Economics Department

Economics Working Papers

The University of Auckland *Year* 2002

A Simple Introduction to Dynamic Programming in Macroeconomic Models

Ian King
University of Auckland, ip.king@auckland.ac.nz

A Simple Introduction to Dynamic Programming in Macroeconomic Models

Ian King[*]

Department of Economics
University of Auckland
Auckland
New Zealand

April 2002
(October 1987)

Abstract

This is intended as a very basic introduction to the mathematical methods used in Thomas Sargent's book *Dynamic Macroeconomic Theory*. It assumes that readers have no further mathematical background than an undergraduate "Mathematics for Economists" course. It contains sections on deterministic finite horizon models, deterministic infinite horizon models, and stochastic infinite horizon models. Fully worked out examples are also provided.

[*] Financial support of the Social Sciences and Humanities Research Council of Canada is gratefully acknowledged. I am indebted to David Backus, Masahiro Igarashi and for comments. I would like to thank Tom McCurdy for his comments on earlier drafts, and for suggesting the last example presented here.

FOREWARD (2002)

I wrote this guide originally in 1987, while I was a graduate student at Queen's University at Kingston, in Canada, to help other students learn dynamic programming as painlessly as possible. The guide was never published, but was passed on through different generations of graduate students as time progressed. Over the years, I have been informed that several instructors at several different universities all over the world have used the guide as an informal supplement to the material in graduate macro courses. The quality of the photocopies has been deteriorating, and I have received many requests for new originals. Unfortunately, I only had hard copy of the original, and this has also been deteriorating.

Because the material in this guide is not original at all – it simply summarizes material available elsewhere, the usual outlets for publication seem inappropriate. I decided, therefore, to simply reproduce the handout as a pdf file that anyone can have access to. This required re-typing the entire document. I am extremely grateful to Malliga Rassu, at the University of Auckland for patiently doing most of this work. Rather than totally reorganize the notes in light of what I've learned since they were originally written, I decided to leave them pretty much as they were – with some very minor changes (mainly references). I hope people continue to find them useful.

INTRODUCTION (1987)

This note grew out of a handout that I prepared while tutoring a graduate macroeconomics course at Queen's University. The main text for the course was Thomas Sargent's *Dynamic Macroeconomic Theory.* It had been my experience that some first year graduate students without strong mathematical backgrounds found the text heavy going, even though the text itself contains an introduction to dynamic programming. This could be seen as an introduction to Sargent's introduction to these methods. It is not intended as a substitute for his chapter, but rather, to make his book more accessible to students whose mathematical background does not extend beyond, say, A.C. Chaing's *Fundamental Methods of Mathematical Economics*.

The paper is divided into 3 sections: (i) Deterministic Finite Horizon Models, (ii) Deterministic Infinite Horizon Models, and (iii) Stochastic Infinite Horizon Models. It also provides five fully worked out examples.

1. DETERMINISTIC, FINITE HORIZON MODELS

Let us first define the variables and set up the most general problem, (which is usually unsolvable), then introduce some assumptions which make the problem tractable.

1.1 *The General Problem*:

$$\underset{\{v(t)\}}{Max} \quad U(x_0, x_1, \cdots, x_T; v_0, v_1, \cdots, v_{T-1})$$

subject to

i) $G(x_0, x_1, \cdots, x_T;\ v_0, v_1, \cdots, v_{T-1}) \geq 0$

ii) $v_t \in \Omega$ for all $t = 0 \cdots T-1$

iii) $x_0 = \bar{x}_0$ given

iv) $x_T \geq 0$

Where: x_t is a vector of state variables that describe the state of the system at any point in time. For example, x_t^i could be the amount of capital good *i* at time *t*.

v_t is a vector of control variables which can be *chosen* in every period by the decision-maker. For example v_t^j could be the consumption of good *j* at time *t*.

$U(\cdot)$ is the objective function which is, in general, a function of all the state and control variables for each time period.

$G(\cdot)$ is a system of intertemporal constraints connecting the state and control variables.

Ω is the feasible set for the control variables – assumed to be closed and bounded.

In principle, we could simply treat this as a standard constrained optimisation problem. That is, we could set up a Lagrangian function, and (under the usual smoothness and concavity assumptions) grind out the Kuhn-Tucker conditions.

In general though, the first order conditions will be non-linear functions of *all* the state and control variables. These would have to be solved *simultaneously* to get any results, and this could be extremely difficult to do if T is large. We need to introduce some strong assumptions to make the problem tractable.

1.2 *Time Separable (Recursive) Problem*:

Here it is assumed that both the $U(\cdot)$ and the $G(\cdot)$ functions are time-separable. That is:

$$U(x_0,\cdots,x_T;v_0,\cdots,v_{T-1}) \equiv U_0(x_0,v_0)+U_1(x_1,v_1)+\cdots+U_{T-1}(x_{T-1},v_{T-1})+S(x_T)$$

where $S(x_T)$ is a "scrap" value function at the end of the program (where no further decisions are made). Also, the $G(\cdot)$ functions follow the Markov structure:

$$\left.\begin{aligned} x_1 &= G_0(x_0,v_0) \\ x_2 &= G_1(x_1,v_1) \\ &\vdots \\ x_T &= G_{T-1}(x_{T-1},v_{T-1}) \end{aligned}\right\} \qquad \text{"Transition equations"}$$

Note: Recall that each x_t is a *vector* of variables x_t^i where *i* indexes different kinds of state variables. Similarly with v_t. Time separability still allows interactions of different state & control variables, but only *within* periods.

