not have weights, then Taylor and Zwicker showed that rough weights may serve as a reasonable substitute. Not all simple games are roughly weighted, and the class of projective games is a prime example. We give necessary and sufficient conditions for a simple game to have rough weights. We define two functions $f(n)$ and $g(n)$ that measure the deviation of a simple game from a weighted majority game and a roughly weighted majority game, respectively. We formulate known results in terms of lower and upper bounds for these functions and improve those bounds. Also we suggest three possible ways to classify simple games beyond the classes of weighted and roughly weighted games. We introduce three hierarchies of games and prove some relationships between their classes. We prove that our hierarchies are true (i.e., infinite) hierarchies. In particular, they are strict in the sense that more of the key "resource" yields the flexibility to capture strictly more games.

Simple games has applications in the theory of qualitative probability orders. The concept of qualitative probability takes its origins in attempts of de Finetti to axiomatise probability theory. An initial segment of a qualitative probability order is a simplicial complex dual to a simple game. We initiate the study of abstract simplicial complexes which are initial segments of qualitative probability orders. This is a natural class that contains the threshold complexes and is contained in the shifted complexes, but is equal to neither. In particular we construct a qualitative probability order on 26 atoms that has an initial segment which is not a threshold simplicial complex. Although 26 is probably not the minimal number for which such example exists we provide some evidence that it cannot be much smaller.

We prove some necessary conditions for this class and make a conjecture as to a characterization of them. The conjectured characterization relies on some ideas from cooperative game theory.

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Contents

Chapter 1

Introduction

1.1 Background and Motivation

In this thesis we are focused on mathematical problems arising in relation to some aspects of game theory and comparative probability orders. Each of these disciplines can contribute significantly to the other: the same objects appear there under different names and interpretations, and different techniques are used to investigate these objects. The subject has broad connections with Economics, Computer Science and Political Science.

This thesis can be divided into the two following (interrelated) topics

1. *Finding an efficient way to compress the information about a game.* The problem arises in games with large number of players. If we keep the information about all winning and losing coalitions of players then we need an exponential amount of computer memory. In some situations it is possible to compress this information by assigning to every player a weight and choosing a threshold, so that a coalition is winning if and only if the sum of players' weights in this coalition reaches the threshold. However it is not always possible to do as most games are not weighted. We are looking at finding reasonable substitutes for weights and criteria of their existence.

2. *Investigating a new class of abstract simplicial complexes.* Abstract simplicial complexes are close relatives of simple games. Indeed the set of losing coalitions of a simple game is an abstract simplicial complex and vice versa. The new class we intend to study is related to certain orders of subsets of a set that arise in the study of probability theories related to decision theory. This

is the theory of qualitative (also known as subjective or comparative) probability. The decision maker may not know the probabilities of the states of the world exactly but for each pair of them knows which one is more likely to happen. Surprisingly the same orders appear in many other areas, e.g., as monomial orders in the theory of Groebner bases in exterior algebras. Obviously these orders generalize the standard probability theory which stipulate that every atom has probability and that the probability is additive. In a way probabilities serve as weights and comparative probability orders that arise from probability measures can be considered as "weighted." Edelman was the first to suggest that we can generalize threshold simplicial complexes by abandoning weights but using a qualitative probability orders instead. The idea is to study initial segments of qualitative probabilities, which by their nature should be combinatorial generalizations of threshold complexes and an important subclass of a broad class of shifted complexes. So far this important class of abstract simplicial complexes has gone virtually unstudied. For instance, the smallest number of atoms that are needed to construct an initial segment of a comparative probability which is not a threshold simplicial complex is not known.

Below, we will give more details about every topic.

1.1.1 Weightedness and Rough Weightedness

In their classical book von Neumann and Morgenstern (1944) outlined the programme of numerical classification and characterisation of all simple games.[1] They viewed the introduction of weighted majority games as the first step in this direction. They noted however[2] that already for six players not all games have weighted majority representation and they also noted that for seven players some games do not have weighted majority representation in a much stronger sense. Therefore one of the most fundamental questions of this programme is to find out what makes a simple game a weighted majority game. The next step is to measure the deviation of an arbitrary game from a weighted majority game in terms of a certain function $f(n)$ of the number of players n and to obtain lower and upper bounds for this function.

[1] See Section 50.2.1, page 433.
[2] Section 5.3 of the same book

The necessary and sufficient conditions that guarantee weightedness of a game are known. Elgot (1960) obtained them in terms of asummability. Taylor and Zwicker (1992) obtained necessary and sufficient conditions later but independently in terms of trading transforms. The advantage of the latter characterisation is that it is constructive in the sense that only finitely many conditions (which depends on the number of players) has to be checked to decide if the game is weighted or not. More precisely, they showed that a simple game is a weighted majority game if no sequence of winning coalitions up to the length 2^{2^n} can be converted into a sequence of losing coalitions by exchanging players.

The sequence of coalitions

$$\mathcal{T} = (X_1, \ldots, X_j; Y_1, \ldots, Y_j) \tag{1.1}$$

is called a trading transform if the coalitions $X_1, \ldots, X_j$ can be converted into the coalitions $Y_1, \ldots, Y_j$ by rearranging players. If game G with n players does not have weights, then the characterisation of Taylor and Zwicker implies that there exists a trading transform (1.1) where all $X_1, \ldots, X_j$ are winning and all $Y_1, \ldots, Y_j$ are losing. We call such a trading transform a certificate of non-weightedness. We denote the minimal length j of such a certificate $\mathcal{T}$ as $f(G)$ and set $f(n) = \max_G f(G)$, where G runs over all games without weighted majority representation. If $f(G) = 2$, then the game G is extremely non-weighted, in fact most games, as noted in (Taylor & Zwicker, 1999), are of this kind. And it is easy to find a certificate of their non-weightedness. Games with the condition $f(G) > 2$ behave in some respects as weighted games, in particular the desirability relation on singletons is a weak order (this is why they are called complete games (Carreras & Freixas, 1996; Taylor & Zwicker, 1999)). For such games Carreras and Freixas (Carreras & Freixas, 1996; Taylor & Zwicker, 1999; Freixas & Molinero, 2009b) obtained a useful classification result. So the larger $f(G)$ the closer the game G to weighted majority games. And it is not surprising that it gets more and more difficult to find a certificate of their non-weightedness. It is important to know what is the maximal length of certificates that has to be checked in order to declare that G is weighted. The function $f(n)$ shows exactly this length. This is a complete analogue to the Fishburn's function $f(n)$ defined for linear qualitative probability orders (Fishburn, 1996).

Many old results can be nicely expressed in terms of this function. In particular the results of Taylor and Zwicker (1992, 1995) (and earlier Gabelman (1961) for small values of n) can be presented as lower and upper bounds for $f(n)$ as follows

$$\lfloor \sqrt{n} \rfloor \leq f(n) \leq 2^{2^n}.$$

A natural desire is to make that interval smaller.

If a simple game does not have weights, then rough weights may serve as a reasonable substitute (see Taylor & Zwicker, 1999). The idea is the same as in the use of tie-breaking in voting in case when only one alternative is to be elected. If the combined weight of a coalition is greater than a certain threshold, then it is winning, if the combined weight is smaller than the threshold, then this coalition is losing. If its weight is exactly the threshold, then it can go either way depending on the "tie-breaking" rule.

In Section 3.4 we obtain a necessary and sufficient conditions for the existence of rough weights. We prove that a game G is a roughly weighted game if for no j smaller than $(n+1)2^{\frac{1}{2}n\log_2 n}$ does there exist a certificate of non-weightedness of length j with the grand coalition among winning coalitions and the empty coalition among losing coalitions. Let us call such certificates potent. For a game G without rough weights we define by $g(G)$ the lengths of the shortest potent certificate of non-weightedness. Then, a function $g(n)$ can be naturally defined which is fully analogous to $f(n)$. It shows the maximal length of potent certificates that has to be checked in order to decide if G is roughly weighted or not. Bounds of this function are particular interest to us.

1.1.2 Generalizations of Rough Weightedness

A simple game is a mathematical object that is used in economics and political science to describe the distribution of power among coalitions of players (von Neumann & Morgenstern, 1944; Shapley, 1962). Recently, simple games have been studied as access structures of secret sharing schemes (Blakley, 1979). They have also appeared, in some cases under others names, in a variety of mathematical and computer science contexts, e.g., in threshold logic (Muroga, 1971). Simple games are closely related to hypergraphs, coherent structures, Sperner systems, clutters, and abstract simplicial complexes. The term "simple" was introduced by von Neumann and Morgenstern (1944), because in this type of games players strive not for monetary rewards but for power, and each coalition is either all-powerful or completely ineffectual. However these games are far from being simple.

An important class of simple games—well studied in economics—is the weighted majority game (von Neumann & Morgenstern, 1944; Shapley, 1962). In such a game every player is assigned a real number, his or her weight. The winning coalitions are the sets of players whose weights total at least q, a certain threshold. However, it is well known that not every simple game has a representation as a weighted majority game (von Neumann & Morgenstern, 1944). The first step in attempting to characterize nonweighted games was the introduction of the class of roughly weighted games (Taylor & Zwicker, 1999). Formally, a simple game G on the player set $P = [n] = \{1, 2, \ldots, n\}$ is *roughly weighted* if there exist nonnegative real numbers $w_1, \ldots, w_n$ and a real number q, called the *quota*, not all equal to zero, such that for $X \in 2^P$ the condition $\sum_{i \in X} w_i > q$ implies X is winning, and $\sum_{i \in X} w_i < q$ implies X is losing. This concept realizes a very common idea in social choice that sometimes a rule needs an additional "tie-breaking" procedure that helps to decide the outcome if the result falls on a certain "threshold." Taylor and Zwicker (1999) demonstrated the usefulness of this concept. Rough weightedness was studied by Gvozdeva and Slinko (2011), where it was characterized in terms of trading transforms, similar to the characterization of weightedness by Elgot (1960) and Taylor and Zwicker (1992).

Before moving on, it is worth mentioning in passing the notion of complete games. In a simple game player, i is said to be at least as desirable as player j (as a coalition partner) if replacing i in a winning coalition with j never makes that coalition losing. This desirability relation was introduced and studied by Isbell (1956). Weighted majority games have the property that players are totally ordered by the desirability relation. Thus another natural extension of the class of weighted majority games is the class of complete games, for which the desirability relation is a total order. This class is significantly larger than the class of weighted majority games since it contains simple games of any dimension (Freixas & Puente, 2008) while, the dimension of a weighted game is always 1. Extensive theoretical and computational results on complete simple games have been obtained by Freixas and Molinero (2009b). The strictness of our hierarchies for that important class of simple games is an interesting open question.

It might seem that nonweighted games and even games without rough weights are somewhat strange. However, an important observation of von Neumann and Morgenstern (1944, Section 53.2.6) states that they "correspond to a different organizational principle that deserves closer study." In some of these games, as von

Neumann and Morgenstern noted, all the minimal winning coalitions are minorities and at the same time "no player has any advantage over any other" (e.g., the Fano game introduced later). This is an attractive feature for secret sharing since in the case of a large number of users it is advantageous to keep minimal authorized coalitions relatively small. This may be why weighted threshold secret sharing schemes were largely ignored and were characterized only recently (Beimel, Tassa, & Weinreb, 2008).

The parameter of the first of the three hierarchies we will discuss reflects the balance of power between small and large coalitions; the larger this parameter the more powerful some of the small coalitions are. Gvozdeva and Slinko (2011) proved that for a game G that is not roughly weighted there exists a certificate of nonweightedness of the form

$$\mathcal{T} = (X_1, \ldots, X_j, P; Y_1, \ldots, Y_j, \emptyset),$$

where $X_1, \ldots, X_j$ are winning coalitions of G, P is the grand coalition, and $Y_1, \ldots, Y_j$ are losing coalitions. However, sometimes it is possible to have more than one grand coalition in the certificate. This may occur when coalitions $X_1, \ldots, X_j$ are small but nonetheless winning.

A certificate of nonweightedness of the form

$$\mathcal{T} = (X_1, \ldots, X_j, P^\ell; Y_1, \ldots, Y_j, \emptyset^\ell)$$

will be called ℓ-potent of length $j + \ell$. Each game that possesses such a certificate will be said to belong to the class of games $\mathcal{A}_q$, where $q = \ell/(j + \ell)$. The parameter q can take values in the open interval $(0, \frac{1}{2})$. We will show that $\mathcal{A}_p \supseteq \mathcal{A}_q$ for any p and q such that $0 < p \le q < \frac{1}{2}$, and that if $0 < p < q < \frac{1}{2}$, then this inclusion is strict, i.e., we have the $\mathcal{A}_p \supsetneq \mathcal{A}_q$.

Another hierarchy emerges when we allow several thresholds instead of just one in the case of roughly weighted games. We say that a simple game G belongs to the class $\mathcal{B}_k$, $k \in \{1, 2, 3, \ldots\}$, if there are k thresholds, $0 < q_1 \le q_2 \le \cdots \le q_k$, and any coalition with total weight of players smaller than q_1 is losing, any coalition with total weight greater than q_k is winning. We also impose an additional condition that, if a coalition X has total weight $w(X)$ that satisfies $q_1 \le w(X) \le q_k$, then $w(X) = q_i$ for some i. All games of the class $\mathcal{B}_1$ are roughly weighted. In fact, as we will prove in Section 4.3, almost all roughly weighted games belong to this class: $\mathcal{B}_1$ is exactly the class of roughly weighted games with nonzero quota. We will show that the

Fano game (Gvozdeva & Slinko, 2011) belongs to $\mathcal{B}_2$ but does not belong to $\mathcal{B}_1$. We will prove that the $\mathcal{B}$-hierarchy is strict, that is,

$$\mathcal{B}_1 \subsetneq \mathcal{B}_2 \subsetneq \cdots \subsetneq \mathcal{B}_\ell \subsetneq \cdots,$$

with the union of these classes being the class of all simple games.

Yet another way to capture more games is by making the threshold "thicker." We here will not use a point but rather an interval $[a, b]$ for the threshold, $a \leq b$. That is, all coalitions with total weight less than a will be losing and all coalitions whose total weight is greater than b will be winning. This time—in contrast with the k limit of $\mathcal{B}_k$—we do not care how many different values weights of coalitions falling in $[a, b]$ may take on. (A good example of this situation would be a faculty vote, where if neither side controls a 2/3 majority—calculated in faculty members or their grant dollars—then the Dean would decide the outcome as he wished.) We can keep weights normalized so that the lower end of the interval is fixed at 1. Then the right end of the interval α becomes a "resource" parameter. Formally, a simple game G belongs to the class C_α if all coalitions in G with total weight less than 1 are losing and every coalition whose total weight is greater than α is winning. We show that the class of all simple games is split into a hierarchy of classes of games $\{C_\alpha\}_{\alpha \in [1,\infty)}$ defined by this parameter. We show that as α increases we get strictly greater descriptive power, i.e., strictly more games can be described, that is, if $\alpha < \beta$, then $C_\alpha \subsetneq C_\beta$. In this sense the hierarchy is strict. This strict hierarchy result, and our strict hierarchy results for hierarchies $\mathcal{A}$ and $\mathcal{B}$, have very much the general flavor of hierarchy results found in computer science: more resources yield more power (whether computational power to accept languages as in a deterministic or nondeterministic time hierarchy theorem, or as is the case here, description flexibility to capture more games).

The strictness of the latter hierarchy was achieved because we allowed games with arbitrary (but finite) numbers of players. The situation will be different if we keep the number of players, n, fixed. Then there is an interval $[1, s(n)]$ such that all games with n players belong to $C_{s(n)}$ and $s(n)$ is minimal with this property. There will be also finitely many numbers $q \in [1, s(n)]$ such that the interval $[1, q]$ represents more n-player games than any interval $[1, q']$ with $q' < q$. We call the set of such numbers the nth spectrum and denote it $\mathrm{Spec}(n)$. We also call a game with n players critical if it belongs to C_α with $\alpha \in \mathrm{Spec}(n)$ but does not belong to any C_β with $\beta < \alpha$. We calculate the spectrum for $n < 7$ and also produce a set of critical games, one for each element of the spectrum. We in addition try to give a reasonably tight upper bound for $s(n)$.

All three of our hierarchies provide measures of how close a given game is to being a simple weighted majority game. That is, they each quantify the nearness to being a simple weighted majority game (e.g., hierarchies $\mathcal{B}$ and $\mathcal{C}$ quantify based on the extent and structure of a "flexible tie-breaking" region). And the main theme and contribution of the chapter is that we prove for each of the three hierarchies that allowing more quantitative distance from simple weighted majority games yields strictly more games, i.e., the hierarchies are proper hierarchies.

1.1.3 Simplicial Complexes

The concept of qualitative (comparative) probability takes its origins in attempts of de Finetti (1931) to axiomatise probability theory. It also played an important role in the expected utility theory of (Savage, 1954, p.32). The essence of a qualitative probability is that it does not give us numerical probabilities but instead provides us with the information, for every pair of events, which one is more likely to happen. The class of qualitative probability orders is broader than the class of probability measures for any $n \geq 5$ (Kraft, Pratt, & Seidenberg, 1959). Qualitative probability orders on finite sets are now recognised as an important combinatorial object (Kraft et al., 1959; Fishburn, 1996, 1997) that finds applications in areas as far apart from probability theory as the theory of Gröbner bases (e.g., Maclagan, 1998/99).

Another important combinatorial object, also defined on a finite set is an abstract simplicial complex. This is a set of subsets of a finite set, called faces, with the property that a subset of a face is also a face. This concept is dual to the concept of a simple game whose winning coalitions form a set of subsets of a finite set with the property that if a coalition is winning, then every superset of it is also a winning coalition. The most studied class of simplicial complexes is the class of threshold simplicial complexes. These arise when we assign weights to elements of a finite set, set a threshold and define faces as those subsets whose combined weight is not achieving the threshold.

Given a qualitative probability order one may obtain a simplicial complex in an analogous way. For this one has to choose a threshold—which now will be a subset of our finite set—and consider as faces all subsets that are earlier than the threshold in the given qualitative probability order. This initial segment of the qualitative probability order will, in fact, be a simplicial complex. The collection of complexes arising as initial segments of probability orders contains threshold complexes and is contained in the well-studied class of shifted complexes (C. Klivans, 2005; C. J. Klivans, 2007). A natural question is therefore to ask if this is indeed a new class of complexes distinct from both the threshold complexes and the shifted ones.

In Chapter 5 we give an affirmative answer to both of these questions. We present an example of a shifted complex on 7 points that is not the initial segment of any qualitative probability order. On the other hand we also construct an initial segment of a qualitative probability order on 26 atoms that is not threshold. We also show that such example cannot be too small, in particular, it is unlikely that one can be found on fewer than 18 atoms.

1.2 Summary of Results

In the following we present the topics and results obtained in each chapter of the thesis. The reader is referred to the corresponding chapters for formal definitions and proofs.

Chapter 2: Preliminaries

This chapter represents a necessary background material on the theory of simple games.

Only a few structures in mathematics appear in different contexts and are exploited by many other areas. One of such structures is hypergraphs.

Definition 2.1.1. A **hypergraph** G is a pair (P, W), where P is a finite set and W is a collection of subsets of P.

We will consider hypergraphs from a voting-theoretic point of view.

Definition 2.1.2. A **simple game** is a hypergraph $G = (P, W)$ which satisfies the **monotonicity condition**: if $X \in W$ and $X \subset Y \subseteq P$, then $Y \in W$. We also require that W is different from $\emptyset$ and P (non-triviality assumption).

Let us consider a finite set P consisting of n elements which we will call **players**. For convenience, the set P can be taken to be $[n] = \{1, 2, \ldots, n\}$. Any set of players is called a **coalition**, and the whole P is usually addressed as the **grand coalition**. Elements of the set W are called **winning coalitions**. We also define the set $L = 2^P - W$ and call elements of this set **losing coalitions**. A winning coalition is said to be **minimal** if every proper subset is a losing coalition. Due to monotonicity, every simple game is fully determined by the set of its minimal winning coalitions. A losing coalition is said to be **maximal** if every proper superset is a winning coalition. A simple game is fully determined by set of maximal losing coalitions.

Simple games play the central role in this thesis. We will see "disguised" simple games in the areas as disparate as secret sharing schemes and qualitative probability orders.

Definition 2.2.1. A simple game is called **proper** if $X \in W$ implies $X^c \in L$, and **strong** if $X \in L$ implies $X^c \in W$. A simple game which is proper and strong is called a **constant sum** game.

Thus, a game is *not proper* if and only if the ground set P can be divided into two disjoint winning coalitions. A game is *not strong* if and only if the ground set P is the union of two disjoint losing coalitions. Note, in a constant sum game with n players there are exactly 2^{n-1} winning coalitions and exactly the same number of losing ones.

One of the key classes of simple games are weighted simple games.

Definition 2.3.1. A simple game G is called **weighted majority game** if there exist non-negative reals $w_1, \ldots, w_n$, and a positive real number q, called quota, such that $X \in W$ iff $\sum_{i \in X} w_i \geq q$. Such game is denoted $[q; w_1, \ldots, w_n]$. We also call $[q; w_1, \ldots, w_n]$ as a **weighted majority representation** for G.

For simplicity, we denote by $w(X)$ the weight $\sum_{i \in X} w_i$ of a coalition X. Note that the weight of a coalition X is equal to the dot product $\mathbf{w} \cdot \chi(X)$ of the weight vector $\mathbf{w}$ with the characteristic vector $\chi(X)$ of X. The **characteristic vector** $\chi(X)$ of X is a vector of $\mathbb{R}^n$ which has ith coordinate equal to 0 if $i \notin X$ and 1 if $i \in X$.

Definition 2.4.1 A sequence of coalitions

$$\mathcal{T} = (X_1, \ldots, X_j; Y_1, \ldots, Y_j)$$

is a **trading transform** if the coalitions $X_1, \ldots, X_j$ can be converted into the coalitions $Y_1, \ldots, Y_j$ by rearranging players or, equivalently, by several trades. It can also be expressed as

$$|\{i : a \in X_i\}| = |\{i : a \in Y_i\}| \qquad \text{for all } a \in P.$$

We say that the trading transform $\mathcal{T}$ has **length j**.

Note that while in (2.1) we can consider that no X_i coincides with any of Y_k, it is perfectly possible that the sequence $X_1, \ldots, X_j$ has some terms equal, the sequence $Y_1, \ldots, Y_j$ can also contain equal subsets. The order of subsets in these sequences is not important.

Definition 2.5.1. A simple game G is called **k-trade robust** if no trading transform $\mathcal{T} = (X_1, \ldots, X_j; Y_1, \ldots, Y_j)$ with $j \leq k$ has all coalitions $X_1, \ldots, X_j$ winning and all $Y_1, \ldots, Y_j$ losing. G is **trade robust** if it is k-trade robust for every k.

Clearly, if a game is not trade robust then it is not weighted. Is it true, however, that trade robustness implies weightedness? Does there exist a positive integer k such that k-trade robustness implies trade robustness? Winder (1962) showed that in general no such k exists. Nevertheless, if we restrict ourselves to games with n players, then the situation changes and such a number k exists for each n. Of course this number k will depend on n. This is an important result contained in the following theorem. (We note though that the equivalence of the first and the second was earlier proved by Elgot (1960).)

Theorem 2.5.1.(Taylor & Zwicker, 1992) The following three conditions are equivalent:

- G is a weighted majority game,
- G is trade robust,
- G is $2^{2^{|P|}}$-trade robust.

Weights represent "power" or "influence" of players. At the same time, there are non-weighted games in which the players can be lined according to their "influence". A natural question arises: what is the influence and how can we measure it? Consider the following situation: two losing coalitions do a one-for-one trade, and after the trade one coalition becomes winning and another one remains losing. Intuitively, the player who turned a losing coalition into winning is more "powerful" than the other player. Isbell (1958) was the first who formalized this idea for simple games. Maschler and Peleg (1966) continued a further generalization of this notion; see also (Muroga, 1971, p. 113).

Definition 2.7.1. Suppose $G = (P, W)$ is a simple game. Then the **individual desirability relation** (for G) is the binary relation $\leq_I$ on P defined by

$$p \leq_I q \text{ iff } \forall X \subseteq P - \{p, q\}, \text{ if } X \cup \{p\} \in W, \text{ then } X \cup \{q\} \in W.$$

Definition 2.7.2. A simple game $G = (P, W)$ is **complete** or **linear** if the individual desirability relation $\leq_I$ for G is a complete preorder.

For simplicity it is convenient to assume that every player is at least as desirable as the previous one (or the other way around), i.e., $i \geq_I j$ iff $i \geq j$. As we already know every simple game is completely defined by the set of minimal winning coalitions. In the case of complete games we need even less information: a game is completely defined by the set of shift-minimal winning coalitions. A minimal winning coalition X is **shift-minimal winning** coalition if $(X - \{i\}) \cup \{j\}$ is losing for any $i \in X$ and $j \notin X$ such that $j <_I i$. In the analogous way we can define shift-maximal losing coalitions.

Every weighted majority game is complete. However, the class of complete games is much broader then the class of weighted games.

The dual objects are very common in all areas of mathematics. The theory of simple games no exception.

Definition 2.9.1. The **dual game** of a game $G = (P, W)$ is defined to be $G^* = (P, L^c)$. This is to say that in the game G^* dual to a game G the winning coalitions are exactly the complements of losing coalitions of G.

The operation of taking the dual is known to preserve weightedness:

Proposition 2.9.1.(Taylor & Zwicker, 1999, Propositions 4.3.10 and 4.10.1)

(i) The simple game G is weighted iff the dual game G^* is weighted.

(ii) For every integer $k \geq 2$, the simple game G is k-trade robust iff G^* is k-trade robust.

Proposition 2.9.2.(Taylor & Zwicker, 1999, Proposition 3.2.8) The individual desirability relation $\leq_I$ is dual symmetric in the sense that $p \leq_I q$ holds in G iff it holds in G^*.

Duality it is a useful tool, and as Shapley (1962) wrote "the usefulness of duality concept depends on the inclusion of improper games in our theory."

There are two natural substructures that arise from the more general notion in threshold logic (see Muroga, 1971, p. 112) which can be traced back to at least (Isbell, 1958). The **subgame** determined by $P' \subseteq P$ is the simple game (P', W_{sg}), where

$$X \in W_{sg} \text{ iff } X \subseteq P' \text{ and } X \in W.$$

The subgame determined by $P' \subseteq P$ is usually denoted by G_A where $A = P - P'$. The **reduced** game determined by P' is the simple game (P', W_{rg}), where

$$X \in W_{rg} \text{ iff } X \subseteq P' \text{ and } X \cup (P - P') \in W.$$

The reduced game determined by P' is usually denoted by G^B, where $B = P - P'$.

Consider the following situation: a group of people P vote in favor of or against a law. A coalition wins if it can pass the law. Intuitively, in this situation the subgame G_A results in assuming that all people of A have already voted against the law. The reduced game G^B results in assuming that people in B have already voted in favor of the law. Hence a Boolean subgame reflects the situation in which some votes are already known.

Proposition 2.10.1.(Taylor & Zwicker, 1999, Proposition 1.4.8) Assume that $G = (P, W)$ is a simple game and that $B \subseteq P$. Then $(G^B)^* = (G^*)_B$, and so $G_B = ((G^*)^B)^*$ and $(G_B)^* = (G^*)^B$.

It is straightforward to show that every subgame and every reduced game of a weighted majority game is also a weighted majority game. For example, the case of subgame one only has to retain the same weights for elements of A^c as in G and the same threshold.

Chapter 3: Weightedness and Rough Weightedness

Results of this chapter are contained in (Gvozdeva & Slinko, 2011).

This chapter is devoted to a more detailed study of weightedness and contains first attempts to generalize this notion.

As we have seen in Chapter 2 some games are almost weighted in the following sense:

- all coalitions with weights below the threshold are losing;
- coalitions with weights above the threshold are winning;
- coalitions with weights exactly on the threshold can be winning or losing.

This is a natural approach as we relax conditions on threshold of a weighted game. Such games are called **roughly weighted** simple games if weight of at least one player is not zero or the quota is not zero.

First we approach simple games from the algebraic point of view. Let $T = \{-1, 0, 1\}$ and $T^n = T \times T \times \dots T$ (n times). For any pair (X, Y) of subsets $X, Y \in [n]$ we define

$$\mathbf{v}_{X,Y} = \chi(X) - \chi(Y) \in T^n,$$

where $\chi(X)$ and $\chi(Y)$ are the characteristic vectors of subsets X and Y, respectively.

Now let $G = (P, W)$ be a game. We will associate an algebraic object with G. For any pair (X, Y), where X is winning and Y is losing, we put in correspondence the following vector $\mathbf{v}_{X,Y}$. The set of all such vectors we denote $I(G)$.

Definition 3.2.1. Let $\mathbf{e}_i = (0, \dots, 1, \dots, 0)$, where the only nonzero element 1 is in the ith position. A subset $I \subseteq T^n$ is called an **ideal** in T^n if

$$(\mathbf{v} \in I \text{ and } \mathbf{v} + \mathbf{e}_i \in T^n) \Longrightarrow \mathbf{v} + \mathbf{e}_i \in I, \text{ for each } i \in [n].$$

There is a natural correspondence between ideals and $I(G)$. More explicitly, $I(G)$ is an ideal. Nevertheless, not every ideal corresponds to a game. Even if a game for an ideal exists it may not be unique.

Proposition 3.2.2. Let G be a finite simple game. Then:

(a) G is weighted iff the system $\mathbf{v} \cdot \mathbf{x} > 0$, $\mathbf{v} \in I(G)$ has a solution.

(b) G is roughly weighted iff the system $\mathbf{v} \cdot \mathbf{x} \geq 0$, $\mathbf{v} \in I(G)$ has a non-zero solution.

The notions of a trading transform and k-trade-robustness can be algebraically reformulated as well.

Taylor and Zwicker (1992) showed that 2^{2^n}-trade robustness implies weightedness of a simple game with n players. As this characterisation of weighted games implies, to show that a game G is not a weighted majority game it is sufficient to present a trading transform $(X_1, \dots, X_j; Y_1, \dots, Y_j)$, where coalitions $X_1, \dots, X_j$ are winning and $Y_1, \dots, Y_j$ are losing. We will call such trading transform a **certificate of non-weightedness** of G. Two interesting questions emerge immediately. What is the maximal length of certificates that we have to check, if we want to check weightedness of a game with n players? If a game is not weighted, then how far away is this game from being weighted?

Definition 3.3.1. Let $G = (P, W)$ be a simple game with $|P| = n$. If G is not weighted we define $f(G)$ to be the smallest positive integer k such that G is not k-trade robust.

If G is weighted we set $f(G) = \infty$. The larger the value $f(G)$ the closer is the game G to a weighted majority game. Let us also define

$$f(n) = \max_G f(G),$$

where the maximum is taken over non weighted games with n players. Equivalently $f(n)$ is the smallest positive integer such that $f(n)$-trade robustness for an n-player game implies its weightedness.

Summarizing results of Taylor and Zwicker (1995), Taylor and Zwicker (1999) and Gabelman (1961) in terms of the function $f(n)$ we obtain:

Corollary 3.3.1. For any $n \geq 2$,

$$\lfloor \sqrt{n} \rfloor \leq f(n) \leq 2^{2^n}.$$

As Taylor and Zwicker noted in (Taylor & Zwicker, 1999), for most non-weighted games the value of $f(G)$ is 2. The closer the game to a weighted majority game the longer is the certificate and it is harder to find it.

By the algebraic means, we improve the existing upper bound. Fishburn (1997) proved the combinatorial lemma which helps us to construct a series of examples. In the light of these examples a better lower bound can be found. The new bounds valid for $n \geq 5$ and can be written in the following way:

$$\left\lfloor \frac{n-1}{2} \right\rfloor \leq f(n) \leq (n+1)2^{\frac{1}{2}n \log_2 n}.$$

Definition 3.4.1. A certificate of non-weightedness which includes P and $\emptyset$ we call **potent**.

We find a criterion for a game to be roughly weighted making use of the idea outlined in (Kraft et al., 1959).

Theorem 3.4.2. The game G with n players is roughly weighted if one of the two equivalent statements holds:

(a) for no positive integer $j \leq (n+1)2^{\frac{1}{2}n \log_2 n}$ do there exist a potent certificate of non-weightedness of length j,

(b) for no positive integer $j \leq (n+1)2^{\frac{1}{2}n\log_2 n}$ do there exist j vectors $\mathbf{v}_1, \ldots, \mathbf{v}_j \in I(G)$ such that

$$\mathbf{v}_1 + \ldots + \mathbf{v}_j + \mathbf{1} = \mathbf{0},$$

where $\mathbf{1} = (1, 1, \ldots, 1)$.

Suppose we have to check if a game G is roughly weighted or not. According to the Criterion of Rough Weightedness (Theorem 3.4.2), we have to check if there is any potent certificates of non-weightedness. We have to know where to stop while checking those. We define a new function for this.

Definition 3.4.4 If the game is roughly weighted let us set $g(G) = \infty$. Alternatively, $g(G)$ is the length of the shortest potent certificate of non-weightedness for G. We also define a function

$$g(n) = \max_G g(G),$$

where the maximum is taken over non roughly-weighted games with n players.

Checking rough weightedness of a game with n players we then have to check all potent certificates of non-weightedness up to a length $g(n)$.

One of the main result of this chapter can be formulated in terms of function $g(n)$ as follows:

Theorem 3.4.7. For any $n \geqslant 5$

$$2n + 3 \leqslant g(n) \leqslant (n+1)2^{\frac{1}{2}n\log_2 n}.$$

We prove the lower bound by constructing examples. We show several easy relations between functions $\mathbf{f}$ and $\mathbf{g}$.

Similarly we can define the function $i(n)$ on complete simple games. In case of complete games it is sufficient to consider only potents certificate of non-weightedness with all winning coalitions being shift-minimal. The value of $i(n)$ is the maximum of the shortest lengths of such potents certificate of non-weightedness, where the maxim is taken over all non roughly-weighted complete games.

We also study rough weightedness of small games. We show that all games with $n \leq 4$ players, all strong and proper games with $n \leq 5$ players, and all constant-sum games with $n \leq 6$ players are roughly weighted. Thus the smallest constant sum game that is not roughly weighted is the game with seven players obtained from the Fano plane (von Neumann & Morgenstern, 1944, p. 470). This game is the smallest representative of the class of projective games (Richardson, 1956). One of the consequences of our characterisation is that all projective games do not have rough weights.

Chapter 4: Generalizations of Rough Weightedness

Results of this chapter are contained in (Gvozdeva, Hemaspaandra, & Slinko, 2010).

This chapter contributes to the program of numerical characterization and classification of simple games outlined in the classic monograph of von Neumann and Morgenstern. We suggest three possible ways to classify simple games beyond the classes of weighted and roughly weighted games. To this end we introduce three hierarchies of games and prove some relationships between their classes. We prove that our hierarchies are true (i.e., infinite) hierarchies. In particular, they are strict in the sense that more of the key "resource" (which may, for example, be the size or structure of the "tie-breaking" region where the weights of the different coalitions are considered so close that we are allowed to specify either winningness or nonwinningness of the coalition) yields the flexibility to capture strictly more games.

Due to Proposition 3.6.1, there is a trivial way to make any game roughly weighted. This can be done by adding an additional player and making him or her a passer. Then we can introduce rough weights by assigning weight 1 to the passer and weight 0 to every other player and setting the quota equal to 0. Note that if the original game is not roughly weighted, then such rough representation is unique (up to multiplicative scaling). In our view, adding a passer trivializes the game but does not make it closer to a weighted majority game; this is why in definitions of our hierarchies $\mathcal{B}$ and $\mathcal{C}$ we do not allow 0 as a threshold value.

The first hierarchy of classes $\mathcal{A}_\alpha$ tries to capture the richness of the class of games that do not have rough weights, and does so by introducing a parameter $\alpha \in (0, \frac{1}{2})$. Our method of classification is based on the existence of potent certificates of nonweightedness for such games (Gvozdeva & Slinko, 2011). We will now show that potent certificates can be further classified. We will extract a very important parameter from this classification.

Definition 4.2.1. A certificate of nonweightedness

$$\mathcal{T} = (X_1, \dots, X_m; Y_1, \dots, Y_m)$$

is called an **ℓ-potent certificate** of length m if it contains at least ℓ grand coalitions among $X_1, \dots, X_m$ and at least ℓ empty sets among $Y_1, \dots, Y_m$.

We will say that for a rational number q a game G **belongs to the class** $\mathcal{A}_q$ of the $\mathcal{A}$-hierarchy if G possesses some ℓ-potent certificate of nonweightedness of length m, such that $q = \ell/m$. If α is irrational, we set $\mathcal{A}_\alpha = \bigcap_{\{q : q<\alpha \wedge q \text{ is rational}\}} \mathcal{A}_q$.

It is easy to see that, if $q \geq \frac{1}{2}$, then $\mathcal{A}_q$ is empty. So our hierarchy consists of a family of classes $\{\mathcal{A}_\alpha\}_{\alpha\in(0,\frac{1}{2})}$. We show that this hierarchy is strict, that is, a smaller parameter captures more games. The following proposition starts us on our path toward showing this.

Proposition 4.2.1. If $0 < \alpha \leq \beta < \frac{1}{2}$, then $\mathcal{A}_\alpha \supseteq \mathcal{A}_\beta$.

We say that a game G is *critical* for $\mathcal{A}_\alpha$ if it belongs to $\mathcal{A}_\alpha$ but does not belong to any $\mathcal{A}_\beta$ with $\beta > \alpha$.

Theorem 4.2.1. For every rational $\alpha \in (0, \frac{1}{2})$ there exists a critical game $G \in \mathcal{A}_\alpha$.

The straightforward corollary of our theorem that for $0 < \alpha < \beta < \frac{1}{2}$ we have $\mathcal{A}_\alpha \supsetneq \mathcal{A}_\beta$. Hence our hierarchy is strict.

The $\mathcal{B}$-hierarchy generalizes the idea behind rough weightedness to allow more "points of (decision) flexibility."

Definition 4.3.1. A simple game $G = (P, W)$ **belongs to** $\mathcal{B}_k$ if there exist real numbers $0 < q_1 \leq q_2 \leq \cdots \leq q_k$, called thresholds, and a weight function $w\colon P \to \mathbb{R}^{\geq 0}$ such that

(a) if $\sum_{i\in X} w(i) > q_k$, then X is winning,

(b) if $\sum_{i\in X} w(i) < q_1$, then X is losing,

(c) if $q_1 \leq \sum_{i\in X} w(i) \leq q_k$, then $w(X) = \sum_{i\in X} w(i) \in \{q_1, \ldots, q_k\}$.

The condition $0 < q_1$ in the definition is essential. If we allow the first threshold q_1 be zero, then every simple game can be represented as a 2-rough game. It is also worthwhile to note that adding a passer does not change the class of the game, that is, a game G belongs to $\mathcal{B}_k$ if and only if the game G' obtained from G by adding a passer belongs to $\mathcal{B}_k$. This is because a passer can be assigned a very large weight. Thus $\mathcal{B}_1$ consists of the roughly weighted simple games with nonzero quota.

Constructing a series of examples we prove strictness of this hierarchy as well.

Theorem 4.3.1. For every natural number $k \in \mathbb{N}^+$, there exists a game in $\mathcal{B}_{k+1} - \mathcal{B}_k$.

Let us consider another extension of the idea of rough weightedness. This time we will use a threshold interval instead of a single threshold or (as in $\mathcal{B}$-hierarchy) a collection of threshold points. It is convenient to "normalize" the weights so that the left end of our threshold interval is 1. We do not lose any generality by doing this.

Definition 4.4.1. We say that a simple game $G = (P, W)$ **is in the class** C_α, $\alpha \in \mathbb{R}^{\geq 1}$, if there exists a weight function $w \colon P \to \mathbb{R}^{\geq 0}$ such that for $X \in 2^P$ the condition $w(X) > \alpha$ implies that X is winning, and $w(X) < 1$ implies X is losing.

The roughly weighted games with nonzero quota form the class C_1. We also note that adding or deleting a passer does not change the class of the game.

Definition 4.4.2. We say that a game G is **critical** for C_α if it belongs to C_α but does not belong to any C_β with $\beta < \alpha$.

It is clear that if $\alpha \leq \beta$, then $C_\alpha \subseteq C_\beta$. However, we can construct an example and show more.

Theorem 4.4.1. For each $1 \leq \alpha < \beta$, it holds that $C_\alpha \subsetneq C_\beta$.

