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Incorporating Ideas from Indian History in the Teaching and Learning of a General Place Value System

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A Degree submitted in fulfilment of the requirements
for the Degree of Doctor of Philosophy in Mathematics Education,
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Abstract

The current decimal place value system is foundational to progress in mathematics and yet research shows that many students, including high school students, experience considerable difficulties with the place value concept. Hence this study, adopting a holistic and multi-pronged approach that included a historical-critical methodology, investigated how the history of mathematics could be used to enhance junior secondary school students’ conceptual understanding of a general positional notation. Situated in mathematics education research, the first part of this study analysed the history of mathematics, in particular the development of the decimal place value system and algebraic symbolism in Indian history, for ideas relevant to understanding the place value construct, and these ideas were then incorporated into a teaching/learning framework.

The second part of the research, a case study, investigated the effectiveness of the framework developed, on junior secondary school students’ understanding of a general place value system. The participants were twenty-nine Year 9 (13 year-old) students from a secondary school in Auckland, New Zealand, who worked cooperatively with the researcher, who was also their classroom teacher, for a full term of the academic year. The teaching intervention, employing multiple (including concrete) representations, took place in a classroom environment that is largely compatible with a socio-cultural view of learning.

The Indian historical analysis revealed key stages in the evolution of the decimal place value system. Beginning around the Vedic period, there was consistent naming of large numbers including powers of ten, which eventually led to the development of the current Hindu-Arabic place value system with zero. The written numeral system appears to have traversed four main stages; that of additive, multiplicative, ‘places’ and finally abstract before its final construction. Consequently, rapid progress was made in algebra paralleling the development of algebraic symbolism, a key feature of which was different names (colours) for different unknowns/variables.

The empirical data was positive, providing evidence indicating that students were successful to a degree, in understanding place value structure. The research suggests that
implementing teaching sequences based on the framework produced could have a positive effect on student understanding of a general place value system.
Dedication

I dedicate this thesis to my family – my parents Sri Santhanam and Smt Lakshmi, my husband Nattu and daughters Mitra and Uttara.
Preface

Personal experience and motivation for the research study

My interest in studying the history of mathematics for ideas to be used in teaching originates from my experiences as a teacher. I have taught high school mathematics for a number of years and in several countries including India, Zambia, Kenya, UK, South Africa and now New Zealand. During the years of my teaching and interacting with students, I realised that some students found place value concepts difficult to understand while for some others the understanding was superficial.

In 2002, I was fortunate to have been awarded a Royal Society Fellowship hosted by the Mathematics Education Unit (MEU) at The University of Auckland. During this year, I investigated Vedic mathematics (in some ways indirectly related to the history of Indian mathematics) and mathematics in South Indian classical music. Although I was born in India and lived and studied there for many years, I knew very little about the history of Indian mathematics and its many successes. Also in 2002, I studied two mathematics education papers, Algebra and Learning, and Research Methods. Studying these papers and the history of mathematics helped me to make numerous links between the two domains. The study caused me to reflect deeply on students’ learning, my teaching practice and how I might help students to achieve sound conceptual understanding and to apply their knowledge. I gradually realised that there were many ideas related to history that could be useful in teaching and learning topics such as the decimal place value system and this eventually led to my PhD enrolment culminating in this thesis.

I enjoyed this project immensely and have gained much in terms of learning and knowledge. The historical analysis enabled me to learn about the progress of mathematical ideas, especially in Indian history, and of the many achievements of past mathematicians. It also brought to the fore several connections between mathematical topics and various other disciplines. I remain grateful for this wonderful opportunity.
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The Royal Society award in 2002 enabled an introductory investigation into history and mathematics education, and the PPTA Study award in 2010 made it possible for me to write a major part of my thesis. I remain indebted to them for their support.

My very special thanks go to the school where I teach. I thank the Principal and the Staff, especially the mathematics department, for their great support over the years, and the students who participated in the research study without whom my thesis would not have been completed.

Finally, I thank God for his love and blessings, especially through this long and sometimes difficult journey.
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CHAPTER 1

INTRODUCTION

1.1 Background

The Indian mathematician Mahavira (850 CE) in his treatise *Ganita Sara-Samgraha* says:

In all transactions which relate to worldly, Vedic or other similar religious affairs, calculation is of use...in the science of wealth, in music and in drama, in the art of cooking, in medicine, in architecture, in poetry, in logic and grammar...and in relation to all that constitutes the peculiar value of the arts, the science of calculation (*ganita*) is held in high esteem...Whatever there is in all the three worlds, which are possessed of moving and non moving beings, cannot exist as apart from *ganita* (measurement and calculation). (as quoted in Datta & Singh, 2001, vol. 1, p. 5)

The concept of number exerts a dual enchantment. When it uncovers the rich relationships between pure magnitudes to the scientist, it fills him with the satisfaction of intellectual insight; for the sake of these relationships the sovereign structure of mathematics rests upon the concept of number. For others it exerts a fascination by its deep interconnection with the daily life of people. Is it not true that every tribe spoke and noted down numbers, that every tribe had to calculate whenever it faced life on this planet? Did man’s relationship to his environment not also necessitate his relationship with numbers? (Menninger, 1969, p. v)

This thesis takes a holistic and multi-dimensional approach to suggesting possible ways to improve junior secondary school students’ *conceptual understanding* of the structure of the current decimal place value numeration system and its generalisation, integrating ideas from the history of mathematics and mathematics education research.

A numeration system is a system for expressing numbers, both oral and written. In a system that incorporates place value, the value of a digit in a numeral is dependent on its *place* or position in the numeral.

The opening quotations indicate the importance of understanding numbers and arithmetic. Central to this understanding, and for developing *proficiency* (Kilpatrick, Swafford, & Findell, 2001) with numbers is the concept of positional notation/place value in the Hindu-Arabic decimal system, and its application in the standard mental and written algorithms. Place value structure is also a key aspect of number sense; it plays a crucial role in estimation (Howe & Epp, 2008) and an efficient use of computational technology. As such, place value and arithmetic need to be meaningful for students (Brownell, 1947;
Van Engen, 1949), as these topics are the foundation of the discipline on which advanced mathematical topics are constructed.

Today, although the number of branches of the discipline has increased and the field of the discipline has been expanded, the foundational status of arithmetic and geometry in mathematics is still unchanged. None of the new branches, whether pure or applied, operates without the basic mathematical rules and computational skills established in arithmetic and geometry (Ma, 1999).

However, in spite of its importance, many researchers and educators (Bednarz & Janvier, 1982; Fuson, 1990b; Hiebert & Carpenter, 1992; Resnick, 1983; Ross, 1990) have reported difficulties that students have in making sense of meanings of multidigit numbers, particularly its multiplicative structure. They have been concerned that many students complete primary and secondary school without developing sufficient knowledge of numeration and place value. This may be due to the fact that while superficially simple, the place value system conceals an unexpected complexity of ideas, as pointed out by Skemp (1971) and which is also revealed in Schmittau and Vagliardo’s (2006, 2009) concept map on positional notation (see Figures 1.1, 1.2, 1.3). Thompson (2000) concluded that the concept takes many years of mental development and is too sophisticated for young children to grasp. In support of this, many studies have reported errors made by students and the difficulties that they experience in regard to the place value concept. In particular, some studies have provided evidence of, counting and reading errors (Bednarz & Janvier, 1982; Cobb & Wheatley, 1988; Resnick, 1983; Sharma, 1993), misconceptions and errors related to zero (e.g. Anthony & Walshaw, 2004; Brown, 1981b; Ginsburg, 1989) and associated with numbers containing decimal fractions (e.g. Brown, 1981b; Hiebert, 1986; Irwin, 1996; Resnick et al., 1989). Researchers have also described student difficulties involving part-whole relationships (e.g. Lamon, 1996; Ross, 1989), English number words (Fuson, 1990b; Miura, Chungsoon, Chang, & Okamoto, 1988) and the structure of numeration system (Baturo, 2002; Fuson, 1990a; N. Thomas, 1998).

Additionally, Howe and Epp (2008) point out that the underlying structure of the place value numeration system is polynomial, and hence implies a need for some knowledge of exponentiation and this makes it a fragile system (Mulligan & Vergnaud, 2006) for young students. This suggests that the place value concept needs to be revisited when students encounter powers in upper primary or junior secondary school and the links between groupings, powers and place value should be made explicit. As well as this, Becker and
Varelas (1993) indicate that the place value system is a *semiotic/sign system* and understanding it involves the integration of both its *conceptual* and *semiotic* aspects. That is, connections have to be made between the *written* number symbols and the recursive grouping structure that is implicit within the system. In this context, as propounded by Vygotsky (1962), there is a deeper appreciation of the concept when place value is understood as a particular instance of a general positional notation. Furthermore, generalisation of multiple bases to the notion of a general base could not only reinforce the place value idea, but also provide for a smoother pathway to algebra. However, thus far very few research studies have reported on *secondary* school students’ understanding of the above relationships. A notable exception is the work of Seah and Booker (2005) on Year 8 Australian students’ numeration and multiplication knowledge.

In response to students’ place value misconceptions and difficulties, a number of researchers have described teaching approaches (e.g. Association of Independent Schools of South Australia, 2004; Bednarz & Janvier, 1988; Cuffel, 2009; Fuson & Briars, 1990; Hiebert & Wearne, 1992; G. A. Jones et al., 1996; Labinowicz, 1985; Price, 2001; Resnick & Omanson, 1987; Ross, 1986; Sharma, 1993) to assist students to make sense of multi-digit numbers. Much of this documented research has focused on *primary* school students’ levels of understanding and hence did not involve place values as powers. For example, Sharma (1993) considers several historical numeration systems and children’s difficulties, and then describes strategies and teaching activities to introduce place value to young children (from kindergarten to third grade). Most of the studies mentioned above have employed concrete materials (mainly Dienes’ base-10 blocks) as representations to facilitate the learning process. Crucial to this understanding, as proposed by M. O. J. Thomas (2004, July), is the importance of shifting attention between multiple representations and, the “interactions of learners with representations of processes and objects, and the links they form between different representations of concepts” (p. 1). However, despite the remediation efforts, student difficulties continue, as shown by many recent studies and reports (e.g. Brown, Küchemann, & Hodgen, 2010; Groves, Mousley, & Forgaz, 2006; Hodgen, Brown, Küchemann, & Coe, 2011, September; Seah & Booker, 2005; Steinle & Stacey, 2004). Further to this, N. Thomas (1998) has highlighted that teaching/learning of numeration as compartmentalised knowledge and processes, which students appear to do, hampers the formation of relationships. Hence educators have
suggested a more holistic/global approach to research (Mulligan & Vergnaud, 2006; N. Thomas, 1998) employing multiple perspectives. Researchers (Greer, 1992) have also called for alternative teaching experiments to teaching the meaning of number, so that students can be helped to build links between their intuitive knowledge of place value, various representational models that might be used, and formal rules of numeration.

In this context, one question to ask is whether, as educators and researchers, we are ‘missing’ (Begg, 2009, p. 1) important aspects that would suggest alternative ways of teaching for better understanding. Begg explains that these missing elements might be related to what we teach, how we teach, how we know and learn, thinking or curriculum. Viewed in this way, it may be that the history of mathematics is a possible ‘missing’ element; its development and the lessons it carries for teaching and learning. In recent years a number of educators and researchers (e.g. Fauvel & van Maanen, 2000; Gupta, 1995b; Katz, 2007; Radford, 2000; Tzanakis & Arcavi, 2000) have turned to the history of mathematics in an attempt to understand student difficulties and to inform practice. Their research has highlighted the usefulness of the history of mathematics as an excellent resource for motivating students to learn and enhance understanding of mathematics. However, historical material may be incorporated into the classroom either directly or indirectly (Fauvel & van Maanen, 2000). One specific example of a direct use of history (Tzanakis & Arcavi, 2000) presenting students with the idea of mathematics as a developing human endeavour, is a study of number systems from different civilizations such as the Egyptian, Babylonian, Greek, Mayan, Chinese and Roman. An implicit or indirect integration might be a historical and epistemological analysis that may help the teacher to understand the various stages in learning and why a certain concept is difficult for the student (Barbin, 2000; Katz, 2007; Puig & Rojano, 2004). In turn, this can guide the teaching strategy and development. An example of such an indirect use of history of mathematics would be harnessing teacher’s knowledge of the historical development of the numeration system in order to develop didactic strategies to encourage a fuller understanding of place value structure.

While a review of history reveals that every civilization had its own numeration system, including the Mayan system with place value and zero, the present day decimal system with ten distinct and arbitrary numerals, the place value principle and the use of zero, originated in India and was then transmitted to Europe by the Arab mathematicians
Hence it is reasonable to suppose that the historical development of this system may hold valuable lessons for today and thus the present research concentrated mainly on the Indian origins of the decimal numeration system. In this study, the researcher undertook an analysis of the history of mathematics in India pertaining to the construction of the Hindu-Arabic decimal numeration system. This aspect of the study was taken up with a view to identifying historical ideas about place value that might prove useful in teaching. These historical ideas were analysed in light of mathematics education research and in turn, integration of ideas of the two domains, aided the development of a teaching sequence for improving students’ understanding of the structure of numeration.

For school students, their first year at secondary school (which is Year 9 in New Zealand) is a key foundational year in the steps towards higher mathematics. According to the number and algebra strand of the New Zealand Curriculum, Year 9 students will learn to use a range of multiplicative strategies when working with whole numbers, fractions, decimals and percentages. They are formally introduced to integers and operations on them; there is also a formal introduction to algebra and students learn to: a) form and solve simple linear equations, b) generalise properties of multiplication and division with whole numbers and c) use graphs, tables and rules to describe linear relationships in patterns (Ministry of Education, 2007). All of the above depend on a sound understanding of decimal place value structure and number sense (students are expected to have understood place value ideas by the end of primary school). Despite the importance and the slow development of place value knowledge, little is known about secondary school students’ understanding of the structure of the numeration system. This thesis addresses this gap in research knowledge and focuses on junior secondary school students’ awareness of a general positional notation. The intention was to revisit and to throw light on a fundamental mathematical topic through a holistic approach comprising multiple perspectives that take into account history of mathematics and mathematics education research. The following section outlines the aims of the research study.

1.2 Main Aim

The main purpose of this study was to develop, on the basis of historical ideas, a didactical framework for teaching the structure of place value, and then evaluate whether it could
enhance Year 9 students’ understanding of the current Hindu-Arabic decimal place value system.

1.3 Specific Aims

A set of related specific aims were:

- To research the theoretical aspects of the application of the history of mathematics in mathematics education.
- To research the educational perspectives on the teaching and learning of the structure of decimal place value system with zero, and its generalisation.

This research examined ideas related to positional notation, such as multiplicative thinking, understanding of powers and number of symbols used in a place value system. It has also considered multiple representations, place values in non-decimal bases and their generalisation, as a way of understanding the structure of numeration. The study sought to identify and consider possible reasons for students’ difficulties in understanding place value structure, and through an intervention that incorporated historical ideas an attempt was made to assist students’ conceptual understanding. The following research questions were addressed in this study.

1.4 General Research Questions

Based on the issues outlined in the background section, the general research questions investigated in this study were:

1. What is the feasibility of a study that involves a two tiered process such as: i) the extraction of historical ideas and the development of a teaching framework and; ii) implementing the framework and evaluating its effectiveness in increasing Year 9 students’ awareness of numeration?

2. What are some of the key historical ideas relevant to understanding the decimal place value system and how might they be useful in teaching and learning?

3. What is the effect of the framework developed on Year 9 students’ understanding of the place value system?
1.5 More Detailed Research Questions

This researcher used several specific questions to frame the study of historical analysis and students’ understanding of the place value system. They were as follows:

1.5.1 On History

Examined in light of mathematics education research:

i. What were the main conceptual stages in the historical construction of the decimal place value numeral system in India, and how could an awareness of these stages be useful in teaching and learning?

ii. What were the key ideas in the development of algebra in the history of Indian mathematics, especially pertaining to variables as a sign system, and how might they be relevant to students’ awareness of a general positional notation?

iii. Based on the analysis of these historical ideas and conceptual stages, examined in light of mathematics education research, can a teaching framework be developed that might enhance Year 9 students’ understanding of a general place value system?

1.5.2 On Teaching

i. How does the framework (developed from historical ideas) combined with the use of multiple representations and modeling with concrete materials, influence Year 9 students’ understanding of decimal place value system and its generalisation?

1.6 Originality of the Research

In terms of teaching and learning, mathematics begins with numbers and counting. Hence, in today’s context, it can be said that the decimal numeration system underpins school mathematics and its teaching. Due to this significance, numerous researchers and educators have addressed perceived problems in teaching and learning place value structure. However, there are many areas that have not yet been explored. This includes, for example, a detailed study of the history of Indian mathematics with a view to enhancing students’ conceptual understanding of both number and algebra. A careful study and analysis of the development of the decimal numeration system, arithmetic operations and algebra in the
history of Indian mathematics, in the light of mathematics learnt at school, has not yet been undertaken. This is one respect in which this research was novel. In addition, this study proposed to construct a framework for the learning of the decimal system and early algebra, as well as investigating the usefulness of this framework. The aim was to base the framework on a combination and synthesis of ideas from a detailed review of both mathematics education research and the history of Indian mathematics. As far as this researcher is aware, students’ understanding of place value has not been researched in the light of historical development and this interpretation is also an original contribution.

1.7 Outline of the Thesis

This thesis is organised into 8 chapters and these are briefly described here. Chapter 1 (the current chapter) presents the introduction and an overview of the study, including aims and research questions. Chapter 2 provides a theoretical orientation to the research study. It discusses some theoretical issues arising in the history and pedagogy of mathematics, and the methodology adopted for the extraction and incorporation of historical ideas. This is followed by theoretical arguments in Chapter 3 on; a) the structural components of the place value system and its understanding, b) multiple bases, literal symbols and algebraic generalisation, and c) representations/signs, related semiotic issues, and matters related to the use of concrete materials in teaching and learning. Chapter 4 is given to an analysis of the history of Indian mathematics, particularly the stages of construction of the Hindu-Arabic decimal system, and the development of signs for variables in algebra. Discussions related to the integration of ideas in the history of mathematics and mathematics education research pertaining to the decimal number system is also presented. This chapter also provides the suggested frameworks for the teaching sequence and answers the research questions related to history of mathematics. The approaches suggested in this chapter are also intended to help inform teachers in charge of primary and junior secondary curriculum. The research design and methodology of the intervention are described in Chapter 5, while Chapter 6 reports the results of the study from the teaching sessions, and pre- and post-tests. Chapter 7 comprises a discussion of the results, in light of the reported research in the literature and the historical findings. Finally Chapter 8 concludes the thesis with a summary of the results, limitations of the study, and implications for; a) further research and, b) teaching, curriculum and teacher education.
Figure 1.1. Concept map on positional notation (Schmittau & Vagliardo, 2009, p. 49).
Figure 1.2. Schmittau and Vagliardo’s concept map (2009) showing “Symbols” and “Many Bases” (p. 50) and “Exponents” (p. 53).
Figure 1.3. Schmittau and Vagliardo’s concept map (2009) showing “Operations” and “Place value” (p. 54), and a pedagogical approach (p. 55)
CHAPTER 2
THEORETICAL PERSPECTIVES ON HISTORY IN
MATHEMATICS EDUCATION

2.1 Introduction

Over many decades, educators, mathematicians and historians have considered the value of integrating elements of history of mathematics in mathematics education. Niels Henrik Abel (1802-1829) wrote in one of his notebooks: “It appears to me that if one [mathematician] wants to make progress in mathematics one should study the masters” (cited in Fauvel & van Maanen, 2000, p. 35). About a century ago, Cajori (1919) saw in history of mathematics an inspiring source of information for teachers. Even so, the question arises as to whether history of mathematics has a role in mathematics education. We do not have to use history in teaching and learning; however history of mathematics can play a valuable role, especially for students who complete their education with less understanding than is useful for them and for students with an anxiety or fear of mathematics. It is then worth exploring alternate pathways (such as incorporating history of mathematics) to improve the process of teaching and learning.

From the time Cajori advocated its use, and particularly in the last thirty years or so, mathematics teachers and educators have increasingly made use of the history of mathematics in their lessons plans. Researchers and educators have asserted that the history of mathematics is a valuable resource for motivating students to learn mathematics, and one of the benefits is in enhancing the understanding of mathematics itself (e.g. Arcavi, Bruckheimer, & Ben-Zvi, 1982; Fauvel & van Maanen, 2000; Gupta, 1995b; Katz, 2007). As a result of increasing interest in history, The International Study Group on the Relations between History and Pedagogy of Mathematics (HPM) was formed in 1972. In view of the potential of history as a didactic tool, many articles and books have appeared in increasing number over time, including educational reports and accounts of teaching experiences (e.g. Arcavi, 1987; National Council of Teachers of Mathematics, 1969). A novel turn in the last two decades has been a search for underlying theories and methodologies for the integration of history in the teaching and learning of mathematics, and this culminated in
the comprehensive report, *History in Mathematics Education: The ICMI Study*, edited by Fauvel and van Maanen (2000). Since then, a number of articles and reports (e.g. Arcavi & Isoda, 2007; Radford & Puig, 2007) have been written on the relationship between history of mathematics and teaching and learning.

In this chapter some theoretical perspectives on ‘whys’ and ‘hows’ (Gulikers & Blom, 2001; Jankvist, 2009; National Council of Teachers of Mathematics, 1969; Tzanakis & Arcavi, 2000) of the use of history of mathematics in mathematics education are discussed. The ‘whys’ of using history fall mainly into two categories: significance for students and significance for teachers.

There are a number of ways of using the history of mathematics. However, two theories that relate to history and teaching are *ontogenesis* and *phylogenesis* and these theories are examined first under the section on the ‘hows’ of history. Subsequently the discussion involves the ways of using history (the ‘hows’) and these may be placed into two categories of approaches; historical information approach and history-based approach.

As can be expected, arguments against the use of history have been raised as well, and these objections are integrated in the sections of this chapter. Arguments for the use of history of mathematics (the ‘whys’), are considered first.

### 2.2 The ‘Whys’ of Using History

A number of reasons have been proposed by researchers (e.g. Fauvel & van Maanen, 2000) for applying history of mathematics in mathematics education. These are discussed below and in order to bring a structure into the different arguments, they have been categorised under ‘significance for students’ and ‘significance for teachers’ (Gulikers & Blom, 2001).

#### 2.2.1 Significance for Students

A main reason for using history of mathematics in teaching and learning is that it can aid *conceptual understanding* of mathematics. It can play the role of a ‘cognitive tool’ (Jankvist, 2009 p. 238) in supporting students’ mathematical conceptual development. Knowledge of historical aspects of a topic (such as the numeration system) in different cultures and at various stages in history can give students more insight with respect to the subject matter under consideration. History can also help to set out a hypothetical learning
trajectory for the classroom in which ‘learning obstacles and smooth progress are in balance’ (Gulikers & Blom, 2001, p. 228). Such a teaching/learning sequence may help students to better understand related concepts of the topic. This is the idea that is developed later in this thesis.

Another key reason for using history in the classroom is that it humanises the subject and helps present mathematics as a dynamic activity; it “gives mathematics a human face” (Fauvel, 1991, p. 4). A similar view is held by Tzanakis and Arcavi (2000) who comment that a greater awareness of history helps students learn that mathematics is an evolving subject and is a human endeavour. History is a powerful tool to counter students’ widespread perception that mathematical truths and methods have never been doubted or disputed. It reveals to students that mathematicians of the past also had doubts and made many mistakes. Students may derive comfort from realising that they are not the only ones with difficulties (Arcavi, 1995; Fauvel, 1991). Hence, they may be less discouraged and disappointed by their own failures and misunderstandings, recognising that these same misunderstandings have been the building blocks of the work of famous mathematicians of the past. Additionally, they may appreciate that the same concept that they are having difficulty understanding, actually took prominent mathematicians hundreds of years to develop into its current form. As a result, students may have a more realistic appreciation of their own attempts at understanding and problem-solving. History can provide role models of human activity and students can learn the value of developing patience and persistence, of posing questions and of striving to develop creative thought (Tzanakis & Arcavi, 2000).

In this connection, an integration of history can play a significant role in uncovering how mathematical concepts and structures have been invented by people (Gulikers & Blom, 2001) as tools to understand phenomena of the world and that the development of ideas did not proceed as smoothly as is suggested by most textbooks. This makes mathematics more concrete for students. Furthermore, historical ideas, origins of problems, methods and proofs may be potentially valuable to motivate and interest students and hence engage them in learning. Historical ideas also may make mathematics less frightening, more enjoyable and inspiring. In this context, Daniel (2000) asserts that not only mainstream students, but also students with learning disabilities, and gifted and talented students can enjoy and be inspired by the history of mathematics. In Daniel’s
(2000) view, the inclusion of history opens up possibilities of helping to cater to the curiosity and thinking needs of gifted and talented students.

Working with the history of mathematics in the classroom can also reveal the multicultural origins of mathematics, and show that it is not just a product from Europe, as many appear to believe (Ernest, 1998; Joseph, 2000). An emphasis on how mathematical theories flourished in many countries can illuminate the diverse contributions of various cultures to contemporary mathematics (Furinghetti & Radford, 2002). Showing how mathematical thinking and applications developed in various cultures in response to the needs and thinking of different societies, enables one to place mathematics in a broader perspective. Not only this, it facilitates a deeper understanding of the concepts embodied in mathematics and also encourages creativity and confidence in using the different aspects of mathematics. For example, studying the multiplication algorithms from various civilizations such as the Egyptian duplication method, Indian Vedic method and the Chinese abacus method can provide fresh insight into the multiplication concept and other elementary concepts in our number system (Arcavi, 1987; Nelson, Joseph, & Williams, 1993). Drawing on the mathematical heritage of different ethnic minority groups and thus recognising and valuing their cultural achievements could help to counter the sometimes historical devaluation of them and develop tolerance and respect among fellow students (Tzanakis & Arcavi, 2000).

A historical (and multicultural) approach such as that described above, if fostered, can demonstrate the links within mathematical topics and between mathematics and a broad range of subjects (Nelson et al., 1993), including history and geography, art and architecture, biology, physics, music and philosophy. According to Furinghetti and Somaglia (1998), one of the problems in education is the phenomenon that they term the “fragmentation of knowledge” (p. 48), due to the fact that school subjects are taught separately from each other. As a result, some students hold a very poor image of mathematics; they appear to think that mathematics is a boring subject with no connection to real life. Furinghetti and Somaglia (ibid) advocate the idea of interdisciplinarity (p. 48), which emphasises the links between disciplines (Nelson et al., 1993), and point out that the history of mathematics is an important element in moving across different subjects. It provides the opportunity to see the genesis of concepts and the underlying ideas of the disciplines, and as a consequence, creates a richer image of mathematics.
2.2.2 Significance for Teachers

While there are benefits for students as outlined above, a study of the history of mathematics can strengthen the didactical background of teachers, and increase their pedagogical repertoire including explanations, examples and alternative approaches to introduce a topic and to problem-solving. In this connection, what follows is a statement from the UK National Curriculum document (cited in Fauvel, 1991, p. 3):

The teacher who knows little of the history of mathematics is apt to teach techniques in isolation, unrelated either to the problems and ideas which generated them or to the further developments which grew out of them... Mathematics can be properly taught only against a background of its own history (Ministry of Education, 1958).

The above quotation suggests that the incorporation of history of mathematics is important as a didactical strategy. Two common reasons that are presented for the inclusion of a historical dimension for teachers are that history of mathematics provides opportunities for a better idea of what mathematics is, and helps teachers to understand concepts and methods better, including their developmental stages (Barbin, 2000). Specifically, a study of history may help teachers to understand key stages in the development of Mathematical Sign System (MSS) in algebra (Puig & Rojano, 2004), as well as its conceptual stages (Katz, 2007) and this knowledge can be put to profitable use in devising classroom strategies. Also, according to Tzanakis and Arcavi (2000), by studying history, teachers may identify the motivation behind the introduction of new mathematical concepts in history in specific mathematical topics. In this context, Gupta (1995b) remarks: “Through [history] one can know more clearly as to how and why mathematics is created, grows, develops, changes, is abstracted, and generalised” (p. 16). A study of history can provide teachers with new opportunities for a deeper and more intuitive grasp of mathematics and can also present fresh insights into the development of ideas (Gulikers & Blom, 2001). In turn, such a historical review can be an inspiration for teachers and can increase their interest and enthusiasm for a topic. Gupta opines that as a result of a sound knowledge and understanding of the subject, “a teacher of mathematics will have more confidence. Consequently, he will become a better teacher” (Gupta, 1995b, p. 16). As suggested by Arcavi and Isoda (2007), reading and interpreting historical texts may result in learning the necessary skills and tools that can serve teachers well in attempting to understand students’ ideas. Through historical review, teachers may become
aware of the difficulties and obstacles that appeared in history and which may also reappear again in the classroom (Tzanakis & Arcavi, 2000). An additional advantage is learning from the mistakes of the past helps to avoid current pitfalls and repetition of errors in the related topics.

Teachers may find that information on the historical construction of a mathematical topic helps them to understand: i) stages in learning, ii) the time taken by students for developing mathematical understanding, iii) why a concept is difficult for students, iv) the epistemological obstacles (this aspect is discussed below) to learning and, v) the ways in which obstacles and difficulties can be overcome (Barbin, 2000; Katz, 2007; Sfard, 1995). In particular, according to Sfard, and reiterated by Thomaidis and Tzanakis (2007), familiarity with the history of mathematics is indispensable to teachers to make them alert to deeply hidden difficulties concerned with new concepts, and this knowledge can be used to modify teaching. As a result, this historical knowledge could bring about an overall change in the teacher’s approach, with the historical element present implicitly or explicitly in the design of teaching strategies.

Considered above are some arguments why we should use history in teaching and learning. However, some objections and difficulties to using history have been raised as well (see e.g. Fauvel, 1991; Fauvel & van Maanen, 2000) and this section ends with some of the main arguments. The first objection is that history may be confusing to students rather than enlightening. This is related to the second objection, which is a lack of teacher expertise on history; teachers may not know enough about the history of mathematics, which is a consequence of a possible lack of appropriate teacher education programmes for incorporating history of mathematics in the classroom (Fauvel, 1991). Thirdly, it is argued that even for those teachers who may want to integrate history in teaching there is limited (or no) access to appropriate resources. However, with respect to this objection, in the last couple of decades, there has been an increasing amount of historical materials available to teachers (e.g. Katz & Michalowicz, 2004). A fourth objection is the lack of time; there is not enough time to teach mathematics as it is and preparation of historical material will require more time (Jankvist, 2009). However, this argument is relative since materials once produced can be used year after year and can still have positive effects on the quality of teaching and learning (Gulikers & Blom, 2001). Also time required at first may have a long-term pay-off in improving the attainment of objectives in the curriculum (Brownell,
1947; Fauvel, 1991). A fifth objection is lack of assessment materials. There are no clear and consistent guidelines for incorporating historical components in students’ assessments. Notwithstanding the objections that have been raised and which Tzanakis and Arcavi (2000, pp. 202-207) have attempted to answer, the discussion above indicates that history can play a positive role in mathematics education (e.g. Arcavi et al., 1982; Katz, 2007; Radford & Puig, 2007). The following section deals with the different ways that history of mathematics can be applied to mathematics education.

2.3 The ‘Hows’ of Using History

Thus far we have considered ‘why’ we might use the history of mathematics in teaching and learning, but an important question is ‘how’ it might be used and this is addressed in this section. However, prior to considering the various ways of incorporating history, it is important to examine the philosophical thesis, that Jankvist (2009) calls a theoretical evolutionary argument; the relationship between ontogeny and phylogeny. That is, the relation between individuals’ learning of mathematical concepts and the historical evolution of the concepts.

2.3.1 Ontogeny and Phylogeny

The link between historical developments in mathematical thinking and students’ learning of mathematics is often made in terms of biological recapitulationism, an idea following Darwin’s writings on the evolution of the species and introduced at the end of the 19th century. According to this idea formulated by the German biologist Ernst Haeckel, the development of the individual (ontogenesis) recapitulates the development of mankind (phylogeny) (Radford, 2000). (A detailed account of this idea is given in Fauvel and van Maanen, 2000, Section 5.1, and in Furinghetti & Radford 2002). Haeckel transferred this ‘biological’ law to the psychological domain, saying that it indicates “… the functioning of the child psyche as something travelling the same path as his or her ancestors” (Furinghetti & Radford, 2002, p. 634), and that the “mathematical development in the individual retraces the history of mathematics itself” (Fauvel, 1991, p. 3). By implication this means that in order to learn mathematics, the student’s mind must go through the same evolutionary stages that earlier generations had experienced. The recapitulation argument not only applies to mathematics as a whole but also to single mathematical topics and
concepts (Jankvist, 2009). However, it soon became clear to psychologists and researchers that such a strong statement cannot be sustained, and hence this idea was questioned by many researchers and psychologists, among them Piaget, Garcia, and Vygotsky. In this context, Furinghetti and Radford (2002, 2008) comment that the problem with the recapitulation argument is due to the fact that psychological recapitulation theory is at odds with history and the development of the individual, because the environmental conditions are not constant. The authors explain:

For evolutionary based recapitulation theories, in contrast, the environment is supposed to play in the development of species a role. The individual is seen as an organism adapting to his or her environment; in the interplay between individual and environment, some of the biological and psychological functions may develop, whereas others may be lost according to the natural selection.... Indeed, from a theoretical point of view, history and recapitulation become difficult to reconcile because, on one hand, Haeckel’s psychological recapitulation supposes that present intellectual developments are to some extent a condensed version of those of the past. On the other hand, natural selection is presented as a function of the environment against which individuals act. For recapitulation to be possible, therefore, such an environment must remain essentially the same, which obviously is not the case. Given that the environment changes, it becomes difficult to maintain that the children’s intellectual development will undergo the same process as the one children experienced in the past (Furinghetti & Radford, 2002, pp. 634-635).

This means that it is difficult to uphold the recapitulation theory ontogenesis and phylogenesis since the social, cultural and economic environments in history are different from the present time. Hence one cannot say that events in history are always exactly reproduced in the classroom. The theoretical difficulties of a superficial view of psychological recapitulation encouraged new reflections concerning the possible relations between phylogenesis and ontogenesis. There have been discussions on whether, and to what extent it is possible to follow a ‘genetic’ teaching model that critically considers the historical roots of mathematical knowledge. Two views, according to Radford (2000), that have been particularly influential in the use of history in mathematics education are those of Piaget and Vygotsky. Radford (2000) and Furinghetti and Radford (2002, 2008) analyse the different stances adopted by the two psychologists and an outline of the discussion follows.

As a reaction to the simplistic psychological recapitulation theory, Piaget and Garcia (1989), rather than seeking a parallelism of contents between historical and psychogenetic
developments, attempted to understand the processes or mechanisms of invention and discovery. Piaget and Garcia (ibid) introduced the concept of genetic development. They argued that the problem of knowledge should be understood in terms of the intellectual instruments and mechanisms allowing its acquisition. In their view, the first of those mechanisms is a general process which takes into account the individual’s assimilation and integration of new knowledge on the basis of previous knowledge. The second mechanism that they identified is a process that leads from the intraobject, or analysis of objects, to the interobject, or analysis of the transformations and relations of objects to the transobject, or construction of structures. The assimilation mechanism has its roots in the conception of knowledge as an extension of the biological nature of individuals. The second mechanism “obeys a structural conception of knowledge and reflects the role that mathematical and scientific thinking played in Piaget’s work” (Furinghetti & Radford, 2002, p. 636). The two mechanisms were thought of as invariants, not only in time but also in space. That is, these mechanisms apply regardless of the historical period, and the location of the individual.

Thus, according to Sierpinska (1994) and Radford (2000), in their approach to investigating the relations between ontogenesis and phylogenesis, Piaget and Garcia did not look for similarities of ideas between historical and psychogenetic developments but of the mechanisms of passage between one historical period and the next period. And they attempted to show that the mechanisms outlined correspond to those of the passage from one psychogenetic stage to the following stage. In addition, the authors make a clear distinction between mechanisms to acquire knowledge and the conception of the object by the individual; “Society can modify the latter, but not the former” (Piaget & Garcia, 1989, p. 267).

However, in the above-mentioned mechanisms, what was found to be missing by some critics was the role of culture and social practices in the formation of knowledge. It is in this respect, as pointed out by Furinghetti and Radford (2002, 2008), that Vygotsky’s approach differs from that of Piaget’s. Instead of understanding the formation of knowledge in terms of mechanisms that are outside of the culture, Vygotsky saw cognitive functions resulting in the production of knowledge as a result of two lines of development: a biological (or natural) process; and a historical (or cultural) one. Related to this, he distinguished between what he termed lower and higher mental functions, and proposed that while the former belong to the biological realm of the individual, the latter are
basically social. In this connection, when discussing higher psychological functions, Vygotsky and Luria state:

... if we include the history of the higher psychological functions in the general context of psychological development and attempt to arrive at an understanding of their source from its laws of development, we cannot but arrive at a new concept of the process itself and its laws. Within this general process of development two qualitatively original main lines can already be distinguished: the line of biological formation of elementary processes and the line of the socio-cultural formation of the higher psychological functions; the real history of child behaviour is born from the interweaving of these two lines (Vygotsky & Luria, 1994, pp. 147-148).

In Vygotsky’s approach, the impact of the merging of the two lines of development (biological and historical) in the intellectual growth of the child is that it cannot include the recapitulation theory (Furinghetti & Radford, 2002, 2008). This is partly due to the variability introduced by the socio-historical conditions, which are different in each period of history. The authors (ibid) explain that for Vygotsky, the intellectual functions of the individual are intrinsically human and this is because human activities are mediated by various tools, languages and other systems of signs which are part of cognitive functions; and importantly, these systems of signs as well as artefacts alter cognitive functioning. Hence, in Vygotsky’s account, knowledge is a result of individual and social construction carried out using cultural tools. Moreover, in this process, sign systems may be modified and new ones may be created. In turn, along with changes in activities and attitudes, there will be changes related to phylogenetic development. Relative to the different approaches adopted by Piaget and Vygotsky, Radford (2000) states:

One of his [Vygotsky’s] fundamental differences with Piaget and Garcia’s approach lies in the epistemological role of culture. For Piaget and Garcia, culture cannot modify the essential instruments of knowledge acquisition, for they saw these instruments as originating in the biological realm of the individual ... In Vygotsky’s approach, though, culture not only provides the specific forms of scientific concepts and methods of scientific inquiry but overall modifies the activity of mental functions through the use of tools - of whatever type, be they artefacts used to write as clay tablets in ancient Mesopotamia, or computers in contemporary societies, or intellectual artefacts such as words, language, or inner speech... (Radford, 2000, p. 146).

Thus, for Piaget, social practices do not play a part in the mechanisms of acquiring knowledge; for Vygotsky, the natural and cultural coincide and form a single order of social-biological formation of the individual psyche. The above discussion on the differing
views of Piaget and Vygotsky on recapitulation shows that the relationship between ontogenesis and phylogenesis is a problematic one. However, when their epistemological viewpoint led them to revisit the parallelism that the recapitulation theory had highlighted, Piaget and Garcia (1989) concluded that while the parallel between history and individual development cannot be overemphasised, in broad terms there certainly are some stages that are the same, a view that Sfard (1995) attempted to corroborate and that is discussed below. This belief is also shared by Radford and Puig (2007) who state that the inadequacy of the recapitulation theory does not mean that individual developments (ontogenesis) are completely devoid of past historic-cultural ones (phylogenesis); it only means that the relationship between ontogenesis and phylogenesis is more complex than predicted by such a theory. Hence, there is an indication that while the notion of ‘strict’ parallelism (Thomaidis & Tzanakis, 2007) is not tenable, there exists the possibility of a restricted form of parallelism between history and students’ learning. It does not mean that history can never be used in teaching. A review of the literature on studies involving history of mathematics and the development of mathematical thinking in individuals shows similarities between the two domains, particularly concerning difficulties and the way these were overcome. Some examples of studies involving such similarities are considered next.

In a comparative analysis of the history of algebra with students’ empirical data, Harper (1987) found a parallelism between the evolution of algebraic symbolism (the rhetorical, syncopated and symbolic stages) (see Section 4.4) proposed by Nesselman in 1842) and the way students understand the use of literal symbols in algebra. The author concluded that “… the sequencing of conceptual acquisition appears to parallel that which is to be detected through the study of the history of mathematics” (p. 85). Harper highlighted the fact that it took a long time in history to make the shift from syncopated to symbolic algebra (with parameters or givens) and suggested the use of history in understanding student difficulties and to make students aware of the various usage of letters. Similarly, Sfard (1995) observes “the huge conceptual difference between equations with numerical coefficients and equations with parameters” (p. 26). Likewise, but from a different perspective, Moreno and Waldegger (1991) found that in situations involving the infinity concept, students’ responses were similar to those of different response schemes given by mathematicians throughout history when they were faced with
the same kind of questions. Another example of historical parallelism is the work of Sfard (1995), who, following a Piagetian epistemological perspective, describes knowledge in terms of genetic structural levels. According to Sfard, it is the nature of the relationship between different levels that accounts for the similarity of phenomena appearing in the historical and in the individual’s construction of knowledge. Using Nesselman’s rhetorical, syncopated and symbolic stages in algebra, Sfard attempted to identify those developmental invariants that ensured the shift from rhetorical and syncopated to symbolic algebra, concluding that one of the invariants is the priority of operational over structural thinking. Operational thinking involves thinking about objects in terms of computational processes and structural thinking is related to abstract objects conceived structurally on a higher level.

While the above examples show historical parallelism (albeit limited) in terms of i) conceptual stages, ii) the mechanisms of passage from one conceptual stage to the next, and iii) and difficulties, a special value of historical analysis is the identification of obstacles, specifically epistemological obstacles (Jankvist, 2009; Sierpinska, 1994), and this will now be briefly discussed. It is an important aspect because, according to Brousseau (1997), many of students’ difficulties centre around these obstacles. The idea of epistemological obstacles was first developed by Bachelard in 1938 and then later introduced into teaching by Brousseau in the 1970s. Bachelard states:

We must pose the problem of scientific knowledge in terms of obstacles. It is not just a question of considering external obstacles... It is in the act of gaining knowledge itself, to know, intimately, what appears, as an inevitable result of functional necessity, to retard the speed of learning and cause cognitive difficulties. It is here that we may find the causes of stagnation and even of regression, that we may perceive the reasons for the inertia, which we call epistemological obstacles. (Bachelard, 1976 cited in Gallardo, 2001)

This means that the study of errors and obstacles is an inseparable counterpart of the study of the development of fundamental concepts, a view also held by Sierpinska (1994). Similarly, Brousseau’s (1997) approach is based on the assumption that an epistemological obstacle is something that is wholly relevant to the sphere of knowledge. In his account, linking historical and psychological phenomena, Brousseau, in the manner of Bachelard, formulates an epistemological obstacle as characterised by its reappearance in both the history of mathematics and in present-day students learning mathematics. It appears as the
source of repeated non-random errors and misconceptions that learners produce when they are trying to solve a problem. According to Brousseau, an epistemological obstacle is a conception or a piece of knowledge that functions well within a certain topic but produces false responses outside of this context, leading to contradictions. A related example given by Hiebert and Wearne (1986) is students’ misconceptions regarding the use of zero in decimal fractions. Further to his definition, Brousseau (1997) states: “Obstacles of really epistemological origin are those from which one neither can nor should escape, because of their formative role in the knowledge being sought. They can be found in the history of the concepts themselves” (p. 87). Similarly, Sierpinska (1994) states that “epistemological obstacles are unavoidable” (p. 126). In this context, Brousseau claims that history is not only useful to identify obstacles, but can also help to overcome them because “certain of the students’ difficulties can be grouped around obstacles attested to by history”. And, as commented by Gallardo: “Historical analysis becomes relevant for mathematical education when it throws light on understanding the emergence and evolution of concepts, when it helps in explaining the conditions which make the problem formable,...” (Gallardo, 2001, p. 124) and by Dorier and Rogers (2000) “…an epistemological reflection on the development of ideas in the history of mathematics can enrich didactical analysis by providing essential clues which may specify the nature of the knowledge to be taught, and explore different ways of access to that knowledge” (p. 169). In addition, organisation of teaching situations that make it possible to overcome epistemological obstacles can pave the way for richer conceptualisations (Radford, Boero, & Vasco, 2000). However, Brousseau (1997) argues against reproducing in the classroom historical situations that led mathematicians to overcome these obstacles. In his view, shared by other researchers (e.g. Katz, Dorier, Bekken, & Sierpinska, 2000; Sierpinska, 1994), historical analyses have to be accompanied by investigations of how mathematical concepts develop in students, and didactical strategies need to be modified according to the classroom situation.

In the above context, an alternative view of the link between ontogenesis and phylogenesis is offered by Radford and Puig (2007) and this was the one found to be most useful for this thesis. One of the difficulties in algebra, according to them is related to students’ understanding of the meaning of signs and the syntax of algebraic language. The authors introduce the Embedment Principle in their study involving a historic semiotic analysis and contemporary students’ understanding of algebraic concepts. According to
this principle, “...our cognitive mechanisms (e.g. perceiving, abstracting, symbolising) are related, in a crucial manner, to a historical conceptual dimension ineluctably embedded in our social practices and in the signs and artifacts that mediate them” (Radford & Puig, 2007, p. 148). The authors explain that the sign systems that students encounter in school are the bearers of the results of cognitive activity of previous generations; this socio-historical and cognitive activity is embedded in the sign (semiotic) systems and their meanings, and the social practices that they mediate offer students lines of conceptual development. The authors’ idea implies that the history of development of system of signs (or MSS) (Puig & Rojano, 2004), which is related to the historical development of a concept, has implications for teaching and learning. It means that the different developmental stages in history are important aspects that may be useful in constructing teaching strategies for the purpose of conceptual understanding. However, the meaning of signs and symbols are often not transparent for students. They need to be actively engaged in the process of sense-making (Arcavi, 1994), and in this respect, the teacher plays a crucial role. In relation to this, it is noted that the majority of the analyses and studies (e.g. Harper, 1987; Katz, 2007; Puig & Rojano, 2004; Sfar, 1995) in mathematics education conducted so far involving stages in historical development and students’ understanding of a sign system, are in the domain of algebra. As far as this researcher is aware, there is virtually none involving the development of the sign system in decimal numeration, and its relation to algebraic symbolism in mathematics education research.

Thus far in this section the recapitulation argument for relating ontogenesis and phylogenesis has been considered. On the basis of the above discussion, it seems that there are some similarities between the difficulties that students might have today and those experienced by mathematicians in the past. This ‘historical parallelism’ extends to obstacles, and the way in which particularly epistemological obstacles identified in history can be recognised in the errors made by present-day students. A study of history also reveals important milestones and their developmental stages, such as the invention of the decimal place value system, and the system of signs in algebra. Hence teachers’ familiarity with the history of mathematics could prove useful for understanding and anticipating obstacles and student difficulties, and for devising classroom strategies to overcome them.

In the section that follows, different ways (the ‘hows’) of using the history of mathematics in the classroom are considered.
2.3.2 The (Direct) ‘Historical Information’ Approach

This approach is a direct method of incorporating history of mathematics in the classroom where the history is ‘out in the open’. In Chapter 7 of the ICMI study (Fauvel & van Maanen, 2000), Tzanakis and Arcavi give an analytical survey of how the history of mathematics can be incorporated into the classroom. In the ‘direct’ approach, the teaching and learning of mathematics is supplemented by historical information, and these supplements may vary in terms of size and range. This includes isolated factual information or historical snippets (Swetz, 1994) such as names, dates, famous works and events, biographies, famous questions, motivating problems, anecdotes and stories (Tzanakis & Arcavi, 2000). One way to think of these smaller supplements suggested by Jankvist (2009) is “as spices added to the mathematics education casserole” (p. 246). These historical snippets can be used in relation to motivational and multi-cultural arguments (Gulikers & Blom, 2001); for example, anecdotes, biographies or famous problems can be used to introduce new concepts to students. In addition, longer instructional units can be prepared using historical materials. The first type is what Tzanakis and Arcavi refer to as “historical packages” (p. 214), which are collections of materials focusing on a small topic and appropriate for two or three lessons. In the middle range we find activities of 10-20 class periods where the packages can be introduced through historical plays and problems. Towards the higher end of the scale there are full courses or books on the history of mathematics which may be an account of historical data or a history of conceptual developments. These approaches could be implemented in other ways such as student research projects on specific topics. From a different perspective, FitzSimons (2000) suggests the use of ideas from ethnomathematics (sometimes related to the history of mathematics) which was first introduced by D’Ambrosio (1985). Commenting on its usefulness in education, FitzSimons, states that ethnomathematics “...is a powerful means of valuing experiences and cultures of members of minority groups while expanding the horizons of all participants” (Fitzsimons, 2000, p. 180).

One of the ideas that is recommended by Tzanakis and Arcavi is that of “Experiential mathematical activities” (Tzanakis & Arcavi, 2000, p. 214). According to the authors, through such an activity students can be guided in an experiential way to a greater understanding and appreciation of the current decimal place value system. The authors give
an example of an activity involving notations. Students can be introduced to ancient numeration systems (Arcavi, 1987; Arcavi & Isoda, 2007) and be given the opportunity to write different decimal numbers in these systems and vice versa. Classroom discussions involving base, place value, zero, number of symbols in the different systems and the related rules could lead to a deeper awareness of the underlying principles of the various numeration systems. As will be seen in Chapter 5, experiences involving various notational systems were incorporated into the teaching sequence in this thesis.

2.3.3 History-Based (Indirect) Approach

This category of using history is what may be called a genetic approach to teaching and learning inspired by the historical evolution of mathematics (Furinghetti & Radford, 2008; Tzanakis & Arcavi, 2000). In this method, introduced by Otto Toeplitz (Furinghetti & Radford, 2008), after a study of history of mathematics, which may include grasping difficulties in the topic and identifying possible obstacles, the teacher identifies crucial steps in the historical development of the mathematics. Hence, historical knowledge helps the teacher to understand stages in learning of a topic (Barbin, 2000) and consequently these important steps may be reconstructed for use in the classroom (Furinghetti & Radford, 2002, 2008), although this might involve modifying the historical order of events. The main aim is not to teach history but to find learning sequences, to adapt historical conceptualisations to facilitate student understanding of mathematics. In this reconstruction, history may be used either explicitly or implicitly (Barbin, 2000). In an explicit integration (Tzanakis & Arcavi, 2000) of history teaching sequences are arranged so that the evolution of the main historical events and the progress of mathematics in a particular period are presented directly to the students, which is referred to as the ‘direct genetic method’ by Toeplitz (1927, cited in Jankvist, 2009).

In contrast to the above, in an implicit use of history, the teacher, after undertaking a historical analysis, modifies the conceptualisations in such a way that they may be no longer related to the history of mathematics. The reconstruction is made, keeping in mind the obstacles and difficulties in history, to smooth the learners’ path and to increase understanding of a topic. Toeplitz (ibid) refers to this method as the ‘indirect genetic method’. According to Furinghetti and Radford (2008), Toeplitz’s method is realistic and may be considered a compromise between the logical and the developmental ways of
thinking about teaching mathematics. Freudenthal (1991), who had a slightly different view of the historical approach from Toeplitz, says: “Children should repeat the learning process of mankind, not as it factually took place but rather as it would have done if people in the past had known a bit more of what we know now” (p. 48). In a historical reconstruction where the history enters implicitly, Tzanakis and Arcavi (2000) suggest a didactic sequence in which concepts and notations that appeared later than the topic under consideration, may be utilised, bearing in mind that the overall teaching goal is to understand mathematics in its present form. In this method, the teaching sequence may or may not follow the historical order; it is neither strictly deductive nor strictly historical. However, Tzanakis and Arcavi (ibid) add that the two possible ways of reconstruction of the historical evolution are not mutually exclusive; both approaches (explicit and implicit) can be used in complementary ways. The authors comment on the two possible approaches:

... in an explicit integration of history, emphasis is on a rough but more or less accurate mapping of the path network that appeared historically and led to the modern form of the subject; in an implicit integration, the emphasis is on the redesigning, shortcutting and signalling of this path network (Vasco 1995, 62). In both cases, historical aspects of famous problems, intuitive arguments, errors and alternative conceptions may be incorporated in teaching ... (Tzanakis & Arcavi, 2000, p. 210)

The possible advantages of the above process of combining the approaches offer interesting potential for a deep and holistic understanding of the subject or a particular topic. For example, possibilities include a mix of explicit and implicit use of history according to the type of students and the topic under consideration. As noted by Tzanakis and Arcavi (ibid), in the indirect method there is no unique specific way of presenting a topic; it is not a strict step-by-step algorithmic method for teaching. A pedagogical sequence may be designed keeping in mind the historical evolution of the subject and the obstacles and difficulties encountered in history, and the reconstruction could be in a modern context and notation, so that they become more accessible to students. The following are some examples of teaching based on history. An interesting example of a teaching sequence inspired by history is Radford’s (1995) analysis of medieval Italian algebra related to the concept of unknown as a ‘hidden’ quantity. Another instance is van Amerom’s (2002) thesis where she enquired into the teaching-learning process pertaining to the transition from arithmetical to algebraic problem solving by drawing on the
historical development of algebra. Similarly, but from the perspective of teacher education, Clark (2006) investigated five teachers’ experiences with logarithms, their efforts to study the topic and to use it in the classroom. An example of epistemological analysis is the work of Puig and Rojano (2004) who scrutinised the different evolutionary stages of the sign system of symbolic algebra and then set out a path of theoretical development for educational algebra (Filloy, Puig, & Rojano, 2008). And adopting a different perspective, Gallardo (2008) analysed cognitive processes displayed by students in their transition from arithmetic to algebra from a historical-epistemological standpoint, choosing three texts from different historical periods for a study of negative numbers and zero.

What has been discussed so far in this section is the viability of two methods; a direct ‘historical information’ approach and an indirect ‘history-based approach’, where history of mathematics is used as a tool for a better and more thorough learning of mathematics. However, these approaches may also be used for learning about features of the history of mathematics that serves as a goal in itself, and which involves meta-issues (Jankvist, 2009). That is, learning about history for the sake of awareness of mathematics as a subject (Tzanakis & Arcavi, 2000). Hence, the focus here is on the developmental and evolutionary aspects of mathematics as a discipline (Jankvist, 2009). As elaborated by Tzanakis and Arcavi (2000), this may involve meta-issues such as social and cultural influences in mathematics, the development of concepts and their associated motivations and mechanisms, the role of notation, modes of expression and representations, the role of paradoxes, intuitions and heuristics concerning specific questions, and motivations for generalising, abstracting and formalising in specific contexts.

In essence, this chapter has considered a number of different possible uses of the history of mathematics in teaching. In addition, reasons as to why we should use history and also perspectives on how history of mathematics can be incorporated in teaching and learning have been presented. Additionally, it has examined the relationship between ontogenesis and phylogenesis, and crucial factors pertaining to epistemological obstacles and difficulties. Also, the study has highlighted some of the advantages of using history of mathematics in helping students to overcome obstacles and difficulties in mathematics and to understand the subject better. Inspired by history, the teacher can prepare appropriate teaching strategies that may be in an approximate historical order. However, Katz et al. (2000) emphasise the importance of reviewing mathematics education literature in order to
understand students’ difficulties and for the development of teaching sequences incorporating history, since in this latter case the fundamental differences between the two domains need to be taken into consideration.

2.4 Methodological Considerations (Historical analysis)

The primary goal of the research study described in this thesis was to improve students’ understanding of the structure of the current Hindu-Arabic numeration system. However, in order to develop a teaching sequence for this, a substantial part of the study dwelled on the history of mathematics (mainly Indian history) in order to extract ideas related to number and algebra that could help enhance awareness of a general place value system with zero.

In this connection what is considered in this section is a historical-critical methodology, which according to Piaget and Garcia (1989) is one of the fundamental tools of an epistemological investigation. As explained by Gallardo: “It is concerned with historical analysis, not as an account of a sequence of discoveries, but as the historical reconstruction of parts of knowledge, in order to carry out a ‘critical’ analysis, in a sense akin to the Kantian idea of criticism” (Gallardo, 2001, p. 122). In Piaget’s work, the reconstruction of the history of science cannot be separated from a psychological analysis, and from a Vygotskian (1978) perspective, historical analysis is important to establish what concepts are most important to teach and the order of these concepts. In this research, the historical-critical analysis (Filloy et al., 2008) forms an important methodological aspect. An example of an application of this methodology is Gallardo’s (2001) work involving the investigation of problems of learning and teaching negative numbers and elementary algebra. A characteristic of using this methodology employed in the theoretical framework given below is a recurring back and forth movement from the analysis of history of mathematics texts to an examination of related theoretical and empirical studies in mathematics education literature. For instance, some aspects mentioned in writings such as persistent student errors in naming numbers, difficulties with the multiplicative structure and zero resulted in a deeper reading of the historical texts. Similarly, the naming of large numbers and number names for powers of ten that occurred from a very early period in Indian history caused a search for articles on large numbers and a review of exponentiation in the mathematics education literature.
Another example of the cyclic process of the historical-critical methodology is that of the place value system as a *system of signs* or a *representational system* (Ernest, 2006). As proposed by Becker and Varelas cognitive growth depends on both conceptual/structural and semiotic/sign aspects and the authors link these ideas to place value numeration. Again, the repeated to and fro readings between the two realms of history and educational psychology foregrounded the importance of understanding the *notation* in the place value system, and its links to the multiplicative grouping structure. On the other hand, some key aspects in the history of mathematics, such as the *multiplicative stage* of the *sign system* could have been bypassed, however were probably brought to the fore due to readings from the educational literature.

Hence, in this research study, the reading of history books in the light of mathematics education writings and vice versa, facilitated bringing to the surface some crucial aspects that might otherwise have been hidden. This feature of alternating between theoretical analysis of historical texts and consideration of empirical work conducted in classroom environments is indicated as *articulation* in the framework that is given in Figure 2.1. Consequently, the to and fro movement between history and classroom studies situates this research in the sphere of mathematics education, in contrast to a pure historical or epistemological inquiry (Gallardo, 2001; Puig & Rojano, 2004).