The problem becomes:

$$\underset{\substack{\{v_t;t=0,1,\ldots,T-1\} \\ v_t \in \Omega}}{Max} \sum_{t=0}^{T-1} U_t(x_t,v_t)+S(x_T)$$

subject to i) $x_{t+1}^i = G_t^i(x_t,v_t)$ $\quad \forall \text{i}=1\cdots\text{n}$ and $t=0,\ldots,T-1$

ii) $x_0^i = \bar{x}_0^i$ given $\quad \forall \text{i}=1\cdots\text{n}$

Once again, in principle, this problem could be solved using the standard constrained optimisation techniques. The Lagrangian is:

$$L = \sum_{t=0}^{T-1} U_t(x_t, v_t) + S(x_T) + \sum_{t=0}^{T-1} \sum_{i=0}^{n} \lambda_t^i \left[G_t^i(x_t, v_t) - x_{t+1}^i \right]$$

This problem is often solvable using these methods, due to the temporal recursive structure of the model. However, doing this can be quite *messy*. (For an example, see Sargent (1987) section 1.2). Bellman's "Principle of Optimality" is often more convenient to use.

1.3 *Bellman's Method (Dynamic Programming):*

Consider the time-separable problem of section 1.2 above, at time $t=0$.

Problem A:

$$\underset{\left\{ \substack{v_t; t=0,1,\ldots,T-1 \\ v_t \in \Omega} \right\}}{Max} \sum_{t=0}^{T-1} U_t(x_t, v_t) + S(x_T)$$

subject to i) $x_{t+1}^i = G_t^i(x_t, v_t) \quad \forall \mathrm{i} = 1 \cdots \mathrm{n}$ and $t = 0, \ldots, T-1$

ii) $x_0^i = \bar{x}_0^i$ given $\quad \forall \mathrm{i} = 1 \cdots \mathrm{n}$

Now consider the same problem, starting at some time $t_0 > 0$.

Problem B:

$$\underset{\left\{ \substack{v_t; t=t_0,\ldots,T-1 \\ v_t \in \Omega} \right\}}{Max} \sum_{t=t_0}^{T-1} U_t(x_t, v_t) + S(x_T)$$

subject to i) $x_{t+1}^i = G_t^i(x_t, v_t) \quad \forall \mathrm{i} = 1 \cdots \mathrm{n}$ and $t = t_0, \ldots, T-1$

ii) $x_{t_0}^i = \bar{x}_{t_0}^i$ given $\quad \forall \mathrm{i} = 1 \cdots \mathrm{n}$

Let the solution to problem B be defined as a value function $V(x_{t_0}, T - t_0)$. Now, Bellman's "*Principle of Optimality*" asserts:

> Any solution to Problem A (i.e. on the range $t = 0 \cdots T$) which yields $x_{t_0} \equiv \bar{x}_{t_0}$ *must* also solve Problem B (i.e.: on the range $t = t_0 \cdots T$).

(Note: This result depends on additive time separability, since otherwise we couldn't "break" the solution at t_0. Additive separability is *sufficient* for Bellman's principle of optimality.)

Interpretation: If the rules for the control variables chose for the t_0 problem are optimal for *any* given $\bar{x}_{t_0}$, then they must be optimal for the $x^*_{t_0}$ of the larger problem.

Bellman's P. of 0. allows us to use the trick of solving large problem A by solving the *smaller* problem B, sequentially. Also, since t_0 is *arbitrary*, we can choose to solve the problem $t_0 = T-1$ first, which is a simple 2-period problem, and then work backwards:

<u>Step 1</u>: Set $t_0 = T-1$, so that Problem B is simply:

$$\underset{\{v_{T-1}\}}{Max}\ U_{T-1}(x_{T-1}, v_{T-1}) + S(x_T)$$

subject to: i) $x_T = G_{T-1}(x_{T-1}, v_{T-1})$

ii) $x_{T-1} = \bar{x}_{T-1}$ given

One can easily substitute the first constraint into the objective function, and use straightforward calculus to derive:

$$v_{T-1} = h_{T-1}(x_{T-1}) \cdots \text{ Control rule for } v_{T-1}$$

This can then be substituted back into the objective fn to characterize the solution as a *value function*:

$$V(x_{T-1}, 1) \equiv U_{T-1}(x_{T-1}, h_{T-1}(x_{T-1})) + S(G_{T-1}(x_{T-1}, h_{T-1}(x_{T-1})))$$