There is a connection between $\mathcal{A}$ hierarchy and C hierarchy:

Theorem 4.4.2 Let G be a simple game that is not roughly weighted and is critical for C_a. Suppose G also belongs to $\mathcal{A}_q$ for some $0 < q < \frac{1}{2}$. Then

$$a \geq \frac{1-q}{1-2q}.$$

The strictness of the latter hierarchy was achieved because we allowed games with arbitrary (but finite) numbers of players. The situation will be different if we keep the number of players, n, fixed. Then there is an interval $[1, s(n)]$ such that all games with n players belong to $C_{s(n)}$ and $s(n)$ is minimal with this property. There will be also finitely many numbers $q \in [1, s(n)]$ such that the interval $[1, q]$ represents more n-player games than any interval $[1, q']$ with $q' < q$. We call the set of such numbers the nth **spectrum** and denote it $\mathrm{Spec}(n)$. We also call a game with n players **critical** if it belongs to C_α with $\alpha \in \mathrm{Spec}(n)$ but does not belong to any C_β with $\beta < \alpha$.

We try to give a reasonably tight upper bound for $s(n)$:

Theorem 4.5.1. For $n \geq 4$, $\frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor \leq s(n) \leq \frac{n-2}{2}$.

In Chapter 3 we showed that all games with four players are roughly weighted. Now let us calculate the spectra for $n \leq 6$. Since we may assume that the game does not have passers we may assume that the quota is nonzero. Hence we have Spec(4) = {1}. So the first nontrivial case is $n = 5$.

Let $G = ([n], W)$ be a simple game. The problem of finding the smallest α such that $G \in \mathcal{C}_\alpha$ holds is a linear programming question. Indeed, we need to find the minimum α such that the following system of linear inequalities is consistent:

$$\begin{cases} w(X) \geq 1 & \text{for } X \in W^{\min}, \\ w(Y) \leq \alpha & \text{for } Y \in L^{\max}. \end{cases}$$

This is equivalent to the following optimization problem:

Minimize: α.
Subject to: $\sum_{i \in X} w_i \geq 1,\ \sum_{i \in Y} w_i - \alpha \leq 0,$ and $w_i \geq 0;\ X \in W^{min}, Y \in L^{max}$.

We used the equivalence of finding spectrum and an optimization problem and wrote Maple code using the "LPSolve" command to find spectrums for five and six players.

Theorem 4.5.2. Spec(5) $= \left\{1, \frac{6}{5}, \frac{7}{6}, \frac{8}{7}, \frac{9}{8}\right\}$.

Theorem 4.5.3. The 6th spectrum Spec(6) contains Spec(5) and also the following fractions:

$$\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{9}{7}, \frac{10}{9}, \frac{11}{9}, \frac{11}{10}, \frac{12}{11}, \frac{13}{10}, \frac{13}{11}, \frac{13}{12}, \frac{14}{11}, \frac{14}{13}, \frac{15}{13}, \frac{15}{14}, \frac{16}{13}, \frac{16}{15}, \frac{17}{13}, \frac{17}{14}, \frac{17}{15}, \frac{17}{16}, \frac{18}{17}.$$

Chapter 5: Initial Segments Complexes Obtained from Qualitative Probability Orders

Results of this chapter are contained in (Edelman, Gvozdeva, & Slinko, 2011).

An important combinatorial object defined on a finite set is an abstract simplicial complex. This concept is dual to the concept of a simple game in a sense that a set of subsets of a finite set is a set of losing coalitions of a game iff this set is an abstract simplicial complex. There are two well-known classes of abstract simplicial complexes, namely threshold simplicial complexes and shifted simplicial complexes. Threshold complexes arise when we assign weights to elements of a finite set, set a threshold and define its elements as those subsets whose combined weight is not achieving the threshold. One can see that any threshold complex correspond to the set of losing coalitions of a simple game. Shifted simplicial complexes are related to complete simple games. More explicitly a set of subtest of a finite set is a set of losing coalition of a complete game if and only if it is a shifted simplicial complex.

In this chapter we initiate the study of abstract simplicial complexes which are initial segments of qualitative probability orders. This is a natural class that contains the threshold complexes and is contained in the shifted complexes, but is equal to neither. In particular we construct a qualitative probability order on 26 atoms that has an initial segment which is not a threshold simplicial complex. Although 26 is probably not the minimal number for which such example exists we provide some evidence that it cannot be much smaller. We prove some necessary conditions for this class and make a conjecture as to a characterization of them. The conjectured characterization relies on some ideas from cooperative game theory.

The structure of this chapter is as follows. In Section 5.1 we introduce the basics of qualitative probability orders. In Section 5.2 we consider abstract simplicial complexes and give necessary and sufficient conditions for them being threshold. In Section 5.3 we give a construction that will further provide us with examples of qualitative probability orders that are not related to any probability measure. Finally in Sections 5.4 and 5.5 we present our main result which is an example of a qualitative probability order on 26 atoms that is not threshold. Section 5.6 concludes with a conjectured characterization of initial segment complexes that is motivated by work in the theory of cooperative games.

An order in this chapter is any reflexive, complete and transitive binary relation. If it is also anti-symmetric, it is called linear order.

Definition 5.1.1. An order $\preceq$ on $2^{[n]}$ is called a **qualitative probability order** on $[n]$ if

$$\emptyset \preceq A$$

for every subset A of $[n]$, and $\preceq$ satisfies de Finetti's axiom, namely for all $A, B, C \in 2^{[n]}$

$$A \preceq B \iff A \cup C \ \preceq \ B \cup C \text{ whenever } \ (A \cup B) \cap C = \emptyset\,.$$

Note that if we have a probability measure $\mathbf{p} = (p_1, \ldots, p_n)$ on $[n]$, where p_i is the probability of i, then we know the probability $p(A) = \sum_{i \in A} p_i$ of every event A. We may now define a relation $\preceq$ on $2^{[n]}$ by

$$A \preceq B \quad \text{if and only if} \quad p(A) \leq p(B);$$

obviously $\preceq$ is a qualitative probability order on $[n]$, and any such order is called **representable** (e.g., Fishburn, 1996; Regoli, 2000). Those not obtainable in this way are called **non-representable**. The class of qualitative probability orders is broader than the class of probability measures for any $n \geq 5$ (Kraft et al., 1959). A non-representable qualitative probability order $\preceq$ on $[n]$ is said to **almost agree** with the measure **p** on $[n]$ if

$$A \preceq B \Longrightarrow p(A) \leq p(B).$$

If such a measure **p** exists, then the order $\preceq$ is said to be **almost representable**.

As before we want to consider all objects from algebraic point of view.

Definition 5.1.2. We say that an order $\preceq$ on $2^{[n]}$ **satisfies the k-th cancellation condition** CC_k if there does not exist a trading transform $(A_1, \ldots, A_k; B_1, \ldots, B_k)$ such that $A_i \preceq B_i$ for all $i \in [k]$ and $A_i \prec B_i$ for at least one $i \in [k]$.

The key result of (Kraft et al., 1959) can now be reformulated as follows.

Theorem 5.1.1. A qualitative probability order $\preceq$ is representable if and only if it satisfies CC_k for all $k = 1, 2, \ldots$.

It was also shown in (Fishburn, 1996, Section 2) that CC_2 and CC_3 hold for linear qualitative probability orders. It follows from de Finetti's axiom and properties of linear orders. It can be shown that a qualitative probability order satisfies CC_2 and CC_3 as well. Hence CC_4 is the first nontrivial cancellation condition. As was noticed in (Kraft et al., 1959), for $n < 5$ all qualitative probability orders are representable, but for $n = 5$ there are non-representable ones. For $n = 5$ all orders are still almost representable (Fishburn, 1996) which is no longer true for $n = 6$ (Kraft et al., 1959).

It will be useful for our constructions to rephrase some of these conditions in vector language. To every such linear order $\preceq$, there corresponds a *discrete cone* $C(\preceq)$ in T^n, where $T = \{-1, 0, 1\}$, as defined in (Fishburn, 1996).

Definition 5.1.3. A subset $C \subseteq T^n$ is said to be a **discrete cone** if the following properties hold:

D1. $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\} \subseteq C$, where $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$,

D2. $\{-\mathbf{x}, \mathbf{x}\} \cap C \neq \emptyset$ for every $\mathbf{x} \in T^n$,

D3. $\mathbf{x} + \mathbf{y} \in C$ whenever $\mathbf{x}, \mathbf{y} \in C$ and $\mathbf{x} + \mathbf{y} \in T^n$.

We note that Fishburn (1996) requires $\mathbf{0} \notin C$ because his orders are anti-reflexive. In our case, condition D2 implies $\mathbf{0} \in C$.

Given a qualitative probability order $\preceq$ on $2^{[n]}$, for every pair of subsets A, B satisfying $B \preceq A$ we construct the characteristic vector of this pair $\chi(A, B) = \chi(A) - \chi(B) \in T^n$. We define the set $C(\preceq)$ of all characteristic vectors $\chi(A, B)$, for $A, B \in 2^{[n]}$ such that $B \preceq A$. The two axioms of qualitative probability guarantee that $C(\preceq)$ is a discrete cone (see Fishburn, 1996, Lemma 2.1).

Following (Fishburn, 1996), the cancellation conditions can be reformulated as well.

Geometrically, a qualitative probability order $\preceq$ is representable if and only if there exists a non-negative vector $\mathbf{u} \in \mathbb{R}^n$ such that

$$\mathbf{x} \in C(\preceq) \Longleftrightarrow (\mathbf{u}, \mathbf{x}) \geq 0 \quad \text{for all } \mathbf{x} \in T^n - \{\mathbf{0}\},$$

where $(\cdot, \cdot)$ is the standard inner product; that is, $\preceq$ is representable if and only if every non-zero vector in the cone $C(\preceq)$ lies in the closed half-space $H_{\mathbf{u}}^+ = \{\mathbf{x} \in \mathbb{R}^n \mid (\mathbf{u}, \mathbf{x}) \geq 0\}$ of the corresponding hyperplane $H_{\mathbf{u}} = \{\mathbf{x} \in \mathbb{R}^n \mid (\mathbf{u}, \mathbf{x}) = 0\}$.

Definition 5.2.1.A subset $\Delta \subseteq 2^{[n]}$ is an **(abstract) simplicial complex** if it satisfies the condition:

$$\text{if } B \in \Delta \text{ and } A \subseteq B, \text{ then } A \in \Delta.$$

Subsets that are in Δ are called *faces*. Abstract simplicial complexes arose from geometric simplicial complexes in topology (e.g., Maunder, 1996). In combinatorial optimization various abstract simplicial complexes associated with finite graphs (Jonsson, 2005) are studied, such as the independence complex, matching complex etc. Abstract simplicial complexes are also in one-to-one correspondence with *simple games* as defined by (von Neumann & Morgenstern, 1944). Obviously the set of losing coalitions L is a simplicial complex. The reverse is also true: if Δ is a simplicial complex, then the set $2^{[n]} - \Delta$ is a set of winning coalitions of a certain simple game.

A well-studied class of simplicial complexes is the *threshold* complexes (mostly as an equivalent concept to the concept of a weighted majority game but also as threshold hypergraphs (Reiterman, Rödl, Šiňajová, & Tůma, 1985)). A simplicial complex Δ is a **threshold complex** if there exist non-negative reals $w_1, \ldots, w_n$ and a non-negativeconstant q, such that

$$A \in \Delta \Longleftrightarrow w(A) = \sum_{i \in A} w_i < q.$$

The same parameters define a *weighted majority game* with the standard notation $[q; w_1, \ldots, w_n]$.

A much larger but still well-understood class of simplicial complexes is *shifted* simplicial complexes (C. Klivans, 2005; C. J. Klivans, 2007). A simplicial complex is **shifted** if there exists an order $\trianglelefteq$ on the set of vertices $[n]$ such that for any face F, replacing any of its vertices $x \in F$ with a vertex y such that $y \trianglelefteq x$ results in a subset $(F - \{x\}) \cup \{y\}$ which is also a face. Shifted complexes correspond to complete games (Freixas & Molinero, 2009b).

One can see that any initial segment of a qualitative probability order is a simplicial complex. Thus we refer to simplicial complexes that arise as initial segments of some qualitative probability order as an initial segment complex.

In a similar manner as for the qualitative probability orders, cancellation conditions will play a key role in our analyzing simplicial complexes.

Definition 5.2.2. A simplicial complex Δ is said to **satisfy $\mathbf{CC_k^*}$** if for no $k \geq 2$ does there exist a trading transform $(A_1, \ldots, A_k; B_1, \ldots, B_k)$, such that $A_i \in \Delta$ and $B_i \notin \Delta$, for every $i \in [k]$.

There is a connection between CC_k and CC_k^*. More precisely, if a qualitative probability order $\preceq$ satisfies CC_k then its initial segment $\Delta(\preceq, T)$ satisfies CC_k^*. This gives us some initial properties of initial segment complexes. Since conditions CC_k, $k = 2, 3$, hold for all qualitative probability orders (Fishburn, 1996) the first three cancelation conditions CC_1^*, CC_2^* and CC_3^* hold for all initial segment complexes.

Using cancellation conditions for simplicial complexes, we will show that this class contains the threshold complexes and is contained in the shifted complexes. Using only these conditions it will be easy to show that the initial segment complexes are strictly contained in the shifted complexes.

Corollary 5.2.1. Every initial segment complex is a shifted complex. Moreover, there are shifted complexes that are not initial segment complexes.

Lemma 5.2.2. Every threshold complex is an initial segment complex.

This leaves us with the question of whether this containment is strict, i.e., are there initial segment complexes which are not threshold complexes. As we know any initial segment of a representable qualitative probability order is a threshold simplex. One might think that evry non-representable qualitative probability order would have at least one initial segmentt that is not threshold. Unfortunately that may not be the case. There are examples of qualitative probability orders such that every initial segment is a threshold complex.

Another approach to finding an initial segment complex that is not threshold is to construct a complex that violates CC_k^* for some small value of k. As noted above, all initial segment complexes satisfy CC_2^* and CC_3^* so the smallest condition that could fail is CC_4^*. We will now show that for small values of n cancellation condition CC_4^* is satisfied for any initial segment. This will also give us invaluable information on how to construct a non-threshold initial segment later.

Definition 5.2.3. Two pairs of subsets (A_1, B_1) and (A_2, B_2) are said to **be compatible** if the following two conditions hold:

$$\begin{aligned}
&x \in A_1 \cap A_2 \Longrightarrow x \in B_1 \cup B_2, \text{ and} \\
&x \in B_1 \cap B_2 \Longrightarrow x \in A_1 \cup A_2.
\end{aligned}$$

Lemma 5.2.3. Let $\preceq$ be a qualitative probability order on $2^{[n]}$, $T \subseteq [n]$, and let $\Delta = \Delta_n(\preceq, T)$ be the respective initial segment. Suppose CC_s^* fails and $(A_1, \ldots, A_s, B_1, \ldots, B_s)$ is a trading transform, such that $A_i \prec T \preceq B_j$ for all $i, j \in [s]$. If any two pairs (A_i, B_k) and (A_j, B_l) are compatible, then $\preceq$ fails to satisfy CC_{s-1}.

This lemma connects cancelation conditions of an initial segment complex and a qualitative probability order from which it arises. Suppose CC_4^* fails, i.e., there is a trading transform $(A_1, \ldots, A_4; B_1, \ldots, B_4)$, $A_i \prec T \preceq B_j$. Every pair $(A_i, B_j), (A_l, B_k)$ is not compatible. Otherwise by Lemma 5.2.3 the order $\preceq$ fails CC_3, which contradicts the fact that every qualitative probability satisfies CC_3. Making you of the fact that there are no compatible pairs we can show:

Theorem 5.2.4. CC_4^* holds for $\Delta = \Delta_n(\preceq, T)$ for all $n \leq 17$.

By the similar argument the following is true:

Theorem 5.2.5. CC_5^* holds for $\Delta = \Delta_n(\preceq, T)$ for all $n \leq 8$.

Our approach to finding an initial segment complex that is not threshold will be to start with a non-linear representable qualitative probability order and then perturb it so as to produce an almost representable order. By judicious breaking of ties in this new order we will be able to produce an initial segment that will violate CC_4^*. The language of discrete cones will be helpful.

We need to understand how we can construct new qualitative probability orders from old ones so we need the following investigation. Let $\preceq$ be a representable but not linear qualitative probability order which agrees with a probability measure $\mathbf{p}$.

Let $S(\preceq)$ be the set of all vectors of $C(\preceq)$ which lie in the corresponding hyperplane $H_{\mathbf{p}}$ through the origin. Clearly, if $\mathbf{x} \in S(\preceq)$, then $-\mathbf{x}$ is a vector of $S(\preceq)$ as well. Since in the definition of discrete cone it is sufficient that only one of these vectors is in $C(\preceq)$ we may try to remove one of them in order to obtain a new qualitative probability order. The new order will almost agree with $\mathbf{p}$ and hence will be at least almost representable. The big question is: what are the conditions under which a set of vectors can be removed from $S(\preceq)$?

What can prevent us from removing a vector from $S(\preceq)$? Intuitively, we cannot remove a vector if the set comparison corresponding to it is a consequence of those remaining. We need to consider what a consequence means formally.

There are two ways in which one set comparison might imply another one. The first way is by means of the de Finetti condition. This however is already built in the definition of the discrete cone as $\chi(A, B) = \chi(A \cup C, B \cup C)$. Another way in which a comparison may be implied from two other is transitivity. This has a nice algebraic characterisation. Indeed, if $C \prec B \prec A$, then $\chi(A, C) = \chi(A, B) + \chi(B, C)$. This leads us to the following definition.

Following (Christian, Conder, & Slinko, 2007) let us define a **restricted sum** for vectors in a discrete cone C. Let $\mathbf{u}, \mathbf{v} \in C$. Then

$$\mathbf{u} \oplus \mathbf{v} = \begin{cases} \mathbf{u} + \mathbf{v} & \text{if } \mathbf{u} + \mathbf{v} \in T^n, \\ \text{undefined} & \text{if } \mathbf{u} + \mathbf{v} \notin T^n. \end{cases}$$

Theorem 5.3.1. Let $\preceq$ be a representable non-linear qualitative probability order on $[n]$ which agrees with the probability measure $\mathbf{p}$. Let $S(\preceq)$ be the set of all vectors of $C(\preceq)$ which lie in the hyperplane $H_{\mathbf{p}}$. Let X be a subset of $S(\preceq)$ such that

- $X \cap \{\mathbf{s}, -\mathbf{s}\} \neq \emptyset$ for every $\mathbf{s} \in S(\preceq)$.
- X is closed under the operation of restricted sum.
- $(S(\preceq) - X) \cap \{e_1, \ldots, e_n\} = \emptyset$.

Then $Y = S(\preceq) - X$ may be dropped from $C(\preceq)$, that is $C_Y = C(\preceq) - Y$ is a discrete cone.

In this section we shall construct an almost representable linear qualitative probability order $\sqsubseteq$ on $2^{[26]}$ and a subset $T \subseteq [26]$, such that the initial segment $\Delta(\sqsubseteq, T)$ of $\sqsubseteq$ is not a threshold complex as it fails to satisfy the condition CC_4^*.

The idea of the example is as follows. We will start with a representable linear qualitative probability order $\preceq$ on [18] defined by positive weights $w_1, \ldots, w_{18}$ and extend it to a representable but nonlinear qualitative probability order $\preceq'$ on [26] with weights $w_1, \ldots, w_{26}$. A distinctive feature of $\preceq'$ will be the existence of eight sets $A'_1, \ldots, A'_4, B'_1, \ldots, B'_4$ in [26] such that:

1. The sequence $(A'_1, \ldots, A'_4; B'_1, \ldots, B'_4)$ is a trading transform.
2. The sets $A'_1, \ldots, A'_4, B'_1, \ldots, B'_4$ are tied in $\preceq'$, that is,
$$A'_1 \sim' \ldots A'_4 \sim' B'_1 \sim' \ldots \sim' B'_4.$$
3. If any two distinct sets $X, Y \subseteq [26]$ are tied in $\preceq'$, then $\chi(X, Y) = \chi(S, T)$, where $S, T \in \{A'_1, \ldots, A'_4, B'_1, \ldots, B'_4\}$. In other words all equivalences in $\preceq'$ are consequences of $A'_i \sim' A'_j$, $A'_i \sim' B'_j$, $B'_i \sim' B'_j$, where $i, j \in [4]$.

Then we will use Theorem 5.3.1 to untie the eight sets and to construct a comparative probability order $\sqsubseteq$ for which
$$A'_1 \sqsubset A'_2 \sqsubset A'_3 \sqsubset A'_4 \sqsubset B'_1 \sqsubset B'_2 \sqsubset B'_3 \sqsubset B'_4,$$
where $X \sqsubset Y$ means that $X \sqsubseteq Y$ is true but not $Y \sqsubseteq X$.

This will give us an initial segment $\Delta(\sqsubseteq, B'_1)$ of the linear qualitative probability order $\sqsubseteq$, which is not threshold since CC_4^* fails to hold.

Theorem 5.4.1. There exists a linear qualitative probability order $\sqsubseteq$ on [26] and $T \subset [26]$ such that the initial segment $\Delta(\sqsubseteq, T)$ is not a threshold complex.

Chapter 2

Preliminaries

2.1 Hypergraphs and Simple Games

Only a few structures in mathematics appear in different contexts and are exploited by many other areas. One of such structures is hypergraphs.

Definition 2.1.1. *A* **hypergraph** *G is a pair (P, W), where P is a finite set and W is a collection of subsets of P.*

The elements of W are called **edges**. Hypergraphs are a very well studied and widely applicable notion. A hypergraph in which every edge consists of exactly two elements is a **graph**. We will consider hypergraphs from a voting-theoretic point of view.

Definition 2.1.2. *A* **simple game** *is a hypergraph $G = (P, W)$ which satisfies the* **monotonicity condition**[1]*: if $X \in W$ and $X \subset Y \subseteq P$, then $Y \in W$. We also require that W is different from $\emptyset$ and P (non-triviality assumption).*

The term "simple" was introduced by von Neumann and Morgenstern (1944) to distinguish the following class of multiperson games: a coalition is either all-powerful or completely ineffectual in such games. Simple games can also be viewed as models of voting systems in which a single alternative, such as a bill or an amendment, is pitted against status quo. Let us consider a finite set P consisting of n elements which we will call **players**. For convenience, the set P can be taken to be $[n] = \{1, 2, \ldots, n\}$. Any set of players is called a **coalition**, and the whole P is

[1] According to Isbell (1958), hypergraphs which satisfy monotonicity condition were introdused by D.B. Gilles in his 1953 Princeton thesis.

usually addressed as the **grand coalition**. Elements of the set W are called **winning coalitions**. We also define the set $L = 2^P - W$ and call elements of this set **losing coalitions**. A winning coalition is said to be **minimal** if every proper subset is a losing coalition. Due to monotonicity, every simple game is fully determined by the set of its minimal winning coalitions. A losing coalition is said to be **maximal** if every proper superset is a winning coalition. A simple game is fully determined by set of maximal losing coalitions.

Example 2.1.1. *The UN Security Council consists of five permanent (China, France, the United Kingdom, Russia, and the United States) and 10 non-permanent countries (on a rotating basis, e.g., in November 1990: Canada, Colombia, Cuba, Ethiopia, Finland, the Ivory Coast, Malaysia, Romania, Yemen, and Zaire). A passage requires an approval of at least nine countries, subject to a veto by any of the permanent members*[2]*. We can represent this situation as a simple game. Let* $\{1, \ldots, 5\}$ *and* $\{6, \ldots, 15\}$ *be five permanent and 10 non-permanent countries respectively. Then the UN Security Council is the following simple game* $G = ([15], W)$*, where* $W = \{X \subseteq 2^{[P]} | X \supset \{1, \ldots, 5\}$ *and* $|X| \geq 9\}$ *is the set of coalitions which can approve a passage.*

Simple games can be dated back as early as to the Dedekind's 1897 work, in which he found the number of simple games with four or fewer players. Since then they have also appeared in a variety of mathematical and computer science contexts under various names, e.g., monotone boolean (Korshunov, 2003) or switching functions, threshold functions (Muroga, 1971), hypergraphs, coherent structures, Spencer systems, and clutters.

Simple games have been widely applied as well. For instance, von Neumann performed early work in the area of reliability theory. Ramamurthy (1990) wrote detailed mathematical and historical connection between the analysis of voting systems and the reliability theory. Simple games found an unexpected application in modeling of neurons in the living organism (see McCulloch & Pitts, 1943), and logical computing devices (see (Muroga, 1971) for historical details).

Simple games play the central role in this thesis. We will see "disguised" simple games in the area of qualitative probability orders.

[2]We consider the simplified model of the UN Security Council in which an abstention is not allowed.

2.2 Constant Sum Games

One can intuitively think that if some group of people form a winning coalition then all remaining people lose. At first glance it also seems as winning coalitions are large coalitions and losing are the small ones. An attentive reader would notice that these "intuitively correct" properties do not follow from the definition of a simple game. Let us formalize that idea, where we denote the complement $P - X$ of X by X^c:

Definition 2.2.1. *A simple game is called* **proper** *if $X \in W$ implies $X^c \in L$, and* **strong** *if $X \in L$ implies $X^c \in W$. A simple game which is proper and strong is called a* **constant sum** *game.*

Thus, a game is *not proper* if and only if the ground set P can be divided into two disjoint winning coalitions. A game is *not strong* if and only if the ground set P is the union of two disjoint losing coalitions. Note, in a constant sum game with n players there are exactly 2^{n-1} winning coalitions and exactly the same number of losing ones.

One can think that not proper games are not valuable enough to consider. From the voting point of view not proper games can lead to a contradictory decision. Shapley (1962, p. 60) goes as far as to say that not proper games "play a role in the theory somewhat analogous to that of 'imaginary' numbers in algebra and analysis". This position certainly has some merit in it, but it excludes some real life situations in which an issue requires an approval of a group of players. A good example of such situations is a *grant of certiorari*.

Example 2.2.1. *For the case to be considered by the U.S. Supreme Court one can get an approval of at least four of the nine Justices. Such approval is called a grant of certiorari.*

Clearly, the grant of certiorari game is not proper. Some authors do not consider not strong games, as in such games the issue could be unsolved. Ramamurthy (1990) says there can be "a paralysis that may result from allowing a losing coalition to obstruct a decision".

2.3 Weighted Majority Games

One of the key classes of simple games are weighted simple games.

Definition 2.3.1. *A simple game G is called a* **weighted majority game** *if there exist non-negative reals $w_1, \ldots, w_n$, and a positive real number q, called quota, such that $X \in W$ iff $\sum_{i \in X} w_i \geq q$. Such game is denoted $[q; w_1, \ldots, w_n]$. We also call $[q; w_1, \ldots, w_n]$ a* **weighted majority representation** *for G.*

For simplicity, we denote by $w(X)$ the weight $\sum_{i \in X} w_i$ of a coalition X. Note that the weight of a coalition X is equal to the dot product $\mathbf{w} \cdot \chi(X)$ of the weight vector $\mathbf{w}$ with the characteristic vector $\chi(X)$ of X. The **characteristic vector** $\chi(X)$ of X is a vector of $\mathbb{R}^n$ which has ith coordinate equal to 0 if $i \notin X$ and 1 if $i \in X$.

Example 2.3.1 (continuation of Example 2.2.1)**.** *The grant of certiorari game can be represented as* $[4; 1, 1, 1, 1, 1, 1, 1, 1, 1]$.

The grant of certiorari game is an example of the simplest weighted majority games, which are often called *symmetric games* or *k-out-of-n games*.

Definition 2.3.2. *Suppose that $G = ([n], W)$ is a simple game. Then G is a* **symmetric game** *or a* **k-out-of-n game** *if $W = \{X \subseteq 2^{[P]} \mid |X| \geq k\}$.*

In such game every player can be assigned the weight one and the quota set at k. The detailed study of symmetric games can be traced at least to Bott (1953). Another interesting example is the European Economic Community.

Example 2.3.2. *By the Treaty of Rome in 1958 the European Economic Community consisted of six countries: France, Germany, Italy, Belgium, the Netherlands, and Luxembourg. Since the countries have different population votes were distributed unequally: France, Germany and Italy got four votes each, Belgium and the Netherlands – two votes each, and Luxembourg – just one. A coalition needed at least twelve of seventeen possible votes to win.*

Some games are weighted majority games even if they appear otherwise. The best-known example is the UN Security Council.

Example 2.3.3 (continuation of Example 2.1.1)**.** *We use the same notation as in Example 2.1.1. The UN Security Council game is a weighted simple game with a weighted majority representation*

$$[39; 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1].$$

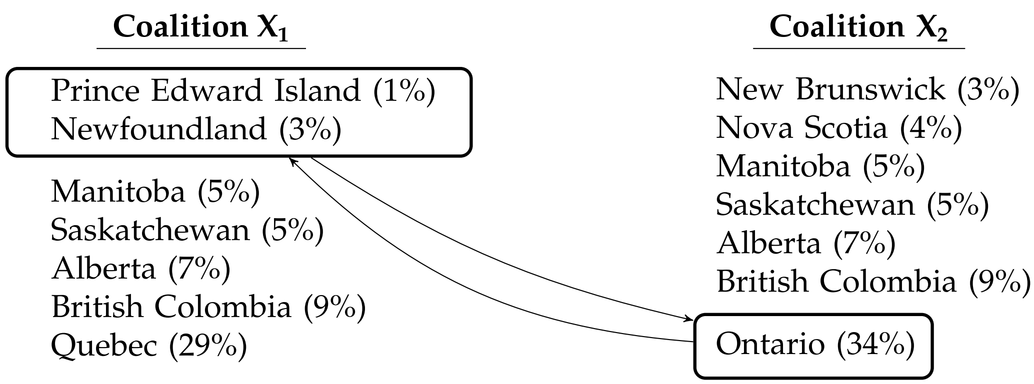

Figure 2.1: Before the trade we have two winning coalitions since there are seven provinces with a total population exceeding 50% of population in each coalition. After the trade both coalitions become losing as one of them lacks population and the other has only six provinces.

Other interesting examples of simple games can be found in (Taylor & Zwicker, 1999; Freixas & Molinero, 2009b).

One can see that choice of weights is not unique and there is a lot of "freedom". For example, two representations $[3; 1, 1, 2]$ and $[4; 1, 2, 3]$ describe the same game.

A weight of a player represents the amount of power or influence[3] of these players. Suppose G is a weighted simple game. Then we can line up all the players in a non-decreasing order, such that every next player is at least as powerful as the previous one. We examine whether the following is true: if a weight carries an influence then influence is represented by weights? Consider the following example:

Example 2.3.4. *(Taylor & Zwicker, 1999, Example 1.2.3)*

Since 1982, an amendment to the Canadian constitution can become law only if it is approved by at least seven of the ten Canadian provinces, subject to the proviso that the approving provinces have, among them, at least half of Canada's population. This voting system was first studied in (Kilgour, 1983)

Clearly, a province with larger population has more influence and is thus more desirable for a coalition's partner. Yet this game is not weighted. Let us assume,

[3] An influence can be measured in many different ways. In Chapter 2.7 we will see the individual desirability relation which also represents influence among players. Alternatively we may define a power index (e.g., Banzhaf or Shapley) for a player and decide an amount of influence according to the index. Information about power indices can be found, e.g., in (Felsenthal & Machover, 1998).

for a moment, that there are non-negative weights and a positive quota q which represent this game. Consider two winning coalitions X_1 and X_2 from Figure 2.1. Imagine that for some reason coalition X_1 decides to "trade" Prince Edward Island and Newfoundland for Ontario. After the "trade" two new coalitions are formed:

$$Y_1 = (X_1 - \{\text{Prince Edward Island, Newfoundland}\}) \cup \{\text{Ontario}\},$$
$$Y_2 = (X_2 - \{\text{Ontario}\}) \cup \{\text{Prince Edward Island, Newfoundland}\}.$$

Both, Y_1 and Y_2, are losing, since Y_1 has only six Canadian provinces and Y_2 has less then 50% of population. Note that $w(X_1) + w(X_2) = w(Y_1) + w(Y_2)$ as the "trade" is a "local" event and does not affect the combined weight of all coalitions participating in it. At the same time X_1, X_2 are winning and Y_1, Y_2 are losing coalitions. Hence if this game were weighted we would have

$$2q \leq w(X_1) + w(X_2) = w(Y_1) + w(Y_2) < 2q,$$

which is a contradiction.

Therefore, even if we can line up players according to their influence it does not mean that a game is weighted. Consequently not all games are weighted.

As von Neumann and Morgenstern noted in their classical book (von Neumann & Morgenstern, 1944, Section 5.3) that already for six players not all games have weighted majority representation and for seven players some games do not have weighted majority representation in a much stronger sense.

Proposition 2.3.1. *(von Neumann & Morgenstern, 1944)*

(a) Each game with 3 or fewer players is weighted.

(b) Each strong or proper game with 4 or fewer players is weighted.

(c) Each constant sum game with 5 or fewer players is weighted.

For six players there are constant sum games that are not weighted (von Neumann & Morgenstern, 1944).

Example 2.3.5. *Let $n = 6$. Let us include in W all sets of cardinality four or greater, 22 sets in total. We want to construct a proper game, therefore we have to choose and include in W at most one set out of each of the 10 pairs (X, X^c), where X is a subset of cardinality three. Suppose we included sets $X_1 = \{1, 2, 4\}$, $X_2 = \{1, 3, 6\}$, $X_3 = \{2, 3, 5\}$, $X_4 = \{1, 4, 5\}$, $X_5 = \{2, 5, 6\}$, $X_6 = \{3, 4, 6\}$ in W (and four other 3-element sets to insure that the game is* *constant sum). If this game had a weighted majority representation* $[q; w_1, \ldots, w_6]$*, then the following system of inequalities (corresponding to the fact that a winning coalition has greater weight than a losing coalition) must have a solution:*

$$\begin{aligned}
w_1 + w_2 + w_4 &> w_3 + w_5 + w_6, \\
w_2 + w_3 + w_5 &> w_1 + w_4 + w_6, \\
w_3 + w_4 + w_6 &> w_1 + w_2 + w_5, \\
w_1 + w_4 + w_5 &> w_2 + w_3 + w_6, \\
w_2 + w_5 + w_6 &> w_1 + w_3 + w_4, \\
w_1 + w_3 + w_6 &> w_2 + w_4 + w_5.
\end{aligned}$$

Nevertheless, this system is inconsistent.

For our purposes the following definition is useful:

Definition 2.3.3. *(Taylor & Zwicker, 1999, p. 6) We say that a player p in a game is a* **dictator** *if p belongs to every winning coalition and to no losing coalition. If all coalitions containing player p are winning, this player is called a* **passer**. *A player p is called a* **vetoer** *if p is contained in the intersection of all winning coalitions. A player who does not belong to any minimal winning coalition is called a* **dummy**. *Such a player can be removed from any winning coalition without making it losing.*

Example 2.3.6. *Consider a game represented by* $[1; 1, 0, 0]$. *The first player is clearly a dictator. The first player in the game* $[5; 5, 3, 2, 2]$ *is a passer since there are winning coalition without the first player, e.g.,* $\{2, 3\}$ *is a winning coalition. Every permanent country from Example 2.1.1 is a vetoer. Finally, we have a very interesting historical fact that Luxembourg from Example 2.3.2 is a dummy.*

2.4 Trading Transform

What is a trade? Lets try to understand the notion first and after that we will give a formal definition. In the world of sports, a trade is very common. For example, soccer team number one has two goalkeepers and no sweepers. At the same time team number two doesn't have a goalkeeper, but has a spare sweeper. Clearly team one can exchange a goalkeeper for a sweeper. In that case both teams would have all needed players. Of course it doesn't mean that the trade was a great deal for in sports there is one very important factor - teamwork.

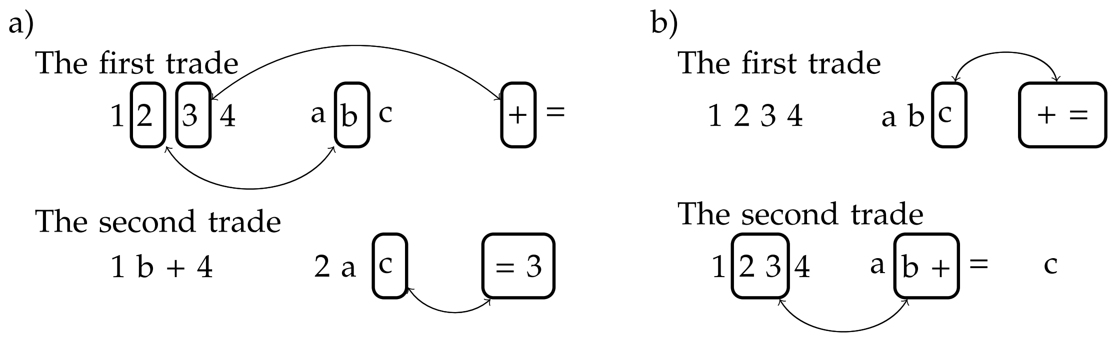

Figure 2.2: The trades a) and the trades b) lead to the same final coalitions

Notice it is possible to have several different trades which give the same final result.

Example 2.4.1. *Let* $X_1 = \{1, 2, 3, 4\}$, $X_2 = \{a, b, c\}$ *and* $X_3 = \{+, =\}$ *be three coalitions. On Figure 2.2 trades on a) and b) are different. However, after two a-trades we have the same resulting coalitions* $Y_1 = \{1, b, +, 4\}$, $Y_2 = \{2, a, =, 3\}$, $Y_3 = \{c\}$ *as after two b-trades.*

In the context of this dissertation we do not take into account which trades we need to obtain the final coalitions. Such situations require us to consider trading matrices. In such a matrix we can follow all trades from the beginning till the final coalition. More information on trading matrixes can be found in, for instance, (Taylor & Zwicker, 1999, Section 2.2).

Definition 2.4.1. *A sequence of coalitions*

$$\mathcal{T} = (X_1, \ldots, X_j; Y_1, \ldots, Y_j) \tag{2.1}$$

is a **trading transform** *if the coalitions* $X_1, \ldots, X_j$ *can be converted into the coalitions* $Y_1, \ldots, Y_j$ *by rearranging players or, equivalently, by several trades. It can also be expressed as*

$$|\{i : a \in X_i\}| = |\{i : a \in Y_i\}| \qquad \text{for all } a \in P.$$

We say that the trading transform $\mathcal{T}$ *has* **length j**.

Note that while in (2.1) we can consider that no X_i coincides with any of Y_k, it is perfectly possible that the sequence $X_1, \ldots, X_j$ has some terms equal, the sequence $Y_1, \ldots, Y_j$ can also contain equal subsets. The order of subsets in these sequences is not important.

2.5 Necessary and Sufficient Conditions for Weightedness

What makes a game a weighted majority game? We have already seen in Example 2.3.4 that if two winning coalitions can be converted into two losing coalitions by trading players then a game is not weighted. If this were a necessary and sufficient condition for a game to be weighted, perhaps our work is done. However, this works only in very special cases and fails in general.

Example 2.5.1 (The Gabelman Games)**.** *Gabelman (1961) constructed "almost weighted" games for small numbers of players. Taylor and Zwicker (1999) rediscovered the same idea. The main trick is using magic squares to construct a game. The simplest example is a* 3×3 *magic square M:*

$$\begin{matrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{matrix}$$

Let $[9]$ *be a set of players and every player i has weight i. The matrix M is a magic square, which means that there exists a constant* $p = 15$ *such that the sum of every row is p and the sum of every column is p. In this example we additionally have the sum of every diagonal equals to* 15*. Note that no other set with* 3 *players except rows, columns, and diagonals is summed to* 15*. Let us define the set of winning coalitions in the following way:*

- *all sets consisting of* 4 *or more players are winning;*
- *all three-player coalitions with weight exceeding* 15 *are winning;*
- *all three-player coalitions with weight exactly* 15*, which are rows of M, are winning.*

This game is not weighted, but is "almost weighted". More explicitly, if the game were weighted then the total weight of three winning coalitions which are rows of M would be strictly greater then the total weight of three losing coalitions which are columns of M. At the same time these weights have to be equal since union of all rows is exactly the union of all columns of M.