The use of the historical-critical methodology was intended to:

- Highlight the motivations behind the development of the place value system with zero and the conditions that made this development possible;
- Shed light on the nature of some of the difficulties and epistemological obstacles related to learning the place value concept;
- Reveal the stages in the evolution of the decimal numeration as a sign system;
- Throw light on the development of the sign system in algebra, generalisation and the variable concept, and the associated difficulties; and
- Shed light on how the obstacles related to the development of a decimal place value system and then symbolic algebra were overcome.

As shown in the framework in Figure 2.1, the historical-critical method led to the integration of ideas from the two domains and then further guided the design of classroom
activities (the related teaching frameworks are given and discussed in Chapter 4). This framework was derived and extended from Radford’s (2000) framework and provided a basis for the scrutiny of the history of mathematics, which led to extraction of ideas in this research. As indicated by the different symbols (triangles and squares), the historical-critical analysis facilitated the incorporation of ideas from both historical and psychological domains into the teaching model, sometimes involving a combination of ideas from the two disciplines.

Figure 2.1. Theoretical framework adapted from Radford (2000) on the use of history in mathematics education.
CHAPTER 3

STRUCTURE OF THE NUMERATION SYSTEM:
EDUCATIONAL PERSPECTIVES

3.1 Introduction

The first chapter in this thesis provided an overview of the research study and Chapter 2 discussed some of the main theoretical issues related to the integration of the history of mathematics in mathematics education. Chapter 2 also outlined the historical-critical methodology adopted in this research for an analysis of history and the extraction of historical ideas.

While the methodology implemented here involved a cyclic process, for the purpose of clarity, the analysis of mathematics education writings (the current Chapter) and the history of mathematics (Chapter 4) have mainly been kept separate. Given the conceptual and semiotic aspects of the place value system, the present chapter focuses on structural features, multiple bases and generalisation, signs/representations and related issues. However, at the end of each section of the history chapter (Chapter 4), the above considerations culminate in a discussion and a synthesis of ideas from the two domains of history and psychology, in order to design an overall teaching sequence and related sub sequences.

In the current chapter, ideas from the mathematics education literature related to the *structure of place value* are examined. Educational perspectives pertaining to *structure in general*, and *structural thinking*, are considered first, since these aspects are also relevant to place value structure. Next, the *key compositional elements* of the structure of place value, that of *multiplication* and *exponentiation* are scrutinised in relation to teaching and learning. The analysis of these primary components was enhanced by travelling back and forth (see Section 2.4) between education writings and the history of mathematics. The second part of the chapter consists of a pedagogical examination of *multiple bases* and *generalisation in algebra*, where again the historical-critical methodology was applied. In the final section a discussion of *signs/representations* and their importance in the
abstraction and formation of concepts, is taken up. Then the various concrete materials referred to in the literature for understanding place value are also examined.

The Chapter is broadly divided into three main sections, decimal place value structure, multiple bases and generalisation, representations and concrete materials.

3.2 Decimal Place Value Structure

Our present place value numeration system, known as the Hindu-Arabic decimal system, facilitates computations involving whole numbers and decimal fractions by making use of the place value properties of the system. In mathematics, place value is a powerful and a vital foundation concept, which students have to understand in order to comprehend the way numbers are named and to develop flexible use of multidigit numbers and their related operations (Bednarz & Janvier, 1982; Fuson, 1990a; Resnick, 1983; Ross, 1989). Underlying the seemingly simple construction of place value are complex concepts that are implicit within the system. Researchers (e.g. Resnick, 1983; Ross, 1990) agree that the place value concept is difficult to learn and difficult to teach and understanding develops gradually over many years. It requires co-ordination of two aspects of quantity; the quantity associated with each digit (face value), and the multi-unit quantity (power of ten) associated with each place. Understanding place value is linked to understanding how a multidigit number is composed and to knowing its relationships to other numbers. In Pengelly’s (1991) view, it is important for students to have access to the structure of the number system in its entirety. This is because learners need to be cognisant of the additive and multiplicative structures within the system and to see how the individual parts fit together as a whole in order to be fully aware of this sophisticated concept. Crucial to this is to understand the multiplicative structure of the place value system, since the place values/powers of ten are multiplicative units. These multiunits are formed by grouping units with a particular multiplier, treating the composite as a unit and then creating further multi-units (Confrey & Smith, 1995). Understanding the composition of these multi-units and the multiplicative relationships within the system is vital in order to appreciate the overall structure. In this section, since structure in general is also relevant to place value structure, it is defined, its importance is discussed and then the key elements of multiplication and exponentiation in the place value structure are considered.
3.2.1 Structure

Over the years, there has been an increasing focus in mathematics education research on the development of ‘structure’ as an important constituent of mathematics teaching and learning (Arcavi, 1994; Hewitt, 1998; Hoch & Dreyfus, 2004; Sfard, 1991; Warren, 2003). In her seminal paper on the dual nature of mathematical conceptions, highlighting the importance of structure, Sfard (1991) states that “seeing a mathematical entity as an object means being capable of referring to it as if it was a real thing – a static structure, existing somewhere in space and time” (p. 4). Also importantly, structure is the basis for understanding mathematical representation, symbolisation, abstraction, generalisation and proof (Mulligan, Vale, & Stephens, 2009) and a structural perception of mathematics (Hiebert, 1986; Sfard, 1991) is crucial for performing operations in both arithmetic and algebra. In the 1930s and 1940s there was a shift towards teaching mathematical concepts in a meaningful way (English & Halford, 1995) and William Brownell (1945), a proponent of teaching mathematical structure, asserted that “meaning is to be sought in the structure, the organisation, the inner relationship of the subject itself” (p. 481) and the goal of instruction was to make the learner aware of arithmetic “as a system of related ideas” (ibid, p. 482). Another proponent of meaningful arithmetic was Van Engen (1993) who described structure “as the search for patterns; patterns which can be used to arrive at solutions to problems...” (p. 93). Van Engen saw differences in the ability to recognise structure as separating good problem solvers from poor problem solvers. Highlighting the link to language, Vygotsky observes that “The connection between word and meaning is no longer regarded as a matter of simple association but as a matter of structure” (Vygotsky, 1962, p. 123). In accordance with these views, Arcavi (1994) propounds the value of making sense of symbols in order to reveal the structure of a problem prior to solving it.

Psychologists Piaget, Dienes, Skemp and Van Hiele have all attempted to define and explain what structure is. Piaget (1968) defines structure as a “system of transformations” (p. 5) and says that it comprises the three key ideas of wholeness, transformation and self-regulation. He also makes the statements that “Inasmuch as it is a system and not a mere collection of elements and their properties, these transformations involve laws: the structure is preserved or enriched by the interplay of its transformation laws, which never
yield results external to the system nor employ elements that are external to it. (Piaget, 1968, p. 5). Piaget notes that structure and its boundaries are maintained by intrinsic laws. However, Van Hiele (1986) deviates from this view and suggests that “the laws a structure is subjected to are more a result than a part of the definition” (p. 28).

Both Dienes and Skemp emphasise the role of relationships and symbolism in the structure of mathematics. As observed by Dienes; “Mathematics will be regarded rather as a structure of relationships, the formal symbolism being merely a way of communicating parts of the structure from one person to another” (Dienes, 1964a, p. 31). In a similar vein, Skemp (1992), states that structure is an essential feature of mathematics and explains its meaning in these terms:

By structure we mean the way in which parts fit together to make a whole. Often this whole has qualities which go far beyond the sum of the separate properties of the parts. Connect together a collection of transistors, condensers, resistances, and the like, most of which will do very little on their own, and you have a radio by which you can hear sounds broadcast from hundreds of miles away. That is, if the connections are right: and this is what we mean by structure in the present example.

In the case of mathematics, the components are mathematical concepts, and the structure is a mental structure.....But the difference between a mathematical structure and a collection of isolated facts is as great as the difference between a radio set and a box of bits. (Skemp, 1992, p. 3)

Thus, Skemp stresses the importance of recognising relationships, and developing an overall holistic view of mathematical structure. This uncovering of the structure of a situation can be done, according to Skemp (1971) and Arcavi (1995, 2008), by using symbols, which are powerful tools in the sense-making process. In this context, Van Hiele (1986) explains structure as an object governed by rules and states that “Structure is what structure does. First see how structures work, and afterward you will understand what structures are. ..a structure is a given thing obeying certain laws” (pp. 5-6) and “with the word pattern we express the same thing as the word structure” (p. 23). Consistent with these views, researchers in recent years (e.g. Mason, Stephens, & Watson, 2009; Warren, 2003) have proposed the development of mathematical structure as the identification and isolation of a set of properties of elements and the use of these properties in particular contexts. Mason et al. (2009) have this to say about structure:
We take mathematical structure to mean the identification of general properties which are instantiated in particular situations as relationships between elements or subsets of elements of a set. These elements can be mathematical objects like numbers and triangles, sets with functions between them, relations on sets, even relations between relations in an ongoing hierarchy. Usually it is helpful to think of structure in terms of an agreed list of properties which are taken as axioms and from which other properties can be deduced. (p. 10)

The above statement means that learners need to acquire knowledge of related general properties of elements of a set and make connections between the central mathematical ideas in order to have an understanding of the overall structure. As part of their definition of structure, Skemp (1992) (as stated in the quote above) and Van Hiele (1986) assert that it is more than a sum of its elements and also stress the importance of seeing it as a totality. In relation to this a useful definition of structure is provided by Hoch and Dreyfus (2004), which although it concerns algebra, can be applied to number and other topics as well. “Any algebraic expression or sentence represents an algebraic structure. The external appearance or shape reveals or if necessary can be transformed to reveal, an internal order. The internal order is determined by the relationships between the quantities and operations that are the component parts of the structure” (Hoch & Dreyfus, 2004, p. 50).

This means that there are conceptual structures involving operations and associations that learners need to be aware of, and that are not always apparent in the superficial view of the expression. This notion is similar to Skemp’s (1989) theory of surface structures and deep structures. This distinction is important due to the difficulty of communicating mathematics to students. Skemp explains that for a number such as 4792, at the surface level we have digits in simple order relationship and at the deep level there are three other ideas; place values of powers of ten (only implied by the position of the digits) and the two operations of addition and multiplication, which are not indicated. The difficulty for learners is that while the surface structure involving symbols is apparent, the deep structure of the mathematical concepts is not always obvious to them, and many students have to be guided in this respect. This aspect is discussed further under the heading ‘Place value structure’ in Section 3.2.3.

There is much written in the research literature about different types of structures. In this context, Freudenthal (1983) describes some structures that are worthwhile for students to develop in his book ‘Didactical Phenomenology of Mathematical Structures’. He
discusses, among other ideas, strong and weak structures, additive and multiplicative structures of the number system, and isomorphism between structures and their teaching implications. Likewise, Van Hiele (1986) speaks of visual structures, rigid and feeble structures, lower and higher level structures and isomorphism of structures. He states that structure is objective and he reasons that other people see and react to a structure just as we do; that is different people continue a pattern/structure in the same way. A structure can be extended due to its composition, although sometimes there are many ways to extend a structure such as in 2,3,5,8... Furthermore, Van Hiele alludes to structure as an important phenomenon and explains that structure enables a person to act in situations that are not exactly the same as those they have met before. Hewitt (1998) agrees with this point and gives the example of being able to say the name of any number such as 13294865 due to the underlying structure, even if this particular number has not been said before.

We now consider some examples of structures that students meet at different levels of learning at school and at university in order to highlight the progressive nature of these structures.

1. As an example of structure in early geometric school mathematics learning, Mulligan and Mitchelmore (2009) consider a rectangular grid of 3 × 5 squares. The implicit structure is increasingly apparent with the transformation of three rows of five squares or five columns of three squares with their sides aligned horizontally or vertically. In this instance, the structure is the relationship between rectangles that represent numbers. The important structural features are repetition (of rows and columns) and spatial relationships such as congruence, parallels and perpendiculars.

2. Middle school numerical examples:
   a. Arithmetic expressions such as 3×(6+3×5) and 18−13+15×4 (Banerjee & Subramaniam, 2005) where the order of operations including brackets impose a structure. The authors believe that appreciating the structure of arithmetic expressions is useful for understanding algebraic expressions.
   b. Understanding that natural numbers and integers are subsets of rationals, which in turn is a subset of real numbers. This structure on the number system is based on the idea of nested subsets. That is, $N \subset Z \subset Q \subset R$. 

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c. Arcavi (2008) provides this example of place value structure: Students are to input a three digit number into their calculators and then to input the same three digit number immediately to the right of the first. So the number arrived at is for instance, 458458. The next step is the process of consecutive divisions of a pair of numbers; 91×11 or 13×77 or 7×143 to arrive at the original number 458. According to Arcavi, some students realise that \(xyzxyz\) is 1001 times \(xyz\). The whole trick is to multiply a 3 digit number by 1001 and then divide by 1001 (which is a product of its prime factors 7,11,13). In this example, decomposition of a number into its prime factors is one aspect of the structure. Moreover, in the above process students are led to realise that forming a six digit number was in fact a multiplication and that dividing by the three possible pairs of decompositions was actually the inverse operation in another form. Some students also realise the general property of dividing by \(p\) and then by \(q\) as being equivalent to dividing by \(p\times q\).

3. Algebraic structure: High school students frequently come across structures in their mathematics learning, particularly in algebra. Structures in the form of algebraic expressions, \(3(x + 5) + 1, 81 - x^2, (x - 3)^4 - (x + 3)^4, 24x^6y^4 - 150z^8, ab + ac + ad, a^2 + 2ab + b^2, (2x + 1)^2 - 3x(2x + 1)\) and in the form of equations \(ax + b = 0, ax^2 + bx + c = 0\) (Hoch & Dreyfus, 2007, February; Novotna & Hoch, 2008; Sfard & Linchevski, 1994; Tall & Thomas, 1991) are often encountered. Knowledge of factorisation and decomposition into factors are some features of these structures. Factorisation involves understanding the relationships between terms and finding common elements.

4. University students come upon structure in their algebra courses, and the term algebraic structure is usually used in abstract algebra. It may be known as a set closed under one or more operations, satisfying some axioms. Examples of algebraic structures: a) the binary operation \((Z, \ast): x \ast y = x - y\) (Novotna & Hoch, 2008) where \(Z\) is the set of integers, that leads to structure. b) The Group \((R^\ast, \times)\) which is a system consisting of the set of non-zero real numbers with the operation of multiplication. Understanding this algebraic structure means an awareness of relationships between elements and understanding its properties of
associativity, existence of identity and of inverse. c) the set of integers, \( \mathbb{Z} \) is a ring and d) the set of real numbers \( \mathbb{R} \) is a field.

The above examples highlight the importance of visual/spatial structures (Van Hiele, 1986) in the beginning years of school and the significance of experiences in arithmetic structures in the middle grades (Linchevski & Livneh, 1999; Warren, 2003) and for understanding of algebraic structures in high school and university. It is evident from the above examples that it is highly valuable that students are structural thinkers in order to advance in their learning of mathematics and this aspect is considered in the next part.

### 3.2.2 Structural Thinking

Structural thinking can be thought of in a number of different ways. As proposed by Sfard (1991), it is the ability to view abstract notions such as number, variable and function both \emph{structurally} as objects and \emph{operationally} as processes. This transition from computations to objects is accomplished in three steps; \emph{interiorisation}, \emph{condensation} and \emph{reification}. Sfard (ibid) explains that thinking structurally involves regarding a concept as an actual entity, separated from the process, and this type of thinking is necessary in order to be able to advance to higher mathematics at school and university. However, for many students this is a long process and difficult to achieve. Paralleling Sfard’s ideas, a useful distinction of symbolisation was made by Gray and Tall (1994). They noted that symbols in arithmetic and algebra had a duality as both \emph{process} (as in the addition of \( 4+3 \)) and as \emph{concept} (the sum \( 4+3 \) is \( 7 \)) which is the result of that process. Gray and Tall (ibid) coined the term \emph{procept} to describe the use of such symbols that represent both a process and the concept and \emph{structural thinking} would involve the ability to see abstract objects as both processes and concepts. Some other viewpoints are that structural thinking can also be seen as similar to \emph{versatile thinking} (Tall & Thomas, 1991), \emph{relational thinking} (Carpenter & Franke, 2001; Stephens, 2006) and as having \emph{structure sense} (Linchevski & Livneh, 1999). Versatile thinking, as described by Tall and Thomas, is about the ability to have a holistic picture of the task at hand and to suggest solution procedures, as well as being able to sense any errors and correct them. The authors believe that “intuition can be honed to link in with the sequential/logical thinking processes and therefore be of value in giving an overall view of the mathematical structure...Versatile thinking requires the availability of
cognitive interaction between concepts represented by imagery as well as symbolically and verbally” (Tall & Thomas, 1991, pp. 4-5).

Similar to the idea of Tall and Thomas, the term structure sense was first used by Linchevski and Livneh (1999) when detailing children’s difficulties with using knowledge of arithmetic structures in the initial stages of algebra learning. Basing their definitions on interviews with mathematics education researchers, Hoch and Dreyfus (2004, 2005) describe structure sense as a collection of abilities, separate from manipulative ability, and making better use of previously learned techniques. They propose that it could be related to experience and, has something in common with intuition. In a similar vein, Mason et al., (2009) define relational or “structural thinking as a disposition to use, explicate and connect these properties in one’s mathematical thinking” (pp. 10-11) and emphasise the importance of such thinking by arguing that “unless students are encouraged to attend to structure and to engage in structural thinking they will be blocked from thinking productively and deeply about mathematics” (p. 10). Furthermore, Mason and his colleagues, as well as Warren (2003), underline the importance of structural awareness in assisting students to move successfully from arithmetic thinking to algebraic thinking.

Related to this, Awareness of Mathematical Pattern and Structure (AMPS) is a new construct proposed by Mulligan and Mitchelmore (2009) and they found a strong positive correlation between students’ conceptual understanding of mathematics and their stage of structural development.

However, being able to think structurally is not easy for many students and there is growing research evidence that students of all ages, including university students (e.g. M. O. J. Thomas, 2008b; Warren, 2003) have a weak grasp of mathematical structure. In Gray and Tall’s (2001) view, students do not perceive structural features and are hampered in learning because they focus on idiosyncratic, non-mathematical aspects. Hence attention to structural features should be an essential and significant aspect of mathematical teaching and learning (Hoch & Dreyfus, 2004; Mason, 2004, July; Mason et al., 2009). There is also increasing evidence that structural thinking develops from the ability to recognise and represent patterns and relationships in the early primary years (Mason, Graham, & Johnston-Wilder, 2005) and multiplication relationships feature strongly in investigations regarding structure in children (Mulligan & Mitchelmore, 2009). In this regard, researchers (e.g. Mulligan, Prescott, Papic, & Mitchelmore, 2006) have shown that young students can
be taught to look for and recognise mathematical structure, with a consequent positive effect on their overall mathematical performance.

In this context, it is the view of Mason and colleagues (Mason et al., 2009) that structural thinking helps to bridge the “mythical chasm” (p. 10) between conceptual and procedural approaches to teaching and learning mathematics. In a similar vein, many researchers (e.g. Hiebert, 1986; Skemp, 1976) have emphasised the need to link conceptual and procedural knowledge as this benefits both conceptual understanding and procedural understanding. In relation to this, Mason and his co-authors take the stance of integrating work on understanding concepts with work on procedural fluency, but keeping the notion of mathematical structure in mind. (Procedural fluency includes both facilities in using procedures and having a toolkit of procedures to use). The authors believe, as do Hiebert and Wearne (1992) and Resnick (1983) that adopting a structural point of view when making pedagogical decisions can overcome many of the difficulties that stem from an exclusive over-dependence on procedural or conceptual thinking.

3.2.3 Place Value Structure

The discussion has so far focused on general ideas of structure and this section of the chapter specifically considers the place value structure of the base ten system, the focus of this thesis. Researchers are in general agreement (e.g. G. A. Jones et al., 1996) that place value concepts are multifaceted, and develop slowly over many years. Understanding the structure of the place value numeration system implies understanding its additive, multiplicative and other related properties (Baturo, 2000; Fuson, 1990a; N. Thomas, 2004). That is, to understand how the various parts of the system fit together to make a whole. A number of researchers have identified a wide range of properties related to place value that students need to acquire in order to use the numeration system effectively (Baturo, 1997; Bednarz & Janvier, 1982; Fuson, 1990a; Hiebert & Wearne, 1992; Resnick, 1983; Ross, 1989; Steffe, Cobb, & Von Glasersfeld, 1988; N. Thomas, 1998). The following categories of properties are adapted from Thomas (1998):

a) Counting properties - the promotion of unitary and skip counting skills;

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1 As will be seen in Chapter 4, historically too the system took a long time to develop, although the oral numeration structure seems to have been established much earlier in Indian history. It is the written place value system that took centuries to construct and which is also the source of difficulty for students.
b) Grouping properties - groupings in the form of powers of ten known as multiunit conceptual structures;

c) Place value properties - the value of a digit (face value) in a numeral is determined by the position it holds in the numeral;

d) Base ten property - ten digits are required (0,1,2,3,4,5,6,7,8,9) and any number, however large or small are written with these digits. The decimal point separates the integer and fractional parts; The values of the positions to the left are given by increasing powers of ten starting with $10^0$; that is, each position to the left is ten times larger and each position to the right is ten times smaller;

e) Zero as a place holder property - Zero is used as a place holder in a positional system to show the absence of a particular power of ten in a number;

f) Multiplicative property - the value of an individual digit is obtained by multiplying the face value of a digit by its place value; also each place value is obtained by multiplying 10 by the preceding place value on the right;

g) Additive property - the value of the whole numeral is the sum of the values represented by the individual digits.

This set of properties was extended in this thesis with the addition of the exponential property.

h) Exponentiation property - each place value or multiunit number is a power of ten; $10^2$, $10^3$ $10^4$, etc; and these representations are associated with groupings, such as groups of groups (100=$10^2$=$10\times10$), groups of groups of groups (1000=$10^3$=$10\times10\times10$) and so on; non-adjacent place values are related by powers of ten.

These properties of the decimal place value numeration system have to be understood and extended during the school years. Students need to build connections between key ideas such as quantifying sets of objects by grouping by ten and then treating the groups as units (e.g. Bednarz & Janvier, 1982; Lamon, 1994; Steffe et al., 1988). Learners also have to construct the multiplicative recursive structure inherent in the whole number system and extend it to decimal fractions (Baturo, 1997; Steinle & Stacey, 2004). As referred to earlier, an awareness of the deep conceptual structures (Skemp, 1989) implicit in the place
value system has to be built up by students. These deep structures are not perceptible to many students. As noted by Mason (1987) students mainly see, and are drawn to, the syntactic surface structure rather than the semantic deep structure. What learners have to do is to interpret the symbols in terms of the relationships (Arcavi, 1994) within the conceptual structure, rather than those of the symbol system. Hence a number such as 684 needs to be interpreted not only as three single digit numbers but as a single number formed by 6 hundreds plus 8 tens plus 4 units. According to Skemp, if the deep conceptual structures are not formed early on, students have little chance of developing meanings for symbols (see Section 3.4 for a discussion on signs/representations). While some initial difficulties may be encountered due to linguistic patterns in the spoken language, as highlighted by some researchers (e.g. Fuson, 1990b), much more important is to understand the meaning (Brownell, 1947) of place value notation and generalise the structure of the system.

**Multiplicative (including exponential) aspects of place value structure**

The decimal numeration system is ostensibly simple but structurally very complex and as can be seen in the list of properties given above, involves the co-ordination of several powerful mathematical concepts (Baturo, 2002). A rich understanding of the base ten system requires an appreciation of its multiplicative structure. In an analysis of the main concepts and processes within the decimal numeration system in terms of the cognitions inherent within, Baturo (1997, 2000) provides a useful definition of its multiplicative structure by naming the salient features: “Place value is continuous (i.e., across whole-number and decimal-fraction places), bi-directional (to the left is 10 times larger in value - \( \times 10 \); to the right is 10 times smaller in value- \( \div 10 \)), and exponential (i.e., nonadjacent places are related by powers of 10 – \( 10^2 \), \( 10^3 \), etc.)” (Baturo, 2000, p. 96).

Highlighted here are the multiplicative relationships within places and between different place values. As can be seen from Fuson’s (1990a) table (see Figure 3.5) for ‘Conceptual structures for multiunit numbers’ (multiunit numbers are the values of positions or powers of ten), the first four rows involve addition and a person can construct the conceptual structure for a multiunit number in isolation using additive thinking. However, the last four rows involve multiplicative thinking, which requires reflection on the structure as a whole. Each position is seen as a result of cumulative trades or
cumulative multiples of ten, which is a crucial point highlighted by Fuson. These successive multiples of ten correspond to the exponential expressions given in the last row. Fuson (1990b) also notes that some aspects of a conceptual structure for the written positional symbols cannot be appreciated until multiplication is understood. This implies that a mature structural understanding of place value is possible only if teaching strategies include students’ experiences in multiplication and exponentiation and their related notation. Given that multiplicative (including exponential) thinking is involved in place value understanding, and given researchers’ agreement on the difficulties of learning and teaching place value, it is surprising that there have been very few studies involving secondary school students knowledge of place value. Much of the research on place value understanding has been conducted within an addition and subtraction context, involving children in the early primary years. While some studies on pre-service teachers’ place value understanding are documented in the literature, there is virtually none on high school students’ awareness of multiplication and exponentiation related to place value, and their understanding of place value as a general positional notation.

In the remaining parts of this section, multiplicative thinking and exponentiation as relevant features of the place value concept and, the generalisation aspect of place value notation are discussed.

a) **Building multiplicative thinking**

In the context of decimal numeration, Bryant and Nunes (2009) note the major role played by multiplicative reasoning in children’s mathematical thinking; “multiplicative reasoning is implicit in understanding place value” (p. 217) - e.g. 345 is interpreted as three hundreds, four tens and five ones; that is \(3 \times 10^2 + 4 \times 10 + 5 \times 1\). The authors state that although children do not need to know multiplication facts to understand this, they need to understand one-to-many correspondence. Thus understanding the structure of numeration is dependent on the following, all of which require multiplicative thinking; a) that the total value of a digit in a numeral is equal to face value × place value, b) the nature of the relationship between adjacent places, c) the nature of the relationship between non-adjacent places, and d) place values themselves as repeated multiples of ten (exponentiation), relative to the ones place (Baturo, 1997; Confrey, 1991; Fuson, 1990a; Price, 1998; N. Thomas, 1998). Hence, multiplicative thinking is fundamental to
understanding the multiplicative relationships within the place value numeration system itself. In contrast, knowledge of the place value concept is required for the fundamental operations including multiplication on whole numbers; in the grouping and regrouping that is involved in mental and written algorithms (Bednarz & Janvier, 1982; Fuson, 1990b; Hiebert & Wearne, 1992; Resnick, 1982; Ross, 1986; Seah & Booker, 2005).

The above discussion highlights the circularity that is involved in understanding the structure of numeration. That is, on the one hand, multiplicative thinking is essential for understanding place value structure and on the other, knowledge of the place value concept is vital to developing multiplicative thinking and understanding multiplication. (This circularity is similar to the “vicious circle” (p. 31) observed by Sfard (1991) in her work on the nature of mathematical notions; on the one hand reification will not occur without higher-level interiorisation, and on the other hand existence of objects on which the higher order processes are performed are necessary for the interiorisation). This circularity (Arcavi, 1994) is one of student difficulties in understanding place value and the multiplication concept. This implies that place value work be reinforced alongside a multiplication (and also addition and subtraction) setting rather than separately before any multiplication work is begun. This is similar to the view of Fuson (1990a) and, Carpenter, Fennema and Franke (1996) who suggest that place value understanding increases concurrently with knowledge of arithmetic operations on multidigit numbers. However, many students seem to rely on additive strategies to solve complex multiplication problems and this suggests that the transition path from additive thinking to multiplicative thinking is not a smooth one for them (Siemon, Izard, Breed, & Virgona, 2006). As pointed out by Confrey (1994), this difficulty is due to the fundamental difference between additive and multiplicative thinking; additive reasoning is used in one-variable problems, where quantities of the same kind are combined or separated, whereas multiplicative reasoning requires two variables connected by a fixed ratio (Bryant & Nunes, 2009) and “they do not refer to the same universes” (Dienes & Golding, 1971, p. 34). Clark and Kamii (1996) distinguish between additive and multiplicative thinking and explain that while additive thinking concerns operation at a single level, multiplicative thinking involves a hierarchical structure. In the example given in Figure 3.1, Clark and Kamii explain that the repeated addition problem of 4+4+4 involves additive thinking and is simple because it involves only ones at one level of abstraction. However, 3×4 involves multiplicative thinking which
has a hierarchical structure. The ‘3’ in $3 \times 4$ refers to ‘3 fours’. To read ‘$3 \times 4$’ correctly, the young student has to be able to transform ‘4 ones’ to ‘1 four’ which is a higher order unit. This higher order thinking is a difficulty for students.

![Diagram](image)

*Figure 3.1. Diagrammatic explanation of additive and multiplicative thinking by Clark and Kamii (1996)*

In her study on young students’ understanding of multiplication, Anghileri (1989) found that many children used repeated addition to solve problems. This is similar to Brown’s (1981a) findings where students chose the easier model of repeated addition over the ‘cartesian product’ model. In the view of some researchers (Anghileri, 1989; F. B. Clark & Kamii, 1996; Steffe et al., 1988; Vergnaud, 1983), the difficulty in progressing from a counting procedure to the use of a number fact could be related to the development that is required in the transition from unary operations (relating to a scalar operator or a function operator) to binary operations (related to areas and arrays). Binary operations can be identified with the fourth and fifth stages in Steffe’s (1988) five counting stages where children are simultaneously aware of the units counted-on as they are being counted (double counting). In this context, Dienes and Golding (1971) explain that in Cartesian product as a multiplication operation we have gone beyond addition; while the answer is the same as in repeated addition, the concept of multiplication involves the new variable of *multiplier* which counts sets and is a property of sets of sets and, the *multiplicand* is a property of sets. In the authors’ words: “In multiplication we are dealing with two different universes at the same time, whereas with addition, we are dealing with the same universe, namely that of sets. Every number refers to sets in addition, whereas in multiplication some refer to sets of sets and others refer to sets” (Dienes & Golding, 1971, p. 34).

While multiplication develops out of addition and is taught as repeated addition in the early primary years, prolonged use of it causes problems later, for instance when students
are asked to multiply two fractions. In this case, repeated addition has no meaning but students apply additive strategies to solve complicated problems involving multiplication. Another misconception noted by Greer (1992) is that as a result of over-emphasis on repeated addition, students appear to hold the misconception that “multiplication always makes bigger, and division smaller, and that division is always division of the larger number by the smaller” (p. 287). Hence, “a fundamental conceptual restructuring is necessary when multiplication and division are extended beyond the domain of positive integers” (Greer, 1992, p. 276). Furthermore, when it becomes necessary to extend multiplication to repeated multiplication and exponentiation, problems are caused by the entrenchedness of the early concept of repeated addition.

Researchers and educators have made suggestions that could help teachers to broaden their students’ understanding of multiplication. For example, Skemp (1971) advocates teaching children multiplication as the combination of two operations. For 6×3, (Skemp suggests reading this as 6 multiplied by 3 for consistency with long multiplication), he says to start with a 6-set and combine 3 of these sets to make 18. When using physical objects, the result can be re-arranged as 1 ten-set and 8 singles. The result is the same as for 6+6+6 and the process appears similar. However, Skemp is of the view that while this is harder initially, students now have a concept that can be expanded, without re-structuring, to include all the other kinds of multiplication (such as multiplication of two fractions) that they are likely to encounter at school. It is interesting to note that Skemp suggests making a set of 6 first for 6×3 and then 3 of that set, as in contrast, some other researchers view this product as 6 threes (e.g. Booker, Bond, Sparrow, & Swan, 2004) and similarly Lampert (1986) views it as 6 groups of 3; i.e. 3+3+3+3+3+3. Alternatively, in recent years, researchers (e.g. Booker et al., 2004; Fuson, 2003; Young-Loveridge, 2005) strongly recommend the use of array/area models in developing a deeper and flexible multiplicative thinking. As stated by the authors, arrays are fundamental to appreciating the two dimensionality of the multiplicative process and to facilitating the development of multiplication as an operation in its own right. Moreover, Booker et al. (2004), and Young-Loveridge (2005) note the usefulness of arrays for demonstrating the commutative law by rotation of arrays. This is illustrated in Figure 3.2.
As with the associative and distributive laws, it is important for students to develop an early intuitive awareness of the commutative law, since the equality of two statements (e.g. $3 \times 4$ and $4 \times 3$) will halve their learning of multiplication facts and also prepare them for algebra. Young-Loveridge (2005) favours developing both counting-based (number line, repeated addition) and collections-based (grouping, arrays, areas) ways of thinking since they help students to appreciate the different conceptions of number. Furthermore, Greer (1992) illustrates a variety of representations for diverse situations with respect to multiplication of whole numbers. Figure 3.3 shows one of the ways to represent 12 possible combinations of elements from a set with 3 elements and a set with 4 elements.

The above consideration clearly indicates that a break from the earlier repeated addition concept and a re-conceptualisation of number (Ma, 1999) is needed to develop multiplicative thinking, which in turn aids the understanding of the place value concept. Other benefits are that it allows the development of proportional reasoning, algebra and higher mathematics that build on these fundamental ideas. Multiplication is more abstract than addition, and is built on experiences with repeated addition, rather than problem
situations (Booker, 2003). Hence, models that reflect the multiplication concept and that provide a representation for multiplication independently of addition are required. Once students appreciate commutativity in multiplication, which many of them appear to do (Brown, 1981a; Greer, 1992), the flexibility gained helps students to view, for example, $1000 \times 3$ as equivalent to $3 \times 1000$. Greer’s representation is similar to Clark and Kamii’s (1996) model (see Figure 3.3) in that the two dimensionality of multiplication is evident; the one-to-many correspondence is shown and involves a hierarchy of thinking. However, the above models do not lend themselves to compression that results from unitising, since each of the individual elements are visible and can be counted or added (the importance of compression is that the compressed object can be manipulated as a procept (Gray & Tall, 1994) or thinkable concept (Gray & Tall, 2007) a crucial factor for abstraction and progression in mathematics). In this connection, through a historical review, a possible improvement has been made on the above discussed models and which has been used as a pedagogical strategy in the research study. An important aspect of this model is unitising (in a bundle) which becomes necessary when one of the numbers is large (for example, $3 \times 1000 = 3000$ in decimal numeration. Figure 3.4 shows the model developed). That is, unlike Figure 3.3, in this model all the possible combinations are not visible.

![Figure 3.4. Suggested model for multiplication that was used in this study.](image)

From what has been considered thus far it is evident that while many researchers highlight the importance of multiplicative thinking there is no clear consensus as to a definition of it. Hence following the review of the literature an attempt is made below to
list some possible ideas that are involved (in no particular order) in attaining multiplicative thinking:

1. The system of reading and writing in the decimal place value system
2. Counting, skip counting
3. Addition – both unequal and equal, with equal addition leading to multiplication and the related notation
4. Grouping experiences - and unitising/initialising
5. Counting different sized groups
6. Rearranging groups – *renaming* including internal zeros that are necessary for algorithms involving fundamental operations
7. Additive part-whole strategies
8. Multiplicative part-whole strategies
9. Multiplication shown with array, area, dot paper models
10. Multiplication tables (rote learn) – knowledge facts
11. Multiplying by 10, 100, 1000 etc or their equivalent forms (powers of ten)
12. Multiplication algorithm
13. Fraction concept and notation
14. Scaling
15. Embedding
16. Grouping involving equal and unequal grouping structures involving multiple bases, and re-unitising/re-initialising leading to multiplication, repeated multiplication and the associated notation
17. Naming, reading, writing and working with *large numbers*

The next part of this chapter involves a consideration of exponentiation.
b) Exponentiation

The idea of powers is inherent in place value and in order to understand the structure of the numeration system, it is essential that students not only understand the multiplication concept, but that they extend their thinking to include exponentiation. Place-value is based on counting of different sized groups (Bednarz & Janvier, 1982) and all these groups consist of powers of the base, which is usually ten (Dienes & Golding, 1971; Skemp, 1989).

Why are understanding exponentiation and its notation important? The concise representation of powers gives meaning to large numbers, which are common in today’s society, particularly in population and financial matters. For example, forty-eight billion is sometimes written as $48 \times 10^9$ and two trillion is written as $2 \times 10^{12}$. In addition, powers of numbers lead to exponential functions, scientific notation, and logarithmic scales which are all tools for understanding key mathematical and cultural ideas. As well, in their concept map on exponents, which forms part of the concept map on positionality, Schmittau and Vagliardo (2006, 2009) show the various ideas that are related to exponentiation, including extension to negative integer and fractional powers and their repeating and non-repeating decimal representations. This means that in a very fundamental way, powers and particularly powers of ten are crucial in understanding the structure of the decimal numeration system. In this context, Osborn, Boyd, DeVault and Houston (1968), state that the ideas of exponents enable one to see this structure more clearly. That is, the grouping structure of place values is reflected in the exponential form of number. Moreover, many researchers (Danielson, 2010; Diedrich & Glennon, 1970; Dienes & Golding, 1971; Schmittau & Vagliardo, 2006; Zazkis & Khoury, 1993) believe that instruction on numeration systems with bases other than ten (including exponentiation in the different bases) increases students’ awareness of the structure of the decimal place value system. Furthermore, exponentiation is crucial to the understanding of algebra and its symbol system (Dienes & Golding, 1971; Osborn et al., 1968; Schmittau & Vagliardo, 2006). These aspects are discussed in Section 3.2. For now, Fuson’s (1990a) work on the analysis of multiunit conceptual structures in the decimal system is considered.

Place values or powers of ten are referred to by Fuson (1990a) as multiunit numbers. In her paper related to the understanding of conceptual structures in place value, she defines
multiunit numbers: “Multiunit numbers are whole numbers composed of one or more kinds of multiunits (collections of single units) and possibly some single units; Multiunit numbers are expressible by number words and by written number marks” (p. 343). In Fuson’s analysis of conceptual structures for multiunit numbers, which were considered useful in structuring a learning sequence, the different conceptions of multiunits range from simple multiunit quantities to abstract expressions of repeated multiples. Figure 3.5 describes the nature of these conceptual structures. The multiunits that are shown in the table have to be constructed by students as collections of single units, as generated by a ten-for-one trade from the next smaller multiunit, as cumulative ten-for-one trades, as cumulative multiples of ten, and as exponential word or symbols/marks of multiples of ten. The system can be extended by using the ten-for-one structure to affirm the multiunit value of any position to the left. According to Fuson, to understand either the words or the symbols of place values/multiunits, at least one of these notions of multiunits given in the table must be understood. The first four conceptual structures given in the table can be developed on additive notions. In contrast, the last four conceptual structures need increasing reflection on the whole multiunit structure immediately above it and also require a movement from additive to multiplicative thinking. A full and mature understanding of the place value concept necessitates a further advancement to exponential thinking; that is, multiunits/place values as exponential symbols for multiples of ten (as given in the last row of the table) have to be constructed.
### Figure 3.5: Fuson's conceptual structures for multiunit numbers (Fuson, 1990a, p. 348)

<table>
<thead>
<tr>
<th>Name of the Conceptual Structure</th>
<th>Nature of the Conceptual Structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Features of the marks</td>
<td>Fifth</td>
</tr>
<tr>
<td>Visual layout</td>
<td>Fourth</td>
</tr>
<tr>
<td>Positions ordered in increasing</td>
<td>Third</td>
</tr>
<tr>
<td>value from the right</td>
<td>Second</td>
</tr>
<tr>
<td>Features of the words</td>
<td>First</td>
</tr>
<tr>
<td>Multiunit names</td>
<td></td>
</tr>
<tr>
<td>Words ordered in decreasing</td>
<td></td>
</tr>
<tr>
<td>value as they are said</td>
<td></td>
</tr>
<tr>
<td>Multiunit structures</td>
<td></td>
</tr>
<tr>
<td>Multiunit quantities</td>
<td></td>
</tr>
<tr>
<td>Regular ten-for-one and one-for-ten trades</td>
<td>Ten thousands one</td>
</tr>
<tr>
<td>Positions/values as cumulative trades</td>
<td>Ten hundreds one</td>
</tr>
<tr>
<td>Positions/values as cumulative multiples of ten</td>
<td>Ten tens one</td>
</tr>
<tr>
<td>Positions/values as exponential words for multiples of ten</td>
<td>Ten ones</td>
</tr>
<tr>
<td>Positions/values as exponential marks for multiples of ten</td>
<td>10⁴</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Note: The diagram shows the visual representation of multiunit structures with corresponding numbers and units.*
However, many learners, including some teachers (e.g. Zazkis & Whitkanack, 1993), experience considerable difficulty with the exponentiation concept (Booker et al., 2004; Stacey & MacGregor, 1994; N. Thomas, 2004) and its notation. In this regard, given its importance, there has been comparatively little research on students’ learning and understanding of exponentiation, particularly in relation to place value. Little is known about the mental constructions that students need to make to develop a meaningful understanding of exponents. Some noteworthy exceptions are the work of Schmittau (2008, July), Weber (2002), and Pitta-Pantazi et al. (2007). Confrey’s (1994) and Smith and Confrey’s work (1994) are key contributions to teaching exponentiation and will be discussed later in this section.

The question that arises is how do we guide students to exponential thinking and to become increasingly aware of place value structure? At the earlier stages of learning about exponents, Dienes and Golding (1971) suggest that students should have varied experiences in different bases formed with different exponents; in other words, they will need to have dealings with groups of objects which number, for instance, 3, 9, 27, 81 objects, that is groups; groups of groups; groups of groups of groups and so on. In other words, groups are powers of three ($3^1, 3^2, 3^3, 3^4$). This leads to the idea of repeated multiplication, which is also a method advocated by Confrey (1994). According to Booker et al. (2004), a way of building up this new notation is to develop the idea of exponent as a shortened means of recording the result of multiplying the same factor many times; that is by repeated multiplication, for example, $3 \times 3 \times 3 \times 3$ is recorded as $3^4$ or $10 \times 10 \times 10$ written as $10^3$ in the case of decimal place value/multiunit. Nonetheless, just as the need arose to extend multiplication beyond its initial meaning of repeated addition, in time exponentiation will need to be widened beyond repeated multiplication (Greer, 1994). For example, multiplication works well as repeated addition for the counting numbers such as $3 \times 4$, but it does not apply to the multiplication of two fractions ($\frac{2}{3} \times \frac{4}{5}$). Hence, the need to progress quickly to area and array models of multiplication, although teaching begins with repeated addition. Similarly, exponentiation of $6^3$ can be computed using repeated multiplication, but the same process cannot be applied to $8.73^{2.45}$. In this regard, a new construct is considered below that has implications for the teaching and learning of exponentiation.
An alternative primitive notion for multiplication has been proposed by Confrey (1994), which she labels *splitting*, on which the concepts of both division and multiplication can be based. As claimed by Confrey, repeated addition is inadequate for describing many multiplicative situations. She argues that there exist two relatively independent primitive structures that learners can use: counting and *splitting*. A *split* is an action of creating *equal parts or copies* of an original. While the counting actions of affixing, joining, annexing and removing create addition, in contrast, the splitting construct is based on actions such as sharing, folding, dividing symmetrically, growing, and magnifying. In a counting structure, the basic operations are addition and subtraction that are inverses of each other, whereas splitting is associated with multiplication and division, where growth happens by a constant multiplier or the *multiplicative unit*, (Confrey & Smith, 1994), for example in a geometric sequence. In the construction of splitting structure of numbers, which is built on the *action of repeated multiplication*, the counting numbers are used as an *index*; counting the number of splitting actions that have taken place and also naming the result of a split. For example, the counting number 3 is used to identify that 3 two-splits are represented, and resulting in 8 pieces as shown in Figure 3.6.

![Figure 3.6. Confrey’s splitting structure. Confrey (1994, p. 310).](image)
Confrey (1994) asserts that this new splitting construct is a precursor to the concept of ratio and proportion, multiplicative rate of change, exponential and logarithmic functions. It is also suggested by Confrey that human beings confront situations in which their quantitative actions are best described by splitting rather than by counting. These situations are ones in which the starting point is conceptualized as a whole; for example in compound interest problems. Confrey suggests tree diagrams (and embedded figures and spiral growth) as effective models for teaching exponentiation. These figures lead to the concept of recursion, which is appropriate for modelling repeated multiplication as it allows for growth in size without an alteration in the basic shape. By comparing counting and splitting further, Confrey suggests that counting can be described in terms of translation, and can be conceived of as movement across our field of vision, whereas splitting has links to the geometric concept of similarity and is useful in describing motion toward and away from us. In Table 3.1, we see the way that Confrey emphasises the distinctions and parallels in the ideas in the splitting (multiplicative) and counting (additive) worlds.

Confrey and Smith (1995) emphasise the importance of repeated multiplication as a beginning strategy for understanding exponentials. They state that students need to engage deeply in related issues in order to understand the splitting structure and exponentiation. The authors explain that their own examination of repeated multiplication led them to fundamental reconsiderations of the multiplication (and function) concept. In Confrey’s view, only in repeated multiplication does the role of the whole unit become visible and this whole unit plays a key role in understanding the concept. An emphasis is laid on a most essential feature of the splitting structure, whether multiplication or division, of equal sized groups. Confrey uses the term reinitialising to one (similar to re-unitising) to describe forming a unit of units at each step/layer of splitting. For example, in a tree diagram (Confrey & Smith, 1995) of a 2 split, after the first split, two options exist. One can reinitialise, forming the two units into a unit, and then split that, reproducing the same figure as before but with a different object – a doubleton rather than a singleton. Alternatively, the authors explain that after the first split, one can treat the two units as two singletons and 2-split each of these. That these two approaches produce the same outcome is an important property of the splitting structure. Reinitialising is an important teaching/learning strategy in the process of repeated multiplication as will be seen in Chapter 5. In Figure 3.7, Confrey demonstrates the two strategies in the tree diagrams.
Table 3.1. A Comparison between Counting and Splitting Structures (based on Confrey, 1994, p. 311; Confrey & Smith, 1995, p. 75)

<table>
<thead>
<tr>
<th>Counting</th>
<th>Splitting</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero is the origin</td>
<td>1 is the origin</td>
</tr>
<tr>
<td>Adding 1 is the successor action</td>
<td>Splitting by n is the successor action</td>
</tr>
<tr>
<td>The unit is 1</td>
<td>The unit (of growth) is n</td>
</tr>
<tr>
<td>0 is the identity element</td>
<td>1 is the identity element</td>
</tr>
<tr>
<td>Addition and subtraction are basic</td>
<td>Multiplication and division are basic</td>
</tr>
<tr>
<td>Commutativity applies to addition</td>
<td>Commutativity applies to multiplication</td>
</tr>
<tr>
<td>Reinitialising to 0</td>
<td>Reinitialising to 1</td>
</tr>
<tr>
<td>Multiplication is constructed as repeated addition</td>
<td>Exponentiation is created as repeated multiplication</td>
</tr>
<tr>
<td>Parts (additive) are created by n-splitting (or dividing)</td>
<td>Parts (multiplicative) are created by n-rooting</td>
</tr>
<tr>
<td>Distributivity is applied to multiplication over addition</td>
<td>Distributivity is applied to exponentiation over multiplication</td>
</tr>
<tr>
<td>Difference is used to describe the interval between two numbers</td>
<td>Ratio is used to describe the interval between two numbers</td>
</tr>
<tr>
<td>Rate is difference per unit time</td>
<td>Rate is the ratio per unit time</td>
</tr>
</tbody>
</table>

Figure 3.7. Comparison of the strategies to powers Confrey (1994, p. 309).
What has been discussed so far has underlined the significance of splitting and reinitialising processes for teaching exponentiation and affirms repeated multiplication as a beginning strategy. Confrey (1991, 1994) proposes that certain topics such as place value and decimal notation require the coordination of counting and splitting worlds and promoting a conceptual understanding of these topics would be a great challenge. Insufficient treatment of splitting will make some topics, including the decimal system, particularly difficult for students to learn. In Confrey’s view, although the splitting construct has a complementary but equally significant role in the elementary curriculum, it appears to be neglected. Hence she suggests the early introduction of splitting structure into the curriculum parallel to the development of counting structures. This implies experiences in counting actions such as adding, joining, removing as well as splitting actions such as multiplying, dividing, magnifying and embedding. In terms of understanding the structure of the place value system, the above discussion also implies sufficient experiences in equal grouping structures and repeated multiplication. As mentioned earlier, students need the opportunity to group objects, groups of groups, groups of groups of groups and so on and also need to represent these in multiple ways.

Taking the above views into consideration, there seems to be no reason why students cannot have early experiences characterising both the counting and splitting/grouping worlds. Confrey (1994) has emphasised the importance of understanding exponentials and the related processes of repeated multiplication and grouping. Her ideas are both useful and important, particularly when one considers the role of powers in the fundamental concept of decimal place value and then its extension to exponential functions, scientific notation and logarithmic scales in multiple disciplines.

For this study, although the tree diagram was not employed, Confrey’s (ibid) ideas on using the counting number as an index (counting the number of splitting/grouping actions that have taken place and also naming the result of a split/group), and reinitialising were particularly useful as pedagogical strategies for developing understanding of the multiunits/place values. As Confrey points out, using the counting numbers to index the splitting numbers is equivalent to mapping the positive whole numbers onto geometric sequences, which is the genesis of the exponential function.
3.3 Multiple Bases and Generalisation

The preceding section has brought into focus the sophistication of the place value system, particularly its multiplicative (and exponential) structure and its roots in grouping ideas. Many psychologists and educators (e.g. Dienes, 1964b; Schmittau & Vagliardo, 2006; Skemp, 1989) believe that this vital and foundational concept of decimal numeration is better understood as a particular instance of a general positional system. This idea brings up the issues of multiple bases and generalisation in algebra, and these topics are examined in this section.

3.3.1 Multiple Bases

During the era of the ‘new mathematics’ movement in the US and UK in the 1960s and 1970s (and due to a flow on effect in many other countries), the teaching of multiple bases to develop the concept of the decimal system was advocated and figured quite prominently in mathematics education curriculum. Curriculum makers believed that primary school children who study multiple bases would achieve a greater understanding of the decimal numeration system. This belief was reinforced by Bruner (1971), who wrote:

“...while many virtues have been discovered for numbers to the base ten, students cannot appreciate such virtues until they recognise that the base ten was not handed down from the mountain by some mathematical god. It is when the student learns to work in different number bases that the base ten is recognised for the achievement that it is” (pp. 71-72).

However, the teaching of multiple bases fell out of favour with the occurrence of the ‘back to basics’ movement which succeeded the ‘new mathematics’ era. The multi-base blocks invented by Zoltan Dienes nearly fifty years ago have become unfashionable and only base ten blocks are in common use in the classrooms at present. Hence, in the last couple of decades, although there are a number of studies (e.g. Danielson, 2010; Hopkins & Cady, 2007; Zazkis & Khoury, 1993) that have been conducted involving teachers’ knowledge of non-decimal numeration, there is a dearth of research reports on primary and secondary school students’ understanding of the topic. Yet, not only Bruner and Dienes, but Skemp (1971, 1989) and Vygotsky (1962) have emphasised the importance of a multi-base approach for the formation of a secondary concept such as the place value system. Vygotsky and Skemp write:
As long as the child operates with the decimal system without having become conscious of it as such, he has not mastered the system but is, on the contrary, bound by it. When he becomes able to view it as a particular instance of the wider concept of a scale of notation, he can operate deliberately with this or any other numerical system. The ability to shift at will from one system to another is the criterion of this new level of consciousness, because it indicates the existence of a general concept of a system of numeration. (Vygotsky, 1962, p. 115)

Concepts of a higher order than those which a person already has cannot be communicated to him by a definition, but only by arranging for him to encounter a suitable collection of examples.

Since in mathematics these examples are almost invariably other concepts, it must first be ensured that these are already formed in the mind of the learner. (Skemp, 1971, p. 32)

Hence in the view of the above authors, a deep concept of positional notation cannot be constructed through the teaching of base ten alone (Schmittau & Vagliardo, 2006). Schmittau and Vagliardo illustrate in their concept map (see Figures 1.1, 1.2, 1.3) the complex web of ideas, leading to and from a consideration of many bases, either directly or indirectly. If students only encounter base ten, Dienes (1964a, 1964b) suggests that students will form associations rather than true abstractions. To provide for a greater degree of generality of the learned concepts, familiarity with non-decimal numerations is essential (Osborn et al., 1968). That is, the idea of decimal place value will be more easily understood if the groups are not always based on ten. In fact, Dienes’s important principle, the “Mathematical Variability Principle” (Dienes, 1964a, p. 44) urges teachers to expose students to the greatest number of examples of all the possible variables. He explains that the meaning of a large number such as 24579 is the expanded notation $2 \times 10^4 + 4 \times 10^3 + 5 \times 10^2 + 7 \times 10^1 + 9 \times 10^0$ and if this is viewed in light of the mathematical variability principle, the three possible variables are i) digits, ii) powers, and iii) the base (Dienes, 1964a, p. 52). Any of these components could be varied without affecting the essential structure of place value. The place value concept is independent of the values of the digits except that their number is constrained according to the base that is chosen. It is also independent of the number of powers that are used, which, in Dienes’s view gives the structure its mathematical openness; it is open towards the infinite. Furthermore, the place value concept is independent of the base. For example:

126 is expanded as $1 \times 10^2 + 2 \times 10^1 + 6 \times 10^0$
and can also be written as \( 1 \times 3^4 + 1 \times 3^3 + 2 \times 3^2 + 0 \times 3^1 + 0 \times 3^0 \) which is 11200 in base 3.

The commonality in all such possible ways of expressing numbers is the place value concept. Powers are inherent in place value notation and the possible variables in a power are the exponent and the base number. Hence, in order to learn about powers, students need to have exposure to a variety of examples in different bases formed with different exponents. And this incorporates the second and third variables in Dienes’s mathematical variability principle. For this purpose, Dienes suggests the use of materials that he is best known for; the Multibase Arithmetic Blocks (MAB) which are sets of wooden blocks with each set representing a different base system. While only the base-ten blocks are used at present, the MAB enabled children to manipulate numbers in different bases and experience the similar patterns of place value across different systems. (MAB is discussed further in the section on concrete materials).

With reference to the first variable in Dienes’s principle, which are the digits, Flegg (1983) and Hughes (1986) refer to the interesting idea of inviting students to create their own symbols and number systems. In Hughes’s study, children even as young as 3 years were able to invent their own symbols, mostly pictographic, in representing quantities. Such an activity could be both amusing and creative and lead to an awareness of structure of numeration (Flegg, 1983). An alternate method of leading students to appreciate the variability of digits/symbols would be to provide them with the opportunity of studying number systems of past civilizations (Ginsburg, 1989) with different bases and values. As well as this, such a study of numeration systems of other cultures, in Ginsburg’s view, might help students to understand that numerals are number symbols as this is a source of confusion for some students. Numerals are written symbols whereas numbers refer to ideas that the symbols stand for. 6 is a symbol that stands for the abstract idea of six. This idea was taken up in this research study; students attempted to create their own numeration system with distinct symbols and also considered the number systems of other cultures such as Egyptian, Babylonian, Greek, Roman and Mayan. This is discussed in more detail in Chapter 4.

Applying Dienes’s mathematical variability principle to multiple bases, once students experience the variation of the digits, powers and the base, they need to abstract, (probably
with guidance), that “In every case we arrange in descending powers of the base, and the value of each digit must always be less than the value of the base: one step to the right means the next power below, one step to the left the next power above. These are the ‘bare bones’ of the place value concept” (Dienes, 1964a, p. 53). In order to be able to abstract place value ideas, one needs to be able to look for similarities and differences when comparing decimal numerals with non-decimal numerals. One research project considering such ideas was that of Zazkis and Khoury (1993) who found more similarities than differences, and they highlighted the similarities, which are:

1. Place value representation of the numbers is dependent on the use of the base related power sequence: \( b^3, b^2, b^1, b^0, b^{-1}, b^{-2} \ldots \) where \( b \) stands for any base.

2. The point separates the integer and the fractional part of the number.

3. The number of distinct digits used in each base \( b \) is \( b \); That is, the digits range from 0 to \( b-1 \).

4. The same grouping-trading rules are applied to perform computations.

5. Zero serves as a placeholder. (adapted from *ibid*, p. 41)

By examining systems in multiple bases, it is more easily seen that the form or structure of non-decimal base systems is the same as it is in the decimal system (Osborn et al., 1968). This idea is corroborated in the work of Diedrich and Glennon (1970) with fourth graders. They concluded that while a study of the decimal system alone is as effective as a corresponding study of non-decimal numeration, a study of non-decimal numeration was more effective than a study of the decimal system alone for understanding a place value system in general. More recently, Slovin and Dougherty (2004) presented an alternate approach to counting based on the work of Davydov, who proposed that a general to specific approach was more beneficial to student understanding than a spontaneous concept approach. In this study, ten second grade students were asked how they counted in multiple bases and specifically how they knew when to go to a new place value and why it was necessary to do so. The researchers report that all ten students were skilled in counting and representing numbers but their responses indicated different levels of generalisation of method and explanation of underlying ideas. Their work showed that competence was not always supported by conceptual understanding. While many are in favour of the above
approach, some educators such as Brownell (1964; cited in Diedrich & Glennon, 1970) have questioned the value of introducing non-decimal systems in the beginning school years and have suggested a later introduction to non-decimal systems after experiences with base ten. This researcher tends to agree with this view since in the early primary years students are still in the process of learning and forming base ten concepts.

In a review of studies involving non-decimal instruction, Cruikshank and Arnold (1969) concluded that while non-decimal systems can be used to enrich the curriculum, to improve attitudes or to present a historical development of number, research has not established whether groupings in non decimals facilitates understanding of decimal system. However, Baroody (1987) indicated that teaching involving other bases was not successful because it was introduced in a highly formal manner that students did not comprehend, and could be used in a more informal and meaningful way. As discussed so far, there are many benefits to learning multiple bases, and Thomas (1998) suggests that maybe it is time to consider the value of other bases when learning the base-ten numeration system, not as a formal content area but as a focus of exploration of alternative (invented and historical) notational systems.

It is a fact that decimal numeration is only one example of a family of positional notation systems. Many concepts are introduced and/or reinforced through multiple examples. Why not place value?