Step 2: Set $t_0 = T-2$ so that Problem B is:

$$\underset{\begin{Bmatrix} v_{T-1} \\ v_{T-2} \end{Bmatrix}}{Max} \{U_{T-2}(x_{T-2}, v_{T-2}) + U_{T-1}(x_{T-1}, v_{T-1}) + S(x_T)\}$$

subject to: i) $x_T = G_{T-1}(x_{T-1}, v_{T-1})$

ii) $x_{T-1} = G_{T-2}(x_{T-2}, v_{T-2})$

iii) $x_{T-2} = \bar{x}_{T-2}$ given

Bellman's P.O. implies that we can rewrite this as:

$$\underset{v_{T-2}}{\text{Max}} \left\{U_{T-2}(x_{T-2}, v_{T-2}) + \underset{v_{T-1}}{Max}\{U_{T-1}(x_{T-1}, v_{T-1}) = S(x_T)\}\right\}$$

subject to (i), (ii) and (iii).

Recall that step 1 has already given us the solution to the inside maximization problem, so that we can re-write step 2 as:

$$\underset{\{v_{T-2}\}}{Max} \{U_{T-2}(x_{T-2}, v_{T-2}) + V(x_{T-1}, 1)\}$$

subject to: i) $x_{T-1} = G_{T-2}(x_{T-2}, v_{T-2})$

ii) $x_{T-2} = \bar{x}_{T-2}$ given

Once again, we can easily substitute the first constraint into the objective function, and use straightforward calculus to derive:

$v_{T-2} = h_{T-2}(x_{T-2})$ $\cdots$ Control rule for v_{T-2}

This can be substituted back into the objective fn. to get a value function:

$$V(x_{T-2}, 2) \equiv \underset{\{v_{T-2}\}}{Max}\{U_{T-2}(x_{T-2}, v_{T-2}) + V(x_{T-1}, 1)\}$$
$$= U_{T-2}(x_{T-2}, h_{T-2}(x_{T-2})) + V(G_{T-2}(x_{T-2}, h_{T-2}(x_{T-2})), 1)$$

Step 3: Using an argument analogous to that used in step 2 we know that, in general, the problem in period T-k can be written as:

$$V(x_{T-k}, k) = \underset{\{v_{T-k}\}}{Max}\{U_{T-k}(x_{T-k}, v_{T-k}) + V(x_{T-k+1}, k-1)\} \quad \frac{\text{"Bellman's"}}{\underline{\text{Equation}}}$$

subject to: i) $x_{T-k+1} = G_{T-k}(x_{T-k}, v_{T-k})$

ii) $x_{T-k} = \bar{x}_{T-k}$ given

This maximization problem, given the form of the value function from the previous round, will yield a control rule:

$$v_{T-k} = h_{T-k}(x_{T-k})$$

Step 4: After going through the successive rounds of single period maximization problems, eventually one reaches the problem in time zero:

$$V(x_0, T) = \underset{\{v_0\}}{Max}\ \{U_0(x_0, v_0) + V(x_1, T-1)\}$$

subject to: i) $x_1 = G_0(x_0, v_0)$

ii) $x_0 = \bar{x}_0$ given

This will yield a control rule:

$$v_0 = h_0(\bar{x}_0)$$

Now, recall that x_0 is *given* a value at the outset of the overall dynamic problem. This means that we have now solved for v_0 as a *number*, independent of the *x's* (except the given $\bar{x}_0$).

Step 5: Using the known x_0 and v_0 and the transition equation:

$$x_1 = G_0(x_0, v_0)$$

it is simple to work out x_1, and hence v_1 from the control rule of that period. This process can be repeated until *all* the x_t and v_t values are known. The overall problem A will then be solved.

1.4 *An Example*:

This is a simple two period minimization problem, which can be solved using this algorithm.

$$\underset{\{v_t\}}{Min} \sum_{t=0}^{1} \left[x_t^2 + v_t^2\right] + x_2^2 \tag{1}$$

subject to: i) $x_{t+1} = 2x_t + v_t$ (2)

ii) $x_0 = 1$ (3)

In this problem, *T=2*. To solve this, consider first the problem in period *T-1* (i.e.: in period 1):

Step 1:

$$\underset{\{v_1\}}{Min} \left\{x_1^2 + v_1^2 + x_2^2\right\} \tag{4}$$

subject to: i) $x_2 = 2x_1 + v_1$ (5)

ii) $x_1 = \bar{x}_1$ given (6)

Substituting 5 and 6 into 4 yields:

$$\underset{\{v_1\}}{Min} \left\{\bar{x}_1^2 + v_1^2 + [2\bar{x}_1 + v_1]^2\right\}$$

FOC: $2v_1 + 2[2x_1 + v_1] = 0$

$\Rightarrow v_1 = -x_1$ Control rule in period 1 (7)

Now substituting 7 back into 4 yields (using 5):

$$V(x_1,1) = x_1^2 + x_1^2 + (2x_1 - x_1)^2$$

$$\Rightarrow V(x_1,1) = 3x_1^2 \qquad (8)$$

Step 2: In period T-2 (i.e.: period 0) Bellman's equation tells us that the problem is:

$$\underset{\{v_0\}}{Min} \; \{x_0^2 + v_0^2 + V(x_1,1)\} \qquad (9)$$

subject to: i) $x_1 = 2x_0 + v_0$ (10)

ii) $x_0 = 1$ (11)