Taylor and Zwicker (1999) proved that for every $n \geq 3$ there exists a rigid magic square[4] *$n \times n$, which we can use to construct a non-weighted game in the same fashion as above. For the more detailed explanation see (Taylor & Zwicker, 1999, Section 2.7).*

From this example we can see that if a number of winning coalitions can be converted into the same number of losing coalitions by rearranging players, then this game is not weighted.

Definition 2.5.1. *A simple game G is called* **k-trade robust** *if no trading transform $\mathcal{T} = (X_1, \ldots, X_j; Y_1, \ldots, Y_j)$ with $j \leq k$ has all coalitions $X_1, \ldots, X_j$ winning and all $Y_1, \ldots, Y_j$ losing. G is* **trade robust** *if it is k-trade robust for every k.*

Clearly, if a game is not trade robust then it is not weighted. Is it true, however, that trade robustness implies weightedness? Does there exist a positive integer k such that k-trade robustness implies trade robustness? Winder (1962) showed that in general no such k exists. Nevertheless, if we restrict ourselves to games with n players, then the situation changes and such a number k exists for each n. Of course this number k will depend on n. This is an important result contained in the following theorem. (We note though that the equivalence of the first and the second was earlier proved by Elgot (1960).)

Theorem 2.5.1. *(Taylor & Zwicker, 1992) The following three conditions are equivalent:*

- *G is a weighted majority game,*
- *G is trade robust,*
- *G is $2^{2^{|P|}}$-trade robust.*

It is hard to trace back who was the first to characterize weightedness. Taylor and Zwicker (1999, p. 68) wrote: "The problem of determining necessary and sufficient conditions for a switching function was known as the synthesis problem in the early days of threshold logic. The question of the extent to which Elgot's result - the earliest of the characterization theorems - is distinct from its algebraic precursors is a difficult one, for which the answer is, at least in part, subjective, Muroga (1971, p. 192) referred to some of these early results as 'forms that are actually restatements of classical theorems known in the theory of linear inequalities.'

[4] A rigid magic square is a $n \times n$ matrix M of integers for which there is a constant p such that the sum of every row and the sum of every column is p, additionally if each set S of entries that sums to p appears as either a row or a column.

The origin of the geometrical argument[5] is less clear. Even von Neumann and Morgenstern (1944, p. 139) speak of separating hyperplanes, but not in the context of simple games. A more recent use of the geometrical approach is Einy and Lehrer (1989)."

It is worthwhile to note that the proof of Theorem 2.5.1 makes no use of monotonicity or the fact that we have only two types of coalitions: either winning or losing. Taylor and Zwicker (1999, Theorem 2.4.2) generalized this characterization to pregames[6].

2.6 At Least Half Property

In the previous section we gave the characterization of weightedness in terms of trade robustness. There exists a related combinatorial characterization in (Taylor & Zwicker, 1999, Section 2.5) which is adapted from the work of Einy and Lehrer (1989).

Definition 2.6.1. *A coalition X is* **blocking** *if its complement X^c is a losing coalition.*

Definition 2.6.2. *(Taylor & Zwicker, 1999, p. 61) A sequence of coalitions $\langle Z_1, \ldots, Z_{2k} \rangle$ is called an* **EL sequence** *if half of its coalitions are winning and half are blocking. A simple game satisfies the* **greater-than-half property** *if every EL sequence has a player occurring in more then a half of the coalitions in the sequence.*

Keeping in mind these definitions, one can show the following theorem:

Theorem 2.6.1. *(Taylor & Zwicker, 1999, Theorem 2.4.6) For a simple game G the following are equivalent:*

- *G is weighted.*
- *G satisfies greater-than-half property.*
- *G is trade robust.*

[5] We visualize the characteristic vector of each coalition as a point in n-dimensional Euclidian space $\mathbb{R}^n$. In this case a simple game is represented by two sets of points in the hypercube, where one corresponds to the winning coalitions while other to the losing coalitions. A simple game is weighted iff W and L can be separated by a hyperplane.

[6] A **pregame** is a tuple (P, W, L), where P is a set of players, $L \subseteq 2^P$ is a set of losing coalitions, and $W \subseteq 2^P$ is a set of winning coalition, which satisfies the monotonicity condition: if $X \in W$ and $Y \supseteq X$ then $Y \notin L$

2.7 Desirability Relation and Complete Games

We have already seen in Section 2.3 that weights represent "power" or "influence" of players. At the same time, there are non-weighted games in which the players can be lined according to their "influence". A natural question arises: what is the influence and how can we measure it? Consider the following situation: two losing coalitions do a one-for-one trade, and after the trade one coalition becomes winning and another one remains losing. Intuitively, the player who turned a losing coalition into winning is more "powerful" than the other player. Isbell (1958) was the first who formalized this idea for simple games. Maschler and Peleg (1966) continued a further generalization of this notion; see also (Muroga, 1971, p. 113).

Definition 2.7.1. *Suppose $G = (P, W)$ is a simple game. Then the* **individual desirability relation** *(for G) is the binary relation $\leq_I$ on P defined by*

$$p \leq_I q \text{ iff } \forall X \subseteq P - \{p, q\}, \text{ if } X \cup \{p\} \in W, \text{ then } X \cup \{q\} \in W.$$

This relation gives rise to the following three relations on P:

- $p \prec_I q$ iff not $q \leq_I p$;
- $p <_I q$ iff $p \prec_I q$ and $p \leq_I q$ both hold;
- $p \equiv_I q$ iff $p \leq_I q$ and $q \leq_I p$ both hold.

The relation $\prec_I$ is usually called the **existential strict ordering**.

Definition 2.7.2. *A simple game $G = (P, W)$ is* **complete** *or* **linear** *if the individual desirability relation $\leq_I$ for G is a complete preorder.*

This notion is well established, and it is difficult to trace back who was the first to recognize the importance of this class. The class was considered, among others, by Winder (1962), Elgot (1960), Muroga, Toda, and Takasu (1961), Muroga (1971), Hammer, Ibaraki, and Peled (1981), Einy and Lehrer (1989), Taylor and Zwicker (1992), Carreras and Freixas (1996), Taylor and Zwicker (1999), and Freixas and Molinero (2009b).

Proposition 2.7.1. *(Taylor & Zwicker, 1999, p. 90) Suppose that G is a simple game. Then the following are equivalent:*

- *G is a complete game.*

- $\prec_I = <_I$ (as relations).

- $<_I$ is transitive.

- $<_I$ is acyclic.

- $\equiv_I$ is an equivalence relation.

For simplicity it is convenient to assume that every player is at least as desirable as the previous one (or the other way around), i.e., $i \geq_I j$ iff $i \geq j$. As we already know every simple game is completely defined by the set of minimal winning coalitions. In the case of complete games we need even less information: a game is completely defined by the set of shift-minimal winning coalitions.

Definition 2.7.3. *In a complete game $G = (P, W)$ a minimal winning coalition X is* **shift-minimal** *if the coalition $(X - \{i\}) \cup \{j\}$ is losing for any $i \in X$ and $j \notin X$ such that $j <_I i$.*

Intuitively, a minimal winning coalition is shift-minimal if an exchange of any player of the coalition for a less desirable one makes that coalition losing. In the analogous way we can define **shift-maximal** losing coalitions:

Definition 2.7.4. *In a complete game $G = (P, W)$ a maximal losing coalition X is* **shift-maximal** *if the coalition $(X - \{i\}) \cup \{j\}$ is winning for any $i \in W$ and $j \notin W$ such that $i <_I j$.*

A maximal losing coalition is shift-maximal if a trade of any player in the coalition for a more desirable one makes it winning. To see the difference between minimal winning coalitions and shift minimal winning coalitions consider the following example:

Example 2.7.1. *Let G be a complete game on 5 players defined in the following way: a coalition is winning iff it contains at least one player out of the first two and at least 3 players in total. In this game we have two equivalence classes $\{1, 2\}$ and $\{3, 4, 5\}$ or, more explicitly, $5 \equiv_I 4 \equiv_I 3 <_I 2 \equiv_I 1$. The set of minimal winning coalitions is $W^{min} = \{\{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}\}$. At the same time the set of shift-minimal winning coalitions is $W^{min} = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}\}$. For instance, $\{1, 2, 3\}$ is not a shift-minimal coalition since $4 <_I 2$ and $(\{1, 2, 3\} - \{2\}) \cup \{4\}$ is winning.*

The information about a complete game can be compressed even more. Carreras and Freixas (1996) illustrate one such technique.

Carreras and Freixas (1996) gave a complete structural characterization of complete games in terms of lattices.

2.8 Weightedness and Complete Games

Every weighted majority game is complete. However, the class of complete games is much broader then the class of weighted games. In Example 2.3.4 we observed a complete non-weighted game. Hence, the class of complete games is a natural generalization of the class of weighted games and is much broader. Freixas and Molinero (2006, 2009b) did a computational analysis of complete games and characterized all complete games with less than eight players. They also performed partial computations for complete games with less than eleven players.

There is a certain interest in weightedness of complete games in the case of integer representations. Minimal integer representations and minimal sum representations were studied, for example, in (Muroga, Tsuboi, & Baugh, 1970; Freixas, Molinero, & Roura, 2007; Freixas & Puente, 2008; Freixas & Molinero, 2009a, 2010).

2.9 Dual Games

The dual objects are very common in all areas of mathematics. The theory of simple games no exeption.

Definition 2.9.1. *The* **dual game** *of a game $G = (P, W)$ is defined to be $G^* = (P, L^c)$. This is to say that in the game G^* dual to a game G the winning coalitions are exactly the complements of losing coalitions of G.*

For a more detailed explanation of duality we refer to (Taylor & Zwicker, 1999, p. 16). We need some standard properties of duality. Shapley (1962) proved:

Theorem 2.9.1. *(Shapley, 1962, p. 62) For any simple game G the following hold:*

(a) $G = G^{**}$.

(b) G^* *is proper if and only if* G *is strong.*

(c) G^* *is strong if and only if* G *is proper.*

The operation of taking the dual is known to preserve weightedness:

Proposition 2.9.1. *(Taylor & Zwicker, 1999, Propositions 4.3.10 and 4.10.1)*

(i) The simple game G is weighted iff the dual game G^* is weighted.

(ii) For every integer $k \geq 2$, the simple game G is k-trade robust iff G^* is k-trade robust.

Proposition 2.9.2. *(Taylor & Zwicker, 1999, Proposition 3.2.8) The individual desirability relation $\leq_I$ is dual symmetric in the sense that $p \leq_I q$ holds in G iff it holds in G^*.*

Duality it is a useful tool, and as Shapley (1962) wrote "the usefulness of duality concept depends on the inclusion of improper games in our theory."

2.10 Substructures

There are two natural substructures that arise from the more general notion in threshold logic (see Muroga, 1971, p. 112) which can be traced back to at least (Isbell, 1958):

Definition 2.10.1. *Let $G = (P, W)$ be a simple game. Then $G' = (P', W')$ is a* **Boolean subgame** *of G iff there exist two disjoint subsets A and B of P such that $P' = P - (A \cup B)$, and for $X \subseteq P'$,*

$$X \in W' \text{ iff } X \cup B \in W.$$

The game G' is called a Boolean subgame **determined** *by A and B.*

Definition 2.10.2. *A simple game G satisfies a property* **hereditarily** *if every Boolean subgame of G satisfies the property as well.*

It is not hard to prove that weightedness is a hereditary property. In this study we are interested in some special cases of Boolean subgames.

Definition 2.10.3. *Consider a simple game $G = (P, W)$, and $A \subseteq P$, and $B = \emptyset$. Then the Boolean subgame determined by A and B is called the* **subgame determined by** *P', where $P' = P - A$.*

Hence, the subgame determined by P' is the simple game (P', W_{sg}), where

$$X \in W_{sg} \text{ iff } X \subseteq P' \text{ and } X \in W.$$

The subgame determined by P' is usually denoted by G_A.

Definition 2.10.4. *Let $G = (P, W)$ be a simple game and $B \subseteq P$, $A = \emptyset$. Then the Boolean subgame determined by A and B is called* **reduced game determined by** *P', where $P' = P - B$.*

Thus, the reduced game determined by P' is the simple game (P', W_{rg}), where

$$X \in W_{rg} \text{ iff } X \subseteq P' \text{ and } X \cup B \in W.$$

The reduced game determined by P' is usually denoted by G^B.

Consider the following situation: a group of people P vote in favor of or against a law. A coalition wins if it can pass the law. Intuitively, in this situation the subgame G_A results in assuming that all people of A have already voted against the law. The reduced game G^B results in assuming that people in B have already voted in favor of the law. Hence a Boolean subgame reflects the situation in which some votes are already known.

Proposition 2.10.1. *(Taylor & Zwicker, 1999, Proposition 1.4.8) Assume that $G = (P, W)$ is a simple game and that $B \subseteq P$. Then $(G^B)^* = (G^*)_B$, and so $G_B = ((G^*)^B)^*$ and $(G_B)^* = (G^*)^B$.*

It is straightforward to show that every subgame and every reduced game of a weighted majority game is also a weighted majority game. For example, the case of subgame one only has to retain the same weights for elements of A^c as in G and the same threshold.

More on substructures can be found, for example, in (Taylor & Zwicker, 1999).

Chapter 3

Weightedness and Rough Weightedness

3.1 Definitions and Examples

As von Neumann and Morgenstern showed (von Neumann & Morgenstern, 1944), there is a weighted majority representation for: every simple game with less than four players, every proper or strong simple game with less than five players, and every constant sum game with less than six players. For six players there are constant sum games with six players that are not weighted (von Neumann & Morgenstern, 1944).

Example 3.1.1 (Continuation of Example 2.3.5)**.** *Six players participate in a game. The set W of winning coalitions consists of all sets of cardinality four or greater and six 3-element sets $X_1 = \{1, 2, 4\}$, $X_2 = \{1, 3, 6\}$, $X_3 = \{2, 3, 5\}$, $X_4 = \{1, 4, 5\}$, $X_5 = \{2, 5, 6\}$, $X_6 = \{3, 4, 6\}$. This game is constant sum. Assume it has is a weighted majority representation $[q; w_1, \ldots, w_6]$. Nevertheless the following system of inequalities*

$$\sum_{i \in X_j} w_i > \sum_{i \in X_j^c} w_i, \qquad j = 1, \ldots, 6. \tag{3.1}$$

is inconsistent. Thence, this game is not weighted. However if we convert all six inequalities (3.1) into equalities, then there will be a 1-dimensional solution space spanned by $(1, 1, 1, 1, 1, 1)$ *which shows that this game "almost" has a weighted majority representation* $[3; 1, 1, 1, 1, 1, 1]$*. Indeed, if we assign weight 1 to every player, then all coalition whose weight falls below the threshold three are in L, all coalitions whose total weight exceeds this*

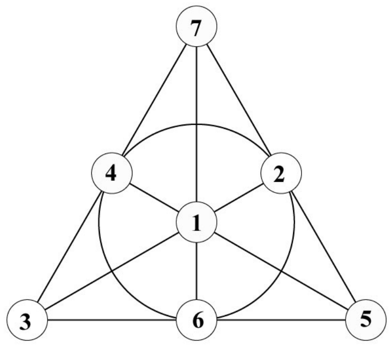

Figure 3.1: The smallest projective plane of order two.

threshold are in W. However, if a coalition has total weight of three, i.e. it is equal to the threshold, it can be either winning or losing.

Definition 3.1.1. *[(Taylor & Zwicker, 1999), p. 78] A simple game G is called* **roughly weighted** *if there exist non-negative real numbers $w_1, \ldots, w_n$ and a real number q, called the* **quota**, *not all equal to zero, such that for $X \in 2^P$ the condition $\sum_{i \in X} w_i < q$ implies $X \in L$, and $\sum_{i \in X} w_i > q$ implies $X \in W$. We say that $[q; w_1, \ldots, w_n]$ is a* **roughly weighted representation** *for G.*

The simple game in Example 3.1.1 is roughly weighted with a roughly weighted representation $[3; 1, 1, 1, 1, 1, 1]$. We will show later (Theorem 3.6.3) that any constant sum game with six players has a roughly weighted representation. In threshold logic roughly weighted games correspond to pseudo-threshold functions (see Muroga, 1971, p. 208). We end this section with one example of a seven players game that cannot be roughly weighted.

Example 3.1.2 (The Fano plane game (von Neumann & Morgenstern, 1944))**.** *Let us denote by $P = [7]$ the set of points of the projective plane of order two, called the Fano plane (see Figure 3.1). Let P be the set of players of the new game. Let us also take the seven lines of this projective plane as minimal winning coalitions:*

$$\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{3, 5, 6\}, \{3, 4, 7\}, \{2, 5, 7\}, \{2, 4, 6\}. \tag{3.2}$$

We will denote them by $X_1, \ldots, X_7$, respectively. This, as it is easy to check, defines a constant sum game, which we will denote Fano. If it had a roughly weighted representation

$[q; w_1, \ldots, w_7]$, *then the following system of inequalities will be consistent:*

$$\sum_{i \in X_j} w_i \geq \sum_{i \in X_j^c} w_i, \qquad j = 1, \ldots, 7.$$

Nevertheless, adding all the equations up we get $\sum_{i=1}^{7} w_i \leq 0$ *which shows that this system does not have solutions with non-negative coordinates other than the zero solution. Since all weights are equal to zero, by the definition, the threshold must be non-zero, coalitions (3.2) cannot be winning. Hence this simple game is not roughly weighted.*

3.2 Games and Ideals

We would like to redefine trading transforms algebraically. Let $T = \{-1, 0, 1\}$ and $T^n = T \times T \times \ldots T$ (n times). With any pair (X, Y) of subsets $X, Y \in [n]$ we define

$$\mathbf{v}_{X,Y} = \chi(X) - \chi(Y) \in T^n,$$

where $\chi(X)$ and $\chi(Y)$ are the characteristic vectors of subsets X and Y, respectively.

Let now $G = (P, W)$ be a game. We will associate an algebraic object with G. For any pair (X, Y), where X is winning and Y is losing, we put in correspondence the following vector $\mathbf{v}_{X,Y}$. The set of all such vectors we will denote $I(G)$.

Definition 3.2.1. *Let* $\mathbf{e}_i = (0, \ldots, 1, \ldots, 0)$, *where the only nonzero element 1 is in the ith position. Then a subset* $I \subseteq T^n$ *will be called an* **ideal** *in* T^n *if for any* $i = 1, 2, \ldots, n$

$$(\mathbf{v} \in I \text{ and } \mathbf{v} + \mathbf{e}_i \in T^n) \Longrightarrow \mathbf{v} + \mathbf{e}_i \in I. \tag{3.3}$$

Proposition 3.2.1. *Let G be a game with n players. Then $I(G)$ is an ideal in T^n.*

Proof. The condition (3.3) follows directly from the monotonicity condition for games. Indeed, if $\mathbf{v}_{X,Y} + \mathbf{e}_i$ is in T^n, then this amounts to either addition of i to X, which was not there, or removal of i from Y. Both operations maintain X winning and Y losing. □

We note that G can be uniquely recovered from $I(G)$ only for games which are proper and strong. Note that in this case we have exactly 2^{n-1} vectors without zeros in I. In general it is easy to construct a counterexample. The key to this recovery is to consider all vectors from $I(G)$ without zeros. Indeed if G is a constant-sum game then for every winning coalition X the complement X^c is losing. It meant that the vector $\mathbf{v}_{X,X^c} \in I(G)$ and we can reconstruct the the set of winning coalitions.

Example 3.2.1. *One can check that simple games $G_1 = ([3], W_1)$ and $G_2 = ([3], W_2)$ have the same ideal*

$$I = \{(0,1,1),(1,1,0),(1,1,1),(-1,1,0),(0,1,-1),(0,1,0),\\ (1,1,-1),(-1,1,1),(0,0,1),(1,0,0),(1,0,1)\},$$

where $W_1 = \{2,12,13,23,123\}$ and $W_2 = \{23,12,123\}$.

One can see that if in addition to $I(G)$ we know that G is strong (or proper) then G can be uniquely recovered as well. Indeed, if G is proper and X is a winning coalition, then X^c is losing and $\mathbf{v}_{X,X^c} \in I(G)$. This vector does not contain zeros and X can be recovered from it uniquely. Thus for a proper game we can reconstruct the set of winning coalitions. Similarly, if G is strong we can reconstruct the set of losing coalitions.

Proposition 3.2.2. *Let G be a finite simple game. Then:*

(a) G is weighted iff the system

$$\mathbf{v}\cdot\mathbf{x} > 0, \qquad \mathbf{v} \in I(G) \tag{3.4}$$

has a solution.

(b) G is roughly weighted iff the system

$$\mathbf{v}\cdot\mathbf{x} \geq 0, \qquad \mathbf{v} \in I(G) \tag{3.5}$$

has a non-zero solution.

Proof. The proof of (a) is contained in (Taylor & Zwicker, 1999) (see Lemma 2.6.5 and comment on page 6 why all weights can be chosen non-negative).

Let us prove (b). Suppose G is roughly weighted, Let $\mathbf{v} = \mathbf{v}_{X,Y} \in I(G)$. Then $X \in W$, $Y \in L$ and $\mathbf{w} = (w_1, \ldots, w_n)$ satisfies

$$\sum_{t\in X} w_t \geq q \geq \sum_{s\in Y} w_s.$$

This implies $\sum_{t\in X} w_t - \sum_{s\in Y} w_s \geq 0$ or $\mathbf{v}\cdot\mathbf{w} \geq 0$ and then $\mathbf{w}$ is a non-zero solution of (3.5) (due to the non-triviality assumption).

On the other hand, any solution to the system of inequalities (3.4) gives us a vector of weights and a threshold. Let $\mathbf{w}$ be such a solution. Then for any two coalitions $X \in W$, $Y \in L$ we will have $\mathbf{v}_{X,Y} \cdot \mathbf{w} \geq 0$ or

$$\sum_{t \in X} w_t \geq \sum_{s \in Y} w_s.$$

Then the smallest sum $\sum_{t \in X} w_t$, where $X \in W$, will still be greater than or equal to the largest sum $\sum_{s \in Y} w_s$, where $Y \in L$. Hence the threshold q can be chosen between them so that

$$\sum_{t \in X} w_t \geq q \geq \sum_{s \in Y} w_s.$$

The only problem left is that $\mathbf{w}$ can have negative components and in the definition of a roughly weighted game all weights must be non-negative. However, due to the monotonicity, if the game G has any rough weights, then it has a non-negative system of rough weights too (with the same threshold). Indeed, if, say, weight w_1 of the first player is negative, then he cannot be pivotal in any winning coalition. Since his weight is negative, his removal from a winning coalition cannot make it losing. By the monotonicity, deleting him from a losing coalition does not make it winning. In this case the weight w_1 can be reset to 0 (or a very small positive weight). We can do this with every negative weight. □

3.3 Criteria for Weighted Majority Games

3.3.1 Trade-robustness and Function f

We remind that a sequence of coalitions

$$\mathcal{T} = (X_1, \ldots, X_j; Y_1, \ldots, Y_j) \tag{3.6}$$

is a trading transform if the coalitions $X_1, \ldots, X_j$ can be converted into the coalitions $Y_1, \ldots, Y_j$ by rearranging players. We will sometimes use the multiset notation and instead of (3.6) will write

$$\mathcal{T} = (X_1^{a_1}, \ldots, X_k^{a_k}; Y_1^{b_1}, \ldots, Y_m^{b_m}),$$

where now $X_1, \ldots, X_k$ and $Y_1, \ldots, Y_k$ are all distinct, $a_1, \ldots, a_k$ and $b_1, \ldots, b_m$ are sequences of positive integers such that $\sum_{i=1}^{k} a_i = \sum_{j=1}^{m} b_j$ and $Z_i^{c_i}$ denotes c_i copies of Z_i with $Z_i \in \{X_i, Y_i\}$, $c_i \in \{a_i, b_i\}$.

We also have the following obvious algebraic reformulation.

Proposition 3.3.1. *Let $X_1, \ldots, X_j$ and $Y_1, \ldots, Y_j$ be two sequences of subsets of $[n]$. Then (3.6) is a trading transform iff*

$$\mathbf{v}_{X_1,Y_1} + \ldots + \mathbf{v}_{X_j,Y_j} = \mathbf{0}.$$

Proposition 3.3.2. *A simple game G is k-trade robust if for no $\mathbf{v}_1, \ldots, \mathbf{v}_m \in I(G)$ and for no non-negative integers $a_1, \ldots, a_m$ such that $\sum_{i=1}^{m} a_i \leq k$, we have*

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_m\mathbf{v}_m = \mathbf{0}. \tag{3.7}$$

Proof. Suppose G is not k-trade robust and there exists a trading transform (3.6) with $X_i \in W$, $Y_i \in L$ for all i and $j \leq k$. Then by Proposition 3.3.1 we have

$$\mathbf{v}_{X_1,Y_1} + \ldots + \mathbf{v}_{X_j,Y_j} = \mathbf{0}$$

with $\mathbf{v}_i = \mathbf{v}_{X_i,Y_i} \in I(G)$ so (3.7) holds. On the other hand, if (3.7) is satisfied for $\sum_{I=1}^{m} a_i \leq k$, then $\mathbf{v}_i = \mathbf{v}_{X_i,Y_i}$ for some $X_i \in W$ and $Y_i \in L$ and the sequence

$$(X_1^{a_1}, \ldots, X_m^{a_m}; Y_1^{a_1}, \ldots, Y_m^{a_m}),$$

where $X_i^{a_i}$ and $Y_i^{a_i}$ mean a_i copies of X_i and Y_i, respectively, is a trading transform violating k-trade robustness. □

Taylor and Zwicker (1992) showed in Theorem 2.5.1 that 2^{2^n}-trade robustness implies weightedness of a simple game with n players. As this characterisation of weighted games implies, to show that the game G is not a weighted majority game, it is sufficient to present a trading transform (3.6), where all coalitions $X_1, \ldots, X_j$ are winning and all coalitions $Y_1, \ldots, Y_j$ are losing. We will call such a trading transform a **certificate of non-weightedness** of G. An interesting question immediately emerges: if we want to check weightedness of a game with n players what is the maximal length of certificates that we have to check?

Definition 3.3.1. *Let $G = (P, W)$ be a simple game with $|P| = n$. If G is not weighted we define $f(G)$ to be the smallest positive integer k such that G is not k-trade robust. If G is weighted we set $f(G) = \infty$. The larger the value $f(G)$ the closer is the game G to a weighted* *majority game. Let us also define*

$$f(n) = \max_G f(G),$$

where the maximum is taken over non weighted games with n players. We can also say that $f(n)$ is the smallest positive integer such that $f(n)$-trade robustness for an n-player game implies its weightedness.

Example 3.3.1. *In Example 3.1.2 the corresponding 7×7 matrix, composed of vectors $\mathbf{v}_{X_i,X_i^c} \in I(G)$, $i = 1, \ldots, 7$, will be:*

$$\begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}.$$

Its rows sum to the vector $(-1,-1,-1,-1,-1,-1,-1)$. If we also add the vector $\mathbf{v}_{P,\emptyset} = (1,1,1,1,1,1,1)$ we will get

$$\sum_{i=1}^{7} \mathbf{v}_{X_i,X_i^c} + \mathbf{v}_{P,\emptyset} = \mathbf{0}.$$

This means that the following eight winning coalitions $(X_1, \ldots, X_7, P)$, where P is the grand coalition, can be transformed into the following eight losing coalitions: $(X_1^c, \ldots, X_7^c, \emptyset)$ (note that $\emptyset = P^c$). The sequence

$$(X_1, \ldots, X_7, P; X_1^c, \ldots, X_7^c, \emptyset) \tag{3.8}$$

is a certificate of non-weightedness of G. This certificate is not however the shortest. Indeed, if we take two lines, say, $\{1,2,3\}$ and $\{3,4,5\}$ and swap 2 and 4, then $\{1,3,4\}$ and $\{2,3,5\}$ will not be lines, hence losing coalitions. Thus, Fano is not 2-trade robust and $f(\text{Fano}) = 2$.

Theorem 2.5.1 gives us an upper bound for $f(n)$. The following theorem gives a lower bound.

Theorem 3.3.1. *(Taylor & Zwicker, 1995) For each integer $m \geq 2$, there exists a game Gab_m, called the Gabelman's game, with $(m+1)^2$ players, that is m-trade robust but not $(m+1)$-trade robust.*

Summarising the results of Theorems 2.5.1 and 3.3.1 in terms of function f we may state

Corollary 3.3.1. *For any $n \geq 2$,*

$$\lfloor \sqrt{n} \rfloor \leq f(n) \leq 2^{2^n}.$$

As Taylor and Zwicker noted in (Taylor & Zwicker, 1999) for most non-weighted games the value of $f(G)$ is 2. The closer the game to a weighted majority game the longer is the certificate and it is harder to find it.

3.3.2 A New Upper Bound for f

In what follows we use the following notation. Let $\mathbf{x} \in \mathbb{R}^n$. Then we write $\mathbf{x} \gg \mathbf{0}$ iff $x_i > 0$ for all $i \in [n]$. We also write $\mathbf{x} > \mathbf{0}$ iff $x_i \geq 0$ for all $i \in [n]$ with this inequality being strict for at least one i, and $\mathbf{x} \geq \mathbf{0}$ iff $x_i \geq 0$ for all $i \in [n]$. In this section we will need the following result which may be considered as a folklore.

Theorem 3.3.2. *Let A be an $m \times n$ matrix with rational coefficients with rows $\mathbf{a}_i \in \mathbb{Q}^n$, $i = 1, \ldots, m$. Then the system of linear inequalities $A\mathbf{x} \gg \mathbf{0}$, $\mathbf{x} \in \mathbb{R}^n$, has no solution iff there exist non-negative integers $r_1, \ldots, r_m$, of which at least one is positive, such that*

$$r_1\mathbf{a}_1 + r_2\mathbf{a}_2 + \cdots + r_m\mathbf{a}_m = \mathbf{0}.$$

A proof can be found in (Taylor & Zwicker, 1999, Theorem 2.6.4, p. 71) or in (Muroga, 1971, Lemma 7.2.1, p. 192).

Theorem 3.3.3. *The following statements for a simple game G with n players are equivalent:*

(a) G is weighted,

(b) G is N-trade robust for $N = (n+1)2^{\frac{1}{2}n\log_2 n}$.

Proof. We only need to prove that (b) implies (a). Suppose G is not weighted. Then by Proposition 3.2.2 the system of inequalities

$$\mathbf{v} \cdot \mathbf{x} > 0, \qquad \mathbf{v} \in I(G),$$

is inconsistent. By Theorem 3.3.2 there exist vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m \in I(G)$ and non-negative integers $r_1, \ldots, r_m$ such that $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_m\mathbf{v}_m = \mathbf{0}$. Let m be minimal with this property. Then all r_i's are non-zero, hence positive. By a standard linear algebra argument (see, e.g. Theorem 2.11 from (Gale, 1960)) we may then assume that $m \leq n+1$ and that the system of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_{m-1}\}$ is linearly independent. We will assume that $m = n+1$ as it is the worst case scenario. Let $A = (\mathbf{a}_1\, \mathbf{a}_2\, \ldots\, \mathbf{a}_{n+1})$ be the $(n+1)\times n$ matrix, which ith row is $\mathbf{a}_i = \mathbf{v}_i$ for $i = 1, 2, \ldots, n+1$. The null-space of the matrix A is one-dimensional, and, since $(\mathbf{r}_1, \ldots, \mathbf{r}_{n+1})$ is in it, the coordinates in any solution are either all positive or all negative. Looking for a solution of the system $x_1\mathbf{v}_1 + \ldots + x_n\mathbf{v}_n = -\mathbf{v}_{n+1}$, by Cramer's rule we find $x_i = \det A_i / \det A$, where $A = (\mathbf{a}_1\, \ldots\, \mathbf{a}_n)$ and A_i is obtained when $\mathbf{a}_i$ in A is replaced with $\mathbf{a}_{n+1}$. Thus

$$\det A_1\, \mathbf{a}_1 + \ldots + \det A_n\, \mathbf{a}_n + \det A\, \mathbf{a}_{n+1} = \mathbf{0}$$

and by the Hadamard's inequality (Hadamard, 1893) we have $\det A_i \leq n^{n/2} = 2^{\frac{1}{2}n\log_2 n}$. The sum of all coefficients is smaller than or equal to $(n+1)2^{\frac{1}{2}n\log_2 n} = N$. Since G is N-trade robust this is impossible by Proposition 3.3.2. □

Corollary 3.3.2. $f(n) \leq (n+1)2^{\frac{1}{2}n\log_2 n}$.

3.3.3 A New Lower Bound for $f(n)$

Let $\mathbf{w} = (w_1, \ldots, w_n)$ be a vector with non-negative coordinates. There may be some linear relations with integer coefficients between the coordinates of $\mathbf{w}$. Let us define those relations that will be important for us. Let X, Y be subsets of $[n]$ such that $X_i \cap Y_i = \emptyset$. If $\mathbf{v}_{X,Y} \cdot \mathbf{w} = 0$, which is the same as $\sum_{i\in X} w_i - \sum_{j\in Y} w_j = 0$, then we say that the coordinates of $\mathbf{w}$ are **in the relation which corresponds to the vector** $\mathbf{v} = \mathbf{v}_{X,Y} \in T^n$.

Given $X \subseteq [n]$ we may then introduce $w(X) = \sum_{i\in X} w_i$. For two subsets $X, Y \subseteq [n]$ we write $X \sim Y$ if $w(X) = w(Y)$. Of course, if this happens, then the coordinates of $\mathbf{w}$ satisfy the equation $\mathbf{v}_{X,Y} \cdot \mathbf{w} = 0$. Suppose $X \sim Y$, then the equivalence $X' \sim Y'$, where $X' = X - (X \cap Y)$ and $Y' = Y - (X \cap Y)$ will be called **primitive** and $X \sim Y$ will be called a **consequence** of $X' \sim Y'$.

Example 3.3.2. *Consider the vector of weights* $\mathbf{w} = (w_1, w_2, w_3, w_4, w_5) = (1, 2, 5, 6, 10)$. *Then*

$$w_1 + w_3 = w_4, \quad w_1 + w_4 = w_2 + w_3, \quad w_2 + w_5 = w_1 + w_3 + w_4, \quad w_3 + w_4 = w_1 + w_5$$

are relations which correspond to vectors

$$(1,0,1,-1,0), \quad (1,-1,-1,1,0), \quad (-1,1,-1,-1,1), \quad (-1,0,1,1,-1),$$

respectively. It is easy to check that there are no other relations between the coordinates of $\mathbf{w}$*. (Note that we do view* $w_1 + w_2 + w_3 = w_2 + w_4$ *and* $w_1 + w_3 = w_4$ *as the same relation.)*

We have[7]

Primitive equivalence	*Total weight of equal subsets*
$13 \sim 4$	*6*
$14 \sim 23$	*7*
$25 \sim 134$	*12*
$34 \sim 15$	*11*

Apart from these four equivalences and their consequences there are no other equivalences.

Definition 3.3.2. *Let* $\mathbf{w} = (w_1, \ldots, w_n)$ *be a vector of non-negative coordinates. We will say that* $\mathbf{w}$ *satisfies the* **kth Fishburn's condition** *if there exist distinct vectors* $\mathbf{v}_i = \mathbf{v}_{X_i,Y_i} \in T^n$, $i = 1, \ldots, k$, *with* $X_i \cap Y_i = \emptyset$, *such that:*

- $X_i \sim Y_i$ *for* $i = 1, \ldots, k$, *that is* $\mathbf{v}_i \cdot \mathbf{w} = 0$ *is a relation for the coordinates of* $\mathbf{w}$.
- *Apart from* $\mathbf{v}_1, \ldots, \mathbf{v}_k$ *there are no other relations.*
- $\sum_{i=1}^{k} \mathbf{v}_i = \mathbf{0}$,
- *No proper subset of vectors of the system* $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ *is linearly dependent.*

For us the importance of this condition is shown in the following:

Theorem 3.3.4. *Let* $\mathbf{w} = (w_1, \ldots, w_n)$, $n > 2$, *be a vector with positive coordinates which satisfies the kth Fishburn condition. Then there exists a simple game on* $n + k$ *players which is* $(k-1)$*-trade robust but not* k*-trade robust.*

Proof. Suppose vectors $\mathbf{v}_i = \mathbf{v}_{X_i,Y_i} \in T^n$, $i = 1, \ldots, k$ are those that are required for the kth Fishburn condition. Then by Proposition 3.3.1 the sequence $\mathcal{T} = (X_1, \ldots, X_k; Y_1, \ldots, Y_k)$ is a trading transform. Let $w(X)$ be the total weight of the coalition X. Then we have $s_i = w(X_i) = w(Y_i)$ for $i = 1, \ldots, k$. Let N be any positive integer greater than $2w(P)$.

[7]Here and below we omit curly brackets in the set notation

We define

$$P' = P \cup \{n+1, \dots, n+k\}, \quad X'_i = X_i \cup \{n+i\}, \quad Y'_i = Y_i \cup \{n+i\}.$$

Then $\mathcal{T}_1 = (X'_1 \dots, X'_k; Y'_1 \dots, Y'_k)$ is obviously also a trading transform. Let us give weight $N - s_i$ to $n+i$. We will call these new elements *heavy*. Then

$$w(X'_1) = \dots = w(X'_k) = w(Y'_1) = \dots = w(Y'_k) = N.$$

Moreover, we are going to show that no other subset of P' has weight N. Suppose there is a subset $Z \subset P'$ whose total weight is N and which is different from any of the $X'_1, \dots X'_k$ and $Y'_1, \dots Y'_k$. Since $N > 2w(P)$ and $2N - s_i - s_j \geq 2N - 2w(P) > N$ holds for any $i, j \in \{1, \dots, k\}$, then Z must contains no more than one heavy element, say Z contains $n+i$. Then for $Z' = Z - \{n+i\}$ we have $w(Z') = N - (N - s_i) = s_i$ which implies $Z' = X_i$ or $Z' = Y_i$, a contradiction.

Let us now consider the game G on $[n]$ with roughly weighted representation $[N; w_1, \dots, w_n]$, where $X'_1, \dots X'_k$ are winning and $Y'_1, \dots Y'_k$ are losing. Since these are the only subsets on the threshold, the game is fully defined. $\mathcal{T}_1$ becomes a certificate of non-weightedness for G so it is not k-trade robust. Let us prove that it is $(k-1)$-trade robust. Suppose, to the contrary, there exists a certificate of non-weightedness for G

$$\mathcal{T}_2 = (U_1, \dots, U_s; V_1, \dots, V_s), \qquad s \leq k-1, \tag{3.9}$$

where $U_1, \dots, U_s$ are all winning and $V_1, \dots, V_s$ are all losing. Then this can happen only if all these vectors are on the threshold, that is,

$$w(U_1) = \dots = w(U_s) = w(V_1) = \dots = w(V_s) = N,$$

hence $U_i \in \{X'_1, \dots X'_k\}$ and $V_j \in \{Y'_1, \dots Y'_k\}$. As was proved, any of U_i and any of V_j contain exactly one heavy player. Suppose, without loss of generality, that $U_1 = X'_{i_1} = X_{i_1} \cup \{n + i_1\}$. Then we must have at least one player $n + i_1$ among the $V_1, \dots, V_s$. Without loss of generality we may assume that $V_1 = Y'_{i_1} = Y_{i_1} \cup \{n + i_1\}$. We may now cancel $n + i_1$ from the trading transform (3.9) obtaining a certificate of non-weightedness

$$\mathcal{T}_3 = (X_{i_1}, U_2, \dots, U_s; Y_{i_1}, V_2, \dots, V_s).$$

Continuing this way we will come to a certificate of non-weightedness

$$\mathcal{T}_4 = (X_{i_1}, \dots, X_{i_s}; Y_{i_1}, \dots, Y_{i_s}),$$

which by Proposition 3.3.1 will give us $\mathbf{v}_1 + \ldots + \mathbf{v}_s = \mathbf{0}$. The latter contradicts the fact that no proper subset of vectors of the system $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly dependent. □

Fishburn (1997) proved the following combinatorial lemma which plays the key role in our construction of games.