3.3.2 Concept of Literal Symbol and Generalisation in Algebra

The foregoing section has examined the value of a consideration of multiple bases for a deep understanding of the place value concept, and argues that there are benefits to helping students see the decimal system as a particular case of a more general concept of positional notation. Hence in this section generalisation in algebra is discussed. This study is about Year 9 students’ understanding of the structure of the place value system. In this crucial first year of secondary school in New Zealand, students are about to learn, among various topics, integer operations, to form and solve linear equations, and to generalise properties of multiplication and division with whole numbers in algebra (Ministry of Education, 2007). For this reason, an introduction to algebraic variables through a generalisation of multiple bases of place values to the idea of a general base \( b \) (and to a general power \( b^m \)) could strengthen the place value concept and also provide for a smoother pathway to
algebra (Dienes, 1964a). However, in order to teach multiple bases and the idea of a
general positional notation, students need some facility with algebraic notation. Hence, in
the ensuing section, important ideas of variable and generalisation are considered.

**Literal symbol**

In a key paper, Usiskin (1988) proposes four conceptions of school algebra: i) algebra as
generalised arithmetic; ii) algebra as a study of procedures for solving certain kinds of
problems; iii) algebra as the study of relationships among quantities; and iv) algebra as the
study of structures. Underpinning all the conceptions in algebra and central to algebraic
thinking and learning is the concept of *variable* as represented by a *literal symbol*. This
concept is one of the most fundamental ideas of mathematics and “understanding the
case provides the basis for the transition from arithmetic to algebra and is necessary for
the meaningful use of all advanced mathematics” (Schoenfeld & Arcavi, 1988, p. 420).
These authors also highlight the role of variables as a means to understand, express and
communicate generalisations and for revealing structure.

...variables are a formal tool in the service of generalisation. Variables are used for making general
statements, characterising general procedures, investigating the generality of mathematical issues,
and handling finitely or infinitely many cases at once. Indeed, the idea of variable came to life as a
notational tool for making generalisations...Developments that followed from the use of this tool-
developments that clearly stretched humankind’s mental power- include algebra, analytic geometry,
calculus and so on. (Schoenfeld & Arcavi, 1988, p. 423)

In his synthesis on different approaches to algebra, Wheeler (1996) considers the
concept of the variable to be one of two ‘big ideas’ in algebra. As specified by Wheeler,
the awareness that students need to acquire is that an unknown number, a general number
and a variable can be symbolised and operated on as if it is a number. Such an awareness
and facility with symbolic literals forms what Arcavi (1994) calls *symbol sense*. In a
detailed discussion of symbol sense as the algebraic component of sense-making in
mathematics, Arcavi (1994) suggests that the process of forming and solving a problem
requires symbol sense, and also suggests a list of attributes that is indicative of its presence.
In this list he includes a feel for the power of symbols, an ability to interpret the meaning
of symbols in a given context and an ability to use symbols to express relationships. Arcavi
proposes that symbol sense is at the heart of what it means to be competent in algebra, and
the teaching of algebra should be geared towards it.
However, research indicates students’ inadequate understandings and misconceptions (e.g. Küchemann, 1981; MacGregor & Stacey, 1997; Tall & Thomas, 1991) of the variable concept and many other algebraic ideas. One reason that algebra is hard is the different ways that letters are used in algebra and the multi-faceted nature of this sophisticated concept (e.g. Schoenfeld & Arcavi, 1988). Some of the difficulties within algebra, as argued by Hewitt (2001), might be due to difficulties with notation rather than the mathematical ideas themselves. Wagner (1981) has described up to ten different uses of literal symbols in mathematics:

Literal variable symbols are used in a multitude of ways in mathematics. Depending on the context in which they occur and the element(s) to which they refer, the role of a variable may be described as that of a name, a placeholder, an index, an unknown, a generalised number, an indeterminate, an independent or dependent variable, a constant, or a parameter. Adding to this complexity is the fact that, generally speaking, different literal symbols can be used to represent the same thing, and the same literal symbol can be used to represent different things...It is no wonder that students have so much difficulty working with literal variables. (S. Wagner, 1981, p. 165)

In terms of the different uses of the variable, Graham and Thomas (2000) state that “This flexibility, while important in mathematics, does indeed make life very hard for students, who often adopt their own multiple interpretations of variables which do not correspond to the meanings conventionally used in mathematics” (p. 266). Students’ different interpretations of letters was reported by Küchemann (1981). In the Concepts in Secondary Mathematics and Science (CSMS) study of students’ interpretation of literal symbols administered to 3000 British high school students (13 to 15 years of age), Küchemann (1981) categorised each item of a 51-item test into one of the following six different ways of interpreting and using letters:

a) Letter evaluated: The letter is assigned a numerical value from the outset;
b) Letter not considered: The letter is ignored or its existence is acknowledged without giving it a meaning;
c) Letter thought of as a concrete object: The letter is regarded as a shorthand for a concrete object or as an object in its own right;
d) Letter used as a specific unknown: The letter is regarded as a specific but unknown number;
e) Letter regarded as a generalised number: The letter is seen as representing, or at least as being able to take on several values rather than just one;

f) Letter considered as a variable: The letter is seen as representing a range of unspecified values, and a systematic relationship is seen to exist between two such sets of values.

Küchemann classified students’ interpretations into four levels which correspond to the Piagetian stages of below late concrete, late concrete, early formal and late formal. The author found that, despite classroom experience in representing number patterns as generalised statements, only a very small percentage of the students were able to use letters as a generalised number. Even fewer students were able to regard letters as variables and a higher number of students were able to consider letters as specific unknowns than as generalised numbers. A majority of the students – 73% of 13 year olds, 59% of 14 year olds, and 53% of 15 year olds either treated letters as concrete objects or ignored them. In Kieran’s (1992) view, in terms of Sfard’s (1991) operational-structural model, these findings suggest that the students tested had not yet begun to interpret expressions as numerical input output procedures; the first phase in Sfard’s evolving process of developing a structural conception of algebraic expressions. In this connection, Sfard and Linchevski (1994) found that students in their study had difficulty with the variable concept. Even older students who had been doing a course in analytical geometry and calculus for almost a year had problems with ‘given’ numbers or parameters. The authors suggest that this is due to the ‘process-object’ dilemma, one which was also faced by mathematicians in history. They explain:

...as we have shown in our historical outline and reinforce with theoretical argumentation, there is an inherent difficulty in the idea of process-object duality – of a recipe which must also be regarded as representing its own product. This difficulty cannot be expected to disappear without a struggle. ...there is no indication that the student distinguishes between the name of a thing and the thing itself; incidentally, the inability to sever a sign from the signified may be one of the reasons why the duality of algebraic expressions is sometimes so difficult to grasp. (Sfard & Linchevski, 1994, p. 209)

Students’ difficulties could also be due to the way the idea is presented to them, and Booth (1995) suggests that the differences in usage of letters are not made explicit in teaching and learning. The idea of using a letter as an object reduces the abstract meaning
of the literal symbol to something more concrete and in Küchemann’s (1981) study, enabled some students to answer some items successfully. However, according to Küchemann, this usage of letter as object frequently occurred when it was not appropriate. Yet, this should not be surprising, since as noted by Booth (1995) and MacGregor and Stacey (1997) some textbooks in an effort to be helpful, encourage this incorrect notion by infamous examples such as *apple + apple = a + a = 2a*. The term ‘fruit salad algebra’ is sometimes used for this misconception in which the letter, rather than being a placeholder for a number, is regarded as being an object, such as *a for apple*. This letter as object misconception may be supported by formulae such as \( A = l \times w \) where \( A \) = Area and \( L = \) Length etc. Booth suggests that despite good reasons for using some specific letters as a mnemonic device, it might be best to avoid such types of alliteration, at least for novices. This is also noted by Quinlan (2001b), who emphasises how important it is for beginning algebra students to learn to view letters as standing for numbers and not objects. In his investigation, Quinlan found that students with better understanding of the meaning of literal symbols are more likely to be successful with algebraic work. Another misconception that is manifested in students’ work is the assigning of alphabetical values to literal symbols. Students count along the alphabets \( (a=1, \) \( b=2, \) \( c=3, \) and so on) and select values at the outset (Arcavi, 1995; MacGregor & Stacey, 1997) of solving a problem. It appears that students import all the features of a symbol system from another system to the new use.

Another problem for students, that is pointed out by Küchemann (1981) and Booth (1995), is the way the blanket term *variable* is used to refer to letters, regardless of whether the actual usage is as a variable, specific unknown, generalised number or place holder. It obscures the real and subtle (e.g. parameter) differences in the meaning given to letters and creates confusion for the students. In addition, many textbooks and syllabuses and teaching approaches typically introduce literal symbols within a context in which their values do not ‘vary’ such as in substitution and in the solving of equations. This serves to reinforce the idea that a letter can only assume one particular value; it is important for students to appreciate that letters can represent a range of values. However, Graham and Thomas (2000) believe that students will be able to use literal symbols successfully in generalised arithmetic, if they first have a firm understanding of ‘letter as place holder’. The authors suggest that learners need to be given the opportunity to see letters as capable of
representing numbers before they can begin to use letters to generalise patterns of numbers. In this study, Graham and Thomas (ibid) used a graphic calculator to provide an environment where students could experience the changing values of the variable with a significant measure of success.

A critical issue and another source of difficulty for students concerns exponential notation. As mentioned before, Stacey and MacGregor (1994) note students’ misuse of the notation, a type of reversal error – students wrote either \( x^4 \) or \( 4^x \) for \( x+x+x+x \) rather than \( 4x \), with minimal improvement from Year 7 to Year 10. The authors conclude from the evidence that a major cause of this difficulty is lack of clear concepts for repeated addition, multiplication and repeated multiplication.

Some recent studies also show that many students still experience difficulty in learning algebra, despite intensive research and the related teaching effort of the last few decades. One example is a large scale survey of attainment in algebra of 11-14 year olds that was conducted in England as part of the first phase of Increasing Student Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS). Tests were administered to a sample of 3000 students in 2008 and used items from the 1970s CSMS study (Hart, 1981) enabling a comparison of students’ attainment with that of 30 years earlier. Preliminary results (Hodgen, Küchemann, Brown, & Coe, 2009) suggest that whilst the practice of teaching algebra earlier confers an initial advantage to students, this increased attainment is not sustained. By age 14, current performance in algebra is broadly similar to that of students in the 70s. The authors note that the sample of students tested in 2008 is a high ability group and state the data provided indicates that increases in examination performance are not matched by an increase in conceptual understanding. In yet another study, conducted in Australia, (Steinle, Pierce, Stacey, Price, & Gvozdenko, 2009) numerical misconceptions related to Year 7 and Year 8 students’ interpretation of letters were reported. Only about 10% of Year 8 students chose answers correctly on a set of six tasks. 48% of students in Year 7 and 32% of students in Year 8 were in a group that included students who treated letters as objects. The authors concluded that for 90% of the Year 8 students who have had more than a year of exposure to algebra, algebra lessons are still difficult; these students are trying to learn procedures, without meaning, carried out on literal symbols with the wrong meaning. In addition, Steinle et al. observed that the possibility of using technology to support students with limited procedural skills is not
particularly feasible, especially if they have inadequate conceptual knowledge. Technology such as computer algebra systems (CAS) still require symbol sense, and as noted by Pierce and Stacey (2001), students need an understanding of the meaning of literal symbols and the correct interpretation of variables.

**Developing understanding of the variable concept**

In the context of students’ difficulties in understanding the role of the literal symbol, some researchers (e.g. Radford, 1996) have highlighted the fundamental, but dual characteristics of the variable. A significant feature that concerns the very nature of the concept of variable related to generalisations is raised by Radford (1996), who examines some key differences inherent in the idea of the variable. The construction of a formula is seen to be built upon the concept of generalised number (Küchemann, 1981) and hence Radford (1996) suggests that “General numbers appear as pre-concepts to the concept of variable” (p. 110). Radford *(ibid)* also shows that the conceptions underlying the notion of variable are related to situations whose objectives are basically different from those related to the concept of unknown (that is the formation of relationships between numbers as opposed to solving equations). The logical bases for the two conceptions are different. “The generalisation way of thinking and the analytic way of thinking that characterises algebraic word problem solving are independent and essentially irreducible, structured forms of algebraic thinking” (Radford, 1996, p. 111). Radford (1996) further explains that “the algebraic concepts of unknowns and equations appear to be intrinsically bound to the problem-solving approach, and that the concepts of variable and formula appear to be intrinsically bound to the pattern generalisation approach” (p. 111). These two approaches seem to be mutually complementary domains in the teaching of algebra. This observation would imply that at the fundamental level, students need to understand the concepts of *both unknown and general-generalised number* in order to progress in algebra. What is shown in Küchemann’s (1981) study is that students, at best, view letters as specific unknown values, and show a lack of familiarity of their role as representing *a range of values*. Booth (1995) opines that generalisation is rarely explicitly addressed in the classroom, and this could possibly account for some of students’ problems. This seems to call for a greater number of experiences in generalisation activities for students than they appear to gain at present.
From the above, it appears that general number is a pre-concept to the concept of variable (Radford, 1996) and according to some authors, can also be considered as a pre-requisite to understanding the parameter or given (Ursini & Trigueros, 2004; Usiskin, 1988). Students often come across parameters in algebra (e.g., in functions such as $y = mx + c$) and many of them, even those at university, experience difficulty (Sfard, 1995) with this important concept. In a theoretical framework for the variable, Trigueros and Ursini (2001) assert that parameters ought to be considered as general numbers since that is the role they play in algebra rather than the usual standpoint of a different use of the variable. Following their analysis of high school and college students’ work on algebraic expressions and problems involving parameters, Ursini and Trigueros (2004) suggest that parameters should be considered as general numbers of the second order, that arise from generalising first order general statements. First order general statements are derived from generalising statements involving only numbers, and parameters emerge when families of first order general statements such as families of equations or functions are represented. The authors found that students initially had great difficulties with parameters but their difficulties decreased when students could assign a clear meaning to them.

From the above discussion we may conclude that many student difficulties with the literal symbol might be related to teaching approaches. In order to acquire a deep understanding of the variable, a vital concept in algebra, beginning algebra students require a strong basis in understanding letter as placeholder as indicated by Graham and Thomas (2000), as well as the ideas of letter as specific unknown, and letter as general number representing a range of values, as suggested by Radford (1996). In order to see how to help students to generalise multiunits/place values in different bases, we now examine the generalisation aspects of algebra.

**Generalisation**

The importance of generalisation as an activity is widely acknowledged within research on the learning and teaching of algebra (see Bednarz, Kieran, & Lee, 1996; Kieran, 2006b). Mason and his colleagues (1996; see also Mason et al., 2005) in particular have developed the point of view that generalisation is also a route to (and at the root of) algebra and Mason has stressed the paramount importance of expressing generality in every mathematics lesson. He states:
Generalisation is the heartbeat of mathematics and appears in many forms...the heart of teaching mathematics is the awakening of pupil sensitivity to the nature of mathematical generalisation and dually, to specialisation; that algebra as it is understood in school is the language for expression and manipulation of generalities; and that the successful teaching of algebra requires attention to the evocation and expression of that natural algebraic thinking. There is no single program for learning algebra through the expression of generality. It is a matter of awakening and sharpening sensitivity to the presence and potential for algebraic thinking.

(Mason, 1996, p. 65)

...a lesson without the opportunity for learners to express a generality is not in fact a mathematics lesson. (Mason et al., 2005, p. 297). (Italics by authors)

Emphasising the value of expressing generality, Booth (1995) points out that “algebra is not so much about letters as about generalisations” and that “....it enables us to operate on the generalisations themselves, in much the same way as the Hindu-Arabic notation system for numbers enabled us to operate on numbers in a way which was not possible under earlier notation systems such as the Roman system” (p. 115). In a similar vein, Lee (1996) highlights pattern generalisation (geometric and numeric) as a central activity involving algebraic symbolism, and is of the view that “indeed all of mathematics is about generalising patterns” (p. 103).

However, research indicates a certain number of obstacles in introducing algebra through a generalisation approach. For example, one of the major problems Lee (1996) found in a teaching experiment was not that of seeing a pattern but that of “perceiving an algebraically useful pattern” (p. 95). That is, expressing a generalisation related to the position of the number in a sequence. Some other researchers (e.g. MacGregor & Stacey, 1995; Quinlan, 2001a) report students’ considerable difficulty with the pattern generalisation approach and in provoking learners to shift from iterative/recursive generalisations to expressing the generality as a function of the position in the sequence. Students seem to be drawn to recursive relations rather than direct formulae. Thomas and Tall (2001) point out that while generalised arithmetic is natural for some students, not all of them are ready for more generalised notions of arithmetic expressions. One difficulty is that the order of operations in algebraic notation does not conform to the usual sequence of reading from left to right as in English. The authors state that even if students can handle the general arithmetic, they may still see algebraic expressions as processes rather than procepts (Gray & Tall, 1994) that can be manipulated. It appears that for learners,
algebraic expressions are harder to accept due to the lack of a well formed answer or lack of closure (Collis, 1975). Thomas and Tall suggest that moving from the process to the object stage requires significant cognitive reconstruction. Given the above difficulties and Radford’s (1996) theory, would one not expect that students have experience with both expressions (which they have to hold in an unevaluated state) and equalities/equations?

An awareness of generality, in Mason’s (1996) view, can be developed by encouraging the abstraction and concretisation of experiences that promote seeing a generality through the particular and seeing the particular in the general. Mason describes a framework (Mason et al., 2005) which he calls MGA (Manipulating-Getting a sense of-Articulating) for the analysis of the process of moving from the particular to the general and vice versa. This framework, describes the phases of manipulation of physical or mental objects which provides the groundwork for a sense of relationships, and this in turn results in articulation of that sense, all in a spiral development. In this context, in a recent keynote paper, Mason and his colleagues (Mason, Drury, & Bills, 2007) describe different forms of generalisation, with examples, and examine pedagogical issues such as how to prompt useful generalisations and the conditions under which proximal (under the surface or imminent) generalisation might occur. Much of what is said is relevant and important to this thesis and hence some of these issues are briefly discussed here. The authors describe generalisation: through sense-making of a collection of objects, relationships become properties, and when these properties become separated (abstracted) from the specific instances and are seen as independent of the particular instances, the objects become cases (Mason, 2003) and these cases become examples of some general class. Other forms of generalisation that Mason et al. (2007) define are, empirical, also known as generalisation from cases, and Generic (sometimes referred to as structural) generalisation. A task such as ‘what is 4 + 7 and then 7 + 4 and again 2 + 3 and 3 + 2’? Which includes several particular instances, and seeing the expansion of \((p + k)^n\) as a generalisation of \((p + k)^2\) and \((p + k)^3\) would be examples of generalisation from cases. In contrast, Generic generalisation comes about when a single example is seen as generic and uncovers the underlying structure. For instance, \(6 \times 0 = 0\); such an example can be used to see through the particular and then expose and articulate the structure \(k \times 0 = 0\).
The authors (ibid) further suggest that generalisation takes place when learners are
guided to *stressing*, or foregrounding, some features and *ignoring*, or backgrounding some
others. Becoming aware of similarities and differences results in stressing and ignoring
which in turn leads to an expression of important features. When looking for what is the
same and what is different, by stressing what is changing while other aspects stay the same,
students can become aware of what is called a *dimension of possible variation* and the
extent of variation permitted in that feature is referred to as *range of permissible change*
(Mason et al., 2007, p. 50; Mason et al., 2005, p. 111). According to Mason, when learning
occurs there is a cognitive change. This cognitive change takes place due to what is
attended to and how it is attended, and Mason (2003, 2004, July) refers to this crucial
aspect of learning as the *structure of attention*. The importance of attention to learning and
teaching is due to the fact that it can be directed to identify critical features, but is also
subject to habit. Mason (2004, July) suggests different forms of attention that are important
in mathematical thinking:

- **Holding Wholes** (gazing)
- **Discerning Details** (features and attributes)
- **Recognising Relationships** (part-part, part-whole)
- **Perceiving Properties** (leading to generalisation)
- **Deducing from Definitions** (axioms etc. stated independently of particular objects)

(Mason, 2004, July, p. 9)

Mason explains that shifts between these forms of attention are rapid and often subtle.
He asserts that many of the mismatches between learners and teachers are due to the
differences in the structures of attention. Awareness of structures of attention alerts
teachers to the consideration that students may be attending in various ways, and also alerts
them to situations where there is a mismatch, or when a student has become stuck in one
aspect. This makes it possible for the teacher to elaborate on some aspect so students can
become confident or direct attention suitable to the situation.

Mason et al., (2007) raise the question as to when it is useful and appropriate for a
teacher to speak a generality that generalises examples that students have met or
constructed for themselves. The authors use the term *Zone of Proximal Generality* to draw
attention to the various states of learners’ generalisations in a particular setting; states in which learners are beginning to be aware of their awareness of a generalisation. This zone of proximal generality was recognised by the authors as a particular case of a Zone of Proximal Awareness. For some learners the generality was already present and it only needed to be confirmed by someone expressing it. This affirmation is often important for learners. However, for some students, the utterance might block generalisation that was imminent, and for still others, what was said passes them by as it was not in their current zone of proximal generality.

In relation to generalisation, Kieran (1989) has argued that noticing generality in the particular may not be sufficient to characterise algebraic generalisation; in addition to seeing the general in the particular, “one must also be able to express it algebraically” (p. 165). In a plenary address on generalisation viewed from a semiotic perspective, Radford (2006a) concurs with this view and states that “Generalising a pattern algebraically rests on the capability of grasping a commonality noticed on some elements of a sequence S, being aware that this commonality applies to all terms of S and being able to use it to provide a direct expression of whatever term of S” (p. 5). While Mason places an emphasis on the role of generalisation in algebra, Radford develops these theories with a framework that combines both semiotics and socio-cultural perspectives. He applies this framework to investigate students’ algebraic thinking in the generalisation of patterns. In Table 3.2 Radford provides a synthesis of the different approaches used by students in dealing with pattern activities in increasing levels of generality.

| Table 3.2. Levels of Generalisation Observed in Students (Radford, 2006a, p. 23) |
|----------------|----------------|----------------|
| **Naive Induction** | **Generalisation** |
| Guessing (Trial and Error) | Arithmetic | Algebraic |
| | Factual | Contextual | Symbolic |

Of the given approaches, two are regarded by Radford (2006a) as non-algebraic; guessing (trial and error) and arithmetic generalisation (when students say ‘going up in twos’ or ‘add 2’ or ‘+2’ instead of $y = 2x + 3$). The process of noticing the general he calls the process of objectification (p. 15), and he theorises that the idea of objectification
is rooted in an ontology according to which the concepts or objects of knowledge are made up of *layers of generality* (p. 15) and there is a semiotic element related to the expression of signs in noticing these generalities. In a reaction paper to Radford’s plenary session, Kieran (2006a) suggests that Radford’s framework offers a powerful means for describing algebraic thinking and that it pinpoints important steps in the transition toward expressing pattern generalisation. However, Kieran proposes that it may not be sufficient to notice a *commonality*, and suggests that becoming simultaneously aware of what is the *same* and also what is *different* as propounded by Mason (Mason et al., 2005) may be critical to students’ reaching deeper levels of symbolic generalisation.

The discussion has so far focused on the importance of understanding literal symbols in algebra, the value of generalising and expressing generalisations and on the role of attention (Mason, 2004, July) in teaching and learning. MacGregor and Stacey (1995) recommend that students need to become familiar with the language of algebra and the conventions of algebraic notation prior to pattern generalisation leading to a formula or rule. This researcher supports the view that it is important for students to develop the understanding of general number (alongside the idea of specific unknown) as advocated by Radford (1996); students need to recognise that the literal symbol can represent a *changing number* or that it can represent a *range of values* before working with formulae and functions that involve linking two variables. To this end (and outlined below), the work of Srinivasan (1989) is useful as it provides a vocabulary related to pattern recognition that could assist students to understand algebraic symbolic language and generalisation. This is important because facility with algebraic notation and being able to generalise are secondary and necessary goals to learn and understand general positional notation.

**Pattern language in pattern recognition**

The use of patterns is worthwhile if it helps learners to construct their understanding of the abstract notion of the ‘variable’. According to Srinivasan (1989), the development of mathematical thinking partly involves understanding symbolic language which is the language of mathematicians and scientists. The author clarifies that this does not mean that novice students necessarily use words such as algebra, variable, expression, constant and generalisation. Rather, they can use pre-symbolic language, which he calls *pattern language* (for algebra) and *design language* (for geometry), that uses words from the
natural language of the learner. In his method, “numerical evaluation of algebraic expressions for given values of variables in them has been reversed by eliciting algebraic expression in the form of pattern language for number patterns and design language for shape arrangements” (p. 3). Hence, rather than using words such as ‘invariant’ etc, Srinivasan (ibid) recommends vocabulary that centres around changing, not changing (or repeating), changing in the same way and changing in different ways, and according to him, all words which are within the experience and easy grasp of students in primary school. He also recommends the use of a new symbol; a ‘wriggly’ line \( \overbrace{\quad\quad\quad\quad\quad} \) as a separator between the number patterns and expression, rather than the use of the usual line segment. Srinivasan’s reason for this is that attention should be paid to the patterns and not to the operational outcome. We will now consider some examples of the use of pattern language in pattern recognition.

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<td>21 + 4</td>
<td>5 + 5</td>
<td>12 + 4</td>
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<td>16 + 4</td>
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<td>3 + 4</td>
<td>9 + 9</td>
<td>1 + 15</td>
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<tr>
<td>12 + 4</td>
<td>46 + 46</td>
<td>87 + 2</td>
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\[\overbrace{\quad\quad\quad\quad\quad}\]

*Figure 3.8. Three examples of setting out numerical patterns for the purpose of generalisation (Srinivasan, 1989).*

In group A, in Figure 3.8, the first number is seen to be changing from one addition to another but not the second number. Thus, as explained by Srinivasan, it is seen that the number is changing from one addition to another and so it is indicated by writing a letter symbol. When a number is repeated, no letter symbol is used but the number itself is retained. When numbers are changing in the same way, it is shown by using the same letter symbol. However, when numbers are changing in different ways, this is shown by using different letter symbols. Hence the pattern language for group A is \( p + 4 \), for group B it is \( p + p \) and for group C it is \( p + q \). This pattern language and the related number patterns
may help students to see that the letter in an expression represents a changing number and provides students with the opportunity to make generalisations. Other possible advantages are an awareness that the letter stands for a number, and not an abbreviated word (object). Furthermore, there is scope for students to consider these number expressions in the unevaluated (unclosed) form without having to perform calculations.

However, as discussed above, recognising patterns and using the pattern language that is under discussion involves structuring students’ attention and the teacher has an important role in assisting learners to attend to salient features of the pattern such as what is same and what is different. That is, students are required to ‘pay attention’ to what is not changing, what is changing and how it is changing in order to symbolise the pattern or general statement.

Also pertinent to what is considered above is the usage of literal symbol as parameter or a given number; a difficult concept for students that was examined before under ‘literal symbol’. In the light of Srinivasan’s recommendation for the use of pattern language for the recognition of patterns and first order generalisations, the same pattern language could perhaps be gainfully employed for the purposes of second order generalisations leading to the idea and notation of the parameter.

This section has reviewed the literature on multiple bases, literal symbols and generalization in algebra, for increasing students’ awareness of a general place value system which is the crucial issue of this thesis. In this context, for years mathematics educators have advocated using a variety of representations in teaching students mathematical ideas such as place value. With its origin in grouping notions, the use of concrete or physical representations in place value instruction is also often suggested. Hence a discussion of, signs/representations and their importance in abstraction and concept formation, is taken up in the next section. In the last part of the section, various concrete materials referred to in the literature are also examined.

3.4 Representations and Concrete Materials

The preceding section discussed the conceptual/structural aspect of numeration and its generalisation via multiple bases, and in the current chapter, signs/representations and their
systems with respect to teaching and learning are considered. This is necessary because the place value system is a sign system, and understanding it involves abstracting from, and co-ordinating the conceptual (related to the grouping structure) and the semiotic (linked to signs/representations) aspects of the system. In addition, as will be seen in Chapter 4 (which is about the historical development of the current Hindu-Arabic numeration, and the extraction of related ideas), progress in mathematics appears to be strongly related to the development of numeration, (and algebra) as systems of signs. The study or science of signs is known as semiotics.

Mathematics depends on different kinds of signs (literal symbols, diagrams and formulas, including the numerals of the place value system) and there are some who see signs as facilitators of thinking and others consider signs and language as the origin of cognition (Radford, 2001, July). Among these and other approaches the potential of signs as psychological tools or meaning-making tools is also recognised (Radford, 2001, July). What is addressed next is a brief review of the research literature on the roles of signs and representations as sense-making tools and as modifiers of cognitive functions, and particularly in relation to the numeration system. The section then examines abstraction for learning mathematics, and ends with a consideration of issues related to concrete materials for understanding the place value system.

3.4.1 Representations

Signs/representations have a specific meaning and play a crucial role in mathematics. Both mathematics and its teaching are basically symbolic practices in which signs are invented, used or re-created (Saenz-Ludlow & Presmeg, 2006). In the discipline of semiotics, a sign is any concrete thing, a mark or token that guides our attention to something different (Otte, 2006). Peirce gives this definition of sign. He says “A sign or representamen, is something which stands to somebody for something in some respect or capacity” (as cited in Radford, 1998, p. 290). Peirce categorised signs as icons, indexes or symbols and for every sign, he proposed a semiotic triad. He wrote: “That for which it

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2 Although sign and representation have the same meaning (see Peirce’s definition) in terms of their usage in mathematics education research, it appears that representation refers to all kinds of signs including number symbols, operation symbols, diagrams, formulas and graphs, and sign mainly refers to number, literal, operational and relational symbols.
[sign] stands is called its object; that which it conveys, its meaning; and the idea to which it gives rise, its interpretant” (CP 1.339) (as cited in Radford, 2006b, p. 44). As pointed out by Radford (ibid), meaning is the relation that links one sign to the next sign in a semiotic chain. Saussure (1959) however, defined sign as a combination of two mental constructs, a signified together with its signifier. Due to the essential role that signs play in the expression of mathematical ideas (Saenz-Ludlow & Presmeg, 2006), in recent years, there has been considerable research interest in mathematics education of the study of signs or semiotics. Some viewpoints of this important domain are presented below.

Following the work of Peirce and Saussure, many researchers have elaborated on the role of signs in cognitive development. As observed by Skemp (1971), a sign can be a sound or something visible and is mentally connected to an idea or concept and this idea or concept is the meaning of the symbol; a sign standing for a particular concept evokes, according to Tall and Vinner (1981) a concept image- all the mental pictures, thoughts, properties and processes that are associated with the concept and which may be different to the concept definition. This means signs/symbols reveal structure; not only symbolic literals (Arcavi, 1994) but also symbols such as operators and the equal symbol (Godfrey & Thomas, 2008), are helpers of, thinking (Radford, 2001, July). In an education context, Kaput (1987) has described any concept of representation as involving “two related but functionally separate entities ....the representing world and the represented world” (p. 23). And according to Goldin (2008), signs/representations can be external or internal; external representations are external to the individual learners (such as concrete manipulatives and notation systems in mathematics) while internal representations refer to the mental constructs or psychological representations of the individuals. Students meet a number of sign systems (or representational registers as termed by Duval (2006)) in their school years, beginning with signs related to numbers, counting and computation (Ernest, 2006) and also signs related to geometry, algebra and statistics. Advanced mathematics students go on to master more abstract systems such as calculus and (axiomatic) group theory. A crucial issue, as pointed out by Duval (2006), is that signs are not only used to designate mathematical objects or to communicate, but also to work on and with mathematical objects.

Similar to Kaput’s (1987) representation systems, the notion of semiotic systems was developed by Radford (1998), however later he widens this idea to a semiotic means to
objectification (Radford, 2006a) which constitute different types of signs such as gestures, inscriptions, words and so on. According to Radford, these signs produce contextual generalisation (see Section 3.3.2 for a more detailed account) where signs are still indexes but are about to become symbols. He also proposes an anthropological perspective (Radford, 2006b) that he calls the semiotic-cultural approach which underlines the central role that culture plays in the creation of objects of knowledge and in cognitive development. Ernest (2006) also defines a semiotic system and presents a useful three component model to explain this term:

- First, there is a set of signs,...
- Second, there is a set of rules of sign production, ...
- Third, there is a set of relationships between the signs and their meanings embodied in an underlying meaning structure. (Ernest, 2006, pp. 69-70)

In his paper, Ernest (ibid) suggests that this model foregrounds the sign system in order to assist construction of mathematical structure and elaborates on how the model can be applied to understand the development of mathematics from a historic, formal and educational point of view. However, recognising the limitations of classical semiotics, and adopting a Vygotskian approach, the above definition of semiotic system is widened by Arzarello to a semiotic set and then semiotic bundle (Arzarello & Robutti, 2008) that include gestures and handling artefacts.

In the context of teaching and learning, Presmeg (2006) raises a significant aspect of semiotic theories, which is their potential use to make connections between different areas of knowledge. The author offers a representational tool that is rooted in semiotics to model students’ learning based on connecting activities. Presmeg’s model supports progressive decontextualisation as the students move through meaningful chains of signification. Furthermore, as noted by Presmeg (ibid), the nested triadic conceptual framework (based on Peirce’s ideas) has the potential to constitute a web of signs; signs that are connected in intricate ways to other signs in several areas of knowledge.

From an epistemological point of view of sign, and related to teaching and learning, a fundamental difference between mathematics and other scientific domains exists and this is highlighted by Duval (2006); while phenomena of astronomy, physics, chemistry and
biology are accessible by perception or by instruments (microscopes, telescopes and other measurement apparatus), mathematical objects are not available through such methods and the only way to have access to mathematical objects and deal with them is by using signs and representations. Hence, according to Duval, there is a double access to knowledge objects in other subject areas, mainly non-semiotic and secondarily semiotic, but only a single access to mathematical objects, which is via semiotic representations. This aspect is also noted by Hoffman (2006) who says that we need signs and representations to experience the object and objectivity of mathematics and that mathematical cognition is mediated by representations; signs are the means to think about mathematical relations and objects but are also products of such thinking. This ability to view signs as both processes and products is called as process/object versatility by M. O. J. Thomas (2004, July). One example given by Thomas (ibid) is that of algebraic expressions, where the process is encapsulated as an object, and these objects themselves are then the subject of further actions. In Thomas’s view, this process/object view is an important aspect of mathematical thinking or “versatile mathematical thinking” (M. O. J. Thomas, 2004, July, p. 10) (See Section 3.2.2).

However, as observed by Duval (2006), learners are faced with two opposite requirements for getting into mathematical thinking and there is a cognitive conflict between these two requirements. Duval (ibid) explains that for any mathematical activity, semiotic representations must necessarily be used, but the mathematical objects must not be confused with the semiotic representations. This means that there is no other access to the mathematical object apart from semiotic representations and still, learners are required to distinguish the represented object from the semiotic representation. This implies that there are two aspects, the conceptual and semiotic, to structural understanding. In this regard, Becker and Varelas (1993) contend that cognitive development involves both conceptual and semiotic achievements. The authors note that cognitive development has predominantly focused on the conceptual side, and “if we do not attend to children’s sign use, we may miss significant aspects of their cognition, including structural-conceptual aspects of their cognition” (Becker & Varelas, 1993, p. 421). The authors argue for a differentiated appreciation of both aspects in research studies. A related study on place

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3 In this connection, Presmeg’s (2006) nested triadic model is useful to teachers to represent the various connections students make in the learning process.
value understanding conducted by Varelas and Becker (1997) is discussed later on in this chapter, and also in Chapter 4 as part of the development of a learning framework.

An added difficulty is that students have to work with a wide variety of semiotic/representational systems. Therefore, in order to build meaning for a concept, mathematical tasks need to provide multiple representations and it is important that the relations between the representations are addressed (Kaput, 1992). In this context, Lesh (1999) describes the ability to establish meaningful links between and among representational forms and to translate meaning from one representation to another as **representational fluency**. Relative to this, M. O. J. Thomas and Hong (2001) introduced the concept of **representational versatility** which includes both representational fluency, which is the fluency of translations within and between representations, and the ability to interact procedurally and conceptually with individual representations. According to M. O. J. Thomas (2008a), representational versatility lies at the heart of mathematical thinking and summarised this as:

In conclusion we might say that because conceptual ideas may be constructed from a number of representations it is a good idea for students to experience a number of these at the time they begin to learn the concept. In particular, the explicit linking of ideas across representations is very useful and important. Hence, teaching should seek to assist representational versatility by concurrently providing, and linking, a number of representations in each learning situation. Further, since student interactions with representations can vary qualitatively, they should be exposed to both procedural and conceptual interactions with the representations. (M. O. J. Thomas, 2004, July, pp. 14-15)

What are the ways in which school students represent the ideas and concepts that they are being taught? Bruner (1971) distinguished between three forms of representation of abstract concepts and proposed that students move through these levels of representation as they learn: **enactive**, **iconic** and **symbolic** (p. 11). These stages are thought to be developmental, each level building on the previous. In the **enactive** stage, the learner directly manipulates objects. In the second stage, learners need to visualise the concrete manipulation by forming images (both internal images and external diagrams and pictures that support these images). Bruner called this an **iconic** mode of representation. In the final stage which is the **symbolic** stage, images and pictures are replaced by symbols that denote objects. Mason and his colleagues (2005) indicate that these stages not only describe different modes, but characterise different ways of thinking, even different worlds of
experience. The authors suggest that learners are best assisted if they are encouraged to overlay the physical with the imagistic (with concrete materials out of reach, helps wean students off the need to have physical models present and strengthens their mental imagery) and then the imagistic with symbols that can be operated on. Likewise, Dienes recommended a ‘learning cycle’ (Dienes & Golding, 1971) in which students progress through a series of cyclic patterns, each incorporating experiences ranging from the concrete to symbolic representations. In his method, students’ activities are systematically structured using physical models.

In essence, for a meaningful understanding of the decimal numeration system, the above discussion on signs and representations would imply the provision of opportunities for students to experience a number of different representations of the place value concept. The representations could include concrete materials (base-10 blocks, bundles of sticks, play money, coloured chips), pictorial, and symbolic representations as advocated by Bruner and Dienes. More importantly, unambiguous and clear connections within representations and between representational systems (both concrete and semiotic) are required to be made for a sound understanding of the structure of place value. Brownell, who promoted teaching for meaning, comments:

> If arithmetic is to be meaningful, children must understand whole numbers in terms of 1’s, and 10’s, and 100’s, and 1000’s, and so on. But they will not understand numbers this way if their activities are restricted to pointing off the places in a few large abstract numbers. Instead, they need abundant experience in actually constructing numbers, many more experiences of course with two-place than with three-place numbers,...And these experiences in constructing numbers need to be supplemented by experiences in recognising totals of objects (bundled sticks or pegs or “tickets”) which are constructed by others...these experiences and the resulting understandings pay large dividends later on,... (Brownell, 1945, p. 485)

The use of representations, including concrete materials, in teaching and learning mathematics, and in particular the concept of place value structure and its notation, leads to abstraction and concept formation and this is addressed next.

### 3.4.2 Abstraction and Concept Formation

Abstraction is a key idea in concept formation (Piaget & Garcia, 1989; Skemp, 1971) and in recent years there has been a marked increase in research writings (e.g. Gray & Tall, 2007; Mitchelmore & White, 2007; White & Mitchelmore, 2010) related to this construct.
In the context of multidigit numbers, based on their experiences and activities with concrete materials, pictorial and semiotic representations, learners are required to *abstract* the meaning of a numeral which is the concept (cardinal number) associated with the numeral. However, a main difficulty for students is that the decimal place value system contains many (as will also be seen in the history chapter) *layers of abstraction* (Dahlke, 1982). Hence *abstraction* and *concept formation* are now considered, prior to the discussion on concrete materials.

In his theory of abstraction, Piaget’s (1970) distinguished between construction of meaning through different types of abstractions; the first is *simple abstraction* (or *empirical abstraction*) which focuses on how students construct meaning for the properties of objects, and the second is *reflective abstraction* which applies to how knowledge is abstracted from, actions on objects, and the co-ordination of these actions and the related properties. Similar to Piaget’s postulation, Dienes (1964b) provides this definition: “A mathematical abstraction is the result of considering some relevant properties of different mathematical situations and discarding at least for the moment all irrelevant ones” (p. 170).

Adopting a different stance, Vygotsky (1962) criticised traditional methods of studying concepts that separated the *word* from the *perceptual material*. He proposed *abstraction* as the final phase in a major three-phase process of concept formation and was of the view that this cognitive growth occurs within a social context including the use of cultural tools such as language. Vygotsky viewed learning as a social process and stressed dialogue and the various roles that language plays in teaching and in cognitive development. He states:

> Only the mastery of *abstraction*, combined with advanced complex thinking, enables the child to progress to the formation of genuine concepts. A concept emerges only when the abstracted traits are synthesised anew and the resulting abstract synthesis becomes the main instrument of thought. The decisive role in this process, as our experiments have shown, is played by the word, deliberately used to direct all part processes of advanced concept formation. (Vygotsky, 1962, p. 78) (italics added)

And Vygotsky emphasises the dual role of the *word/sign* in concept formation and in representing the concept:

> All the higher psychic functions are mediated processes, and signs are the basic means used to master and direct them. The mediating sign is incorporated in their structure as an indispensable,
indeed the central, part of the total process. In concept formation, that sign is the \textit{word}, which at first plays the role of means in forming a concept and later becomes its symbol. (Vygotsky, 1962, p. 56)

Hence, for Vygotsky, both language and real life experiences are an integral part of cognitive growth and abstraction, whereas for Piaget, language played a subordinate role and he focussed almost exclusively on the psychological processes leading to concept formation.

In a later work, and building on Piaget’s and Dienes’s theories, Skemp (1989) uses the term \textit{abstraction} for the process of concept formation and describes a \textit{concept} as “A mental representation of common properties” (p. 52). According to Skemp (\textit{ibid}), concepts represent regularities abstracted from experiences; that is, an awareness of something in common between one’s experiences and activities. By directing our attention to commonalities of a set of objects (which may also be mental objects) or the actions on the objects, we are ‘pulling out’ or \textit{abstracting} this common property (Skemp, 1989) and the result of this mental activity is what the author calls a concept. Some concepts involve more levels of abstraction resulting in a hierarchy of lower-order (less abstract) and higher-order (more abstract) concepts. For students the lower-order abstractions/concepts need to be grasped before higher order concepts are abstracted. The author states: “.....in the building up of the structure of successive abstractions, if a particular level is imperfectly understood, everything from then on is in peril. This dependency is probably greater in mathematics than in any other subject” (Skemp, 1971, pp. 34-35).

Subsequently, abstractions/concepts are \textit{assimilated} or \textit{accommodated} (Piaget & Garcia, 1989) into a cognitive \textit{schema} (a mental structure) and this schema integrates existing knowledge as well as functioning as a mental tool for the attainment of new knowledge. \textit{Assimilation} characterises when a new concept is integrated with previous knowledge (and the new concept expands the existing network of ideas) and, if the new information does not fit in with the mental schema, Piaget and Garcia (\textit{ibid}) state: “...in particular, if the content is new, the assimilatory schemata are somewhat modified by means of \textit{accommodations} – that is, differentiations in accordance with the object to be assimilated” (p. 269). In this context, Hiebert and Carpenter (1992) provide a similar definition: “A mathematical idea or procedure or fact is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and the strength of the connections” (p. 67).
As asserted by Skemp (1971), the abstraction process is a very natural human trait. This view is echoed by Gray and Tall (2007) who suggest that abstraction has a multi-modal meaning as a process, property or concept, and describe mathematical abstraction as arising through a natural mechanism of the human brain in which complicated phenomena are compressed into what the authors call *thinkable concepts*. According to Gray and Tall (*ibid*), the brain copes with information overload by suppressing extraneous data and focusing only on a few important features at any given time and the authors call this *compression*. However, highlighting the importance of *language*, the authors state that once properties and connections are noticed, this compressed phenomenon (*thinkable concept*) is *named* (example of a thinkable concept is *procept*; *procept* is discussed in Section 3.2.2) so that one can talk and think about it and manipulate it.

In terms of the relationship of abstraction to place value, Skemp (1971) asserts that place value concerns secondary concepts (as opposed to primary concepts) involving abstraction at a high level. Understanding the place value idea, in Skemp’s view is often mistakenly considered as elementary, and relies on all of the following concepts:

- the natural numbers
- order
- counting
- unit objects
- sets of objects
- sets as single entities
- sets of sets
- numerals, and the distinction between these and numbers
- numeration
- bases for numeration

(Skemp, 1989, p. 61)

As pointed out by Skemp (1989) all of the above are themselves secondary concepts, some of them being of quite a high order of abstraction. The implication of this theory is significant for the teaching of the place value system. For example, in order to understand
‘sets of sets’ which is given in the above list of secondary concepts, students need experiences at first with groupings such as bundles of sticks or arrays of dots, counting (and notating) these groups, then groupings of these groups as equal grouping structures and their symbolic forms; a process involving many levels of abstraction.

From a pedagogical point of view, the above analysis on representations and abstraction has highlighted the importance of both psychological and semiotic approaches (Ernest, 2006) for a deep awareness of the place value concept. The next and the last section of this chapter, involves a review of the literature related to the use of concrete manipulatives in teaching and learning, particularly those related to place value understanding.

3.4.3 Concrete Materials

Students regularly come across representations in mathematics and some of these are concrete embodiments of mathematical ideas and procedures, such as bundles of sticks or Multibase Arithmetic Blocks (MAB or also known as Dienes’ blocks) and others are representations inherent in the discipline of mathematics; e.g. number lines and symbols. As the representations serve basically in an analogical capacity, Boulton-Lewis and Halford (1992) refer to the representations as analogues. An analogy can be defined as a mapping from one structure, the base or source, to another structure which is the target (Gentner, 1983. Cited in English & Halford, 1995). The authors point out that in the case of mathematical analogues, the concrete or pictorial representations are the source and the concept to be acquired is the target. They state that “The value of these analogs is that they can mirror the structure of the concept and thus enable the child to use the structure of the representation to construct a mental model of the concept” (pp. 97-98). Another reason that concrete representations are useful, in M. O. J. Thomas’s (2008a) view, is that representational versatility lies at the core of mathematical thinking. The versatility arises in the ability to translate between representations of a concept and to interact with them in different ways; by observing them, by noticing the properties of the representations or the associated concepts and objects, or by acting on them. In so doing, students become aware of the concept. Students need to be made aware of the parallel relationships between concrete materials and number symbols. They need to develop meaning for the symbols and Resnick and Omanson (1987) believe that “…mapping between blocks and writing
may play an important role in learning by helping children to develop an abstraction – a higher level representation – that encompasses both blocks and writing” (p. 90).

Many types of concrete materials have been documented in the literature as well suited for use in teaching place value concepts (e.g. Baroody, 1990; Cauley, 1988; English & Halford, 1995; Hiebert & Carpenter, 1992; Labinowicz, 1985). These materials include a variety of objects (including pictures of objects) such as counters, lollies, beans, Unifix cubes, Multilink material, bean sticks, string lengths, base-10 dot materials, play money, coloured chips, abaci, base-ten blocks, cuisenaire rods and place value charts. While most studies have preferred Dienes’s base-ten blocks to model place value ideas, other research studies have employed different manipulative materials. The following examples show some of the variety of concrete materials that have been used in classroom studies and the different pedagogic approaches taken for teaching the concept of place value numeration.

A constructivist teaching experiment carried out by Bednarz and Janvier (1982, 1988) focused on the groupings inherent in the numeration system. The materials used included cereal boxes grouped by six into cases, peppermints in rolls of 10 and bags of 10 rolls and paper flowers (each made of 10 sheets of paper) grouped into bouquets of 10 flowers. The materials were used in context in which the operations had meaning and tasks created in order to expose the grouping structure of the situation. A more structured approach to teaching place value was advocated by Fuson and Briars (1990). They reported a teaching/learning setting where first and second grade children successfully used base-ten blocks and digit cards to construct multiunit conceptual structures. In this approach the teacher continually reinforced the links between written symbols, number names and concrete materials. After such experiences the children were introduced to four-digit addition and then subtraction and traditional algorithms were developed through instruction using base-ten blocks, verbalising of actions (both block words and number words used) and recording of numerals after each action with the blocks.

A semiotic (and often confusing) feature of the base ten system is the fact that sometimes the numeral 4 represents four units; at other times it represents four tens; and still other times it represents four hundreds. Varelas and Becker (1997) call this aspect ‘multiplicity of value’. They also coined the terms ‘complete value’ (face value multiplied by the quantity that it represents) and ‘composition value’ which is the sum of all the
complete values of a multi-digit number. According to Varelas and Becker (1997), students’ difficulties lie with these two characteristics of the place-value system, particularly with the ‘multiplicity of value’ feature, and say that “They [students] fail to differentiate between the face value of each symbol in a number and the complete value of the same symbol” (p. 265). They state that research has shown that students may have facility with conceptual, grouping and computational aspects and still have difficulties with the place-value system. Resnick and Omanson (1987) found a similar difficulty and state “although they [children] could use blocks to represent two and three digit total quantities, they could not use them reliably to represent individual digits [in a multidigit number]” (p. 64). Varelas and Becker explored a system using a semiotic perspective that is intermediate between the written place-value system (which has multiplicity of value) and base 10 blocks (no multiplicity of value), and whether it helped children to understand the place-value system. In their system, there are pieces and a board, where the upper sides of all the pieces are one colour, indicating face value. The undersides are another colour (usually hidden) and present complete value. For example, a number 333 will have the undersides as 3, 30, and 300. The results show that instruction and practice in this system (for 2nd, 3rd and 4th grade students) helped them to differentiate between face value and the complete values of individual digits and also to grasp that the sum of all the complete values add up to the compositional/total value.

Some researchers advocate the teaching of place value in a calculation context, particularly addition and subtraction exercises as a means to gaining proficiency in understanding and using multidigit numbers. This teaching approach was shown by both Resnick and Omanson (1987), who used base ten blocks and Thompson (1992), whose research was in a base ten blocks and computerised microworld setting. In tasks that focused on digit-correspondence, Ross (1986) used beans, sticks and base ten blocks and Cauley (1988) examined second and third graders’ ability to construct part/whole logical structure with the aid of bundles of sticks. The above studies are but a few examples from a number of studies that have used concrete materials and that have been conducted (with mainly primary school students) since Dienes’s (1964a) and Bruner’s (1971) publications; the most commonly used materials being the base-ten blocks. However, the results are mixed. Some studies (Resnick & Omanson, 1987; P. W. Thompson, 1992) found that using base ten blocks had little impact on primary students’ facility with algorithms. In terms of
understanding the place value concept, Labinowicz (1985) described the considerable difficulties that the students in his study had in making sense of base ten blocks. However, other studies (Fuson & Briars, 1990; Hiebert, Wearne, & Taber, 1991) reported consistent success with these materials on children’s understanding of, and skill with, decimal numeration and multidigit operations.

Given these varied findings, English and Halford (1995) examined some common analogues for their potential for representing a particular concept. They state that some manipulatives can be distracting, ambiguous and open to interpretation while others were more suited to the particular topic under consideration. English and Halford (ibid) classified manipulatives as unstructured analogues (e.g. counters), semistructured analogues (such as bundling sticks, unifix cubes, and coloured chips) and structured analogues (base ten blocks, place value chart, abacus, Cuisenaire rods, money, the number line and area models). They suggest that while unstructured analogues such as counters are useful at the initial stages of understanding basic number concepts, abstract aids such as coloured chips were more suitable for enrichment work since the chips are not amenable to grouping and other place value ideas. As well, multiple mapping processes were required such as chip to colour and colour to value and this resulted in an additional processing load. Base ten blocks scored highly in their view; the blocks facilitate the trading process and display clarity of the source structure. However, English and Halford point out some of the limitations. The effectiveness of the blocks will be limited if the blocks are not arranged in accordance with the place value scheme and arranged in any order, and this would result in failure of students to form connections. Secondly, there is an added complexity when representing decimal fractions; values of the blocks need to be changed in order to accommodate decimal fractions. When the blocks take on new values, as a consequence, students are faced with additional mapping processes and hence a higher processing load (Boulton-Lewis & Halford, 1992). Furthermore, a problem noted by Stacey, Helme, Archer and Condon (2001) is that students using base-10 blocks experienced difficulty generalising to numbers beyond the base-10 blocks model, and this appeared to be due to their difficulties with volume and apparent dimensional shifts in their notions of the components.

In a similar analysis, Labinowicz (1985) compared what he called discontinuous, groupable objects (such as sticks beans and unifix cubes) and continuous, pre-formed
objects (such as Dienes blocks). One of the important differences highlighted by the author is that, while students can group sticks and decompose these bundles of sticks through direct action, the pre-structured blocks were built up through indirect action through trading one form of representation for another. In this sense, there is more involvement in the grouping of discrete objects such as sticks. In Labinowicz’s (1985) view, while the continuous nature of the base-10 blocks together with the indirect activity induces learners to count any block as one, the direct activity of composing and decomposing groups of sticks seems to make relations more accessible to students. Additionally, discrete objects permit groupings of different sizes and also increase learners’ awareness of the need for uniform or equal grouping structures. In contrast, unless a variety of multibase blocks are readily available, a particular arrangement tends to be imposed on students when using continuous materials such as base-10 blocks. For these reasons, Labinowicz suggested using materials conducive to grouping, and at more than one level, as in grouping of groups, and groupings of groupings of groups, and so on. Likewise, in his investigation with Year 3 students’ understanding of place value, Price (2001) recommends the use of grouped single material such as bundling sticks, rather than base-ten blocks, at least in primary school. While using different concrete embodiments for one concept provides the opportunity for students to make connections by relating various representations of concepts to one another, it can result in students being confused and seeing many different concepts. Hence, students should be encouraged to use a particular model regularly and become familiar with the materials in order to use them effectively (Boulton-Lewis & Halford, 1992).

The considerations presented above suggest that it is important that teachers choose the appropriate material to assist students in making links related to the concept but it seems that they are offered little advice (Ball, 1992; Hiebert & Wearne, 1992). However, this aspect of the choice of the material for teaching is only one part of the picture. The other significant factors are the accompanying language and the procedures in manipulating the materials. English and Halford (1995) indicate that irrespective of a material’s potential for representing a concept, it will not succeed if the accompanying explanation is unclear and if the manipulations of the materials do not reflect the target concept or procedure. Resnick and Omanson (1987) agree with this point of view and mention that materials themselves cannot impart meaning. Mathematical ideas do not actually reside in the models and cannot
be directly seen through the use of concrete materials (Ball, 1992) and students need assistance in building links between physical, pictorial, verbal and symbolic representations of the concept (Bednarz & Janvier, 1982; Fuson & Briars, 1990; Hiebert & Wearne, 1992; P. W. Thompson, 1992). There can be a difference between how students experience physical materials and how they perceive pictures or diagrams but the difference resides in how they are used (P. W. Thompson, 1994). In the context of teaching place value concepts, it is commonly agreed that students need to think actively (Ross, 1986) about the quantities represented by place value materials and engage in problem-solving tasks, as otherwise there is a risk that “just as with symbols, pupils can learn to use manipulatives mechanically to obtain answers” (Baroody, 1989, p. 4).

Hart (1989) reports a gap that seems to exist in the minds of many students between the two systems of number representations (concrete materials and written symbols) in the preliminary results of a research project entitled “Children’s Mathematical Frameworks”. From the poor performance of the project’s participants, she arrived at the conclusion that they were making little connection between concrete manipulatives and written algorithms. Hart (ibid) believed that students see the use of manipulatives and written procedures as disconnected processes and suggested a subtitle for the project report as; “Sums are Sums and Bricks are Bricks” (p. 139). She summed up her report in these words:

Many of us have believed that in order to teach formal mathematics one should build up to the formalisation by using materials, and that the child will then better understand the process. I now believe that the gap between the two types of experience is too large and we should investigate ways of bridging that gap by providing a third transitional form. (p. 142)

As is evident from the studies discussed above, the use of concrete materials and in particular, base ten blocks is not always effective and it is unclear as to why we sometimes get positive outcomes and at other times negative ones. Labinowicz (1985) and Thompson (1994) suggest that this could probably be related to instruction. Thompson (ibid) states:

These apparent contradictions probably are due to aspects of instruction and students’ engagement to which studies did not attend. Evidently, just using concrete materials is not enough to guarantee success. We must look at the total instructional environment to understand effective use of concrete materials –especially teachers’ images of what they intend to teach and students’ images of the activities in which they are asked to engage. (p. 556)
It is clear from the above that it is important for teachers to a) assess the suitability of a particular model (Baroody, 1990; English & Halford, 1995) before it is used as a number representation and b) guide students to think about the materials, construct meanings for the various actions with them and make the relevant connections. The realisation that the place value principle involves a special kind of grouping is crucial (Bednarz & Janvier, 1982; Hiebert & Carpenter, 1992). Given the importance of groupings (in different bases) and the varied findings with respect to base-10 blocks, bundles of craft matchsticks (as concrete aids) appear to be particularly useful in teaching the multiplicative structure of the place value system. However this researcher tends to agree with Hart’s observation that the gap between the concrete and abstract forms is too big and needs to be bridged by the use of a third transitional form (or forms) that integrates both the conceptual/concrete representations and the semiotic representations (Arcavi, 1987; Becker & Varelas, 1993).

This chapter has reviewed educational theories and perspectives on place value structure, multiple bases and generalisation, representations, abstraction, and concrete manipulatives. It has also underlined the importance of assisting students to generalise the multiumits/powers in various bases, and to construct links between multiple representations, in order to abstract the notion of place value. With this background of mathematics education research, we now turn to the history of mathematics, and in particular Indian history, to trace the development of the current Hindu-Arabic decimal numeration and algebraic symbolism. History is researched for ideas that could be useful in constructing a teaching framework for a deep awareness of the place value system.
CHAPTER 4

HISTORY OF MATHEMATICS

4.1 Introduction

This thesis presents the results of an investigation of how historical ideas might be used to enhance Year 9 students’ understanding of the structure of the current numeration system. The preceding chapters have reviewed the literature on the theoretical perspectives involving a) history and mathematics education, and b) teaching and learning numeration. In this chapter, we turn to the history of mathematics, and explore historical content for ideas that will help strengthen students’ understanding of the decimal numeration system. The key focus is on the history of Indian mathematics since the current Hindu-Arabic decimal numeration system originated in India (Datta & Singh, 2001; Menninger, 1969). The main goal of this chapter was to answer the following general research question related to the history of mathematics.