Substituting 8 into 9, and then 10 and 11 into 9 yields:

$$\underset{\{v_0\}}{Min} \; \{1 + v_0^2 + 3[2 + v_0]^2\}$$

FOC: $2v_0 + 6[2 + v_0] = 0$

$\Rightarrow \boxed{v_0 = \frac{-3}{2}}$ Control value in period 0 (12)

Step 5: Substituting 11 and 12 into 10 gives:

$$x_1 + 2 + \left[\frac{-3}{2}\right] \Rightarrow \boxed{x_1 = \frac{1}{2}} \qquad (13)$$

Now substitute 13 into 7 to get:

$\boxed{v_1 = \frac{-1}{2}}$ Control value in period 1 (14)

Finally, substitute 13 and 14 into 5 to get:

$$x_2 = 1 - \frac{1}{2} \Rightarrow \boxed{x_2 = \frac{1}{2}} \qquad (15)$$

Equations 11-15 characterize the full solution to the problem.

2. DETERMINISTIC, INFINITE HORIZON MODELS

2.1. *Introduction:*

One feature of the finite horizon models is that, in general the functional form of the control rules vary over time:

$$v_t = h_t(x_t)$$

That is, the *h* function is different for each *t*. This is a consequence of two features of the problem:

i) The fact that T is finite

ii) The fact that $U_t(x_t, v_t)$ and $G_t(x_t, v_t)$ have been permitted to depend on time in arbitrary ways.

In *infinite* horizon problems, assumptions are usually made to ensure that the control rules to have the *same* form in every period.

Consider the infinite horizon problem (with time-separability):

$$\underset{\left\{\substack{v_t\,;t=0,\ldots,\infty \\ v_t \in \Omega}\right\}}{Max} \sum_{t=0}^{\infty} U_t(x_t, v_t)$$

subject to: i) $x_{t+1} = G_t(x_t, v_t)$

ii) $x_0 = \bar{x}_0$ given

For a unique solution to *any* optimization problem, the objective function should be bounded away from infinity. One trick that facilitates this bounding is to introduce a discount factor β_t where $0 \le \beta_t < 1$.

A convenient simplifying assumption that is commonly used in infinite horizon models is *stationarity*:

Assume i) $\beta_t = \beta \; \forall$t

ii) $U_t(x_t, v_t) = \beta^t U(x_t, v_t)$

iii) $G_t(x_t, v_t) = G(x_t, v_t)$

A *further* assumption, which is *sufficient* for boundedness of the objective function, is boundedness of the payoff function in each period:

Assume: $0 \leq U(x_t, v_t) < M < \infty$ where M is any finite number.

This assumption, however, is not necessary, and there are many problems where this is not used.

The infinite horizon problem becomes:

$$\underset{\left\{\substack{v_t\,;t=0,\ldots,\infty \\ v_t \in \Omega}\right\}}{Max} \sum_{t=0}^{\infty} \beta^t U(x_t, v_t)$$

subject to: i) $x_{t+1} = G(x_t, v_t)$

ii) $x_0 = \bar{x}_0$ given

The Bellman equation becomes:[1]

$$V_t(x_t) = \underset{\{v_t\}}{Max}\left\{\beta^t U(x_t, v_t) + V_{t+1}(x_{t+1})\right\}$$

subject to (i) and (ii)

This equation is defined in *present value* terms. That is, the values are all discounted back to time $t = 0$. Often, it is more convenient to represent these values in *current value* terms:

Define: $W_t(x_t) = \dfrac{v_t(x_t)}{\beta^t}$

Multiplying both sides of the Bellman equation by β^{-t} yields:

$$W_t(x_t) = \underset{\{v_t\}}{Max}\left\{U(x_t, v_t) + \beta W_{t+1}(x_{t+1})\right\} \qquad (*)$$

subject to (i) and (ii).

[1] A subtle change in notation has been introduced here. From this point on, $V_t(x_t)$ represents the form of the value function in period *t*.

It can be shown that the function defined in (*) is an example of a *contraction mapping*. In Sargent's (1987) appendix, (see, also, Stokey *et al*, (1989)) he presents a powerful result called the "contraction mapping theorem", which states that, under certain regularity conditions,[2] iterations on (*) starting from any bounded continuous W_0 (say, $W_0 = 0$) will cause *W* to *converge* as the number of iterations becomes large. Moreover, the $W(\cdot)$ that comes out of this procedure is the *unique optimal* value function for the infinite horizon maximization problem. Also, associated with *W*(.) is a unique control rule $v_t = h(x_t)$ which *solves* the maximization problem.

This means is that there is a unique time invariant value function W(·) which satisfies:

$$W(x_t) = \underset{\{v_t\}}{Max}\{U(x_t, v_t) + \beta W(x_{t+1})\}$$

subject to: i) $x_{t+1} = G(x_t, v_t)$

ii) $x_0 = \bar{x}_0$ given.