Lemma 3.3.1 (Fishburn, 1997)**.** *For every $n \geq 5$ there exists a vector of weights $\mathbf{w} = (w_1, \dots, w_n)$ which satisfies the $(n-1)$-th Fishburn condition.*

Proof. See (Fishburn, 1996, 1997). □

Corollary 3.3.3. *For each integer $n \geq 5$, there exists a game with $2n-1$ players, that is $(n-2)$-trade robust but not $(n-1)$-trade robust. Moreover, $n-1 \leq f(2n-1)$. For an arbitrary n*

$$\left\lfloor \frac{n-1}{2} \right\rfloor \leq f(n). \tag{3.10}$$

Proof. The first part follows immediately from Theorem 3.3.4 and Lemma 3.3.1. Indeed, in this case the length of the shortest certificate of non-weightedness is $n-1$. We also trivially have $n-1 \leq f(2n)$. These two inequalities can be combined into one inequality (3.10). □

Example 3.3.3 (Continuation of Example 3.3.2)**.** *Suppose $P = [9]$. The first five players get weights $(w_1, w_2, w_3, w_4, w_5) = (1, 2, 5, 6, 10)$. The other four players get weights $(w_6, w_7, w_8, w_9) = (106, 105, 100, 101)$. Then we get the following equivalences:*

Equivalence	*Total weight of subsets*
$136 \sim 46$	*6+106= 112*
$147 \sim 237$	*7+105= 112*
$258 \sim 1348$	*12+100= 112*
$349 \sim 159$	*11+101= 112*

We define

- *Coalitions whose total weight is > 112 are winning.*
- *Coalitions whose total weight is < 112 are losing.*

- $46, 237, 1348, 159$ *are winning.*
- $136, 147, 258, 349$ *are losing.*

This gives us a game with a shortest certificate of length 4, that is, $f(9) \geq 4$. *Gabelman's example (see Example 2.5.1) gives* $f(9) \geq 3$.

Fishburn (1996, 1997) conjectured that a system of n weights cannot satisfy the n'th Fishburn condition for $n' \geq n$. This appeared to be not the case. Conder and Slinko (2004) showed that a system of 7 weights can satisfy the 7th Fishburn condition. Conder[8] checked that this is also the case for $7 \leq n \leq 13$. Marshall (2007) introduced a class of optimus primes and showed that if p is such a prime then a system of p weights satisfying the pth Fishburn condition exists. Although computations show that optimus primes are quite numerous (Marshall, 2007), it is not known if there are infinitely many of them. The definition of an optimus prime is too technical to give here.

Corollary 3.3.4. *For each integer* $7 \leq n \leq 13$ *and also for any* n *which is an optimus prime, there exists a game with* $2n$ *players, that is* $(n-1)$*-trade robust but not* n*-trade robust. Moreover,* $n \leq f(2n)$ *for such* n.

Proof. Follows from Theorem 3.3.4 along the lines of Corollary 3.3.3. □

3.4 Criteria for Roughly Weighted Games

3.4.1 A Criterion for Rough Weightedness

The following result that we need in this section is not new either. Kraft et al. (1959) outlined the idea of its proof without much detail. Since this result is of fundamental importance to us, we give a full proof in the appendix (see Appendix A). On its our rights this theorem is also known and "Theorem of the Alternative", which is an important tool in convex analysis (for details see Goldman, 1956; Tucker, 1956).

Theorem 3.4.1. *Let* A *be an* $m \times n$ *matrix with rational coefficients. Let* $\mathbf{a}_i \in \mathbb{Q}^n$, $i = 1, \ldots, m$ *be the rows of* A. *Then the system of linear inequalities* $A\mathbf{x} \geq \mathbf{0}$ *has no non-negative solution* $\mathbf{x} \geq \mathbf{0}$, *other than* $\mathbf{x} = \mathbf{0}$, *iff there exist non-negative integers* $r_1, \ldots, r_m$ *and a vector* $\mathbf{u}$ *whose all entries are positive integers such that*

$$r_1\mathbf{a}_1 + r_2\mathbf{a}_2 + \cdots + r_m\mathbf{a}_m + \mathbf{u} = \mathbf{0}. \tag{3.11}$$

[8] Reported in (Marshall, 2007)

Definition 3.4.1. *A certificate of non-weightedness which includes P and $\emptyset$ we will call* **potent***.*

We saw such a certificate in (3.8) for the Fano plane game. Now we can give a criterion for a game to be roughly weighted.

Theorem 3.4.2 (Criterion of rough weightedness)**.** *The game G with n players is roughly weighted if one of the two equivalent statements holds:*

(a) for no positive integer $j \le (n+1)2^{\frac{1}{2}n\log_2 n}$ does there exist a potent certificate of non-weightedness of length j,

(b) for no positive integer $j \le (n+1)2^{\frac{1}{2}n\log_2 n}$ do there exist j vectors $\mathbf{v}_1, \ldots, \mathbf{v}_j \in I(G)$ such that

$$\mathbf{v}_1 + \ldots + \mathbf{v}_j + \mathbf{1} = \mathbf{0}, \tag{3.12}$$

where $\mathbf{1} = (1, 1, \ldots, 1)$.

Proof. Since $\mathbf{1} = \mathbf{v}_{P,\emptyset}$, by Proposition 3.3.1 we know that (a) and (b) are equivalent. We also note that, as in Theorem 3.3.3, it can be shown that if a relation (3.12) holds in an n-player game for some j, then there is another such relation with $j \le (n+1)2^{\frac{1}{2}n\log_2 n}$.

Suppose that (3.12) is satisfied but G is roughly weighted. By Proposition 3.2.2 this means that the system

$$\mathbf{v}_i \cdot \mathbf{x} \ge 0, \qquad i = 1, 2, \ldots, j \tag{3.13}$$

has a non-zero non-negative solution, let us call it also $\mathbf{x}_0$. Then

$$0 = (\mathbf{v}_1 + \ldots + \mathbf{v}_j + \mathbf{1}) \cdot \mathbf{x}_0 \ge |\mathbf{x}_0| > \mathbf{0},$$

where $|\mathbf{x}|$ denotes the sum of all coordinates of $\mathbf{x}$. This is a contradiction.

Let us suppose now that a system of rough weights for the game G does not exist. Then the system (3.13) has no solution and by Theorem 3.4.1 there exist vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m \in I(G)$ and a vector $\mathbf{u}$ whose all coordinates are positive integers and

$$\mathbf{v}_1 + \ldots + \mathbf{v}_m + \mathbf{u} = \mathbf{0} \tag{3.14}$$

(where not all of the vectors $\mathbf{v}_i$ may be different). Let us consider the relation (3.14) with the smallest sum $|\mathbf{u}| = u_1 + \ldots + u_n$ of coordinates of $\mathbf{u}$. If $|\mathbf{u}| = n$ we are done. Suppose $u_i > 1$ for some $i \in [n]$. Then we can find $j \in [n]$ such that the ith coordinate of $\mathbf{v}_j$ is -1. Then $\mathbf{v}'_j = \mathbf{v}_j + \mathbf{e}_i \in I(G)$ and we can write

$$\mathbf{v}_1 + \ldots + \mathbf{v}'_j + \ldots + \mathbf{v}_m + \mathbf{u}' = \mathbf{0},$$

where $\mathbf{u}' = \mathbf{u} - \mathbf{e}_i$. Since all coordinates of $\mathbf{u}'$ are positive integers and their sum is $|\mathbf{u}| - 1$ this contradicts the minimality of $|\mathbf{u}|$. □

The game Fano can be generalised in several different ways. We will consider two such generalisations.

Example 3.4.1 (Hadamard games)**.** *An Hadamard matrix H of order $n \times n$ is a matrix with entries ± 1 such that $H^T H = HH^T = I_n$, where I_n is the identity matrix of order n. The latter condition is equivalent to the system of rows of H as well as the system of columns being orthogonal. The standard example of Hadamard matrices is the sequence*

$$H_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad H_{k+1} = \begin{pmatrix} H_k & H_k \\ H_k & -H_k \end{pmatrix}$$

discovered by Sylvester (1867). Here H_k is an $2^k \times 2^k$ matrix. It is known that the order of an Hadamard matrix must be divisible by four and the hypothesis is being tested that for any k an Hadamard matrix of order $4k$ exists. However it has not been proven and the smallest k number for which it is not known whether or not an Hadamard matrix of order $4k$ exists is currently 167 (Kharaghani & Tayfeh-Rezaie, 2005).

Suppose now that an Hadamard matrtix of order $n > 4$ exists. In a usual way (by multiplying certain rows and columns by -1, if necessary) we may assume that all integers in the first row and in the first column of H are 1. Then we consider the matrix $\overline{H}$ which is H without its first row and its first column. The game $HG_{n-1} = (P, W)$ will be defined on the set of players $P = [n-1]$. We consider the rows of $\overline{H}$ and view them as the characteristic vectors of subsets $X_1, \ldots, X_{n-1}$. Any two rows of H are orthogonal which implies that the number of places where these two rows differ are equal to the number of places where they coincide. However, if $X_i \cap X_j = \emptyset$, then the number of places where the two rows differ would be $2(n/2 - 1) = n - 2$ which is greater than $n/2$ for $n > 4$. Hence $X_i \cap X_j \neq \emptyset$ for any i, j. Let us consider $X_1, \ldots, X_{n-1}$ as minimal winning coalitions of HG_{n-1}. It is easy to see that the Hadamard game HG_7 obtained from H_3 is the Fano plane game.

Definition 3.4.2. *A game with n players will be called* **cyclic** *if the characteristic vectors of minimal winning coalitions consist of a vector $\mathbf{w} \in \mathbb{Z}_2^n$ and all its cyclic permutations. We will denote it by $C(\mathbf{w})$.*

It is not difficult to see that the game Fano in Example 3.1.2 is cyclic.

Theorem 3.4.3. *Suppose that the Hamming weight of* $\mathbf{w} \in \mathbb{Z}_2^n$ *is smaller than* $n/2$ *and the game* $C(\mathbf{w})$ *is proper. Then it is not roughly weighted.*

Proof. Let $X_1, \ldots, X_n$ correspond to the characteristic vectors, which are $\mathbf{w}$ and all its cyclic permutations. Suppose that the Hamming weight of $\mathbf{w}$ is k. Then the sequence

$$\mathcal{T} = (X_1, \ldots, X_n, \underbrace{P, \ldots, P}_{n-2k}; X_1^c, \ldots, X_n^c, \underbrace{\emptyset, \ldots, \emptyset}_{n-2k})$$

is a trading transform. Since the game is proper, $X_1, \ldots, X_n \in W$ and $X_1^c, \ldots, X_n^c \in L$. Thus by Theorem 3.4.2 the game $C(\mathbf{w})$ is not roughly weighted. □

Richardson (1956) studied the following class of games that generalise the Fano game. Let $q = p^r$, where p is prime. Let $GF(q)$ be the Galois field with q elements and $PG(n, q)$ be the projective n-dimensional space over $GF(q)$. It is known (Richardson, 1956; Hall, 1986) that $PG(n, q)$ contains $\frac{q^{n+1}-1}{q-1}$ points and any its $(n-1)$-dimensional subspace consists of $\frac{q^n-1}{q-1}$ points. Any two such subspaces have $\frac{q^{n-1}-1}{q-1}$ points in their intersection.

We define a game $Pr_{n,q} = (PG(n, q), W)$ by defining the set W^m of all minimal winning coalition be the set of all $(n-1)$-dimensional subspaces of $PG(n, q)$. This class of games is known as **projective games**. These games are cyclic by Singer's theorem (see, e.g., Hall, 1986, p. 156).

Corollary 3.4.1. *Any projective game is not roughly weighted.*

Proof. We note that, since any two winning coalitions of $Pr_{n,q}$ intersect, this game is proper. By Singer's Theorem any projective game $Pr_{n,q}$ is cyclic. Now the statement follows from Theorem 3.4.3. □

3.4.2 The AT-LEAST-HALF Property

We can also characterise rough weightedness in terms of EL sequences similar to Theorem 2.6.1 of Taylor and Zwicker (1999). We remind to the reader that a coalition is **blocking** if it is a complement of a losing coalition. A sequence of coalitions $(Z_1, \ldots, Z_{2k})$ is called an **EL sequence of degree** k (see Taylor & Zwicker, 1999, p. 61) if half of its coalitions are winning and half are blocking.

Definition 3.4.3. *A simple game satisfies the* **at-least-half property of degree** k *if any EL sequence of degree k or less has some player occurring in at least half of the coalitions in the sequence.*

Theorem 3.4.4. *For a simple game G the following are equivalent:*

(i) G is roughly weighted.

(ii) G has the at-least-half property of degree $(n+1)2^{\frac{1}{2}n\log_2 n}$.

Proof. Suppose G is roughly weighted and let $[q; w_1, \ldots, w_n]$ be its roughly weighted representation. Let $Z = (Z_1, \ldots, Z_{2k})$ be an EL sequence. Then any winning coalition in Z has weight of at least q and any blocking coalition in Z has weight of at least $\Sigma - q$, where $\Sigma = \sum_{i=1}^{n} w_i$. The total weight of coalitions in Z is therefore at least $kq + k(\Sigma - q) = k\Sigma$. If (ii) is not satisfied, then any player occurs in the sequence less than k times and the total weight of coalitions in Z is therefore strictly less than $\sum_{i=1}^{n} kw_i = k\Sigma$, which is a contradiction. Hence (i) implies (ii).

Suppose now that (ii) is satisfied but G is not roughly weighted. Then there exist a potent certificate of non-weightedness

$$\mathcal{T} = (X_1, \ldots, X_k, P; Y_1, \ldots, Y_k, \emptyset),$$

where $k \leq (n+1)2^{\frac{1}{2}n\log_2 n}$. Then the sequence $Z = (X_1, \ldots, X_k; Y_1^c, ..., Y_k^c)$ is an EL sequence. Consider an arbitrary player a. For a certain positive integer s it occurs s times in the subsequence $Z' = (X_1, \ldots, X_k, P)$ and s times in the subsequence $Z'' = (Y_1, \ldots, Y_k, \emptyset)$. Thus in Z it will occur $(s-1) + (k-s) = k-1$ times, which is less than half of $2k$ and the at-least-half property of degree k does not hold. By the Criterion of rough weightedness (Theorem 3.4.2) k can be chosen in the interval $[1, (n+1)2^{\frac{1}{2}n\log_2 n}]$, a contradiction. □

3.4.3 Function g.

Suppose now that we have to check if a game G is roughly weighted or not. According to the Criterion of Rough Weightedness (Theorem3.4.2) we have to check if there are any potent certificates of non-weightedness. We have to know where to stop while checking those. We will define a new function for this.

Definition 3.4.4. *If the game is roughly weighted let us set* $g(G) = \infty$. *Alternatively,* $g(G)$ *is the length of the shortest potent certificate of non-weightedness for G. We also define a* *function*

$$g(n) = \max_G g(G),$$

where the maximum is taken over non roughly-weighted games with n players.

Checking rough weightedness of a game with n players we then have to check all potent certificates of non-weightedness up to a length $g(n)$.

For the Fano plane game in Example 3.1.2 we have a potent certificate of non-weightedness (3.8) which has length 8. We will prove that this is the shortest potent certificate for this game.

Theorem 3.4.5. *$g(Fano) = 8$.*

Proof. We claim that any $\mathbf{v} \in I(G)$ has the sum of coefficients $|\mathbf{v}| = v_1 + \ldots + v_n \geq -1$. Indeed, such a vector would be of the form $\mathbf{v} = \mathbf{v}_{X,Y}$, where X is winning and Y is losing. Since X is winning $\mathbf{v}$ has at least three positive ones and since Y is losing it has at most four negative ones (as all coalitions of size five are winning).

Suppose now there is a sum

$$\mathbf{v}_1 + \ldots + \mathbf{v}_j + \mathbf{1} = \mathbf{0},$$

where $\mathbf{v}_i \in I(G)$, which represents a potent certificate of non-weightedness of length less than eight. In this case $j \leq 6$. By the observation above the sum of coefficients of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_6$ is at least -6. Since the sum of coefficients of $\mathbf{1}$ is seven, we obtain a contradiction. $\square$

Theorem 3.4.6. *$f(HG_n) = 2$ and $g(HG_n) = n + 1$ for all n.*

Proof. Repeats the respective proofs for the Fano game. $\square$

Let us now deal with the lower and upper bounds for g.

Theorem 3.4.7. *For any $n \geqslant 5$*

$$2n + 3 \leqslant g(n) \leqslant (n+1)2^{\frac{1}{2}n\log_2 n}.$$

Proof. Due to Theorem 3.4.2 we need only to take care of the lower bound. For this we need to construct a game G with n players such that $g(G) = 2n + 3$.

Let us define the game $G_{n,2} = ([n], W)$ where

- $\{1, 2\} \in W$ and $\{3, 4, 5\} \in W$;

- if $|S| > 3$ then $S \in W$.

Note that all losing coalition have cardinality at most three.

We note that the trading transform

$$\mathcal{T} = \{\{1,2\}^n, \{3,4,5\}^{n+2}, P; \{2,3,5\}^3, \{2,3,4\}^3, \underbrace{\{2,3,6\}, \ldots, \{2,3,n\}}_{n-5}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}^{n-1}, \emptyset\}$$

is a potent certificate of non-weightedness for G. Its length $2n+3$ is minimal. To prove this we will use the idea introduced in the proof of Theorem 3.4.5. Since all losing coalition have cardinality at most three, any $\mathbf{v} \in I(G)$ has the sum of coordinates $v_1 + \ldots + v_n \geq -1$ and all such vectors $\mathbf{v}$ with $v_1 + \ldots + v_n = -1$ have the form $\mathbf{v}_{\{1,2\},Y}$ for Y being a losing 3-player coalition.

Suppose now there is a sum

$$\mathbf{v}_1 + \ldots + \mathbf{v}_k + \mathbf{1} = \mathbf{0},$$

where $\mathbf{v}_i \in I(G)$, which represents a potent certificate $\mathcal{T} = (X_1, \ldots, X_k, P; Y_1, \ldots, Y_k, \emptyset)$. Due to the comment above, at least n vectors among $\mathbf{v}_1, \ldots, \mathbf{v}_k$ must have the sum of coordinates -1 and hence be of the form $\mathbf{v}_{\{1,2\},Y}$, where Y is a losing 3-player coalition. This means that there are at least n sets $\{1,2\}$ among $X_1, \ldots, X_k$. Add the grand coalition and we obtain that the union $X_1 \cup \ldots \cup X_k \cup P$ has at least $n+1$ elements 1 and at least $n+1$ elements 2. At the same time no losing coalition can contain both 1 and 2. Hence we will need at least $2n+2$ losing coalitions $Y_1, \ldots, Y_k$ to achieve the equality $X_1 \cup \ldots \cup X_k \cup P = Y_1 \cup \ldots \cup Y_k \cup \emptyset$. Hence $\mathcal{T}$ is minimal. □

3.4.4 Further Properties of Functions f and g.

What can we say about the relation between $f(n)$ and $g(n)$? One thing that can be easily observed is given in the following theorem.

Theorem 3.4.8. $f(n) \leq g(n) - 1$.

Proof. Suppose $g(n)$ is finite and there is a sum

$$\mathbf{v}_1 + \ldots + \mathbf{v}_m + \mathbf{1} = \mathbf{0}, \qquad m = g(n) - 1,$$

where $\mathbf{v}_i \in I(G)$, which represents a potent certificate of non-weightedness of length $g(n)$. We will show that $\mathbf{v}_1, \ldots, \mathbf{v}_m$ can absorb $\mathbf{1} = \mathbf{e}_1 + \ldots + \mathbf{e}_n$ and remain in $I(G)$.

Let us start with $\mathbf{e}_1$. One of the vectors, say $\mathbf{v}_i$, will have -1 in the first position. Then we replace $\mathbf{v}_i$ with $\mathbf{v}_i + \mathbf{e}_1$. The new vector is again in $I(G)$. It is clear that we can continue absorbing $\mathbf{e}_i$'s until all are absorbed. □

Let us talk about the duality in games. The operation of taking the dual is known to preserve both weightedness and rough weightedness (Taylor & Zwicker, 1999).

Theorem 3.4.9. *Let g be a simple game, then $f(G) = f(G^*)$ and $g(G) = g(G^*)$.*

Proof. Firstly, we shall prove the statement about f. Let $G = (P, W)$ be a simple game and $\mathcal{T} = (X_1, \ldots, X_k; Y_1, \ldots, Y_k)$ be a certificate of non-weightedness of G, then a sequence of even length $\mathcal{T}^* = (Y_1^c, \ldots, Y_k^c; X_1^c, \ldots, X_k^c)$ will be a trading transform for G^*. Indeed, it is not difficult to see that $X_1^c, \ldots, X_k^c$ are losing coalitions in G^* and $Y_1^c, \ldots, Y_k^c$ are winning. Hence $f(G) \le f(G^*)$. However, due to Theorem 2.9.1 we have $f(G^*) \le f(G^{**}) = f(G)$. Proof of the second part of the theorem is similar. □

3.5 Complete Simple Games

Complete games are a very natural generalisation of weighted games. At the same time this class is much larger so measures of non-weightedness for such games are important and interesting.

Suppose a complete simple game G is not weighted with $|P| = n$. Theorem 4.3 of (Freixas & Molinero, 2009b) states that then there exists a certificate of non-weightedness

$$\mathcal{T} = (X_1, \ldots, X_k; Y_1, \ldots, Y_k) \tag{3.15}$$

for G such that all coalitions $X_1, \ldots, X_k$ are shift-minimal winning coalitions. The shortest certificate of this kind may not be the shortest in the class of all certificates of non-weightedness. So if G is complete and not weighted we define $i(G)$ to be the smallest positive integer k such that G has a certificate of non-weightedness (3.15) with all $X_1, \ldots, X_k$ being shift-minimal winning coalitions. If G is weighted we set $i(G) = \infty$. Let us also define

$$i(n) = \max_G i(G),$$

where the maximum is taken over all non-weighted complete games G with n players.

Theorem 3.5.1. *For an odd $n \geq 9$*

$$\left\lceil \frac{n}{2} \right\rceil \leq i(n) \leq (n+1)2^{\frac{1}{2}n\log_2 n}.$$

Proof. The upper bound follows from Corollary 3.3.2. The lower bound follows from Theorem 5.1 of (Freixas & Molinero, 2009b). □

3.6 Games with a Small Number of Players

Proposition 3.6.1. *Suppose G is a simple game with n players. Then G is roughly weighted if any one of the following three conditions holds:*

(a) G has a passer.

(b) G has a vetoer.

(c) G has a losing coalition that consists of $n-1$ players.

Proof. To prove (a) we simply give weight 1 to the passer and 0 to everybody else. The rough quota must be set to 0 (this is possible since we have a non-zero weight). To prove (b) suppose that v is a vetoer for this game and that there exists a potent certificate of non-weightedness

$$\mathcal{T} = (X_1, \ldots, X_k, P; Y_1, \ldots, Y_k, \emptyset), \tag{3.16}$$

Then v belongs to all winning coalitions $X_1, \ldots, X_k, P$ of this trading transform but it cannot belong to all losing coalitions since $\emptyset$ does not contain v. This contradiction proves (b). Part (c) follows from (b) since if Y is a losing coalition of size $n-1$, then the player v such that $\{v\} = P - Y$ is a vetoer. □

Theorem 3.6.1. *Let G be a proper simple game with n players. If G has a two-player winning coalition, then G is a roughly weighted game.*

Proof. Suppose G has a two-player winning coalition. Players from this coalition we will call heavy. Since G is proper every winning coalition X must contain at least one heavy player and any losing coalition Y can contain at most one. Then a potent certificate of non-weightedness (3.16) cannot exist since coalitions $X_1, \ldots, X_k, P$ will contain at least $k+2$ heavy players while coalitions $Y_1, \ldots, Y_k, \emptyset$ will contain at most k. Therefore G is roughly weighted. □

Corollary 3.6.1. *Let G be a simple strong game with n players. If G has a losing coalition of cardinality $n-2$, then G is a roughly weighted game.*

Proof. The dual game G^* is proper and it will have a two-player winning coalition, hence Theorem 3.6.1 implies this corollary since the operation of taking the dual game preserves rough weightedness. □

Proposition 3.6.2. *Every game with $n \leqslant 4$ players is a roughly weighted game.*

Proof. It is obvious for $n = 1, 2, 3$ as all games in this case are known to be weighted. Let G be a 4-player simple game. If G has a one-player winning coalition it is roughly weighted by Proposition 3.6.1(a). Also, if G has a 3-player losing coalition, it is roughly weighted by Proposition 3.6.1(c). Thus we may assume that all winning coalitions have at least two players and all losing coalitions have at most two players.

Suppose that there is a potent certificate of non-weightedness of the form (3.16). As we mentioned, we may assume that $|X_i| \geq 2$ and $|Y_i| \leq 2$ for every $i = 1, \ldots, k$. But in this case the multisets $X_1 \cup \ldots \cup X_k \cup P$ and $Y_1 \cup \ldots \cup Y_k \cup \emptyset$ have different cardinalities and cannot be equal which is a contradiction. □

Theorem 3.6.2. *Any simple game G with five players, which is either proper or strong, is a roughly weighted game.*

Proof. Assume, to the contrary, that there exists a game $G = ([5], W)$, which is not roughly weighted. It means that there is a potent certificate of non-weightedness (3.16). As in the proof of the previous theorem we may assume that $|X_i| \geq 2$ and $|Y_i| \leq 3$ for all $i = 1, \ldots, k$. We note that there must exist $i \in [k]$ such that $|X_i| = 2$. If this does not hold, then $|X_i| \geq 3$ for all i and the multisets $X_1 \cup \ldots \cup X_k \cup P$ and $Y_1 \cup \ldots \cup Y_k \cup \emptyset$ cannot be equal. Similarly, there must exist $j \in [k]$ for which $|Y_j| = 3$. Now by Theorem 3.6.1 in the proper case and its corollary in the strong case we deduce that G is roughly weighted. □

The following example shows that the requirement for the game to be strong or proper cannot be discarded.

Example 3.6.1 (Game with five players that is not roughly weighted)**.** *We define the game $G = (P, W)$, where $P = [5]$, by defining the set of minimal winning coalitions to be*

$$W^m = \{\{1, 2\}, \{3, 4, 5\}\}.$$

Then the trading transform

$$\mathcal{T} = \{\{1,2\}^5, \{3,4,5\}^7, P; \{2,3,5\}^4, \{2,3,4\}^2, \{1,3,4\}^2, \{1,4,5\}^4, \emptyset\}.$$

is a potent certificate of non-weightedness. Indeed, all four coalitions $\{2,3,5\}$*,* $\{2,3,4\}$*,* $\{1,3,4\}$*,* $\{1,4,5\}$ *are losing since they do not contain* $\{1,2\}$ *or* $\{3,4,5\}$*.*

We note that any simple constant sum game with five players is weighted (von Neumann & Morgenstern, 1944).

Theorem 3.6.3. *Any simple constant sum game with six players is roughly weighted.*

Proof. Assume, to the contrary, that G doesn't have rough weights. Then there is a a potent certificate of non-weightedness of the form (3.16). Because G is proper, by Theorem 3.6.1 every X_i has at least three elements. As G is also strong, so by Corollary 3.6.1 every Y_j contains at most three elements. However, it is impossible to find such trading transform $\mathcal{T}$ under these constraints since multisets $X_1 \cup \ldots \cup X_k \cup P$ and $Y_1 \cup \ldots \cup Y_k \cup \emptyset$ have different cardinalities. □

Example 3.6.2 (Proper game with six players that is not roughly weighted)**.** *We define* $G = (P, W)$*, where* $P = [6]$*. Let the set of minimal winning coalitions be*

$$W^m = \{\{1,2,3\}, \{3,4,5\}, \{1,5,6\}, \{2,4,6\}, \{1,2,6\}\}.$$

A potent certificate of non-weightedness for this game is

$$\mathcal{T} = (\{1,2,3\}, \{3,4,5\}^2, \{1,5,6\}, \{2,4,6\}, \{1,2,6\}, P; \{1,2,4,5\}^2, \{1,3,4,6\}^2, \{2,3,5,6\}^2, \emptyset).$$

An example of a not roughtly weighted constunt-sum game is not known.

3.7 Conclusion and Further Research

In this chapter we proved a criterion of existence of rough weights in the language of trading transforms similar to the criterion of existence of ordinary weights given byTaylor and Zwicker (1992). We defined two functions $f(n)$ and $g(n)$ which measure the deviation of a simple game from a weighted majority game and a roughly weighted majority game, respectively. We improved the upper and lower bounds for $f(n)$ and obtain upper and lower bounds for $g(n)$. However the gap between lower and upper bounds remain large and further investigations are needed to reduce it.

We believe the class of complete games—as a natural generalisation of weighted games—must be thoroughly studied. This class is much larger than the class of weighted games so measures of non-weightedness for complete games are important and interesting. Section 3.5 only scratches the surface of this problem.

Chapter 4

Generalizations of Rough Weightedness

4.1 Preliminaries

In Example 3.3.1 we saw that the Fano plane game is not roughly weighted. The following eight winning coalitions corresponding to the lines of the smallest projective plane—$X_1, \ldots, X_7, P$—can be transformed into the following eight losing coalitions: $X_1^c, \ldots, X_7^c, \emptyset$. So the sequence

$$(X_1, \ldots, X_7, P; X_1^c, \ldots, X_7^c, \emptyset)$$

is a potent certificate of nonweightedness for this game. So the game is not roughly weighted, thanks to Theorem 3.4.2.

Due to Proposition 3.6.1, there is a trivial way to make any game roughly weighted. This can be done by adding an additional player and making him or her a passer. Then we can introduce rough weights by assigning weight 1 to the passer and weight 0 to every other player and setting the quota equal to 0. Note that if the original game is not roughly weighted, then such rough representation is unique (up to multiplicative scaling). In our view, adding a passer trivializes the game but does not make it closer to a weighted majority game; this is why in definitions of our hierarchies $\mathcal{B}$ and $\mathcal{C}$ we do not allow 0 as a threshold value.

As in Chapter 3.2 we would like to use algebraic representation of trading transforms and coalitions.

4.2 The $\mathcal{A}$-Hierarchy

This hierarchy of classes $\mathcal{A}_\alpha$ tries to capture the richness of the class of games that do not have rough weights, and does so by introducing a parameter $\alpha \in (0, \frac{1}{2})$. Our method of classification is based on the existence of potent certificates of nonweightedness for such games (Gvozdeva & Slinko, 2011). We will now show that potent certificates can be further classified. We will extract a very important parameter from this classification.

Definition 4.2.1. *A certificate of nonweightedness*

$$\mathcal{T} = (X_1, \dots, X_m; Y_1, \dots, Y_m)$$

is called an ℓ**-potent certificate of length m** *if it contains at least* ℓ *grand coalitions among* $X_1, \dots, X_m$ *and at least* ℓ *empty sets among* $Y_1, \dots, Y_m$*.*

Obviously, every ℓ-potent certificate of length m is also an ℓ'-potent certificate of the same length for any $\ell' < \ell$.

Definition 4.2.2. *Let q be a rational number. A game G* **belongs to the class $\mathcal{A}_\mathbf{q}$** *of the $\mathcal{A}$-hierarchy if G possesses some ℓ-potent certificate of nonweightedness of length m, such that $q = \ell/m$. If α is irrational, we set $\mathcal{A}_\alpha = \bigcap_{\{q : q<\alpha \,\wedge\, q \text{ is rational}\}} \mathcal{A}_q$.*

It is easy to see that, if $q \geq \frac{1}{2}$, then $\mathcal{A}_q$ is empty. Why? Well, suppose $q \geq \frac{1}{2}$ and $\mathcal{A}_q$ is not empty. Then there is a game G with a certificate of nonweightedness

$$\mathcal{T} = (X_1, \dots, X_k, P^m; Y_1, \dots, Y_k, \emptyset^m) \tag{4.1}$$

with $m \geq k$. This is not possible since m copies of P contain more elements than are contained in the sets $Y_1, \dots, Y_k$ taken together and so (4.1) is not a trading transform. So our hierarchy consists of a family of classes $\{\mathcal{A}_\alpha\}_{\alpha \in (0, \frac{1}{2})}$. We would like to show that this hierarchy is strict, that is, a smaller parameter captures more games. The following proposition starts us on our path toward showing this, and after proving this proposition we will soon establish actual strictness when α is strictly smaller than β.

Proposition 4.2.1. *If $0 < \alpha \leq \beta < \frac{1}{2}$, then $\mathcal{A}_\alpha \supseteq \mathcal{A}_\beta$.*

Proof. It is sufficient to prove this statement when α and β are rational. Suppose that we have a game G in $\mathcal{A}_\beta$ that possesses a certificate of length n_1 with k_1 grand coalitions and $\beta = k_1/n_1$. Let $\alpha = k_2/n_2$. We can then represent these numbers as $\beta = m_1/n$ and $\alpha = m_2/n$, where $n = \operatorname{lcm}(n_1, n_2)$. Since $\alpha \le \beta$, we have $m_2 \le m_1$. Since $n = n_1 h$ and $m_1 = k_1 h$ for some integer h, we can now combine h certificates for G to obtain one with length n and m_1 grand coalitions. As $m_1 \ge m_2$ we will get a certificate for G of length n with m_2 grand coalitions. So $G \in \mathcal{A}_\alpha$.

□

We say that a game G is **critical** for $\mathcal{A}_\alpha$ if it belongs to $\mathcal{A}_\alpha$ but does not belong to any $\mathcal{A}_\beta$ with $\beta > \alpha$.

Theorem 4.2.1. *For every rational $\alpha \in (0, \frac{1}{2})$ there exists a critical game $G \in \mathcal{A}_\alpha$.*

Proof. First, we will construct a two-parameter family of simple games. For any integers $a \ge 2$ and $b \ge 2$, let $G = ([a^2 + a + b + 1], W)$ be a simple game for which a coalition X is winning exactly if $|X| > a^2 + 1$ or X contains a subset whose characteristic vector is a cyclic permutation of $(\underbrace{1, \ldots, 1}_{a+1}, \underbrace{0, \ldots, 0}_{a^2+b})$.

Let $X_1, \ldots, X_{a^2+a+b+1}$ be winning coalitions, whose characteristic vectors are cyclic permutations of $(\underbrace{1, \ldots, 1}_{a+1}, \underbrace{0, \ldots, 0}_{a^2+b})$. Also let $Y_1, \ldots, Y_{a^2+a+b+1}$ be losing coalitions, whose characteristic vectors are cyclic permutations of

$$(\underbrace{1, \ldots, 1}_{a}, 0, | \underbrace{1, \ldots, 1}_{a}, 0, | \underbrace{1, \ldots, 1}_{a}, 0, | \ldots, | \underbrace{1, \ldots, 1}_{a}, 0, | 0, 1, \underbrace{0, \ldots, 0}_{b-1}),$$

where there are a groups of symbols $\underbrace{1, \ldots, 1}_{a}, 0$. Regarding the $b-1$ of the rightmost part, it is important to keep in mind that $b - 1 \ge 1$.

This game possesses the following potent certificate of nonweightedness

$$\mathcal{T} = (X_1, \ldots, X_{a^2+a+b+1}, P^{a^2-a}; Y_1, \ldots, Y_{a^2+a+b+1}, \emptyset^{a^2-a}).$$

One can see that $\mathcal{T}$ is a valid potent certificate. By symmetry losing coalitions in $\mathcal{T}$ each contain $a^2 + 1$ copies of every player and winning coalitions $X_1, \ldots, X_{a^2+a+b+1}$ have only $a+1$ copies of every player. Hence we need to add $a^2 - a$ grand coalitions to make it a trading transform. Clearly the condition $a \ge 2$ is necessary, because otherwise the certificate $\mathcal{T}$ will not be potent.

So $G \in \mathcal{A}_{\frac{a^2-a}{2a^2+b+1}}$. Let us prove that G is critical for this class, that is, it does not belong to any $\mathcal{A}_{q'}$ for $q' > q$. Note that the vectors $\mathbf{v}_i = \mathbf{v}_{X_i,Y_i}$ belong to the ideal of this game. Note also that the sum of all coefficients of $\mathbf{v}_i$ is $\mathbf{v}_i \cdot \mathbf{1} = a - a^2$ and that for any other vector $\mathbf{v} \in I(G)$ from the ideal of this game we have $\mathbf{v} \cdot \mathbf{1} \geq a - a^2$.

Suppose G also has a potent certificate of nonweightedness

$$(A_1, \ldots, A_s, P^t; B_1, \ldots, B_s, \emptyset^t), \tag{4.2}$$

with $q' = \frac{t}{t+s} > \frac{a^2-a}{2a^2+b+1} = q$. The latter is equivalent to $\frac{a^2+a+b+1}{a^2-a} > \frac{s}{t}$. Let $\mathbf{u}_i = \mathbf{v}_{A_i,B_i} \in I(G)$; then (4.2) can be written as

$$\mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_s + t \cdot \mathbf{1} = \mathbf{0}.$$

Since $\mathbf{u}_i \cdot \mathbf{1} \geq a - a^2$, taking the dot product of both sides with $\mathbf{1}$ we get $t(a^2+a+b+1) \leq s(a^2-a)$, which is equivalent to $\frac{a^2+a+b+1}{a^2-a} \leq \frac{s}{t}$, so we have reached a contradiction.

We will now show that any rational number between 0 and $\frac{1}{2}$ is representable as $\frac{a^2-a}{2a^2+1+b}$ for some positive integers $a \geq 2$ and $b \geq 2$. Let $\frac{p}{q} \in (0, \frac{1}{2})$. Then $q - 2p > 0$ and it is possible to choose a positive integer k such that $k^2p(q-2p) - kq - 3 > 0$. By the choice of k one can see that $kp > 1 + \frac{3+2pk}{k(q-2p)} \geq 2$. Take $a = kp$ and $b = k^2p(q-2p) - kq - 1$. Substituting these values we get $\frac{a^2-a}{2a^2+1+b} = \frac{p}{q}$. □

Corollary 4.2.1. *If* $0 < \alpha < \beta < \frac{1}{2}$, *then* $\mathcal{A}_\alpha \supsetneq \mathcal{A}_\beta$.

Example 4.2.1. *Let us illustrate this proof by an example. Suppose a game G is defined on the set of players* $P = [10]$ *with* $a = 2$ *and* $b = 3$. *Let us include in W all sets of cardinality greater than five and all coalitions with three consecutive players (we think of players as situated on the circle so that* 10 *and* 1 *are neighbors). The* 3*-player minimal winning coalitions, denoted* $X_1, \ldots, X_{10}$, *are*

$$\{1,2,3\}, \{2,3,4\}, \{3,4,5\}, \{4,5,6\}, \{5,6,7\},$$
$$\{6,7,8\}, \{7,8,9\}, \{8,9,10\}, \{9,10,1\}, \{10,1,2\}.$$

Let $Y_1, \ldots, Y_{10}$ *be the losing coalitions, whose characteristic vectors are cyclic permutations of* $(1,1,0,1,1,0,0,1,0,0)$, *respectively. For example,* $Y_1 = \{1,2,4,5,8\}$, $Y_2 = \{2,3,5,6,9\}$, *and* $Y_{10} = \{10,1,3,4,7\}$.

Then the potent certificate of nonweightedness

$$\mathcal{T} = (X_1, \ldots, X_{10}, P^2; Y_1, \ldots, Y_{10}, \emptyset^2)$$

shows that this game belongs to $\mathcal{A}_{1/6}$. As we saw in the proof of Theorem 4.2.1, for no β satisfying $1/6 < \beta < 1/2$ *does G belong to $\mathcal{A}_\beta$.*

As was mentioned before, the larger parameter α is, the more relatively "small" winning coalitions and relatively "large" losing coalitions the game has. To see this, consider a simple game $G = ([n], W)$ with an ℓ-potent certificate of length $j + \ell$,

$$\mathcal{T} = (X_1, \ldots, X_j, P^\ell; Y_1, \ldots, Y_j, \emptyset^\ell).$$

Set $\alpha = \ell/(\ell + j)$. Then $G \in \mathcal{A}_\alpha$, and the average number of players in winning coalitions $X_1, \ldots, X_j$ is σ/j where $\sigma = \sum_{i \in [j]} |X_i|$. At the same time the average number of players in losing coalitions $Y_1, \ldots, Y_j$ is $(\sigma + n\ell)/j$. On average a losing coalition in $\mathcal{T}$ contains $n\ell/j$ more players than a winning coalition in $\mathcal{T}$. From above we know that $j > \ell$. The bigger $\ell/(\ell + j)$ is the bigger the ratio ℓ/j is and hence the bigger $n\ell/j$ is. This means that when α is increasing some winning coalitions become smaller and some losing coalitions become larger.

4.3 $\mathcal{B}$-Hierarchy

The $\mathcal{B}$-hierarchy generalizes the idea behind rough weightedness to allow more "points of (decision) flexibility."