What are some of the key historical ideas relevant to understanding the decimal place value system and how might they be useful in teaching and learning?

In order to address the above research question, the present chapter comprises the following sections: a) an overview of the history of Indian mathematics; b) different stages in the construction of the Hindu-Arabic numeration system; c) algebraic ideas in Indian history; and d) numeration systems in different civilizations. Following each section, there is a reflection on and articulation of the historical content in the context of mathematics education literature (the cyclic process of historical-critical methodology) that was reviewed in Chapters 3 and 4. As a result, ideas from both history and mathematics education were combined, resulting in a teaching and learning framework. First, in the ensuing section, an overview of the history of Indian mathematics is outlined.

4.2 An Overview of the History of Indian Mathematics

A significant and fundamental contribution of India to mathematics is the place value decimal numeration system with zero. (India as mentioned in this thesis refers to the Indian subcontinent or South Asian subcontinent that encompasses most of present-day India,
Pakistan, Nepal, Bangladesh, Bhutan and Sri Lanka rather than only the modern country of India). The earliest literature contained number names for powers of ten, and according to Datta and Singh (2001), the numeration system was developed sometime around 200 BCE. The following section briefly outlines the history of mathematics in India until the medieval period ending in the 16th century. In this way the mathematics related activities that led to the invention of the place value system are highlighted along with its impact on the development of mathematics after its creation. This overview is followed by reflection and articulation of historical and mathematics education ideas.

In India, mathematics was, and is, deeply linked to culture, and mathematics is part of an integrated and holistic system of knowledge that generally contributed to the advancement of thought. The history of mathematics in India, as it is generally regarded at present, begins with the Indus valley civilisation followed by the Vedic period, the post-Vedic period consisting of Jaina and Buddhist mathematics, the period of the Siddhantas (astronomical works), and ends with the mathematics of the classical and medieval eras.

An important aspect of mathematics in Indian history was that it was developed and used very much for practical purposes (Bag, 1979; Rao, 1998). In the earliest period of the Indus valley civilisation, archaeological evidence suggests that mathematics was mainly concerned with weights, measuring scales and architecture in which an advanced brick technology utilised ratios (Joseph, 2000; Pearce, 2002). In the succeeding Vedic period, according to Sen (1971), mathematics grew out of religion – there was a need for the construction of various kinds of sacrificial altars, the reckoning of time for calendrical purposes and the study of intricacies of the science of language. Sen (ibid) further points out that in India, as in some other cultures, much of mathematics (such as numeration system and arithmetic, geometry, algebra and trigonometry) developed as a result of progress in astronomy and, a sizeable part of post-Vedic mathematics⁴ (arithmetic) can be found in astronomical works. Essentially, the relationship between astronomy and mathematics involved a feed-back process. Commercial factors also played a major part from early times, as indicated by Bag (1979), and led to the inclusion of problems on

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⁴ Vedic mathematics, as mentioned in this thesis, refers to the mathematical content in the Indian Vedic texts. Recently, however, the term Vedic mathematics has come to mean mental calculation algorithms published in 1965 in a book entitled Vedic mathematics (Tirthaji, 1992). The author Sri Bharati Krsna Tirthaji Maharaja describes these algorithms as reconstructed from the Atharva Veda.
capital and interest, profit and loss, barter, computation of gold, mixture, and wages and payments.

The evidence of the use of mathematics in India appears first in the Indus valley, which dates back to about 3000 BCE (Datta & Singh, 2001; Joseph, 2000). Excavations of the Indus Valley at Mohenjodaro, Harappa and Lothal reveal an urban civilization that had advanced town planning, built brick houses and used metals such as gold, silver, copper and bronze. As noted by Joseph (2000), it was a highly organised society where the archaeological finds point to geometric sophistication, skills in practical arithmetic and mensuration. Written documents, seals and inscriptions of this period give the earliest examples and specimens of writing in India (Srinivasengar, 1967). However, this Indus script has not yet been deciphered and hence an evaluation of the mathematical proficiency is based on excavated artefacts. Scales and instruments show accurate gradations and significantly, a decimal system was already established during this period, as indicated by an analysis of weights and measures in the Harappa area (Sen, 1971). Weights corresponding to ratios of 0.05, 0.1, 0.2, 0.5, 1, 2, 5, 10, 20, 50, 100, 200 and 500 have been identified along with scales with decimal divisions. Joseph comments that “Such standardisation and durability is a strong indication of a numerate culture with a well-established, centralised system of weights and measures” (Joseph, 2000, p. 222). The next period is known as what is thought to be the Vedic period. What follows is a brief outline of the Vedic literature and the mathematics contained in it.

The Vedas (literally meaning knowledge) is the sacred literature of the Hindus; it is a vast body of integrated knowledge with instructions on all aspects of life. Some scholars and researchers (e.g. Joseph, 2000; Macdonell, 1956) have suggested that these monumental collections are probably the oldest surviving literary documents of mankind. The Vedic literature, in Sanskrit verse, is thought to have been composed around 2000 BCE (see for e.g. Datta & Singh, 2001; Eves, 1969; Van Der Waerden, 1954) and possibly documented around the start of the first millennium BCE (Joseph, 2000). These Sanskrit texts comprising the Vedic literature were transmitted orally from generation to generation, without being written down (V. Kameshwari, personal communication, December 28, 2005).

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5 There is confusion as to when and how this Indus (or Harappan) culture ended, when and how the Vedic (pertaining to the Vedas) culture began, or whether the Vedic culture was earlier, and the links between the two cultures.
2006). As pointed out by Plofker (2009), the Vedic texts were meant to be recited, heard and rote-learned, and even technical subjects were thought of as knowledge that had to be learnt by heart. The importance given to orality and memorisation may have inspired the interest in, and the sophisticated development of phonetics and grammar among Indian intellectuals. Techniques of recitation, memorisation and conservation were developed which preserved the ancient texts and used by pandits who still practise this art at the present time (Filliozat, 2004). Due to this early oral tradition, Kak (2000) argues that the Vedic literature was probably composed at a much earlier date; perhaps even well before 4000 BCE.

Figure 4.1. Map of India and (inset) South-east Asia (Joseph, 2000, p. 220).
The Vedas are four in number (Rigveda, Samaveda, Yajurveda and Atharvaveda) and extended to Upa Vedas and Vedangas. Of these, the Rigveda is the earliest, and according to Menninger (1969), it is “...the oldest collection of hymns in existence” (p. 102). The Vedas underwent four stages of development namely, Samhitas, Brahmanas, Aranyakas and Upanishads with Samhitas being the earliest. The Vedic texts are generally known as sruti, that which is heard via divine revelation, on the other hand the other texts such as the Vedangas and Upa Vedas are called smriti, which is remembered from human tradition (Plofker, 2009). According to Sen (1971), there is enough material in this literature to help form a good idea of the mathematical ability of the early Indians although these are scattered and diffused in the writings. Referring to the earlier sections of the literature, Datta and Singh (2001) state:

In these works are to be found well-developed systems of metaphysical, social and religious philosophy, as well as the germs of most of the sciences and arts which have helped to make up the modern civilisation. It is here that we find the beginnings of the science of mathematics (arithmetic, geometry, algebra, etc) and astronomy...the Brahmanas...devoted themselves, from one generation to another, to the cultivation of the sciences and the arts, religion and philosophy. (vol.1, pp. 1-2)

With regard to mathematics, a study of Indian texts reveals the key role that Sanskrit played in the development of mathematics. Sanskrit’s facility for scientific expression has been highlighted by researchers (e.g. Filliozat, 2004; Joseph, 2000; Staal, 1995) and according to Joseph, Sanskrit served as a useful medium, not only to compose and record the early religious and philosophical texts, such as the Vedas and Upanishads, but also scientific literature such as the Vedangas. Its capacity for scientific use was increased as a result of Panini’s (c. 500 BCE) systematisation of its grammar in his Ashtadhyayi. In this connection, Joseph states:

Panini offered what must be the first attempt at a structural analysis of a language. On the basis of just 4000 sutras, he built virtually the whole structure of the Sanskrit language, whose general ‘shape’ hardly changed for the next two thousand years... An indirect consequence of Panini’s efforts to increase the linguistic facility of Sanskrit soon became apparent in the character of scientific and mathematical literature. It may be brought out by comparing the grammar of Sanskrit with the geometry of Euclid- a particularly apposite comparison since, whereas mathematics grew out of philosophy in ancient Greece, it was, as we shall see, partly an outcome of linguistic developments in India.
The geometry of Euclid’s elements starts with a few definitions, axioms and postulates and then proceeds to build up an imposing structure of closely interlinked theorems, each of which is in itself logically coherent and complete. In a similar fashion, Panini began his study of Sanskrit by taking about 1700 basic building blocks – some general concepts, vowels and consonants, nouns, pronouns and verbs, and so on – and proceeded to group them into various classes. With these roots and some appropriate suffixes and prefixes, he constructed compound words by a process not dissimilar to the way in which one specifies a function in modern mathematics. (Joseph, 2000, p. 217)

The association between language and mathematics that is highlighted above is possibly true of all cultures. Barton (2008) in his recent book has suggested that mathematics and language evolved together and has stated that “it is not possible to have mathematics without language” (p. 159). He also comments with regard to the facility of some languages in expressing scientific thought:

Mathematics arises after, not before, human activity, in response to human thinking about quantity, relationships, and space within particular sociocultural environments. Thus the factors determining the choices made in the development of mathematics are primarily social and cultural...Mathematical language is more consonant with some languages, and less consonant with others. (Barton, 2008, p. 173)

However, in ancient Indian mathematics, this link is particularly close. More specifically, the development of the decimal place value numeration is the result of a confluence of linguistic and mathematical thought in India, as shown by Kadvany (2007) and Pandit (1993). It will be seen that in the earliest times (period of the Rig Veda) arbitrary names for powers of ten were given in Sanskrit. Furthermore, Sanskrit played a major part in the development of nomenclature in algebra.

In the context of mathematics in early Vedic literature, a thorough analysis of the mathematical and numerical references that appear in the Vedic Samhitas is presented by Pandit (1993). The author gives a list of all the cardinal number-words that are recorded in the earliest literature of the nine Vedic Samhitas. The number names for 1 to 100, 200, 300, 400, 500, 1000, 3000, 4000, 6000 are presented. In addition to this, in the various recensions of the Yajur Veda Samhita, number names for powers of ten up to a trillion \((10^{12})\) are given, with each successive number noted ten times higher than the previous one (large numbers are discussed in more detail below).

Other examples given in the texts: 27 is named as both saptavimsati \((20 + 7)\) and trinava \((3 \times 9)\), and 21 is also referred to as trisapta \((3 \times 7)\) rather than ekavimsati.
(Pandit, 1993). Interestingly, the subtractive principle is also used in the number nomenclature illustrating the relationship with the closest power of ten. For example, 19 which is nava-dasa (10 + 9) is also named as ekonna-vimsati (20 – 1). Pandit (1993) explains that the above number names are expressed in simple, non-compounded wordstructures whereas the numbers above ten which are not radix (not root) or underived words (e.g. 10, 100 etc) are expressed in compound word-structures. Thus, as stated by Ifrah, “...the Sanskrit numeral system contained the very key to the discovery of the place-value system,” (Ifrah, 1998, p. 429) and in Dutta’s (2009) view, a major step was taken when the ancients gave single word-names for successive powers of ten. This nomenclature possibly impacted on the development of numerals as will be seen in Section 4.3.3.

Other numerical references in the Samhitas involve classification of odd and even numbers (Rao, 1998), forward and backward counting, ordinal numbers, innumerable (asamkhyata) numbers, numbers as adjectives, signs and sign words for mathematical operations, concept of sets, squares, fractions, arithmetic and geometric progression and by implication, the concept of position, infinity and linguistic zero inherent in the Sanskrit language (Pandit, 1993). These last aspects are taken up in subsequent sections. According to Rao (1998), fractions such as half (ardha), quarter (pada), one-eighth (sapha) and one-sixteenth (kala) are referred to for the first time in history, in the Rig Veda.

The next division of the Vedic literature, known as the Brahmanas, consists of commentaries on the mantras or chants of the Samhitas, and descriptions of sacrificial rites and their methods (Sen, 1971; Srinivasiengar, 1967). With the passage of time, there arose the necessity of a meticulous preservation of the mantras, their meaning and knowledge of the construction of fire altars (Joseph, 2000). Thus emerged the Vedangas (literally limbs to the Vedas) which are important sources of mathematics, were classified into six branches of knowledge: siksha (phonetics), vyakarana (grammar), niruktha (etymology), chandas (prosody or metronomy), jyotisha (astronomy) and kalpa (rituals) (Rao, 1998). The knowledge in these Vedangas was composed in the form of sutras (Joseph, 2000), a poetic style of writing that aims at the utmost brevity in capturing the essence of an argument, a method or a result. Joseph (ibid) also points out that an important reason for the study of phonetics, metronomy and grammar was to ensure perfect accuracy in
enunciating every syllable in a hymn or mantra, while etymology was studied for the preservation of meaning of the chants. This tradition of composing terse *sutras* in the metrical form, which could be easily memorised, ensured that the knowledge was properly safeguarded and orally transmitted to successive generations. An important aspect of the rituals was an accurate calculation for times of certain festivals and auspicious times for sacrifices and acts of worship. This required correct knowledge of times of rising and setting of the sun and moon and the occurrences of solar and lunar eclipses (Srinivasiengar, 1967). Hence the study of astronomy; the Vedanga *jyotisha*, which essentially deals with the science of astronomy is the culmination of accumulated knowledge of preceding centuries (Rao, 1998); it consists of records of astronomical knowledge needed for the day-to-day life of the people of those times. This implies, as pointed out by Srinivasaiengar (1967) knowledge of arithmetic, plane and spherical geometry and trigonometry and the construction of simple astronomical instruments. Although mathematics was considered as a hand-maiden to astronomy, it was recognised as an important subject and held in high esteem as is evident in the following quotation: “Like the crest of a peacock and like the gem on the hood of a snake, so is mathematics at the head of all sciences”. (Vedanga *jyotisha*, 4) (Datta & Singh, 2001, vol. 1, p. 7)

The prime preoccupation of the Vedic people was to perform sacrifices and hence there was a need to construct altars (*vedis*) in line with prescribed shapes and sizes (Rao, 1998), using special bricks. According to Rao (*ibid*), in the process of prescribing and constructing such definite shapes and sizes involving specific areas and volumes, geometry, arithmetic, mensuration and algebra developed in India. A rich source of mathematical knowledge, especially geometry is provided in the *Sulba-sutras* (800-500 BCE) which form part of the Vedanga *kalpa* dealing with rituals (Sarasvati Amma, 1979). These are manuals for the geometrical constructions of the various sacrificial altars and they state related mathematical facts already known in the *Samhitas* and *Brahmanas*. The designs of these brick-altars were quite involved; for example one of the more elaborate ones was shaped like a falcon in flight (Joseph, 2000). An altar had symbolic significance and had to be accurately constructed and hence involved a thorough understanding of geometrical and algebraic features of rectilinear figures and circles. Specifically, mathematical results compiled in the *Sulba-sutras* concern enumeration, arithmetic operations, properties of rectilinear figures, surds, irrational numbers, quadratic and
indeterminate equations (Sen, 1971). In addition, it is interesting to note that the Pythagoras theorem was known in India by the 8th century BCE (D. E. Smith & Karpinski, 1911) and stated in four of the nine Sulba-sutras including the earliest – Baudhayana Sulba-sutra (Sarasvati Amma, 1979). Furthermore, as indicated by Sen (1971) and Joseph (2000), the Sulba-sutras give approximate values of irrational numbers such as \( \sqrt{2} \) to a high degree of accuracy. The Sulba-sutra period ended about the middle of the first millennium BCE and sacrificial rituals gradually began to decline. This meant that the occasions for the construction of altars that required geometric knowledge were very few.

Overlapping the period of the sutras is the literature of the Jaina in the 6th century BCE and Joseph (2000) observes that a notable aspect of this period is that mathematics became an abstract subject cultivated for its own sake. Among the many mathematical topics that were developed by the Jaina, theory of numbers, indices, permutations and combinations, and geometry remain distinctive (Bag, 1979; Joseph, 2000). As indicated by Sarasvati Amma (1979), correct calculations in their cosmography were very important to the Jainas; their enumeration of large numbers and concept of infinity seem to have played a major part in the development of the decimal place value system. Following the work of the Jainas, in the first few centuries of the Common Era, is the period of astronomical works or Siddhantas. Among the 18 Siddhanthic works that were composed during this period, only about five have survived and these major works are summarised in Varahamihira’s (505 CE) Pancha-Siddhantika (Plofker, 2009). This includes the Surya Siddhanta which contains important changes in astronomical practices requiring innovative techniques of plane and spherical trigonometry. Important Babylonian and Hellenistic ideas were integrated into Indian astronomy at this time (Joseph, 2000; Plofker, 2009).

In the context of decimal place value, an interesting feature is highlighted by many authors (e.g. Datta & Singh, 2001; Joseph, 2000; D. E. Smith & Karpinski, 1911). In the Siddhantas as well as the Puranas of the same period, there was the use of symbolic word numerals (with place value) or bhuta-sankhya (Ayyangar, 1928) in the expression of large numbers. Word numerals offered a convenient method of expressing large numbers in verse and allowed for easy memorisation. Additionally, different words standing for the same number provided “scope for the poetic expression of mathematical results” (Joseph, 2000, p. 243). Thus, among many other synonyms (Datta & Singh, 2001), zero was
represented by sunya (void) or ambara, akasa (heavenly space) and one by rupa or bhumi and three by agni (fire because there were three fires in Indian mythology) or rama (because Rama had three brothers) (Flegg, 1989; Joseph, 2000), and so on. To give an example of a number, 1230 may be expressed as: akasa-kala-netra-dhara (Datta & Singh, 2001, vol. 1, p. 54) where akasa is 0, kala is 3, netra is 2 and dhara is 1 and the digits are expressed in the right to left order, a common practice in Indian history. Another example given by Datta and Singh is kha (0), kha (0), asta (8), muni (7), rama (3), asvi (2), netra (2), asta (8), sara (5), ratripah (1) = 1 582 237 800. Due to easy versification, this ancient practice of using word numerals to denote numbers continued till the 10th century, even after the invention of the decimal place value numerals.

By the beginning of the fifth century mathematics had acquired a certain standing and was separated from exclusively practical and religious requirements. The practice of writing Siddhantas or astronomical works continued during what is known as the classical period (400 CE-1200 CE) in India and this period produced a “galaxy” (Pearce, 2002) of astronomer-mathematicians. A substantial amount of mathematical development occurred within these treatises (Datta & Singh, 2001). Arithmetic, algebra and geometry developed in a significant way. The decimal place value notation as it is currently known that had developed by this time was to have far reaching consequences in mathematics as will be seen in the achievements given below.

The mathematics of the Bakshali Manuscript and some of the key mathematical achievements of four major mathematicians are now considered. This discussion will highlight the richness of the mathematics that was developed during what is known as the classical era of Indian mathematics as a result of the invention of the decimal place value system.

Bakshali Manuscript: This work by unknown author(s) written on birch bark was discovered in 1881 in north-west India. Its dating is contentious; it is thought to be a copy of the original written in the 7th century but could be much earlier (Joseph, 2011; Srinivasiengar, 1967). The manuscript is devoted mainly to arithmetic and algebra with a few problems on geometry and mensuration. The arithmetic topics include fractions, square roots, progressions, profit and loss, interest and the rule of three and the algebra topics discussed are simple, simultaneous and quadratic equations, surds, unknown
quantities, negative signs and the method of false position (Sen, 1971). From the point of view of numeration, numerals are in decimal place value notation with zero represented by a dot (Plofker, 2009). Interestingly, unknown in an equation is also represented as zero. Some examples are given later in this chapter.

Aryabhatta I (b. 476 CE): He is well known for his work *Aryabhatiya* which is mainly a summary of results obtained in the older *Siddhantas* (astronomical treatises). The mathematics section, as was the practice, begins with an invocatory verse praising God, earth and other celestial bodies. The second verse sets out the names of places or powers of ten for place value notation. The remainder of the mathematics chapter contains details of an alphabet system of notation, general methods of operations on numbers including squaring and cubing, extraction of square roots and cube roots and general methods of solving simple and quadratic equations (Joseph, 2000; Srinivasiengar, 1967). According to Srinivasiengar, Aryabhatta’s methods of finding square roots and cube roots indicate his knowledge of the place value principle with zero. Aryabhatta gave a good approximation of π as 3.1416. He also gave correct rules for computing the sum of natural numbers and of their squares and cubes (Bag, 1979). Aryabhatta introduced the sine and versine (1−cosine) and constructed a sine table. It was during his time that algebra grew into a distinct branch of mathematics and perhaps one of the more important contribution of Aryabhatta is his solution to the indeterminate equation of the first degree, \( ax - by = c \) (Bag, 1979; Datta & Singh, 2001). This opened up a favourite line of investigation among later Hindu mathematicians (Murthy, Rangachari, & Baskaran, 1985).

Brahmagupta (b. 598): One of the celebrated astronomer-mathematicians of his time, Brahmagupta wrote two works, the first and more important being *Brahma Sputa Siddhanta* (Srinivasiengar, 1967). Again, this is an astronomy text with several chapters on mathematics. Brahmagupta had a thorough understanding of the decimal numeration and gave an advanced technique for multiplication. He gave general rules of operations with zero as well as for negative numbers. Brahmagupta was the first to attempt to divide by zero and although he was not successful he demonstrated a deep understanding of an extremely abstract concept (Pearce, 2002). He developed a clear algebraic terminology and one of the most outstanding contribution of Brahmagupta to mathematics is his solution to the indeterminate equation \( N x^2 + 1 = y^2 \), often mistakenly called Pell’s equation (Bag,
1979; Srinivasiengar, 1967). He was interested in the determination of rational right-angled triangles and rational cyclic quadrilaterals. As pointed out by Eves (1969) and Cajori (1919), Brahmagupta not only gave Heron’s formula for the area of a triangle but also its notable extension; the area of a cyclic quadrilateral having sides $a$, $b$, $c$, $d$ and semi perimeter $s$ as $\sqrt{(s-a)(s-b)(s-c)(s-d)}$. In Cajori’s view, a most remarkable result in Hindu geometry is Brahmagupta’s theorem for diagonals in a cyclic quadrilateral. Another contribution of his is a method of obtaining sines of intermediate angles from a given table of sines. According to Joseph (2000), Brahmagupta holds a special place in the history of mathematics and it was partly through a translation of his work that the Arab mathematicians and then the West, became aware of Indian astronomy and mathematics.

Mahavira (c. 850): Mahavira, a mathematician from South India, was a Jain by religion and in keeping with the tradition, he studied mathematics for its own sake and not in association with astronomy, as was the vogue then (Sen, 1971). His tribute to arithmetic is given in the opening quotation of this thesis. Mahavira was familiar with Jaina mathematics on infinity and infinitesimals and included it in his work (Plofker, 2009). The Ganita-sara-samgraha concerns operations on numbers including determination of square, square roots, cube and cube roots, arithmetic and geometric series, fractions, rule of three, mensuration and algebra, including quadratic and indeterminate equations. He uses the place value decimal system, mentions 24 notational places and also uses word-numerals, as was the established practice. There is a separate chapter on technical terms since according to Mahavira, “it is not possible to understand the meaning of anything without its name” (Plofker, 2009, p. 163). As pointed out by Sen (1971), he was the first among Indian mathematicians to have used the method of lowest common multiple, termed by him as niruddha in order to shorten the process. He was also one of the rare Indian mathematicians who studied the ellipse but he was unsuccessful in deriving its area.

Bhaskaracharya II (c. 1114 CE): Bhaskara was a distinguished mathematician astronomer from the school at Ujjain. He is well-known for his three works; Lilavati (literally means “beautiful”), Bijaganita (seed arithmetic) and Siddhanta Siromani (Crest-jewel of the Siddhantas). The lilavati which is thought to have been named after Bhaskaracharya’s daughter, shows a profound understanding of arithmetic (Joseph, 2000). In this work he gives properties of zero, particularly division by zero (a culmination of
hundreds of years of progress in mathematics), numerical work on integers and surds, fundamental operations, rules of three, five, seven, nine, eleven, and permutations and combinations. In the *Bijaganita*, which is a treatise on algebra, is contained, a comprehensive algebraic notation, generalisations involving zero and negative numbers, evaluation of surds, solutions of simple, quadratic and equations, and general methods of solution of indeterminate equations of the first and second degree. Bhaskara also gave methods of solution for the biquadratic (Ayyangar, 1938). On the topic of algebra, Bhaskara is most famous for his *chakravala* (or cyclic) method of solving indeterminate equations of the form \( ax^2 + bx + c = y \) where the same set of operations are repeated over and over again. This method is given in detail in Bag (1979, pp. 217-223). This cyclic method was rediscovered in the west by William Brouncker in 1657 (Joseph, 2011).

In the *Siddhanta Siromani*, the author presents the sine table and relationships involving different trigonometric functions. According to Joseph (2000, 2009), in this work one can trace certain preliminary concepts of the infinitesimal calculus and analysis, which were taken up by the Kerala mathematicians work on infinite series about two hundred years later. Bhaskara won such great acclaim that a temple inscription refers to him in glowing terms.

The twelfth century in India is generally believed to be the end of the Classical period and it is mostly thought that mathematics came to a stop after Bhaskaracharya. However, Joseph (2000, 2009) states that this view has had to be amended in the light of research (e.g. Rajagopal & Rangachari, 1978, 1986) on medieval Indian mathematics, mainly originating from Kerala. A key figure in this period is Madhava of Sangamagramma (c. 1340-1425), an astronomer-mathematician whose contributions include infinite-series expansions of trigonometric functions and finite-series expansions. In one of his recent books, Joseph (2009) explores the possibility of the transmission of medieval Indian mathematics from Kerala to Europe in the sixteenth and seventeenth centuries and their impact on European mathematics.

4.2.1 Reflection and Articulation on the Overview of Indian History

A brief outline of the history of Indian mathematics has been presented above, and as argued in Chapter 2, a study of history is rich in lessons applicable to the present. The study of history of mathematics is especially rewarding in this way (Fauvel & van Maanen,
2000), since it can direct our attention to the processes of evolution of mathematics and suggest to us ways of enriching students’ experience of mathematics, particularly numbers.

In relation to this, Flegg (1983) observes:

All history involves history of ideas...The history of mathematics is, amongst other things, a library of ideas and methods which have worked with varying degrees of success in the past. For this reason alone, it should be studied seriously by every practising and would-be teacher of mathematics, no matter how elementary the level of teaching to be undertaken. (Flegg, 1983, pp. 278-279)

What are the implications for teaching from the overview of Indian history of mathematics? a) First, it is seen that the naming of large numbers and their use in calculation was a common practice in Indian mathematics. Mathematics, including the place value system (with a decimal base) and the related arithmetic developed in a realistic context and was used for practical purposes. This indicates that teaching is likely to be more successful if numbers are related to everyday experiences and calculation exercises are set within the context of real-life problems, giving them some meaning. b) As is evident in the construction of altars from very early times, the basis for Indian mathematics was geometry. The beginnings of algebra are to be traced to the constructional geometry of the Vedic Sulbasutras (Sarasvati Amma, 1979). Down the ages, the Indian mathematicians have shown a preference for demonstrating arithmetic operations and algebraic truths geometrically. Hence, as suggested by Flegg (1983), students’ understanding of algebraic proofs could be considerably reinforced by the introduction of visual geometric demonstrations. In this connection, it is revealing that the early works mention 72 sciences and arts to be studied (Datta & Singh, 2001; D. E. Smith & Karpinski, 1911) and further mention lipi or lekha (alphabets, reading and writing), rupa (drawing and geometry), and ganana (arithmetic) as the main subjects of study (Datta & Singh, 2001). c) It is interesting to note optional forms of historical number words for some specific numbers highlighting relationships between numbers; for instance 19 was known as both 10 + 9 and also as 20 – 1. d) In this connection, a crucial factor is the development of flexible approaches to operations on numbers and to become aware of those that are efficient. For example, in a multiplication problem such as 37 \times 19, students could choose the more efficient strategy (Anghileri, 1999; Young-Loveridge, 2005) from 37(10 + 9) or 37(20 – 1). As can be seen from the nomenclature, the multiplication principle is inherent in the way numbers
were named, and this suggests the fundamental importance of multiplicative reasoning in understanding place value numeration, a fact highlighted by Bryant and Nunes (2009). In view of the significance of multiplicative thinking, and by implication the mastery of multiplication (highlighted in Section 3.2.3) basic facts, perhaps there is a role for drill exercises and rote-learning in mathematics and its teaching, especially in the case of multiplication tables. Once the idea of multiplication is demonstrated and understood for positive integers with the aid of manipulative materials, learning the tables by heart can facilitate the acquisition of basic facts and skills (Fuson, 2003). The Indian scholars appear to have been well aware of rote memory and versification (rhythm) from ancient times, even from the Vedic era. In this context, Dehaene (1997) and Devlin (2000) note the huge potential of verbal memory and present thought-provoking arguments on the brain’s strategy in recording arithmetic facts in verbal memory and the idea that tables are learnt using our ability to remember patterns of sound. That is, tasks that require a precise answer depend on our linguistic faculty. Related to this however, is the problem of circularity in mathematics learning that is indicated by Sfard (1991) and Arcavi (1994), and in this thesis the difficulty of circularity is highlighted by the teacher/researcher concerning place value and multiplicative thinking. This implies that the properties of the decimal place value numeration need to be linked to fundamental operations on numbers and place value concept be taught alongside operations on numbers. e) In this connection, it can be seen that all the Hindu astronomer-mathematicians began their section on mathematics with several methods of operations on numbers before solving problems in arithmetic; in the case of multiplication seven distinct methods of multiplication are given. This seems to indicate that students will benefit from a review of fundamental operations prior to problem solving in the classroom. f) What is also evident from history is the impact of the invention of the decimal place value system on advances in mathematics. Although its creation took thousands of years, once developed, some consequences within a few centuries were: zero considered as a number; ease of fundamental operations; arithmetic of integers; general methods of operations on numbers; generalisation of number properties; algebraic notation; general methods of solution of linear, quadratic, cubic and quartic equations; and the capstone of Indian mathematics, namely indeterminate analysis, culminating in finding general methods of solutions of ‘Diophantine’ equations of the second degree. The transmission of the Indian numerals to Europe via the Arab
mathematicians appears to have facilitated the rapid progress of modern mathematics as well as science and technology (Flegg, 1983; Ifrah, 1998). The implication for teaching and learning of mathematics is the importance of a solid understanding of the foundational concept of decimal place value numeration which can pave the way for greater progress in mathematics.

4.3 Construction of the Decimal Numeration System in India

This section attempts to answer the following research question.

*What were the main conceptual stages in the historical construction of the decimal place value numeral system in India, and how could an awareness of these stages be useful in teaching and learning?*

An overview of the history in India of the decimal place value system indicates that it was the result of the confluence of several aspects: a consideration of very large numbers and infinity; the emergence of writing; number symbolism; denominations known as ‘places’; and the development of the concept of zero. It should be noted that this happened over very many centuries, and the different segments in its development can be broadly divided into two phases: i) Consideration of large numbers and infinity; and ii) evolution of numerals including zero in a place value system. Within this second phase there were several stages of development. These phases are discussed below and each is followed by a section on reflection and articulation of the historical and pedagogical ideas related to students’ understanding of positional numeration and Indian history of mathematics.

4.3.1 Large Numbers and Infinity in Indian History

A study of Indian texts reveals a number of descriptions of large (and small) numbers that are outlined below, along with comments related to the nature and development of the numbers. While most numbers were probably expressed verbally, they are shown here in numerical notation.

1. A major milestone in the development of the Hindu-Arabic place value system is a (surprisingly very early) set of number names for powers of ten. In the *Vajasaneyi (Sukla Yajurveda) Samhita* (17.2) (c. 2000 BC) of the Vedas, the following list of arbitrary number names is given by Medhatithi in Sanskrit verse: *Eka* (1), *Dasa*
(10), Sata (10\(^2\)), Sahasra (10\(^3\)), Ayuta (10\(^4\)), Niyuta (10\(^5\)), Prayuta (10\(^6\)), Arbuda (10\(^7\)), Nyarbuda (10\(^8\)), Samudra (10\(^9\)), Madhya (10\(^10\)), Anta (10\(^11\)), Parardha (10\(^12\))

(e.g. Bag & Sarma, 2003; Datta & Singh, 2001). Each of these named denominations is 10 times the preceding one, so that they were later aptly called the *dasagunottara samjna* (decuple terms) confirming that there was a definite systematic mode of arrangement in the naming of numbers. The same list of names of powers of ten occurs in two places in the *Taittiriya Samhita* (4.40.11.4 and 7.2.20.1), and in the latter this list was extended to *usas* (10\(^{13}\)), *vyusti* (10\(^{14}\)), *udesyat* (10\(^{15}\)), *udyat* (10\(^{16}\)), *udita* (10\(^{17}\)), *suvarga* (10\(^{18}\)) and *loka* (10\(^{19}\)) (Gupta, 1987, 2001). As will be seen in later examples, sometimes a centesimal scale was occasionally used to codify large numbers in the Vedic literature. For example, in the *Taittiriya Upanishad* (2.8), a base-100 scheme is adopted to describe different orders of bliss; it is said that Brahmananda (the bliss of Brahman or Self-knowledge) is 100\(^{10}\) times a unit of human bliss (Dutta, 2006). In view of such high numbers, it is not entirely clear why such large numbers were named, however, Ifrah (1998) notes that these names for powers of ten were given in an environment which was at once philosophical, cosmological and metaphysical.

2. The same list up to *parardha* (one trillion) as in the *Vajanaseyi Samhita* is repeated in the *Pancavimsa Brahmana* (17.14.1.2) with further extensions and the following passage from it will give an idea of the context and the manner in which large numbers were introduced:

...By offering the *agnistoma* sacrifices, he becomes equal to one who performs a sacrifice of a thousand cows as sacrificial fee. By offering ten of these, he becomes equal to one who performs a sacrifice with ten thousand *daksinas* (fee). By offering ten of these, he becomes equal to one who sacrifices with a sacrifice of a hundred thousand *daksinas*... By offering ten of these, he becomes equal to one who sacrifices with a sacrifice of 100 000 million *daksinas*. By offering ten of these he becomes the cow [one trillion]. (Sen, 1971, p. 141)

3. In the Buddhist work *Lalitavistara* (c. 100 B.C.E), there are examples of series of number names based on the centesimal scale. For example, in a test, the mathematician Arjuna asks how the counting would go beyond *koti* (10\(^7\)) on the centesimal scale, and Bodhisattva (Gautama Buddha) replies: Hundred *kotis* are called *ayuta* (10\(^9\)), hundred *ayutas* is *niyuta* (10\(^11\)), hundred *niyutas* is *kankara*
(10^{13}),...and so on to *sarvajna* \(10^{60}\), *vibhutangama* \(10^{51}\), *tallaksana* \(10^{53}\). It is to be noted that there are 23 names from *ayuta* to *tallaksana*. Then follows 8 more such series starting with \(10^{53}\) and leading to the truly enormous number \(10^{53+8\times46} = 10^{421!}\) (Gupta, 1987; Ifrah, 1998; Menninger, 1969). Arjuna gave Buddha a further test: to name all the divisions or primary atoms of a *yoyana* (a mile). Buddha replied with a system of number names: 7 primary atoms make a very minute particle, 7 of these make a minute particle, 7 minute particles make... and so on and multiplying all the factors concluded that a mile contains \(4 \times 10^3 \times 8 \times 12 \times 7^{10} = 108\,470\,495\,616\,000\) atoms (Ifrah, 1998; Menninger, 1969). Similarly, (Buddha adds) one could also number all the atoms in all the real and mythical lands in this world and even in the 3000 thousand worlds contained in the universe. In Menninger’s (1969) view, this enumeration of particles (p. 138) is very likely the oldest example of computation with vast numbers. What is also seen is that the centesimal scale was used for very large numbers and interestingly, the (rare) use of base 7 for calculating number of atoms in a mile.

4. In the epic *Ramayana*, written by *Valmiki*, there is mention of the size of *Rama’s* army as being \(N = 10^{10} + 10^{14} + 10^{20} + 10^{24} + 10^{30} + 10^{34} + 10^{40} + 10^{44} + 10^{52} + 10^{57} + 10^{62} + 5\) (Joseph, 2000). Even though these numbers are fantastic, the fact that names for such numbers existed indicates that the Vedic Indians were quite at home with very large numbers. Interestingly in this example, the initial counting proceeds by the scale factor of 100 000 (one *lakh*).

Related to the enormous numbers detailed in this section, particularly the large numbers in examples 3 and 4, Menninger (1969) offers an explanation. The author comments that counting was the ancient Indians’ way of honouring sacred divinity, and numbers were “sanctified by their very magnitude” (p. 136). He states:

Their importance lies not in any clear conception of numbers (for who can visualise a number like \(10^{53}\)?) but in their underlying recognition that a tower can be built up, level upon level, and that the tower of numbers rises high above the world of mortals to the realm of the gods and the “Enlightened”...conceiving such numbers verbally does not mean visualising them; it means mastering the undefined, the innumerable, the obscure “many” by using the clear, verbally defined concept of the counted, of the number which falls within the scope of the short number sequence created by the rudiments of ordinary life. The significance of such an intellectual
attitude for the entire culture of a people, and the difference between the Hindu approaches to numbers and those of other peoples can be seen here very distinctly (Menninger, 1969, pp. 137-138)

5. Another example of variation of scale is an interesting series of number names increasing by the 10 million or $10^7$ scale that is found in Kaccayana’s Pali Grammar. The number names go up to koti (10 million) in multiples of 10 and then by multiples of ten millions. It says that: hundred-hundred thousand kotis ($10^7$) give pakoti ($10^{14}$), hundred-hundred thousand pakotis is kotipakoti ($10^{21}$), similarly we have nahuta ($10^{28}$),.......kathana ($10^{126}$), mahakathanas, asankhyeya ($10^{140}$) (Datta & Singh, 2001).

6. In the Vedic literature, time is reckoned in terms of yugas or time cycles. The four yugas are Krta-yuga, Treta yuga, Dwapara yuga and Kali yuga. According to Hindu cosmology, the time-span of these four yugas is said to be 1728000, 1296000, 864000, and 4320000 years, respectively, in the ratio 4:3:2:1. The total of these four yugas was considered as one yuga-cycle or Mahayuga and was thus 4320000 years (Rao, 1998; Srinivasiengar, 1967). Moreover, it is believed that 1000 such yuga-cycles comprise one day in the life of Brahma, which is 4,320,000,000 years is called kalpa. Thus one day and night period is 8.64 billion years. In Vedic cosmology Brahma is thought to live for a 100 years, which in human years is approximately equal to 311 trillion or $311 \times 10^{12}$ years (Ifrah, 1998; Kak, 2000, 2008; Wilson, 1972). As pointed out by Plofker (2009) time in the astronomical works is bound by cosmological concepts. In one kalpa which is 432000000 years, all celestial objects are considered to complete integer number of revolutions about the earth.

7. In the Anuyogadvara-sutra (c. 100 BCE), a Jaina text, the number of human beings in the world is given as $2^{96}$. This is another incredible number which is an example of the use of a base other than 10. It is also in this work that the first mention of the word ‘place’ is used for a denomination. Another number that occurs in the Jaina works is $756 \times 10^{11} \times (8 \ 400 \ 000)^{28}$ which is the number of days representing a period of time called SirsapraHELika (Datta & Singh, 2001; Joseph, 2000). One of the Jaina methods of forming large numbers rapidly was called vargita samvargita
(literal meaning: a quantity raised to the power of itself). For example from the number two we get

1st vargita samvargita = \(2^2 = 4\)

2nd vargita samvargita = \(4^4 = 256\)

3rd vargita samvargita = \(256^{256}!\) (Gupta, 1987)

8. Both Vishnu Purana and the Vayu Purana specify eighteen places or powers of ten. In addition, the Vayu Purana makes the generalisation; “These are eighteen places (sthana) of calculation; the sages say that in this way the number of places can be hundreds” (Datta & Singh, 2001, vol. 1, p. 84). In the Amalasidhi (unknown author) is given a list of what can be described as the world’s longest list of decuple terms; extended after \(10^{18}\) from a previous list, and starting from \(ksiti\) \((10^{19})\) to \(dasa-ananta\) \((10^{96})\), thus producing 97 names for powers of ten (Gupta, 2001, pp. 87-88).

9. The naming of decuple terms or powers of ten continued throughout the classical period and up to the 12th century as is seen in the works of Sridhara (c. 750 CE) who named 18 decuple terms, Mahavira (c. 850 CE) who listed 24 decuple terms and which eventually was taken to 36 names of powers of ten by Mallana who lived in the 12th century (Datta & Singh, 2001; Gupta, 2001)

10. In the Paitamaha-siddhanta, an astronomical treatise which forms part of a Purana text, this large number is given: “The orbit of heaven is 18 712 069 200 000 000 [yojanas]...The diameter of the orbit equals the circumference divided by the square root of ten” (Plofker, 2009, p. 69)

11. While there are more instances of large numbers, from the microscopic side some examples of very small numbers are also found in the early works. The Satapatha Brahmana (12.3.2.1) (c. 2000 BCE) gives a minute subdivision of time. According to this work, there are 30 muhurta in a day, 15 ksipra in a muhurta, 15 itarhi in a ksipra, 15 idani in an itarhi, and 15 prana in an idani. Thus one prana is approximately equivalent to one-seventeenth of a second. In the Lalitavistara, a system of very small linear measures are also given, the smallest being the diameter
of a molecule namely *paramanu*. One *paramanu* = $1.3 \times 10^{-10}$ inches (Datta & Singh, 2001).

According to Sen (1971), the naming of such large numbers in history implies a proficiency in developing a scientific vocabulary of number names, in which the principles of addition, subtraction and multiplication were conveniently used. There appears to be an early awareness of the effect of repeated multiplication by a quantity. The nomenclature developed by the ancient Indians for a decimal system consisted of:

i) The first nine digits, e.g. *eka*, *dvi*, *tri*, *catur*, *panca*, *sat*, *sapta*, *asta* and *nava*;

ii) The second set of nine numbers acquired by multiplying each of the above digits by 10, e.g. *dasa*, *vimsati*, *trimsat*, *catvarimsat*, *pancasat*, *sasti*, *saptati*, *asiti*, and *navati*;

iii) The third group of 11 numbers beginning with *sata* ($10^2$) and ending with *parardha* ($10^{12}$) (Sen, 1971, p. 142).

The multiplicative principle was already indicated in forming the numbers of the second and third group above. [Another example of a number is *sastim sahasrani* ($60000 = 60 \times 1000$). The additive principle is used in naming a number in which the numbers of the first and second group participate, for instance *asta-trimsat* (38). Additive and multiplicative principles are simultaneously used when, in the number concerned, members of the third group participate along with those of the second or first; example *sapta satani vimsatih* (720)].

Due to the Jaina interest in large numbers sometimes problems were generated that had a potential for calculation, such as: “*a raju* is the distance travelled by a god in six months if he covers a hundred thousand *yojana* (approximately a million kilometres) in each blink of his eye” and “*a palya* is the time it will take to empty a cubic vessel of side one *yojana* filled with the wool of newborn lambs if one strand is removed every century (Joseph, 2011, p. 350). It may be that these problems were created to form an idea of the size of quantities.

**The idea of infinity in Indian history**

While most of the numbers described above are often extremely large, it is interesting to note that there was also an early notion of infinity. The peace song in the *Isa* Upanishad
of the Vedas (Gupta, 2003) mentions \textit{purna} (also a word numeral for zero). The Upanishadic verse says:

\textit{Om purnam adah purnam idam purnat purnam udacyate}

\textit{Purnasya purnam adaya purnam evavasisyate.}

[Meaning]: ‘that is \textit{purna}, this is \textit{purna}; \textit{purna} has been evolved from \textit{purna}. Taking the \textit{purna} from the \textit{purna}, what remains is still \textit{purna}’.

Philosophically, \textit{purna} (‘perfect’) here may be taken to mean ‘infinite fullness’. Mathematically, the second half implies: \textit{purna} – \textit{purna} = \textit{purna}, or \(x - x = x\) which is always satisfied by \(x = 0\) but can also be satisfied when \(x\) is infinity in many limiting cases. (Gupta, 2003, pp. 20-21)

Like the word \textit{purna}, some of the other synonyms for infinity are \textit{ananta} (eternity), \textit{akasa} (space, ether or void), \textit{ambara} (atmosphere), \textit{kha} (sky or space), \textit{gagana} (canopy of heaven, firmament), \textit{sunya} (void, space, atmosphere) etc, and interestingly, these same Sanskrit words were used to denote zero as well (Ifrah, 1998). Hence there appears to be a close link between the concepts of infinity and zero (Bag & Sarma, 2003; Datta & Singh, 2001). This interest in large numbers and infinity was carried forward by the Jains (600 BCE) in India who classified all numbers into three groups, and which were further subdivided into three orders.

1. \textit{Enumerable} (samkhya): lowest, intermediate and highest;
2. \textit{Innumerable} (asamkhya): nearly innumerable, truly innumerable and innumerably innumerable;

(Bag, 1979; Joseph, 2000; Srinivasiengar, 1967)

In order to form an idea or a sense of quantity of an \textit{Enumerable} number, one is advised to count all the numbers from 2 (interestingly 1 was ignored) until the highest number was reached. For example, in the \textit{Anuyogadwara-sutra} of Jaina mathematics this example is given:

Consider a trough whose diameter is that of the Earth (100 000 yojanas) [1 yojana \(\approx 10\) kilometers]. Fill it up with white mustard seeds counting one after another. Similarly fill up with mustard seeds other troughs of the sizes of the various lands and seas. Still it is difficult to reach the highest enumerable number. (Srinivasiengar, 1967, p. 24)
Once this number, which can be called N is attained, infinity is reached through the following sequence of operations:

\[ N + 1, N + 2, \ldots, (N + 1)^2 - 1 \]

\[ (N + 1)^2, (N + 2)^2, \ldots, (N + 1)^4 - 1 \]

\[ (N + 1)^4, (N + 2)^4, \ldots, (N + 1)^8 - 1 \]

\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \] (Joseph, 2000, p. 251)

In the third group, Joseph (ibid) describes how Jaina mathematics recognized five different kinds of infinity: *infinity in one direction; infinity in two directions; infinite in area; infinite everywhere; and infinite perpetually*. Thus the idea of infinity was combined with that of dimension defining infinity in one, two, three and infinite dimensions (Srinivasiengar, 1967). Joseph further mentions that the Jains were the first to discard the idea that all infinities are the same, an idea held in Europe until Georg Cantor’s work in the late 19th century delineated countable infinities, such as \( \aleph_0 \) (aleph-zero, which is also called the first transfinite number) from the uncountable. According to Bag (1979) the highest enumerable number in the first group above corresponds to aleph-zero of modern mathematics.

### 4.3.2 Reflection and Articulation on Large Numbers and Infinity in Indian History

What can we learn for today’s classroom from the above examples?

**General aspects**

As can be seen above, enormous (and very small) numbers were dealt with from a very early time period in India and it is important to note the extent to which these numbers were taken. According to Sarma (2009), “such large series did not exist anywhere outside India” (p. 207). Additionally, the above examples indicate a variety of contexts used in the naming and computation of large numbers. An interesting aspect that could be put to use pedagogically is that examples 6, 7 and 10 given above all involve time measures, which are more abstract than distance calculations and the counting of physical objects. Hence it could be beneficial for students if problems involving time followed distance problems in the teaching design. What is also intriguing is that these number names for *powers of ten*
were given in the earliest Vedic literary works. The very early number names for powers of

ten affirms Dehaene’s (1997) thesis that doing mathematics is language-based. Sanskrit
language played a major role in the growth of mathematics in India (Staal, 1995) in
agreement with Barton’s (2008) comment: “It is in the communication of the experience of
*quantity* that the concept of *number* is formed, and this occurs with the development of
language” (p. 70). This implies that quantity sense is developed prior to number sense and
the ancient Indians appear to have had a sense of magnitude of the quantities that they
named. From the perspective of the number names of powers of ten, the naming and
handling of such high numbers in Sanskrit, according to Ifrah (1998), Staal (1995) and
Filliozat (2004), had much to do with the close link between Sanskrit grammar and number
names.

In relation to numbers as powers, some of the numbers developed in India remind one
of Archimedes and his *Sand-Reckoner* (Eves, 1969; Menninger, 1969). In it he describes a
method for the representation of large numbers to the finding of an upper limit to the
number of grains of sand ($10^{63}$) which would fill a sphere with earth as the centre and
radius reaching out to the sun. However, Archimedes was one of the very few Greek
mathematicians to count the seemingly ‘uncountable’ (Menninger, 1969) whereas, in
ancient and medieval India there was a consistent and systematic consideration of large
numbers by most mathematicians for many centuries. As suggested by some historians and
researchers (e.g. Ifrah, 1998; Joseph, 2000; Joseph, 2009; Menninger, 1969) the handling
of such large numbers for a long period of time (at first in the oral tradition), probably
spurred the development of the current decimal number system in India, although several
stages involving symbolization, concept of place value and zero were crossed before the
final construction and representation of the numeration system (Datta & Singh, 2001).
Historians explain how this might have happened. Joseph (2011) suggests that adoption of
a series of arbitrary Sanskrit names for successive powers of ten led to a word numeral or
named place-value multiplicative system (similar to the Chinese system) with the
associated nomenclature. For example, 8 *mahapadma* 6 *padma* 7 *vyarbuda* 8 *koti* 9
*prayuta* 3 *laksa* 2 *ayuta* 5 *sahasram* 1 *satam* 7 *dasa* 8 means $8 \times 10^{10} + 6 \times 10^{9} + 7 \times 10^{8} +$
\[ \cdots \cdots \cdots \cdot +7 \times 10^{1} + 8 \] (Menninger, 1969, p. 142). Naming the number in this way where
each multiunit was specified, possibly aided the development of the place value system.
Also perhaps due to the dearth of writing materials and the length of time taken to write out
these large numbers in full with the ranks/multiunits, and also given their propensity for conciseness, the ancient Indians seem to have progressed to *suppressing* the names of powers of ten (Ifrah, 1998). The powers of ten were assigned *positions or places* resulting in a decimal place value system (the above number would then become 86 789 325 178). As Joseph (2000, 2011), Ifrah (1998) and Ayyangar (1928) point out, this process was made easier since the place values were assigned according to the order of ascending powers of ten. This confirms the importance of having a *name* for each and every numerical *rank* or power of ten, and according to Pandit (1993), and as stated by Ifrah (1998), “...the Sanskrit numeral system contained the very key to the discovery of the place-value system” (p. 429). Similar to the situation in India, large numbers were also considered in other cultures such as the Mayan in Central and South America (Joseph, 2002), who like the Indians were interested in astronomy, accurate measurement of time and development of calendars.

**Naming of large numbers and working with large numbers in context**

Large numbers clearly held a special fascination for the ancient Indians. Still, such numbers were mostly employed in a practical/realistic context such as astronomy (Bag, 1979; Plofker, 2009) and time measures (for calendar purposes) in Indian history, according to the needs related to social and cultural practices of the time periods. This suggests that students, many of whom are also fascinated by large numbers (M. Jones et al., 2007), may also be more motivated if a meaningful, realistic context is used. In today’s world of space travel and research, where distances are in millions of light years, computer technology involving memory in gigabytes (1 gigabyte = 1 billion bytes), and huge government budgets running into the trillions (Ronau, 1988), large numbers are relatively common. Students may come across such numbers in everyday life and in their secondary schooling. For example, they see Avogadro’s number in chemistry (6.022×10²³) and come across the use of exponential functions, with large values, to model many scientific and social phenomena. In some economic situations, due to hyperinflation, money transactions may involve trillions or quadrillions of a specific currency, although this is a rare event. Not only large numbers, technological advancements now allow us to explore phenomena within the smallest (nanoscale, where nano denotes 1 billionth) realms of science (M. Jones et al., 2007). Other examples that might be employed are disease prediction, radioactive
waste and national debt, all of which involve very large (or very small) numbers and exponential growth/decay. Hence there is a need to understand large numbers and students could benefit from *naming, reading, writing and computing* with them (Fuson, 1990a; Hewitt, 1998; Labinowicz, 1985) in different contexts. Understanding the *naming convention* is a prerequisite for a grasp of large numbers and their structure, and may help students to form a link between the place value chart (or abacus) and the place value system. Students may gain an understanding of the current nomenclature with its auxiliary bases (with digits in clusters/periods of 3) from reading and writing numbers involving million \((10^6)\), billion \((10^9)\), trillion \((10^{12})\) and so on to decillion \((10^{33})\). Numbers such as thousand, million and trillion play the role of auxiliary bases since we say ten thousand, hundred thousand, hundred million, ten billion etc (Ifrah, 1998). Some examples of numbers that could be considered in the classroom are 1 quadrillion; 7 quintillion; 4 trillion and 855 million and 36. These numbers could be written in words, as a combination of words and digits, and in full.

**Quantity sense, grouping and representation involving powers**

However, apart from the nomenclature, it is also important to develop a *sense* for the *size* of the numbers and, the structure of the powers. Both *quantity sense* (defined below) and understanding of powers could, among other ways, be achieved through grouping experiences (including higher order grouping) as in repeated multiplication. The above historical instances taken from Indian history show that, in naming large numbers, the multiplicative and exponential structures were already present from early Vedic times. Pedagogically, this implies the value of an early and explicit teaching of powers in order to create a *foundational base* for a deep understanding of the decimal place value system. This suggestion of an early teaching of powers of ten as signpost numbers is also made by Reddy and Srinath (2001). They advocate experiences in naming numbers such as 1000, 10000, 3000, 5000, 200, 400 that highlight the multiplicative structure of numeration. The second historical example from *Pancavimsa Brahmana* indicates repeated multiplication as a strategy, both to understand concept of powers and to develop quantity sense. In this context, another teaching strategy would be to assist students to produce examples of large numbers with a *potential* for calculation which sometimes occurred in Indian history, such as the examples given in Jaina mathematics. Such problems would involve many skills such as conversion, estimation, calculations and rounding, and working across topics such
as measurement. In relation to *quantity sense*, Wagner and Davis (2010) define it as a *feel* for amounts and magnitudes. However, in their view, quantity sense appears to be separated from number sense in school mathematics. According to the authors, quantity sense needs to be ingrained in number sense with connections to personal experiences. The importance of educating quantity sense is not insignificant since, as Dehaene (1997) observes, humans have only a limited capacity to imagine quantity. However, Wagner and Davis state that such a sense can be developed.

...our experience - is that it is indeed possible, but efforts must be anchored in individual students making comparisons that relate closely to their experiences. Simply giving students an image to associate with a large quantity is not nearly as effective as prompting them to draw on their experiences to develop their own images (D. Wagner & Davis, 2010, p. 45).

Hence, students could benefit from experiential activities related to large amounts and compare the relative sizes for a feel or *sense* of large quantities. As Wagner and Davis (*ibid*) explain, students have active roles in linking experiences of quantity with their *computation* skills, in contrast to being only provided with visual images. With very large numbers, students do calculations first and seek to connect to their experiences, drawing on past events. In this context, Ronau (1988) suggests that it could be useful if students are helped to create a *conceptual referent* for a number such as 10000, since once this is established, the conceptual referent is used as a basis to discuss and appreciate larger amounts. Wagner and Davis give the example of estimating and calculating the number of grains of rice in a bowl and then using this as a basis for further computations. Another example is to calculate the height of a million dollars in 1000 dollar bills and to use this to calculate the height for a billion and a trillion dollars.

In terms of learning numeration, students arrive at secondary school having already used the base-ten system, and yet, they often have little understanding of its structure. A pedagogical aim is to assist them to develop a deeper appreciation of the decimal place value system and its meaning by reinforcing and building on their prior knowledge. This could be approached by initially providing grouping experiences of the first order and leading to multiplication (Anghileri, 1999; Greer, 1992; Kamii & Joseph, 2004) followed by groupings of the higher order (groupings of groupings and leading to repeated multiplication) accompanied by a consideration of their outcomes (Fuson, 1990a). In addition, students could be assisted to move from the collection organised as groupings to
the representation (Bednarz & Janvier, 1982) of the number in various forms, especially in exponential form. Examples: $2+2+2+2+2$ as $2 \times 5$ (or $5 \times 2$) and $2 \times 2 \times 2 \times 2 \times 2$ as $2^5$, and $10+10+10$ as $3 \times 10$ and $10 \times 10 \times 10$ as $10^3$. This aligns well with Dienes’s variability principle and Vygotsky’s recommendation in relation to learning other bases for mastering place value structure. As reviewed in the literature in Chapter 3, students’ attention could be guided to some key aspects of higher order grouping although not in so many words: a) reinitialising to one (Confrey, 1994) which is forming a unit of units at each layer or step in the process; b) a second important feature is the counting of splits (the exponent number) rather than computing the number which was advocated by Confrey and Smith (1994). Thus explicitly focusing on the meaning and notation of powers, both in base ten and other bases via multiple representations (M. O. J. Thomas & Hong, 2001) may help establish a conceptual base for a more thorough knowledge of the place value system and also has the foundational applicability to decimal numbers, general positional notation and also later, polynomials.

The historical record shows that large numbers were often embedded in problems, either real or carefully constructed recreational puzzles or games. Thus students today may benefit from activities in games contexts. As suggested by Reddy and Srinath (Personal Communication, 2004), games involving the construction of large numbers (even for younger primary school students) may assist students to abstract the structure of numeration system and to understand the role of powers of ten as landmark or signpost numbers.