Associated with this value function is a unique time invariant control rule:

$$v_t = h(x_t)$$

2.2 *How To Use These Results:*

There are several different methods of solving infinite horizon dynamic programming problems, and three of them will be considered out here. In two, the key step is finding the form of the value function, which is unknown at the outset of any problem even if the functional forms for U(·) and G(·) are known. This is not required in the third.

[2] Stokey *et al.*, (1989) chapter 3 spell these conditions out. Roughly, these amount to assuming that $U(x_t, v_t)$ is bounded, continuous, strictly increasing and strictly concave. Also, the production technology implicit in $G(x_t, v_t)$ must be continuous, monotonic and concave. Finally, the set of feasible end of period *x* must be compact

Consider the general problem:

$$\underset{\left\{\substack{v_t;t=0,\dots,\infty \\ v_t\in\Omega}\right\}}{Max} \sum_{t=0}^{\infty} \beta^t U(x_t, v_t)$$

subject to: i) $x_{t+1} = G(x_t, v_t)$ (also regularity conditions)

ii) $x_0 = \bar{x}_0$ given

<u>Method 1</u>: Brute Force Iterations

First, set up the Bellman equation:

$$W_t(x_t) = \underset{\{v_t\}}{Max}\{U(x_t, v_t) + \beta W_{t+1}(x_{t+1})\}$$

s.t. i) $x_{t+1} = G(x_t, v_t)$

ii) $x_0 = \bar{x}_0$ given.

Set $W_{t+1}(x_{t+1}) = 0,$ and solve the maximization problem on the right side of the Bellman equation. This will give you a control rule $v_t = h_t(x_t).$ Substitute this back into $U(x_t, v_t),$ to get:

$$W_t(x_t) = U(x_t, h_t(x_t))$$

Next, set up the Bellman equation:

$$W_{t-1}(x_{t-1}) = \underset{\{v_{t-1}\}}{Max}\{U(x_{t-1}, v_{t-1}) + \beta U(x_t, h(x_t))\}$$

s.t. i) $x_t = G(x_{t-1}, v_{t-1})$

ii) $x_0 = \bar{x}_0$ given.

Substitute constraint (i) into the maximand to get:

$$W_{t-1}(x_{t-1}) = \underset{\{v_{t-1}\}}{Max}\ \{U(x_{t-1}, v_{t-1}) + \beta U(G(x_{t-1}, v_{t-1}), h(G(x_{t-1}, v_{t-1})))\}$$

Solve the maximization problem on right, to get: $v_{t-1} = h_{t-1}(x_{t-1})$

Now substitute this back into the right side to get: $W_{t-1}(x_{t-1})$

.

Continue this procedure until the $W(\cdot)$'s *converge*. The control rule associated with the $W(\cdot)$ at the point of convergence *solves* the problem.

Method 2: Guess and Verify.

This method is a variant of the first method, but where we make an informed guess about the functional form of the value function. If available, this method can save a lot computation. For certain classes of problems, when the period utility function $U(x_t, v_t)$ lies in the HARA class (which includes CRRA, CARA, and quadratic functions), the value function takes the same general functional form.[3] Thus, for example, if the period utility function is logarithmic, we can expect the value function will also be logarithmic.

As before, we first set up the Bellman equation:

$$W(x_t) = \underset{\{v_t\}}{Max} \left\{U(x_t, v_t) + \beta W(x_{t+1})\right\}$$

subject to: i) $x_{t+1} = G(x_t, v_t)$

ii) $x_0 = \bar{x}_0$ given.

Second, *guess* the form of the value function, $W^G(x_{t+1})$, and substitute the guess into the Bellman equation:

$$W(x_t) = \underset{\{v_t\}}{Max} \left\{U(x_t, v_t) + \beta W^G(x_{t+1})\right\} \qquad \text{s.t. (i) and (ii)}$$

[3] See Blanchard and Fischer (1989), chapter 6, for a discussion.

Third, perform the maximization problem on the right side. Then substitute the resulting control rules back into the Bellman equation to (hopefully) verify the initial guess. (i.e.: The guess is verified if $W(x_t) = W^G(x_{t+1})$.)

If the initial guess was *correct*, then the problem is solved. If the initial guess was *incorrect*, try the form of the value function that is suggested by the first guess as a second guess, and repeat the process. In general, successive approximations will bring you to the unique solution, as in method 1.

<u>Method 3</u>: Using the Benveniste-Schienkman Formula.