Definition 4.3.1. *A simple game $G = (P, W)$* **belongs to $\mathcal{B}_k$** *if there exist real numbers $0 < q_1 \le q_2 \le \cdots \le q_k$, called thresholds, and a weight function $w\colon P \to \mathbb{R}^{\ge 0}$ such that*

(a) if $\sum_{i \in X} w(i) > q_k$, then X is winning,

(b) if $\sum_{i \in X} w(i) < q_1$, then X is losing,

(c) if $q_1 \le \sum_{i \in X} w(i) \le q_k$, then $w(X) = \sum_{i \in X} w(i) \in \{q_1, \ldots, q_k\}$.

*Games from $\mathcal{B}_k$ will be sometimes called k-*rough.

The condition $0 < q_1$ in Definition 4.3.1 is essential. If we allow the first threshold q_1 be zero, then every simple game can be represented as a 2-rough game. To do this we assign weight 1 to the first player and 0 to everyone else. It is also worthwhile to note that adding a passer does not change the class of the game, that is, a game G belongs to $\mathcal{B}_k$ if and only if the game G' obtained from G by adding a passer belongs to $\mathcal{B}_k$. This is because a passer can be assigned a very large weight. Thus $\mathcal{B}_1$ consists of the roughly weighted simple games with nonzero quota.

Example 4.3.1. *We know that the Fano game is not roughly weighted. Let us assign weight 1 to every player of this game and select two thresholds, $q_1 = 3$ and $q_2 = 4$. Then each coalition whose weight falls below the first threshold is in L, and each coalition whose total weight exceeds the second threshold is in W. If a coalition has total weight of three or four, i.e., its weight is equal to one of the thresholds, it can be either winning or losing. Thus the Fano is a 2-rough game.*

Example 4.3.2. *Let $n = 8$ and assume we have four types of players with players $2i - 1$ and $2i$ forming the ith type. Let us include in W all sets that contain two elements from the same type. Minimal winning coalitions for this game are $\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}$. The trading transform*

$$\mathcal{T} = (\{1, 2\}^2, \{3, 4\}^2, \{5, 6\}^2, \{7, 8\}^2, P; \{1, 3, 5, 7\}^3, \{2, 4, 6, 8\}^3, \emptyset^3)$$

is the potent certificate of nonweightedness. So by Theorem 3.4.2, G is not roughly weighted.

On the other hand, if we assign weight 1 to every player, then G is a 3-rough game with thresholds 2, 3, and 4. Let us show that G is not 2-rough. Assume, to the contrary, that we have weights $w_1, \dots, w_8$ and two positive thresholds q_1 and q_2 that make this game 2-rough. Without loss of generality we can assume that $w_{2i-1} \geq w_{2i}$ for every type i. Two players of the same type form a winning coalition. This means that the weight of this coalition is at least q_1. Moreover the players w_1, w_3, w_5, w_7 with the biggest weight in each type have weight not smaller than $\frac{q_1}{2}$ each. Let us consider the following three losing coalitions with strictly increasing weights

$$\{1, 3\} \subsetneq \{1, 3, 5\} \subsetneq \{1, 3, 5, 7\}.$$

The coalition $\{1, 3\}$ has weight at least $\frac{q_1}{2} + \frac{q_1}{2} = q_1$. In the worst-case scenario this coalition lies exactly on the first threshold q_1. So the weight of coalition $\{1, 3, 5\}$ is greater than or equal to q_2. We can see that in every possible scenario the losing coalition $\{1, 3, 5, 7\}$ has weight strictly greater than q_2, a contradiction.

Let us generalize the idea of Example 4.3.2.

Theorem 4.3.1. *For every natural number $k \in \mathbb{N}^+$, there exists a game in $\mathcal{B}_{k+1} - \mathcal{B}_k$.*

Proof. We will construct a simple game that is a $(k + 1)$-rough but not k-rough. Let $G_{k+1,n} = ([n], W)$ be a simple game with $n = 2k + 4$ players. We have $k + 2$ types of players with the ith type consisting of two elements $2i - 1$ and $2i$. The set of minimal winning coalitions of this game is $W^m = \{\{2i - 1, 2i\} \mid i = 1, 2, \dots, k + 2\}$.

If we assign weight 1 to every player, then $G_{k+1,n}$ is $(k+1)$-rough game with thresholds $q_1 = 2$, $q_2 = 3$, $\dots$, $q_{k+1} = k+2$. Let us assume that this game is j-rough for some $j < k+1$, and let w be the new weight function. Without loss of generality we may assume that the players are ordered so that $w(2i-1) \geq w(2i)$. Since the coalition $\{2i-1, 2i\}$ is winning we have $w(2i-1) \geq q_1/2 > 0$ for any $i = 1, 2, \dots, k+2$. The coalitions $L_j = \{2i-1 \mid 1 \leq i \leq j\}$ are losing, their weights are different, and each of them has weight at least $jq_1/2$ for all $2 \leq j \leq k+2$. Thus at least $k+1$ coalitions of different weights lie in the tie-breaking region.

This is a contradiction. Thus $G_{k+1,n}$ is not j-rough for any $j < k+1$. □

An obvious upper bound on the number of thresholds is $K - k + 1$, where K is the cardinality of the largest losing coalition and k the cardinality of the smallest winning coalition. Indeed, it can be made $(K - k + 1)$-rough by choosing weights $w(i) = 1$ for all $i \in [n]$ and setting thresholds $k, k+1, \dots, K$. However this bound is not tight as is seen from the following example.

Example 4.3.3. *Let $G = ([7], W)$ be a simple game with minimal winning coalitions $\{1,2\}, \{6,7\}, \{3,4,5\}$ and all coalitions of four players except $\{2,3,4,6\}$. This game is not roughly weighted, because we have the following potent certificate of nonweightedness*

$$\begin{aligned}\mathcal{T} =&\{\{1,2\}^7, \{3,4,5\}^9, P; \{2,3,5\}^3, \{2,3,4\}^3,\\ &\{2,3,6\}, \{2,3,7\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}^6, \emptyset\}.\end{aligned}$$

Let us assign weight 0 to the third player and $\frac{1}{2}$ to everyone else. Then the following four statements hold:

- $w(\{1,2\}) = w(\{6,7\}) = w(\{3,4,5\}) = 1$ *and* $w(\{2,3,4,6\}) = \frac{3}{2}$.
- *If X is winning coalition with four or more players, then $w(X) \geq \frac{3}{2}$.*
- *If X is losing coalition with three players, then $w(X) \in \{1, \frac{3}{2}\}$.*
- *If X is losing coalition with fewer than three players, then $w(X) \leq 1$.*

Thus G is a 2-rough game with thresholds 1 and $\frac{3}{2}$. Note that the third player has weight zero but is not a dummy.

4.4 $\mathcal{C}$-Hierarchy

Let us consider another extension of the idea of rough weightedness. This time we will use a threshold interval instead of a single threshold or (as in $\mathcal{B}$-hierarchy) a collection of threshold points. It is convenient to "normalize" the weights so that the left end of our threshold interval is 1. We do not lose any generality by doing this.

Definition 4.4.1. *We say that a simple game $G = (P, W)$* **is in the class** $\mathcal{C}_\alpha$*, $\alpha \in \mathbb{R}^{\geq 1}$, if there exists a weight function $w\colon P \to \mathbb{R}^{\geq 0}$ such that for $X \in 2^P$ the condition $w(X) > \alpha$ implies that X is winning, and $w(X) < 1$ implies X is losing. Games from $\mathcal{C}_\alpha$ will sometimes be called rough$_\alpha$.*

The roughly weighted games with nonzero quota form the class $\mathcal{C}_1$. From Example 4.3.1 we can conclude that the Fano game is in $\mathcal{C}_{4/3}$ (by giving each player weight $1/3$). We also note that adding or deleting a passer does not change the class of the game.

Definition 4.4.2. *We say that a game G is* **critical** *for $\mathcal{C}_\alpha$ if it belongs to $\mathcal{C}_\alpha$ but does not belong to any $\mathcal{C}_\beta$ with $\beta < \alpha$.*

It is clear that if $\alpha \leq \beta$, then $\mathcal{C}_\alpha \subseteq \mathcal{C}_\beta$. However, we can show more.

Proposition 4.4.1. *Let c and d be natural numbers with $1 < d < c$. Then there is a simple game G that is rough$_{c/d}$, but that for each $\alpha < c/d$ is not rough$_\alpha$.*

Proof. Define a game $G = (P, W)$, where $P = [cd]$. Similarly to the proof of Theorem 4.3.1, we have c types of players with d players in each type and the different types do not intersect. Winning coalitions are sets with at least $c + 1$ players and also sets having all d players from the same type. By i_j we will denote the ith player of jth type.

If we assign weight $1/d$ to each player, then the lightest winning coalition (d players from the same type) has weight 1 and the heaviest losing coalition has weight c/d. Thus G belongs to $\mathcal{C}_{c/d}$.

Let us show that G is not rough$_\alpha$ for any $\alpha < c/d$. Suppose G is rough$_\alpha$ relative to a weight function w. Let $\max\{1_j, \ldots, d_j\}$ be the element of the set $\{1_j, \ldots, d_j\}$ that has the biggest weight relative to w.

For any type j we know that $w(\max\{1_j, 2_j, \ldots, d_j\}) \geq \frac{1}{d}$. The coalition

$$Y = \{\max\{1_1, \ldots, d_1\}, \ldots, \max\{1_c, \ldots, d_c\}\}$$

is losing by definition. Moreover, it has weight $w(Y) \geq c/d$. So c/d is the smallest number that can be taken as α so that G is rough$_\alpha$. $\square$

Theorem 4.4.1. *For each $1 \leq \alpha < \beta$, it holds that $\mathcal{C}_\alpha \subsetneq \mathcal{C}_\beta$.*

Proof. We know that $\mathcal{C}_\alpha \subseteq \mathcal{C}_\beta$. If β is a rational number, then by Proposition 4.4.1 there exists a game G that is rough$_\beta$ but is not rough$_\alpha$. If β is an irrational number, then choose a rational number r, such that $\alpha < r < \beta$. By Proposition 4.4.1 there exists a game G that is rough$_r$ but is not rough$_\alpha$. So $\mathcal{C}_\alpha \subsetneq \mathcal{C}_r$. All that remains to notice is that $\mathcal{C}_r \subseteq \mathcal{C}_\beta$. $\square$

Theorem 4.4.2. *Let G be a simple game that is not roughly weighted and is critical for $\mathcal{C}_a$. Suppose G also belongs to $\mathcal{A}_q$ for some $0 < q < \frac{1}{2}$. Then*

$$a \geq \frac{1-q}{1-2q}.$$

Proof. Obviously we can assume that q is rational. Since G is in $\mathcal{A}_q$, it possesses a certificate of nonweightedness $\mathcal{T}$ of the kind

$$\mathcal{T} = (X_1, \ldots, X_t, P^s; Y_1, \ldots, Y_t, \emptyset^s).$$

Suppose we have a weight function $w\colon P \to \mathbb{R}^{\geq 0}$ instantiating $G \in \mathcal{C}_\alpha$. Then since $w(X_i) \geq 1$ and $w(P) \geq a$, we have

$$w(X_1) + \cdots + w(X_t) + sw(P) \geq t + sa.$$

On the other hand, $w(Y_i) \leq a$ and

$$w(Y_1) + \cdots + w(Y_t) \leq ta.$$

From these two inequalities we get $t + sa \leq ta$ or $a \geq \frac{t}{t-s}$. Since $q = \frac{s}{t+s}$ we obtain $a \geq \frac{1-q}{1-2q}$, which proves the theorem.

$\square$

4.5 Degrees of Roughness of Games with a Small Number of Players

Suppose there is a fixed number n of players. Then there is an interval $[1, s(n)]$ such that all games with n players belong to $C_{s(n)}$ and $s(n)$ is minimal with this property. There will be also finitely many numbers $q \in [1, s(n)]$ such that the interval $[1, q]$ represents more n-player games than any interval $[1, q']$ with $q' < q$. We call the set of such numbers the **nth spectrum** and denote it $\mathrm{Spec}(n)$.

First, we will derive bounds on the largest number $s(n)$ of the spectra $\mathrm{Spec}(n)$.

Theorem 4.5.1. *For $n \geq 4$, $\frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor \leq s(n) \leq \frac{n-2}{2}$.*

Proof. Let G be a game with n players. Without loss of generality we can assume that G does not contain passers. Moreover the maximal value of $s(n)$ is achieved on games that are not roughly weighted. By Proposition 3.6.1 the biggest losing coalition contains at most $n-2$ players and the smallest winning coalition has at least two players. If we assign weight $\frac{1}{2}$ to every player, then G is in $C_{(n-2)/2}$.

We can use a game similar to the one from Theorem 4.3.1 to prove the lower bound. Suppose our game has n players. If n is odd, then the nth player will be a dummy. The remaining $2\left\lfloor \frac{n}{2} \right\rfloor$ players will be divided into $\left\lfloor \frac{n}{2} \right\rfloor$ pairs: $\{1, 2\}, \{2, 3\}, \ldots,$ $\{m-1, m\}$, where $m = 2\left\lfloor \frac{n}{2} \right\rfloor$. These pairs are declared minimal winning coalitions. Given any weight function w we have $w(\max\{2i-1, 2i\}) \geq \frac{1}{2}$ for each i. Then

$$w(\{\max\{1, 2\}, \ldots, \max\{m-1, m\}) \geq \frac{m}{2},$$

while this coalition is losing. So $s(n) \geq m/2$ which proves the lower bound.

$\square$

Now let us calculate the spectra for $n \leq 6$. By Proposition 3.6.2 all games with four players are roughly weighted. Since we may assume that the game does not have passers we may assume that the quota is nonzero. Hence we have $\mathrm{Spec}(4) = \{1\}$. So the first nontrivial case is $n = 5$.

Let $G = ([n], W)$ be a simple game. The problem of finding the smallest α such that $G \in C_\alpha$ holds is a linear programming question. Indeed, let $W^{\min}$ and $L^{\max}$ be the set of minimal winning coalitions and the set of maximal losing coalitions, respectively. We need to find the minimum α such that the following system of linear inequalities is consistent:

$$\begin{cases} w(X) \geq 1 & \text{for } X \in W^{\min}, \\ w(Y) \leq \alpha & \text{for } Y \in L^{\max}. \end{cases}$$

This is equivalent to the following optimization problem:

Minimize: α.
Subject to: $\sum_{i \in X} w_i \geq 1,\ \sum_{i \in Y} w_i - \alpha \leq 0,$ and $w_i \geq 0;\ X \in W^{min}, Y \in L^{max}$.

Theorem 4.5.2. *Spec*$(5) = \left\{1, \frac{6}{5}, \frac{7}{6}, \frac{8}{7}, \frac{9}{8}\right\}$.

Proof. Let G be a critical game with five players. If G has a passer, then as was noted, the passer can be deleted without changing the class of G, hence $G \in C_1$. If G has no passers and does not belong to C_1, then it is not roughly weighted. By Theorem 3.6.2 each game that is not roughly weighted is not strong and is not proper. Thus we have a winning coalition X such that X^c is also winning and a losing coalition Y such that Y^c is also losing.

By Proposition 3.6.1 we may assume that the cardinalities of both X and Y are two. Without loss of generality we assume that $X = \{1, 2\}$ and $X^c = \{3, 4, 5\}$. Note that Y cannot be contained in X^c as otherwise Y^c contains X and is not losing. So without loss of generality we assume that $Y = \{1, 5\}, Y^c = \{2, 3, 4\}$.

We have two levels of as yet unclassified coalitions, which can be set either losing or winning:

$$\begin{aligned} &\text{level 1} : \{1,3,4\}, \{1,3,5\}, \{1,4,5\}, \{2,3,5\}, \{2,4,5\}, \\ &\text{level 2} : \{1,3\}, \{1,4\}, \{2,5\}, \{3,5\}, \{4,5\}. \end{aligned}$$

We wrote Maple code using the "LPSolve" command. First we choose losing coalitions on level 1 and delete all subsets of them from level 2. We add every unclassified coalition from level 1 to winning coalitions. After that we choose losing coalitions on level 2. We run through all possible combinations of losing coalitions on both levels and solve the respective linear programming problems. The results of these calculations are displayed in Table 4.1.

□

α	Minimal winning coalitions and maximal losing coalitions	Weight representation
$\frac{9}{8}$	$W^{min} = \{\{1,2\},\{1,3,5\},\{1,4,5\},\{3,4,5\}\}$, $L^{max} = \{\{1,5\},\{1,3,4\},\{2,3,4\},\{2,3,5\},\{2,4,5\}\}$	$w_1 = \frac{5}{8}, w_2 = \frac{3}{8}, w_5 = \frac{4}{8}$, $w_3 = w_4 = \frac{2}{8}$
$\frac{8}{7}$	$W^{min} = \{\{1,2\},\{2,5\},\{1,3,4\},\{3,4,5\}\}$, $L^{max} = \{\{1,3,5\},\{1,4,5\},\{2,3,4\}\}$	$w_1 = w_5 = \frac{3}{7}, w_2 = \frac{4}{7}$, $w_3 = w_4 = \frac{2}{7}$
$\frac{7}{6}$	$W^{min} = \{\{1,2\},\{1,4,5\},\{3,4,5\}\}$, $L^{max} = \{\{1,3,4\},\{1,3,5\},\{2,3,4\},\{2,3,5\},\{2,4,5\}\}$	$w_1 = w_2 = \frac{3}{6}$, $w_3 = w_4 = w_5 = \frac{2}{6}$
$\frac{6}{5}$	$W^{min} = \{\{1,2\},\{1,3\},\{1,4\},\{2,5\},\{3,5\},\{4,5\}\}$, $L^{max} = \{\{1,5\},\{2,3,4\}\}$	$w_1 = w_5 = \frac{3}{5}$, $w_2 = w_3 = w_4 = \frac{2}{5}$

Table 4.1: Examples of critical simple games for every number of the 5th spectrum

Theorem 4.5.3. *The 6th spectrum Spec*(6) *contains Spec*(5) *and also the following fractions:*

$$\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{9}{7}, \frac{10}{9}, \frac{11}{9}, \frac{11}{10}, \frac{12}{11}, \frac{13}{10}, \frac{13}{11}, \frac{13}{12}, \frac{14}{11}, \frac{14}{13}, \frac{15}{13}, \frac{15}{14}, \frac{16}{13}, \frac{16}{15}, \frac{17}{13}, \frac{17}{14}, \frac{17}{15}, \frac{17}{16}, \frac{18}{17}.$$

Proof. Let G be a critical game with six players. If G has a passer, then $G \in C_\alpha$ where $\alpha \in$ Spec(5). In the other words Spec(5) $\subseteq$ Spec(6). If G doesn't have a passer, then assume it is not roughly weighted. By Theorem 3.6.3 we know that every game with six players that is not roughly weighted is either not strong ($Y, Y^c \in L$ for some $Y \in 2^P$) or is not proper ($X, X^c \in W$ for some $X \in 2^P$). By Proposition 3.6.1 we can restrict ourselves to the consideration of games for which every coalition with less than two players is losing and every coalition with more that four players is winning. Since G is not roughly weighted there is a potent certificate of nonweightedness $\mathcal{T} = (X_1, \ldots, X_k, P; Y_1, \ldots, Y_k, \emptyset)$, where the coalitions $X_1, \ldots, X_k$ are winning and the coalitions $Y_1, \ldots, Y_k$ are losing. The latter absorb all players in $X_1, \ldots, X_k$ and the grand coalition. Then there exists a losing coalition Y_j among $Y_1, \ldots, Y_k$ with more players than in the smallest winning coalition X_i among $X_1, \ldots, X_k$. If X_i consists of two players, then Y_j has at least three players. If X_i has three players, then Y_j has four players and any subset of it with three players is also losing. Clearly X_i cannot have four players or more. So in any case we have a losing coalition with three players and a winning coalition with three players. Without loss of generality we need to check only six possible cases:

- If G is not proper:

  1. $\{1,2\}, \{3,4,5,6\} \in W$ and $\{1,3,4\} \in L$;

2. $\{1,2\},\{3,4,5,6\} \in W$ and $\{3,4,5\} \in L$;
3. $\{1,2,3\},\{4,5,6\} \in W$ and $\{1,4,5\} \in L$.

- If G is not strong:

4. $\{1,2\},\{3,4,5,6\} \in L$ and $\{1,3,4\} \in W$;
5. $\{1,2\},\{3,4,5,6\} \in L$ and $\{1,2,3\} \in W$;
6. $\{1,2,3\},\{4,5,6\} \in L$ and $\{1,2,4\} \in W$.

For each case the algorithm considers all possible assignments of the attributes "winning" and "losing" to coalitions that are not yet classified. Let level 1 consists of all 4-element coalitions, level 2 consists of 3-element coalitions, and level 3 consists of 2-element coalitions. As in the code discussed in the proof of Proposition 4.5.2, first the algorithm selects losing coalitions at level 1 (everything else at level 1 will be winning) and classifies all subsets of these coalitions from levels 2 and 3 as losing. Next it selects losing coalitions among coalitions of level 2, which are not yet classified, and classifies all subsets of them from level 3 as losing. Finally, it selects losing coalitions among remaining coalitions of level 3 and solves the linear programming problem using "LPSolve" in Maple, which tries to assign weights to players consistent with the classification of coalitions. We repeat everything for each possible combination of losing coalitions at all levels. The code and the list of critical games can be found in Appendix.

□

4.6 Conclusion and Further Research

Economics has extensively studied weighted majority games. This class was previously extended to the class of roughly weighted games (Taylor & Zwicker, 1999; Gvozdeva & Slinko, 2011). However, many games are not even roughly weighted, and some of these games are important both for theory and applications.

In this chapter we introduced three hierarchies, each of which partitions the class of games without rough weights according to some parameter that can be viewed as capturing some resource—either a measure of our flexibility on the size and structure of the tie-breaking region or allowing certain types of certificates of nonweightedness.

It is important to look for further connections between the classes of the three hierarchies, and we commend that direction to the interested reader.

Also, in this chapter we studied only the $\mathcal{C}$-spectrum. Some interesting questions about this spectrum still remain; for example, the bounds for $s(n)$ are of considerable interest. It would be interesting to study both the $\mathcal{A}$-spectrum and $\mathcal{B}$-spectrum as well.

As we mentioned in the introduction, another important generalization of a class of weighted majority games is the class of complete games. All questions that we investigated in this article can be reformulated for games in this class: strictness of the three hierarchies, description of spectra, etc. This is an important direction for future research.

Chapter 5

Initial Segments Complexes Obtained from Qualitative Probability Orders

5.1 Qualitative Probability Orders and Discrete Cones

Definition 5.1.1. *An order*[9] $\preceq$ *on* $2^{[n]}$ *is called a* **qualitative probability order** *on* $[n]$ *if*

$$\emptyset \preceq A \tag{5.1}$$

for every subset A *of* $[n]$*, and* $\preceq$ *satisfies de Finetti's axiom, namely for all* $A, B, C \in 2^{[n]}$

$$A \preceq B \iff A \cup C \preceq B \cup C \text{ whenever } (A \cup B) \cap C = \emptyset\,. \tag{5.2}$$

Note that if we have a probability measure $\mathbf{p} = (p_1, \ldots, p_n)$ on $[n]$, where p_i is the probability of i, then we know the probability $p(A) = \sum_{i \in A} p_i$ of every event A. We may now define a relation $\preceq$ on $2^{[n]}$ by

$$A \preceq B \quad \text{if and only if} \quad p(A) \leq p(B);$$

obviously $\preceq$ is a qualitative probability order on $[n]$, and any such order is called **representable** (e.g., Fishburn, 1996; Regoli, 2000). Those not obtainable in this way are called **non-representable**. The class of qualitative probability orders is broader than the class of probability measures for any $n \geq 5$ (Kraft et al., 1959). A non-representable qualitative probability order $\preceq$ on $[n]$ is said to **almost agree**

[9] An order in this chapter is any reflexive, complete and transitive binary relation. If it is also anti-symmetric, it is called linear order.

with the measure $\mathbf{p}$ on $[n]$ if

$$A \preceq B \Longrightarrow p(A) \leq p(B). \tag{5.3}$$

If such a measure $\mathbf{p}$ exists, then the order $\preceq$ is said to be **almost representable**. Since the arrow in (5.3) is only one-sided it is perfectly possible for an almost representable order to have $A \preceq B$ but not $B \preceq A$ while $p(A) = p(B)$.

We begin with some standard properties of qualitative probability orders which we will need subsequently. Let $\preceq$ be a qualitative probability order on $2^{[n]}$. As usual the following two relations can be derived from it. We write $A \prec B$ if $A \preceq B$ but not $B \preceq A$ and $A \sim B$ if $A \preceq B$ and $B \preceq A$.

Lemma 5.1.1. *Suppose that $\preceq$ is a qualitative probability order on $2^{[n]}$, $A, B, C, D \in 2^{[n]}$, $A \preceq B$, $C \preceq D$ and $B \cap D = \emptyset$. Then $A \cup C \preceq B \cup D$. Moreover, if $A \prec B$ or $C \prec D$, then $A \cup C \prec B \cup D$.*

Proof. Firstly, let us consider the case when $A \cap C = \emptyset$. Let $B' = B - C$ and $C' = C - B$ and $I = B \cap C$. Then by (5.2) we have

$$A \cup C' \preceq B \cup C' = B' \cup C \preceq B' \cup D$$

where we have $A \cup C' \prec B' \cup D$ if $A \prec B$ or $C \prec D$. Now we have

$$A \cup C' \preceq B' \cup D \Leftrightarrow A \cup C = (A \cup C') \cup I \preceq (B' \cup D) \cup I = B \cup D.$$

Consider the case when $A \cap C \neq \emptyset$. Let $A' = A - C$. By (5.1) and (5.2) we now have $A' \preceq B$. Since now we have $A' \cap C = \emptyset$ so by the previous case

$$A \cup C = A' \cup C \preceq B \cup C \preceq B \cup D.$$

One can check that if either $A \prec B$ or $C \prec D$ we will get a strict inequality $A \cup C \prec B \cup D$ in this case as well. $\square$

A weaker version of this lemma can be found in (Maclagan, 1998/99, Lemma 2.2).

Definition 5.1.2. *We say that an order $\preceq$ on $2^{[n]}$* **satisfies the k-th cancellation condition** *CC_k if there does not exist a trading transform $(A_1, \ldots, A_k; B_1, \ldots, B_k)$ such that $A_i \preceq B_i$ for all $i \in [k]$ and $A_i \prec B_i$ for at least one $i \in [k]$.*

The key result of (Kraft et al., 1959) can now be reformulated as follows.

Theorem 5.1.1 (Kraft-Pratt-Seidenberg)**.** *A qualitative probability order $\preceq$ is representable if and only if it satisfies CC_k for all $k = 1, 2, \ldots$.*

It was also shown in (Fishburn, 1996, Section 2) that CC_2 and CC_3 hold for linear qualitative probability orders. This follows from de Finetti's axiom and properties of linear orders. It can be shown that a qualitative probability order satisfies CC_2 and CC_3 as well. Hence CC_4 is the first nontrivial cancellation condition. As was noticed in (Kraft et al., 1959), for $n < 5$ all qualitative probability orders are representable, but for $n = 5$ there are non-representable ones. For $n = 5$ all orders are still almost representable (Fishburn, 1996) which is no longer true for $n = 6$ (Kraft et al., 1959).

It will be useful for our constructions to rephrase some of these conditions in vector language. To every such linear order $\preceq$, there corresponds a *discrete cone* $C(\preceq)$ in T^n, where $T = \{-1, 0, 1\}$, as defined in (Fishburn, 1996).

Definition 5.1.3. *A subset $C \subseteq T^n$ is said to be a* **discrete cone** *if the following properties hold:*

D1. $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\} \subseteq C$, *where $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$,*

D2. $\{-\mathbf{x}, \mathbf{x}\} \cap C \neq \emptyset$ *for every $\mathbf{x} \in T^n$,*

D3. $\mathbf{x} + \mathbf{y} \in C$ *whenever $\mathbf{x}, \mathbf{y} \in C$ and $\mathbf{x} + \mathbf{y} \in T^n$.*

We note that Fishburn (1996) requires $\mathbf{0} \notin C$ because his orders are anti-reflexive. In our case, condition D2 implies $\mathbf{0} \in C$.

Given a qualitative probability order $\preceq$ on $2^{[n]}$, for every pair of subsets A, B satisfying $B \preceq A$ we construct the characteristic vector of this pair $\chi(A, B) = \chi(A) - \chi(B) \in T^n$. We define the set $C(\preceq)$ of all characteristic vectors $\chi(A, B)$, for $A, B \in 2^{[n]}$ such that $B \preceq A$. The two axioms of qualitative probability guarantee that $C(\preceq)$ is a discrete cone (see Fishburn, 1996, Lemma 2.1).

Following Fishburn (1996), the cancellation conditions can be reformulated as follows:

Proposition 5.1.1. *A qualitative probability order $\preceq$ satisfies the k-th cancellation condition CC_k if and only if there does not exist a set $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ of nonzero vectors in $C(\preceq)$ such that*

$$\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_k = \mathbf{0} \tag{5.4}$$

and $-\mathbf{x}_i \notin C(\preceq)$ *for at least one i.*

Geometrically, a qualitative probability order $\preceq$ is representable if and only if there exists a non-negative vector $\mathbf{u} \in \mathbb{R}^n$ such that

$$\mathbf{x} \in C(\preceq) \iff (\mathbf{u}, \mathbf{x}) \geq 0 \quad \text{for all } \mathbf{x} \in T^n - \{\mathbf{0}\},$$

where $(\cdot, \cdot)$ is the standard inner product; that is, $\preceq$ is representable if and only if every non-zero vector in the cone $C(\preceq)$ lies in the closed half-space $H_{\mathbf{u}}^+ = \{\mathbf{x} \in \mathbb{R}^n \mid (\mathbf{u}, \mathbf{x}) \geq 0\}$ of the corresponding hyperplane $H_{\mathbf{u}} = \{\mathbf{x} \in \mathbb{R}^n \mid (\mathbf{u}, \mathbf{x}) = 0\}$.

Similarly, for a non-representable but *almost* representable qualitative probability order $\preceq$, there exists a vector $\mathbf{u} \in \mathbb{R}^n$ with non-negative entries such that

$$\mathbf{x} \in C(\preceq) \Longrightarrow (\mathbf{u}, \mathbf{x}) \geq 0 \quad \text{for all } \mathbf{x} \in T^n - \{\mathbf{0}\}.$$

In the latter case we can have $\mathbf{x} \in C(\preceq)$ and $-\mathbf{x} \notin C(\preceq)$ despite $(\mathbf{u}, \mathbf{x}) = 0$.

In both cases, the normalised vector $\mathbf{u}$ gives us the probability measure, namely $\mathbf{p} = (u_1 + \ldots + u_n)^{-1} (u_1, \ldots, u_n)$, from which $\preceq$ arises or with which it almost agrees.

5.2 Simplicial Complexes and Their Cancellation Conditions

In this section we will introduce the objects of our study, simplicial complexes that arise as initial segments of a qualitative probability order. Using cancellation conditions for simplicial complexes, we will show that this class contains the threshold complexes and is contained in the shifted complexes. Using only these conditions it will be easy to show that the initial segment complexes are strictly contained in the shifted complexes. Showing the strict containment of the threshold complexes will require more elaborate constructions which will be developed in the rest of the chapter.

Definition 5.2.1. *A subset* $\Delta \subseteq 2^{[n]}$ *is an* **(abstract) simplicial complex** *if it satisfies the condition:*

$$\textit{if } B \in \Delta \textit{ and } A \subseteq B, \textit{ then } A \in \Delta.$$

Subsets that are in Δ are called **faces**. Abstract simplicial complexes arose from geometric simplicial complexes in topology (e.g., Maunder, 1996). Indeed, for every geometric simplicial complex Δ the set of vertex sets of simplices in Δ is an abstract simplicial complex, also called the **vertex scheme** of Δ. In combinatorial optimization various abstract simplicial complexes associated with finite graphs (Jonsson, 2005) are studied, such as the independence complex, matching complex etc. Abstract simplicial complexes are also in one-to-one correspondence with simple games as defined by (von Neumann & Morgenstern, 1944). Obviously the set of losing coalitions L is a simplicial complex. The reverse is also true: if Δ is a simplicial complex, then the set $2^{[n]} - \Delta$ is a set of winning coalitions of a certain simple game.

A well-studied class of simplicial complexes is the *threshold* complexes (mostly as an equivalent concept to the concept of a weighted majority game but also as threshold hypergraphs (Reiterman et al., 1985)). A simplicial complex Δ is a **threshold complex** if there exist non-negative reals $w_1, \ldots, w_n$ and a non-negative constant q, such that

$$A \in \Delta \iff w(A) = \sum_{i \in A} w_i < q.$$

The same parameters define a weighted majority game with the standard notation $[q; w_1, \ldots, w_n]$.

A much larger but still well-understood class of simplicial complexes is *shifted* simplicial complexes (C. Klivans, 2005; C. J. Klivans, 2007). A simplicial complex is **shifted** if there exists an order $\trianglelefteq$ on the set of vertices $[n]$ such that for any face F, replacing any of its vertices $x \in F$ with a vertex y such that $y \trianglelefteq x$ results in a subset $(F - \{x\}) \cup \{y\}$ which is also a face. Shifted complexes correspond to complete[10] games (Freixas & Molinero, 2009b).

Let $\preceq$ be a qualitative probability order on $[n]$ and $T \in 2^{[n]}$. We denote

$$\Delta(\preceq, T) = \{X \subseteq [n] \mid X \prec T\},$$

where $X \prec Y$ stands for $X \preceq Y$ but not $Y \preceq X$, and call it an **initial segment** of $\preceq$.

Lemma 5.2.1. *Any initial segment of a qualitative probability order is a simplicial complex.*

Proof. Suppose that $\Delta = \Delta(\preceq, T)$ and $B \in \Delta$. If $A \subset B$, then let $C = B - A$. By (5.1) we have that $\emptyset \preceq C$ and since $A \cap C = \emptyset$ it follows from the de Finetti's axiom (5.2) that $\emptyset \cup A \preceq C \cup A$ which implies that $A \preceq B$. Since Δ is an initial segment, $B \in \Delta$ and $A \preceq B$ implies that $A \in \Delta$ and thus Δ is a simplicial complex. $\square$

[10]sometimes also called linear

We will refer to simplicial complexes that arise as initial segments of some qualitative probability order as an **initial segment complex**.

In a similar manner as for the qualitative probability orders, cancellation conditions will play a key role in our analyzing simplicial complexes.

Definition 5.2.2. *A simplicial complex Δ is said to* **satisfy $\mathbf{CC}^*_\mathbf{k}$** *if for no $k \geq 2$ does there exist a trading transform $(A_1, \dots, A_k; B_1, \dots, B_k)$, such that $A_i \in \Delta$ and $B_i \notin \Delta$, for every $i \in [k]$.*

One can show the connection between CC_k and CC^*_k.

Theorem 5.2.1. *Suppose $\preceq$ is a qualitative probability order on $2^{[n]}$ and $\Delta(\preceq, T)$ is its initial segment. If $\preceq$ satisfies CC_k then $\Delta(\preceq, T)$ satisfies CC^*_k.*

This gives us some initial properties of initial segment complexes. Since conditions CC_k, $k = 2, 3$, hold for all qualitative probability orders (Fishburn, 1996) we obtain

Theorem 5.2.2. *If an abstract simplicial complex $\Delta \subseteq 2^{[n]}$ is an initial segment complex, then it satisfies CC^*_k for all $k \leq 3$.*

From this theorem we get the following corollary, due to Caroline Klivans (personal communication):

Corollary 5.2.1. *Every initial segment complex is a shifted complex. Moreover, there are shifted complexes that are not initial segment complexes.*

Proof. Let Δ be a non-shifted simplicial complex. then it is known to contain an obstruction of the form: there are $i, j \in [n]$, and $A, B \in \Delta$, neither containing i or j, so that $A \cup i$ and $B \cup j$ are in Δ but neither $i \cup B$ nor $j \cup A$ are in Δ (C. Klivans, 2005). But then $(A \cup i, B \cup j; B \cup i, A \cup j)$ is a trading transform that violates CC^*_2. Since all initial segments satisfy CC^*_2 they must all be shifted.

On the other hand, there are shifted complexes that fail to satisfy CC^*_2 and hence can not be initial segments. Let Δ be the smallest shifted complex (where shifting is with respect to the usual ordering) that contains $\{1, 5, 7\}$ and $\{2, 3, 4, 6\}$ Then it is easy to check that neither $\{3, 4, 7\}$ nor $\{1, 2, 5, 6\}$ are in Δ but

$$(\{1, 5, 7\}, \{2, 3, 4, 6\}; \{3, 4, 7\}, \{1, 2, 5, 6\})$$

is a trading transform in violation of CC^*_2. $\square$

Similarly, the **terminal segment**

$$G(\preceq, T) = \{X \subseteq [n] \mid T \preceq X\}$$

of any qualitative probability order is a complete simple game.

Theorem 2.4.2 of the book (Taylor & Zwicker, 1999) can be reformulated to give necessary and sufficient conditions for the simplicial complex to be a threshold.

Theorem 5.2.3. *An abstract simplicial complex $\Delta \subseteq 2^{[n]}$ is a threshold complex if and only if the condition CC_k^* holds for all $k \geq 2$.*

Above we showed that the initial segment complexes are strictly contained in the shifted complexes. What is the relationship between the initial segment complexes and threshold complexes?

Lemma 5.2.2. *Every threshold complex is an initial segment complex.*

Proof. The threshold complex defined by the weights $w_1, \ldots, w_n$ and a positive constant q is the initial segment of the representable qualitative probability order where $p_i = w_i$, $1 \leq i \leq n$ and where the threshold set T has the property that $w(A) \leq w(T) < q$ for all $A \in \Delta$. □

This leaves us with the question of whether this containment is strict, i.e., are there initial segment complexes which are not threshold complexes. As we know any initial segment of a representable qualitative probability order is a threshold simplex. One might think that evry non-representable qualitative probability order would have at least one initial segmentt that is not threshold. Unfortunately that may not be the case.

Example 5.2.1. *This example, adapted from (Maclagan, 1998/99, Example2.5, Example 3.9), gives a non-representable qualitative probability order for which every initial segment complex is threshold. Construct a representable qualitative probability order on $2^{[5]}$ using the p_i's $\{7, 10, 16, 20, 22\}$. The order begins*

$$\emptyset \prec 1 \prec 2 \prec 3 \prec 12 \prec 4 \prec 5 \prec \cdots$$

where 1 *denotes the singleton set* $\{1\}$ *and by* 12 *we mean* $\{1, 2\}$. *Since the qualitative probability order is representable, every initial segment is a threshold complex. Now suppose we interchange the order of* 12 *and* 4. *The new ordering, which begins*

$$\emptyset \prec 1 \prec 2 \prec 3 \prec 4 \prec 12 \prec 5 \prec \cdots,$$

is still a qualitative probability order but it is no longer representable (Maclagan, 1998/99, Example 2.5). With one exception, all of the initial segments in this new non-representable qualitative order are initial segments in the original one and, thus, are threshold. The one exception is the segment

$$\emptyset \prec 1 \prec 2 \prec 3 \prec 4$$

which is obviously a threshold complex.

Another approach to finding an initial segment complex that is not threshold is to construct a complex that violates CC_k^* for some small value of k. As noted above, all initial segment complexes satisfy CC_2^* and CC_3^* so the smallest condition that could fail is CC_4^*. We will now show that for small values of n cancellation condition CC_4^* is satisfied for any initial segment. This will also give us invaluable information on how to construct a non-threshold initial segment later.