4.3.3 Development of Written Numerals, Place Value and Zero in Indian History

The consistent naming of large numbers and consideration of infinity described above eventually led to the development of decimal numeration with place value and zero in India. In this context, Laplace (1814), acknowledging the Indian origin of the numerals and their importance, states:

It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value; a profound and important idea which appears so simple to us now that we ignore its true merit. But its very simplicity and the great ease which it has lent to all computations put our arithmetic in the first rank of useful inventions; and we shall appreciate the grandeur of this achievement the more when we remember
that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity. (Laplace (1814), cited in Dantzig, 1967, p. 19)

Consequent to the naming of large numbers, a detailed examination of the historical development of the written decimal place value numeration system in India suggests the following broad classification of the stages that occurred:

- Verbal/Initial/Additive stage in number notation;
- Interim stage (involving mainly multiplicative) in number notation;
- ‘Places’ stage (denominations/powers of ten known as places); concept of zero;
- Final abstract stage with full place value and zero.

Each of these stages is considered below. (See Figure 4.8 for a summary diagram of the above stages).

**Initial (Verbal/Additive) stage in number notation**

During this stage, numbers (both small and large) were written down in words without the principle of place value, in the same way as they were spoken, for example:

i. *sahasrani sata dasa* (= one thousand + one hundred + one ten, i.e. 1110) Rig Veda (2.1.8) (Dutta, 2006)

ii. *sasthim sahasrani panca satani navatim nava* (= sixty thousand + five hundred + nine ten + nine, i.e. 60599) (Bag, 2003, p. 160).

iii. *dvi-navaka* (= twice nine, 2×9, i.e. 18).

iv. *trini satani trisahasrani trimsa ca nava* (= three thousands + three hundreds + three tens + nine, i.e., 3339) (Datta & Singh, 2001, vol. 1, p. 15).

v. *ekonna vimsati* (= one less than twenty, 20 −1, i.e. 19) (Sen, 1971, p. 142).

**Interim (Multiplicative) stage in number notation**

In the early stages of numerical symbolism, according to Datta and Singh (2001), while numbers were written out in full in words, symbols were also used for the smaller units and words for the larger units. In this intermediate stage, before the establishment of the place value principle, numbers were written in numerical symbols with the application of the
additive, multiplicative, and a combination of the additive and multiplicative principles. The following are some examples.

a) The examples in Figure 4.2 are some numbers in Kharoshthi numerals inscribed in the stone pillars of Asoka (c. 300 BCE). In Kharoshthi script the numerals are written from right to left.

![Figure 4.2. Kharoshthi Numerals (Datta & Singh, 2001, vol. 1, p. 105).](image)

As can be seen the additive principle alone is applied for numerals 40, 50, 60, 70, and 80. Reading 50 from the right we have 20+20+10, and for 60 we have 20+20+20.

b) Both multiplicative and additive principles have been applied for the numbers 100, 200, 300, 122 and 274 in Kharoshthi numerals. Reading the numeral 274 from the right we have 2×100+20+20+20+10+4.

c) Figure 4.3 shows numbers in Brahmi numerals, the ancestors of the Hindu Arabic numerals (Ayyangar, 1928), found in a cave in the Nanaghat hills near Poona in India. It is to be noted that the Brahmi numerals are written from the left to right. The multiplicative principle is applied here. For example, for 6000 the symbols for 6 and 1000 are conjoined. Therefore 6000 = 1000×6 and not 1000+1000+1000 …six times.
d) Another example of the multiplicative system in writing numbers is the composite number 24400 (see Figure 4.4) written in Brahmi numerals. The number given is both multiplicative and cipherised similar to the early Chinese and Greek systems respectively.

Reading the numeral 24400 from left to right we have, 20000+4000+400 where the numerals for 20000, 4000 and 400 are cipherised. However, the numerals are also multiplicative, in view of the fact that the symbols are conjoined, as mentioned before and seen in Figure 4.4. Hence $24400 = 2 \times 10000 + 4 \times 1000 + 4 \times 100 = 20000 + 4000 + 400$. 
‘Places’ stage (powers of ten assigned ‘places’)

An important stage in the structure development was that the denominations of *eka* (1), *dasa* (10), *sata* (100), *sahasra* (1000), etc. were known as ‘places’ (the Sanskrit word for ‘place’ is *sthana*). In India the decimal place value system developed when numbers in successive powers of ten were associated with the values of the ‘places’ or ‘positions’ of the numbers arranged from left to right, or right to left. Consequently, the names of these ‘places’ or powers of ten were concealed in the numerical expressions resulting only in the units/digits forming the coefficients of powers.

The first use of the world ‘place’ for the denomination is encountered in the Jaina canonical work *Anuyogadvara-sutra* (c. 100 BCE). In this work, the total number of human beings in the world is given thus: “a number which when expressed in terms of the denominations, *koti*-*koti*, etc, occupies 29 ‘places’ (*sthana*) or it is beyond the 24th ‘place’ and within the 32nd ‘place’, or it is a number obtained by multiplying the sixth square (of two) by its fifth square, (i.e. $2^{96}$) or it is a number divided by two ninety-six times”. According to Hema Candra (b. 1089 CE) the number *Sirsaprahelika* is so large as to

![Figure 4.4. Composite numbers in Brahmi numerals. (Datta & Singh, 2001, vol. 1, p. 117).](image-url)
occupy 194 notational ‘places’ or \(anka-sthanehi\) (Datta & Singh, 2001). Aryabhatta I (499 CE) states that the denominations are names of ‘places’. He says: “Eka (unit), dasa (ten), sata (hundred), sahasra (thousand), ayuta (ten thousand), niyuta (hundred thousand), prayuta (million), koti (ten million), arbuda (hundred million), and vrnda (thousand million) are respectively from ‘place’ to ‘place ‘each ten times the preceding’ (decuple terms). In most of the mathematical works, the denominations are called ‘names of places’ and eighteen are generally enumerated. Sridhara (750) gives names of eighteen places and adds that the decuple names proceed even beyond this.

In relation to the above, some interesting analogies have been employed in Indian history that hint at the concept of place value. For example, a commentary, probably of 5\(^{th}\) century on the ancient Yoga-sutra of Patanjali, states that: “Just as a line in the hundreds place [means] a hundred, in the tens place ten, and in the ones place, so one and the same woman is called mother, daughter, and sister [by different people]” (Plofker, 2009, p. 46). A similar analogy, also given in Plofker (2009), is used by the Buddhist philosopher Vasumitra involving merchants’ counting pits, where clay markers were used to keep track of quantities in transactions. Vasumitra says when the same clay counting piece is in the units place it is denoted as one and when in the hundreds place, it is denoted as hundred. The implication, as Plofker points out, is that there was the expectation for the audience to be familiar with the concept of numerical symbols representing different powers of ten according to their relative positions.

With the advent of denominations as ‘places’, “the names indicating the base and its powers were suppressed in the body of the numerical expressions ... and only retained the names of the units forming the coefficients of the diverse consecutive powers...” (Ifrah, 1998, p. 429) and according to Bag (1979) numerals took values \(\times 1, \times 10, \times 100, \ldots\) when placed in the first, second, third places... respectively. In this connection, Smith and Karpinski demonstrate with an example as to why the notation adopted by the Indians tended to bring out the place value idea. They consider the large number 8 443 682 155 and how the Hindus wrote and read it and then by contrast, as the Arabs and Greeks would have read it.

*Modern reading*: 8 billion 443 million 682 thousand 155
Hindu: 8 padmas 4 vyarbudas 4 kotis 3 prayutas 6 laksas 8 ayutas 2 sahasra 1 sata 5 dasa 5.

[Meaning: \(8 \times 10^8 + 4 \times 10^7 + 3 \times 10^6 + 6 \times 10^5 + 8 \times 10^4 + 2 \times 10^3 + 1 \times 10^2 + 5 \times 10^1 + 5 \times 10^0\)]

Arabic and early German: eight thousand thousand thousand and four hundred thousand thousand and forty-three thousand thousand, and six hundred thousand and eighty-two thousand and one hundred fifty-five (or five fifty).

Greek: eighty-four myriads of myriads and four thousand three hundred sixty-eight myriads and two thousand and one hundred fifty-five. (D. E. Smith & Karpinski, 1911, pp. 41-42). According to Smith and Karpinski, the introduction of the myriads in the Greek system and thousands as in Arabic and in modern numeration is a step away from the decimal system.

Hence when working with large numbers, given the Indian tendency for conciseness of expression, consideration of the powers of ten as ‘places’ and their suppression seems to have been a natural step. Numbers and word numerals were soon written with the place value principle, as in the following example.


![Figure 4.5. Place value numerals with zero (Menninger, 1969, p. 397).](image-url)
The concept of zero in Indian history

With the suppression of powers of ten, the need arose to record absence of certain powers and their units. In this context, Sarma (2003) and Joseph (2009) call attention to the fact that, while a place-value system can and did exist without a zero symbol, zero never existed without place value. Given students’ difficulties with zero (e.g. Anthony & Walshaw, 2004; Brown, 1981b; Ginsburg, 1989), it might be now useful to consider the origins and use of zero.

The different names for zero in its journey from one culture to another are one of interest. As explained by many authors (e.g. Menninger, 1969; D. E. Smith & Karpinski, 1911) the Indian Sanskrit name sunya meaning ‘void’ or ‘empty’ was passed over into Arabic as as-sifr or sifr. When the Leonardo of Pisa (1202) wrote about the Hindu numerals he spoke of it as zephirum and the next step is the shortened version zero. Along the way, it was also called chiffre in French, ziffer in German and as cipher in English. In the context of its creation, Ifrah (1985, 1998) states that the significance of the invention of zero is appreciated when it is realised that it was made only three times in history: by Babylonian scholars, by Mayan astronomer-priests and by the Indian mathematicians and astronomers. However, Joseph (2002) remarks that the Egyptian hieroglyph nfr, dating back about 4000 years is a zero signifying a direction separator. The zero appeared late in Mesopotamia; initially for about fifteen centuries, when the Babylonian scholars left an empty space for zero. This signifying of absence by absence (empty space), as Rotman (1987) indicates, resulted in ambiguity since the empty space of zero merged with the space used to separate words or numbers from each other. Also, the author points out that the empty space is not ‘transportable’ (Rotman, 1987, p. 59) meaning that it cannot be operated on, and this could possibly be the reason it was not conceived of as a number in Babylonia (Ifrah, 1985). Interestingly, when the zero was introduced in Babylonia, mathematicians used it in the medial position only, while astronomers used it in the medial and final positions. In the case of the Maya, Joseph (2002) considers their system of numerical notation as an economical system using only three symbols including a special sign for zero. The zero was used both in the medial and final positions. However, due to

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6 According to Menninger (1969), although the Chinese mathematicians conceived the place-value principle, the zero symbol was not introduced until the 13th century, and, possibly from India. However the absence of zero did not prevent it from being integrated into a computational system in China.
the irregularity in their place value numeration (see Section 4.5.1), the Mayan zero could not be operated on (Ifrah, 1985). By contrast in India, the concept of sunya or zero gave rise to a fully developed decimal place value system. It was invented sometime in the pre-
Common Era (Datta & Singh, 2001) and it is considered to be one of the more important
Indian contributions in mathematics (Ifrah, 1998; Joseph, 2002). According to Joseph
(2002) and Gupta (2003), the zero is a multi-faceted object; it is a symbol, a place-holder, a
number, a magnitude, a reference point and a direction separator. The following remarks
relate to its significance to the numeration system and its impact on the progress of
mathematics:

The importance of the creation of the zero mark can never be exaggerated. This giving to airy
nothing, not merely a local habitation and name, a picture, a symbol but helpful power, is
characteristic of the Hindu race whence it sprang. It is like coining the Nirvana into dynamos. No
single mathematical creation has been more potent for the general on-go of intelligence and power.
(Halstead as quoted in Srinivasiengar, 1967)

In conjunction with the place value principle, discovery of the zero marks the decisive stage in a
process of development without which we cannot imagine the progress of modern mathematics,
science and technology. The zero freed human intelligence from the counting board that held it
prisoner for thousands of years, eliminated all ambiguity in the written expression of numbers,
revolutionised the art of reckoning, and made it accessible to everyone. (Ifrah, 1985, p. 433)

In India, sunya or zero as a concept(s) probably pre-dated zero as a number by
hundreds of years (Joseph, 2002). As indicated by Ifrah (1998), the word sunya was not
invented specifically for the place value system and the word and its synonyms (with
different nuances) possibly existed well before the development of the positional system. It
has a very long history and had varied meanings in the different dimensions of philosophy,
language, mathematics and science. Zero in the philosophical, linguistic and social
contexts appears to have paved the way for the development of the corresponding
mathematical concept (Bag & Sarma, 2003). Some of the various perspectives in Indian
history are considered below.

1. The word sunya, according to Joseph (2002), is possibly a combination of shuna
(past particle of the root word svi, ‘to grow’) and its meaning of “the sense of lack
or deficiency” as occurring in the Rigveda. Joseph explains that it is possible that
the two different words were fused to give sunya as a single sense of absence or
emptiness with the potential for growth. Hence as elaborated by Joseph, its
derivative *sunyata* characterised the Buddhist theory of Emptiness, “being the spiritual practice of emptying the mind of all impressions...a course of action prescribed in a wide range of creative endeavours” (p. 112) such as poetry, music, painting and architecture.

2. In a literary and philosophical context, early uses of the word *sunya* were related to its dictionary meanings such as void, empty and vacant. Gupta gives the example of an ancient Indian saying: *Avidyam jivanam sunyam* which means “Without learning life is void (or zero)” (Gupta, 1995a, p. 45). However, as pointed out by Mukhopadhyay (2009), *sunya* not only conveys the sense of void or nothingness but also that of perfect fullness or *purna* (mentioned earlier under large numbers and infinity), and an infinitude, that is *ananta*, through its connection with the idea of expanse of space; this association is found in a wide variety of word numerals or synonyms such as *kha*, *gagana*, *akasa*, *jaladharapatha*, *ambara*, *vyoma*, *bindu*, *antariksa*, *vishnupada* (Bag & Sarma, 2003, pp. 59-69) etc. Thus zero and infinity shared many common synonyms and the concept of *sunya* in Indian antiquity was “a dichotomy as well as a simultaneity between nothing and everything, the ‘zero’ of the void and that of an all pervading ‘fathomless’ infinite” (Mukhopadhyay, 2009). On the other hand, interestingly, the word *ksudra* (meaning very small or trifling) which is found in the *Atharvaveda* (XIX, 22-23) is thought to refer to zero, along with *anrca* as negative number (Gupta, 1995a).

3. Many scholars (e.g. Bag & Sarma, 2003) have written about the possible contribution of philosophical ideas in the development of the concept and symbol for zero. These include the Vedic philosophy of void and *maya*, Buddhist doctrine of *sunyavada* or zeroism as formulated by Nagasena (c. 100 BCE) and Nagarjuna (c. 200 CE), and the notion of *abhava* or ‘absence’ of the Indian *Nyaya* (logic) system Gupta (1995a). More specifically, as mentioned by Gupta (ibid), *sunyavada* is related to the absence of any single principle or entity as the root-cause of worldly phenomenon, and in Pandit’s (1993) view, is comparable to the notion of zero as a number, such as in the example $x - x = 0$.

4. Similarly, since written mathematical treatises of Indian antiquity are not extant, historians have looked for mathematical clues in non-mathematical texts namely
the Vedangas of Sanskrit grammar and prosody. In this connection, it has sometimes been suggested that the Sanskrit grammatical system of Panini (c. 500 BCE) stimulated the concept of zero in a place value numeration system (Pandit, 1993, 2003). As specified by Pandit (ibid), Panini, in his treatise *Astadhyayi*, gave a comprehensive and scientific theory of phonetics, phonology and morphology. He presented a system of formal production rules and definitions to describe the whole of Sanskrit grammar, thus systematising it (Joseph, 2000). According to Pandit (ibid), Panini used many techniques in his grammar including *lopa* or zero for descriptive purposes; his sutra *adarsanam lopah* means non-appearance of a sound or morpheme. In an interesting comparison involving positional analysis, operation of subtraction and the process of going from maximum to minimum, Pandit (1993, 2003) suggests that *lopa* or elision constituting the grammatical zero directly parallels the concept of mathematical zero.

5. A more direct evidence of the use of zero is found in Pingala’s (c. 200 BCE) *Chanda-sutra*, a work on prosody or metrics. According to Sarma (2003), this work “constitutes an important landmark in the history of the decimal place value system for it mentions, for the first time, zero (*sunya*) and its symbol” (p. 126). In this work, as indicated by Datta and Singh (2001), Pingala gives the solution of the problem of finding the total number of arrangements of possible metres of two kinds of syllables ‘long’ and ‘short’ (denoted by *l* and *g*) in *n* places, repetitions being allowed. The method is enunciated in four pithy sutras: “Place two when halved, when unity is subtracted then place zero, multiply by two when zero and, square when halved” (p. 76). This procedure is illustrated in the following example for the *Gayatri* metre which contains six syllables.

The calculation begins from the last number in column B. Taking 1 as the first operand, double it at 0. This gives 2; at 2 square this 2 and the result is $2^2$; then at 0 double $2^2$, the result is $2^3$ and finally at 2 square this $2^3$ and the result is $2^6$ (=64) which gives the total number of ways in which two things can be arranged in 6 places. As observed by Plofker (2009), this method would be useful, especially when *n* is large. A main aspect which Datta and Singh (2001) point out is the zero symbol is employed as a marker to record the absence of halving or subtraction by 1; a common practice at that time in Indian mathematics.
Table 4.1. Pingala’s (c. 200 BCE) Rule for Metrics using Symbols for One and Zero

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Write the number of syllables</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td>Halve 6</td>
<td>3</td>
<td>Place 2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Square: $(2\times2^2)^2$</td>
</tr>
<tr>
<td>3.</td>
<td>3 cannot be halved; therefore reduce it by 1</td>
<td>2</td>
<td>Place 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Double: $2\times2^2 = 2^3$</td>
</tr>
<tr>
<td>4.</td>
<td>Halve 2</td>
<td>1</td>
<td>Place 2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Square: $2^2$</td>
</tr>
<tr>
<td>5.</td>
<td>1 cannot be halved; therefore reduce it by 1</td>
<td>0</td>
<td>Place 0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Double 1 = 2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Stop</td>
</tr>
</tbody>
</table>

6. Zero was also employed in the system of word numerals (word numerals were mentioned earlier) in the astronomy texts. Bag (1979) gives this example from the Pulisaśiddanta (200 CE) which includes zero:

\[kha-kha-ru\text{-}pa-asta\]

0  0  1  8 which is the number 8100 by reading from right to left.

7. The Jaina cosmological text *Lokavibhaga* (458 CE) is thought to be the oldest known Indian text to use zero and the place-value system with word-symbols. The number 13,107,200,000 is indicated (in the reverse order) as:

*five voids, then two and seven sky one and three and form* (Ifrah, 1985).

8. Aryabhata (499 CE) created a novel method (alphabetic system) which required a sound understanding of zero and the place-value system. Rules in his *Aryabhatiya* for finding square roots and cube roots use the place-value system. The description of his method for finding the square root is: “Always divide the even place by twice the square-root (up to the preceding odd place); after having subtracted from the odd place the square (of the quotient), the quotient put down at the next place (in the line of the root) gives the root” (Datta & Singh, 2001, vol. 1, p. 170).
9. Zero was also used as a symbol to denote the unknown or the absence of a term in 'rule of three' problems.

10. In terms of the form of the symbol for zero, Datta and Singh (2001) suggest that the earliest symbol was a solid dot and not a small circle, and they give some examples. The Bakshali Manuscript (7th century, possibly earlier), a mathematical work, explicitly uses the decimal place-value system and employs a dot for zero. Datta and Singh further state that the term bindu (dot) has been used for zero in word numerals, as well as in later literature when a small circle was in use to denote zero. However, Menninger (1969) suggests that the small circle could have resulted from the Brahmi numeral for ten or, as abbreviation of the Greek word ouden meaning 'nothing'. In this context, Pingree (2003) is of the view that when the Greeks invaded India at the beginning of the Common Era, they brought with them astronomical tables in which, in the sexagesimal fractions, the empty places were occupied by a circular symbol and this symbol was adopted by the Indian mathematicians instead of writing sunya. However, Ifrah (1998) argues a system that was only used to record fractions, moreover with base 60, could not have influenced a decimal place value system originally developed for recording whole numbers. Further, Joseph (2011) states that the link between the concept of zero and its symbol was already established in the early centuries of the Common Era and gives this quotation by the writer Subandhu “The stars shone forth, like zero dots [sunya-bindu] scattered in the sky as if on a blue rug...(Vasavadatta, c. AD 400)” (p. 346). And Mukhopadhyay (2009) comments that the symbol for zero was a solid circular dot to begin with, and then later only the peripheral circle was drawn, perhaps to save time required for darkening its interior.

11. Bhaskara I uses zero to denote places. In his commentary on the Aryabhatiya, he states: “Writing down the places we have, 0 0 0 0 0 0 0 0” (Datta & Singh, 2001, vol. 1, p. 82). In this connection, the authors state that in all works on arithmetic (patiganita), zero has been used to denote the unknown. Datta and Singh write:

This use of zero can be traced back to the third century A. D. It is used for the unknown in the Bakshali arithmetic. In algebra, however, letters or syllables have been always used for the unknown. It seems that zero for the unknown was employed in arithmetic, really to denote the
absence of a quantity, and was not a symbol in the same sense as the algebraic $x\ (ya)$, for it does not appear in subsequent steps as the algebraic symbols do. This use of zero is mostly found in problems on proportion- the Rule of five, Rule of seven, etc. (Datta & Singh, 2001, vol. 1, p. 83)

The authors give the example of a problem taken from the *Lilavati*: If the interest of a hundred in one month be five, what will be the interest of 16 in 12 months? And the problem was set out in the following way:

<table>
<thead>
<tr>
<th></th>
<th>100</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

12. Attempts were made by Indian mathematicians to describe the complete arithmetic of zero (Datta & Singh, 2001); in the early centuries of the Common Era, zero was gradually conceived as a *number* (with operation of addition and subtraction of zero) and this is evident in Varahamihira’s (505 CE) work.

13. The development of negative numerals was made possible by the incorporation of zero in mathematics (Joseph, 2002). In the 18th chapter of his treatise, Brahmagupta (628 CE) gives what is probably the first set of rules for the arithmetic of zero, and positive and negative numbers (Colebrooke, 1817; Datta & Singh, 2001). According to Datta and Singh (2001) and Joseph (2002), most Sanskrit texts on mathematics contain a section called *sunya-ganita* or computations involving zero. While the arithmetic texts were restricted to only addition, subtraction and multiplication, the discussion in algebra sections covered the arithmetic of integers and zero. Some examples: “The sum of cipher and negative is negative; of affirmative and nought is positive;...negative taken from cipher becomes positive,...” (Colebrooke, 1817, p. 339). Regarding division, as indicated by Joseph (2002) and Datta and Singh (2001), while Brahmagupta observes that the results of the operations (in modern notation) $x \div 0$ and $0 \div x$ should be written as $\frac{x}{0}$ and $\frac{0}{x}$ respectively, the idea of zero as an infinitesimal and the relation between zero and infinity (*ananta*) is more evident in the work of
Bhaskara II (1150). And, Bhaskara II attempted to rectify Brahmagupta’s statement that \( \frac{0}{0} = 0 \) (Datta & Singh, 2001).

**The final abstract stage with full place value and zero**

With the advent of zero to denote the absence of a specific multiunit/power of ten, an abstract place-value system was born. Thus in the decimal system we have ten arbitrary, distinct numerals including zero (see Figure 4.6), with no direct visual reference and each of which is drawn in one stroke of a pen (the first order digits are not additive), operating within a place value system (Dehaene, 1997; Ifrah, 1985, 1998). It is also noted that although these numerals are derived from the Brahmi script, they are different to the phonetic alphabets (Datta & Singh, 2001; Menninger, 1969) and hence appear to have avoided potential ambiguities. Figure 4.7 shows the genealogy of the current digits; from Brahmi through to the current Devanagari, to Arabic and to finally the European versions.

![Figure 4.6. The ten numerals including zero (D. E. Smith, 1958, p. 70).](image)

Examples of the final symbolic stage in number notation from the Gwalior inscription (876 CE) are the numbers 50 and 270, given as:

![Examples of the final symbolic stage in number notation from the Gwalior inscription (876 CE).](image)

Similarly, in the Bakshali Manuscript the following numbers are found:

![Similarly, in the Bakshali Manuscript the following numbers are found.](image)

(Joseph, 2000, p. 241)

And, an inscription in Banka shows 608.

![And, an inscription in Banka shows 608.](image)

(Sen, 1971, p. 178)
To summarise this section, what is seen above is the gradual evolution of the written numeration system in a broad sense; from the additive form of the symbols to the multiplicative one which enabled a separation of the face values and the powers of ten (ranks or multi-units of powers of ten) from the total values characterised by powers of ten called as ‘places’ or ‘positions’. Once these powers of ten (which were systematic) were known as ‘places’ it was not necessary to write or symbolise them, thus leading to an abstract place value structure. Such a structure requires a zero to denote absent powers of ten and since the concept existed from early times in Indian history, its inclusion was a natural progression to a full place value system with zero. A pictorial representation, (using bundles of sticks) of the four developmental stages of written numerals is shown in Figure 4.8. These representations paralleling the stages were used in classroom teaching (see Chapter 5).

With regard to zero, it can be seen that the concept evolved through several meanings such as void, fullness, relation to infinity, absence, subtraction, places, place-holder and, as a number. It appeared relatively late in the evolution of number systems after a long process of trial and error, reflection and invention; due to the fact that it was only in a
place-value system (Sarma, 2003) that a clear need for it arose. The story of the
transmission of the Indian place value system including zero to south-east Asia and
westwards to Europe have been well documented (e.g. Cajori, 1919; Datta & Singh, 2001;
Eves, 1969; Flegg, 1989). However, this system with zero was not particularly easy to grasp when the idea was first introduced in Europe. Flegg argues that its acceptance in
Europe was slow because the zero presented problems of comprehension: “People found it
hard to understand how it was that a symbol which stood for nothing could, when put next
to a numeral, suddenly multiply its value ten-fold” (Flegg, 1983, p. 72). As Rotman (1987)
declares, zero is a sign about signs; that is, it is a meta-sign (p. 12) and its meaning is that it
indicates the absence of other names or signs 1,2,3,...9. Again, according to Rotman, zero
is also a meta-number (p. 14), a sign indicating the potentially infinite progression of integers. This dual nature of zero, which presented difficulties historically, could also be contributing to students’ difficulties.

4.3.4 Reflection and Articulation on the Development of Written Numerals, Place
Value and Zero in Indian History

What can we, as teachers and educators, learn from the above historical development of
written numerals and zero?

The historical development of the numeration system has been briefly discussed and it
is observed that it advanced through a number of different stages and took a very long time
before the decimal positional system attained its current form. The different stages
correspond to gradually evolving sets of signs or semiotic systems; and each of these sign
systems (Radford & Puig, 2007) was governed by a set of rules and a set of relationships
between the signs (Ernest, 2006) with corresponding meanings integrated into the
structure. In a broad sense, the various stages appear to involve spoken number words,
written number words, development of numerical symbols marked by an early additive
principle, a slow replacement with more efficient multiplicative and cipherised systems
and the current form of ten abstract symbols including zero, reflecting place value
structure. What are the pedagogical implications of these stages?

Given the early historical awareness of the exponentiation concept evident from
number names of powers of ten in the Vedas, one lesson from history is that students
would likely need some preliminary groundwork involving multiplication, and its
extension exponentiation, since the place value system depends (Confrey, 1991) on both. For example: $345 = 3 \times 10^2 + 4 \times 10^1 + 5 \times 10^0$. As discussed earlier in Chapter 3, multiplicative thinking, which is higher order thinking (Kamii & Joseph, 2004), is possibly the key to understanding numbers and their properties. This is because, as stated by many researchers (e.g. Bryant & Nunes, 2009; Carpenter et al., 1996; Greer, 1992), a sound grasp of number depends on number operations and the place value structure, both of which rely on thinking multiplicatively. As can be seen in the historical development of Indian history, multiplicative representation of the numbers enabled a detachment of the place values from the face values (face values indicate the number of multiunits). Hence at some stage, and perhaps as soon as it is feasible, students need to break away from additive thinking and re-conceptualise number in terms of multiplication (Ma, 1999). With regard to exponentiation, it is seen that repeated multiplication was not an easy concept and it is quite likely students today will experience the same difficulty as mathematicians did in history (Sfard, 1995). This suggests that early grouping experiences (e.g. Bednarz & Janvier, 1982; Dienes, 1964a), beginning with repeated addition and then moving to repeated multiplication with the corresponding use of a notation for the understanding of powers of ten, which are the landmark or signpost (Reddy & Srinath, 2001; Rubin & Russell, 1992) numbers in the numeration system, would be beneficial for students. Hence, in this researcher’s view, students may profit from grouping objects in the form $a \times c$ and $a^c$ and also working out the values of these numbers in order to understand the difference between the two concepts and notations (see Appendix 5). For example, experiences in the grouping and notation of $10+10+10+10$ as $10\times4$ and $10\times10\times10\times10$ as $10^4$, and in other bases such as $5 + 5 + 5 + 5$ as $5 \times 4$ and $5 \times 5 \times 5 \times 5$ as $5^4$ may be helpful to students in understanding these vital foundational concepts (Dienes & Golding, 1971). In this connection, Smith and Karpinski’s (1911) example of a comparison involving the reading of large numbers such as 8 443 682 155 in different numeration systems could be of value. Apart from reading the number the modern way, the Indian method of reading the number could highlight that the number indicates counting of different sized groups; 8 lots of $10^8$ and 4 lots of $10^9$, and so on.

In order to assist students to construct the compositional structure of the place value system, if we were to try and follow the historical stages in our teaching, it would involve the following. First, numbers should be written out in full in words, the way we say them,
such as two thousand, four hundred and thirty five, and large numbers should be named in this way. Alongside grouping experiences such as bundling of sticks, the next few steps would involve transforming concrete representations (bundles of sticks) to reflect the main stages (from additive to multiplicative to the ‘places’ stage) and to symbolise these with numerals rather than words, but keeping the structure the same. This order can be summarised in the following way for the number 2435.

a. \[1000 + 1000 + 100 + 100 + 100 + 10 + 10 + 10 + 1 + 1 + 1 + 1\] (additive principle)

b. \[2 \times 1000 + 4 \times 100 + 3 \times 10 + 5 \times 1\] (multiplicative principle)

c. Powers of ten known as ‘places’:

<table>
<thead>
<tr>
<th>1000</th>
<th>100</th>
<th>10</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

Multiple representations of powers of ten

<table>
<thead>
<tr>
<th>Thousand</th>
<th>Hundred</th>
<th>Ten</th>
<th>Ones</th>
</tr>
</thead>
<tbody>
<tr>
<td>10^2 \times 10</td>
<td>10^2</td>
<td>10^1</td>
<td>10^0</td>
</tr>
<tr>
<td>10^1 \times 100</td>
<td>10 \times 10</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>10^3</td>
<td>1000</td>
<td>10</td>
<td>1</td>
</tr>
</tbody>
</table>

| 2 | 4 | 3 | 5 |

d. Abstract stage with place value:

2435

The above set of stages (or representations) offers a route to understanding the place value concept. In the context of base-10 representations, most textbooks use base-10 Dienes’ blocks to represent a multidigit number (corresponding to the first additive stage above) in order to help students form links between the concrete blocks and the symbolic form of the number (final abstract stage above). Therefore, what are demonstrated in textbooks are only two stages; the first and the last of the stages given above. However, students’ failure to form connections have been reported (e.g. Resnick & Omanson, 1987; P. W. Thompson, 1992) and Hart (1989) has stated that the gap between the concrete and
abstract forms is too big and needs to be bridged by the use of a third transitional form. Hence what is now gained from a study of Indian history is the idea of two intermediate conceptual stages reflecting the multiplicative and ‘places’ stages. These bridging stages can be useful for the following reasons. As Hart (ibid) observed, the gap between the additive and final stage is too wide; a feature of the gap is that in the first stage the place values are explicit and in the form of blocks/sticks, and in the final stage they are hidden and only the face values are visible in the form of numerals. Also in the base-10 manipulatives, spatial position is not important in determining the value of a piece; the value is known from its appearance (Varelas & Becker, 1997). As well, in the first additive stage, the face values and place values are merged in the blocks; that is, there is no separation between them. However, the second stage in the model above allows a move to a multiplicative interpretation thus also enabling a separation between the face value and place value. And in the ‘places’ stage, there is a complete separation of the face and place values, with the place values still visible, but occupying a position above the corresponding face value. In the final abstract stage the place values are implicit with only the face values (numeral/digits) shown. Hence, the above model can be seen as providing a route from the conceptual (related to grouping structures) to the semiotic (related to numerals as a system of signs) aspects of the place value system, both of which, according to Becker and Varelas (1993), are important to the cognitive development of the place value idea. Although the equivalent form of the ‘places’ stage above is not recorded as such in the historical documents, it could prove useful in scaffolding students’ learning of the system, and hence its inclusion. The final step would be to introduce a number (such as two thousand and thirty five) that has a missing power (hundred) and as a result, students see the need for a placeholder symbol such as zero. Using the idea of place value, and following similar steps to those given above, the number is written as 2035. (The issue of the number of symbols required in the system is crucial and one that needs discussion in class and paving the way for a consideration of the number of symbols required for numbers written in other bases. See Appendix 4). The choice of numbers for groupings, multiple representations, and the related debate could include decimal fractions, as given below for 2435.32, thus extending digits to the right of the decimal point and leading students to comprehend negative powers of ten (see Figure 4.8 for a representation paralleling these stages of the written system).
The above process that is described involves two representational systems. That is concrete representations (in the form of sticks), and the symbolic representations of numerals. The four historic stages (see Section 4.3.3 for an explanation of these stages) represented in each system and links made within and between the two systems may help students to comprehend the place value idea. Similarly, the above process could involve pictorial and symbolic representations and connections made between representations.

a. \[1000 + 1000 + 100 + 100 + 100 + 10 + 10 + 10 + 1 + 1 + 1 + 1 + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{100} + \frac{1}{100} + \text{(additive principle)}\]

b. \[2 \times 1000 + 4 \times 100 + 3 \times 10 + 5 \times 1 + 3 \times \frac{1}{10} + 2 \times \frac{1}{100} \text{ (multiplicative principle)}\]

c. Powers of ten known as ‘places’:

<table>
<thead>
<tr>
<th></th>
<th>Thousand</th>
<th>Hundred</th>
<th>Ten</th>
<th>Ones</th>
<th>Tenth</th>
<th>Hundredth</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>10⁴ × 100</td>
<td>10³</td>
<td>10²</td>
<td>10¹</td>
<td>10⁻¹</td>
<td>10⁻²</td>
</tr>
<tr>
<td>100</td>
<td>10²</td>
<td>10¹</td>
<td>10⁰</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>10 × 10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0.01</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.01</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Multiple representations of powers of ten:

<table>
<thead>
<tr>
<th>Thousand</th>
<th>Hundred</th>
<th>Ten</th>
<th>Ones</th>
<th>Tenth</th>
<th>Hundredth</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
<td>3</td>
<td>5</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

d. Abstract stage with place value:

2435.32
1. Initial/Additive stage

2. Interim/Multiplicative stage

3. ‘Places’ stage

4. Abstract stage

*Figure 4.8.* Representations paralleling development of the stages of the *written* place value system.
Another important aspect of the above pedagogical strategy is the linking not only of representations across systems, such as bundles of sticks and written notation, but also helping students to connect multiple representations, such as the powers of ten shown above. Furthermore, (although in Indian history, numeration system was mostly decimal) the above process of grouping, notations and their transformations could also be applied to numbers in non-decimal multiple bases, thus paving the way for a generalisation of the place value system. Vygotsky (1962) declares that without attaining knowledge of a more general concept of positional notation one cannot master the base ten system. That is, the teaching of base ten alone will not suffice for the development of the concept of positional notation. In view of this, and as suggested by Dienes (1964a), Baroody (1987) and Schmittau and Vagliardo (2006), experiences involving groupings in non-decimal bases and their number representations may lead students to grasp the place value structure and to understand decimal numeration as a particular instance of a general positional system. In a non-decimal base such as *six*, the grouping and number representation would be similar to base ten since the grouping structures in the bases are the same. What is given below is a summary of the stages for 4534.23 (in base six), however, with a base ten reference.

a. \[216 + 216 + 216 + 36 + 36 + 36 + 36 + 6 + 6 + 6 + 1 + 1 + 1 + \frac{1}{6} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36}\] (additive principle)

b. \[4 \times 216 + 5 \times 36 + 3 \times 6 + 4 \times 1 + 2 \times \frac{1}{6} + 3 \times \frac{1}{36}\] (multiplicative principle)

c. Powers of six known as 'places'

<table>
<thead>
<tr>
<th>216</th>
<th>36</th>
<th>6</th>
<th>1</th>
<th>\frac{1}{6}</th>
<th>\frac{1}{36}</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>
d. Multiple representations of powers of six:

<table>
<thead>
<tr>
<th>Two hundred and sixteen</th>
<th>Thirty six</th>
<th>Six</th>
<th>Ones</th>
<th>Sixth</th>
<th>Thirty-sixth</th>
</tr>
</thead>
<tbody>
<tr>
<td>$6^3$</td>
<td>$6^2$</td>
<td>$6^1$</td>
<td>$6^0$</td>
<td>$6^{-1}$</td>
<td>$6^{-2}$</td>
</tr>
<tr>
<td>$6 \times 6 \times 6$</td>
<td>$6 \times 6$</td>
<td>6</td>
<td>1</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{36}$</td>
</tr>
<tr>
<td>216</td>
<td>36</td>
<td>6</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

While the symbol 6 does not exist in base six, its employment in the table above rather than 10 (one and zero) is useful to highlight the structure of the place value system. Writing the place values in terms of powers and pointing out the similarities in place values of base ten ($10^3$, $10^2$, $10^1$...), base six ($6^3$, $6^2$, $6^1$...) and other bases (Zazkis & Khoury, 1993), for example base eight and base thirteen may help students to understand the structure. Demonstrating the counting process in each base (such as 1,2,3,4,5, 10,11, 12... in base six and 1,2,3,4,5,6,7,10, 11 etc in base eight) and the resulting discussion on the number of symbols required in various bases (see Appendix 4) could help students to see 10 as not a symbol for ten but as one ten (or base value) and no units, or one ten and zero units. Subsequently, students can be gradually led to see 10 (to be read as one and zero) in base six as one six and zero units.

Instead of the development described above, students today are often presented with the final structural form of the number system rather than experiencing the different developmental stages during their learning, and this could be contributing to their difficulty. In addition, a crucial aspect of the above strategy involving historical stages is classroom discussion, particularly involving linking of ideas and concepts across and between representations (such as base six and other bases) and representational systems (M. O. J. Thomas, 2004, July; M. O. J. Thomas & Hong, 2001). Specifically, students’ attention (Mason, 2004, July) needs to be guided to see the relationships and connections (Presmeg, 2006) between bundles of sticks and their transformations, pictorial representations, and respective notations.
The historical developmental stages also have implications for teaching of the place value concept in a computational setting. History suggests that the intermediate conceptual stages, if incorporated as a teaching strategy could prove useful in understanding corresponding algorithms performed on numerals. The following example illustrates this:

A problem representation given in Figure 4.9 is as it usually appears in textbooks. Operations on numbers are usually shown alongside manipulations with the blocks in textbooks. (In the diagrams given below blocks and counters are used instead of bundles of sticks for ease of presentation).

<table>
<thead>
<tr>
<th>Problem</th>
<th>Additive representation of the problem with blocks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Hundreds 100s</td>
</tr>
<tr>
<td>346</td>
<td></td>
</tr>
<tr>
<td>+ 423</td>
<td></td>
</tr>
</tbody>
</table>

*Figure 4.9. Additive representation of the problem.*

What is given below is the problem representation with intermediate (main) conceptual stages as learnt from history. It may be that students need this scaffolding; that is, to be shown manipulations on each of the intermediate representations given in Figure 4.10 rather than only on the bundles of sticks and the numbers. Students may find it helpful to be shown manipulations on the counters (representing face value) with the teacher explaining that what they are operating on for example, is the 3 and the 4 in 3×100 and 4×100 instead of operating on 300 and 400 as represented above where the face value and the place value are merged in the representation.
<table>
<thead>
<tr>
<th>Problem</th>
<th>Multiplicative representation of the problem with blocks and counters</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Hundreds 10s</td>
</tr>
<tr>
<td></td>
<td>●●●</td>
</tr>
<tr>
<td></td>
<td>●●●</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Problem</th>
<th>‘Places’ representation of the problem with blocks and counters</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>❑ Hundreds 10s</td>
</tr>
<tr>
<td></td>
<td>●●●</td>
</tr>
<tr>
<td></td>
<td>●●●</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Problem</th>
<th>‘Places’ representation of the problem with blocks and numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>❑ Hundreds 10s</td>
</tr>
<tr>
<td></td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Final abstract representation of the problem without visible written place values or place value manipulatives</th>
</tr>
</thead>
<tbody>
<tr>
<td>346</td>
</tr>
<tr>
<td>+423</td>
</tr>
<tr>
<td>769</td>
</tr>
</tbody>
</table>

*Figure 4.10. Multiplicative, places and abstract representation of the problem*
The key idea in the above stages is the separation of face values and place values (powers of ten) from the total values and showing manipulations in each of these stages. Most textbooks do not recommend showing manipulations on the intermediate stages for performing operations; that is the additive, followed by multiplicative which in turn is followed by ‘powers of ten as place values’ stage and then the final abstract form of number. It is suggested by this researcher that for some students providing such scaffolding for at least a few initial problems could lead to a better grasp of place value structure.

In relation to zero, students do not find this concept easy to grasp, particularly the zero in the place value system and with decimal numbers (Hiebert & Wearne, 1986; M. Hughes, 1986; Seah & Booker, 2005) and this difficulty has its parallel in history when zero first made its appearance in Europe. In view of this difficulty, Pandit’s (1993, 2003) analysis of related aspects of Panini’s grammar Ashtadhyayi and similar comparison with the numeration system could prove pedagogically useful. First, Pandit proposes that in terms of positional analysis of the Sanskrit words, ten belongs to the succeeding series from 11-19 rather than the single digit series. Hence 10 is the first number in the two-digit series rather than the last number of 1, 2, 3, etc. (most textbooks mainly show groupings such as 1, 2, 3,.....10, and 11,12,13,......20, etc). Secondly, if 10 is read as one ten and zero rather than just ten, this may help students to understand the structure of the place value concept and role of zero in the system better. A further implication involving non-decimal bases is that 10 in base 7 should be read as one seven and zero and 10 in base 8 as one eight and zero. Another student difficulty related to zero concerns internal zeros in a place value system. Seah and Booker (2005) have reported that many students have a problem with internal zeros of a number in number operations, and when reading and writing numbers. Pandit’s analysis of Panini’s grammatical rules regarding number words implies the value of comparison of numbers as shown in Table 4 for a better understanding of the role of zero in the positional system.

Table 4.2. Comparison of Numbers With and Without Zero Based on Pandit’s Analysis

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>873</td>
<td>7438</td>
<td>4319</td>
<td>59562</td>
</tr>
<tr>
<td>803</td>
<td>7408</td>
<td>4009</td>
<td>50560</td>
</tr>
</tbody>
</table>
A key aspect of place value which is also difficult for students, pointed out by Seah and Booker (ibid) is ‘renaming’ (non-standard representations of a number). This refers to naming a set of numbers according to its value. For a question as to how many tens are there in 4009, some students would probably say that there are none. Grouping and regrouping experiences could show that while there are no separate groups of ten left over, there are 400 tens contained in 4000. Hence renaming the above numbers in multiple ways could highlight the place value structure for students and alleviate some of the difficulties of zero in whole numbers and subsequently in decimal fractions.

Bhaskara’s method of writing zeros for ‘places’ could prove useful for students when writing large numbers. For example, in order to write twenty three trillion, seven hundred and eighty-six million and forty-nine, it may be helpful to first write 23 followed by twelve zeroes since one trillion is $10^{12}$; that is $23\ 000\ 000\ 000\ 000$ (a variation could be 23 followed by twelve small lines and then to write $23\ 000\ 786\ 000\ 000$ (since a million has six zeros) and finally $23\ 000\ 786\ 000\ 049$.

Given the many misconceptions, cognitive obstacles and difficulties involving zero that students face, Joseph asks this question; “Should we follow the example of early Indian mathematics and bring ‘shunyaganita’ [arithmetic of zero] back into the curriculum?” (Joseph, 2002, p. 125). It may be that revision of rules of operations on zero prior to the start of a related lesson or topic could alleviate some of students’ problems.

Thus far in this chapter, an overview of the history of Indian mathematics has been considered, as well as the different stages in the creation of the place value decimal system as traced through the development of numerals. The generalisation aspect, as well as the idea of general number/generalised number in the history of Indian algebra is considered next in order to search for ideas for a better understanding of the literal symbol that may help students to generalise the place values in multiple bases. That is, in this study, algebraic generalisation is a supporting idea in the generalisation of a place value system.

4.4 Generalisation, the Variable Concept and its Notation in Indian Algebra

The research question addressed in this section on algebra is the following:
What were the key ideas in the development of algebra in the history of Indian mathematics, especially pertaining to variables as a sign system, and how might they be relevant to students’ awareness of a general positional notation?

In India, rapid progress was made in algebra in the centuries after the Common Era, and algebraic nomenclature was developed by Brahmagupta’s (628 CE) time. According to Colebrooke, the early development of algebraic symbolism in India and related generalisations was probably due to the invention of the place value decimal system with zero and its associated arithmetic. The author states: “The Hindus had the benefit of a good arithmetical notation: the Greeks, the disadvantage of a bad one. Nearly allied as algebra is to arithmetic, the invention of the algebraic calculus was more easy and natural where arithmetic was best handled” (Colebrooke, 1817, p. xxii)

In this section, the development of Indian algebra from early Vedic times up to the time of Bhaskara II (1150 CE) is considered. The focus is on some of the notational and generalisation aspects of Indian history in a search for ideas related to understanding a general place value numeration system. What follows is the meaning and origin of algebra, a discussion on the development of algebraic terminology and related issues, followed by reflection on and articulation of historical ideas and mathematics education ideas.

The word algebra appears in the title of one of the works by Al-Khwarizmi (c. 825 CE): al-jabr w’al-muqabalah. In a 16th century English work it is found as algiebar and almachabel, amongst various forms, but was finally shortened to algebra. The words mean restoration and opposition (D. E. Smith, 1958, p. 388). As further elaborated by Smith,

Given \[ bx + 2q = x^2 + bx - q, \]

_al-jabr_ gives \[ bx + 2q + q = x^2 + bx, \]

And _al-muqabalah_ gives \[ 3q = x^2 \]

Explaining this process, Smith says that _al-jabr_ has the idea of transposition of a negative quantity and _al-muqabalah_ the transposition of a positive quantity and the simplification of each term. It appears that eventually _al-jabr w’al-muqabalah_ was shortened to _al-jabr_ or algebra. Although algebra was named as given above, it is suggested that along with the Indian numerals, some of the Indian algebraic ideas may also
have influenced Arabic thought and consequently, European mathematics (Cajori, 1919; Colebrooke, 1817; Dantzig, 1967). Cajori states:

Both the form and spirit of the arithmetic and algebra of modern times are essentially Indian. Think of our notation of numbers, brought to perfection by the Hindus, think of the Indian arithmetical operations nearly as perfect as our own, think of their elegant algebraical methods, and then judge whether the Brahmins on the banks of the Ganges are not entitled to some credit. Unfortunately, some of the most brilliant results in indeterminate analysis, found in Hindu works, reached Europe too late to exert the influence they would have exerted, had they come two or three centuries earlier. (Cajori, 1919, pp. 97-98)

The origin of algebra in India, in the sense of finding solutions of equations, can be traced back to the period of the Sulbasutras (800 BCE) and even the Brahmana (c. 2000 BCE) period when it was mostly related to geometry (Datta & Singh, 2001). Geometrical methods (such as transformation of a square into a rectangle having a given side and equivalent to the solution of a linear equation) were employed in solving algebraic problems involving linear, simultaneous and even indeterminate equations, which were related to the construction of different types of sacrificial altars and their respective brick arrangements. It was only from the time of Aryabhata I that algebra grew into a definite branch of mathematics, separated from arithmetic and geometry and from then on came to be regarded as a science of great importance. As noted earlier, by Aryabhatta’s era, the decimal place value system was already in place. Various names were given to algebra such as kuttaka-ganita or bijaganita and later Bhaskara II considers the relation between the two branches of arithmetic and algebra to be this: “The science of calculation with unknowns is the source of the science of calculation with knowns” (Datta & Singh, 2001, vol. 2, p. 3) meaning that algebra is the source for arithmetic. The authors note that the science of algebra is divided into two parts; one part deals with analysis and the other part consists of subjects that are essential for analysis. These are the laws of signs, the arithmetic of zero (and infinity), operations with unknowns, surds, the pulveriser (indeterminate equation of the first degree), and the Square-nature (indeterminate equations of the second degree) (p. 5 Vol II). Hence, it was important to study these prior to solving equations in algebra. Bhaskara II says: “(The section of) this science of calculation which is essential for analysis has been briefly set forth. Next I shall propound analysis, which is the source of pleasure to the mathematician.” (Datta & Singh, 2001, vol. 2, p. 6).
In the sense that algebra is a science of “generalised computational processes” (Sfard, 1995, p. 18), a crucial aspect is the development of symbolism (a system of signs) which enables one not only to represent the structure of the problem but also to calculate at a syntactic level (Puig & Rojano, 2004). In this connection, as a preliminary to this discussion on algebraic symbolism, it is useful to note Nesselman’s (1842) characterisation of three stages in the historical development of algebraic notation. First is rhetorical algebra, in which the solution of a problem is written out in full without abbreviations or symbols; the second stage is syncopated algebra in which abbreviations are adopted for some of the quantities and operations. As the last stage, we have symbolic algebra, in which solutions largely appear in symbols (e.g. Eves, 1969; Puig & Rojano, 2004; D. E. Smith, 1958) with little connection to the semantic content of the problem. (More recently, and from a different perspective, Katz (2007) has argued that besides these three stages of expression, four conceptual stages have happened alongside the notational stages; these are the geometric stage, the static equation-solving stage, the dynamic function stage and finally the abstract stage. As stated by Katz, these stages are not completely separate from Nesselman’s notational stages, there is some overlap. However, in this study, the focus is on the notational stages).

In the context of Nesselman’s stages, Eves (1969) states that Indian algebra, beginning with Brahmagupta (628 CE), or perhaps earlier, was syncopated; that is, some symbols and abbreviations of frequently occurring words were adopted. The use of symbols and a well-developed sign system (Puig & Rojano, 2004) which had different names for different unknown quantities, contributed to the development of algebra. In this context, Joseph (2000) highlights the key role that Sanskrit played in the development of notational symbolism in India. He states:

... the linguistic facility of the language [Sanskrit] came to be reflected in the character of mathematical literature and reasoning in India. Indeed, it may even be argued that the algebraic character of ancient Indian mathematics is but a by-product of the well-established linguistic tradition of representing numbers by words.

---

7 However, in a recent paper Hieffer (2007) has cast some doubt on the historical reality of Nesselman’s proposed three stages. Heiffer points out, for example, that the syncopated stage supposedly appears in Diophantus’ work, but that the later Islamic mathematicians who studied Diophantus in the middle ages still continued using rhetorical algebra.
...A significant feature of early algebra which distinguishes it from other mathematical traditions was the use of symbols, such as a dot (in the Bakshali Manuscript) or the letters of the alphabet, to denote unknown quantities. In fact it is this very feature of algebra that one immediately associates with the subject today. The Indians were probably the first to make systematic use of this method of representing unknown quantities. (Joseph, 2000, pp. 217-219, 272)

What follows is a consideration of the algebraic nomenclature since this development led to rapid progress in algebra and in mathematics in India. A study of Indian historical texts reveals a variety of technical terms and symbols as part of an evolving sign system. However, this development occurred over many centuries and was not an easy process, as is highlighted in the following discussion. At the beginning of the syncopated stage of algebraic notation, zero was employed by some authors as a symbol to denote any unknown quantity. The absence of an adequate symbolism for the unknown quantities led to a certain amount of ambiguity in the representation of equations, and had to be interpreted suitably according to the context. For example in the Bakshali Manuscript the following equations are given:

\[
\begin{array}{ccc|ccc}
0 & 5 & yu & mu & 0 & sa & 0 & 7+ & mu & 0 \\
1 & 1 & & 1 & & 1 & 1 & & 1 \\
\end{array}
\]


This means \(\sqrt{x + 5} = s\) and \(\sqrt{x - 7} = t\). In the above equation, 0 (solid dot) with 1 written underneath is ‘a certain unknown’, \(yu\) stands for addition and \(mu\) for root. Unusually, in this treatise, + is used for denoting a negative! As is seen, 0 is the symbol that is used for different unknown quantities, \(x\), \(s\) and \(t\). In later treatises, even when an adequate notation was developed, a zero sign was used to mark a vacant place. For example, in the \(Trisatika\) written by Sridhara (c. 750), the following statement of an arithmetical progression whose first term \(adih\) is 20, number of terms \(gacchah\) 7, sum \(ganitam\) 245 and whose common difference \(uttarah\) is unknown is given:

\[
\begin{array}{ccc|cc}
adih & 20 & u & 0 & gacchah & 7 & ganitam & 245 \\
\end{array}
\]

In the authors’ view, the zero sign was considered necessary as algebraic symbols could not be used in arithmetic treatises. However, when there are a number of unknowns, the symbol 0 is dropped, and the first syllables of prathama, dvitiya, triiya (first, second, third etc) and so on are employed. Thus in the above treatise,

\[
\begin{array}{cccccc}
9 \text{ pra} & 7 \text{ dvi} & 10 \text{ tri} & 8 \text{ cha} & 11 \text{ pan} \\
7\text{dvi} & 10 \text{ tri} & 8 \text{ cha} & 11 \text{ pan} & 9 \text{ pra}
\end{array}
\]

\textit{yutam jatam pratyeka (kramena) 16|17|18|19|20}

(Srinivasiengar, 1967, p. 33)

This means 9 of the first (\textit{pra}) plus 7 of the second (\textit{dvi}) is equal to 16, 7 of the second (\textit{dvi}) and 10 of the third (\textit{tri}) equals 17 and so on. In modern notation this would mean that the solutions of the equations

\[
\begin{align*}
x_1 + x_2 &= 16 \\
x_2 + x_3 &= 17 \\
x_3 + x_4 &= 18 \\
x_4 + x_5 &= 19 \\
x_5 + x_1 &= 20
\end{align*}
\]

are given by \(x_1 = 9, \ x_2 = 7, \ x_3 = 10, \ x_4 = 8, \ x_5 = 11\).

In the same manuscript, the unknown was sometimes called \textit{yadrcchca, vancha} or \textit{kamika} (any desired quantity). Aryabhatta I (499 CE) called it \textit{gulika} (Bag, 1979; Datta & Singh, 2001). According to Datta and Singh, the term \textit{gulika} was probably due to a \textit{coloured shot} (shot means small ball or bead) being used to represent an unknown. Although Brahmagupta (628 CE) mentions the term \textit{avyakta} (literal meaning unknown), the unknown quantity was called \textit{yavat-tavat} (as many as or so much as, meaning an arbitrary quantity) by him and shortened to \textit{ya}. Subsequently, in the case of more unknowns, the first unknown came to be denoted as \textit{yavat-tavat} (\textit{ya}) and the other unknowns by colours. Says Bhaskaracharya II (1150 CE):
“So much as” and the colours “black, blue, yellow and red” and others besides these, have been selected by venerable teachers for names of values of unknown quantities, for the purpose of reckoning therewith... Among quantities so designated, the sum or difference of two or more which are alike must be taken: but such as are unlike, are to be separately set forth. (Colebrooke, 1817, p. 139)

In another chapter, Bhaskara II again introduces many colours to represent unknown quantities and suggests that letters can also be employed to represent them:

This is analysis by equation comprising several colours. In this, unknown quantities are numerous, two, three or more. For which yavat-tavat and the several colours are to be put to represent the values. They have been settled by the ancient teachers of the science: viz. “so much as” (yavat-tavat), black (calaca), blue (nilaca), yellow (pitaca), red (lohitaca), green (haritaca), white (swetaca), variegated (chitraca), tawny (capilaca), tan-coloured (pingala) grey (d’humraca), pink (patalaca), white (savala), another black (mechaca) and so forth. Or letters are to be employed; that is the literal characters c &c. as names of the unknown, to prevent the confounding of them. (Colebrooke, 1817, pp. 228-229)

As explained by Datta and Singh, the Sanskrit word varna means colour as well as the letters of the alphabet so that unknowns are generally represented by letters of the alphabet or by colours such as calaca (black) or simply ca and nilaka or ni (Colebrooke, 1817; Datta & Singh, 2001). However, the first unknown yavat-tavat or ya denotes measure or quantity. In essence, in the syncopated stage the unknown quantity was first named (such as avyakta or gulika) and sometimes symbolised as 0, and the same symbol (0) was employed for different unknowns. Afterwards different symbols such as pra, dvi and tri were used to denote different unknown quantities and finally yavat-tavat (or ya) and different colours and their abbreviations ca, ni, pi were used for the different unknowns.

The extent of this difficulty or obstacle of naming other unknown quantities in Indian history (described above) is realised when the same obstacle is observed in the mathematics of other civilisations. Diophantus (c. 250 CE) converted all problems to equations in one unknown, and he used the word arithmos and the symbol ζ to denote the unknown. As described by Van Der Waerden (1954):

Diophantus reduces all his problems no matter how complicated, to equations in one unknown [or variable]. Either he expresses the other unknowns [variables] in terms of that one, or he gives them arbitrary values to be modified afterwards, if necessary, in order to satisfy the other conditions of the problem. Al-khwarizmi also reduces all his problems systematically to equations in one unknown. (Van Der Waerden, 1954, p. 281).
Similarly, *shay* (thing) in Arabic treatises and *res* in the Italian works denoted unknown and functioned as a *common name* and not as a proper name. Highlighting the difficulty in history of naming other unknown quantities, Puig and Rojano (2004) state:

A competent user of Al-Khwarizmi’s sign system names an unknown quantity as ‘thing’ and has to be careful in referring to other unknown quantities by compound expressions because a different name for them is not available in this system. He uses in fact a common name as a proper name. (p. 201).