This is Sargent's preferred method of solving problems in his Dynamic Macro Theory textbook. The advantage of using this method is that one is not required to know the form of the value function to get some results. From Sargent [1987] p. 21, the B-S formula is:

$$\frac{\partial W(x_t)}{\partial x_t} = \frac{\partial U(x_t, h(x_t))}{\partial x_t} + \frac{\beta \partial G(x, h(x))}{\partial x_t} \frac{\partial W(G(x, h(x))}{\partial x_{t+1}} \qquad \textit{B-S Formula } 1$$

Sargent shows that if it is possible to re-define the state and control variables in such a way as to *exclude* the state variables from the transition equations, the B-S formula simplifies to:

$$\frac{\partial W(x_t)}{\partial x_t} = \frac{\partial U(x_t, h(x_t))}{\partial x_t} \qquad \textit{B-S Formula } 2$$

The steps required to use this method are:

(1) Redefine the state and control variables to exclude the state variables from the transition equations.

(2) Set up the Bellman equation with the value function in general form.

(3) Perform the maximization problem on the r.h.s. of the Bellman equation. That is, derive the F.O.C.'s.

(4) Use the B-S Formula 2 to substitute out any terms in the value function.

As a result, you will get an Euler equation, which may be useful for certain purposes, although it is essentially a second-order difference equation in the state variables.

2.3. *Example 2*: The Cass-Koopmans Optimal Growth Model.

$$\underset{\{c_t, k_{t+1}\}}{Max} \sum_{t=0}^{\infty} \beta^t \ell n C_t \quad \text{s.t.} \qquad \text{i)} \quad k_{t+1} = Ak_t^{\alpha} - C_t$$

$$\text{ii)} \quad k_0 = \bar{k}_0 \ \text{ given.}$$

Where : $C_t \equiv$ consumption

$k_t \equiv$ capital stock

$0 < \beta < 1$

$0 < \alpha < 1$

Since the period utility function is in the HARA class, it is worthwhile to use the guess-verify method.

The Bellman equation is:

$$W(k_t) = \underset{\{C_t, k_{t+1}\}}{Max} \left\{\ell n C_t + \beta W(k_{t+1})\right\} \quad \text{s.t. (i) and (ii)} \tag{1}$$

Now *guess* that the value fn. is of this form:

$$W(k_t) = E + F \ \ell \text{n k}_\text{t} \tag{2}$$

$$\Rightarrow W(k_{t+1}) = E + F\ell n k_{t+1} \tag{2'}$$

where *E* and *F* are coefficients to be determined.

Substitute constraint (i) and (2') into (1) to get:

$$W(k_t) = \underset{\{k_{t+1}\}}{Max} \left\{\ell n\left[Ak_t^{\alpha} - k_{t+1}\right] + \beta\left[E + F \ \ell n \ k_{t+1}\right]\right\} \tag{3}$$

$$\text{FOC:} \quad \frac{1}{Ak_t^{\alpha} - k_{t+1}} = \frac{\beta F}{k_{t+1}}$$

$$\Rightarrow k_{t+1} = \frac{\beta F}{1+\beta F} A k_t^{\alpha} \tag{4}$$

Now substitute 4 back into 3:

$$W(k_t) = \ell n\left[A k_t^{\alpha} - \frac{\beta F}{1+\beta F} A k_t^{\alpha} \right] + \beta E + BF\ \ell \mathrm{n}\left[\frac{\beta \mathrm{F}}{1+\beta \mathrm{F}} A k_t^{\alpha} \right]$$

$$\Rightarrow W(k_t) = \left[\ell n\ A - \ell n(1+\beta F) + \beta E + \beta F\ \ell \mathrm{n}\ A + \beta F\ \ell n\left[\frac{\beta F}{1+\beta F} \right] \right] + \alpha(1+\beta F)\ell n\ \mathrm{k_t} \tag{5}$$

Now, comparing equations (2) and (5), we can determine the coefficient for $\ell n\ \mathrm{k_t}$ *:*

$$F = \alpha(1+\beta F)$$

$$\Rightarrow F = \frac{\alpha}{1-\alpha\beta} \tag{6}$$

Similarly, *E* can be determined from (2) and (5), knowing (6).

Since both the coefficient E and F have been determined, the initial guess (equation 2) has been *verified.*

Now (6) can be substituted back into (4) to get:

$$k_{t+1} = \alpha\beta A k_t^{\alpha} \qquad \text{and using constraint (ii):} \tag{7}$$

$$C_t = [1-\alpha\beta] A k_t^{\alpha} \tag{8}$$

Given the initial $\bar{k}_0$ equations (7) - (8) completely characterize the solution paths of *C* and *k.*

2.4. *Example 3*: Minimizing Quadratic Costs

This is the infinite horizon version of example 1.

$$\underset{\{v_t\}}{Min} \sum_{t=0}^{\infty} \beta^t \left[x_t^2 + v_t^2\right]$$

subject to i) $x_{t+1} = 2x_t + v_t$

ii) $x_0 = \bar{x}_0$ given

Since this is a linear-quadratic problem, it is possible to use the guess-verify method.

The Bellman equation is:

$$W(x_t) = \underset{\{v_t\}}{Min} \left\{ x_t^2 + v_t^2 + \beta W(x_{t+1}) \right\} \tag{1}$$

s.t. i) $x_{t+1} = 2x_t + v_t$

ii) $x_0 = \bar{x}_0$ given

Now *guess* that the value fn. is of this form:

$$W(x_t) = Px_t^2 \tag{2}$$

$$\Rightarrow W(x_{t+1}) = Px_{t+1}^2 \tag{2'}$$

where P is a coefficient to be determined.