Definition 5.2.3. *Two pairs of subsets (A_1, B_1) and (A_2, B_2) are said to* **be compatible** *if the following two conditions hold:*

$$x \in A_1 \cap A_2 \Longrightarrow x \in B_1 \cup B_2, \text{ and}$$
$$x \in B_1 \cap B_2 \Longrightarrow x \in A_1 \cup A_2.$$

Lemma 5.2.3. *Let $\preceq$ be a qualitative probability order on $2^{[n]}$, $T \subseteq [n]$, and let $\Delta = \Delta_n(\preceq, T)$ be the respective initial segment. Suppose CC_s^* fails and $(A_1, \ldots, A_s, B_1, \ldots, B_s)$ is a trading transform, such that $A_i \prec T \preceq B_j$ for all $i, j \in [s]$. If any two pairs (A_i, B_k) and (A_j, B_l) are compatible, then $\preceq$ fails to satisfy CC_{s-1}.*

Proof. Let us define

$$\bar{A}_i = A_i - (A_i \cap B_k), \qquad \bar{B}_k = B_k - (A_i \cap B_k),$$
$$\bar{A}_j = A_j - (A_j \cap B_l), \qquad \bar{B}_l = B_l - (A_j \cap B_l).$$

We note that

$$\bar{A}_i \cap \bar{A}_j = \bar{B}_k \cap \bar{B}_l = \emptyset. \tag{5.5}$$

Indeed, suppose, for example, $x \in \bar{A}_i \cap \bar{A}_j$, then also $x \in A_i \cap A_j$ and by the compatibility $x \in B_k$ or $x \in B_l$. In both cases it is impossible for x to be in $x \in \bar{A}_i \cap \bar{A}_j$. We note also that by Lemma 5.1.1 we have

$$\bar{A}_i \cup \bar{A}_j \prec \bar{B}_k \cup \bar{B}_l. \tag{5.6}$$

Now we observe that

$$(\bar{A}_i, \bar{A}_j, A_{m_1}, \ldots, A_{m_{s-2}}; \bar{B}_k, \bar{B}_l, B_{r_1}, \ldots, B_{r_{s-2}}).$$

is a trading transform. Hence, due to (5.5),

$$(\bar{A}_i \cup \bar{A}_j, A_{m_1}, \ldots, A_{m_{s-2}}; \bar{B}_k \cup \bar{B}_l, B_{r_1}, \ldots, B_{r_{s-2}})$$

is also a trading transform. This violates CC_{s-1} since (5.6) holds and $A_{m_t} \prec B_{r_t}$ for all $t = 1, \ldots, s-2$. □

Corollary 5.2.2. *Let $\preceq$ be a qualitative probability order on $2^{[n]}$. Suppose CC_s^* fails and $(A_1, \ldots, A_s, B_1, \ldots, B_s)$ is a trading transform, such that $A_i \preceq B_i$ for all $i \in [s]$ and $A_i \prec B_i$ for at least one i. If any two pairs (A_i, B_i) and (A_j, B_j) are compatible, then $\preceq$ fails to satisfy CC_{s-1}.*

Proof. Without loss of generality we can assume $i = 1$ and $j = 2$. Let

$$\bar{A}_1 = A_1 - B_1, \qquad \bar{A}_2 = A_2 - B_2,$$
$$\bar{B}_1 = B_1 - A_1, \qquad \bar{B}_2 = B_2 - A_2.$$

Following the proof of Lemma 5.2.3 and taking into account Lemma 5.1.1 we have a trading transform

$$(\bar{A}_1 \cup \bar{A}_2, A_3, \ldots, A_s; \bar{B}_1 \cup \bar{B}_2, B_3, \ldots, B_s),$$

where $\bar{A}_1 \cup \bar{A}_2 \preceq \bar{B}_1 \cup \bar{B}_2$ and $A_k \preceq B_k$ for all $k = 3, \ldots, s$.

To show that the trading transform above witnesses a failure of CC_{s-1}, we need to prove that at least one inequality out of $\bar{A}_1 \cup \bar{A}_2 \preceq \bar{B}_1 \cup \bar{B}_2$ and $A_k \preceq B_k$, where $k \in \{3, \ldots, s\}$, is strict. Note that, in the initial trading transform, there exists $t \in [s]$ such that $A_t \prec B_t$. If $t > 2$, then CC_{s-1} is clearly fails. If $t \in \{1, 2\}$, then by Lemma 5.1.1 we have $\bar{A}_1 \cup \bar{A}_2 \prec \bar{B}_1 \cup \bar{B}_2$ and CC_{s-1} fails as well. □

By definition of a trading transform we are allowed to use repetitions of the same coalition in it. However we will show that to violate CC_4^* we need a trading transform $(A_1, \ldots, A_4; B_1, \ldots, B_4)$ where all A's and B's are different.

Lemma 5.2.4. *Let $\preceq$ be a qualitative probability order on $2^{[n]}$, $T \subseteq [n]$, and let $\Delta = \Delta_n(\preceq, T)$ be the respective initial segment. Suppose CC_4^* fails and $(A_1, \ldots, A_4, B_1, \ldots, B_4)$ is a trading* *transform, such that* $A_i \prec T \preceq B_j$ *for all* $i, j \in [4]$. *Then*

$$|\{A_1, \ldots, A_4\}| = |\{B_1, \ldots, B_4\}| = 4.$$

Proof. Note that pairs $(A_i, B_j), (A_l, B_k)$ are not compatible for every $i \neq l$ and $j \neq k$. Otherwise by Lemma 5.2.3 the order $\preceq$ fails CC_3, which contradicts the fact that every qualitative probability satisfies CC_3. Assume, to the contrary, that we have at least two identical coalitions among $A_1, \ldots, A_4$ or $B_1, \ldots, B_4$. Without loss of generality we can assume $A_1 = A_2$. Clearly all A's or all B's cannot coincide and there are at least two different A's and two different B's. Suppose $A_1 \neq A_3$ and $B_1 \neq B_2$. The pair $(A_1, B_1), (A_3, B_2)$ is not compatible. It means one of the following two statements is true: either there is $x \in A_1 \cap A_3$ such that $x \notin B_1 \cup B_2$ or there is $y \in B_1 \cap B_2$ such that $y \notin A_1 \cup A_3$. Consider the first case. The second case is similar. We know that $x \in A_1 \cap A_3$ and we have at least three copies of x among $A_1, \ldots, A_4$. At the same time $x \notin B_1 \cup B_2$ and there could be at most two copies of x among $B_1, \ldots, B_4$. This is a contradiction. □

Theorem 5.2.4. CC_4^* *holds for* $\Delta = \Delta_n(\preceq, T)$ *for all* $n \leq 17$.

Proof. Let us consider the set of column vectors

$$U = \{\mathbf{x} \in \mathbb{R}^8 \mid x_i \in \{0, 1\} \text{ and } x_1 + x_2 + x_3 + x_4 = x_5 + x_6 + x_7 + x_8 = 2\}. \tag{5.7}$$

This set has an involution $\mathbf{x} \mapsto \bar{\mathbf{x}}$, where $\bar{x}_i = 1 - x_i$. Say, if $\mathbf{x} = (1, 1, 0, 0, 0, 0, 1, 1)^T$, then $\bar{\mathbf{x}} = (0, 0, 1, 1, 1, 1, 0, 0)^T$. There are 36 vectors from U which are split into 18 pairs $\{\mathbf{x}, \bar{\mathbf{x}}\}$.

Suppose now a trading transform $\mathcal{T} = (A_1, A_2, A_3, A_4; B_1, B_2, B_3, B_4)$ witnesses a failure of CC_4^*. It means that $A_i \prec T \preceq B_j$ and no two coalitions in $\mathcal{T}$ coincide. Let us write the characteristic vectors of $A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4$ as rows of $8 \times n$ matrix M, respectively. Since $\preceq$ satisfies CC_3, by Lemma 5.2.3 we know that no two pairs (A_i, B_a) and (A_j, B_b) are compatible. The same can be said about the complementary pair of pairs (A_k, B_c) and (A_l, B_d), where $\{a, b, c, d\} = \{i, j, h, l\} = [4]$. We have

$$A_i \prec B_a,\ A_j \prec B_b,\ A_h \prec B_c,\ A_l \prec B_d,$$

Since (A_i, B_a) and (A_j, B_b) are not compatible one of the following two statements is true: either there exists $x \in A_i \cap A_j$ such that $x \notin B_a \cup B_b$ or there exists $y \in B_a \cap B_b$ such that $x \notin A_i \cup A_j$. As $\mathcal{T}$ is a trading transform in the first case we will also have $x \in B_c \cap B_d$ such that $x \notin A_h \cup A_l$; in the second $y \in A_h \cap A_l$ such that $y \notin B_c \cup B_d$.

Let us consider two columns M_x and M_y of M that correspond to elements $x, y \in [n]$. The above considerations show that both belong to U and $M_x = \bar{M}_y$.

In particular, if $(i, j, k, l) = (a, b, c, d) = (1, 2, 3, 4)$, then the columns M_x and M_y will be as in the following picture

$$M = \begin{bmatrix} \chi(A_1) \\ \chi(A_2) \\ \chi(A_3) \\ \chi(A_4) \\ \chi(B_1) \\ \chi(B_2) \\ \chi(B_3) \\ \chi(B_4) \end{bmatrix} = \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{cc} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ \hline 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{array}\right] \end{array}$$

(we emphasize however that we have only one such column in the matrix, not both). We saw that one pairing of indices $(i, a), (j, b), (k, c), (k, d)$ gives us a column from one of the 18 pairs of U. It is easy to see that a vector from every pair of U can be obtained by the appropriate choice of the pairing of indices. This means that the matrix contains at least 18 columns. That is $n \geq 18$. $\square$

Lemma 5.2.3 can be easily generalised in the following way:

Lemma 5.2.5. *Let $\preceq$ be a qualitative probability order on $2^{[n]}$, $T \subseteq [n]$, and let $\Delta = \Delta_n(\preceq, T)$ be the respective simplicial complex. Suppose CC_k^* fails and $(A_1, \ldots, A_k; B_1, \ldots, B_k)$ is a trading transform, such that $A_i \prec T \preceq B_j$ for all $i, j \in [k]$. If any m disjoint pairs are compatible, then $\preceq$ fails to satisfy CC_{k-m}.*

Before showing that CC_5^* holds for small values of n, we need to investigate how many identical coalitions we may have in a trading transform that violates CC_5^*. More specifically, we will prove that the trading transform $(A_1, \ldots, A_5, B_1, \ldots, B_5)$, where $A_i \prec T \preceq B_j$ for all $i, j \in [4]$, can contain either two identical A's or two identical B's, but not two identical A's and B's at the same time.

Lemma 5.2.6. *Let $\preceq$ be a qualitative probability order on $2^{[n]}$, $T \subseteq [n]$, and let $\Delta = \Delta_n(\preceq, T)$ be the respective initial segment. Suppose $\mathcal{T} = (A_1, \ldots, A_5, B_1, \ldots, B_5)$ is a trading transform violating CC_5^*, and $A_i \prec T \preceq B_j$ for all $i, j \in [4]$. Assume the pairs $(A_i, B_j), (A_k, B_l)$ are not compatible for any $i, j, k, l \in [5], i \neq k$ and $j \neq l$. Then*

$$|\{A_1, \ldots, A_5, B_1, \ldots, B_5\}| \geq 9.$$

Proof. Note that $A_i \neq B_j$ for all $i, j \in [5]$ and

$$|\{A_1, \ldots, A_5, B_1, \ldots, B_5\}| = |\{A_1, \ldots, A_5\}| + |\{B_1, \ldots, B_5\}|.$$

Without loss of generality assume $A_1 = A_2 = A_3$. We know, that the situations where $A_1 = \cdots = A_5$ or $B_1 = \cdots = B_5$ are not possible. Hence, we can assume that $A_1 \neq A_4$ and $B_1 \neq B_2$. Pairs $(A_1, B_1), (A_4, B_2)$ are not compatible. So either there is $x \in A_1 \cap A_4$ such that $x \notin B_1 \cup B_2$, or there is $y \in B_1 \cap B_2$ such that $y \notin A_1 \cup A_4$. Consider the first situation (the second can be done in the same way). We have $x \in A_1 \cap A_4$ and, thus, x is contained in at least four sets out of $A_1, \ldots, A_5$. On the other hand, $x \notin B_1 \cup B_2$. Hence, we have at most three copies of x among $B_1, \ldots, B_5$, a contradiction. Therefore,

$$|\{A_1, \ldots, A_5, B_1, \ldots, B_5\}| \geq 6.$$

Assume there are two pairs of identical sets among the A's or B's. Without loss of generality we have $A_1 = A_2$, and $A_3 = A_4$, and $B_1 \neq B_2$. By the above we know that $A_1 \neq A_3 \neq A_5$. The pairs $(A_1, B_1), (A_3, B_2)$ are not compatible. Hence, there is $x \in A_1 \cap A_3$ such that $x \notin B_1 \cup B_2$ or there is $y \in B_1 \cap B_2$ such that $y \notin A_1 \cup A_3$. From here we consider only the first case and the second one can be done similarly. Thence $x \in A_1 \cap A_3$. There are at least four copies of x among the A's. At the same time $x \notin B_1 \cup B_2$ and there are at most three copies of x among the B's, a contradiction. Thus

$$|\{A_1, \ldots, A_5, B_1, \ldots, B_5\}| \geq 8.$$

Suppose, to the contrary, that $|\{A_1, \ldots, A_5, B_1, \ldots, B_5\}| = 8$ or, equivalently, there are unique $i, j, k, l \in [5]$ such that $A_i = A_j$ and $B_k = B_l$. Without loss of generality we may assume that $i = k = 1, j = l = 2$. Pairs $(A_1, B_1), (A_3, B_3)$ are not compatible. Hence, either there is $x \in A_1 \cap A_3$ such that $x \notin B_1 \cup B_3$, or there is $y \in B_1 \cap B_3$ such that $y \notin A_1 \cup A_3$. As above we can consider only the first case. Therefore $x \in A_1 \cap A_3$ and we can meet x at least three times among $A_1, \ldots, A_5$. At the same time $x \notin B_1 \cap B_3$ and x can be in at most two coalitions among $B_1, \ldots, B_5$. We know $\mathcal{T}$ is a trading transform and, hence, we have a contradiction.

$\square$

In the proof of Theorem 5.2.4 we exploit the fact, that there are no compatible pairs in a trading transform. However, if $(A_1, \ldots, A_5; B_1, \ldots, B_5)$ witnesses the failure of CC_5^* then there could be compatible pairs in this trading transform.

If $(A_i, B_j), (A_k, B_m)$ and $(A_a, B_b), (A_c, B_d)$ are compatible pairs then $\{A_i, B_j, A_k, B_m\} \cap \{A_a, B_b, A_c, B_d\} \neq \emptyset$. Note that if this intersection is empty then, by Lemma 5.2.5, an order $\preceq$ fails to satisfy CC_2 or CC_3. Suppose our trading transform has compatible pairs $(A_i, B_k), (A_j, B_l)$. Let

$$\bar{A}_i = A_i - B_k, \bar{A}_j = A_j - B_l \text{ and } \bar{B}_k = B_k - A_i, \bar{B}_l = B_l - A_j.$$

Then by Theorem 5.2.3 we have a trading transform

$$T = (\bar{A}_i \cup \bar{A}_j, A_{m_1}, A_{m_2}, A_{m_3}; \bar{B}_k \cup \bar{B}_l, B_{r_1}, B_{r_2}, B_{r_3}),$$

where $\bar{A}_i \cup \bar{A}_j \prec \bar{B}_k \cup \bar{B}_l$ and $A_{m_t} \prec T \preceq B_{r_s}$. Let us show that coalitions $\bar{A}_i \cup \bar{A}_j, A_{m_1}, A_{m_2}, A_{m_3}$ are different and so are $\bar{B}_k \cup \bar{B}_l, B_{r_1}, B_{r_2}, B_{r_3}$.

Lemma 5.2.7. *Let $\preceq$ be a qualitative probability order on $2^{[n]}$. Suppose $(A_1, \dots, A_4, B_1, \dots, B_4)$ is a trading transform, where $A_i \prec T \preceq B_j$ for all $i, j \in [3]$ and $A_4 \prec B_4$. Then*

$$|\{A_1, \dots, A_4\}| = |\{B_1, \dots, B_4\}| = 4.$$

Proof. Clearly pairs $(A_i, B_j), (A_k, B_l)$ and $(A_4, B_4), (A_i, B_j)$ are not compatible, for all $i, j, k, l \in [3]$. If $(A_i, B_j), (A_k, B_l)$ or $(A_4, B_4), (A_i, B_j)$ are compatible then after an appropriate enumeration, by Corollary 5.2.2, CC_3 fails, a contradiction.

Assume, to the contrary, that we have at least two identical coalitions among A's and B's. Without loss of generality we can consider two following cases: $A_1 = A_2$; $A_1 = A_4$.

Arguments similar to the ones used in the proofs of Lemma 5.2.4 and Lemma 5.2.6 show that these two cases are contradictory.

□

Theorem 5.2.5. *CC_5^* holds for $\Delta = \Delta_n(\preceq, T)$ for all $n \leq 8$.*

Proof. Suppose that $(A_1, \dots, A_5; B_1, \dots, B_5)$ is a trading transform and $A_i \prec T \preceq B_j$ for each $i, j \in [5]$. Then by Lemma 5.2.6 it is sufficient to consider the following cases:

1. there are no compatible pairs, and all A's and B's are different;

2. there are no compatible pairs, and exactly two coalitions coincide;

3. there is one different set of compatible pairs.

Consider the set of column vectors

$$E = \{\mathbf{x} \in \mathbb{R}^{10} \mid x_i \in \{0,1\} \text{ and } x_1 + \cdots + x_5 = x_6 + \cdots + x_{10} = k, k \in \{2,3\}\}. \tag{5.8}$$

This set has an involution $\mathbf{x} \mapsto \bar{\mathbf{x}}$, where $\bar{x}_i = 1 - x_i$. Say, if $\mathbf{x} = (1,1,0,0,0,0,0,0,1,1)^T$, then $\bar{\mathbf{x}} = (0,0,1,1,1,1,1,1,0,0)^T$. There are 200 vectors from E which are split into 100 pairs $\{\mathbf{x}, \bar{\mathbf{x}}\}$. Let us write the characteristic vectors of $A_1, \ldots, A_5, B_1, \ldots, B_5$ as rows $M_1, \ldots M_{10}$ of a $10 \times n$ matrix M, respectively. Every vector $\mathbf{x} \in E$ shows that six different pairs are not compatible if $\mathbf{x}$ is a column of M. For example, if $\mathbf{x} = (1,1,0,0,0,0,0,0,1,1)^T$ then pairs $(A_1, B_1), (A_2, B_2)$; $(A_1, B_1), (A_2, B_3)$; (A_1, B_2), (A_2, B_3); $(A_3, B_4), (A_4, B_5)$; $(A_3, B_4), (A_5, B_5)$; $(A_4, B_4), (A_5, B_5)$ are not compatible. One can see that any vector from E shows the langest possible number of non-compatible pairs.

Case (1). There are no compatible pairs and all A's and B's are different. Therefore we need to show that $\binom{5}{2} \cdot \binom{5}{2} = 100$ pairs are not compatible. To achieve the smallest number of columns in M we need to add columns that show the largest possible number of non-compatible pairs, i.e. vectors of E. Assume that every new vector $\mathbf{x} \in E$ cancels out new 6 non-compatible pairs. Hence we need at least $\lceil \frac{100}{6} \rceil = 17$ columns in M or equivalently $n \geq 17$.

Case (2). There are no compatible pairs and exactly two coalitions coincide. Without loss of generality we may assume $A_1 = A_2$. As we know, pairs $(A_1, B_i), (A_3, B_j)$ are not compatible for every $i, j \in [5]$. To cancel out all such pairs we need 10 vectors from E, because every vector $\mathbf{x} \in E$ shows that $(A_1, B_i), (A_3, B_j)$ are not compatible only for the one pair of indexes i, j. Hence, we need at least 10 columns in M to cover pairs $(A_1, B_i), (A_3, B_j)$ for all $i, j \in [5]$. Moreover, pairs $(A_1, B_i), (A_4, B_j)$ are not compatible for every $i, j \in [5]$. Note that in all the ten columns of M every $x \in [10]$ belongs to exactly one coalition A_1 or A_4. Therefore none of the existing 10 columns of M can show that $(A_1, B_i), (A_4, B_j)$ are not compatible for some i, j. As before we need at least 10 more vectors of E to cancel out all non-compatible pairs $(A_1, B_i), (A_4, B_j)$ for all $i, j \in [5]$. Hence, M has at least 20 columns or, equivalently, $n \geq 20$.

Case (3). Without loss of generality we may there is one compatible pair $(A_4, B_4), (A_5, B_5)$. Let

$$\bar{A}_4 = A_4 - B_4, \bar{A}_5 = A_5 - B_5 \text{ and } \bar{B}_4 = B_4 - A_4, \bar{B}_5 = B_5 - A_5.$$

By Lemma 5.2.3, the trading transform

$$T = (A_1, A_2, A_3, \bar{A}_4 \cup \bar{A}_5; B_1, B_2, B_3, \bar{B}_4 \cup \bar{B}_5),$$

shows the failure of CC_4, where $A_i \prec B_j$ for every $i, j \in [3]$ and $\bar{A}_4 \cup \bar{A}_5 \prec \bar{B}_4 \cup \bar{B}_5$. By Lemma 5.2.7

$$|\{A_1, A_2, A_3, \bar{A}_4 \cup \bar{A}_5\}| = |\{B_1, B_2, B_3, \bar{B}_4 \cup \bar{B}_5\}| = 4.$$

Denote $\bar{A}_4 \cup \bar{A}_5$ and $\bar{B}_4 \cup \bar{B}_5$ by A'_4 and B'_4 respectivly. Pairs $(A_i, B_j), (A_k, B_l)$ and $(A'_4, B'_4), (A_i, B_j)$ are not compatible for all $i, j, k, l \in [3]$. If $(A_i, B_j), (A_k, B_l)$ or $(A'_4, B'_4), (A_i, B_j)$ are compatible then after an appropriate change of notation by Corollary 5.2.2 CC_3 fails, a contradinction. Note that to prevent reducing CC_4 to CC_3 by Corollary 5.2.2 we need at least $\binom{3}{2}^2 + 3^2 = 18$ non-compatible pairs. To see this we consider the following situation:

$$B'_4 \prec A_i \text{ and } B'_4 \prec (A'_4 - B'_4) \cup (A_i - B_j) \text{ for } i, j \in [3].$$

Let $(A'_4, B_i), (A_j, B_k)$ be compatible pairs for some $i, j, k \in [3]$ and

$$\bar{A}'_4 = A'_4 - B'_4, \bar{A}_i = A_i - B_j \text{ and } \bar{B}'_4 = B'_4 - A'_4, \bar{B}_j = B_j - A_i.$$

Then

$$(\bar{A}'_4 \cup \bar{A}_j, A_{m_1}, A_{m_2}; \bar{B}_i \cup \bar{B}_k, B_r, B'_4)$$

is a trading transform, but it doesn't show the failure of CC_3. More specifically, $B'_4 \prec A_{m_s}$ for all $s \in [2]$ and $B'_4 \prec \bar{A}'_4 \cup \bar{A}_i$. However, either $A_{m_s} \prec B_r$ for all $s \in [2]$ and $\bar{A}'_4 \cup \bar{A}_i \prec B_r$, or $A_{m_s} \prec \bar{B}_i \cup \bar{B}_k$ for all $s \in [2]$ and $\bar{A}'_4 \cup \bar{A}_i \prec \bar{B}_i \cup \bar{B}_k$. It means there is no change of notations $\{\bar{A}'_4 \cup \bar{A}_j, A_{m_1}, A_{m_2}\} = \{X_1, X_2, X_3\}$ and $\{\bar{B}_i \cup \bar{B}_k, B_r, B'_4\} = \{Y_1, Y_2 Y_3\}$ such that for the trading transform $(X_1, X_2, X_3; Y_1, Y_2 Y_3)$ either $X_s \preceq Y_s$, for every $s \in [3]$, or $Y_s \preceq X_s$, for every $s \in [3]$. Hence CC_3 holds even if all pairs $(A'_4, B_i), (A_j, B_k)$ are compatible for all $i, j, k \in [3]$.

Let $(A_i, B_j), (A_k, B'_4)$ be compatible pairs for some $i, j, k \in [3]$. Then $B'_4 \prec A_i, A_k$ and $A_i, A_k \prec B_j$. We do not know the order of $(A_i - B_j) \cup (A_k - B'_4)$ and $(B_j - A_i) \cup (B'_4 - A_k)$. Hence, this order may prevent the failure of CC_3 even if $(A_i, B_j), (A_k, B'_4)$ are compatible pairs.

Therefore 18 is the possible smallest number of non-compatible pairs. More explicitly, there are at least 18 non-compatible pairs in a trading transform

$$(A_1, A_2, A_3, A_4'; B_1, B_2, B_3, B_4').$$

If we have less than 18 non-compatible pairs then it leads to the failure of CC_3.

Consider the set of column vectors

$$V = \{\mathbf{x} \in \mathbb{R}^8 \mid x_i \in \{0,1\} \text{ and } x_1 + x_2 + x_3 + x_4 = x_5 + x_6 + x_7 + x_8 = 2\}.$$

By the proof of Theorem 5.2.4 there are 36 vectors from V which are split into 18 pairs $\{\mathbf{x}, \bar{\mathbf{x}}\}$. Let us write the characteristic vectors of $A_1, A_2, A_3, A_4', B_1, B_2, B_3, B_4'$ as rows $N_1, \ldots, N_8$ of a $8 \times n$ matrix N, respectively.

Pairs $(A_1, B_i), (A_2, B_j)$ are not compatible for every $i, j \in [3]$. Note that if a column of N is a vector $\mathbf{x} \in V$, satisfying $x_1 + x_2 \in \{0, 2\}$, then $(A_1, B_i), (A_2, B_j)$ are not compatible for the only one pair $i, j \in [3]$. To cover all pairs $(A_1, B_i), (A_2, B_j)$ we need at least three different vectors of V as columns of N. Clearly $x_1 + x_3 = x_2 + x_3 = 1$ in every vector out of those three. Hence, in order to show that pairs $(A_1, B_i), (A_3, B_j)$ and $(A_2, B_i), (A_3, B_j)$ are not compatible for every $i, j \in [3]$, we need at least six more different vectors of V.

Pairs $(A_4', B_4'), (A_i, B_j)$ are not compatible for all $i, j \in [3]$. However in the nine columns of N we have already showed that all of them are not compatible, because if $(A_i, B_j), (A_k, B_l)$ are not compatible then $(A_4', B_4'), (A_s, B_t)$ are not compatible as well for all $\{i, k, s\} = \{j, l, t\} = [3]$. $\square$

While no initial segment complex on fewer than 18 points can fail CC_4^*, there is such an example on 26 points which will show that the initial segment complexes strictly contain the threshold complexes. The next three sections are devoted to constructing such an example. The next section presents a general construction technique for producing almost representable qualitative probability orders from representable ones. This technique will be employed in Section 5.4 to construct our example. Some of the proofs required will be done in Section 5.5.

5.3 Constructing Almost Representable Orders from Nonlinear Representable Ones

Our approach to finding an initial segment complex that is not threshold will be to start with a non-linear representable qualitative probability order and then perturb it so as to produce an almost representable order. By judicious breaking of ties in this new order we will be able to produce an initial segment that will violate CC_4^*. The language of discrete cones will be helpful and we begin with a technical lemma that will be needed in the construction.

Proposition 5.3.1. *Let $\preceq$ be a non-representable but almost representable qualitative probability order which almost agrees with a probability measure* $\mathbf{p}$. *Suppose that the mth cancellation condition CC_m is violated, and that for some non-zero vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_m\} \subseteq C(\preceq)$ the condition (5.4) holds, i.e., $\mathbf{x}_1 + \cdots + \mathbf{x}_m = \mathbf{0}$ and $\mathbf{x}_i \notin C(\preceq)$ for at least one $i \in [m]$. Then all of the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_m$ lie in the hyperplane $H_\mathbf{p} = \{\mathbf{x} \in \mathbb{R}^n \mid (\mathbf{p}, \mathbf{x}) = 0\}$.*

Proof. First note that for every $\mathbf{x} \in C(\preceq)$ which does not belong to $H_\mathbf{p}$, we have $(\mathbf{p}, \mathbf{x}) > 0$. Hence the condition (5.4) can hold only when all $\mathbf{x}_i \in H_\mathbf{p}$. □

We need to understand how we can construct new qualitative probability orders from old ones so we need the following investigation. Let $\preceq$ be a representable but not linear qualitative probability order which agrees with a probability measure $\mathbf{p}$.

Let $S(\preceq)$ be the set of all vectors of $C(\preceq)$ which lie in the corresponding hyperplane $H_\mathbf{p}$. Clearly, if $\mathbf{x} \in S(\preceq)$, then $-\mathbf{x}$ is a vector of $S(\preceq)$ as well. Since in the definition of discrete cone it is sufficient that only one of these vectors is in $C(\preceq)$ we may try to remove one of them in order to obtain a new qualitative probability order. The new order will almost agree with $\mathbf{p}$ and hence will be at least almost representable. The big question is: what are the conditions under which a set of vectors can be removed from $S(\preceq)$?

What can prevent us from removing a vector from $S(\preceq)$? Intuitively, we cannot remove a vector if the set comparison corresponding to it is a consequence of those remaining. We need to consider what a consequence means formally.

There are two ways in which one set comparison might imply another one. The first way is by means of the de Finetti condition. This however is already built in the definition of the discrete cone as $\chi(A, B) = \chi(A \cup C, B \cup C)$. Another way in which a comparison may be implied from two other is transitivity. This has a nice algebraic characterisation. Indeed, if $C \prec B \prec A$, then $\chi(A, C) = \chi(A, B) + \chi(B, C)$. This leads us to the following definition.

Following (Christian et al., 2007) let us define a restricted sum for vectors in a discrete cone C. Let $\mathbf{u}, \mathbf{v} \in C$. Then

$$\mathbf{u} \oplus \mathbf{v} = \begin{cases} \mathbf{u} + \mathbf{v} & \text{if } \mathbf{u} + \mathbf{v} \in T^n, \\ \text{undefined} & \text{if } \mathbf{u} + \mathbf{v} \notin T^n. \end{cases}$$

It was shown in (Fishburn, 1996, Lemma 2.1) that the transitivity of a qualitative probability order is equivalent to closedness of its corresponding discrete cone with respect to the restricted addition (without formally defining the latter). The axiom D3 of the discrete cone can be rewritten as

D3. $\mathbf{x} \oplus \mathbf{y} \in C$ whenever $\mathbf{x}, \mathbf{y} \in C$ and $\mathbf{x} \oplus \mathbf{y}$ is defined.

Note that a restricted sum is not associative.

Theorem 5.3.1 (Construction method)**.** *Let $\preceq$ be a representable non-linear qualitative probability order on $[n]$ which agrees with the probability measure $\mathbf{p}$. Let $S(\preceq)$ be the set of all vectors of $C(\preceq)$ which lie in the hyperplane $H_{\mathbf{p}}$. Let X be a subset of $S(\preceq)$ such that*

- $X \cap \{\mathbf{s}, -\mathbf{s}\} \neq \emptyset$ *for every* $\mathbf{s} \in S(\preceq)$.
- *X is closed under the operation of restricted sum.*
- $(S(\preceq) - X) \cap \{e_1, \ldots, e_n\} = \emptyset$.

Then $Y = S(\preceq) - X$ may be dropped from $C(\preceq)$, that is $C_Y = C(\preceq) - Y$ is a discrete cone.

Proof. We first note that if $\mathbf{x} \in C(\preceq) - S(\preceq)$ and $\mathbf{y} \in C(\preceq)$, then $\mathbf{x} \oplus \mathbf{y}$, if defined, cannot be in $S(\preceq)$. So due to closedness of X under the restricted addition all axioms of a discrete cone are satisfied for C_Y. On the other hand, if for some two vectors $\mathbf{x}, \mathbf{y} \in X$ we have $\mathbf{x} \oplus \mathbf{y} \in Y$, then C_Y would not be a discrete cone and we would not be able to construct a qualitative probability order associated with this set. □

Example 5.3.1 (Positive example)**.** *The probability measure*

$$\mathbf{p} = \frac{1}{16}(6, 4, 3, 2, 1).$$

defines a qualitative probability order $\preceq$ on $[5]$*:*

$$\emptyset \prec 5 \prec 4 \prec 3 \prec 45 \prec 35 \sim 2 \prec 25 \sim 34 \prec 1 \prec 345 \sim 24 \prec 23 \sim 15 \prec 245 \prec 14 \sim 235 \ldots$$

(Here only the first 17 terms are shown, since the remaining ones can be uniquely reconstructed. See (Kraft et al., 1959, Proposition 1) for details. There are only four equivalences here

$$35 \sim 2, \quad 25 \sim 34, \quad 23 \sim 15 \quad \textit{and} \quad 14 \sim 235,$$

and all other follow from them, that is:

$$\begin{aligned} &35 \sim 2 \textit{ implies } 345 \sim 24,\ 135 \sim 12;\\ &25 \sim 34 \textit{ implies } 125 \sim 134;\\ &23 \sim 15 \textit{ implies } 234 \sim 145;\\ &14 \sim 235 \textit{ has no consequences} \end{aligned}$$

Let $\mathbf{u}_1 = \chi(2,35) = (0,1,-1,0,-1)$, $\mathbf{u}_2 = \chi(34,25) = (0,-1,1,1,-1)$, $\mathbf{u}_3 = \chi(15,23) = (1,-1,-1,0,1)$ *and* $\mathbf{u}_4 = \chi(235,14) = (-1,1,1,-1,1)$. *Then*

$$S(\preceq) = \{\pm\mathbf{u}_1, \pm\mathbf{u}_2, \pm\mathbf{u}_3, \pm\mathbf{u}_4\}$$

and $X = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ *is closed under the restricted addition as* $\mathbf{u}_i \oplus \mathbf{u}_j$ *is undefined for all* $i \neq j$. *Note that* $\mathbf{u}_i \oplus -\mathbf{u}_j$ *is also undefined for all* $i \neq j$. *Hence we can subtract from the cone* $C(\preceq)$ *any non-empty subset* Y *of* $-X = \{-\mathbf{u}_1, -\mathbf{u}_2, -\mathbf{u}_3, -\mathbf{u}_4\}$ *and still get a qualitative probability. Since*

$$\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 + \mathbf{u}_4 = \mathbf{0}.$$

it will not be representable. The new order corresponding to the discrete cone C_{-X} *is linear.*

Example 5.3.2 (Negative example)**.** *A certain qualitative probability order is associated with the Gabelman game of order 3. Nine players are involved each of whom we think as associated with a certain cell of a* 3×3 *square:*

1	*2*	*3*
4	*5*	*6*
7	*8*	*9*

The ith player is given a positive weight w_i, $i = 1, 2, \ldots, 9$, *such that in the qualitative probability order, associated with* $\mathbf{w} = (w_1, \ldots, w_9)$,

$$147 \sim 258 \sim 369 \sim 123 \sim 456 \sim 789.$$

Suppose that we want to construct a qualitative probability order $\preceq$ *for which*

$$147 \sim 258 \sim 369 \prec 123 \sim 456 \sim 789.$$

Then we would like to claim that it is not weighted since for the vectors

$$\mathbf{x}_1 = (0, 1, 1, -1, 0, 0, -1, 0, 0) = \chi(123, 147),$$
$$\mathbf{x}_2 = (0, -1, 0, 1, 0, 1, 0, -1, 0) = \chi(456, 258),$$
$$\mathbf{x}_3 = (0, 0, -1, 0, 0, -1, 1, 1, 0) = \chi(789, 369)$$

we have $\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 = \mathbf{0}$. *Putting the sign* $\prec$ *instead of* $\sim$ *between* 369 *and* 123 *will also automatically imply* $147 \prec 123$, $258 \prec 456$ *and* $369 \prec 789$. *This means that we are dropping the set of vectors* $\{-\mathbf{x}_1, -\mathbf{x}_2, -\mathbf{x}_3\}$ *from the cone while leaving the set* $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ *there. This would not be possible since* $\mathbf{x}_1 \oplus \mathbf{x}_2 = -\mathbf{x}_3$. *So every* $X \supset \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ *with* $X \cap \{-\mathbf{x}_1, -\mathbf{x}_2, -\mathbf{x}_3\} = \emptyset$ *is not closed under* $\oplus$.

5.4 An Example of a Nonthreshold Initial Segment of a Linear Qualitative Probability Order

In this section we shall construct an almost representable linear qualitative probability order $\sqsubseteq$ on $2^{[26]}$ and a subset $T \subseteq [26]$, such that the initial segment $\Delta(\sqsubseteq, T)$ of $\sqsubseteq$ is not a threshold complex as it fails to satisfy the condition CC_4^*.

The idea of the example is as follows. We will start with a representable linear qualitative probability order $\preceq$ on [18] defined by positive weights $w_1, \ldots, w_{18}$ and extend it to a representable but nonlinear qualitative probability order $\preceq'$ on [26] with weights $w_1, \ldots, w_{26}$. A distinctive feature of $\preceq'$ will be the existence of eight sets $A_1', \ldots, A_4', B_1', \ldots, B_4'$ in [26] such that:

1. The sequence $(A_1', \ldots, A_4'; B_1', \ldots, B_4')$ is a trading transform.

2. The sets $A_1', \ldots, A_4', B_1', \ldots, B_4'$ are tied in $\preceq'$, that is,

$$A_1' \sim' \ldots A_4' \sim' B_1' \sim' \ldots \sim' B_4'.$$

3. If any two distinct sets $X, Y \subseteq [26]$ are tied in $\preceq'$, then $\chi(X, Y) = \chi(S, T)$, where $S, T \in \{A_1', \ldots, A_4', B_1', \ldots, B_4'\}$. In other words all equivalences in $\preceq'$ are consequences of $A_i' \sim' A_j'$, $A_i' \sim' B_j'$, $B_i' \sim' B_j'$, where $i, j \in [4]$.

Then we will use Theorem 5.3.1 to untie the eight sets and to construct a comparative probability order $\sqsubseteq$ for which

$$A_1' \sqsubset A_2' \sqsubset A_3' \sqsubset A_4' \sqsubset B_1' \sqsubset B_2' \sqsubset B_3' \sqsubset B_4',$$

where $X \sqsubset Y$ means that $X \sqsubseteq Y$ is true but not $Y \sqsubseteq X$.

This will give us an initial segment $\Delta(\sqsubseteq, B_1')$ of the linear qualitative probability order $\sqsubseteq$, which is not threshold since CC_4^* fails to hold.

Let $\preceq$ be a representable linear qualitative probability order on $2^{[18]}$ with positive weights $w_1, \ldots, w_{18}$ that are linearly independent (over $\mathbb{Z}$) real numbers in the interval $[0, 1]$. Due to the choice of weights, no two distinct subsets $X, Y \subseteq [18]$ have equal weights relative to this system of weights, i.e.,

$$X \neq Y \Longrightarrow w(X) = \sum_{i \in X} w_i \neq w(Y) = \sum_{i \in Y} w_i.$$

Let us consider again the set U defined in (5.7). Let M be a subset of U with the following properties: $|M| = 18$ and $\mathbf{x} \in M$ if and only if $\bar{\mathbf{x}} \notin M$. In other words M contains exactly one vector from every pair into which U is split. By M we will also denote an 8×18 matrix whose columns are all the vectors from M taken in arbitrary order. By $A_1, \ldots, A_4, B_1, \ldots, B_4$ we denote the sets with characteristic vectors equal to the rows $M_1, \ldots, M_8$ of M, respectively. The way M was constructed guarantees that the following lemma is true.

Lemma 5.4.1. *The subsets $A_1, \ldots, A_4, B_1, \ldots, B_4$ of $[18]$ satisfy:*

1. *$(A_1, \ldots, A_4; B_1, \ldots, B_4)$ is a trading transform;*
2. *for any choice of $i, k, j, m \in [4]$ with $i \neq k$ and $j \neq m$ the pair $(A_i, B_j), (A_k, B_m)$ is not compatible.*

We shall now embed $A_1, \ldots, A_4, B_1, \ldots, B_4$ into $[26]$ and add new elements to them forming $A_1', \ldots, A_4'$, $B_1', \ldots, B_4'$ in such a way that the characteristic vectors $\chi(A_1'), \ldots, \chi(A_4')$, $\chi(B_1'), \ldots, \chi(A_1')$ are the rows $M_1', \ldots, M_8'$ of the following matrix

$$M' = \begin{array}{c} \begin{array}{c|c|c} 1 \ldots 18 & 19\ 20\ 21\ 22 & 23\ 24\ 25\ 26 \end{array} \\ \left[\begin{array}{c|c|c} \chi(A_1) & & \\ \chi(A_2) & I & I \\ \chi(A_3) & & \\ \chi(A_4) & & \\ \hline \chi(B_1) & & \\ \chi(B_2) & J & I \\ \chi(B_3) & & \\ \chi(B_4) & & \end{array}\right] \end{array}.$$

Here I is the 4×4 identity matrix and

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Note that if X belongs to [18], it also belongs to [26], so the notation $\chi(X)$ is ambiguous as it may be a vector from $\mathbb{Z}^{18}$ or from $\mathbb{Z}^{26}$, depending on the circumstances. However the reference set will be always clear from the context and the use of this notation will create no confusion.