However, while this difficulty was overcome almost immediately during Brahmagupta’s (628 CE) time in India, European algebra had to wait till the 16th century before *other unknowns* were named.

In the development of algebraic symbolism, other technical terms (such as powers which are particularly relevant for this thesis) that were used are highlighted by historians (e.g. Bag, 1979; Cajori, 1919; Colebrooke, 1817; Datta & Singh, 2001; Joseph, 2000; Sen, 1971) and these are given below:

**Coefficient**: *anka* (number), *gunaka* (multiplier) and *prakrti* (multiplier)

**Equation**: *sama-karana* or *sami-karana* (making equal), *sama* (equation) and *samya* and *samatva* (equality)

**Classification of equations**: *varga* or *va* denotes square or quadratic, *ghana* or *gha* denotes cubic, and *varga-varga* means biquadratic

**Absolute term**: *drsyā* (visible), *rupa* (appearance); that is, the absolute term represents the known or visible part of the equation.

**Symbols of operation**: addition is denoted by *yuta* or *yu*, subtraction by *ksaya*, multiplication by *gunita* or *gu*, and division by *bhaga* or *bhajita* or simply *bha*;

**Powers (and roots)**: A key aspect of algebraic development in India is that, unlike Diophantus and the medieval Arabic and Italian mathematicians, unknown quantities and their powers (types of numbers) were differentiated. In relation to powers, the earlier works (e.g. *Uttaradhyayana-sutra*, c. 300 BCE) applied both the multiplicative and additive principles in naming them, Datta and Singh indicate that Brahmagupta formulated a scientifically better method of expressing powers. For example, he called the fifth power *panca-gata* (raised to the fifth) and *sad-gata* (raised to the sixth). Similarly any power is
produced by adding the suffix *gata* to the name of the number indicating that power. Similarly, root is referred to as *mula* (*mu*) or *karani* (*ka*) and is usually placed before the quantity affected.

Furthermore, the product of two or more unknown quantities was indicated by writing *bhavita*, abbreviated to *bha* or *bh*. For example, *yava-kagha-bha* ...means \((ya)^2(ka)^3\) or \(x^2y^3\). Bhaskara states certain rules of multiplication that facilitates the writing of algebraic expressions with more than one unknown quantity:

When absolute number and colour (or letter) are multiplied one by the other, the product will be colour (or letter). When two, three or more homogeneous quantities are multiplied together, the product will be the square, cube or other (power) of the quantity. But if unlike quantities be multiplied, the result is their (bhavita) ‘to be’ product or factum. (Colebrooke, 1817, pp. 140-141)

The above terminology appears to have enabled the Indian mathematicians to write algebraic expressions and equations. As Puig and Rojano (2004) explain, the representation of what is needed for the solution of the problem is done by means of two categories of tools; different names for different unknown quantities and, types or species of numbers (such as square and cube) which were present in Brahmagupta’s and Bhaskara’s sign systems. Some examples are given below:

*a)* In Pruthudakaswami’s commentary on Brahmagupta’s *Brahma Sputa Siddhanta* there appears the following representation of an equation:  

\[
\begin{align*}
yava \quad & 0 \\
yava \quad & 1
\end{align*}
\]

\[
\begin{align*}
& ya \\
& \text{ru} \quad 8
\end{align*}
\]

\[
\begin{align*}
& yava \\
& \text{ru} \quad 1
\end{align*}
\]

Here *ya* is an abbreviation for *yavat tavat* (the unknown quantity, or *x*) and *yava* is an abbreviation for *yavat tavat* and *varga* (the square of the unknown quantity or \(x^2\); and *ru* stands for *rupa* (the constant term). In other words this is what we would now write as \(10x + 8 = x^2 + 1\). (Joseph, 2000, pp. 272-273)

*b)* Bhaskara II writes the algebraic expression \(x^3 + 3x^2y + 3xy^2 + y^3\) as follows:  

\[
\begin{align*}
yah \quad & 1 \\
yah \quad & v.
\end{align*}
\]

\[
\begin{align*}
& ca \\
& bh \quad 3 \\
& ca \\
& ya \\
& bh \quad 3 \\
& ca \\
& gh \quad 1
\end{align*}
\]


Here *ya* represents an unknown quantity, *ca* is the abbreviation of *calaca* and represents another unknown quantity, *v* is the beginning of *varga* (square), *gh* is the abbreviation of *ghana* or cube, and *bh* is the beginning of *bhavita* or product. As pointed
out by Puig and Rojano (2004) in their analysis, numbers were always present, even if the coefficient is the number 1, and we find names of unknown quantities, the names of the species of numbers (powers and constants) and the name of the indicated product. In comparison, the authors state that this distinction between names of unknown quantities and the types of numbers was not present in Al-Khwarizmi’s (Arabic) and Fibonacci’s (Latin) systems. The term mal (Arabic) and dynamis or census (Latin) represents square and is written on its own without being accompanied by the name of the unknown quantity. While in Brahmagupta (628 CE) and Bhaskara’s sign systems, varga (square) for example is accompanied by the unknown quantity yavat tavat (ya) and ghana (cube) is accompanied by calaca (ca). “Thus, Bhaskara’s sign system has no difficulty in representing different unknown quantities by different signs, precisely by keeping the representations of quantities and types of numbers [powers] differentiated” (Puig & Rojano, 2004, p. 203).

This achievement (names for other unknowns), the use of which was a key aspect of progress in algebra, is significant when it is observed that after Brahmagupta (628 CE), nearly a thousand years elapsed before Viete (1540-1603) in Europe employed different signs to represent different unknown quantities or variables (Puig & Rojano, 2004). In this context, it may be that after the arrival of the decimal place value system with zero from India, alphabetic letters that were employed for numerals were freed up for use in algebra (Dantzig, 1967). In terms of algebraic symbolism, where Viete differed from the Indian mathematicians was in using symbols for also parameters or givens (such as coefficients or constants) in equations. The Indian astronomer-mathematicians, while representing the coefficients or parameters as numbers in equations and expressions, had a general name such as gunakara which is multiplier (Datta & Singh, 2001) or anka which is numeral for referring to coefficients (Colebrooke, 1817); and they alluded to the constant term as rupa when stating general methods of solutions for equations. However, what is surprising is that Bhaskara II, unlike Viete, did not directly designate different parameters by different signs (although a general name or compound sign such as ‘multiplier of x³’ was given) especially when he says:

...the letters of alphabets beginning with ka, should be taken as measures of the unknowns in order to prevent confusion... (The maxim), ‘colours such as yavat-tavat, etc, should be assumed for the unknowns’, gives (only) one method of implying (them). Here, denoting them by names, the equations may be formed by the intelligent (calculator). (Datta & Singh, 2001, vol. 2, pp. 19-20)
It may be that Bhaskara II had the problem of finding a set of names with unlike initial letters (to that of the colours) to avoid confusion. The algebraic symbolism that was developed enabled the Indian mathematicians to solve linear and quadratic equations and to state general methods of solutions. The mathematicians admitted negative and irrational numbers and recognised that a quadratic has two formal roots (Eves, 1969). The familiar method of completing the square known as the *Hindu method* (Cajori, 1919; Eves, 1969) is attributed to Sridhara (c. 750 CE). However, greater progress appears to have been made by the Indians in indeterminate analysis (e.g. Cajori, 1919; Sen, 1971). They gave general methods of solution of indeterminate equations such as \( ax + c = by \) and \( Nx^2 + 1 = y^2 \) and according to Cajori (1919) these problems occurred in Astronomy. The latter equation was solved first by Brahmagupta by the method of *samastabhavana* (e.g. Bag, 1979; Srinivasiengar, 1967) and then Bhaskara II by what is known as the *chakravala* or cyclic method (e.g. Bag, 1979; Cajori, 1919; Datta & Singh, 2001; Srinivasiengar, 1967). The first equation is called Diophantine equation (Van Der Waerden, 1954) and usually has infinitely many solutions; its general solution was first given by Aryabhata (499 CE) and then further extended by Brahmagupta. Interestingly, in comparing the Indian and Diophantus’s (Greek) methods, Van Der Waerden states that Diophantus admitted only *positive rational* solutions and was content when he had found any one solution. However, the Indian mathematicians sought *all possible integral* solutions. As a possible explanation for the development of a notational symbolism, Cajori (1919) suggests that Indian mathematics did not make the distinction between numbers and magnitudes as was the case in Greek mathematics. In Indian algebra, methods were invented to designate *abstract* magnitudes (both known and unknown, variable or constant and indeterminate constants such as parameters).

The analysis thus far in this section shows the development of algebraic symbolism and among others, the key idea of assigning *different signs* to *different unknowns*. Invention of the other unknowns and the naming of *colours* for different unknowns (as also general names for parameters) in Indian mathematics constituted a major milestone in algebraic history. This achievement is appreciated when one considers the subsequent impact on mathematics in general as outlined in Section 4.2. The above analysis has also served to highlight some of the epistemological obstacles and difficulties in the development of algebraic language and which are central elements that link the domains of history and
mathematics education. Furthermore, reflecting on history and understanding how these obstacles were overcome in history can be put to profitable use in the development of didactic strategies and this is considered next.

4.4.1 Reflection and Articulation on Generalisation, the Variable Concept and its Notation in Indian Algebra

What has been discussed so far in this section are some of the relevant features of Indian algebra, such as notational aspects, for the generalisation of a place value system. It was seen that a terminology was developed, whose key features are different names for different unknowns as well as powers of unknowns (Filloy et al., 2008; Puig & Rojano, 2004), which enabled Indian mathematicians to generalise computational processes and to develop general methods of solutions for whole classes of both determinate and indeterminate equations. The review also highlighted the vital link between notation for powers of numbers (base in the place value system) and powers of unknowns/variables. This raises the following question: What role could the preceding historical analysis play in the generalisation of the place value system and the modern day teaching of algebra, particularly beginning algebra? This is debated in the remainder of this section.

A particular use of history is that awareness of historical development alerts teachers to students’ attempts at notation and to see it as a specific stage in the students’ understanding of algebraic symbolism. Another use of history as a cognitive tool is the identification of epistemological obstacles (Radford, 2000) such as zero and how these were overcome. Milestones in history are indicative of such epistemological obstacles (van Amerom, 2002). A major obstacle in history was the existence of only one name for the unknown quantities for a lengthy period; as elaborated by Puig and Rojano (2004) shay or thing in the Arabic works and res in the Italian treatises functioned as a common name and not as a proper name for the unknown. And as seen in the preceding section on Indian algebra, in the Bakshali Manuscript, 0 was used as a symbol for any unknown quantity and the meaning had to be deduced from the context. Puig and Rojano (ibid) observe the same difficulty for today’s students, thus indicating a historical parallelism. The authors remark:

Students taught to name the unknown of a word problem with an x, frequently see the x as a common name meaning ‘unknown’ and not a proper name referring to a specific unknown quantity, labelling then any unknown quantity with an x. The meaning they give to x does not correspond to
its meaning in the current MSS [Mathematical sign system] of algebra, but to the meaning of a less abstract MSS. (Puig & Rojano, 2004, p. 201)

Besides the variable concept, other similar hurdles that students also face include the need to operate on the unknown in linear equations with literals on both sides of the equation (see e.g. Puig & Rojano, 2004) and the use of parameter or givens in equations (Harper, 1987; Sfard, 1995). Due to these difficulties, researchers (e.g. Harper, 1987) have appealed for more awareness of the levels of notation in the teaching and learning of algebra; teachers need to be alert to students’ attempts at notation and its range of levels. Teachers need to prepare students for the various usage of letters that students are required to assimilate by making the differences explicit to them.

Historically in India it is seen that the very early use of yavat-tavat or ya (so much as, meaning an arbitrary quantity) indicates an equation solving approach to algebra. As mentioned in Section 4.4, in the arithmetical works, in the rule of three (proportion) problems, a symbol (sometimes 0) was used to denote the specific unknown (Datta & Singh, 2001) or the absence of a term. Nevertheless other unknowns (or variables) were denoted by different signs (colours) as early as 7th century. The implication is that while students can be (and mostly are) introduced to algebra through problem solving involving specific unknown(s), it may be beneficial if they meet generalised numbers and variables sooner than its induction into the curriculum at present. The need for generalisation as a way of thinking (Mason, 1996; Radford, 1996) suggests that at the fundamental level students need to understand both concepts and these have to be made explicit to students; that of the unknown and also that of the general/generalised numbers leading to the idea of the variable (and maybe extended to the idea of a parameter or given). Instead, students tend to view letters as specific unknowns mainly through experiences that involve substitution and equation solving, and show a lack of familiarity of letters representing a range of values.

How, then, can generalised numbers be introduced to students? In this connection, the idea of colours (as signs) in Indian history to denote different unknowns could prove useful. The CSMS study (Küchemann, 1981) involving high school students’ interpretation of literal symbols showed that a majority of the students - 73% of 13 year olds, 59% of 14 year olds and 53% of 15 year olds either treated letters as concrete objects or ignored them. Thus it may be that the abstract nature of colours (without reference to any particular
object) may help to alleviate the problem for beginning algebra students. Another beneficial feature is that colours stress the visual aspect in recognising changing numbers. And, Srinivasan’s method of setting out of patterns and the associated pattern language which is built on the natural language of the learner could prove helpful to understand symbolic language better. Hence, based on the historical and psychological perspectives discussed above, a pedagogical aim could be to assist students to acquire a deeper awareness of the symbolic language of algebra by combining ideas of pattern language, and colours/signs from Indian history within a pattern generalisation activity. Such a combination is outlined below.

<table>
<thead>
<tr>
<th>Method advocated by Srinivasan:</th>
</tr>
</thead>
<tbody>
<tr>
<td>28 − 5 × 9</td>
</tr>
<tr>
<td>457 − 5 × 38</td>
</tr>
<tr>
<td>3.4 − 5 × 653</td>
</tr>
<tr>
<td>3062 − 5 × 7/8</td>
</tr>
<tr>
<td>6 − 5 × 86003</td>
</tr>
</tbody>
</table>

changing − not × changing differently number changing to the first number

or

\[ x − 5 \times y \]

<table>
<thead>
<tr>
<th>Possible method involving combination of ideas:</th>
</tr>
</thead>
<tbody>
<tr>
<td>28 − 5 × 9</td>
</tr>
<tr>
<td>457 − 5 × 38</td>
</tr>
<tr>
<td>3.4 − 5 × 653</td>
</tr>
<tr>
<td>3062 − 5 × 7/8</td>
</tr>
<tr>
<td>6 − 5 × 86003</td>
</tr>
</tbody>
</table>

changing − not × changing differently number changing to the first number

Red − 5 × Green

R − 5 × G

Or \[ R − 5 \times G \]

Figure 4.11. Pattern generalisation using a combination of historical ideas and Srinivasan’s method.
In terms of understanding a general place value system, similar examples may be employed to prepare students for understanding the variable concept and to help generalise place values in multiple bases (as given below) for a deeper understanding of a general positional notation. In this connection, Mason’s (2004, July) idea of the *structure of attention* is crucial in directing students’ attention to identify critical features of the pattern under consideration.

\[
\begin{array}{ccccccc}
6^3 & 6^2 & 6^1 & 6^0 & 6^{-1} & 6^{-2} & 6^m \\
7^3 & 7^2 & 7^1 & 7^0 & 7^{-1} & 7^{-2} & 7^m \\
25^3 & 25^2 & 25^1 & 25^0 & 25^{-1} & 25^{-2} & 25^m \\
\end{array}
\]

The possible value of helping students to generalise place values in different bases for example, to $p^m$ is related to a better understanding of the structure of numeration through a general positional notation as well as a deeper awareness of the concept and notation of powers of unknowns in algebra. Apart from different names for different unknowns/variables (Puig & Rojano, 2004), the other important tool is powers of unknowns in the development of algebraic symbolism, which is vital to progress in algebra. Hence, powers are a crucial link between arithmetic and algebraic notation and concepts and should be exploited in teaching and learning. It may be that generalising place values could provide students with the opportunity to develop a depth of understanding not only of a concept as fundamental as the positional numeration system but also a deeper awareness of key ideas in algebraic symbolism- different symbols for different unknowns, and powers of unknowns.

Up to now, the focus was on Indian history of mathematics, and more specifically, the development of the decimal numeration system and algebraic symbolism. Indian history was examined for ideas related to the construction of a teaching framework for understanding a general place value system. Figure 4.12 shows the historical development traced in this chapter.
In the next section, numeration systems from different civilisations are considered, again for ideas related to classroom strategies for a better understanding of the current Hindu-Arabic place value numeration system.

### 4.5 Numeration Systems of the World

Numbers and counting are the foundation on which “the whole structure of mathematics has been built” (Flegg, 1983, p. 3) with whole numbers as the basis of all other numbers. All cultures created their own system of counting (for example the *Inca quipu* in South America and the *Yoruba* number system in Africa) in response to the concerns of people; priests, astronomers, astrologers, accountants, and in the last instance mathematicians (Ifrah, 1998; Joseph, 2000). Eventually most of these cultures gradually developed symbols for numbers known as numerals (Menninger, 1969). Over time more symbols (or signs) and more sophisticated systems were developed for using the symbols to denote numbers, and the systems of numeration included rules for using these symbols. While some numeration systems such as the Roman are still in use to a limited extent, virtually all the early systems have given way to the universally accepted Hindu-Arabic numeration system (Flegg, 1983; Ifrah, 1985) since the early systems were not convenient for representing large numbers or for use in calculations. Much can be learnt from these early inventions (Tzanakis & Arcavi, 2000), such as the types of numerals used and the
principles associated with the numerals of a particular system. Hence an overview of some of these notation systems and a discussion of their possible use in the classroom is presented below.

4.5.1 Some Ancient Numerations Systems

A selection of the numeration systems that are considered in this section are the Egyptian (hieroglyphics), the Babylonian, Greek, Roman, Chinese and Mayan systems. Some of the main features of these systems are highlighted and compared with other numeration systems.

**Egyptian system:** The Egyptian hieroglyphics (pictorial) was one of the earliest numeral systems based on a scale of 10 and dates back to about 3000 BCE. It is what is called a *simple grouping or additive system* (Eves, 1969) without place value (similar to the Roman which was also additive) and yet, according to Joseph (2000), was sufficient to serve the economic and technological needs of the time. In such a system, which was mainly recorded on papyrus, symbols are adopted for 1, 10, 100, 1000, and so on and any number is expressed by repeating each symbol the required number of times. As seen in Figure 4.13, the Egyptians used a vertical staff to denote 1, a heel bone for 10 and a coil of rope for 100, and in this way they developed symbols for up to a million. The value of a numeral is given by adding the values of all the symbols represented in the numeral. An example is given in Figure 4.13. As observed by Joseph (2000), the lack of a zero or place-holder in this system did not pose any difficulties and hence the symbols could be written in any order. However, the practice was generally to arrange them from right to left in descending order of magnitude, as seen in Figure 4.13.

**Babylonian system:** The ancient Babylonian numeration system (with base 60) evolved sometime around the middle of the third millennium BCE and one of the outstanding achievements of Babylonian mathematics was the creation of a place-value numeration system (Ifrah, 1998; Joseph, 2000). Interestingly, Babylonians wrote by pressing into clay with a stylus, the result being wedge-shaped (cuneiform) characters (D. E. Smith, 1958) as shown in Figure 4.13. These clay tablets were then dried in the sun and became relatively permanent records. An important feature highlighted by Eves (1969) and Ifrah (1985) is that this sexagesimal (base 60) system had special signs only for 1 and 10, and hence proceeded by additive repetitions of these two signs for all other numbers below 60. In
contrast to the Babylonian system, the Egyptians created newer symbols for denoting powers up to a million. However, Joseph (2000) compares the simplicity of using only two symbols (for one and ten) with the awkwardness of representing a number such as 59, which in Babylonian notation would require fourteen symbols. Another disadvantage, as pointed out by Eves is there was no symbol for zero until after 300 BCE; a space might be left to indicate a missing sexagesimal place, but this resulted in ambiguity (this aspect was discussed earlier in Section 4.3.3). When the zero was introduced in Babylonian numeration, it was not used in the terminal position of a number and hence the system was not an absolute positional system (Ifrah, 1985). Related to zero, another point raised by Rotman (1987) is that signifying zero by an empty space meant that it was not manipulable; which is possibly the reason that zero was not considered as a number in Babylonia.

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<th><strong>Greek system</strong></th>
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**Babylonian numerals**

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<td><img src="image3" alt="Babylonian numerals" /></td>
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**Figure 4.13.** Egyptian (Joseph, 2000, p. 62), Greek (National Council of Teachers of Mathematics, 1969, p. 27) and Babylonian (Ifrah, 1985, pp. 374, 375 & 379) numerals.

**Greek system:** The Greek numeral system with base 10 is a ciphered numeral system (Osborn, Boyd, De Vault, & Houston, 1969) using the additive principle, and can be traced
back to 450 BCE. It consists of twenty-seven symbols; twenty-four from the Greek alphabet and three obsolete symbols: digamma for 6, kappa for 90, and sampi for 900 (National Council of Teachers of Mathematics, 1969). The symbols and their values and some examples are given in Figure 4.13. Although number representations are compact in this system, the main disadvantage was the necessity of memorizing so many symbols. Another issue was the confusion that might be created having the same symbol represent a letter as well as a number as was the case. In this context, Dantzig (1967) comments that the use of all the letters as numerals was an obstacle to the discovery of positional notation.

**Roman system:** The numerals in this system are still in use to a limited extent at the present time, for example in the Film and TV industry for denoting years. This is a simple grouping or additive system (Eves, 1969), similar to the Egyptian, again to base 10 using letters of the alphabet as symbols; I, X, C, M denote 1, 10, 100, 1000 respectively and V, L, D stand for 5, 50, and 500. This system did not have place value but the order of the number symbols was important as sometimes the subtraction principle was applied. Some examples in this system: a) 387 = CCCLXXXVII, b) 279 = CCLXXIX. In the second example, the subtraction principle is applied. 279 was written as above rather than CCLXXXVIII. As in the Egyptian system, a major disadvantage was that many symbols had to be used to express a large number. Furthermore, according to Skemp (1971) and Dehaene (1997), number operations are inconvenient in Roman numeration. Skemp explains that nothing betrays that the number D (500) is ten times greater than the number L (50) and which in turn is ten times greater than the number V (5). However, as Dehaene points out, magnitude relations between 5, 50, 500 and 5000 are completely transparent in the Hindu-Arabic place value numeration, thus simplifying operations. The complexity of multiplication is reduced to memorizing a table of products from 2×2 up to 9×9.
**Chinese system (pictorial)**

Chinese system: This is an instance where a simple grouping scheme developed into what is called as a multiplicative grouping system (Eves, 1969) or a named place-value system. Ifrah (1985) refers to this type of system as a hybrid system. In this early base-10 system (circa 1000 BCE), which was later adopted in Japan, the Chinese employed nine symbols as well as additional symbols for the powers of ten up to 10 000. Numbers were represented as shown in Figure 4.14. Around 200 BCE, the Chinese calculated by means of

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8 The hybrid principle is also evident in the Tamil and Malayalam numeration systems of Southern India (Ifrah, 1985)
sticks or rods alternately laid vertical or horizontal on a table and these arrangements led to the second written form of numeration as can be seen in Figure 4.14. As there was no symbol for zero, a vacant space was left for the absence of a digit which sometimes caused confusion (Ifrah, 1998). Zero in the terminal position had to be deduced from the fact that there are no vertical lines. According to Menninger (1969), the zero was transmitted from India to China around the middle of the 13th century, although Ifrah (1985) suggests that the Indian influence possibly occurred around the 8th century. Ifrah also observes that, similar to the Babylonian and Mayan systems, in the Chinese rod system, the additive principle was used to represent units of the first order. As well, unlike the current Indian system with its single series of signs for numbers 1 to 9, the Chinese system has two different series of nine signs for those numbers.

*Mayan system:* The Mayan culture (300-900 CE) of Central America was highly advanced in the areas of astronomy, calendar, architecture and commerce (Joseph, 2000; National Council of Teachers of Mathematics, 1969). This remarkable civilization is thought to have developed in isolation a numeration system of base 20 with special signs for 1 and 5 and repeated those two signs for numbers below 20. Hence the Maya used a sub-base 5 and the additive principle to represent the units of each order. As seen above, a similar system was used by the Babylonians whose system used symbols for 1 and 10 and they wrote numbers using an additive base ten notation. The Mayan system had place value and a zero symbol; the Mayan zero was a full-fledged digit (unlike the Chinese zero which was an empty space) and it was used both in the medial and the terminal positions in contrast to the Babylonian zero which was only used in the medial positions by the mathematicians. However, due to the Mayan method of time counting, there was an irregularity in the third positional notation as shown in the example in Figure 4.14; the third place value is $18 \times 20 = 360$ rather than $20 \times 20 = 400$. The place values are 1, 20, $18 \times 20$, $18 \times 20^2$, $18 \times 20^3$ and so on instead of 1, 20, $20^2$, $20^3$ etc. In Ifrah’s view, due to this mixed-base system, operations on numbers including zero was not possible. The Mayan zero did not function as an operator, since placing a zero at the end of a numerical expression did not multiply the corresponding number by 20 (Joseph, 2002). As Ifrah (1985) explains, the Mayan system was not designed for the needs of everyday computation but was created for use in astronomy and time counting.
In the context of numerals, Dehaene (1997) points to a “unique innovation” (p. 100) in the Hindu-Arabic notation that was lacking in the other historical place-value systems; the selection of ten arbitrary, distinct digits whose forms are unrelated to one another, and the number they represented. The digits are also unlike the phonetic (Brahmi) alphabets from which they were developed. In this respect, consider two dots for 2 and three dots for 3 in the Mayan system, pictorial symbols in the old Chinese numeration, and the letters of the alphabet that were used for numerals in the Greek system (Eves, 1969). As Dehaene (1997) elaborates, one might think that arbitrary shapes are a disadvantage and a series of strokes (Chinese) or dots (Mayan) appear to provide a more transparent way of representing numbers, and which is easier to learn. This could have been the reasoning behind creation of the digits by the Babylonian, Chinese and Mayan scientists. As can be seen in the above figures, in these three systems, a sub-base was used and the additive principle, using mainly wedges, lines and dots, was employed in representing units (digits) of the first order (at times giving rise to ambiguities that often caused errors (Ifrah, 1985)). However, Dehaene highlights in his book that the human brain takes longer to count five objects than to recognise an arbitrary shape and assign it a meaning. He states:

The peculiar disposition of our perceptual apparatus for quickly retrieving the meaning of an arbitrary shape, which I have dubbed the “comprehension complex”, is admirably exploited in the Indian-Arabic place-value notation. This numeration tool, with its ten easily discernible digits, tightly fits the human visual and cognitive system (Dehaene, 1997, p. 100)

Thus, the nine distinct numerals and zero, drawn in just one stroke of a pen or pencil is one of the characteristics of our numeral system, a view also expressed by Ifrah (1985, 1998).

4.5.2 Reflection and Articulation on Numeration Systems of the World

What are the implications for teaching from the above examples of different numeration systems? First, as indicated by Tzanakis and Arcavi (2000), students can become familiar with ancient numeration systems through their notations. History used in this way can be called as the ‘direct’ approach where history is brought into the open. Through what is known as experiential activities that consist of reviewing notations in the past, students can be given the opportunity to study the rules for writing numerals in the above mentioned ancient number systems and then to translate the numerals into the current Hindu-Arabic
system (and vice versa). As indicated by Ginsburg (1989) and Hughes (1986), such experiences may lead students to appreciate the variety of ways in which numbers were symbolised in several cultures and also importantly, to develop understanding of numerals as representations of numbers. At times, according to Flegg (1983), this aspect is a source of confusion for some students. Numerals are written symbols, whereas the word number refers to the idea that the symbols stand for. Thus, 6 is a symbol that stands for the abstract idea of six (which is also a symbol). In addition, studying other number systems helps students to see how the structure in other systems is built up from different bases (Flegg, 1983) and to relate the number of symbols used within a particular base. And providing opportunities to extend some of the above systems could bring out the creativity in students and further help understand the structure of a system. Additionally, Tzanakis and Arcavi (2000) opine that by comparing and contrasting crucial features such as place value and zero, and by considering the advantages and disadvantages of various systems (M. Hughes, 1986), students can analyse the underlying principles of the decimal positional numeration and its efficiency. Such comparisons involving other historical number systems and their underpinning concepts could lead students to a greater understanding of the current numeration system and the ease it lends for calculations. Tasks such as described above, could help students to realise that the efficiency of the number system comes from the combined use of the concepts of place value, base ten, and the use of nine distinct arbitrary digits and zero and not one of the concepts alone. In addition, as discussed in Chapter 2, historical activities expose the multicultural origins of the subject and reaffirms the human face of mathematics (Ernest, 1998; Joseph, 2000). For our students, the activities may aid the recognition that our numeration system is not the only possible way of representing numbers, and studying the contributions from other cultures may increase their respect for peers. Introducing students to the above systems, may motivate them to investigate further and ask questions such as where did the symbols in a system originate? How did the symbols develop? And why did they evolve in a particular way? These questions serve to highlight the multidisciplinary (Cooper & Tomayko, 2011) aspects of numeration systems. Furthermore, in order to appreciate the structure of numeration, students can also be given the opportunity to create their own numeration system (Krusen, 1999) and thereby engage with the key principles involved in the structure of numeration.
Thus far in this chapter, Indian history was examined for ideas pertinent to the understanding of the place value system and its generalisation such as the naming of large numbers, the construction of the written numerals and zero, and the development of algebraic terminology. Other ancient numeration systems were also briefly reviewed for related ideas on base, place value and the number of symbols and the types of symbols in respective systems. The rest of this chapter outlines the teaching frameworks developed from the historical analysis for implementation in the classroom.

**4.6 Teaching Frameworks based on Historical and Educational Perspectives**

This study is about enhancing Year 9 students’ understanding of a general place value system using historical ideas. In this section the following research question is addressed:

*Based on the analysis of these historical ideas and conceptual stages, examined in light of mathematics education research, can a teaching framework be developed that might enhance Year 9 students’ understanding of a general place value system?*

Following the historical analysis and the discussion in light of mathematics education research in this chapter, given below are teaching frameworks that were developed for implementation in the classroom.

a) Overarching teaching framework (Figure 4.15)

b) Large numbers teaching sequence (Figure 4.16)

c) Generalisation – teaching sequence (Figure 4.17)

d) Compositional structure of a number – teaching sequence (Figure 4.18)
a) The overarching teaching framework developed and implemented in the study.

Figure 4.15. The overarching teaching framework developed and implemented in the study.

b) Teaching sequence for large numbers.

Reading and writing powers of ten. E.g. trillion = $10^{12}$ or $1000000000000$ or $10 \times 10 \times 10 \ldots$.

Reading writing large numbers. E.g. 532 609 418 056 or 532 billion 609 million... or $5 \times 10^{11}$.

Grouping experiences: $4 + 4 + 4 + 4 = 4 \times 5$, $4 \times 4 \times 4 \times 4 = 4^5$, $10 \times 10 \times 10 = 10^3$.

Working with large numbers and increasing quantity sense: How long does it take to count to a million, and to a billion, etc? How many grains of sand in all beaches.

Generating problems with a potential for calculation: If on average, a Year 9 student speaks 15000 words per day, how many words will be spoken by 210 students in 3 years?

Calculator work: Comparing values of $4 \times 6$ and $4^6$, $7 \times 6$, $20 \times 6$ and $20^6$ and so on. Also, evaluating $4^7$, $4^8$, $4^{10}$, $4^{11}$, $4^{29}$... and $4^5$, $5^5$, $9^6$, $15^5$ etc.

Figure 4.16. Teaching sequence for large numbers.
Various uses of literal symbols in algebra

- **Specific unknown(s) as in equation**
  - $3p + 5 = 23$
  - $p^2 - 2p = 3$

- **General/generalised number**
  - (use of colours)
  - *in expressions*: e.g., $2k + 6$, $5p + 4 - 3y$, $m^2$, $6^k$
  - *in equations*: e.g., $p \times 1 = p$, $h \times k = k \times h$

- **Variables in formula or rule**
  - (use of colours)
  - $g = 2r + 5$
  - independent variable
  - dependent variable

- **Parameters or givens in**
  - (colours)
  - $m^2$, $ap + c$
  - $g = cr + w$, $y = ax^2 + bx + c$

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*Figure 4.17. Teaching sequence involving algebraic generalisation.*
Structure of numeration – composition of number

Review grouping reflecting repeated multiplication and its notation
Both decimal and non-decimal bases

Grouping and modeling with sticks as shown in Figure 4.8 the four stages of written numerals thus exposing the implicit multiplicative structure for base-ten numbers such as:

\[4537\]
\[4037\]
\[4537.24\]
\[4537.04\]
\[4037.04\]

Linking of concrete and symbolic representations

Pictorial representations of the numbers above
Linking of pictorial and symbolic representations

Similar process as rows 3 and 4 above in non-decimal bases such as 6, 7 or 12

*Figure 4.18. Teaching sequence for the compositional structure of numbers and generalisation of place values.*
What is given in Figures 4.15 to 4.18, is an overarching framework including four main modules, with related teaching sequences for understanding the numeration system. The modules, which were developed from historical ideas, involve working with large numbers including grouping and notation of powers of ten, experiential activities comparing ancient numeration systems with the current system, and algebraic generalisation using colours in preparation for generalising the place values in multiple bases. The final teaching sequence involves grouping with craft matchsticks reflecting the compositional structure of the place value numerals paralleling historical development.

In addition to the above pedagogical sequences, the following framework (1 to 5) developed by G. A. Jones et al. (1996) and broadened to (6) by N. Thomas (1998) is now further extended to 7.

Framework for instruction and assessment of children’s thinking in multidigit number situations:

1. Pre-place value
2. Initial place value
3. Developing place value
4. Extended place value
5. Essential place value
6. System place value
7. General place value widens the above to very large (and small) numbers, requires understanding of the grouping structure (including grouping reflecting powers with integer exponents) and composition of numbers and their representations (both integers and rational) in multiple bases and, generalising these ideas.

This thesis extends the framework from system place value to the notion of general place value (7).

As discussed in Chapter 3 and the current chapter, key ideas were combined from both history and mathematics education research resulting in the teaching sequences for learning the place value concept. To answer the research question, the teaching sequences that were
developed indicate that, construction of such frameworks from a combination of historical and mathematics education ideas, is achievable.

4.7 Summary and Conclusion

To summarise this chapter, the research analysis of Indian history found that, in a broad sense, large numbers (including powers of ten) were named as number words and used in calculations at a very early stage, around 2000 BCE, and subsequently the written place value system appears to have evolved over many centuries through four stages (additive, multiplicative, places and abstract including zero) in the construction of the decimal place value system. Once the decimal place value system was created, rapid advancement was made in mathematics, setting off a chain of mathematical developments such as rules for operations on zero and negative number, and general methods of solutions of equations. In particular, in the realm of algebra, what is seen is that the development of nomenclature (sign system) played a major part in its overall development. A key finding is that other variables/unknowns were invented and, different names for different variables were a landmark occurrence and this terminology was developed as early as the 7th century. In Indian history, different colours were used for different names of different variables. In addition, in relation to numeration systems of other civilizations, a review of ancient cultures revealed the types of symbols used and the structure of these systems. We may conclude that the above related discussions involving history and teaching, and leading to the pedagogical sequences, show the possibility of using history in the development of teaching and learning frameworks for understanding a general place value system.
CHAPTER 5

TEACHING PROGRAMME METHOD

In this chapter, the method for the teaching phase, the second part of the research study, is described. It specifies the teaching approach taken, the pedagogical sequence employed and describes the related tasks and questionnaires. The teaching framework built on the basis of the historical-epistemological and critical analysis of the decimal place value system, early arithmetic and algebra is also given. This framework was developed in order to attempt to produce in students a sound awareness of the structure of the current Hindu-Arabic place value system, to generalise the system to multiple bases and to understand it as a particular case of a general positional notation.

5.1 Teaching Approach (Methodological Considerations)

The pedagogical orientation of the study was based on sociocultural theory with an emphasis also on constructivist principles, since as pointed out by Becker and Varelas (1993), cognitive development involves both conceptual, and semiotic (related to signs and a key aspect of sociocultural theory) achievements. That is, in order to make sense of the place value structure, students have to integrate their knowledge of the conceptual and, semiotic aspects of the system. Although learning theories such as constructivism are not teaching strategies, the theories inform teaching and lead to a particular teaching approach. Many of the ideas associated with this theory resonated with this researcher and were useful in constructing the teaching framework of the study. These are briefly examined next.

In the learning context, many theories have been developed in the last century (Begg, 2009) (see Figure 5.1) and it is well known that the two leading forces in conceptualising learners’ cognitive development have been Piaget (1970) and Vygotsky (1978). The constructivist perspective, derived from the work of Piaget and extended by Vygotsky and von Glasersfeld (1990) asserts that knowledge cannot be transferred but must be actively constructed by each individual entirely on the basis of their own experience. This construction occurs through the processes of empirical and reflective abstraction. Piaget formulated that in response to new information, existing schemas are accommodated and
new knowledge is *assimilated* by the learner. Following constructivist techniques, and as will be seen in the following section, concrete materials were used in this study to facilitate students’ appreciation of the grouping structure in the place value system. In relation to this, Piaget’s ideas were useful in constructing the didactic sequence; the conceptual aspects of place value were able to be stressed with the use of concrete materials (craft matchsticks) and attempts were made to assist students to *abstract* the multiplicative structure of the system.

From another perspective, sociocultural theories suggest that an individual’s cognition originates in social interactions and according to John-Steiner and Mahn (1996), they “emphasise the interdependence of the social and individual processes in the co-construction of knowledge” (p. 191). Socioculturists, in the tradition of Vygotsky and his activity theory, consider learning communities, where individuals are engaged in discourse, discuss new ideas, make revisions and agree upon the new resulting mathematical knowledge. Some of the main elements of Vygotsky’s (1962, 1978) psychological framework are: a) that development is led by learning; b) the importance of the social origins of individual development; c) that the individual plane is formed by the process of *internalisation*; d) that learning takes place in the *zone of proximal development* (ZPD); e) the importance of the notion of *mediation* by tools or *semiotic mediation* (John-Steiner & Mahn, 1996; Lerman, 1996a). From the pedagogical point of view, all these ideas were valuable in this research, particularly that of semiotic mediation. This is because the numeration system and algebraic symbolism, the main areas of focus in this study are semiotic systems that students encounter in their primary and middle school years.

Other writers such as John-Steiner and Mahn (1996) have identified the key Vygotskian theme of semiotic mediation. “Semiotic mediation is the key to all aspects of knowledge co-construction” (p. 192). That is, human action, on both the social and individual planes, is mediated by tools and signs. In Vygotsky’s approach, the (school) culture provides the student with the cognitive tools for development. These tools could include artefacts as diverse as clay tablets from ancient Mesopotamia or computers in contemporary societies (Radford, 2000).
Figure 5.1. A list of learning theories from Begg (2009).

Also incorporated in this are intellectual artefacts, be they history of mathematics, culture, language, the social context or systems of signs. Vygotsky has argued that these tools are more than aids to learning; they become internalised and are crucial in supporting and transforming mental activities of our cognitive functions. That is, cultural tools are appropriated. Tools that are available to students in their sociocultural setting are internalised to be employed later.

From a didactic perspective, Vygotsky (1978) put forward the notion that a learners’ development is shaped by their social influence. Learners’ social interaction with knowledgeable members of society, such as parents, teachers and other adults is important. In this connection, Vygotsky’s zone of proximal development (ZPD) is possibly his best-known concept and he defines it thus:

The zone of proximal development ... is the distance between the actual development level as determined by independent problem solving and the level of potential development as determined
through problem solving under adult guidance or in collaboration with more capable peers (Vygotsky, 1978, p. 86).

That is, the ZPD is the gap between what the student has already mastered and what he/she could achieve with guidance from more knowledgeable others, such as parents and teachers. And what learners can do with assistance today they can do by themselves tomorrow. Additionally, it is proved that the collaborative nature of ZPD motivates the student to work beyond their potential. In the context of ZPD, in a classroom setting, the teacher is responsible for planning learning sequences, structuring interactions and guiding learners through tasks associated with learning a concept. This form of assistance is often referred to as scaffolding. Vygotsky’s socialculturalist theory allows much more room for an active, involved teacher (than Piaget’s theory) although in various guises; the teacher may be a more informed peer or a parent who has no definite intention to teach. According to Arzarello and Paola (2007), in a classroom setting, the role of the teacher is one of a mediator for the students’ cognitive growth and appropriation of scientific concepts. This role is central and crucial, particularly when the history of mathematics is employed in a pedagogical situation, as is the case in this study. According to Ernest (1994), guiding the learner’s development through the use of cultural tools is an essential activity for the teacher, since only under explicit guidance can the student master, internalise and appropriate mathematical knowledge.

The above discussion has outlined some of the key features of Vygotsky’s sociocultural theory and underlined the importance of the teacher’s task in assisting students in the cognitive process. For this study, sociocultural theory was relevant since, teaching based on such an approach follows naturally from considerations of mathematics, history, language and numerals as part of semiotic systems, which constitute the core of this thesis. First, semiotic systems such as place value numeration and the symbol system in algebra arise out of historical and cultural developments in mathematics (as seen in Chapter 4), are socially entrenched, and provide the underlying structure to the mathematical theories that students learn at school or university (Ernest, 2006). Students have to unpack (Furinghetti & Radford, 2008) the signs, objects, concepts and methods of mathematics that they meet at school. Secondly, mathematics as a discipline can be described as socially constructed. More specifically, all sign use is socially located and is part of social and historical practice. Furthermore, as highlighted by Ernest (2006), the Hindu-Arabic numeration
system, which is the main focus of this study, is the primary semiotic system encountered in school mathematics and foundational to almost all mathematical topics (Ma, 1999) learned later. (Although this research involved 13 year old students, the learners revisited the place value topic). Learning the semiotic system of number and the counting process means that a learner is able to participate in certain sociocultural activities, since, the socially widespread semiotic system of numeration is utilised throughout practical, educational and mathematical applications (Ernest, 2006). Hence, in this study a sociocultural approach was considered advantageous since it goes beyond the traditional psychological concentration on mental structures and functions ‘inside’ an individual (Ernest, 2006) and takes into account personal appropriation of representations/signs by individuals within their social contexts of learning and sign use.

In the constructivist and sociocultural theories just described, there is a gap between the ideas of individual construction formulated in constructivist theory and internalisation of cultural tools as theorised in socioculturalism (Lerman, 1996b). However, as pointed out by Linsell (2005), despite fundamental differences between the two theories, there is much in common in terms of their recommendations for teaching such as building on prior knowledge, use of multiple representations and the importance of social perspectives. As highlighted earlier, intellectual development depends on both conceptual and semiotic achievements (Becker & Varelas, 1993). Hence, in this study on students’ understanding of place value structure, while aspects of Piaget’s (constructivist) ideas about logical forms and deductive reasoning have been retained in the teaching modules, an emphasis was placed on the Vygotskian (sociocultural) perspective on semiotic systems and sign use. This approach was adopted in the teaching part of the study, which is considered next.

5.2 The Case Study

The second part of this research, attempting to answer the general research questions and the related sub-questions (Sections 1.4, 1.5) of the teaching aspect of the research, comprised a case study that involved first naming and working with large numbers and then using concrete materials (craft matchsticks) to model the different stages in the historical development of the Hindu-Arabic numeration system, as well as multiple bases and multiple representations for the place value concept.
5.2.1 Participants

The students who participated in this research were the researcher’s own students in a state high school. This is because this was easier than collecting data from another class in the same school (or different) during school time. The 29 students (13 years-old) were members of a Year 9 class in a multicultural secondary school in Auckland, New Zealand. All the students were representative of a wide range of socio-economic and cultural backgrounds representing Korean, Chinese, Indian, European, Filipino, Maori, Pacific Island and New Zealand European ethnicities. Most students had their Intermediate schooling in New Zealand and hence were reasonably proficient in English. However six students were attending literacy classes and two students were attending an ESOL (English for Speakers of Other Languages) class. A preliminary test given to the participants involving place value concepts and skills (e.g. meaning of zero in 30407, place value of 2 in 8.207, ordering decimals up to 3 decimal places) showed that the structure of the system was initially a problem for many students in the class.

5.3 Teaching Method

In this section a description of the teaching sequence is given. The teaching sequence was designed keeping in mind the New Zealand Curriculum, the focus of the study, students’ prior understanding, historical development and difficulties, students’ misconceptions and other concerns raised in the mathematics education literature, all the while “with an eye on the mathematical horizon” (Ball, 1993). An attempt was made to maintain a balance with regard to the above factors that have been raised thus far pertaining to the study (Anthony & Walshaw, 2007).

5.3.1 Teaching Design

What is presented below is a rough sequence of the teaching method for the study, albeit with considerable overlapping. The frameworks developed (see Figure 4.8 and figures in Section 4.6) were applied in classroom teaching. What is also indicated is where the tests fitted in the sequence.

1. a. Pre-test on numeration system
   b. Pre-test on large numbers
2. **Motivational task:**
   Students to create a numeration system of their own by working in groups

3. **Consideration of large numbers including powers:**
   a. **Large numbers:** reading, writing, naming and calculating with large numbers.
   b. **Grouping experiences reflecting powers:** experiences related to modelling (using matchsticks) of repeated addition, repeated multiplication and their abbreviated notations, both in base 10 and other bases such as 3, 8 and 23.

4. **Experiential activities:** consideration of numeration systems from past civilisations- Egyptian, Babylonian, Greek, Roman and the Mayan.

5. a. Post-test on large numbers
   b. Pre-test on algebra

6. **Compositional structure of number using concrete and semiotic representations:** The teaching sequence as shown in Figure 4.18 was followed for this unit of work. Modeling of numbers such as 4769 and 12046 and 2458.62 using matchsticks – first with whole numbers and then numbers with fractional parts. The concrete (and semiotic) representations to be made first in base ten and then in other bases and to be accompanied by explanations from the teacher and discussion between teacher and students. The structure to be gradually exposed by modelling the number as per the following historical stages (see Figures 4.8, 4.9, 4.10 in Section 4.3.4) and accompanied by discussion:
   i. **Initial stage** – additive stage where the number of multi-units or powers of ten (or another base) are clearly visible and countable as bundles
   ii. **Interim stage** – multiplicative stage where the number of multi-units or powers is indicated as a product
   iii. ‘Places’ stage – separation and vertical placement of the bundles (powers) thus indicating positions
iv. *Abstract stage* – removal of multi-units or powers thus showing the number in the final abstract form. Numbers to be chosen first without zero and then with zero.

7. **Generalisation in algebra**: letter as general/generalised number (and as specific unknown). Generalisation of number patterns using *different colours for different variables* as in Indian history and which was described in Section 4.4.1 (see Figures 4.11 and 4.17). This is in order to lead to a generalisation of place values of any base, \( y^p \).

8. a. Post-test on algebra

b. Post-test on numeration system

The teaching programme took place in Term 1 (of four) in the school year. This researcher, who is also the students’ mathematics teacher in the school (e.g. Price, 2001) (related methodological issues are discussed in Section 8.1), taught all the sessions and these consisted of approximately 26 lessons (each of 1 hour duration) and included the time taken for testing. Only one part of each lesson, about 20 minutes, was utilised for the unit of work concerned with the study and the rest of the lesson was taught according to the curriculum. However, when the lessons involved the use of concrete materials, the whole lesson period (one hour) was taken up for teaching. The teacher explained to the students what was going to be taught and why it was important to their learning; also it was important that everyone in the class cooperated and supported each other in their learning. The teacher encouraged the students to discuss and negotiate meanings, and to help each other. Students were also urged to ask questions for clarification, when aspects of what was taught were not very clear to them, and to persist in their efforts. Attempts were made to establish a climate of cooperation, intellectual patience (Arcavi, 2005) and perseverance throughout the unit of work. Students were informed about when the teacher was available in the classroom during lunchtimes so students could ask questions, seek explanations and use and discuss the concrete materials available (e.g. sticks, base ten blocks, coloured counters for integers, algebra blocks). Throughout the teaching, the mandatory ‘Do now’ activity (usually for about 6-7 minutes) at the start of the lesson involved about 5 questions on baseline knowledge; reinforcing, also building on what students know and what they were about to do in that lesson and in the following lessons.
To assess the effect of the intervention the pre- and post-test on number systems (see Appendix 3) were given to the students before and after the teaching sessions in order to identify differences in students’ conceptual understanding of place value numeration before and after the teaching phase. A pre-test on large numbers (see Appendix 1) was also given to the students that addressed their current knowledge involving the naming and writing of and working with large numbers.

**Task on number system**

The teaching module was based on a number of tasks/activities with the first task asking students to create their own number system (see Figure 5.2). The students formed their own groups and each group was given a large number of coloured sticks (educational craft matchsticks) to aid their thinking; they had to discuss and decide on how the sticks were going to be grouped, representations of numbers, how many symbols were needed and to show an example of addition of numbers. This was a short motivational group task (without teacher intervention) designed to get the students thinking about the need for a numeration system with a specific base and how it might have been developed.

<table>
<thead>
<tr>
<th>Task on Number system</th>
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</thead>
<tbody>
<tr>
<td>Here is your chance to put your creative talents to use!</td>
</tr>
<tr>
<td>Your task is to create a new number system.</td>
</tr>
<tr>
<td>You can work in groups of 3, 4 or 5.</td>
</tr>
<tr>
<td>1. Brainstorm first</td>
</tr>
<tr>
<td>2. Decide how you are going to group the numbers (4’s or 6’s or 13’s etc).</td>
</tr>
<tr>
<td>3. Then discuss as a group how many symbols you need and make up your own symbols and write them down.</td>
</tr>
<tr>
<td>4. Give three or four examples of how you would represent numbers in your new system.</td>
</tr>
<tr>
<td>5. If you have time, show how you would add two numbers in your system.</td>
</tr>
<tr>
<td>6. What were your thoughts when you created this new number system.</td>
</tr>
<tr>
<td>7. Make sure that all your names are on the paper before handing it in.</td>
</tr>
</tbody>
</table>

*Figure 5.2. Questions used for the task on number system.*

In the next three weeks (over approximately 11 lessons) (see Figure 4.15) the teaching involved large numbers as well as a study of numeration systems from other cultures. The unit on large numbers comprised the following and used the framework developed for large numbers (see Figure 4.16).
Naming of large numbers including powers of ten

Large numbers were named, read and written in order to draw attention to the relation between the current system of spoken number words and the named-value system of number words where each multi-unit (place value) is explicitly stated, thus emphasising structure: that each place value on the left increases by a multiple of ten and each place value on the right decreases by a multiple of ten. Number names such as million to billion to trillion... to decillion and their multiple representations were read and written so as to highlight for students the base ten multiplicative structure, and the auxiliary base of thousand-unit chunks (Fuson, 1990a) in the numeration system. Thus the seventh through ninth positions are named one million, ten million and hundred million and the tenth position which is a thousand million is called a billion. The names were written out in words, in the expanded form of \(10 \times 10 \times 10 \times 10 \times 10 \times 10\) and as \(10^6\). Other examples of numbers include numbers such as 7,648,309,105 that was read and written out as 7 billion, 648 million, 309 thousand and 105 and also as \(7 \times 10^9 + 6 \times 10^8 + 4 \times 10^7 + 8 \times 10^6 + 3 \times 10^5 + 0 \times 10^4 + 9 \times 10^3 + 1 \times 10^2 + 0 \times 10^1 + 5 \times 10^0\) (Initially \(10^0\) was written as 1).

Grouping experiences using craft matchsticks

In this study, grouping experiences were thought to be important for students to understand multiplication, exponentiation and the power notation. Grouping was also thought to be an essential part of developing a conceptual referent for understanding the relative sizes of large numbers. This part of the unit was based on Bruner’s three modes of representations of enactive, iconic and symbolic (elaborated in Chapter 3) in an attempt to chart a progression path for learners by moving them from manipulating objects to imagining manipulating objects and then denoting manipulations of objects by symbols, all the while making the links explicit through questions and classroom discussion. The teaching took into account (see Figure 4.16): a) unequal grouping structures- for example, students grouped 5+6+12+23, 9+4+17+9 etc and 3×7, 4×2×3, 3×3×2 and the discussion highlighted the fact that one cannot write these numbers as a single multiple or as a single power b) Equal grouping structures – students grouped craft matchsticks (enactive) in the form, for example 6+6+6+6+6 and depicted this as a picture (iconic) and then wrote the shortened notation 5×6 (symbolic). As well, sticks were grouped in the form 5+5+5+5+5+5 or 6×5.
The commutative law was stressed (which many students were already aware of) and reinforced with more examples. Similarly, students were taught to group for instance 3 lots of 3 lots of 3 lots of 3 sticks and to write the related notation gradually in steps; as $3 \times 3$ ($3^2$), then $3 \times 3 \times 3$ ($3^3$) and finally leading to $3 \times 3 \times 3 \times 3$ and the abbreviated notation of $3^4$. Through grouping experiences, students were also guided to see that $3^4$ is not the same as $4^3$. Other grouping examples included $10 \times 3$ and $10^3$. The above experiences involving grouping were completed in two full lessons of one hour each. An additional experience was with the use of calculators in computation; a common misconception amongst students is to write powers such as $5^4$ as $5 \times 4 = 20$. Hence students were asked to use calculators to draw up a table and compute and compare the values of $5 \times n$ and $5^n$ (and also $k^5$) for various values for $n$ (see Appendix 5). This was done in order to highlight the difference and the meaning of the two notations and, to increase awareness on how quickly the values of powers (repeated multiplication) increased to a very high number compared with multiplication (repeated addition).

**Formative assessment of understanding grouping reflecting repeated multiplication**

During the teaching intervention after two lessons of grouping with the craft sticks (that reflected, for example, $6^3$, $3^4$ and $10^3$), students were asked to work in groups and demonstrate the grouping structure involved in $3 \times 5$ and $3^5$. The purpose of this was to assess how students were learning and to make any needed changes in the instruction. While all the students were able to successfully show $3 \times 5$, despite the teaching, many students struggled to demonstrate $3^5$. The following are students’ responses in the various groups:

1. Group A: The students in this group said that they were showing $5 \times 5 \times 5$ with the sticks but proceeded to present 3 groups of 20 (sets of 5) and when asked if they thought they had got it right, they rearranged to show 5 groups of 5 bundles of 5 sticks. (See Figure 5.3).

2. Group B: This group showed 3 groups of 9 (in sets of 3).

3. Group C: The students in this group demonstrated 3 groups of 25 (sets of 5) but said they had made 3 groups of 15.
4. Group D: This group showed 5 groups of 15 (in sets of 3) and then rearranged to show 3 groups of 27 (sets of 3)

5. Group E: Students demonstrated 5 groups of 27 (sets of 3) with the sticks.

6. Group F: Students showed 3 groups of 15 (sets of 3)

![Figure 5.3. Group A’s attempt at grouping 3^3.](image)

Only one of the groups of students self-corrected after being questioned by the teacher researcher and after discussion within their group. This task and students’ attempts revealed to this teacher/researcher the difficulty of understanding the (nested) grouping structure in repeated multiplication. Subsequently, in the next lesson students were again given the craft sticks and they grouped the sticks that reflected repeated multiplication with the teacher going around to the different groups questioning, discussing and clarifying related issues. Later they were again given the task of demonstrating the grouping structure involved in 5^3, and this time the different groups were successful in doing so.

**Working with large numbers**

In order to promote a (quantity) sense of the enormous magnitudes involved, that is to have a *feel* for the size of large amounts, students were given the opportunity to work with large numbers (see Figure 4.16 for the sequence). Questions such as the following were asked and discussed in class. How long does it take to count to a million if we assume that it takes 2 seconds on average to say each number? And how long does it take to count to a billion if it takes 3 seconds on average to say each number? How far does light travel in a year if the speed of light is approximately 300 000 km/sec? How many grains of sand are there in Muriwai beach if it is given that the length of the beach is 50km, its width is 50m and the depth is 4m? After some discussion it was agreed that 1 cm^3 contains approximately 8000 grains of sand. These questions involved discussion on the relative
sizes of numbers, calculations with time and distance, and also conversions, estimation, area and volume calculations. As well, in order to foster a better conception of decimal scale the film “Powers of Ten” was shown and subsequently there was classroom discussion regarding, size and distance, small and large scales.

**Numeration systems of past civilizations**

Alongside the unit on large numbers, students also investigated numeration systems of past civilizations to see what can be learnt from them, such as their systems of signs. Students worked through different numeration systems from around the world and from different time periods, including Primitive, Egyptian, Babylonian, Roman, Greek and Mayan civilizations (Leimbach & Leimbach, 1990) and finally the current Hindu-Arabic system. The students converted from different numerals to the present system and vice versa in order to compare the ancient systems with the current system. They were encouraged to help each other and to discuss their work when working through the sheets. Parts of the sheets were completed for homework. The teacher went around the class to assist students if and when they needed help. Following the completion of each worksheet, in the next lesson, the students and teacher discussed the symbols in the system, along with general features such as place value and zero, the system’s advantages and disadvantages and then wrote down their observations on the system. Subsequently, after the work on large numbers and numeration systems, students were given the post-test on large numbers. Almost immediately, students were given the algebra pre-test (see Appendix 2).

**Concrete materials and compositional structure of the base 10 system**

The teaching sequence in Figure 4.8 and Figure 4.18 was used for this module. The activity again involved concrete materials (craft matchsticks as used earlier) in order to help students understand the meaning of base 10 place value representational system (and other bases). Students were given sticks and elastic bands to make groups of ten, ten of these to make a hundred, and then a thousand (students even managed one ten thousand) in order to model numbers such as 436, 406 and 12658.23, with parts of the sticks used for numbers involving decimal fractions. One of the reasons that the sticks were useful for this activity was that one-tenth and one-hundredth could be represented by breaking off parts of the sticks and both whole numbers and, thus decimal numbers could be represented without changing the unit. There was discussion on the relative sizes of powers of ten and links
were made between the grouped concrete materials and the written notation. The sticks were used to model the number according to the four historical stages (see Figure 4.8 and Section 4.3.4). First, sticks were modelled as we say the number; that is one lot of ten thousand and two lots of thousand etc. That is \((10 \times 10 \times 10 \times 10) + (10 \times 10 \times 10) + (10 \times 10 \times 10) + (10 \times 10) + (10 \times 10) + (10 \times 10) \ldots\) or 10000+1000+1000+100+100+100...which employs the *additive* principle. This is usually how base-10 blocks are represented for students. Secondly, the same number was modelled by applying the *multiplicative* principle (e.g. \(2 \times 1000\) rather than \(1000+1000\)) which is similar to the historical development evidenced in the *Kharoshti* and *Brahmi* numerals.