Substitute constraint (i) and (2'), then the result into (1) to get:

$$W(x_t) = \underset{\{v_t\}}{Min} \left\{ x_t^2 + v_t^2 + \beta P[2x_t + v_t]^2 \right\} \tag{3}$$

FOC: $2v_t + 2\beta P[2x_t + v_t] = 0$

$$\Rightarrow v_t[1 + \beta P] = -2\beta P x_t$$

$$\Rightarrow v_t = -\frac{2\beta P}{1 + \beta P} x_t \tag{4}$$

Substitute (4) back into (3) to get:

$$W(x_t) = x_t^2 + \left[-\frac{2\beta P}{1+\beta P} x_t \right]^2 + \beta P \left[2x_t - \frac{2\beta P}{1+\beta P} x_t \right]^2$$

$$\Rightarrow W(x_t) = \left[1 - \left[\frac{2\beta P}{1+\beta P} \right]^2 \right] x_t^2 + \beta P \left[2 - \frac{2\beta P}{1+\beta P}^{\,2} \right] x_t^2$$

$$\Rightarrow W(x_t) = \left\{ 1 - \left[\frac{2\beta P}{1+\beta P} \right]^2 + \beta P \left[2 - \frac{2\beta P}{1+\beta P} \right]^2 \right\} x_t^2 \qquad (5)$$

P can now be determined by comparing the equations (2) and (5):

$$P = \left\{ 1 - \left[\frac{2\beta P}{1+\beta P} \right]^2 + \beta P \left[2 - \frac{2\beta P}{1+\beta P} \right]^2 \right\} \qquad (6)$$

The P that solves (6) is the coefficient that was to be determined. For example, set $\beta = 1,$ and equation (6) implies:

$$P = 4.24$$

Equation (4) becomes : $v_t = -1.62 x_t$ (7)

Constraint (i) becomes: $x_{t+1} = 0.38 x_t$ (8)

For a given $\bar{x}_0$, equations (7) and (8) completely characterize the solution. Note from equation 8 that the system will be stable since 0.38 < 1.

2.5. *Example 4*: Life Cycle Consumption

$$\underset{\{c_t\}_{t=0}^{\infty}}{Max} \quad \sum_{t=0}^{\infty} \beta^t U(C_t) \tag{1}$$

s.t. i) $A_{t+1} = R_t\left[A_t + y_t - C_t\right]$ Transition equation (2)

ii) $A_0 = \overline{A}_0$ given (3)

Where: $C_t \equiv$ consumption

$y_t \equiv$ known, exogenously given income stream

$A_t \equiv$ non - labour wealth beginning in time t

$R_t \equiv$ known, exogenous sequence of one-period rates of return on A_t

To rule out the possibility of infinite consumption financed by unbounded borrowing, a present value budget constraint is imposed:

$$C_t + \sum_{j=1}^{\infty}\left[\prod_{k=0}^{j-1} R_{t+k}^{-1}\right] C_{t+j} = y_t + \sum_{j=1}^{\infty}\left[\prod_{k=0}^{j-1} R_{t+k}^{-1}\right] y_{t+j} + A_t \tag{4}$$

Here, we are not given an explicit functional form for the period utility function. We can use method 3 in this problem to derive a well-known result.

To use the B-S formula 2, we need to define the state and control variables to exclude state variables from the transition equation.

Define: State Variables: $\{A_t, y_t, R_{t-1}\}$

Control Variable: $v_t \equiv \dfrac{A_{t+1}}{R_t}$ (5)

Now we can re-write the transition equation as:

$$A_{t+1} = R_t v_t \tag{2'}$$

Notice that (2') and (2) imply: $C_t = A_t + y_t - v_t$ (6)

The problem becomes:

$$\underset{\{v_t\}_{t=0}^{\infty}}{Max}\sum_{t=0}^{\infty}\beta^t U\left(A_t + y_t - v_t\right) \quad (1')$$

s.t. i) $A_{t+1} = R_t v_t$ (2')

ii) $A_0 = \overline{A}_0$ given (3)

} Reformulated problem

The Bellman equation is:

$$W(A_t, y_t, R_{t-1}) = \underset{\{v_t\}}{Max}\ \{U(A_t + y_t - v_t) + \beta W(A_{t+1}, y_{t+1}, R_t\} \qquad \text{s.t. 2' and 3.}$$

Substituting (2') into the Bellman equation yields:

$$W\left(A_t, y_t, R_{t-1}\right) = \underset{\{v_t\}}{Max}\ \left\{U\left(A_t + y_t - v_t\right) + \beta W\left(R_t v_t, y_{t+1}, R_t\right)\right\}$$

FOC: $-U'\left(A_t + y_t - v_t\right) + \beta R_t W_1\left(R_t v_t, y_{t+1}, R_t\right) = 0$ (7)