One can see that $(A'_1, \dots, A'_4; B'_1, \dots, B'_4)$ is again a trading transform and there are no compatible pairs $(A'_i, B'_j), (A'_k, B'_m)$, where $i, k, j, m \in [4]$ and $i \neq k$ or $j \neq m$. We shall now choose weights $w_{19}, \dots, w_{26}$ of new elements $19, \dots, 26$ in such a way that the sets $A'_1, A'_2, A'_3, A'_4, B'_1, B'_2, B'_3, B'_4$ all have the same weight N, which is a sufficiently large number. It will be clear from the proof how large it should be.

To find weights $w_{19}, \dots, w_{26}$ that satisfy this condition we need to solve the following system of linear equations

$$\begin{pmatrix} I & I \\ J & I \end{pmatrix} \begin{pmatrix} w_{19} \\ \vdots \\ w_{26} \end{pmatrix} = N\mathbf{1} - M \cdot \mathbf{w}, \tag{5.9}$$

where $\mathbf{1} = (1, \dots, 1)^T \in \mathbb{R}^8$ and $\mathbf{w} = (w_1, \dots, w_{18})^T \in \mathbb{R}^{18}$.

The matrix from (5.9) has rank 7, and the augmented matrix of the system has the same rank. Therefore, the solution set is not empty, moreover, there is one free variable (and any one can be chosen for this role). Let this free variable be w_{26} and let us give it value K, such that K is large but much smaller than N. In particular, $126 < K < N$. Now we can express all other weights $w_{19}, ..., w_{25}$ in terms of $w_{26} = K$

as follows:

$$
\begin{aligned}
w_{19} &= N{-}K - (\chi(A_4) - \chi(B_1) + \chi(A_1)) \cdot \mathbf{w} \\
w_{20} &= N{-}K - (\chi(A_4) - \chi(B_1) + \chi(A_1) - \chi(B_2) + \chi(A_2)) \cdot \mathbf{w} \\
w_{21} &= N{-}K - (\chi(A_4) - \chi(B_1) + \chi(A_1) - \chi(B_2) + \chi(A_2) - \\
&\qquad \chi(B_3) + \chi(A_3)) \cdot \mathbf{w} \\
w_{22} &= N{-}K - \chi(A_4) \cdot \mathbf{w} \\
w_{23} &= K{-}(-\chi(A_4) + \chi(B_1)) \cdot \mathbf{w} \\
w_{24} &= K{-}(-\chi(A_4) + \chi(B_1) - \chi(A_1) + \chi(B_2)) \cdot \mathbf{w} \\
w_{25} &= K{-}(-\chi(A_4) + \chi(B_1) - \chi(A_1) + \chi(B_2) - \chi(A_2) + \chi(B_3)) \cdot \mathbf{w}.
\end{aligned} \tag{5.10}
$$

By choice of N and K weights $w_{19}, ..., w_{25}$ are positive. Indeed, all "small" terms in the right-hand-side of (5.10) are strictly less then $7 \cdot 18 = 126 < \min\{K, N-K\}$.

Let $\preceq'$ be the representable qualitative probability order on [26] defined by the weight vector $\mathbf{w}' = (w_1, \ldots, w_{26})$. Using $\preceq'$ we would like to construct a linear qualitative probability order $\sqsubseteq$ on $2^{[26]}$ that ranks the subsets A'_i and B'_j in the sequence

$$A'_1 \sqsubset A'_2 \sqsubset A'_3 \sqsubset A'_4 \sqsubset B'_1 \sqsubset B'_2 \sqsubset B'_3 \sqsubset B'_4. \tag{5.11}$$

We will make use of Theorem 5.3.1 now. Let $H_{\mathbf{w}'} = \{x \in \mathbb{R}^n | (\mathbf{w}', x) = 0\}$ be the hyperplane with the normal vector $\mathbf{w}'$ and $S(\preceq')$ be the set of all vectors of the respective discrete cone $C(\preceq')$ that lie in $H_{\mathbf{w}'}$. Suppose

$$X' = \{\chi(C, D) \mid C, D \in \{A'_1, \ldots, A'_4, B'_1, \ldots, B'_4\} \text{ and } D \text{ earlier than } C \text{ in (5.11)}\}.$$

This is a subset of T^{26}, where $T = \{-1, 0, 1\}$. Let also $Y' = S(\preceq') - X'$. Since by construction $e_i \notin S(\preceq')$ for every $i \in [26]$ to use Theorem 5.3.1 with the goal to achieve (5.11) we need to show, that

- $S(\preceq') = X' \cup -X'$ and
- X' is closed under the operation of restricted sum.

If we could prove this, then $C(\sqsubseteq) = C(\preceq') - Y'$ is a discrete cone of a linear qualitative probability order $\sqsubseteq$ on [26] satisfying (5.11). Then the initial segment $\Delta(\sqsubseteq, B'_1)$ will not be a threshold complex, because the condition CC_4^* will fail for it.

Let Y be one of the sets $A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4$. By $\check{Y}$ we will denote the corresponding superset of Y from the set $\{A'_1, A'_2, A'_3, A'_4, B'_1, B'_2, B'_3, B'_4\}$.

Proposition 5.4.1. *The subset*

$$X = \{\chi(C, D) \mid C, D \in \{A_1, \ldots, A_4, B_1, \ldots, B_4\} \text{ with } \check{D} \text{ earlier than } \check{C} \text{ in (5.11)}\}.$$

of T^{18} is closed under the operation of restricted sum.

Proof. Let $\mathbf{u}$ and $\mathbf{v}$ be any two vectors in X. As we will see the restricted sum $\mathbf{u} \oplus \mathbf{v}$ is almost always undefined. Without loss of generality we can consider only five cases.

Case 1. $\mathbf{u} = \chi(B_i, A_j)$ and $\mathbf{v} = \chi(B_k, A_m)$, where $i \neq k$ and $j \neq m$. In this case by Lemma 5.4.1 the pairs (B_i, A_j) and (B_k, A_m) are not compatible. It means that there exists $p \in [18]$ such that either $p \in B_i \cap B_k$ and $p \notin A_j \cup A_m$ or $p \in A_j \cap A_m$ and $p \notin B_i \cup B_k$. The vector $\mathbf{u} + \mathbf{v}$ has 2 or -2 at pth position and $\mathbf{u} \oplus \mathbf{v}$ is undefined. This is illustrated in the table below:

	$\chi(B_i)$	$\chi(B_k)$	$\chi(A_j)$	$\chi(A_m)$	$\chi(B_i, A_j)$	$\chi(B_k, A_m)$	$\mathbf{u} + \mathbf{v}$
pth	1	1	0	0	1	1	2
coordinate	0	0	1	1	-1	-1	-2

Case 2. $\mathbf{u} = \chi(B_i, A_j)$, $\mathbf{v} = \chi(B_i, A_m)$ or $\mathbf{u} = \chi(B_j, A_i)$, $\mathbf{v} = \chi(B_m, A_i)$, where $j \neq m$. In this case choose $k \in [4] - \{i\}$. Then the pairs (B_i, A_j) and (B_k, A_m) are not compatible. As above, the vector $\chi(B_i, A_j) + \chi(B_k, A_m)$ has 2 or -2 at some position p. Suppose $p \in B_i \cap B_k$ and $p \notin A_j \cup A_m$. Then B_i has a 1 in pth position and each of the vectors $\chi(B_i, A_j)$ and $\chi(B_i, A_m)$ has a 1 in pth position as well. Therefore, $\mathbf{u} \oplus \mathbf{v}$ is undefined because $\mathbf{u} + \mathbf{v}$ has 2 in pth position. Similarly, in the case when $p \in A_j \cap A_m$ and $p \notin B_i \cup B_k$ the pth coordinate of $\mathbf{u} + \mathbf{v}$ is -2. The case when $\mathbf{u} = \chi(B_j, A_i)$ and $\mathbf{v} = \chi(B_m, A_i)$ is similar.

Case 3. $\mathbf{u} = \chi(B_i, B_j)$, $\mathbf{v} = \chi(B_k, B_m)$ or $\mathbf{u} = \chi(A_i, A_j)$, $\mathbf{v} = \chi(A_k, A_m)$, where $\{i, j, k, m\} = [4]$. By construction of M there exists $p \in [18]$ such that $p \in B_i \cap B_k$ and $p \notin B_j \cup B_m$ or $p \notin B_i \cup B_k$ and $p \in B_j \cap B_m$. So there is $p \in [18]$, such that $\mathbf{u} + \mathbf{v}$ has 2 or -2 in pth position. Thus $\mathbf{u} \oplus \mathbf{v}$ is undefined.

Case 4. $\mathbf{u} = \chi(B_i, B_j)$, $\mathbf{v} = \chi(B_k, B_m)$ or $\mathbf{u} = \chi(A_i, A_j)$, $\mathbf{v} = \chi(A_k, A_m)$, where $i = k$ or $j = m$. If $i = k$ and $j = m$, then $\mathbf{u} \oplus \mathbf{v}$ is undefined. Consider the case $i = k$, $j \neq m$ and $\mathbf{u} = \chi(B_i, B_j)$, $\mathbf{v} = \chi(B_i, B_m)$. Let $s = [4] - \{i, j, m\}$. By construction of M either we have $p \in [18]$ such that $p \in B_i \cap B_s$ and $p \notin B_j \cup B_m$ or $p \notin B_i \cup B_s$ and $p \in B_j \cap B_m$. In both cases $\mathbf{u} + \mathbf{v}$ has 2 or -2 in position p.

Case 5. $\mathbf{u} = \chi(B_i, B_j)$, $\mathbf{v} = \chi(B_k, B_m)$ or $\mathbf{u} = \chi(A_i, A_j)$, $\mathbf{v} = \chi(A_k, A_m)$, where $j = k$ or $i = m$. Suppose $j = k$. Since $i > j$ and $j > m$ we have $i > m$. This implies that $\chi(B_i, B_m)$ belongs to X. On the other hand $\mathbf{u} + \mathbf{v} = \chi(B_i) - \chi(B_m) = \chi(B_i, B_m)$. Therefore $\mathbf{u} \oplus \mathbf{v} = \mathbf{u} + \mathbf{v} \in X$. □

Corollary 5.4.1. *X' is closed under restricted sum.*

Proof. We will have to consider the same five cases as in the Proposition 5.4.1. As above in the first four cases the restricted sum of vectors will be undefined. In the fifth case, when $\mathbf{u} = \chi(B'_i, B'_j)$, $\mathbf{v} = \chi(B'_k, B'_m)$ or $\mathbf{u} = \chi(A'_i, A'_j)$, $\mathbf{v} = \chi(A'_k, A'_m)$, where $j = k$ or $i = m$, we will have $\mathbf{u} + \mathbf{v} = \chi(B'_i) - \chi(B'_m) = \chi(B'_i, B'_m) \in X'$ or $\mathbf{u} + \mathbf{v} = \chi(A'_i) - \chi(A'_m) = \chi(A'_i, A'_m) \in X'$. □

To satisfy conditions of Theorem 5.3.1 we need also to show that the intersection of the discrete cone $C(\preceq')$ and the hyperplane $H_{\mathbf{w}'}$ equals to $X' \cup -X'$. More explicitly we need to prove the following:

Proposition 5.4.2. *Suppose $C, D \subseteq [26]$ are tied in $\preceq'$, that is $C \preceq' D$ and $D \preceq' C$. Then $\chi(C, D) \in X' \cup -X'$.*

Proof. Assume, to the contrary, that there are two sets $C, D \in 2^{[26]}$ that have equal weights with respect to the corresponding system of weights defining $\preceq'$ but $\chi(C, D) \notin X' \cup -X'$. The sets C and D have to contain some of the elements from $[26] - [18]$ since $w_1, \ldots, w_{18}$ are linearly independent. Thus $C = C_1 \cup C_2$ and $D = D_1 \cup D_2$, where $C_1, D_1 \subseteq [18]$ and $C_2, D_2 \subseteq [26] - [18]$ with C_2 and D_2 being nonempty. We have

$$0 = \chi(C, D) \cdot \mathbf{w}' = \chi(C_1, D_1) \cdot \mathbf{w} + \chi(C_2, D_2) \cdot \mathbf{w}^+,$$

where $\mathbf{w}^+ = (w_{19}, \ldots, w_{26})^T$. By (5.10), we can express weights $w_{19}, \ldots, w_{26}$ as linear combinations with integer coefficients of N, K and $w_1, \ldots, w_{18}$ obtaining

$$\chi(C_2, D_2) \cdot \mathbf{w}^+ = \left(\sum_{i=1}^{4} \gamma_i \chi(A_i) + \sum_{i=1}^{4} \gamma_{4+i} \chi(B_i) \right) \cdot \mathbf{w} + \beta_1 N + \beta_2 K,$$

where $\gamma_i, \beta_j \in \mathbb{Z}$.

Clearly the expression in the brackets on the right-hand-side is just a vector with integer entries. Let us denote it α. Then

$$\chi(C_2, D_2) \cdot \mathbf{w}^+ = \alpha \cdot \mathbf{w} + \beta_1 N + \beta_2 K, \tag{5.12}$$

where $\alpha \in \mathbb{Z}^{18}$. We can now write $\chi(C, D) \cdot \mathbf{w}'$ in terms of $\mathbf{w}, K$ and N:

$$0 = \chi(C, D) \cdot \mathbf{w}' = (\chi(C_1, D_1) + \alpha) \cdot \mathbf{w} + \beta_1 N + \beta_2 K.$$

We recap that K was chosen to be much greater then $\sum_{i\in[18]} w_i$ and N is much greater then K. So if β_1, β_2 are different from zero then $|\beta_1 N + \beta_2 K|$ is a very big number, which cannot be canceled out by $(\chi(C_1, D_1) + \alpha) \cdot \mathbf{w}$. Weights $w_1, \ldots, w_{18}$ are linearly independent, so for arbitrary $\mathbf{b} \in Z^{18}$ the dot product $\mathbf{b} \cdot \mathbf{w}$ can be zero if and only if $\mathbf{b} = \mathbf{0}$. Hence

$$w(C) = w(D) \text{ iff } \chi(C_1, D_1) = -\alpha \text{ and } \beta_1 = 0, \beta_2 = 0.$$

Taking into account that $\chi(C_1, D_1)$ is a vector from T^{18}, we get

$$\alpha \notin T^{18} \Longrightarrow w(C) \neq w(D).$$

We need the following two claims to finish the proof, their proofs are delegated to the next section.

Claim 1. *Suppose $\chi(C_1, D_1)$ belongs to $X \cup -X$. Then $\chi(C, D)$ belongs to $X' \cup -X'$.*

Claim 2. *If $\alpha \in T^{18}$, then α belongs to $X \cup -X$.*

Now let us show how with the help of these two claims the proof of Proposition 5.4.2 can be completed. The sets C and D have the same weight and this can happen only if α is a vector in T^{18}. By Claim 2 $\alpha \in X \cup -X$. The characteristic vector $\chi(C_1, D_1)$ is equal to $-\alpha$, hence $\chi(C_1, D_1) \in X \cup -X$. By Claim 1 we get $\chi(C, D) \in X' \cup -X'$, a contradiction. $\square$

Theorem 5.4.1. *There exists a linear qualitative probability order $\sqsubseteq$ on $[26]$ and $T \subset [26]$ such that the initial segment $\Delta(\sqsubseteq, T)$ is not a threshold complex.*

Proof. All weights are positive so $\{e_1, \ldots, e_{26}\} \cap S(\preceq') = \emptyset$. Then by Corollary 5.4.1 and Proposition 5.4.2 all conditions of Theorem 5.3.1 are satisfied. Therefore $C(\preceq')-(-X')$ is a discrete cone $C(\sqsubseteq)$, where $\sqsubseteq$ is an almost representable linear qualitative probability order. By construction $A'_1 \sqsubset A'_2 \sqsubset A'_3 \sqsubset A'_4 \sqsubset B'_1 \sqsubset B'_2 \sqsubset B'_3 \sqsubset B'_4$ and thus $\Delta(\sqsubseteq, B'_1)$ is an initial segment, which is not a threshold complex. $\square$

Note that we have a significant degree of freedom in constructing such an example. The matrix M can be chosen in 2^{18} possible ways and we have not specified the linear qualitative probability order $\preceq$.

5.5 Proofs of Claim 1 and Claim 2

Lets fix some notation first. Suppose $\mathbf{b} \in \mathbb{Z}^k$ and $\mathbf{x}_i \in \mathbb{Z}^n$ for $i \in [k]$. Then we define the product

$$\mathbf{b} \cdot (\mathbf{x}_1, \ldots, \mathbf{x}_k) = \sum_{i \in [k]} b_i \mathbf{x}_i.$$

It resembles the dot product (the difference is that the second argument is a sequence of vectors) and is denoted in the same way. For a sequence of vectors $(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ we also define $(\mathbf{x}_1, \ldots, \mathbf{x}_k)_p = (\mathbf{x}_1^{(p)}, \ldots, \mathbf{x}_k^{(p)})$, where $\mathbf{x}_i^{(j)}$ is the jth coordinate of vector $\mathbf{x}_i$.

We start with the following lemma.

Lemma 5.5.1. *Let* $\mathbf{b} \in \mathbb{Z}^6$. *Then*

$$\mathbf{b} \cdot (\chi(B_1, A_4), \chi(B_2, A_1), \chi(B_3, A_2), \chi(A_2, A_1), \chi(A_3, A_1), \chi(A_4, A_1)) = \mathbf{0}$$

if and only if $\mathbf{b} = \mathbf{0}$.

Proof. We know that the pairs (B_1, A_4) and (B_2, A_1) are not compatible. So there exists an element p that lies in the intersection $B_1 \cap B_2$ (or $A_1 \cap A_4$), but $p \notin A_4 \cup A_1$ ($p \notin B_1 \cup B_2$, respectively). We have exactly two copies of every element among $A_1, \ldots, A_4$ and $B_1, \ldots, B_4$. Thus, the element p belongs to $A_2 \cap A_3$ ($B_3 \cap B_4$) and doesn't belong to $B_3 \cup B_4$ ($A_2 \cup A_3$). The following table illustrates this:

	$\chi(A_1)$	$\chi(A_2)$	$\chi(A_3)$	$\chi(A_4)$	$\chi(B_1)$	$\chi(B_2)$	$\chi(B_3)$	$\chi(B_4)$
pth	0	1	1	0	1	1	0	0
coordinate	1	0	0	1	0	0	1	1

Then at pth position we have

$$(\chi(B_1, A_4), \chi(B_2, A_1), \chi(B_3, A_2), \chi(A_2, A_1), \chi(A_3, A_1), \chi(A_4, A_1))_p = \pm(1, 1, -1, 1, 1, 0)$$

and hence

$$b_1 + b_2 - b_3 + b_4 + b_5 = 0.$$

From the fact that other pairs are not compatible we can get more equations relating

$b_1, \ldots, b_6$:

$$\begin{aligned}
b_1 - b_2 + b_3 - b_4 - b_6 &= 0 && \text{from} \quad (B_1, A_4), (B_3, A_2);\\
-b_1 + b_2 + b_3 + b_5 + b_6 &= 0 && \text{from} \quad (B_1, A_4), (B_4, A_3);\\
b_2 + b_5 + b_6 &= 0 && \text{from} \quad (B_1, A_1), (B_2, A_2);\\
b_4 + b_6 &= 0 && \text{from} \quad (B_1, A_1), (B_3, A_3);\\
b_3 + b_5 + b_6 &= 0 && \text{from} \quad (B_1, A_1), (B_3, A_2).
\end{aligned}$$

The obtained system of linear equations has only the zero solution. □

Lemma 5.5.2. *Let* $\mathbf{a} = (a_1, \ldots, a_8)$ *be a vector in* $\mathbb{Z}^8$ *whose every coordinate* a_i *has absolute value which is at most* 100. *Then* $\mathbf{a} \cdot \mathbf{w}^+ = 0$ *if and only if* $\mathbf{a} = \mathbf{0}$.

Proof. We first rewrite (5.10) in more convenient form:

$$\begin{aligned}
w_{19} &= N - K - (-\chi(B_1, A_4) + \chi(A_1)) \cdot \mathbf{w}\\
w_{20} &= N - K - (-\chi(B_1, A_4) - \chi(B_2, A_1) + \chi(A_2)) \cdot \mathbf{w}\\
w_{21} &= N - K - (-\chi(B_1, A_4) - \chi(B_2, A_1) - \chi(B_3, A_2) + \chi(A_3)) \cdot \mathbf{w}\\
w_{22} &= N - K - \chi(A_4) \cdot \mathbf{w}\\
w_{23} &= K - \chi(B_1, A_4) \cdot \mathbf{w}\\
w_{24} &= K - (\chi(B_1, A_4) + \chi(B_2, A_1)) \cdot \mathbf{w}\\
w_{25} &= K - (\chi(B_1, A_4) + \chi(B_2, A_1) + \chi(B_3, A_2)) \cdot \mathbf{w}\\
w_{26} &= K
\end{aligned} \tag{5.13}$$

We calculate the dot product $\mathbf{a} \cdot \mathbf{w}^+$ substituting the values of $w_{19}, \ldots, w_{26}$ from (5.13):

$$\begin{aligned}
0 = \mathbf{a} \cdot \mathbf{w}^+ = N \sum_{i \in [4]} a_i - K \left(\sum_{i \in [4]} a_i - \sum_{i \in [4]} a_{4+i} \right)&\\
- \Big[\chi(B_1, A_4) \left(\sum_{i=5}^{7} a_i - \sum_{i=1}^{3} a_i \right) + \chi(B_2, A_1) \left(\sum_{i=6}^{7} a_i - \sum_{i=2}^{3} a_i \right)&\\
+ \chi(B_3, A_2)(-a_3 + a_7) + \sum_{i \in [4]} a_i \chi(A_i) \Big] \cdot \mathbf{w}.&
\end{aligned} \tag{5.14}$$

The numbers N and K are very big and $\sum_{i \in [18]} w_i$ is small. Also $|a_i| \leq 100$. Hence the three summands cannot cancel each other. Therefore $\sum_{i \in [4]} a_i = 0$ and $\sum_{i \in [4]} a_{4+i} = 0$. The expression in the square brackets should be zero because the coordinates of $\mathbf{w}$ are linearly independent.

We know that $a_1 = -a_2 - a_3 - a_4$, so the expression in the square brackets in (5.14) can be rewritten in the following form:

$$\begin{aligned} &b_1\chi(B_1, A_4) + b_2\chi(B_2, A_1) + b_3\chi(B_3, A_2)+ \\ &a_2\chi(A_2, A_1) + a_3\chi(A_3, A_1) + a_4\chi(A_4, A_1), \end{aligned} \tag{5.15}$$

where $b_1 = \sum_{i=5}^{7} a_i - \sum_{i=1}^{3} a_i$, $b_2 = \sum_{i=6}^{7} a_i - \sum_{i=2}^{3} a_i$ and $b_3 = a_7 - a_3$.

By Lemma 5.5.1 we can see that expression (5.15) is zero iff $b_1 = 0, b_2 = 0, b_3 = 0$ and $a_2 = 0, a_3 = 0, a_4 = 0$ and this happens iff $\mathbf{a} = \mathbf{0}$. □

Proof of Claim 1. Assume, to the contrary, that $\chi(C_1, D_1) \in X \cup -X$ and $\chi(C, D)$ does not belong to $X' \cup -X'$. Consider $\chi(\check{C}_1, \check{D}_1) \in X' \cup -X'$. We know that the weight of C is the same as the weight of D, and also that the weight of $\check{C}_1$ is the same as the weight of $\check{D}_1$. This can be written as

$$\begin{aligned} &\chi(C_1, D_1) \cdot \mathbf{w} + \chi(C_2, D_2) \cdot \mathbf{w}^+ = 0, \\ &\chi(C_1, D_1) \cdot \mathbf{w} + \chi(\check{C}_1 - C_1, \check{D}_1 - D_1) \cdot \mathbf{w}^+ = 0. \end{aligned}$$

We can now see that

$$(\chi(\check{C}_1 - C_1, \check{D}_1 - D_1) - \chi(C_2, D_2)) \cdot \mathbf{w}^+ = 0.$$

The left-hand-side of the last equation is a linear combination of weights $w_{19}, \ldots, w_{26}$. Due to Lemma 5.5.2 we conclude from here that

$$\chi(\check{C}_1 - C_1, \check{D}_1 - D_1) - \chi(C_2, D_2) = \mathbf{0}.$$

But this is equivalent to $\chi(C, D) = \chi(\check{C}_1, \check{D}_1) \in X' \cup -X'$, which is a contradiction. □

Proof of Claim 2. We remind the reader that α was defined in (5.12). Sets C and D have the same weight and we established that $\beta_1 = \beta_2 = 0$. So

$$\chi(C_2, D_2) \cdot \mathbf{w}^+ = \alpha \cdot \mathbf{w}.$$

If we look at the representation of the last eight weights in (5.13), we note that the weights w_{19}, w_{20}, w_{21}, w_{22} are much heavier than the weights w_{23}, w_{24}, w_{25}, w_{26}.

Hence $w(C) = w(D)$ implies

$$\begin{aligned}|C_2 \cap \{19,20,21,22\}| &= |D_2 \cap \{19,20,21,22\}| \text{ and}\\ |C_2 \cap \{23,24,25,26\}| &= |D_2 \cap \{23,24,25,26\}|.\end{aligned} \tag{5.16}$$

That is C and D have equal number of super-heavy weights and equal number of heavy ones.

Without loss of generality we can assume that $C_2 \cap D_2$ is empty. Similar to derivation in the proof of Lemma 5.5.2, the vector α can be expressed as

$$\alpha = a_1\chi(B_1, A_4) + a_2\chi(B_2, A_1) + a_3\chi(B_3, A_2) + \sum_{i\in[4]} b_i\chi(A_i) \tag{5.17}$$

for some $a_i, b_j \in \mathbb{Z}$. The characteristic vectors $\chi(A_1), \ldots, \chi(A_4)$ participate in the representations of super-heavy elements $w_{19}, \ldots, w_{22}$ only. Hence $b_i = 1$ iff element $18 + i \in C_2$ and $b_i = -1$ iff element $18 + i \in D_2$. Without loss of generality we can assume that $C_2 \cap D_2 = \emptyset$. By (5.16) we can see that if C_2 contains some super-heavy element $p \in \{19, \ldots, 22\}$ with $\chi(A_k)$, $k \in [4]$, in the representation of w_p, then D_2 has a super-heavy $q \in \{19, \ldots, 22\}$, $q \neq p$ with $\chi(A_t), t \in [4] - \{k\}$ in representation of w_q. In such case $b_k = -b_t = 1$ and

$$b_k\chi(A_k) + b_t\chi(A_t) = \chi(A_k, A_t).$$

By (5.16) the number of super-heavy element in C_2 is the same as the number of super-heavy elements in D_2. Therefore (5.17) can be rewritten in the following way:

$$\alpha = a_1\chi(B_1, A_4) + a_2\chi(B_2, A_1) + a_3\chi(B_3, A_2) + a_4\chi(A_i, A_p) + a_5\chi(A_k, A_t),$$

where $a_1, a_2, a_3 \in \mathbb{Z}$; $a_4, a_5 \in \{0, 1\}$ and $\{i, k, t, p\} = [4]$.

Now the series of technical facts will finish the proof.

Fact 1. *Suppose* $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{Z}^3$ *and* $|\{i, k, t\}| = |\{j, m, s\}| = 3$. *Then*

$$a_1\chi(B_j, A_i) + a_2\chi(B_m, A_k) + a_3\chi(B_s, A_t) \in T^{18}$$

if and only if

$$\mathbf{a} \in \{(0,0,0),\ (\pm1,0,0),\ (0,\pm1,0),\ (0,0,\pm1),\ (1,1,1),\ (-1,-1,-1)\}. \tag{5.18}$$

Proof. The pairs $((B_j, A_i), (B_m, A_k))$, $((B_j, A_i), (B_s, A_t))$ and $((B_m, A_k), (B_s, A_t))$ are not compatible. Using the same technique as in the proofs of Proposition 5.4.1 and Lemma 5.5.1 and watching a particular coordinate we get

$$(a_1 + a_2 - a_3),\ (a_1 - a_2 + a_3),\ (-a_1 + a_2 + a_3) \in T,$$

respectively. The absolute value of the sum of every two of these terms is at most two. Add the first term to the third. Then $|2a_2| \leq 2$ or, equivalently, $|a_2| \leq 1$. In a similar way we can show that $|a_3| \leq 1$ and $|a_1| \leq 1$. The only vectors that satisfy all the conditions above are those listed in (5.18). □

Fact 2. *Suppose* $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{Z}^3$ *and* $|\{i, k, t\}| = |\{j, m, s\}| = 3$. *Then*

$$a_1\chi(B_j, A_i) + a_2\chi(B_m, A_k) + a_3\chi(B_s, A_t) + \chi(A_k, A_t) \in T^{18}$$

if and only if

$$\mathbf{a} \in \{(0,0,0),\ (0,1,0),\ (0,0,-1),\ (0,1,-1)\}.$$

Proof. Considering non-compatible pairs $((B_m, A_k), (B_s, A_t))$, $((B_j, A_i), (B_m, A_k))$, $((B_j, A_i), (B_s, A_t))$, $((B_j, A_k), (B_s, A_i))$, $((B_j, A_t), (B_m, A_i))$, we get the inclusions

$$(-a_1 + a_2 + a_3),\ (a_1 + a_2 - a_3 - 1),\ (a_1 - a_2 + a_3 + 1),\ (a_1 - 1),\ (a_1 + 1) \in T,$$

respectively. We can see that $|2a_2 - 1| \leq 2$ and $|2a_3 + 1| \leq 2$ and $a_1 = 0$. So a_2 can be only 0 or 1 and a_3 can have values -1 or 0. □

Fact 3. *Suppose* $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{Z}^3$ *and* $\{i, k, t, p\} = [4]$ *and* $|\{j, m, s\}| = 3$. *Then*

$$a_1\chi(B_j, A_i) + a_2\chi(B_m, A_k) + a_3\chi(B_s, A_t) + \chi(A_i, A_p) \in T^{18}$$

if and only if

$$a \in \{(0,0,0),\ (1,0,0),\ (1,1,1),\ (2,1,1)\}.$$

Proof. Let $\ell \in [4] - \{j, m, s\}$. From consideration of the following non-compatible pairs

$$((B_j, A_i), (B_m, A_k)),\ ((B_j, A_i), (B_s, A_t)),\ ((B_m, A_k), (B_s, A_t)),\ ((B_j, A_i), (B_m, A_t)),$$
$$((B_j, A_i), (B_m, A_p)),\ ((B_j, A_i), (B_s, A_p)),\ ((B_s, A_t), (B_\ell, A_i))$$

we get the following inclusions

$$(a_1 + a_2 - a_3 - 1),\ (a_1 - a_2 + a_3 - 1),\ (-a_1 + a_2 + a_3),$$
$$(a_1 - 1),\ (a_1 - a_3),\ (a_1 - a_2),\ (a_2 - a_3 + 1) \in T,$$

respectively. So we have $|2a_3 - 1| \leq 2$ (from the second and the third inclusions) and $|2a_2 - 1| \leq 2$ (from the first and the third inclusions) from which we immediately get $a_2, a_3 \in \{1, 0\}$. We also get $a_1 \in \{2, 1, 0\}$ (by the forth inclusion).

- If $a_1 = 2$, then by the fifth and sixth inclusions $a_3 = 1$ and $a_2 = 1$.
- If $a_1 = 1$, then a_2 can be either zero or one. If $a_2 = 0$ then we have $\chi(B_j, A_i) + a_3\chi(B_s, A_t) + \chi(A_i, A_p) = \chi(B_j, A_p) + a_3\chi(B_s, A_t)$. By Fact 1, a_3 can be zero only. On the other hand, if $a_2 = 1$, then $a_3 = 1$ by the seventh inclusion.
- If $a_1 = 0$ then a_2 can be a 0 or a 1. Suppose $a_2 = 0$. Then $a_3 = 0$ by the first two inclusions. Assume $a_2 = 1$. Then $a_3 = 0$ by the third inclusion and on the other hand $a_3 = 1$ by the second inclusion, a contradiction.

This proves the statement. □

Fact 4. *Suppose* $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{Z}^3$ *and* $\{i, k, t, p\} = [4]$ *and* $|\{j, m, s\}| = 3$. *Then*

$$a_1\chi(B_j, A_i) + a_2\chi(B_m, A_k) + a_3\chi(B_s, A_t) + \chi(A_i, A_p) + \chi(A_k, A_t) \notin T^{18}.$$

Proof. Let $\ell \in [4] - \{j, m, s\}$. Using the same technique as above from consideration of non-compatible pairs

$$((B_j, A_i), (B_m, A_t)),\ ((B_s, A_t), (B_j, A_k)),\ ((B_j, A_i), (B_s, A_t)),$$
$$((B_m, A_k), (B_s, A_t)),\ ((B_j, A_i), (B_m, A_p)),\ ((B_j, A_i), (B_\ell, A_k))$$

we obtain inclusions:

$$a_1,\ a_3,\ (a_1 - a_2 + a_3),\ (-a_1 + a_2 + a_3),\ (a_1 - a_3),\ (a_1 - a_3 - 2) \in T,$$

respectively.

From the last two inclusions we can see that $a_1 - a_3 = 1$. This, together with the first and the second inclusions, imply $(a_1, a_3) \in \{(1, 0), (0, -1)\}$. Suppose $(a_1, a_3) = (1, 0)$. Then

$$\chi(B_j, A_i) + a_2\chi(B_m, A_k) + \chi(A_i, A_p) + \chi(A_k, A_t) = \chi(B_j, A_p) + a_2\chi(B_m, A_k) + \chi(A_k, A_t).$$

By Fact 3, it doesn't belong to T^{18} for any value of a_2.

Suppose now that $(a_1, a_3) = (0, -1)$. Then by the third and the forth inclusions a_2 can be only zero. Then $\mathbf{a} = (0, 0, -1)$ and

$$-\chi(B_s, A_t) + \chi(A_i, A_p) + \chi(A_k, A_t) = -\chi(B_s, A_k) + \chi(A_i, A_p).$$

However, by Fact 3 the right-hand-side of this equation is not a vector of T^{18}. □

Fact 5. *Suppose* $\mathbf{a} \in \mathbb{Z}^5$ *and*

$$\mathbf{v} = a_1\chi(B_j, A_i) + a_2\chi(B_m, A_k) + a_3\chi(B_s, A_t) + a_4\chi(A_i, A_p) + a_5\chi(A_k, A_t).$$

If $a_4, a_5 \in \{0, 1, -1\}$ *and* $v \in T^{18}$*, then* v *belongs to* X *or* $-X$*.*

Proof. First of all, we will find the possible values of $\mathbf{a}$ in case $\mathbf{v} \in T^{18}$. By Facts 1 –4 one can see that $\mathbf{v} \in T^{18}$ iff $\mathbf{a}$ belongs to the set

$$\begin{aligned} Q = \{&(0,0,0,0,0),\ (\pm1,0,0,0,0),\ (0,\pm1,0,0,0),\ (0,0,\pm1,0,0),\ (1,1,1,0,0),\\ &(0,0,0,\pm1,0),\ (\pm1,0,0,\pm1,0),\ (\pm1,\pm1,\pm1,\pm1,0),\ (\pm2,\pm1,\pm1,\pm1,0),\\ &(0,0,0,0,\pm1),\ (0,\pm1,0,0,\pm1),\ (0,0,\mp1,0,\pm1),\ (0,\pm1,\mp1,0,\pm1)\}. \end{aligned}$$

By the construction of $\preceq$ the sequence $(A_1, \ldots, A_4; B_1, \ldots, B_4)$ is a trading transform. So for every $\{i_1, \ldots, i_4\} = \{j_1, \ldots, j_4\} = [4]$ the equation

$$\chi(B_{i_1}, A_{j_1}) + \chi(B_{i_2}, A_{j_2}) + \chi(B_{i_3}, A_{j_3}) + \chi(B_{i_4}, A_{j_4}) = 0. \tag{5.19}$$

holds. Taking (5.19) into account one can show, that for every $\mathbf{a} \in Q$, vector $\mathbf{v}$ belongs to X or $-X$. For example, if $\mathbf{a} = (2, 1, 1, 1, 0)$ then

$$\begin{aligned} &2\chi(B_j, A_i) + \chi(B_m, A_k) + \chi(B_s, A_t) + \chi(A_i, A_p) = \\ &\qquad\qquad \chi(B_j, A_i) - \chi(B_\ell, A_p) + \chi(A_i, A_p) = \chi(B_j, B_\ell), \end{aligned}$$

where $\ell \in [4] - \{j, m, s\}$. □

One can see that $\mathbf{v}$ from the Fact 5 is the general form of α. Hence $\alpha \in T^{18}$ if and only if $\alpha \in X \cup -X$ which is Claim 2. □

5.6 Acyclic Games and a Conjectured Characterization

So far we have shown that the initial segment complexes strictly contain the threshold complexes and are strictly contained within the shifted complexes. In this section we introduce some ideas from the theory of simple games to formulate a conjecture that characterizes initial segment complexes. The idea in this section is to start with a simplicial complex and see if there is a natural linear order available on $2^{[n]}$ which gives a qualitative probability order and has the original complex as an initial segment. We will follow the presentation of Taylor and Zwicker (1999).

Let $\Delta \subseteq 2^{[n]}$ be a simplicial complex. Define the **Winder desirability relation**, $\leq_W$, on $2^{[n]}$ by $A \leq_W B$ if and only if for every $Z \subseteq [n] - ((A - B) \cup (B - A))$ we have that

$$(A - B) \cup Z \notin \Delta \Rightarrow (B - A) \cup Z \notin \Delta.$$

Furthermore define the **Winder existential ordering**, $\prec_W$, on $2^{[n]}$ to be

$$A \prec_W B \iff \text{It is not the case that } B \leq_W A.$$

Definition 5.6.1. *A simplicial complex Δ is called* **strongly acyclic** *if there are no k-cycles*

$$A_1 \prec_W A_2 \prec_W \cdots A_k \prec_W A_1$$

for any k in the Winder existential ordering.

Theorem 5.6.1. *Suppose $\preceq$ is a qualitative probability order on $2^{[n]}$ and $T \in 2^{[n]}$. Then the initial segment $\Delta(\preceq, T)$ is strongly acyclic.*

Proof. Let $\Delta = \Delta(\preceq, T)$. It follows from the definition that $A \prec_W B$ if and only if there exists a $Z \in [n] - ((A - B) \cup (B - A))$ such that $(A - B) \cup Z \in \Delta$ and $(B - A) \cup Z \notin \Delta$. From the definition of Δ it follows that

$$(A - B) \cup Z \prec (B - A) \cup Z$$

which, by de Finneti's axiom 5.2, implies

$$A - B \prec B - A$$

and hence, again by de Finneti's axiom,

$$A \prec B.$$

Thus a k-cycle

$$A_1 \prec_W \cdots \prec_W A_k \prec_W A_1$$

in Δ would imply a k-cycle

$$A_1 \prec \cdots \prec A_k \prec A_1$$

which contradicts that $\prec$ is a total order. □

Conjecture 1. *A simplicial complex Δ is an initial segment complex if and only if it is strongly acyclic.*

We will return momentarily to give some support for Conjecture 1. First, however, it is worth noting that the necessary condition of being strongly acyclic from Theorem 5.6.1 allows us to see that there is little relationship between being an initial segment complex and satisfying the conditions CC_k^*.

Corollary 5.6.1. *For every $M > 0$ there exist simplicial complexes that satisfy CC_M^* but are not initial segment complexes.*

Proof. Taylor and Zwicker (Taylor & Zwicker, 1999) construct a family of complexes $\{G_k\}$, which they call Gabelman games that satisfy CC_{k-1}^* but not CC_k^*. They then show (Taylor & Zwicker, 1999, Corollary 4.10.7) that none of these examples are strongly acyclic. The result then follows from Theorem 5.6.1. □

Our evidence in support of Conjecture 1 is based on the idea that the Winder existential order can be used to produce the related qualitative probability order for strongly acyclic complexes. Here are two lemmas that give some support for this belief:

Lemma 5.6.1. *If Δ is a simplicial complex with $A \in \Delta$ and $B \notin \Delta$ then $A \prec_W B$.*

Proof. Let $Z = A \cap B$. Then

$$(A - B) \cup Z = A \in \Delta$$
$$(B - A) \cup Z = B \notin \Delta,$$

and so $A \prec_W B$. □

Lemma 5.6.2. *For any Δ, the Winder existential order $\prec_W$ satisfies the property*

$$A \prec_W B \Longleftrightarrow A \cup D \prec_W B \cup D$$

for all D disjoint from $A \cup B$.