And in the third stage of the task, the multiunits or powers of ten were called ‘*places*’ or positions. Hence the single bundle of sticks representing power of ten was moved to a place above the face value digit. That is,

\[
\begin{align*}
1000 \\
2
\end{align*}
\]

instead of \(2 \times 1000\). This was done for all the digits. Again this paralleled development in history when denominations of powers of ten were known as *places*. And in the final stage, the bundle of sticks representing powers of ten (and *place value*) was removed, and students had to imagine the value of the *place*, and saw the numbers in their final abstract form. The teacher went around the class to the different groups and through discussion with the group, and using the sticks that they had bundled, explained and demonstrated the various stages of the evolution of number representation. Subsequently, this was reinforced with iconic representations on the whiteboard of the various stages and these in turn were linked with semiotic (symbol) representations involving whole class discussion.

The cognitive thinking required for these representations is a key step in the construction of the system. There was also discussion on the need for a symbol for zero when we consider a number such as 6008. This was done by first considering 6528 and then comparing it to 6008. During the intervention, different numbers were written up on the board with their place values (including their multiple representations) written on top of each digit leading to a discussion of exponential multiplication and place value. For example, one thousand was also written as \(1000, 10\times10\times10, 10\times100, 10^3, 10^2\times10\), and in words. This was done so that students could not only see 1 thousand as a thousand ones,
but as 10 groups of 10 groups of 10 and also as 10 groups of 100, which has implications for a better grasp of structure and also operations on numbers. Figure 5.3 is an example of what was written up for each of the positions. Through attention to the pattern in power notation, students were able to determine that 10 is $10^1$, 1 is $10^0$ and $1/10$ is $10^{-1}$. The discussion also involved the number of symbols required and why with the use of only ten symbols any number could be written. At this stage, numbers from other systems (e.g. Egyptian and Mayan) were compared and contrasted for better understanding. As well, the number 10 was put up on the board and was re-read as 1 ten and 0 ones (or no singles), rather than ten, which enabled students to see 10 in a new light.

<table>
<thead>
<tr>
<th>$10^2 \times 10$</th>
<th>$10^1$</th>
<th>$10 \times 100$</th>
<th>$10 \times 10 \times 10$</th>
<th>$1000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thousand</td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tbody>
</table>

5 6 4 8 . 7

*Figure 5.4. Multiple representations of place values written up on the board.*

**Other bases**

In the second part of this activity (see Figure 4.18), exactly the same procedure was followed, but this time the students grouped the sticks in sets of 6s, 36s, 216s etc and hence different numbers were represented in base six (see figures in Section 4.3.4). Again this was written on the board in different representations: in words, exponential and full forms. E.g.: 3 lots of 216 ($6^3$), 5 lots of 36 ($6^2$) etc. There was discussion on the word base and how many symbols were needed for a particular base. It was then pointed out to students, that although the symbol 6 does not exist in base six, it was denoted to indicate a bundle of 6 things, such as sticks. Parallels were drawn with base ten to assist understanding. The counting procedure was shown; 1, 2, 3, 4, 5, 10 (6), 11 (7) and so on (Appendix 4). 10 in base six was read as one six and zero ones. The same counting procedure was also followed for base 4 and base 12 in order to assist students to grasp the counting pattern.
Generalisation

Towards the end of the unit on numeration system, students were introduced to generalisation in algebra, in order to help them to generalise the place value structure (see Figure 4.17). As part of the ‘Do Now’ activity at the beginning of the lesson students already had some experience in solving for the unknown number in number sentences. The teacher explained what was going to be taught in algebra and the conversation centred on what algebra was about and why it was central to their learning. The teacher and students also discussed why it was important to have a clear understanding of the various uses of letters. Students had intermittent guidance in the observation of invariants in number patterns put up on the board, including whole class discussion using vocabulary previously presented. The teacher guided students’ attention to similarities and differences in the pattern and allowed students to express the structure of the pattern, thus providing pedagogical (semiotic) mediation. The evolution of signs was encouraged in the classroom from the personal senses that students gave to them (“the number in that column is always 5!”), “the first number is always the same”, “3 is always multiplied by a different number”) towards a shared sense. Through classroom discussion, the teacher guided the students towards the use of vocabulary suggested by Srinivasan such as changing, not changing, changing in the same way and changing in a different way as shown in Figure 5.5. The teacher suggested to the students that perhaps colour could be used for the changing number (and the non-changing/repeating number could be recorded as such). The teacher invited the students to choose a colour and a blob of colour (e.g. Red) was shown underneath the changing number as given in the pattern in Figure 5.5. Subsequently the colour red was written out in words and finally to symbol $R$, all the while discussing the process with students. Students’ attention was guided to invariants and variation in patterns such as $p + 8$, $5 + b - g$, $y + 4 - 2 \times y$. If numbers changed the same way, the same colour/letter was used and different colours were used for numbers that changed differently. Words such as expression, equation, specific unknown number, general number, variable, exponent, power, repeated multiplication, symbol and solving were discussed and clarified.

In a reversal of the process, the teacher also asked students the meaning of expressions such as $7 + Y$, $B - R + 2B$ and asked the students to generate patterns from them. The
meaning and notation of exponential forms such as $4^5, 10^6, m^4, 4^m$ and $r^p$ were presented too. In addition, the differences in the meanings of $p$ in the expression $p + 8$ and in the equation $p + 8 = 12$ were discussed with the students.

<p>| | | | | |</p>
<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>28</td>
<td>5</td>
<td>×</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>457</td>
<td>5</td>
<td>×</td>
<td>38</td>
<td></td>
</tr>
<tr>
<td>3.4</td>
<td>5</td>
<td>×</td>
<td>653</td>
<td></td>
</tr>
<tr>
<td>3062</td>
<td>5</td>
<td>×</td>
<td>7/8</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>×</td>
<td>86003</td>
<td></td>
</tr>
</tbody>
</table>

changing – not × changing differently
number changing to the first number

Red – 5 × Green

R – 5 × G

Or $R = 5 \times G$

*Figure 5.5. Generalisation using colours as in Indian history.*

In this chapter, the method for the teaching part of the study is outlined. What was described included the pedagogical approach taken, and how the teaching models for large numbers, algebraic generalisation, and the structure of the place value numeration system were implemented in the classroom. Students’ responses to the formative assessment involving craft sticks are also described. Following this teaching unit the students were given the post test on algebra, and then the post test on the numeration system. In the next chapter the pre- and post-test results before and after the teaching intervention are considered.
CHAPTER 6

RESULTS

One of the main purposes of the teaching intervention in this study described in the previous chapter is to seek to improve students’ understanding of the structure of the numeration system by using ideas from the history of mathematics. In this chapter the results are outlined and examined. Evidence from the written answers and comments concerning students’ understanding of large numbers and powers, their ability to write the meaning of numbers in multiple bases, and their ability to generalise number patterns and powers/multiunits are analysed.

6.1 Large Numbers

The questions in the large number test (see Appendix 1) involved different ways of writing large numbers, students’ ability to compute with large numbers, their ability to produce questions that use large numbers with a potential for calculations, exponentiation, and the concept of infinity. While all of the above related questions contribute to understanding students’ thinking in regard to the numeration system, in the ensuing analysis, particular attention was paid to students’ answers to questions related to exponentiation. This is due to the specific importance of exponentiation (discussed in previous chapters) in understanding place value structure.

6.1.1 Overall Performance in the Large Numbers Test

When the participants’ answers to the questionnaires were analysed, from the pre-test to the post-test every student except two improved their score leading to a significant improvement. (Mean\(_{pre}\)= 2.85, Mean\(_{post}\)=10.02, \(t=10.46\) and \(p<0.0001\)).

6.1.2 Different Ways of Writing Large Numbers and Evidence of Exponentiation

In the pre-test, all students except one were able to give examples of large numbers. For the question, “what is the largest number that you can say in five seconds”? the answers varied in size and format, with some students writing them as numerals, others in words or number names and some as a mix of words and numerals. Some examples were: S9 (Student 9 – each student is assigned a number in what follows) wrote 42 101 420 500; S14...
gave 900 900 999; S5 wrote 9 999 999 999 999. S5, who by writing 9 into every place, appeared to be seeking to maximise the value of the number. There was also a mixture of words and numerals from some students with S8 giving 100 million, S16 writing 9.9 billion and 11 billion from S26. Other students demonstrated a knowledge of number names, writing fifty thousand billion (S21), nine hundred trillion (S11), googol (S17). S17 also explained that this number is “1 and one hundred zero [sic]”, and googolplex (S15). Interestingly S30 wrote infinity as a word for this question. Although he did not state anything further he explained his idea of infinity in answer to Question 15, which was “what is infinity”? Here he wrote that infinity is “infinite number of an item. E g. Never run out of apples. Infinite = no end”, thereby indicating that for him infinity was more than an extremely large number. With regard to the post-test, the answers showed students’ increasing knowledge of number names; For Q1, S1 and S8 writing 1 nonillion, S5 and S18 writing 1 quadrillion, a million quadrillion written by S25 and 1 decillion written by S6 and S15. This growing awareness of number names and their numerical forms was also reflected in the answers to Q8 which was “Write one quadrillion, five hundred and thirty six million, eight hundred and seven in numerals”. While 4 students were able to provide the correct answer in the pre-test, in the post-test, 10 students were able to do so thereby revealing an understanding of the naming convention and the periodicity involved in naming numbers greater than thousand (see Figure 6.1 for S21 and S4’s answers. The first answer is from S21 and the second from S4. The answers are in order from the top in this figure and in what follows in this chapter).

Figure 6.1. S21 and S4’s answers to Q8.
Evidence of exponentiation was looked for in both pre-test and post-test results. Since this study sought to increase students’ understanding of a general place value system through historical ideas, an awareness of the meaning and notation of powers is crucial, that was a key concept in Indian history. In the event, for this first question, no student answered in terms of powers in the pre-test. On the other hand, in the post-test, although not specifically asked to do so, 11 out of 26 students gave their answer in an exponential form. The following are some examples: S4 wrote $99^{99}$, $50^{150}$ was written by S20, and $368^{50}$ was written by S8. Some students wrote in terms of a much larger base and exponential numbers. Examples: $50250640^{100}$ (S23), $480910^{80}$ (S24), $1000000^{1000}$ (S14) who also wrote it in words (see Figure 6.2 for S21 and S3’s answers). Some other students gave their answers in powers with a mix of number names and numerals, as the following show: 4 billion$^{700}$ was written by S9, and S7 wrote “900 sextillion to the power of 1 sextillion”. These examples show that although it is unlikely that the students understood the enormous size of the numbers that they had created, they appeared to be aware that writing the number in the exponential form can rapidly increase the value of the number.

The next question that involved exponentiation (although this was not explicitly stated in the question) was Question 4: Write one trillion in numerals in two different ways. In the pre-test, 15 out of 26 students gave the answer in the correct numeral form: that is $1\ 000\ 000\ 000$ $000$. However, all students except one gave the answer only in one way. S13 was the only student who gave the answer as a power but then made an error in writing 1 followed by fifteen zeros. In the post-test, however, it is noteworthy that 17 students gave the correct answer in two different ways including the power form; $1000\ 000\ 000\ 000$ and $10^{12}$. Of these students, three (S13, S16 and S18) further elaborated that $10^{12} = 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$. Of these students, three (S13, S16 and S18) further elaborated that $10^{12} = 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$ (see Figure 6.2 for S18 mad S16’s answers). Interestingly, two students, S5 and S22, while giving the correct power form of one trillion as $10^{12}$, made an error in writing the number of zeros after the one. Both S12 and S21 did not answer correctly; still, they made an attempt to write one trillion in exponential form; S21 wrote $10^{13}$ and $10 \ 000\ 000\ 000\ 000$ and S12 gave the answer as $10^{15}$ and $1\ 000\ 000\ 000\ 000\ 000$. These students were not able to associate one trillion with $10^{12}$. 197
6.1.3 Calculating with Large Numbers and Producing Questions Involving Large Numbers

As discussed previously, an important aspect of understanding large numbers is developing a sense of the quantities that they represent. That is, in order to grasp the enormity of a large number, both number and quantity sense are used with links to past experiences. In this connection, Qs 5, 6, and 7 concerned the length of time it would take to count to a million, a billion and one trillion respectively. The students were told to assume that on average, it would take 2 seconds to count each number in a million, 3 seconds in a billion and 5 seconds in one trillion. In the pre-test, almost all students were unsuccessful (one student experienced partial success) but in the post-test many students attempted and some
provided the method involving multiple operations and then the correct answers to the questions. S9 and S22’s answers are given in Figure 6.3.

It is one thing to be able to calculate using large numbers given the problem/question as in Qs 5, 6 and 7 but it is quite another matter to generate them and Q16 involved this process. In Q16i, an example of a large number from Indian mathematics history (more specifically, Jaina mathematics) was given (see Appendix 1) and students were asked to compute this number. In the second part of the question: “Can you think up a similar example of a large number, but which is more relevant to the present?” students were asked to produce a similar question but were not asked to calculate. Among the students who responded S10 gave this answer: “If I were to speak 172800 words a week, how many words would I speak in 2 years?” and S9 gave a version of a question involving how far light travels in 20 years if its speed is 300 000 km/sec. Another response from S21 was: “I play the computer for 5 hours per day. How many months do I play in 100 years?”. These questions gave evidence of the ability to see a problem with a multiplicative solution of more than two steps.

6.1.4 Understanding Exponentiation

The next five questions that were analysed all involve students’ understanding of exponentiation as repeated multiplication, and its generalisation, which was a primary focus of this study. The question facilities and the results in the pre- and post-tests are given in Table 6.1.

<table>
<thead>
<tr>
<th>Question</th>
<th>% correct in Pre-test</th>
<th>% correct in Post-test</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(N=26)</td>
<td>(N=26)</td>
</tr>
<tr>
<td>Q9</td>
<td>What does 7(^11) mean to you?</td>
<td>23.1 (6)</td>
</tr>
<tr>
<td>Q11</td>
<td>Write what you understand by 2(^96)</td>
<td>19.2 (5)</td>
</tr>
<tr>
<td>Q12</td>
<td>What does (a^7) mean to you?</td>
<td>26.9 (7)</td>
</tr>
<tr>
<td>Q13</td>
<td>What is the meaning of 2(^y)?</td>
<td>3.8 (1)</td>
</tr>
<tr>
<td>Q14</td>
<td>Write what you understand by (a^y)?</td>
<td>3.8 (1)</td>
</tr>
</tbody>
</table>
Understanding exponentiation as repeated multiplication is a crucial step on the way to number system construction and there was some evidence of numbers written in the expanded form. For Question 9 in the pre-test, 6 out of the 26 students wrote the correct answer. Only 3 of these students were able to transfer their understanding of repeated multiplication for the higher powers (Question 11). 50% of the students made the classic error, writing $7 \times 11 = 77$ or a similar answer, while two students did not provide an answer. However, in the post test, 21 out of the 26 students were successful in giving the correct answer; $7 \times 7 \times 7 \times \ldots \times 7$ (11 times) (see Figure 6.4). Of these 21 successful students, 19 were able to extend their understanding to higher powers. In terms of errors, it is noteworthy that no student made the above mentioned mistake of writing $7 \times 11$ for $7^{11}$ for any of the questions.
Questions 12 to 14 sought to examine whether students could extend the idea of repeated multiplication to the general form. That only one student (S17) was able to get Questions 13 and 14 correct in the pre-test shows that this was difficult. Many students did not attempt these two questions in the pre-test. What is particularly interesting to note is that for some students, writing the expanded form for large numbers was harder than making generalisations. For Question 12, seven students were able to give the expanded form $a \times a \times a \times a \times a$, however, only 5 were able to write that $2^{96}$ is $2 \times 2 \times 2 \ldots \times (96 \text{ times})$ (see Figure 6.4). As seen in Table 6.1, questions involving powers such as $2^{96}$, $2^y$ and $a^y$ appear to have been problematic for students in the pre-test, possibly because the exponents, rather than the base either involve a large number or a literal symbol. Comparing these results with the results from the post-test, students’ performance improved in these questions (see Table 6.1). More specifically, in both the pre- and post-tests, students performed better in Question 12 (involving $a^7$) than in Question 13 (involving $2^7$). Given that a main focus of this thesis is on the grouping structure and
notation of powers such as $10^4$ (and their generalisation) it is of note that many students were able to experience some success in the questions on exponentiation. However, some interesting errors did surface and this is considered next.

‘Commutative’ property error: During the intervention one of the classroom tasks involved grouping (using sticks) and evaluation of terms such as $a \times m$ and $a^m$. A table was drawn up, with $a \times m$ in one column and $a^m$ in another column (see Section 5.3.1). Using a calculator, students compared values, for example, of $3 \times 4$ and $3^4$, and $3 \times 6$ and $3^6$ (see Appendix 5). Similar comparisons were made with different base values and a discussion ensued to alleviate the classic error of thinking $a^m$ as equal to $a \times m$. This appears to have been effective since no student made this error in the post-test.

![Image](image.png)

*Figure 6.5. S1 and S26’s comments on grouping work with craft sticks.*

Also, what was also explicitly highlighted during intervention with numerical examples was that while $a \times m$ and $m \times a$ are equal, $a^m$ and $m^a$ were not the same. Despite the teaching effort, two students appeared to think otherwise. S16’s answer to Question 9 in
the post-test was to write $7 \times 7 \times 7 \ldots 11$ times and that this is “11 to the power of 7”. Similarly, in answering Question 11, S21 wrote “96×96 or 2×2×2×\ldots96$ times”. It seems that some students need more experience in grouping, evaluating powers and, comparing and contrasting the answers. Many students did however comment (e.g. see Figure 6.5) that they found the experience useful in understanding grouping in powers reflecting repeated multiplication and to realise the enormity of the quantities involved. S13 (an ESOL student) “Sticks helped me to understand powers more easy”. S21: “The sticks helped me to understand powers like $3^4$ or $4^3$. It also helped me to understand how high numbers can get”.

In the next section, the algebraic generalisation results from the pre- and post-tests are analysed.

6.2 Generalisation in Algebra

The unit of work on generalisation in algebra was developed in order to help students generalise the place value structure. Prior to the teaching of this unit, a pre-test was administered and upon completion of the unit of work a post-test was given. In order to understand their thinking with regard to generalisation, students were also asked to write their comments on the questions in the test and their views on generalisation. The pre-test and post-test results are outlined below.

6.2.1 Overall Performance in the Generalisation Test

When the answers to the generalisation questionnaires were analysed, every student improved their score from the pre-test to the post-test and hence there was also an improvement on the mean score on the test (Mean (pre-test) = 0.89 and Mean (post-test) = 9.93, $t=14.19$, and $p<0.0001$). Students found most of the questions difficult in the pre-test and many of the questions were left unanswered. However, in the post-test, most of the students attempted almost all of the questions.

6.2.2 Algebraic Notation for Denoting Variation and Invariance

The pre- and post-test questions in algebra involved students’ understanding of the generalised/general number. Questions 1, 2, 3, 4, 7 and 8 (Q7 was adapted from Küchemann’s (1981) study and Qs 1, 4 and 8 were adapted from Srinivasan (1989)) were
designed to see whether students could recognise quantities that are changing (i.e. variation) and could distinguish them from those that are invariant. While Qs 1b and 4a involved different changing quantities, Qs 8a and 8b consisted of numbers that changed the same way, and all were expressions. Q4b involved an equation. In most of the questions in the tests, the numerical expressions were set out one underneath the other (see Appendix 2) in order to direct students’ attention to variation and invariants in the pattern, and ability to generalise. However, as is usual, place values in a particular base are set out in horizontal fashion, and hence Question 2 was designed to see if students could generalise numerical patterns that are set out horizontally. Questions 5 and 6 tested students’ conceptual understanding of specific unknown and general number and the ability to distinguish between the two, in equation and expression contexts. In the event students’ answers in the generalisation questionnaire were categorised in the following way:

- Not attempted: Student did not attempt the question or wrote “I don’t know”, “this is hard” or something similar. Awarded a score of 0.

- Attempted but showed no understanding of the structure in a pattern: Student attempted the question however, the response was unrelated to the question and instead involved activities such as calculating when it was not required. Awarded a score of 0.

- Attempted and recognised variation: Some student responses showed that they were acquiring ideas on generalisation that were yet to be fully formed. In particular variation was often recognised but students were unable to symbolise it correctly. Some examples of this developing understanding were: S16 wrote for Q8b: “not changing ×changing +not changing”. Similarly, in response to Q6a S23 wrote “3 times to any high number”. S30 answered Q5i) as “if its p=1 then it is 1+5”. These answers showed that students were gradually building their ideas about the general/generalised number. Awarded a score of 0.5.

- Able to symbolise variation: Some students were successful in symbolising variation, thus demonstrating an understanding of general number and even of the variable: The following are some examples of symbolisation: In response to Q3, S1 wrote “ 4×yellow, not changing ×changing” thus using a colour (as a word) to
represent a changing number. When answering Q2, S3 wrote $A^3$, choosing not to use a colour. And S11 wrote: 8+ changing number

8+Blue

8+B

for Q1 and S6’s answer to Q1b demonstrated a deep understanding. He wrote: “$3 \times y - p^2$”. Awarded a score of 1.

Students’ responses to these questions were first analysed quantitatively. Table 6.2 and Table 6.3 give the numbers and percentages of the 28 students who were able to recognise and symbolise changing quantities for the pre- and post-test questions.

Table 6.2. Facilities for Questions on Recognition of Variation and Invariance (Pre-test)

<table>
<thead>
<tr>
<th>Pre-test Questions</th>
<th>1a</th>
<th>1b</th>
<th>2</th>
<th>3</th>
<th>4a</th>
<th>4b</th>
<th>7</th>
<th>8a</th>
<th>8b</th>
</tr>
</thead>
<tbody>
<tr>
<td>Not attempted (N=28)</td>
<td>11</td>
<td>17</td>
<td>17</td>
<td>15</td>
<td>18</td>
<td>21</td>
<td>16</td>
<td>25</td>
<td>23</td>
</tr>
<tr>
<td>%</td>
<td>39.3</td>
<td>60.7</td>
<td>60.7</td>
<td>53.6</td>
<td>64.2</td>
<td>75</td>
<td>57.1</td>
<td>89.3</td>
<td>82.1</td>
</tr>
<tr>
<td>Attempted but no understanding</td>
<td>14</td>
<td>10</td>
<td>10</td>
<td>11</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>%</td>
<td>50</td>
<td>35.7</td>
<td>35.7</td>
<td>39.3</td>
<td>21.4</td>
<td>17.9</td>
<td>14.3</td>
<td>10.7</td>
<td>14.3</td>
</tr>
<tr>
<td>Recognising variation</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>%</td>
<td>10.7</td>
<td>3.6</td>
<td>3.6</td>
<td>7.1</td>
<td>14.3</td>
<td>7.1</td>
<td>7.1</td>
<td>0</td>
<td>3.6</td>
</tr>
<tr>
<td>Symbolising variation</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>%</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>21.4</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

In the pre-test, only around 9% of the students demonstrated the ability to recognise variation (partial understanding) or symbolise it using symbolic literals. In contrast to this, in the post-test, 80.2% of the students displayed either partial or full understanding of varying quantities. 71% of the students were not only able to distinguish between variation and invariance but also could symbolise the variation using a letter. This is especially noteworthy since, in the pre-test, apart from 6 students (21.4%) who answered Q7, no student was able to symbolise changing quantities for any of the other questions. This could perhaps be due to their familiarity with this type of question. For Q2, numbers were written in words and in a horizontal fashion and this could possibly explain that only
46.4% of the students were successful in symbolising the changing number. However, for the same question another 25% of the students were able to recognise variation but without symbolising it.

Table 6.3. Facilities for Questions on Recognition of Variation and Invariance (Post-test)

<table>
<thead>
<tr>
<th>Post-test Questions</th>
<th>1a</th>
<th>1b</th>
<th>2</th>
<th>3</th>
<th>4a</th>
<th>4b</th>
<th>7</th>
<th>8a</th>
<th>8b</th>
</tr>
</thead>
<tbody>
<tr>
<td>Not attempted (N=28)</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>%</td>
<td>0</td>
<td>3.6</td>
<td>14.3</td>
<td>14.3</td>
<td>14.3</td>
<td>17.9</td>
<td>14.3</td>
<td>14.3</td>
<td>14.3</td>
</tr>
<tr>
<td>Attempted but no understanding</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>6</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>%</td>
<td>10.7</td>
<td>7.1</td>
<td>14.3</td>
<td>3.6</td>
<td>3.6</td>
<td>3.6</td>
<td>21.4</td>
<td>3.6</td>
<td>3.6</td>
</tr>
<tr>
<td>Recognising variation</td>
<td>3</td>
<td>3</td>
<td>7</td>
<td>6</td>
<td>6</td>
<td>3</td>
<td>3</td>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>%</td>
<td>10.7</td>
<td>10.7</td>
<td>25</td>
<td>21.4</td>
<td>21.4</td>
<td>10.7</td>
<td>10.7</td>
<td>28.6</td>
<td>10.7</td>
</tr>
<tr>
<td>Symbolising variation</td>
<td>22</td>
<td>22</td>
<td>13</td>
<td>17</td>
<td>17</td>
<td>19</td>
<td>15</td>
<td>15</td>
<td>20</td>
</tr>
<tr>
<td>%</td>
<td>78.6</td>
<td>78.6</td>
<td>46.4</td>
<td>60.7</td>
<td>60.7</td>
<td>67.9</td>
<td>53.6</td>
<td>53.6</td>
<td>71.4</td>
</tr>
</tbody>
</table>

**Qualitative Analysis of Student Performance in the Questions**

For Qs 1a and 1b in the pre-test, all the students except one, found the question difficult and either answered wrongly or did not attempt to answer. The students' tendency to compute was evident in the answers. Nine students, S5, S9, S15, S16, S18, S21, S23, S24 and S26 calculated across each line of the pattern and some of them totalled these values to give one final answer. For example, S8 continued the pattern numerically in Q4a, and S11 turned the expression $G - 2 + B$ in Q5ii into an equation; He wrote $G - 2 + B = 2$, S17 commented when attempting to answer Q1b: “I can’t get the solving method”. Students appear to find it difficult to hold an abstraction and seem to have a compulsion to calculate and solve equations.
Figure 6.6. S26’s answers to Qs 1a and 1b.

Figure 6.7. S6’s answers to Qs 1a and 1b.

However, in the post-test, all except three students (S4, S10 and S15) were successful (partially or fully) in answering Qs 1a and 1b. These students were able to observe what was changing and what was not changing and then use a colour to denote a changing number. Students were able to distinguish variation from invariance and to symbolise the variation using a symbolic literal $r$. In Figure 6.6 we see that for Q1b which involved a power in the numerical pattern, S26 has written one colour (and letter) for a changing number and a different colour (letter) for a number that is changing differently, and writing $3 \times p$ for the generalisation in Q1b. However, some students (S6, S16, S21, S23) were more specific as seen in S6’s work (see Figure 6.7), where he wrote the generalisation for Q1b as $3 \times y - p^2$ and stated explicitly that the exponent is not changing.
Both Questions 2 and 3 also involved generalisation but were set out horizontally. While the horizontal structure of the pattern did not generally pose a problem, in the post-test some students were maybe able to ‘see’ the pattern better when it was set out in a vertical manner (which was mainly done during the intervention) and perhaps for this reason re-wrote the question/pattern (See Figure 6.9).

For both questions, S5, S16 and S29 observed what was varying and what was invariant and did not write anything further, while others went on to represent the changing number with a colour and then literal symbol. Still some other students went on to writing a symbolic literal without first writing as a colour. It appears that students need more guidance and practice with horizontal structure of patterns.

Figure 6.8. S26’s facilities with Qs 4a and 4b.

Figure 6.9. S3’s answer to Q3, and S17’s answers to Qs 2 and 3.
Question 4 was similar in construction to Q1 except that Q4b involved the equals sign and each line in the pattern was given as an arithmetic equality. On the whole, students performed well in this question in the post test and wrote a correct generalisation, for example, \(5 \times P - 2 = B\) (S26) (Figure 6.8). However, S17 went on to simplify it by removing the multiplication sign, which perhaps enabled him to ‘see’ the structure better, and he wrote, \(5R - 2 = G\).

Although Q8 was structured similarly to Q1 and Q4, albeit with different changing quantities, it proved difficult for some students. While they were able to observe what was changing and what was not, only a few were able to notice that the second general number in Q8a was changing in the same way as the first general number. Instead of using the same letter/colour, many students represented the second changing number with a different letter. However, for this question (Q8) students S6, S8, S11, S13, S17, S18, S21, S22, S23, S24, S25, S26, S27, S28 and S29 were all able to write the correct expression in a form such as \(9 \times p + 8 + 7 \times p\) and S13, S17 and S22 were able to simplify it to \(9p + 8 + 7p\), with S17 and S22 even able to extend this to remove the explicit multiplication signs, add the like terms and write \(16p + 8\). Furthermore, S17, the only student to do so, denoted the changing number in Q8a and Q8b with the same letter \(y\) and simplified both expressions correctly to \(16y + 8\). And in response to Q8c, S17 commented: “… we can add or subtract the same types of numbers!” (see Figure 6.10).

As can be seen in Tables 6.2 and 6.3 which summarises the results on generalisation, while six students provided an answer to Q7 similar to “p+3” in the pre-test, 15 students (e.g. S2, S4, S5, S6) were successful in the post-test. S9’s response showed his ideas were still developing; he wrote “what he had plus 3”. He was still not able to symbolise what he was able to say in words. However, the abstraction was a challenge for some students (e.g. S16) who assigned a numerical value (3) for the number of sweets and responded with the answer 6. The difficulty of symbolising with a letter is clearly evident in S12’s answer; “? Im [sic] not sure because it doesn’t say how much sweets he has it only says a certain number”. 
Questions 5i and 5ii, tested students’ ability to reverse the process, that is, given symbolic generalisations students had to explain the meaning of the literal symbols. Many of the responses in 5i showed that students had the idea of general number although some expressed this generality by giving possible values (and some only gave possible values); “it could be any number...5+5, 0.5+5, 20+5, ...”(S17), “it’s an expression. P is a general number which means that any number can be the value of P” (S22), “11+5, 8+5, 20+5, 35+5” (S25). For Q5ii, student responses demonstrated they understood that the generalised numbers G and B can vary differently, writing: “G and B are general numbers” (S3), “G and B are unspecific unknowns and their value could be anything” (S4), “G and B are different changing numbers” (S21), “G and B can be any number but sometimes it can be the same number” (S24). However, this question with two variables did lead S14 to the wrong conclusion that G and B cannot take the same value: “G= changing number. B is also a changing number but changes differently from G. B and G can be any number just not the same”.

Questions 6a and 6b were examples addressing whether students could make a distinction between generalised number and specific unknown in expression and equation contexts respectively. Many of the responses showed that students had the idea of the general number and were also able to distinguish between a general number and a specific
unknown. Additionally, student answers also demonstrated that they had a sense of the difference between an expression and equation by stating the nature of the variable in both forms. S6 wrote “$3 \times y$ is an expression, $y$ can take any number. $3 \times y = 12$ is an equation, the $y$ is a specific number there is only 1 possible number to hold, $y=4$”. And S27 commented: “i $(3 \times y)$ is an expression and ii $(3 \times y = 12)$ is an equation. $y$ in the first one can be any number but $y$ in the second is a specific unknown number”. And S11 wrote; “$3 \times y$ is a generalisation, $y$ is a general number, it can take any number. $3 \times y = 12$ is a equation. $y$ is a specific number”.

**Students’ written comments on generalisation**

Students’ notions on algebraic generalisation before and after the units of work on the topic and place value structure are revealing, both in terms of their attitude as well as their understanding. It appears that this idea was not easy, possibly because the students were unfamiliar with the concept.

Students’ comments in the pre-test:

S3: “I don’t get anything. I don’t understand generalisation”.

S7: I found it really hard because I don’t understand the question. The whole thing about it. I don’t know what generalisation means”.

S9: “I am lost I don’t get this”.

S24: “I didn’t understand the word generalisation and general”.

S12 mentioned this test was hard five times in the paper. For students whose first language is not English the difficulty was compounded.

S17 commented: “…and I am not strong in English so I don’t know”.

In contrast to the difficulties described in the above comments, students’ comments in the post-test were more positive and reveal their views on generalisation. Some of their comments are given below and in Figure 6.11.

S11: “…its easy now. We can use it to solving [sic] some problems”

S6: “When I first saw generalisation I didn’t know a thing but now I understand generalisation”.

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S2: “I understand that when the numbers that change is turned into a colour or any letter”.

S7: “I also understand the concept of generalisation is the number that is changing...to group a changing number with a letter or symbol”.

S13: “generalisation is when you find some numbers changing like $3^1$, $3^2$, you first chose [sic] any alphabet and put it as $3^b$ so you know those numbers are changing”.

Figure 6.11. S4’s and S27’s comments on generalisation.

### 6.3 Place Value Structure Of The Decimal Numeration System And Its Generalisation

This section presents the results associated with student facility with questions on both base-ten and non-decimal numeration and generalisation of place values. The questions in the pre- and post-test on numeration (see Appendix 3) involved: i) converting base-ten numbers given in words to numerals, ii) writing numerals in expanded notation or in words, iii) the number of symbols used in a system, iv) writing the expanded notation of
numerals in different bases, and v) generalising the place value structure. The pre-test was given at the start of the year. After a term’s work that consisted of groupings in multiplication and exponentiation, tasks on large numbers, consideration of historical numeration systems from other cultures, work on generalisation, and tasks on groupings in numbers in multiple bases and writing their expanded notation, the post-test was administered at the end of Term 1.

The following aspects are examined in this section: i) Students’ overall performance in the tests and ii) their thinking on exponentiation which, as discussed in the preceding chapters is a key aspect contributing to a deep understanding of place value structure. Also considered are students’ views on the whole unit of work particularly on grouping with concrete materials (coloured matchsticks) in base ten and other bases.

6.3.1 Overall Performance in the Numeration System Tests

The answers to the questions in the tests were categorised in the following way:

*Not attempted (NA):* Students either did not attempt the question or wrote “I don’t understand” or equivalent. Awarded a score of 0.

*Attempted but showed no understanding of structure (ANU):* Student attempted the question, however, they either provided the wrong answer or wrote something unrelated to the question. Two examples in this category would be: When asked to write 100005 in words or their meaning, S15 wrote “One million and five” and when asked to write place values for numbers with base 29, S25 responded with “Th H Te O. O”. Awarded a score of 0.

*Partial (developing) understanding of structure (DU):* Some student responses showed that students were still in the process of forming ideas about place value structure. Sometimes they made an error in writing the answer and at other times the answers appeared incomplete. For instance, for Q9c, students were asked to write the meaning of the number 324.15 (base six) or to write it in words. In addition to writing the numeral in expanded notation, S11’s response was: “three thirty six, two six, four, point one upon six, five one upon hundred”. Similarly for Q9a, which was to write the meaning of 4523 (base six), S20 wrote $6^3, 6^2, 6^1$, and $6^0$ on top of the digits. While this is correct, this is only part of the answer. Awarded a score of 0.5.
Good understanding of structure (GU): Students’ answers demonstrated a good understanding of the structure of the place value system. Some examples: S13’s answer to Q1a which was to write the meaning of 7905; He wrote “$7 \times 10^3 + 9 \times 10^2 + 0 \times 10^1 + 5 \times 10^0$”. For Q8 which was on the number of symbols required for a number system with base forty-three, S14 wrote 43. And S22’s response to Q10a was: $4 \times 7^3 + 6 \times 7^2 + 3 \times 7^1 + 2 \times 7^0$. Awarded a score of 1.

The tables in the next two pages show a quantitative analysis of students’ performance in the pre-test and post-test.
Table 6.4. Facilities for Questions related to the Structure of Numeration (Pre-test)

<table>
<thead>
<tr>
<th>Pre Test Qs</th>
<th>1a</th>
<th>1b</th>
<th>2a</th>
<th>2b</th>
<th>3a</th>
<th>3b</th>
<th>3c</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9a</th>
<th>9b</th>
<th>9c</th>
<th>10a</th>
<th>10b</th>
<th>10c</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N=25)</td>
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<td>8</td>
<td>0</td>
<td>7</td>
<td>12</td>
<td>8</td>
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<td>0</td>
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Table 6.5. Facilities for Questions related to the Structure of Numeration (Post-test)

<table>
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<tr>
<th>Post Test Qs</th>
<th>1a</th>
<th>1b</th>
<th>2a</th>
<th>2b</th>
<th>3a</th>
<th>3b</th>
<th>3c</th>
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<th>5</th>
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<th>7</th>
<th>8</th>
<th>9a</th>
<th>9b</th>
<th>9c</th>
<th>10a</th>
<th>10b</th>
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<td>12</td>
<td>13</td>
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<td>4</td>
<td>12</td>
<td>16</td>
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<td>48</td>
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### Table 6.4. Continued

<table>
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<th>11b</th>
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<th>12b</th>
<th>13a</th>
<th>13b</th>
<th>14a</th>
<th>14b</th>
<th>14c</th>
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<td>NA or ANU</td>
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<td>25</td>
<td>25</td>
<td>25</td>
<td>25</td>
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<td>%</td>
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<td>100</td>
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<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
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<td>0</td>
<td>0</td>
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<td>0</td>
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<td>%</td>
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<td>0</td>
<td>0</td>
<td>0</td>
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### Table 6.5. Continued

<table>
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<th>Post-Test Qs (N=25)</th>
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<th>11b</th>
<th>12a</th>
<th>12b</th>
<th>13a</th>
<th>13b</th>
<th>14a</th>
<th>14b</th>
<th>14c</th>
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</thead>
<tbody>
<tr>
<td>NA or ANU</td>
<td>6</td>
<td>8</td>
<td>5</td>
<td>8</td>
<td>7</td>
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<td>16</td>
<td>15</td>
</tr>
<tr>
<td>%</td>
<td>24</td>
<td>32</td>
<td>20</td>
<td>32</td>
<td>28</td>
<td>32</td>
<td>44</td>
<td>64</td>
<td>60</td>
</tr>
<tr>
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<td>1</td>
<td>1</td>
<td>2</td>
<td>7</td>
<td>7</td>
<td>10</td>
</tr>
<tr>
<td>%</td>
<td>4</td>
<td>4</td>
<td>8</td>
<td>4</td>
<td>4</td>
<td>8</td>
<td>28</td>
<td>28</td>
<td>40</td>
</tr>
<tr>
<td>GU</td>
<td>18</td>
<td>16</td>
<td>18</td>
<td>16</td>
<td>17</td>
<td>15</td>
<td>7</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>%</td>
<td>72</td>
<td>64</td>
<td>72</td>
<td>64</td>
<td>68</td>
<td>60</td>
<td>28</td>
<td>8</td>
<td>0</td>
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</tbody>
</table>
All the students improved their score from the pre-test to the post-test (Mean\textsubscript{pre} = 4.8, Mean\textsubscript{post} = 18.34, t = 10.87, and p < 0.0001). There was improvement on almost every question on the tests (except Q1a) but especially on Qs 5, 6, 7 and 8 (from 0, 1, 0, 0 students correct in the pre-test to scores of 25, 17, 19, 21 in the post-test). Questions 7 and 8 asked how many symbols are needed for bases 6 and 43, and this generalisation was clearly better understood after the module of work on grouping and bases. Questions 1, 2, 3, 4, 5, 6 and 11 involved the decimal numeration system and Qs 3 and 11 were designed to see if students could write the place values in base ten. In the pre-test, for Qs 3a, 3b and 3c, 52%, 64% and 20% respectively were able to (partially or fully) write the place values. While 3a and 3b were on whole numbers, 3c involved a decimal fraction. However, in contrast to this, for the respective questions in the post-test, 84%, 84% and 72% of the students provided the correct place values. These results are surprising given that these students have had 8 years of exposure to decimal place value and related number work in primary school.

For Q11a, which involved writing decimal place values on top of given boxes, only 4 (16%) students were either partially or fully correct in the pre-test compared to the 76% in the post-test. And no student was able to generalise the place values of base 10 in the pre-test which was required for Q11b with only 5 even attempting to answer the question, but in the post-test, for the same question, 4% were partially successful and 64% were fully correct. The analysis showed significant improvement for Qs 12a, b, 13a, b and 14a (from 0 on every question to scores from 14 to 20 correct) which comprised writing place values of non-decimal bases (8 and 29) and generalising these. It was pleasing to see that by the end of the module of work, 20 of the students could write the place-values (as powers) for base 8 (Q12a), and 17 of these students could generalise the place values to 8\textsuperscript{p} (Q12b) or equivalent. Similarly 18 students could do the same for base 29 (Q13a) with 17 able to generalise the place value to a version of 29\textsuperscript{p} (Q13b) and 14 students (Q14a) could write in a different base such as 9\textsuperscript{p} and 7 of these could even take this to any base and write \(g^p\) or similar. Students found Qs 14b and 14c that involved the nature of numbers in the bases and exponents quite difficult and most did not attempt these in the pre-test (related concepts were not covered in the teaching intervention due to limitations on time) and yet, the results showed slight improvement; the students scored 0 in both questions in the pre-test but were partially successful in both questions (7 in Q14b and 10 in Q14c) in the post-
test, with two students also achieving good understanding in Q14b. Furthermore, it is one thing to employ the place values and write the meaning of numbers in base-10 but another when they have to be used to write the meaning of numbers (such as expanded notation) in non-decimal bases. There was a significant improvement for the Qs 9a,b,c and 10a,b,c that concerned meaning of numerals in bases 6 and 7 (from 0 on every question to scores from 12 to 16 correct).

Many of the questions in the pre-test were unattempted and in this regard, some of the students’ comments are revealing. S3 wrote: “I don’t understand most of it, that’s why some questions I didn’t answer. I found this hard...” S10 “All the questions were very hard”. S22 “I didn’t get it. But I tried [sic]”. Referring to Qs 11 to 14 she wrote; “This was the hardest”. And there was difficulty with the word ‘base”; S7 “I don’t understand what base means”; S9 “I don’t get the base things”. However, the comments in the post-test were more positive after the unit of work as will be seen at the end of the chapter.

The type of questions in the numeration system test can be broadly classified as involving i) base-ten numbers and ii) numbers in other bases. What follows is an analysis of students’ performance in exponentiation (see Table 6.6) which is a key idea highlighted in this thesis as a requirement for a deep understanding of a general positional system and its notation, and also for algebraic symbolism.

6.3.2 Classification of student answers on exponentiation

What follows is a description of the different categories involving powers, their abbreviation and also examples from these categories.

1. Writing the meanings of numerals in words or standard numerals such as 100, 1000 etc (Words or Std numerals): This category applies to responses where students used words or abbreviations of words or standard numerals such as 10, 100, 1000 etc to denote place values. Some examples: “One hundred thousand and five” for 100 005 (S1 in Pre-test), and writing “7 × 1000 + 9 × 100 + 0 × 10 + 5 × 1” for 7905.

2. Writing place values as powers in base ten with exponent ≥ 0 (Base-10 powers with Exp ≥ 0): It is one thing to write place values in words (ones, tens, hundreds) or standard numerals such as 10, 100 etc and another to write them as powers of
ten. Here students symbolised place values in power notation and were able to do this for exponents ≥ 0. An example of this category is S6’s response to Q3a (involving the value of the digits of 35275), who wrote 100, 101, 102, 103 and 104.

3. **Writing place values as powers in base ten with integer exponents (Base-10 powers with Int Exp):** This classification relates to students who were not only able to write place values for whole numbers but also for fractional parts of a base-ten number as powers of ten. That is, they were also able to indicate powers with exponents < 0.

4. **Using place values or powers of ten in writing meaning of numbers (Using base-10 powers):** Students were able to employ power notation in order to write what numbers such as 35275 mean or their expanded notation.

5. **Generalising place values or powers in base 10 (Gen base-10 powers):** Students in this category were able to generalise exponents of the place values of base ten numbers; they wrote 10^x or equivalent. The letter/exponent is seen as being able to take several values rather than just one.

6. **Writing the meanings of numerals in words (not as powers) in non-decimal bases (Words or Std numerals in non-decimal bases):** Examples in this category would be a) S11’s answer to Q9a; he wrote 4523base six as “Four two hundred sixteen, five thirty six two six and three” and b) S9’s response to Q10a which was to write the meaning of 4632base 7. He wrote “4 lots of 343, 8 lots of 49, 3 lots of 7 and 2 lots of ones.

7. **Writing place values as powers in non-decimal bases with exponent ≥ 0 (Non-decimal base powers with Exp ≥ 0):** Students symbolised place values in power notation for exponents ≥ 0. An example would be to indicate place values for 4523base six as 63, 62, 61 and 60 (S8).

8. **Writing place values as powers in non-decimal bases with integer exponents (Non-decimal base powers with Int Exp):** In this category students were not only able to write place values as powers for whole numbers but also for fractional parts of a non-decimal number. That is, they were able to indicate powers with exponents < 0, such as 8^4, 8^3, 8^2, 8^1, 8^0, 8^-1, 8^-2 for a base-eight number (S9).
9. Using powers (place values) in non-decimal bases in writing meaning of numbers (Using Non-decimal base powers): This categorisation applies to answers where powers in non-decimal bases were employed to convey the meaning of numbers or their expanded notation. For instance, for the number $20035_{\text{base six}}$, S3 wrote $2 \times 6^4 + 0 \times 6^3 + 0 \times 6^2 + 3 \times 6^1 + 5 \times 6^0$.

10. Generalising place values in non-decimal bases (Gen Non-decimal base powers): In this category the letter is seen as representing a range of different values; students were able to generalise exponents of powers of non-decimal bases; for example, S20 wrote $29^B$ when asked to generalise place values in base 29.

11. Generalising both bases and exponents (Gen powers): Under this classification both base and exponent were seen as representing a range of values and students were able to generalise both exponents and bases: For instance, S13 wrote $B^P$ in response to the question “Make a generalisation and write the place value for base with any number”.

12. Understanding the numeral/symbol 10 in different bases (Meaning of 10 in multiple bases): In this final category another type of generalisation is made; the numeral 10 is seen as representing ‘one × base number + zero units’ in a specific base under consideration. For example 10 in base six is understood as representing $1 \times 6 + 0 \times 1$.

Results on Exponentiation

The Table 6.6 displays an analysis of students’ performance in the number/numeration system pre-test and post-test with respect to exponentiation within the categories just described.

Overall, with respect to exponentiation, the students in this study did very well on post-test measures when compared to their pre-test performance. Table 6.6 shows that while there was no evidence of exponentiation in the pre-test, most students could symbolise powers in the post-test; with the exception of 2 students (one of whom (S25) was absent for about 5 weeks and returned to class just before the post-test) all were able to signify the exponential form. However, the results given in the above table are not mutually exclusive; there was an overlap both between and within base-10 and non-
decimal bases on the number of students who were able to symbolise powers and employ them in expanded notation. For example, for Q1 in the post-test, although 23 out of the 25 students wrote the given number in words or in standard numerals, this result also included students who provided the exponential form for the place values. 10 students were able to employ the power notation for place values when they provided the expanded form of the number in Q1, although as in the pre-test, students were not specifically asked to give the expanded form or exponential notation (See Figure 6.17 for S8 S6 and S11’s work).

Table 6.6. Facilities Related to Exponentiation

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<tr>
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<th>Pre-test</th>
<th>Post-test</th>
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</tr>
<tr>
<td></td>
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</tr>
<tr>
<td>2. Base-10 powers with Exp ≥ 0</td>
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<tr>
<td></td>
<td>0</td>
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</tr>
<tr>
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</tr>
<tr>
<td></td>
<td>0</td>
<td>88</td>
</tr>
<tr>
<td>4. Using base-10 powers</td>
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<td>5. Gen base-10 powers</td>
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<tr>
<td><strong>Non-decimal base</strong></td>
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</tr>
<tr>
<td>6. Words or Std numerals in non-decimal base</td>
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As shown in the Table 6.6, facilities in the categories 2, 3, and 5 in base 10 are almost the same as categories 7, 8, and 10 in non-decimal base; students were able to write powers with integer exponents and generalise them (E.g.: $10^9$, $8^6$) in base-10 and non-decimal bases with equal ease (see Figures 6.13 and 6.14). S7 was even able to use the abstract notation $A$ such as $29^4$ and not use the first letter of a colour to denote the changing number (see Figure 6.14). (As mentioned previously, students were taught to generalise numbers
using abbreviations of colours such as \( p \) for purple and \( b \) for blue). Surprisingly, however, while 40\% used these powers to explain the meaning of numbers in base 10, more students (60\%) did so in a non-decimal base.

The reason could be that in a decimal base, some students resorted to responding in a familiar way, to their previous experiences in primary school, and hence some of their answers were possibly automatic even though the teaching sequence in the study involved base-10 grouping and its exponential notation. This would explain why S3 and S20, although able to correctly write powers for base 10 for Q11a, responded with standard numerals (1, 10, 100 etc) for the expanded notation in Q1. On the other hand, in a non-decimal base which they were less familiar with, students possibly related better to their more recent experiences and answered accordingly. In addition, the grouping structure is more evident in power notation and it may have been easier and simpler to refer to \( 4523_{\text{base six}} \) as \( 4 \times 6^2 + 5 \times 6^1 + 2 \times 6^0 \) than in words. This may be seen in S17’s justification when writing \( 4523 \) (base six) in powers: “I understand that number but it

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example.png}
\caption{S8, S6, and S11’s facilities with categories 1 and 4.}
\end{figure}
doesn’t fit like... four hundred and six teens five thirty sixes... it doesn’t fit as a word” and then proceeded to write the expanded notation in powers.

In the context of generalisation, 7 students were able to progress to the next stage and make a second order generalisation; these students were able to write the correct answer $B^6$ or its equivalent in response to Q14a (category 11). One insightful response reflecting structural thinking was S17’s answer to Q10 where he was able to recognise the symbol 10 (one and zero) as the base number (one $\times$ base number $+$ zero $\times$ units) under consideration. That is, he did not read it only as ten but appears to have thought of it as one ten and zero units. For Q10a he gave the answer as shown in Figure 6.18 and provided the explanation “one zero = seven, 10=7”. That is, he seems to have recognised 10 as one seven and zero units. Interestingly, S17 ‘saw’ 10 as ‘one seven and zero ones’ in base 7 (see Q10 in Appendix 3) in the related question where different symbols were used to denote most of the numbers 1 to 6 and 0 (1, $\Gamma$, $\Delta$, $\Box$, $\Psi$, $\sigma$ and 0) rather than in Q9 which, while it did involve a different base (base six), had familiar symbols 1, 2, 3, 4, 5 and 0. It may be that for some students, stepping away from what is familiar and viewing it from a new perspective could lead to structural thinking and understanding.

The penultimate category 11 (of generalisation) given in Section 6.3.2 (of being able to generalise both base and exponent) was also reached by S27, S28 and S29, all of whom joined about four weeks after the start of the year and missed the pre-test (Hence their post-test results are not included in the data analysis above). They were given help (including grouping experiences with the sticks) during lunchtimes to be able to catch up with missed work. All three students were not only able to write powers in base 10 and non-decimal bases, they were also able to use them to give the meaning of numbers in expanded notation in multiple bases. Also, like S7 (mentioned above), S29 was able to do away with letters as abbreviations of colours and provided the answer $y^z$ when asked to generalise both base and exponent in Q14a.
Figure 6.13. S12 and S26’s facility with categories 2, 3, 5 and, S9, S13’s answers with respect to category 4.
The post-test results in Table 6.6 show improvement with respect to understanding the place value system, however, not all students achieved a high level of awareness of a general positional notation. As mentioned previously, many of the students’ answers showed that they were in a transitional state with respect to a structural understanding of the place value concept; they were still grappling with some related notions and in the process of forming others. The examples of students’ answers (in different categories) in the post-test discussed in this section (also see Figures 6.15, 6.16 and 6.17) illustrate their developing understanding.
6.3.3 Students’ Developing Understanding of Exponentiation

Outlined in this section are various examples of students’ answers that show that they are in transition from an understanding of place value at a surface level to a full awareness of place value structure such as demonstrated by S17 (see Figure 6.18). The examples are categorised under a) Base-ten place values and their generalisation, b) Using base-10 place value powers, c) Non-decimal base place values and their generalisation and d) Using non-decimal place value powers.
Base-ten place values (powers) and their generalisation

What surfaced in the analysis is that there are many sub-levels of understanding within each category of the exponentiation concept; being able to write place values for powers ≥ 0, for integer powers and to generalise these. One example is S10’s answer to Q3a, b, c. While she started correctly in Q3a and wrote the place values as 10^0, 10^1 ....etc, in Q3b she indicated (wrongly) the units place as 10^1 rather than 10^0. Moreover, she did not attempt Q3c which involved decimal fractions. While this was an improvement on the pre-test where there was no evidence of exponentiation or repeated multiplication, S10 was not yet able to link fractional parts of a number to negative exponents.

Figure 6.16. S9’s facility with category 9.

Another example is the work of two other students, S18 and S23, both of whom demonstrated exponential thinking, but again made the classic error (e.g. Hiebert & Wearne, 1986) of treating the decimal point as a point of symmetry, but as it appears, for a
different reason. While providing the correct place values for the fractional parts, that is $10^{-1}$ and $10^{-2}$, they wrote the place value for units as $10^1$, either treating the place values on both sides of the decimal point as symmetrical about the decimal point, or associating the 1 in the exponent with units (this error is also seen in the history of mathematics (Puig & Rojano, 2004)). And at the next level, S1, S14, S16, S19 and S24, all of whom were able to write the place values successfully both for whole number part and the fractional part of a number, were not able to make the generalisation and write $10^B$ or its equivalent.

While the above students were not able to generalise, S5 did and in the process, provided an unusual and interesting answer for Q11a. He appears to have first written the place values as $10^3$  $10^2$  $10^1$  $10^0$  .  $10^{-1}$  $10^{-2}$

And, after crossing out the exponents, he wrote $P$ in all of them as shown below and wrote that “$P = \text{changing number}$”

$10^p$  $10^p$  $10^p$  $10^p$  .  $10^p$  $10^p$

S5 further explained: “11b, 12b, and 13b were confusing because it said to generalise and I thought I got what it meant. So I switched my answers in 11a, 12a, 13a”. Although S5’s answer is not wrong, he appears confused about the general and the specific. His answer seems to be the reverse of what Küchemann (1981) identified as the “letter evaluated” category where a letter is assigned a numerical value. However S5 assigned a letter (same) to each of the numbers (exponents). The examples outlined so far show that these students’ ideas about exponentiation were still developing and needed reinforcement.
Using base-10 place value powers

For some students a stage prior to writing the expanded notation of a number in the power form is to first denote the place values on top of the digits. Three students S5, S8 and S18 appeared to be in this transitional stage as they clearly indicated the positions in the exponential form but did not yet use them to write the expanded notation. All three of them wrote the meaning of the numbers in base-10 in words. Although S8 and S18 were able to use the exponential form in base 6 and base 7. So their answer for base-10 may have been given more out of habit than any other reason. Interestingly, S18, while correctly stating the place values for numbers in Q1 (10^0, 10^1, 10^2, ...) which did not involve decimal fractions, was unable to transfer the same understanding for question (Q11a) which included decimal fractions. He appeared to see the decimal point as a point of symmetry (mentioned above) and made the error of writing the place values for the whole number part starting from 10^1 rather than from 10^0. Despite the practice, he was confused which indicates that exponentiation is not an easy concept, particularly with zero as an exponent. It would appear that some students need more practice and time to engage with some of the concepts related to the idea of positional notation.
Non-decimal base place values (powers) and their generalisation

A key aspect of generalising powers is being able to denote powers in various bases (in this test base 8 and base 29) and Qs 12, 13 and 14 sought to examine this. As in base ten, the post-test results revealed stages in students’ understanding such as being able to write powers where exponents are $\geq 0$, to write powers with integer exponents and then to generalise these. At the first level, S15, although achieving some success in writing meaning of numbers in words, did not attempt any of the questions in this section involving other bases. S15’s struggle was evident in his comment “This page is so hard I don’t know what to do”. At the next level, while S10 showed some knowledge of powers in base ten, however, left these questions on non-decimal bases unanswered. It appears that the students struggled when faced with questions formed in a different way from what they were used to. In relation to power notation, S25 (who had missed nearly 5 weeks of school but returned just before sitting the post-test) provided an interesting answer in that he showed he was thinking proportionally. He wrote the place values for 8 as follows:
Although this is not correct, this student has demonstrated multiplicative thinking. He seems to have multiplied 8 by the place values of base ten showing that he is still in the process of developing place value concepts. And S18 and S23, although able to write place values for integer exponents, made the error (as in base 10) of writing place value for units as $8^1$ and place value for 8s as $8^2$ and so on. At the next level of understanding non-decimal base powers, five students (S1, S14, S16, S19 and S24) were able to write the place values for base 8 and 29 but were unable to make the generalisation. Despite the instructional strategies, these students found generalisation quite challenging and this is evident in some of their comments. For example, this is what S14 wrote when asked to write place values for base 8 and base 29.

$$\begin{align*}
8^4 & \quad 8^3 & \quad 8^2 & \quad 8^1 & \quad 8^0 & \quad 8^{-1} & \quad 8^{-2} \\
29^3 & \quad 29^2 & \quad 29^1 & \quad 29^0 & \quad 29^{-1}
\end{align*}$$

Accompanying this she stated: “I don’t understand this still. I don’t understand base and generalise. It’s really hard”. It may be that she requires more guidance in recognising invariants and variation when items in the pattern are set across the page rather than in a vertical fashion. Still, 14 students were able to progress to the next level; they were not only able to specify the place values correctly, they were also able to express the correct generalisations such as $29^p$ or something similar.