Proof. See (Taylor & Zwicker, 1999, Proposition 4.7.8). $\square$

This pair of lemmas leads to a slightly stronger version of Conjecture 1.

Conjecture 2. *If Δ is strongly acyclic then there exists an extension of $\prec_W$ to a qualitative probability order.*

What are the barriers to proving Conjecture 2? The Winder order need not be transitive. In fact there are examples of threshold complexes for which $\prec_W$ is not transitive (Taylor & Zwicker, 1999, Proposition 4.7.3). Thus one would have to work with the transitive closure of $\prec_W$, which does not seem to have a tractable description. In particular we do not know if the analogue to Lemma 5.6.2 holds for the transitive closure of $\prec_W$.

5.7 Conclusion

In this chapter we have begun the study of a class of simplicial complexes that are combinatorial generalizations of threshold complexes derived from qualitative probability orders. We have shown that this new class of complexes strictly contains the threshold complexes and is strictly contained in the shifted complexes. Although we can not give a complete characterization of the complexes in question, we conjecture that they are the strongly acyclic complexes that arise in the study of cooperative games. We hope that this conjecture will draw attention to the ideas developed in game theory which we believe to be too often neglected in the combinatorial literature.

Appendix A

Proof of Theorem 3.4.1

Proof of Theorem 3.4.1. In one direction the statement is clear: if a linear combination (3.11) with coefficients $r_1, \dots, r_m$ exists, then there are no non-negative solutions for $A\mathbf{x} \geq \mathbf{0}$, other than $\mathbf{x} = \mathbf{0}$. Indeed, suppose $\mathbf{s} = (s_1, \dots, s_n)$ is such a solution. Denote $\mathbf{r} = (r_1, \dots, r_m)$. If $\mathbf{x} = (x_1, \dots, x_n) \geq \mathbf{0}$ is a non-zero solution, then (3.11) implies $(\mathbf{r}A)\mathbf{x} = s_1x_1 + \dots + s_nx_n < 0$, which is impossible since $\mathbf{r}(A\mathbf{x}) \geq 0$.

Let us prove the reverse statement by induction. For $n = 1$ the matrix A is an $m \times 1$ matrix system reduces to $a_{11}x_1 \geq 0, \dots, a_{m1}x_1 \geq 0$. Since it has no positive solutions we have $a_{i1} < 0$ for some i. Suppose $a_{i1} = -\frac{s}{t}$, where s, t are integers and the fraction $\frac{s}{t}$ is in lowest possible terms. Then we can take $r_i = t$ and $r_j = 0$ for $j \neq i$ and obtain $r_1\mathbf{a}_1 + r_2\mathbf{a}_2 + \cdots + r_m\mathbf{a}_m = ta_{i1} = -s < 0$ and s is an integer.

Suppose now that the statement is proved for all $m \times k$ matrices A with $k < n$. Suppose now A is an $m \times n$ matrix and the system $A\mathbf{x} \geq \mathbf{0}$ has no non-negative solutions other than $\mathbf{x} = \mathbf{0}$.

Suppose that a certain column, say the jth one, has no positive coefficients. Then we may drop this column and the resulting system will still have no non-zero non-negative solutions (otherwise we can take it, add $x_j = 0$, and obtain a non-zero non-negative solution for the original system). By the induction hypothesis for the reduced system we can find non-negative integers $r_1, \dots, r_m$ and a vector $\mathbf{u}$ such that (3.11) is true. Then the same $r_1, \dots, r_m$ will work also for the original system.

We may now assume that any variable has positive coefficients. Let us consider the variable x_1. Its coefficients are not all negative but they are not all positive either (otherwise the system would have a non-zero non-negative solution $\mathbf{x} = (1, 0, \dots, 0)$). Multiplying, if necessary, the rows of A by positive rational numbers, we find that our system is equivalent to a system of the form

$$\begin{array}{rcl} x_1 - f_i(x_2, , \dots, x_n) & \geq & 0, \\ -x_1 + g_j(x_2, , \dots, x_n) & \geq & 0, \\ h_p(x_2, , \dots, x_n) & \geq & 0, \end{array}$$

$i = 1, \dots, k,\ j = 1, \dots, m,\ p = 1, \dots, \ell$, where $f_i(x_2, , \dots, x_n)$, $g_j(x_2, , \dots, x_n)$ and $h_p(x_2, , \dots, x_n)$ are linear functions in $x_2, , \dots, x_n$, and $k \geq 1$, $m \geq 1$. The matrix of such system has the rows: $\mathbf{U}_i = (1, \mathbf{u}_i)$, $\mathbf{V}_j = (-1, \mathbf{v}_j)$, $\mathbf{W}_s = (0, \mathbf{w}_s)$, where $i = 1, \dots, k,\ j = 1, \dots, m,\ s = 1, \dots, \ell$. Then the following system of $km + m + \ell$ inequalities

$$\begin{array}{rcl} g_i(x_2, , \dots, x_n) & \geq & f_j(x_2, , \dots, x_n), \\ g_i(x_2, , \dots, x_n) & \geq & 0, \\ h_s(x_2, , \dots, x_n) & \geq & 0, \end{array}$$

$i = 1, \dots, k,\ j = 1, \dots, m,\ s = 1, \dots, \ell$, has no non-negative solutions other than $x_2 = \dots = x_n = 0$. Indeed, if such a solution $(x_2, \dots, x_n)$ is found then we can set $x_1 = \min g_i(x_2, , \dots, x_n)$ and since this minimum is non-negative to obtain a non-zero solution of the original inequality.

By the induction hypothesis there exist non-negative integers $a_{i,j}$, d_j, e_t, s, where $i = 1, \dots, k,\ j = 1, \dots, m,\ t = 1, \dots, \ell$ such that at least one of these integers positive and

$$\sum_{j=1}^{m} \sum_{i=1}^{k} a_{i,j}(\mathbf{u}_i + \mathbf{v}_j) + \sum_{j=1}^{m} d_j \mathbf{v}_j + \sum_{t=1}^{\ell} e_t \mathbf{w}_t + \mathbf{s}_1 = \mathbf{0}, \qquad \text{(A.1)}$$

where $\mathbf{s} \in \mathbb{R}^{n-1}$ has all its coefficients non-negative. Let $B_i = \sum_{j=1}^{m} a_{i,j}$, $C_j = \sum_{i=1}^{k} a_{i,j}$, $d = \sum_{j=1}^{m} d_j$. Then $\sum_{i=1}^{k} B_i = \sum_{j=1}^{m} C_j$, and (A.1) can be rewritten as

$$\sum_{j=1}^{k} B_i \mathbf{U}_i + \sum_{j=1}^{m} (C_j + d_j) \mathbf{V}_j + \sum_{t=1}^{\ell} e_t \mathbf{W}_t + d\mathbf{e}_1 + (0, \mathbf{s}_1) = \mathbf{0}.$$

If $d \neq 0$, then we have finished the proof. If not, then we found the numbers $r_1^1, \dots, r_m^1$ such that

$$r_1^1 \mathbf{a}_1 + \dots + r_m^1 \mathbf{a}_m + (0, \mathbf{s}_1) = \mathbf{0}.$$

Similarly, we can find the numbers $r_1^2, \ldots, r_m^2$ such that

$$r_1^2\mathbf{a}_1 + \ldots + r_m^2\mathbf{a}_m + (\mathbf{s}_2, 0) = \mathbf{0}.$$

But then we can set $r_i = r_1^1 + r_1^2$, $i = 1, \ldots, m$ and $\mathbf{s} = (0, \mathbf{s}_1) + (\mathbf{s}_2, 0)$ and obtain (3.11). Now all coordinates of $\mathbf{s}$ are positive, hence the statement is proved. □

Appendix B

Examples of critical simple games for every number of the 6th spectrum

α	Minimal winning coalitions and maximal loosing coalitions	Weight representation
$\frac{18}{17}$	$W^{min} = \{\{2,3,4\},\{2,3,6\},\{2,4,6\},\{2,5,6\},$ $\{1,2,4,5\}\{1,3,4,6\},\{1,3,5,6\}\},$ $L^{max} = \{\{1,2,4\},\{1,2,6\},\{1,3,6\},\{2,4,5\},$ $\{1,2,3,5\},\{1,3,4,5\},\{1,4,5,6\},\{3,4,5,6\}\}$	$w_1 = 3\backslash 17, w_2 = 8\backslash 17,$ $w_3 = 5\backslash 17, w_4 = 4\backslash 17,$ $w_5 = 2\backslash 17, w_6 = 7\backslash 17$
$\frac{17}{16}$	$W^{min} = \{\{1,2\},\{2,4,6\},\{2,5,6\},\{1,3,4,6\},$ $\{1,3,5,6\},\{1,4,5,6\},\{2,3,4,5\},\{3,4,5,6\}\},$ $L^{max} = \{\{1,3,6\},\{1,4,6\},\{1,5,6\},\{2,3,4\},\{2,3,5\},$ $\{2,3,6\},\{2,4,5\},\{3,4,6\},\{3,5,6\},\{4,5,6\},\{1,3,4,5\}\}$	$w_1 = 7\backslash 16, w_2 = 9\backslash 16,$ $w_4 = w_5 = 4\backslash 16,$ $w_3 = 2\backslash 16, w_6 = 6\backslash 16$
$\frac{16}{15}$	$W^{min} = \{\{1,2\},\{1,3,5\},\{1,3,6\},\{1,4,5\},\{1,4,6\},$ $\{1,5,6\},\{2,3,4\},\{2,4,6\},\{2,5,6\},\{4,5,6\},\{3,4,5,6\}\},$ $L^{max} = \{\{1,5\},\{1,6\},\{1,3,4\},\{2,3,5\},\{2,3,6\},$ $\{2,4,5\},\{3,4,5\},\{3,4,6\},\{3,5,6\}\}$	$w_1 = 8\backslash 15, w_2 = 7\backslash 15,$ $w_3 = 3\backslash 15, w_4 = 5\backslash 15,$ $w_5 = 4\backslash 15, w_6 = 6\backslash 15$
$\frac{15}{14}$	$W^{min} = \{\{1,3\},\{1,6\},\{1,2,4\},\{1,2,5\},$ $\{2,3,6\},\{3,5,6\},\{4,5,6\},\{2,3,4,5\}\},$ $L^{max} = \{\{1,2\},\{1,4,5\},\{2,3,4\},\{2,3,5\},\{2,4,5\},$ $\{2,4,6\},\{2,5,6\},\{3,4,5\},\{3,4,6\}\}$	$w_1 = 9\backslash 14, w_2 = 3\backslash 14,$ $w_3 = 5\backslash 14, w_4 = 2\backslash 14,$ $w_5 = 4\backslash 14, w_6 = 8\backslash 14$
$\frac{14}{13}$	$W^{min} = \{\{3,6\},\{1,2,3\},\{1,2,5\},\{1,2,6\},$ $\{1,3,5\},\{1,4,6\},\{1,5,6\},\{2,3,4\},\{2,5,6\},$ $\{3,4,5\},\{4,5,6\},\{2,3,5,6\}\},$ $L^{max} = \{\{1,6\},\{5,6\},\{1,2,4\},\{1,3,4\},\{1,4,5\},$ $\{2,3,5\},\{2,4,5\},\{2,4,6\}\}$	$w_1 = 5\backslash 13, w_3 = 6\backslash 13,$ $w_2 = w_5 = 4\backslash 13,$ $w_4 = 3\backslash 13, w_6 = 7\backslash 13$

Table B.1: Examples of critical simple games for every number of the 6th spectrum

α	Minimal winning coalitions and maximal loosing coalitions	Weight representation
$\frac{13}{12}$	$W^{min} = \{\{1,3\},\{3,5\},\{3,6\},\{1,5,6\},\{2,4,5\},$ $\{2,5,6\},\{4,5,6\},\{1,2,4,6\},\{2,3,4,6\}\},$ $L^{max} = \{\{5,6\},\{1,2,4\},\{1,2,5\},\{1,2,6\},$ $\{1,4,5\},\{1,4,6\},\{2,3,4\},\{2,4,6\}\}$	$w_1 = 3\backslash 12, w_3 = 9\backslash 12,$ $w_2 = w_4 = 2\backslash 12,$ $w_5 = 8\backslash 12, w_6 = 5\backslash 12$
$\frac{12}{11}$	$W^{min} = \{\{1,2\},\{1,6\},\{1,3,4\},\{1,3,5\},$ $\{2,4,5\},\{2,4,6\},\{2,5,6\},\{4,5,6\}\},$ $L^{max} = \{\{1,3\},\{1,4,5\},\{2,3,4\},\{2,3,5\},$ $\{2,3,6\},\{3,4,5\},\{3,4,6\},\{3,5,6\}\}$	$w_1 = 6\backslash 11, w_3 = 2\backslash 11$ $w_2 = w_6 = 5\backslash 11,$ $w_4 = w_5 = 3\backslash 11$
$\frac{12}{11}$	$W^{min} = \{\{1,2\},\{1,6\},\{1,3,4\},\{1,3,5\},$ $\{2,4,5\},\{2,4,6\},\{2,5,6\},\{4,5,6\}\},$ $L^{max} = \{\{1,3\},\{1,4,5\},\{2,3,4\},\{2,3,5\},$ $\{2,3,6\},\{3,4,5\},\{3,4,6\},\{3,5,6\}\}$	$w_1 = 6\backslash 11, w_3 = 2\backslash 11$ $w_2 = w_6 = 5\backslash 11,$ $w_4 = w_5 = 3\backslash 11$
$\frac{11}{10}$	$W^{min} = \{\{1,2\},\{1,3,5\},\{1,3,6\},\{1,4,5\},\{1,4,6\},$ $\{2,3,4\},\{2,3,6\},\{2,4,5\},\{2,4,6\},\{4,5,6\}\},$ $L^{max} = \{\{2,4\},\{1,3,4\},\{1,5,6\},\{2,3,5\},$ $\{2,5,6\},\{3,4,5\},\{3,4,6\},\{3,5,6\}\}$	$w_1 = w_2 = 5\backslash 10,$ $w_3 = 2\backslash 10, w_4 = 4\backslash 10,$ $w_5 = w_6 = 3\backslash 10$
$\frac{10}{9}$	$W^{min} = \{\{1,3\},\{3,5\},\{1,2,5\},\{1,4,6\},\{1,5,6\},$ $\{2,3,6\},\{2,4,5\},\{3,4,6\},\{4,5,6\}\},$ $L^{max} = \{\{3,6\},\{1,2,4\},\{1,2,6\},\{1,4,5\},$ $\{2,3,4\},\{2,4,6\},\{2,5,6\}\}$	$w_1 = w_5 = 4\backslash 9,$ $w_3 = 5\backslash 9, w_4 = 2\backslash 9,$ $w_2 = w_6 = 3\backslash 9$
$\frac{17}{15}$	$W^{min} = \{\{1,2,3\},\{1,3,5\},\{1,5,6\},\{2,3,5\},$ $\{3,4,5\},\{3,5,6\},\{4,5,6\},\{1,3,4,6\}\},$ $L^{max} = \{\{3,5\},\{1,3,4\},\{1,3,6\},\{1,4,5\},\{2,5,6\},$ $\{1,2,4,5\},\{1,2,4,6\},\{2,3,4,6\}\}$	$w_1 = 4\backslash 15, w_2 = 3\backslash 15,$ $w_3 = 8\backslash 15, w_4 = 1\backslash 15,$ $w_5 = 9\backslash 15, w_6 = 5\backslash 15$
$\frac{15}{13}$	$W^{min} = \{\{1,2\},\{1,3,6\},\{1,4,5\},\{1,4,6\},$ $\{2,3,5\},\{2,4,5\},\{2,5,6\},\{3,4,6\},\{1,3,4,5\}\},$ $L^{max} = \{\{2,5\},\{1,3,4\},\{1,3,5\},\{1,5,6\},\{2,3,4\},$ $\{2,3,6\},\{2,4,6\},\{3,4,5\},\{3,5,6\},\{4,5,6\}\}$	$w_1 = 7\backslash 13, w_2 = 6\backslash 13,$ $w_3 = w_4 = 4\backslash 13,$ $w_5 = 3\backslash 13, w_6 = 5\backslash 13$
$\frac{13}{11}$	$W^{min} = \{\{1,2,3\},\{1,5,6\},\{2,5,6\},\{3,4,5\},$ $\{3,4,6\},\{4,5,6\}\},$ $L^{max} = \{\{1,3,4\},\{1,3,5\},\{1,3,6\},\{1,4,5\},\{2,3,4\},$ $\{2,3,5\},\{2,3,6\},\{3,5,6\},\{1,2,4,5\},\{1,2,4,6\}\}$	$w_1 = w_2 = w_4 = 3\backslash 11,$ $w_3 = 5\backslash 11,$ $w_5 = w_6 = 4\backslash 11$
$\frac{17}{14}$	$W^{min} = \{\{1,2,3\},\{1,2,6\},\{1,3,6\},\{2,5,6\},$ $\{3,5,6\},\{4,5,6\},\{2,3,4,5\}\},$ $L^{max} = \{\{1,4,5\},\{1,4,6\},\{1,5,6\},\{2,3,5\},$ $\{1,2,4,5\},\{1,3,4,5\},\{2,3,4,6\}\}$	$w_1 = 4\backslash 14, w_4 = 1\backslash 14,$ $w_2 = w_3 = 5\backslash 14,$ $w_5 = 7\backslash 14, w_6 = 6\backslash 14$

Table B.2: Examples of critical simple games for every number of the 6th spectrum

α	Minimal winning coalitions and maximal loosing coalitions	Weight representation
$\frac{11}{9}$	$W^{min} = \{\{1,2\},\{2,5\},\{1,3,4\},\{1,3,6\},$ $\{1,4,5\},\{1,4,6\},\{3,4,6\},\{4,5,6\}\},$ $L^{max} = \{\{1,4\},\{1,3,5\},\{1,5,6\},\{2,3,4\},$ $\{2,3,6\},\{2,4,6\},\{3,4,5\},\{3,5,6\}\}$	$w_1 = w_5 = 4\backslash 9,$ $w_2 = 5\backslash 9,$ $w_3 = w_4 = w_6 = 3\backslash 9$
$\frac{16}{13}$	$W^{min} = \{\{1,2,3\},\{1,2,4\},\{1,4,6\},\{2,4,5\},$ $\{3,4,5\},\{4,5,6\},\{1,3,4,6\}\},$ $L^{max} = \{\{1,3,4\},\{1,4,5\},\{1,2,5,6\},$ $\{1,3,5,6\},\{2,3,4,6\},\{2,3,5,6\}\}$	$w_1 = w_5 = 5\backslash 13,$ $w_4 = 6\backslash 13, w_6 = 2\backslash 13,$ $w_2 = w_3 = 4\backslash 13$
$\frac{5}{4}$	$W^{min} = \{\{1,2\},\{1,3,6\},\{1,4,6\},\{1,5,6\},\{2,3,5\},$ $\{2,4,5\},\{2,4,6\},\{3,5,6\},\{4,5,6\}\},$ $L^{max} = \{\{1,6\},\{1,3,4\},\{2,3,4\},\{2,3,6\},$ $\{2,5,6\},\{3,4,6\},\{1,3,4,5\}\}$	$w_1 = w_2 = w_6 = 2\backslash 4,$ $w_3 = w_4 = w_5 = 1\backslash 4$
$\frac{14}{11}$	$W^{min} = \{\{1,2\},\{4,5\},\{4,6\},\{1,3,5\},\{1,3,6\},$ $\{1,5,6\},\{2,3,4\},\{3,5,6\}\},$ $L^{max} = \{\{1,5\},\{1,6\},\{2,4\},\{1,3,4\},\{2,3,5\},$ $\{2,3,6\},\{2,5,6\}\}$	$w_1 = 7\backslash 11, w_2 = 4\backslash 11,$ $w_3 = 1\backslash 11, w_4 = 6\backslash 11,$ $w_5 = w_6 = 5\backslash 11$
$\frac{9}{7}$	$W^{min} = \{\{1,2\},\{3,5\},\{5,6\},\{1,4,5\},\{1,4,6\},$ $\{2,3,4\},\{2,3,6\},\{2,4,6\},\{3,4,6\}\},$ $L^{max} = \{\{1,5\},\{2,3\},\{2,6\},\{4,6\},\{1,3,4\},$ $\{1,3,6\},\{2,4,5\}$	$w_1 = w_3 = w_6 = 3\backslash 7,$ $w_2 = w_5 = 4\backslash 7,$ $w_4 = 1\backslash 7$
$\frac{13}{10}$	$W^{min} = \{\{2,3\},\{1,3,6\},\{1,5,6\},\{3,4,5\},$ $\{3,4,6\},\{4,5,6\}\}$ $L^{max} = \{\{1,3,4\},\{1,3,5\},\{1,4,5\},\{2,5,6\},$ $\{3,5,6\},\{1,2,4,5\},\{1,2,4,6\}\}$	$w_1 = w_4 = 2\backslash 10,$ $w_2 = w_3 = 5\backslash 10,$ $w_5 = 4\backslash 10, w_6 = 4\backslash 10$
$\frac{17}{13}$	$W^{min} = \{\{1,2\},\{2,3,5\},\{2,3,6\},\{2,4,5\},$ $\{3,4,6\},\{4,5,6\}\},$ $L^{max} = \{\{1,3,4\},\{1,4,6\},\{2,3,4\},\{2,4,6\},\{2,5,6\},$ $\{1,3,4,5\},\{1,3,5,6\}\}$	$w_1 = 6\backslash 13, w_2 = 7\backslash 13,$ $w_3 = w_5 = 3\backslash 13,$ $w_4 = w_6 = 5\backslash 13$
$\frac{4}{3}$	$W^{min} = \{\{1,2\},\{2,5\},\{1,3,6\},\{1,4,6\},\{1,5,6\},$ $\{2,3,4\},\{3,4,6\},\{4,5,6\}\},$ $L^{max} = \{\{1,6\},\{2,3,6\},\{2,4,6\},\{3,4,5\},\{3,5,6\},$ $\{1,3,4,5\}\}$	$w_1 = w_3 = 1\backslash 3,$ $w_2 = 2\backslash 3,$ $w_4 = w_5 = w_6 = 1\backslash 3$
$\frac{3}{2}$	$W^{min} = \{\{1,6\},\{2,3\},\{2,5\},\{2,6\},\{3,4\},$ $\{4,6\},\{1,3,5\}\},$ $L^{max} = \{\{1,3\},\{1,2,4\},\{1,4,5\},\{3,5,6\}\}$	$w_1 = w_2 = w_3 = 1\backslash 2,$ $w_4 = w_5 = w_6 = 1\backslash 2$
$\frac{9}{8}, \frac{8}{7}, \frac{7}{6}, \frac{6}{5}$	add dummy player with zero weight to games from Table 4.1	

Table B.3: Examples of critical simple games for every number of the 6th spectrum

Appendix C

Maple's Codes

C.1 Code for the 5th Spectrum

```
with(Optimization):
with(combinat):
```

The procedure "All" generates the set of all subsets of $\{1, .., n\}$ with cardinality #in $[i1, i2]$.

```
All:=proc(i1::integer,i2:: integer, n::set):: set;
local i, p, allcomb;

 allcomb:={}:
 for i from i1 by 1 to i2 do
  allcomb:=allcomb union choose(n,i):
 end do:
 allcomb:
end proc;
```

The procedure "supersetsminus" removes all supersets (not necessary strict) #of a set *seta* from a set of subsets *all*.

```
supersetsminus:=proc(seta::set, all::set)::set;
local i,ss;
 ss:=all;
 for i in all do
  if verify(`intersect`(seta,i), seta, { `set`, `equal`})
```

```
   then ss:=ss minus {i}: end if:
 end do:
 ss;
end proc;
```

The procedure "subsetsminus" removes all subsets (not necessary strict) of a #set *seta* from a set of subsets *all*.

```
subsetsminus:=proc(seta::set, all:: set):: set;
local i,ss;
 ss:=all;
 for i in all do
  if (i subset seta) then ss:=ss minus {i}: end if:
 end do:
 ss;
end proc;
```

The procedure "converttoconditions" returns the set fw of vectors $\chi(j)x$, where #$\chi(j)$ is the characteristic vector of a set j of a (note that every set from a is a subset #of $\{1, .., n\}$) and x is a vector of undermined variables.

```
converttoconditions:=proc(a::set,n::integer)::set;
local uslx,p,fw,i1,i2;
 uslx :={seq(x[i], i = 1 .. n)};
 p:=0;
 fw:={};
 for i1 in a do
  for i2 in i1 do
   p:=p+uslx[i2];
  end do;
  fw:= fw union {p};
  p:=0;
 end do;
 fw;
end proc;
```

#The procedure "rough" returns the list $final$, which consists of two sets $final1$ #and $final2$, where $final1$ is the set of all possible parameters α. The set fw is a #set of known winning coalitions. The set fl is a set of known losing coalitions.

#For every classification, say classification number r, it solves linear programming #problem and find value p, such that this game is critical in C_p. If we had the #same p for some previous situation (or equivalently $p \in final1$) we go to the next #classification. If we found p for the first time then we add it to $fanal1$ and add the #list $[p, \{conditions\}]$ to $final2$, where $\{conditions\}$ is a set of conditions: $\chi(j)x \leq p$ if j #is losing and $\chi(j)x \geq 1$ if j is winning.

```
rough:=proc(fw::set, fl::set, g:: set, n:: integer)::list;
local k, k1, p,p1, i,j, i1,i2, final1,final2, final3,indexall,
wincond,looscond,strangecond,indexw, indexl, uslx, condition,
addcond, final;


 final1:= {};
 final2:={};
 final3:={};
```

#Form idex set.

```
 indexall := {seq(i, i=1..nops(g))};
```

#Form known conditions from fw≥ 1 and fl $\leq z$.

```
 wincond:= converttoconditions(fw,n);
 looscond:= converttoconditions(fl,n);
 strangecond:= converttoconditions(g,n);
 condition:={};
 for k1 in wincond do
  condition:= `union`(condition, {k1 >= 1});
 end do;
 for k1 in looscond do
  condition:= `union`(condition,{k1<=z});
 end do;

 addcond:={};
```

#All additional conditions for winning coalitions.

```
 for k1 in strangecond do
  addcond:= `union`(addcond,{k1>=1});
 end do;
 p := LPSolve(z, `union`(`union`(condition, addcond), {z>=1}));
 p1 := convert(p[1], rational);

 if evalb(p1 in final1) then
  else final1 := `union`(final1, {p1});
  final2:=final2 union {[p, {condition,addcond}]};
 end if;
```

#First one of the additional conditions is loosing, after that two, three and so on.

```
 for i in indexall do
  addcond:={};
  for j in All(i,i,g) do
   indexl:=j:
   indexw:= g minus j:
   for k in indexw do
    addcond:= `union`(addcond,{ converttoconditions({k},5)[1]>=1});
   end do;
   for k in indexl do
    addcond:= `union`(addcond,{ converttoconditions({k},5)[1]<=z});
   end do;
   p := LPSolve(z, `union`(`union`(condition, addcond), {z>=1}));
   p1 := convert(p[1], rational);

   if evalb(p1 in final1) then
    else final1 := `union`(final1, {p1});
    final2:=final2 union {[p, {condition,addcond}]};
   end if;
  end do;
 end do;

 final:=[final1,final2];
end proc;

######################
```

```
final1:={};
final2:={};
final3:={};
fw := {{1,2}, {3,4,5} };
fl:={ {1,5}, {2,3,4} };
level3:={ {1,3,4} , {1,3,5} , {1,4,5}, {2,3,5}, {2,4,5} };
level2:= {{1,3}, {1,4}, {2,5}, {3,5}, {4,5}};



for i1 from 0 to 4 do
 p:=All(i1,i1,level3):
 for i2 in p do
  fwtemp:=fw:
  fltemp:=fl:
  g:=i2;
  l2:=level2;
  for i3 in g do
   l2:=subsetsminus(i3,l2);
  end do;

  fltemp:=fltemp union g;
  fwtemp:=fwtemp union (level3 minus  g);
  P:= rough(fwtemp , fltemp, l2, 5);
  final1:=`union`(final1,P[1]);
  final2:=`union`(final2,P[2]);
 end do;
end do;
```

C.2 Code for the 6th Spectrum

```
with(Optimization):
with(combinat):
#Digits:=50;
```

#The procedure "All" generates a set of all subsets of $\{1, .., n\}$ with cardinality in #$[i1, i2]$.

```
All:=proc(i1::integer,i2:: integer, n::set):: set;
local i, p, allcomb;

 allcomb:={}:
 for i from i1 by 1 to i2 do
  allcomb:=allcomb union choose(n,i):
 end do:
 allcomb:
end proc;
```

The procedure "supersetsminus" removes all supersets (not necessary strict) #of a set *seta* from the set of subsets *all*.

```
supersetsminus:=proc(seta::set, all::set)::set;
local i,ss;
 ss:=all;
 for i in all do
  if verify(`intersect`(seta,i), seta, { `set`, `equal`})
   then ss:=ss minus {i}: end if:
 end do:
 ss;
end proc;
```

The procedure "subsetsminus" removes all subsets (not necessary strict) of a# set *seta* from the set of subsets *all*.

```
subsetsminus:=proc(seta::set, all:: set):: set;
local i,ss;
 ss:=all;
 for i in all do
  if (i subset seta) then ss:=ss minus {i}: end if:
 end do:
 ss;
end proc;
```

#The procedure "subsetsofsetminus" removes all subsets (not necessary strict) #of sets in *setsa* from the set of subsets *all*.

```
subsetsofsetminus:=proc(setsa::set, all::set)::set;
local i,ss;
 ss:=all;
 for i in setsa do
  ss:=subsetsminus(i, ss);
 end do;
 ss;
end proc;
```

#The procedure "supersetsofsetminus" removes all supersets (not necessary #strict) of sets in *setsa* from the set of subsets *all*.

```
supersetsofsetminus:=proc(setsa::set, all::set)::set;
local i,ss;
 ss:=all;
 for i in setsa do
  ss:=supersetsminus(i, ss);
 end do;
 ss;
end proc;
```

The procedure "converttoconditions" returns the set fw of vectors $\chi(j)x$, where #$\chi(j)$ is the characteristic vector of a set j of a (note that every set from a is a subset #of $\{1, .., n\}$) and x is a vector of undermined variables.

```
converttoconditions:=proc(a::set,n::integer)::set;
local uslx,p,fw,i1,i2;
 uslx :={seq(x[i], i = 1 .. n)};
 p:=0;
 fw:={};
 for i1 in a do
  for i2 in i1 do
   p:=p+uslx[i2];
  end do;
  fw:= fw union {p};
  p:=0;
 end do;
 fw;
end proc;
```

#The procedure “rough” returns the list $final$, which consists of two sets $final1$ #and $final2$, where $final1$ is the set of all possible parameters α. The set fw is #a set of known winning coalitions, fl is a set of known losing coalitions, g is a #3-element list, where $g[i]$ consists of all unclassified coalitions of cardinality $i+1$. #Note that $g[i]$ doesn't contain subsets of sets from fl and supersets of sets from #fw. The algorithm classifies sets from $g[1]$, $g[2]$ and $g[3]$ as it is explained in the #chapter. For every such classification, say classification number x, it solves a linear #programming problem and finds value p, such that this game is in C_p. If we have #the same p for some previous situation (or, equivalently, $p \in final1$) we go to the #next classification. If we find p for the first time then we add it to $fanal1$ and add #the list $[p, \{conditions\}]$ to $final2$, where $\{conditions\}$ is a set of conditions: $\chi(j)x \leq p$ #if j is losing, and $\chi(j)x \geq 1$ if j is winning.

```
rough:=proc(fw::set, fl::set, g:: list, n:: integer)::list;
local final1, final2, l4,l2temp1,l3temp1, l2temp2, i1,i2,i3,k1,k2,
k3,j1,fwtemp,fltemp,fwtemp2,fltemp2,fwtemp3,fltemp3,p,p1,condition,
final;

 final1:= {};
 final2:={};
 l4:=g[3];

 for i1 from 0 by 1 to nops(l4) do
  for k1 in All(i1,i1,l4)  do
   fwtemp:=fw;
   fltemp:=fl;
   l2temp1:=g[1];
   l3temp1:=g[2];
   l2temp1:=subsetsofsetminus(k1,l2temp1);
   l3temp1:=subsetsofsetminus(k1,l3temp1);
   fwtemp:=fwtemp union (l4 minus k1);
   fltemp:=fltemp union k1;
    for i2 from 0 to nops(l3temp1) do
     for k2 in All(i2,i2,l3temp1) do
      l2temp2:=subsetsofsetminus(k2,l2temp1);
      fwtemp2:=fwtemp union (l3temp1 minus k2);
      fltemp2:=fltemp union k2;
```

```
      for i3 from 0 to nops(l2temp2) do
       for k3 in All(i3,i3,l2temp2) do
        fwtemp3:=fwtemp2 union (l2temp2 minus k3);
        fltemp3:=fltemp2 union k3;
        fltemp3:= converttoconditions(fltemp3,n);
        fwtemp3:= converttoconditions(fwtemp3,n);
        condition:={};
        for j1 in fwtemp3 do
         condition:= `union`(condition, {j1 >= 1});
        end do;
        for j1 in fltemp3 do
         condition:= `union`(condition,{j1<=z});
        end do;
        p := LPSolve(z, condition union {z>=1}, assume = nonnegative);
        p1 := convert(p[1], rational);

        if evalb(p1  in final1) then  else
         final1 := `union`(final1, {p1});
         final2:=final2 union {[p, {condition}]};
        end if;
      end do;
     end do;
    end do;
   end do;
  end do;
 end do;

 final:=[final1,final2];
end proc;
```

The main code.
The not proper case 1.

```
final1:={}:
final2:={}:

fw := {{1,2}, {3,4,5,6} }:
fl:={{1,3,4}}:
```

```
P:={seq(i,i=1..6)} :
level4:=All(4,4,P) :
level3:=All(3,3,P) :
level2:= All(2,2,P) :
level4:=supersetsofsetminus(fw,level4) :
level4:=subsetsofsetminus(fl,level4) :
level3:=supersetsofsetminus(fw,level3) :
level3:=subsetsofsetminus(fl,level3) :
level2:=supersetsofsetminus(fw,level2) :
level2:=subsetsofsetminus(fl,level2) :

g:=[level2,level3,level4] :
final:=rough(fw,fl,g,6) :
final1:=final1 union final[1] :
final2:=final2 union final[2] :
k := final1: k1 := k:
for i to nops(k) do
 for j from i+1 to nops(k) do
  if abs(k[i]-k[j]) < 10^(-7) then k1 := `minus`(k1, {k[j]})
 end if
end do
end do:
final1:=k1:
k := final2: k1 := k:
for i to nops(k) do
 for j from i+1 to nops(k) do
  if abs(k[i][1][1]-k[j][1][1]) < 10^(-7)
   then k1 := `minus`(k1, {k[j]})
  end if
 end do
end do:
final2 := k1;
```

The not proper case 2.

```
fw := {{1,2}, {3,4,5,6} }:
fl:={{3,4,5}}:
P:={seq(i,i=1..6)} :
```

```
level4:=All(4,4,P) :
level3:=All(3,3,P) :
level2:= All(2,2,P) :
level4:=supersetsofsetminus(fw,level4) :
level4:=subsetsofsetminus(fl,level4) :
level3:=supersetsofsetminus(fw,level3) :
level3:=subsetsofsetminus(fl,level3) :
level2:=supersetsofsetminus(fw,level2) :
level2:=subsetsofsetminus(fl,level2) :

g:=[level2,level3,level4] :
final:=rough(fw,fl,g,6) :
final1:=final1 union final[1] :
final2:=final2 union final[2] :
k := final1: k1 := k:
for i to nops(k) do
 for j from i+1 to nops(k) do
  if abs(k[i]-k[j]) < 10^(-7)
   then k1 := ‘minus‘(k1, {k[j]}) end if
 end do
end do:
final1:=k1:
k := final2: k1 := k:
for i to nops(k) do
 for j from i+1 to nops(k) do
  if abs(k[i][1][1]-k[j][1][1]) < 10^(-7)
   then k1 := ‘minus‘(k1, {k[j]}) end if
 end do
end do:
final2 := k1;
```

The not proper case 3.

```
fw := {{1,2,3}, {4,5,6} }:
fl:={{1,4,5} }:
P:={seq(i,i=1..6)} :
level4:=All(4,4,P) :
level3:=All(3,3,P) :
```

```
level2:= All(2,2,P) :
level4:=supersetsofsetminus(fw,level4) :
level4:=subsetsofsetminus(fl,level4) :
level3:=supersetsofsetminus(fw,level3) :
level3:=subsetsofsetminus(fl,level3) :
level2:=supersetsofsetminus(fw,level2) :
level2:=subsetsofsetminus(fl,level2) :

g:=[level2,level3,level4] :
final:=rough(fw,fl,g,6) :
final1:=final1 union final[1] :
final2:=final2 union final[2] :
k := final1: k1 := k:
for i to nops(k) do
 for j from i+1 to nops(k) do
  if abs(k[i]-k[j]) < 10^(-7)
   then k1 := `minus`(k1, {k[j]}) end if
 end do
end do:
final1:=k1:
k := final2: k1 := k:
for i to nops(k) do
 for j from i+1 to nops(k) do
  if abs(k[i][1][1]-k[j][1][1]) < 10^(-7)
   then k1 := `minus`(k1, {k[j]}) end if
 end do
end do:
final2 := k1;
```

#The not strong case 1.

```
finalk:={} :
final1k:={} :
final2k:={} :

fl := {{1,2}, {3,4,5,6} }:
fw:={} :
P:={seq(i,i=1..6)} :
```

```
level4:=All(4,4,P) :
level3:=All(3,3,P) :
level2:= All(2,2,P) :
level4:=subsetsofsetminus(fl,level4) :
level3:=subsetsofsetminus(fl,level3) :
level2:=subsetsofsetminus(fl,level2) :

g:=[level2,level3,level4] :
finalk:=rough(fw,fl,g,6) :
final1k:=final1k union finalk[1] :
final2k:=final2k union finalk[2] :
k := final1k: k1 := k:
for i to nops(k) do
 for j from i+1 to nops(k) do
  if abs(k[i]-k[j]) < 10^(-7)
   then k1 := `minus`(k1, {k[j]}) end if
 end do
end do:
final1k:=k1:
k := final2k: k1 := k:
for i to nops(k) do
 for j from i+1 to nops(k) do
  if abs(k[i][1][1]-k[j][1][1]) < 10^(-7)
   then k1 := `minus`(k1, {k[j]}) end if
 end do
end do:
final2k := k1:
```

#The not strong case 2 (cases 2 and 3 from the chapter)

```
fl := {{1,2,3}, {4,5,6} }:
fw:={ }:
P:={seq(i,i=1..6)} :
level4:=All(4,4,P) :
level3:=All(3,3,P) :
level2:= All(2,2,P) :
level4:=subsetsofsetminus(fl,level4) :
level3:=subsetsofsetminus(fl,level3) :
```

```
level2:=subsetsofsetminus(fl,level2) :

g:=[level2,level3,level4] :
finalk:=rough(fw,fl,g,6) :
final1k:=final1k union finalk[1] :
final2k:=final2k union finalk[2] :
k := final1k: k1 := k:
for i to nops(k) do
 for j from i+1 to nops(k) do
  if abs(k[i]-k[j]) < 10^(-7)
   then k1 := `minus`(k1, {k[j]}) end if
 end do
end do:
final1k:=k1:

k := final2k: k1 := k:
for i to nops(k) do
 for j from i+1 to nops(k) do
  if abs(k[i][1][1]-k[j][1][1]) < 10^(-7)
  then k1 := `minus`(k1, {k[j]}) end if
 end do
end do:
final2k:=k1:
```

$final1k$ is a set of all possible α.
$final2k$ is set of $[p, \{conditions\}]$, where p $\in final1k$.

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