The next level required the students to make a generalisation and write the place value for any base, 7 of the above 14 successful students appear to have interpreted it as any number other than the given 8 and 29. Hence these students’ response to Q14a was $34^p$ (S22) or equivalent which is a transitional step to full generalisation. However, the other 7 students were able to extend this idea; it was pleasing to see them advance to make the full second order generalisation. These students wrote the general place value as $B^y$ (S6) or equivalent. However, S3’s answer indicates a transitional phase in which he wrote examples of specific base numbers with the exponent $R$ and demonstrating that what he has symbolised as $B$ is the changing base number (See Figure 6.17 for S6 and S3’s work). A similar explanation was provided by S6 reflecting a sound grasp of the concept.
Using non-decimal place value powers

A major advance made by students is to employ powers in non-decimal bases, and Questions 9 and 10 targeted students’ understanding of the structure of numeration in other bases, and their respective notations. Q9 involved base 6 and Q10 concerned base 7. Both whole numbers and numbers with decimal fractions were included, and while the current number symbols were used in base 6, in base 7 different number symbols were given. Many of the student answers displayed intermediate levels of understanding and also movement between these levels as shown by the following examples.

For Q9 on base six, 7 students (S5, S10, S14, S15 S16, S19 and S25) were not successful and resorted to familiar base-ten notation to convey the meaning of the numbers. However, 4 of the above 7 students (S5, S14, S16 and S19) did demonstrate their familiarity and knowledge of exponential notation both in base ten and other bases (and S10 for base ten in Q3) in Qs 11, 12 and 13. It might be that these students were answering in familiar ways that was known to them; on hindsight, these questions might have elicited more correct responses if students had been asked to write the place values on top of the digits as a first step in the process. The other 18 students demonstrated that they were thinking structurally; all of them wrote the place values in terms of powers for one or more of the questions. Of these students, three (S12, S20 and S23) wrote only the place values on top of the digits and not the expanded notation; doing so seems to be a key transitional phase. While S12 and S20 provided the correct answer, S23 repeated her earlier classic historical error of writing the units place as $6^1$. It seems that these students’ understanding was still at the developing stage and they were not able to make some of the connections between earlier (grouping) work on manipulatives, the symbols, the nature of the base and, exponential notation. Three other students (S9, S11 and S22) (of the 18) wrote the correct expanded notation both in words and in exponential notation. One student (S6) however, gave the notation only in words but made the same error mentioned previously; of starting the first place value as $6^1$. He wrote: “4 lots of 216 + 5 lots of 36+ 2 lots of 12 + 3 lots of 6”. Although his answer is incorrect, he appears to have a sense of the grouping structure in base 6 and he provided correct answers to questions involving base 7. The other 11 students, except for a couple of minor errors, were able to write the correct answers for Q9a and Q9b.
Q10 involving base 7 was difficult for 5 students and one source of difficulty appears to be the different symbols employed in the question. Of the 18 students who were successful in denoting place values in base 6, 13 of them experienced success in writing the expanded notation for base 7. All the 13 students (except S17) first translated the symbols to familiar number symbols (e.g. $\Box$ to 4632) and then wrote the expanded notation using powers. However, S12, S20 and S23, who were able to write the place values for base 6 in powers, and S3 and S24, who were both able to use the powers to write the expanded notation, were unable to do so for base 7 in Q10. The unfamiliar symbols in this question seem to have been a stumbling block. Interestingly, however, while the new symbols presented an obstacle for the above five students, these very same symbols (and using a non-decimal base) appear to have facilitated S17’s structural understanding of positional notation (see Figure 6.18). Understanding the sub-concepts and their connections possibly helped him to achieve a state of readiness whereby he was able to understand the structure.

The evidence in the different categories above shows that while a few students have understood some of the fundamental ideas of the decimal place value system, many were still in the process of internalising the related sub-concepts and also the chain of relationships and connections leading to full place value structure. That it was not easy at the start is seen in students’ comments in the pre-test (Section 6.3.1). However, their comments were more positive after the current module of work. It appears that first grouping (reflecting repeated multiplication) and writing the place values in the exponential form (and other representations) may have helped students to ‘see’ the structure and for some to write the general form. Their comments are instructive and the following are some examples. S9 “I didn’t get the powers [before] but now I do. The sticks made it easier because we put them in groups”. S29 “The board [work] helped me more than sticks, but sticks helped the power... The sticks helped about base”. This student also remarked on the drawings on the board reflecting groups of 10, 100, 1000 and wrote “The drawing and the writing helps how to group”. S7 indicated that although he knew place values before, the work with the sticks enhanced his appreciation of place value. He wrote: “The sticks didn’t help because I already understood. But it helped me understand a lot better. It was more clear to me. E.g. doing a physical activity with something to help me learn. I also understand the concept of generalisation is the number thats changing and
Base is how we group things”. Other comments written by students are given in Figure 6.19.

Figure 6.19. Students’ comments on work with sticks for the compositional structure of place value notation.

The results presented in this chapter focused on students’ understanding of a general positional notation. The analysis of the results involved students’ thinking related to large numbers, exponentiation, generalisation of patterns and the compositional structure of a general place value system. While the results were positive and many students were able to demonstrate their knowledge of the multiplicative structure of the system (and some even able to generalise to a base $b$) the evidence suggests that some students’ understandings
are still developing. Students’ written comments indicate that they found the use of multiple representations including concrete representations helpful to understand the place value system.

Comparing the results of the current intervention with a previous class, the results of this study is an improvement on the results of the preceding Year 9 class (Nataraj & Thomas, 2009a). A similar teaching framework was implemented with the previous class. One example of the improvement is that in the current research, S17’s answer with regard to base seven (see Figure 6.18) was insightful and was not provided by any of the students in the previous study. The improved results may have been due to some modifications that were made to the teaching design of the previous year such as an increased focus on grouping and notation of powers and their various representations, generalisations involving place value structure, number of symbols and counting in multiple bases as shown in Appendix 4. In the next chapter, the results presented in this chapter are discussed.
CHAPTER 7

DISCUSSION AND SYNTHESIS OF TEST RESULTS

7.1 Introduction

In the preceding chapter the results of an examination of many aspects of students’ thinking pertaining to the generalisation of decimal place value notation were presented and analysed. These results were based on students’ test answers on large numbers, generalisation and the structure of the numeration system. The goal of the current chapter however, is to discuss the results in the light of the research questions and hence evaluate the effectiveness of the framework developed for students’ conceptual understanding of a general place value system, doing so in relation to mathematics education literature and the historical research findings. In view of this, the general research questions presented in the introductory chapter are re-stated first.

General research questions

1. What is the feasibility of a study that involves a two tiered process such as: i) the extraction of historical ideas and the development of a teaching framework and; ii) implementing the framework and evaluating its effectiveness in increasing Year 9 students’ awareness of numeration?

2. What are some of the key historical ideas relevant to understanding the decimal place value system and how might they be useful in teaching and learning?

3. What is the effect of the framework developed on Year 9 students’ understanding of the place value system?

The second general research question and more detailed questions arising out of it on history of mathematics were answered in Chapter 4. In this chapter, the third general research question is addressed concentrating on a discussion of the results related to this research question on the teaching of the numeration system, and doing so in the context of the historical results/ideas. The aim of this chapter is to also answer the first general
research question regarding the feasibility of the study, and the following specific research question:

How does the framework (developed from historical ideas), combined with the use of multiple representations and modeled with concrete materials, influence Year 9 students’ understanding of decimal place value system and its generalisation?

The discussion will be presented in three main sections; involving student responses to test questions on: a) large numbers; b) algebraic generalisation; and lastly c) the structure of the numeration system. At the start of each section, both the test results and historical findings are briefly summarised.

7.2 Students’ Ideas About Large Numbers

As seen from the responses to the large number questions given in the Results chapter, there was a significant improvement in the students’ performance in being able to write examples of large numbers, compute with them and also to give examples of situations that use large numbers. Students appeared to have a sense of the meaning of powers (reflecting repeated multiplication) and its notation. There was also evidence of the development of quantity sense and many of the answers showed multiplicative thinking.

With regard to the historical findings, a study of the history of Indian mathematics revealed systematic and consistent naming of large numbers including powers of ten [eka(1), dasa(10), sata, sahasra...parardha (10^{12}) and then extended to loka (10^{19})] for many centuries beginning from the Vedic era (Datta & Singh, 2001; Gupta, 1987, 2001; Joseph, 2000). Naming of large numbers also included ideas about infinity (see Section 4.3.1) for a detailed list and discussion of large numbers in Indian history), as well as problem statements involving large numbers such as example 7 in Section 4.3.1 with a potential for calculation and with a multiplicative solution of more than two steps.

As indicated in the results, many students demonstrated their acquisition of knowledge of large number names, including powers in the post-test, as well as their ability to translate a large number from words to numerals. The positive results indicate the usefulness of the overarching framework and the related large number teaching sequence
which were constructed from historical ideas. The encouraging results also tend to substantiate researchers’ (Fuson, 1990a; Irwin & Burgham, 1992; Reddy & Srinath, 2001; Ronau, 1988) views that, by naming large numbers and working with them, students imbibe the naming convention and the base-ten structure of numeration.

Somewhat parallel to the historical development in India, and also noted by Nataraj and M. O. J. Thomas (2009a, 2009b), most students in the post-test changed their scale of counting when asked to count faster. Some examples include S24 who changed 1000 to 1000 000, S11 altered his scale from 100 000 to 100 000 000 and S3 who changed it from trillions to quadrillions to count to the number he had suggested, which was 1 billion to the power of 1 quadrillion. These answers and other responses to Q4 tend to confirm Zazkis’s (2001) theory that contemplation of large numbers encourages students towards a sense of structure. Some students’ answers showed structural understanding reflected in the general statements they made when asked to give an example of a large number and the responses demonstrated that some students were able to see a problem with a multiplicative solution of more than two steps. The emphasis on operations without actual computing, as proposed by Hewitt (1998) may help students towards abstraction. In support of Wagner and Davis’s (2010) view that calculating and connecting to previous experiences may be beneficial to develop a sense of quantity involved in large numbers, students’ performance improved in Questions 5, 6, 7 and 16i that concerned computing with big numbers. There is the added bonus, according to Wagner and Davis, that computing with large numbers incorporate ideas from other topics in the curriculum such as fractions, time, area and volume (and even across disciplines e.g. science and geography).

As highlighted several times in this research study, a crucial aspect of understanding place value structure would be an awareness of powers (Dienes, 1964a) as repeated multiplication, the grouping structure (Bednarz & Janvier, 1982; Fuson, 1990a) that reflects this, and the related quantity sense. The examples in Indian history consisting of number names for powers of ten indicate an early awareness of exponentiation. This is particularly evident in the second example on large numbers from the *Pancavimsa Brahmana* of the Vedic literature in Indian history, which explains the grouping structure involved in a trillion ($10^{12}$). The results show the degree of success experienced by students pertaining to the exponentiation concept and again corroborates Zazkis’s (2001) hypothesis that consideration of large numbers particularly powers, leads learners to structure sense.
Students’ experiences in grouping, and working not only in base 10 but also in other non-decimal bases that have been advocated by educators and psychologists (e.g. Dienes & Golding, 1971; Schmittau & Vagliardo, 2006; Skemp, 1989; Vygotsky, 1962) may have contributed to their success in responding to questions that involved a generalisation of powers. The test results and students’ comments (see Section 6.1 and Figure 6.5) tend to confirm these psychologists’ and researchers’ views. The comments also illustrate that, working with manipulatives, and the visualisation aspect are important factors in the early stages of concept development. Working with the different representations as suggested by researchers (e.g. M. O. J. Thomas, 2004, July) including concrete materials (e.g. Bednarz & Janvier, 1982) may have helped some students to generalise the meaning to larger numbers ($2^{96}$) and to powers with literal symbols ($a^7$, $2^y$, and $a^y$) as seen in the results.

However, surprisingly, despite the instruction and practice, some students still found grouping in powers confusing and difficult (see Sections 5.3.1 and 6.1.4). It seems that grouping reflecting repeated multiplication (as opposed to and built on grouping mirroring repeated addition) is not particularly easy. What surfaced in this study is the importance of two ideas related to grouping in repeated multiplication. One key aspect that was adapted is reinitialisation (or regrouping as a whole unit) as proposed by Confrey (1994) at each step in the process and keeping a record of the transformation. Another crucial idea is the establishment of links between manipulatives in the grouped form and their representation in the written form (English & Halford, 1995), again at every stage of the whole procedure.

What also came to the fore in students’ work was the presence of some ‘epistemological obstacles’ (Brousseau, 1997; Sierpinska, 1994), that were similar to obstacles in the history of mathematics. In the case of multiplication and repeated multiplication, students’ way of knowing multiplication (and its notation) appears to sometimes stand in the way (obstacle) of their grasp of repeated multiplication, and this sometimes result in the classic mistake of writing $2 \times y$ instead of $2^y$. The reason could be that although students had experiences in grouping reflecting repeated multiplication, they did not as yet make generalisations using literal symbols. In another instance, one student’s answer to $2^{96}$ was to write $96 \times 96$. Despite an attempt during instruction to anticipate this very obstacle and modify the teaching strategy (to address it for example by the use of calculators to compute $3^8$ and $8^3$), this student seems to have applied a form of the rule of commutativity in exponentiation. This confirms Brousseau’s view that epistemological
obstacles are sometimes unavoidable, but they can be used for clarifications and to correct misconceptions that students might acquire.

In the above context, what was also uncovered in the study is that although exponentiation is an extension of multiplication, it has its own associated difficulties. It is true that just as multiplication is given its initial meaning of repeated addition, exponentiation is given its initial meaning of repeated multiplication. Even so, as seen in this study, this initial meaning of repeated multiplication is challenging for some students. As pointed out by Greer (1994) (and this researcher’s view is consistent with this), epistemological and ontological questions of comparable interest and value are raised at every stage of re-conceptualisation throughout the long development of a particular conceptual field. As in the case of multiplication and exponentiation, this aspect was found to be true in this study. On reflection, students in this study may have profited more from additional classroom time spent on activities involving grouping (with craft matchsticks) mirroring repeated multiplication in several bases.

In summary, while some students were still developing the exponentiation concept, for most the early naming of and working with large numbers including powers of ten (and its related grouping) appears to have helped them to understand (to an extent) the naming convention in large numbers and the equal grouping structures involved in powers of ten and other bases.

7.3 Students’ Ideas About Generalisation

The results from the generalisation tests indicate that most students were able to recognise and symbolise variation, distinguish between specific unknown and generalised number, and were able to generalise numerical patterns by using different letters for different variables.

Historical results indicated that a major milestone in the evolution of algebraic symbolism was not only the use of the letter as specific unknown but also in its use as a general number (in the sense of something that varies). Indian history also revealed the assigning of different names (colours) such as kalaka, nilaka, pitaka for different variables, which were then abbreviated to the first letters (ka, ni, pi) of the word. This terminology for different variables (x, y etc) and powers of variables (x^3, y^2, xy^3) was historically
difficult but once achieved by the 7th century in India, substantial progress in algebra and in mathematics was able to be made.

The test results of this study, which integrated the above historical ideas in teaching, show that a method of observing patterns and noting the variation and invariants was accessible to most students (80.2% in the post-test in contrast to the 8% in the pre-test who were able to recognise variation or symbolise it using colours) and they were able to classify them with varying degrees of success. This supports Mason’s (Mason et al., 2005) and Srinivasan’s (1989) contention that observing patterns and classifying/generalising may be inherent in children. Even students who struggled with many of the concepts were able to experience a certain measure of success with questions which involved recognising variation and symbolising it. For example, one student (S23), who found many of the questions in the large numbers and structure of numeration tests challenging, was able to successfully denote invariance, and also variation with names of colours (and shortened to first letter of the colour) for the above questions on generalisation. As mentioned above, a chief landmark in algebraic history is designating different letters for different variables and Qs 1b and 4b tested this understanding. For these questions, it was noted that around 73% of the students were not only able to recognise that two quantities were varying differently, but were also able to symbolise this with different colours and then the first letter of the colour. However, resembling history, four students appeared to find the variable concept challenging. These students struggled to express the correct generalisation and their responses showed that they were still grappling with the notion of the variable. This agrees with the idea of ‘historical parallelism’; as indicated by Harper (1987) and Sfard (1995), both phylogeny (historical) and ontogeny (individual development) of mathematics reveal similarities in difficulties and in this case it is that of understanding the variable concept. On the other hand, some students were thinking beyond their curriculum level. In this connection S17, in response to Q8a, first generalised the pattern to “9 × y + 8 + 7 × y”, then wrote it without the multiplication symbol, “9y+8+7y” and further simplified it to “16y+8”. Likewise he simplified the pattern generalisation in Q8b to “16y+8”. He was able to add like terms although it was not taught during the intervention. Subsequently, comparing the two answers S17 commented that “we can add or subtract the same types of numbers!” This supports Arcavi’s (1995) statement that, similar to the historical development of mathematics, symbols arise to express ideas in a concise way but
once expressed, this sometimes results in pushing the ideas further than they were originally intended for, as exemplified above. As Arcavi also points out, one advantage of looking into history is the potential benefit of insight in regard to the nature of learning algebra.

A key concept highlighted in this thesis is that of the importance of exponentiation in the learning of place value and Q2 is related to this aspect. There was a large increase in the number of students able to recognise or symbolise variation after the intervention. In the pre-test, only one student was able to recognise variation (but not symbolise it), but in the post-test, 71% could either recognise the changing quantities or write as symbols using letters. Still, only 46.4% could symbolise variation of powers. As noted by MacGregor and Stacey (1997) and Pitta-Pantazi et al. (2007), exponentiation is not an easy idea for many students and they need more time to come to grips with some of the related notions. In hindsight, in order to generalise, students in this study needed to have been provided with more experiences in: i) the interpretation of the exponential form and ii) the recognition and symbolisation of patterns where the elements of the sequence are set out in a horizontal structure similar to positional values.

In terms of the meaning of the literal symbol, the post-test results indicate that students were able to successfully identify letter as specific unknown and as general number in equation and expression situations respectively. However, for four students the notion of letter as specific unknown seemed to interfere with their interpretation of generalised number. For example, S24 wrote: “y means 4. like i) 3×4 (for 3×y) and ii) 3×4=12 (for 3×y=12)”. This supports Malisani and Spagnolo’s (2009) finding that the conception of letter as unknown is an ‘epistemological obstacle’ and interferes with the conception of general number and variable as standing for multiple values. As noted by Küchemann (1981), the letter as a specific unknown is simpler than generalised number and poses less difficulty for the students. Nevertheless, the fact that around 82% of the students in this study were able to recognise specific unknowns and general numbers, and showed understanding of the difference between the two uses is notable. This is especially so, given that Küchemann’s large-scale study reported only 17% of 13 year-old students able to use letter as specific unknown and less than 2% could deal with letter as generalised number. These two views of the literal symbol belong to two different conceptual formations; as stated by Radford (1996), and shown by the historical-epistemological
analysis, they are crucial for progress in algebra. If students are to make sense of the many uses of letters in their secondary school algebra learning, then they have first to make sense of these fundamental uses as specific unknown and generalised/general number. This is in agreement with the conclusion of Puig and Rojano (2004) whose study of the history of algebra identified conceptual understanding of names/letters for different variables, and powers of variables as two vital categories for advancement in algebra.

In this study, a historical analysis highlighted the difficulties associated with the variable concept and its notation, and how these were overcome in India, such as through the use of *different colours* for *different variables*. The results in the algebraic generalisation test show that explicit teaching (incorporating historical and mathematics education ideas) of the foundational notions of letters as specific unknown and generalised numbers can yield positive results.

7.4 Students’ Understanding of the Structure of the Numeration System

The results from the structure of numeration tests suggest that many of them were able to relate the multiple representations of number, in the concrete, pictorial and symbolic forms across different bases. The results also indicate that many of the students had acquired a sense of the meaning of multiunit/place values reflecting repeated multiplication, and an appreciation of the place value structure with its inherent composition of values.

In the historical context, a review of Indian history revealed that even as the Vedic oral numeration structure was multiplicative and exponential, the development of the *written* decimal numeration sign system indicated a pattern (Datta & Singh, 2001; Ifrah, 1985; D. E. Smith & Karpinski, 1911) involving Kharoshti and Brahmi numerals; the evolution can be characterised (ignoring false paths and dead-ends) as follows:

1. *additive* or *verbal* stage;
2. *multiplicative* stage;
3. ‘*places*’ stage when powers of ten were thought of as places or positions;
4. final *abstract* stage (with inherent place value and integrating the *zero* concept).

That is, the *written* numeration sign system appears to have traversed a set of transformations: from the *additive* to the *multiplicative*, then to the *places* stage and finally
to the *abstract* stage. In the context of teaching models, researchers had in the past looked at increasing students’ place value understanding through a mapping from the physical manipulatives such as base-ten blocks (e.g. Fuson, 1990a; Price, 2001; Resnick & Omanson, 1987) to the number symbol. However, one of the contributions of this research is that the model in this research extends previous ones by incorporating *two intermediate conceptual stages* extracted from Indian history within a four stage model. The *two intermediate conceptual stages* are the *multiplicative* and ‘places’ stages where the places or positions are given in the form of *powers*. The above set of stages (or representations), in accordance with Radford and Puig’s *Embedment Principle* (Radford & Puig, 2007) offered a route to understanding the place value concept. Hence it was incorporated into the teaching sequence (See Figure 4.8) along with concrete manipulatives. As evidenced in the results and their comments, most students appear to have found the scaffolding or *structuring* (Freudenthal, 1991; Piaget, 1968) that was developed and modified from a historical-cultural analysis (Vygotsky, 1978), helpful in constructing the place value idea. After the teaching intervention, there was a substantial increase in the number of students who were able to:

- Write place values as powers with integer exponents in different bases.
- *Use* place values or powers in writing *meaning* of numbers in expanded notation in both decimal and non-decimal bases.
- Generalise exponents in powers with base-10 and other bases.
- Generalise both bases and exponents.

As shown in the results analysis (see Table 6.6), there was no evidence of exponentiation in the pre-test on the structure of numeration, however, in the post-test at least 84% of the students denoted place values/multiunits in terms of *powers* for questions involving base-10 and non-decimal bases. Further, when asked to write the meaning of a multidigit number in base-10, without being specifically asked to do so, 16 (68%) students gave the answer in expanded notation and of these, 10 employed a power notation. For a similar question in non-decimal bases, 15 (60%) students provided the correct answer in expanded notation, including the *exponential form*. Although, in the tests, some students also provided the expanded notation in numbers without power notation (e.g. 4523 in base
six: 4 lots of 216 + 5 lots of 36 + 2 lots of 12 + 3 lots of 6), however a few students made mistakes. This is consistent with Zazkis and Khoury’s (1993) suggestion that fewer errors are made with the power form of place values. With respect to generalisation, as seen in the results, 68% of the students were able to generalise place values in decimal and non-decimal bases and 28% were able to extend the generalisation to both bases and exponents by using different colours to notate the different variables. Students’ ability to generalise powers supports Vygotsky’s (1962) theory of teaching the place value concept where he proposed teaching the concept in multiple bases for the development of the idea of positional notation. Furthermore, as seen previously in the generalisation test, the results in this structure test where many students were able to generalise place values, also tend to confirm the observation made by Mason et al. (2005), that observing patterns and generalising may be natural to children.

In addition to the positive test results, students’ feedback suggested that they seemed to find the demonstration and discussion on the different representations (of each of the above stages) during intervention valuable in forming the link between conceptual and the semiotic, both of which, as highlighted by Becker and Varelas (1993), are important to the cognitive development of the idea of place value. The positive outcomes from the intervention which included aspects of Dienes’s (1964a) perceptual variability principle, Bruner’s three modes (enactive, iconic and symbolic) of representations, and Skemp’s (1971) theories on abstraction and concept formation tend to support the importance of these ideas. Students’ success in the tests also stress the importance of: a) construction of multiunit quantities using concrete materials and linking them with symbols and words (Fuson, 1990a; Fuson & Briars, 1990); b) representational versatility as proposed by M. O. J. Thomas (2008a) on being able to shift between multiple representations of the same concept (place values/powers and expanded notation in the present case) and notice their features in order to gradually become aware of the concept; and c) connections (Presmeg, 2006) that have to be made within and between the conceptual stages in their various representational/semiotic systems.

An important aspect foregrounded in this study is the usefulness of labelling the positions/place values of the digits in power form (corresponding to the 3rd ‘places’ phase in the above historical stages) rather than in words and their abbreviations, or writing in
ordinary form such as 10, 100, 1000 etc. In addition to the fact that such notation (words and 10, 100 and 1000) could be cumbersome, it may also give rise to ambiguities. This is seen in the following examples from the student answers in the pre- and post-tests: ones is written as 0 which is shorthand for ones. Both thousand, and tenth written in shortened form th, ten thousand and tenth both written as ten th. For Q3 one student labelled ones as 0, and both hundred thousand and hundredth as Hth. Possibly realising that this is unclear, she (correctly) wrote all the place values also in power form, which may have seemed less ambiguous to her. Hence eventually writing the place values in the power form may not only aid the understanding of the grouping structure of the place value idea, but could also help to avoid confusion. As well as this, this research highlights other benefits to teaching the meaning of powers. The results show that many students were not only able to symbolise 1000 as $10^3$, 100 as $10^2$ and 10 as $10^1$, but also write 1 as $10^0$, 1/10 as $10^{-1}$ and so on thus moving to powers with zero and negative exponents. As pointed out by Zazkis and Khoury (1993) and Confrey (1991), exponentiation can be difficult for many (even teachers) especially when negative exponents are involved.

In relation to powers, some of the students were able to translate their understanding of base-10 positional values to non-decimal bases. Similar to the column chart strategy highlighted in Zazkis and Khoury’s (1993) study on teachers’ place value understanding, some students in this research seemed to prefer to first label the places (as $6^2$, $6^1$, $6^0$, $6^{-1}$ or $7^1$, $7^0$) in the power form before writing the meaning of the number or expanded form. In the context of history (Datta & Singh, 2001), identifying multiunits/powers as positional values corresponds to the third places stage, and the expanded notation corresponds to the second multiplicative stage in the historical development of the decimal system. This is a reversal of the historical stages and may have implications for teaching; when the meaning of a number such as 7835 is asked for, it may be a useful strategy for students to label the positional values above the digits first before writing the expanded notation. This is confirmed by Zazkis and Khoury’s (1993) finding that (even) elementary school teachers found labelling a ‘cognitively less demanding’ strategy than expanded notation. On the same note, Skemp (1992) has advocated against hastily doing away with marking place values when performing operations on numbers. In identifying the place values, while some students employed ordinary numbers (10, 100, 1000 etc) to label the decimal positional values, no student labelled the non-decimal places as such (for example, 216, 36,
6, and 1 in base 6), most preferring to use the power form in non-decimal bases. Possibly, the grouping structure was more ‘visible’ in the power form for the students, an aspect that has implications for teaching. This is again in agreement with Zazkis and Khoury’s (1993) observation that exponential notation presents the major similarity among representations in different bases, and it helps to generate correct strategies on questions involving place value; use of ordinary numbers or other ‘explicit’ representations produce more frequent incorrect responses.

What was also evidenced in the results comments was that many students were able to appreciate the multiplicative nature of the place value system and the related conciseness in the representation. For example one student commented on the exponential form of writing place values: “...... instead of 1000 you can write $10^3$ showing that it can be made smaller...”. And another student made the following comment on the multiplicative stage: “The sticks helped a lot, because I could see what was happening. I understood better when the teacher replaced the bundles with single sticks (it was made less complicated). I understood place value” [italics added]. Hence moving from the additive to the multiplicative stage helped this student to develop insight in relation to place value structure. Reflecting on the compositional feature of place value structure, students may have found it more beneficial if classroom activities had included added experiences involving the reinforcement of the place value idea in an operational (such as addition or subtraction) setting as discussed in Section 4.3.4 (See Figures 4.8, 4.9, 4.10).

Moving now to the algebra aspect of place value composition, some students may have found generalisation easier due to the initial ground preparation in multiplicative grouping structures and numerical pattern generalisation as per the teaching framework (see Figure 4.17 in Chapter 4). This is seen in a few students’ successful performance in Qs 11 to 14 in the post-test. However, the results show that for some others it was not an easy task and they struggled to make a generalisation. This is in agreement with Küchemann’s (1981) finding that very few students were able to consider literal symbols as generalised numbers and also supports MacGregor and Stacey’s (1995) observation related to students’ difficulty when two variables are involved. Generalisation may not have been easy for some students for a number of reasons and this may also explain why they were not able to progress to this level. Some of these are outlined below. One aspect of the related questions
was that students had to first write the place values before making a generalisation; it was not given to them as such. In addition, the place values were written horizontally whereas their experience in generalising was mostly when the patterns were set out in a vertical structure. In addition, students were asked to generalise the power form and most of the examples discussed during intervention involved ordinary form. Furthermore, generalising both bases and exponents involved second order generalisation of the power notation with two differently changing numbers. This is not surprising, since, as seen previously in history, a similar difficulty was experienced by mathematicians in the past. Assigning two different names for two variables was a major milestone in the history of mathematics (e.g. Colebrooke, 1817; Datta & Singh, 2001) and one that was achieved with great difficulty. It appears that students need more practice with generalising different types of patterns and this aspect needs to be taken into account for future studies.

**Summary and conclusion**

This chapter has discussed the results pertaining to students’ performance in large numbers, algebraic generalisation and the structure of numeration tests, examined within the background of the history of mathematics and mathematics education research and related to a conceptual understanding of a general positional notation system. The positive results tend to affirm the idea of incorporating a (more or less) historical order in teaching (Barbin, 2000; Tzanakis & Arcavi, 2000), and also substantiate the argument on early naming of large numbers (in order to understand the rule for the same) prior to work on structural composition of a multi-digit number using multiple representational systems. Additionally, the results tend to support the case for teaching (the grouping structure and notation) of multiple bases and their generalisation. The results also confirm, as highlighted in the historical analysis, the importance of exponentiation reflecting repeated multiplication for the understanding of place value structure, and its role as a key link between place value notation and symbolisation in algebra.

A main aim of the research study was to enhance Year 9 students’ conceptual awareness of the place value system and its generalisation, and in order to realise this, an overarching didactic framework was developed including secondary teaching sequences. That such a framework, constructed from a holistic, multi-dimensional approach and out of a combination of historical and educational ideas was developed and implemented in the
classroom, corroborates the *feasibility* of the framework. Further, the positive test results subsequent to the teaching intervention, affirm the efficacy of the research design and the said framework (See Section 4.6 for the framework and its related teaching sequences). Answering the first general research question, this case study suggests that it is possible to develop such pedagogical sequences using history and implement these within the context of the normal school programme.
CHAPTER 8

LIMITATIONS, IMPLICATIONS AND CONCLUSION

In this chapter, the limitations of the study, recommendations for further research in history of mathematics and mathematics education, and implications for curriculum and teaching are outlined. These are considered with respect to both the historical and didactic aspects of the study. The chapter ends with a summary and conclusion to the thesis.

8.1 Limitations

This thesis is a first tentative attempt at a holistic approach to enhancing students’ understanding of the structure of the place value system from multiple perspectives, including the history of mathematics. However, some constraints were unavoidable and these are briefly addressed in this section.

8.1.1 Historical Analysis

Available time was an issue in the historical research. Although time did not adversely affect the investigation and extraction of historical ideas, given the vastness of the topic and the spread of ancient Indian mathematics over historical, philosophical, religious, grammatical, astronomical and other texts there was insufficient time for a more in-depth examination of wide-ranging but related historical-cultural books and manuscripts. Another time-related issue is the accessibility of historical materials and publications. While The University of Auckland library has a substantial collection of books on Indian history, some publications could only be procured from India and as a result, this took more time. A further limitation is this researcher’s lack of expert knowledge of Sanskrit. Although this researcher is able to read and write Sanskrit and is familiar with some terms and phrases in the language, much of the research material was obtained from an analysis of history books, translations of texts and journal papers.

8.1.2 Teaching Intervention

Time was also a factor in the teaching part of the study. A limitation on time was imposed by undertaking the research at the same time as being a practising teacher, with the obligation to teach the curriculum topics, assessment responsibilities and other factors
typical of a modern classroom. More time would have allowed for a consolidation of the place value concept through operations on numbers using multiple representational systems as discussed in this study (see Section 4.3.4 and Figures 4.8, 4.9, 4.10).

The research was not designed as a controlled experiment allowing for generalisation of the results. It was a case study involving 13 year-old students from one class in a secondary school. The results obtained are specific to a particular context and cannot be generalised for all Year 9 students. The research was also limited by the small sample size of students. A larger number of participants would have enabled a deeper understanding of students’ thinking. With a larger number of students, clearer patterns of students’ thinking and their abilities might have emerged from the data. Furthermore, the researcher in this study was also the students’ class teacher. This aspect raises some methodological issues. Perhaps the time spent on the study was more than what would otherwise have been the case, only because the teacher/researcher was more motivated or more informed about historical and mathematics education research. Also, the approach described in this thesis is that of one teacher/researcher. Not only that history does not have to be used in teaching, but several distinct approaches are possible both using history of mathematics and without using history.

With regard to the results from the tests, these were positive as seen in Chapter 6. However, one shortcoming is that it is difficult to pinpoint which aspect of the research contributed most to students’ success in the structure of numeration test – whether it was the work on large numbers or their experiences with respect to the grouping structure using concrete materials or the generalisation of the place values or, a combination of some or all of these aspects. Hence the account must be viewed as one that is experimental and subject to adjustment according to various educational factors, including the type of students, and as the influence of each will be clarified as further research evidence accumulates. A similar experiment, such as described in this study could be conducted with Years 7, 8 and 9 students across different schools. Some other research possibilities are described next.
8.2 Implications for Research

The research study was conducted in terms of students’ understanding of the structure of the decimal place value system and its generalisation using historical ideas. However, in this section, several considerations for further research is advanced that would increase knowledge in these areas and could be useful for teaching and learning.

8.2.1 Historical Analysis

The suggested topics involve, either directly or indirectly, some of the historical aspects of large numbers, exponentiation, numeration system and algebra (all of which were considered in this study) in Indian history. In turn, these studies may be didactically useful in structuring classroom strategies. In order to maximise the benefits of the research for education, analysis of the proposed topics could take into account mathematics education readings on the topic (could perhaps apply the historical-critical methodology that was valuable for this research study) and also possibly highlight the pedagogic potential of the topic, as demonstrated in some recent research and historical materials (e.g. Dutta, 2005; Fauvel & van Maanen, 2000; Joseph, 2002; Katz & Michalowicz, 2004). This would perhaps involve a collaborative effort and a synthesis of ideas from historians of mathematics, mathematicians and mathematics educators in universities, across disciplines at the school level and, between mathematics departments in universities and schools. In the context of Indian history of mathematics, except for very few research articles such as Dutta’s and Joseph’s, there is a gap in the educational literature with regard to such research that needs to be addressed. It goes without saying that it is not claimed that research should be restricted to Indian history, however, given the focus of this research study, the suggestions given below mainly concern Indian history of mathematics.

Recommendations for future historical research (in an education context) are suggested in the following areas:

- Exploration of a wide range of texts for examples of large numbers (both oral and written numerals) including powers, and their rationale in Indian history, and comparisons with large numbers in other ancient histories such as the Egyptian, Babylonian, Greek, Chinese and Mayan.
• Comparisons of number signs in oral and written numeration of different cultures.

• Instances of large numbers and infinity and their associated concepts in Jaina literature.

• Examination of scientific Sanskrit texts (Vedic literature) for instances of multiplicative thinking, exponentiation, mathematical patterns and their generalisation, infinity and zero.

• Ideas related to zero and infinity, links between them, and comparisons with zero and infinity in other cultures.

• Various multiplication methods in Indian history.

• General methods of operations on numbers and their links to generalisation in Indian algebra.

• General methods of solutions of equations in Indian algebra.

• The evolution of other concepts in number such as negative numbers in Indian history.

• The nature of abstract thinking in Indian history of mathematics.

8.2.2 Teaching and Learning

This thesis has raised the importance of working with large numbers, developing multiplicative (including exponential) thinking, flexible use of multiple representations, and the generalisation of place values. As discussed in this study, all these ideas are key factors required to understand the numeration system. Given the slow development and difficulty of the place value idea acknowledged in the educational literature, students may find it helpful if some of these ideas, such as grouping reflecting powers and their notation, were to be introduced in the earlier years (upper primary and intermediate, Years 6 to 8). Additionally, due to the limitations outlined above, similar studies could be conducted with Year 9 students (even Year 10) and for larger number of student participants. Also, the research design could be replicated in a controlled setting with students in two classes perhaps taught by the same teacher or different teachers, allowing for comparisons and investigation of the effectiveness of the framework.
In view of this, a number of research possibilities that could contribute to further understanding arise, and the following recommendations are made for further research that may continue to address some of these research topics.

- Naming, reading and writing of, and computing with large numbers (and very small numbers) including powers in the decimal system, and whether this would assist with the enhancement of quantity sense, the naming convention and structure of numeration for multidigit numbers.

- Experiences in multiple representational systems including concrete materials such as craft match sticks as used in this study and their linking, for grouping and regrouping in powers reflecting repeated multiplication in multiple bases and to investigate their effect for understanding the multiplicative (exponential) structure of the place values.

- Experiences involving the structural composition of multi-digit numbers in both decimal and non-decimal bases as per the (more or less) historical order detailed in this study, through the use of concrete materials and other representations. The opportunity for these experiences also in the context of operations. To investigate the effect of such experiences for the enhancement of the structural composition of multidigit numbers.

- Pattern recognition and generalisation of patterns using different colours to symbolise different variables.

- Comparing different historical numeration systems and answering non-routine questions and to investigate the extent to which these comparisons can enhance students’ understanding of number symbols and their role in a system of numeration.

- Implementing a teaching framework that includes a combination of some or all of the above topics, such as in this research study, and to investigate its effect on students’ awareness of a general place value system.

- Students’ experience of structuring according to a (modified) historical sequence can be applied to other number concepts as well as other mathematical topics and would be worth investigating.
Such teaching experiments as described above may help researchers and teachers to understand students’ thinking in response to the implementation of alternative frameworks. With regard to further research studies, this researcher intends to continue examining historical texts for other notions involving numeration systems, number concepts and algebraic ideas and to refine the developed models in this research. I am also aiming, (possibly as a longitudinal study) to replicate the current study design for larger number of students taught by different teachers, both in the upper primary and junior secondary schools. A further intention is to investigate teachers’ and students’ understanding of, multiplication and exponentiation, of the idea of variable, structure of a general positional notation, and general methods of solving equations in algebra. This research is already underway.

8.3 Implications for Teaching and Curriculum

In this final part of the thesis some questions are raised as a result of the study, and some observations and suggestions are outlined related to teaching and curriculum. These are made with respect to: i) the history of mathematics; and ii) teaching the decimal place value structure and its generalisation, and implicate learning in both primary and secondary school, especially number and algebra.

8.3.1 Using the History of mathematics

One recommendation is to use the history of mathematics in mathematics education. Although it is not absolutely essential to use the history of mathematics in teaching, historical scrutiny was very useful for the current research on student understanding of the generalisation of the place value system.

As a teacher and researcher, reviewing the history of mathematics, among other ideas I learned about:

1. the very early historical awareness of the multiplication and exponentiation concepts, perhaps resulting in the naming of large numbers including powers of ten, again, very early in history;
2. the length of time it took for some key concepts to emerge (and the related sign/symbol systems), such as place value and zero, and names for different algebraic variables;

3. the various stages, including the twists, turns and false paths in the evolution of ideas including the place value and algebraic variable concepts and;

4. some of the difficulties and obstacles that past mathematicians encountered in the construction of the above ideas.

Consequently, I was in a better position to understand students’ difficulties, the errors they make, and the obstacles that they face when trying to understand key ideas in the curriculum. The historical review also enabled me to develop teaching sequences for large numbers, generalisation and place value structure, and to propose problems inspired by history. Other benefits that surfaced during the intervention were the increased motivation, and interest shown by many students, particularly in the direct use of history such as other ancient historical number systems. In relation to the use of historical materials in the classroom, in the last couple of decades many resources have become available to teachers (e.g. Joseph, 2011; Katz & Michalowicz, 2004; Nelson et al., 1993; Swetz, 1994).

8.3.2 Teaching and learning

Revisit the place value idea in secondary (or intermediate) school

In this research, the concept of place value was revisited in junior secondary school in order to enhance understanding of the idea. Revisiting, is similar to Bruner’s (1961) ideas involving ‘Spiral Curriculum’ and to the ‘folding back’ feature in Pirie and Kieren’s (1994) model of ‘Growth in mathematical understanding’. As many teachers will confirm, students often struggle with some basic notions, even at secondary school. Hence revisiting fundamental ideas such as positional notation from the high school standpoint (or Year 7/8, 11-12 year-olds in Intermediate school) may be beneficial. This was the case in this research as reflected in the positive results. When students revisit topics/ideas formed earlier, it is not identical to the original cognitive levels; it is now informed and influenced by their enhanced levels of understandings. Hence reviewing an earlier concept at junior secondary school when students are about to learn other related topics may help not only to reinforce, but also to extend and deepen their understanding. It may also be a necessity;
many researchers have pointed out that the place value concept is difficult for primary students to grasp and takes time to learn. Revisiting may also facilitate the formation of vital connections and establishing continuity, and to appreciate mathematics as a system.

**Provide opportunities to read, write, name and compute with, large numbers**

Are we spending too much time on small whole numbers in our mathematics classes even though today’s society is inundated with very large (and very small) numbers? Reading and writing large numbers deserves special treatment in primary and junior secondary school. By doing so, students will be able to develop a sense of number structure, the periodicity of the system and the naming convention. Students need to rote-learn period names (such as thousand, million, billion and so on), place value positions and their relationship to ten. When using large numbers one can see the advantage of expressing a number in exponential form and hence students can be introduced to exponential notation. Furthermore, working with large numbers and comparing numbers of different magnitudes develops *number sense* and aids the sense of relative sizes of numbers or *quantity sense*.

**Provide experiences in grouping reflecting multiplication, repeated multiplication, and their shortened notation using concrete representations**

**Develop positive attitudes towards learning mathematics – includes intellectual patience towards partial understandings, and perseverance**

The place value system is based on the counting of different sized groups, these groups having elements in them that are all powers of the base ten. Students need experiences that bring out the recursive nature of these repeated groupings employing manipulatives such as craft matchsticks or other similar materials. Multiunits need to be constructed for example, through groups of groups (10×10=10²) and groups of groups of groups (10×10×10=10³) and linked to the related exponential notation (see Sections 4.3.2 and 4.3.4). In order that students learn about powers, they should have experiences in grouping and notation not only in base ten, but with *different bases* formed with *different exponents* (see Appendix 5) This is because the exponential form of the multiunits reflect the grouping structure of the place value system, and is useful when calculating and when dealing with very large or very small numbers. Moreover, it is good preparation for algebra. Crucial to this process is the teachers’ role in providing assistance to students to form links between the concrete
materials, pictures and symbols. Given the importance of exponentiation, the question arises; At what level can students be introduced to equal grouping multiplicative structures reflecting powers and its notation? As seen during the intervention in this study, this recursive multiplicative structure is far more difficult to represent than simple groups which can be easily be represented using repeated addition.

Since the above suggestion involving powers and related groupings is dependent on multiplication, it implies that multiplication as it relates to grouping and partitioning should perhaps be introduced sooner rather than later, alongside the learning of basic facts including multiplication tables. That students rote-learn their multiplication tables is important. However, this memorisation (and recall) may be more effective if students have been provided opportunities to move through several representational experiences involving the multiplication concept. As well, the multiplication concept should be founded on an understanding of the operations strategies (Fuson, 2003; Young-Loveridge, 2005). (Interestingly, this feature is consonant with the memorisation aspect in the traditional Indian method of learning (mathematics or other topics); the facility of ancient Indian mathematicians to rote-learn a vast number of mathematical rules in the Vedic literature in verse form). However, Dienes and Golding (1971) have pointed out that one cannot fully comprehend multiplication without understanding exponents. This foregrounds the notion of circularity (Arcavi, 1994; Sfard, 1991) which is sometimes unavoidable in learning and understanding mathematics. That is, on the one hand, a student is required to give meaning to multiplication in order to understand place values, and on the other hand, has to understand place values in order to make sense of the multiplication idea. Hence awareness of the multiplicative structure of place value develops gradually over time alongside operations on numbers. In relation to the circularity aspect, Arcavi (2005) suggests, that sense making, instead of being considered mainly as a cognitive issue, could be more of a matter of attitude towards knowledge and learning; part of the answer has to do with developing intellectual patience and perseverance, within a classroom culture that supports reflection alongside drill and practice, for example involving multiplication tables, and operations on numbers (and symbols in algebra). Competence may also include the ability to live with partial understandings for periods of time until meanings are linked and the big picture emerges. In view of these considerations, it goes without saying that when planning and developing teaching
frameworks, teachers have to take into account not only students’ knowledge and ability levels, and related research results but also students’ attitudes, what is valued and what is expected of them.

**Expose the compositional structure of the place value system in base-10 and non-decimal bases with the use of multiple representations**

Once the ground is prepared with the experience and knowledge of naming and working with large numbers, and groupings in multiplication and exponentiation, classroom activities could focus on the demonstration and discussion of the composition of the place value system, perhaps structured (in decimal and non-decimal bases) according to the historical order as described in Chapters 4 and 5. Again, each of the historical stages may be represented in the concrete, pictorial and symbolic representational systems. Forming connections within and between these various representational systems (M. O. J. Thomas, 2008a) is crucial to make sense of the multiplicative structure. As well as this, performing the traditional algorithms on numbers with the numbers structured according to the historical sequence could further cement students’ understanding (See Section 4.3.4 and Figures 4.8, 4.9, 4.10).

**Make explicit the various uses of literal symbols in algebra**

Teaching should attempt to prepare students for the various usage of letters in algebra that they will need to assimilate, including specific unknown, general number, variable and parameter. The concepts of specific unknown in an equation-solving approach and general number in a generalisation approach are independent but mutually complementary fields in teaching algebra. They are irreducible, and general number is a pre-concept to the idea of variable and parameter. This study has highlighted that historically, a sign system with different letters for different variables and powers of variables are major landmarks in the advancement of mathematics. Hence, in beginning algebra, instruction should make explicit the use of literal symbol as specific unknown and general number, and highlight the difference between the two concepts for enhanced understanding. Furthermore, exponentiation acts as a bridge between the place value system and algebra (in terms of polynomials and notation) and students should be given opportunities to generalise the place values or powers in the system. The above experiences involving groupings in
various bases and its notation, and such generalisations should pave the way for an easier
passage to algebra.

In order to facilitate a discussion of the ideas described above and for their possible
implementation in the classroom, teachers, both pre-service and in-service, will require
professional development. What has been discussed in this section also has implications for
university courses including teacher education (e.g. P. Hughes, 2010; Tee, 2001), and
teacher education research (e.g. K. Clark, 2006).

8.4 Summary and Conclusion

This study sought to improve Year 9 students’ conceptual awareness of the structure of the
decimal place value numeration system through the notion of a general positional notation.
To this end, the first, and major, part of the study consisted of an analysis of the historical
development in India of the current Hindu-Arabic decimal system and the evolution of
algebraic notation. A historical-critical examination conducted in the light of mathematics
education research revealed, among other developments, systematic and early
consideration of large numbers including powers of ten for many centuries. It also enabled,
in a broad sense, the identification of key stages (additive, multiplicative, places and
abstract) in the development of the written decimal place value system with zero. What
was also revealed from Indian history is the growth in algebraic ideas and the related
symbolism characterised by names for several variables as early as the 7th century. That is,
different colours (and their abbreviation) were used for different variables and names were
given for powers of variables. The historical review highlighted the (almost immediate)
impact of the creation of the decimal place value system in India, namely: development of
rules for operations on zero and operations on integers; solving of equations; development
of algebraic symbolism and in turn; the development of general methods of solutions of
equations of the first, second and higher degrees. A similar development appears to have
occurred in Europe subsequent to the arrival of the decimal place value system. The
historical analysis also brought to the fore the unifying role of exponentiation in number
and algebra; that of expressing place values/powers in a numeral as well as powers of
variables. One of the key contributions of this study is the identification and foregrounding
of the above ideas from history and the construction of a (approximate) sequence of
historical events.
The historical development in India described above provided the inspiration and impetus for the second part of the research study. This study involved an *ontological investigation* of Year 9 students’ development of a meaningful understanding of place value, the structure of the numeration system and its generalisation, mediated by the experience of exponentiation. What was considered was how understanding of place value notation might be improved using a combination of historical perspectives, concrete materials, linked *multiple representations* and multiple bases with a focus on understanding of *structure*, and recognition that the numerals that students deal with on a daily basis are number symbols forming part of a *conceptual system*. This process entailed the development of an overarching *framework* and several, but connected *teaching sequences* related to understanding specific place value concepts and their supporting ideas, which again are major contributions of this research.

The results suggest that an understanding and use of large numbers, and of exponentiation (even infinity) is accessible to students. It is also seen that students can achieve a certain measure of success in constructing a sense of the structure of the numeration system to the extent that they were able to: a) write the expanded notation of a number in base ten and non-decimal bases in power form; and b) generalise its multiplicative (including exponential) structure. However, due to the limitations in the design of the study as outlined above, any conclusions drawn can only be tentative.

This study indicates the *feasibility* of a process that involved extraction of ideas from history, development of a teaching framework incorporating the ideas, and implementation of this framework in the classroom. It showed that it is possible to apply such a *teaching framework* within the course of the school curriculum. The study also showed that students respond well when extended beyond what they are responsible for in terms of learning, in order to conceptualise what they *have* to learn in the curriculum and this in turn may have implications for mathematics curriculum development and teaching. It may be that teaching this crucial concept through ideas structured more or less according to the historical order, with an activity, task-based approach may help students to construct the structure of numeration. In summary, it appears that historical ideas may reveal important novel approaches to understanding the concept of the place value system, notation for variables and their powers in algebra.
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APPENDICES
Appendix 1

(Large Number Questions)
Large number questions

Name: __________________________

Answer the following questions as best as you can.

1. What is the largest number that you can say in five seconds?

2. Suppose you are asked to count to the number in 1 above, what would be your method?

3. If you are asked to count in less time to the number in 1 above, what would be your method then?

4. Write one trillion in numerals in two different ways.

5. Estimate how long it would take to count to 1 million.

6. Estimate how long it would take to count to 1 billion. Round your answer sensibly.

7. Estimate how long it would take to count to 1 trillion. Round your answer sensibly.

8. Write one quadrillion, five hundred and thirty six million, eight hundred and seven in numerals.

______________________________________________________________

9. What does $7^{11}$ mean to you? .........................................................
10. Without working out the answer, write which is the larger of the two given numbers.

\[ 348694 \times 4392 \quad \text{or} \quad 8^{14} \]

11. Write what you understand by \(2^{96}\).

12. What does \(a^7\) mean to you?

13. What is the meaning of \(2^{17}\)?

14. Write what you understand by \(a^3\).

15. What is infinity?

16. The following is an example of a large number taken from an ancient Indian text.

A rajju is the distance travelled by a bird in six months if it covers a hundred thousand yojana (approximately a million kilometers) in each blink of its eye.

(i) Work out this number as best as you can.

(ii) Can you think up a similar example of a large number, but which is more relevant to the present?
Appendix 2

(Algebra Questions)
Questions on Algebra

Name:________________________

1. In the following patterns, observe what is changing and what is not changing and then make a generalisation:

   a)  
   \[
   \begin{align*}
   8 + 2 \\
   8 + 5 \\
   8 + 0.34 \\
   8 + \frac{5}{4}
   \end{align*}
   \]

   b)  
   \[
   \begin{align*}
   8 \times 0.6 - 7^2 \\
   8 \times 5 - 4^2 \\
   8 \times \frac{1}{2} - 5^2 \\
   8 \times 4538 - 1.5^2 \\
   8 \times 76 - 840^2 \\
   \end{align*}
   \]

2. (Eight)\(^3\), (six)\(^3\), (four)\(^3\), (seven)\(^3\) ........ .......................Can you generalise this pattern?

3. 4\(\times\) 7, 4\(\times\) 9, 4\(\times\) 12, 4\(\times\) 6........ ................ .................Generalise the given pattern.

4. Observe what is changing and not changing for the following patterns and then make a generalization

   a)  
   \[
   \begin{align*}
   20 + 1 \times 15 \\
   20 + 2 \times 15 \\
   20 + 3 \times 15 \\
   20 + 4 \times 15 \\
   20 + 5 \times 15
   \end{align*}
   \]

   b)  
   \[
   \begin{align*}
   5 \times 1 - 2 = 3 \\
   5 \times 2 - 2 = 8 \\
   5 \times 3 - 2 = 13 \\
   5 \times 4 - 2 = 18 \\
   5 \times 5 - 2 = 23
   \end{align*}
   \]
5. i) Write down what you understand by $P + 5$ and what are the possible values for $P$?

ii) Write down what you understand by $G - 2 + B$. What are possible values that $G$ and $B$ can take?

6. a) Compare and then explain what you understand by i) $3 \times y$ and ii) $3 \times y = 12$.

b) Explain the meaning that you would give $y$ in $3 \times y$ and also in $3 \times y = 12$

7. John has a certain number of sweets in his pocket and to these he adds 3 more. How many does he have altogether now? Can you write this down? ........................................

8. Observe what is changing and not changing. Then make a generalisation and simplify.

<table>
<thead>
<tr>
<th>a)</th>
<th>b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$9 \times 1 + 8 + 7 \times 1$</td>
<td>$16 \times 1 + 8$</td>
</tr>
<tr>
<td>$9 \times 2 + 8 + 7 \times 2$</td>
<td>$16 \times 2 + 8$</td>
</tr>
<tr>
<td>$9 \times 3 + 8 + 7 \times 3$</td>
<td>$16 \times 3 + 8$</td>
</tr>
<tr>
<td>$9 \times 4 + 8 + 7 \times 4$</td>
<td>$16 \times 4 + 8$</td>
</tr>
</tbody>
</table>

c) What do you notice about the generalisation in the above columns?
Appendix 3

(Questions on Number Systems)
Questions on number systems (A) Name:________________________

1. Write the following in words just as you would say them or write the meaning of the numbers:
   a) 7905 __________________________________________________________
   b) 100005 ________________________________________________________

2. Write the following in numerals
   a) fifty three thousand eight hundred and ninety two ______________________
   b) sixty two thousand and nine__________________________

3. For the following numbers, what is the actual value for each of the digits?
   a) 3 5 2 7 5
   b) 6 0 0 8
   c) 7 6 5 8 . 3 2

4. What is the meaning of zero in question 3b?
   ______________________________________________________________________
   ______________________________________________________________________
   ______________________________________________________________________

5. How many symbols do we have in the number system that we use? ____________

6. What is the base of the number system that we use?____________________

7. How many symbols do we need for a number system with base six? _________

8. How many symbols do we need for a number system with base forty- three?____
Questions on number systems (B)  
Name: ______________________

9. Suppose we consider a number system with base six. Write the following numbers in words as you would say them or write what you understand by these numbers.
   a) 4523 _____________________________________________________
   b) 20035 _____________________________________________________
   c) 324.15 _____________________________________________________

10. Suppose we consider a number system with base seven where

    1 is written as 1
    2 is written as Λ
    3 is written as Δ
    4 is written as ♡
    5 is written as Ξ
    6 is written as ☐
    0 is written as 0

Then write the following numbers in words as you would say them or write the meaning of these numbers.
   a) ♡ΔΛ _____________________________________________________
   b) Δ0☐ ☐ _____________________________________________________
   c) ΛΔ0.Ξ☐ _____________________________________________________

Comments:
Questions on number systems (C)  
Name:____________________

11 a) Write only the values of the places for numbers with base 10 on top of the given boxes.

b) In general, what is the value of any place with base 10?____________________

12 a) Write the values of the places for numbers with base 8 on top of the given boxes.

b) Now generalise and write the place value for numbers with base 8.__________

13 a) Write place values for numbers with base 29 on top of the given boxes.

b) Generalise and write the place value for base 29.____________________

14 a) Make a generalization and write the place value for base with any number__________

b) In terms of place value, are there any numbers that you cannot have as the base?
   ______________________________________________________________________
   ______________________________________________________________________

   c) In terms of place value, what types of numbers can the exponent have?
   ______________________________________________________________________

Comments:________________________________________________________________
   ______________________________________________________________________
   ______________________________________________________________________
Appendix 4

(Counting in Different Bases)
## Counting in different bases

<table>
<thead>
<tr>
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<th>Base Six</th>
<th>Base Eight</th>
<th>Base Thirteen</th>
</tr>
</thead>
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<td>1</td>
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<td>1</td>
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</tr>
<tr>
<td>2</td>
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<td>B</td>
</tr>
<tr>
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<td>14</td>
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</tr>
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<td>15</td>
<td>10</td>
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<td>36</td>
<td>28</td>
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Appendix 5

(Calculations with Powers)
Calculations with Powers

Students used their calculators to compute answers to the questions given in the following tables.

5A)

<table>
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<th>Number $n$</th>
<th>$5 \times n$</th>
<th>$5^n$</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>$5 \times 1 = 5$</td>
<td>$5^1 = 5$</td>
</tr>
<tr>
<td>2</td>
<td>$5 \times 2 = 10$</td>
<td>$5^2 = 5 \times 5 = 25$</td>
</tr>
<tr>
<td>6</td>
<td>$5 \times 6 = 30$</td>
<td>$5^6 = 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 15625$</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>$5 \times 14 = 70$</td>
<td>$5^{14} = 5 \text{ multiplied } 14 \text{ times} = 6103515625$</td>
</tr>
<tr>
<td>22</td>
<td>$5 \times 22 = 110$</td>
<td>$5^{22} = 5 \text{ multiplied } 22 \text{ times} = 2.38 \times 10^{15}$</td>
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### 5B)

<table>
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<th>$a^m$</th>
<th>Meaning and value</th>
<th>$m^a$</th>
<th>Meaning and value</th>
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<td>$4^6$</td>
<td>$4 \times 4 \times 4 \times 4 \times 4 = 4096$</td>
<td>$6^4$</td>
<td>$6 \times 6 \times 6 \times 6 = 1296$</td>
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