Now recall that the B-S formula 2 implies:

$$W_1\left(R_t v_t, y_{t+1}, R_t\right) = U'\left[A_{t+1} + y_{t+1} - \frac{A_{t+2}}{R_{t+1}}\right] \equiv U'\left(C_{t+1}\right) \qquad (8)$$

Substituting (8) into (7) gives:

$$U'\left(C_t\right) = \beta R_t U'\left(C_{t+1}\right) \qquad \underline{\text{Euler Equation}} \qquad (9)$$

Now, to get further results, let's impose structure on the model by specifying a particular utility function:

Let $U\left(C_t\right) \equiv \ell n\, C_t$

$$\Rightarrow U'\left(C_t\right) = \frac{1}{C_t} \qquad (10)$$

Substituting 10 into 9 yields:

$$C_{t+1} = \beta R_t C_t \qquad \text{now updating:} \qquad (11)$$

$$\Rightarrow C_{t+2} = \beta R_{t+1}[\beta R_t C_t]$$

$$\Rightarrow C_{t+2} = \beta^2 \prod_{k=0}^{1} R_{t+k} C_t$$

In general: $C_{t+j} = \beta^j \left[\prod_{k=0}^{j-1} R_{t+k} \right] C_t$ (12)

Substituting 12 back into 4 yields:

$$C_t + \sum_{j=1}^{\infty} \left[\prod_{k=0}^{j-1} R_{t+k}^{-1} \right] \beta^j \left[\prod_{k=0}^{j-1} R_{t+k} \right] C_t = y_t + \sum_{j=1}^{\infty} \left[\prod_{k=0}^{j-1} R_{t+k}^{-1} \right] y_{t+j} + A_t$$

$$\Rightarrow \quad C_t + \sum_{j=1}^{\infty} \beta^j C_t = \quad " \quad " \quad "$$

$$\Rightarrow \quad \sum_{j=0}^{\infty} \beta^j C_t = \quad " \quad " \quad "$$

$$\Rightarrow \quad \frac{C_t}{1-\beta} = \quad " \quad " \quad "$$

$$\Rightarrow C_t = (1-\beta)\left[y_t + \sum_{j=1}^{\infty} \left[\prod_{k=0}^{j-1} R_{t+k}^{-1} \right] y_{t+j} + A_t \right]$$

Life Cycle Consumption Function

3. STOCHASTIC INFINITE HORIZON MODELS

3.1. *Introduction:*

As with deterministic infinite horizon problems, it is convenient to assume time separability, stationary, and boundedness of the payoff function. However, stochastic models are more general than deterministic ones because they allow for some uncertainty.

The type of uncertainty that is usually introduced into these models is of a very specific kind. To preserve the recursive structure of the models, it is typically assumed that the stochastic shocks the system experiences follow a homogeneous first order Markov process.

In the simplest terms, this means that the value of this period's shock depends only on the value of last period's shock, not upon any earlier values. (White noise shocks are a special case of this: they do not even depend upon last period's shock.)

Consider the stochastic problem:

$$\underset{\left\{\substack{v_t;t=0,\dots,\infty \\ v_t\in\Omega}\right\}}{Max} \; E_0 \sum_{t=0}^{\infty} \beta^t U(x_t, v_t)$$

subject to: i) $x_{t+1} = G(x_t, v_t, \varepsilon_{t+1})$

ii) x_0 given

where $\{\varepsilon_t\}_{t=0}^{\infty}$ is a sequence of random shocks that take on values in the interval $[\underline{\varepsilon}, \bar{\varepsilon}]$ and follow a homogeneous first order Markov process with the conditional cumulative density function

$$F(\varepsilon', \varepsilon) = Pr\{\varepsilon_{t+1} \le \varepsilon' \mid \varepsilon_t = \varepsilon\} \qquad \text{(iii)}$$

Also, E_t denotes the mathematical expectation, given information at time *t*, I_t. We assume:

$$I_t = \left(\{x_k\}_{k=0}^{t}, \{v_k\}_{k=0}^{t}, \{\varepsilon_k\}_{k=0}^{t}, G(.), U(.), F(.)\right)$$

The assumed sequence of events is:

(1) x_t is observed

(2) Decision maker chooses v_t

(3) Nature chooses ε_{t+1}

(4) Next period occurs.

The Bellman equation for this problem is:

$$W(x_t, \varepsilon_t) = \underset{\{v_t\}}{Max}\{U(x_t, v_t) + \beta E_t W(x_{t+1}, \varepsilon_{t+1})\}$$

As in the deterministic infinite horizon problem, under regularity conditions[4] iterations on

$$W_t(x_t, \varepsilon_t) = \underset{\{v_t\}}{Max}\{U(x_t, v_t) + \beta E_t W_{t+1}(x_{t+1}, \varepsilon_{t+1})\}$$

starting from any bounded continuous W_{t+1} (say, $W_{t+1} = 0$) will cause $W(.)$ to converge as the number of iterations becomes large. Once again, the $W(\cdot)$ that comes out of this procedure is the *unique* optimal value function for the above problem.

Furthermore, associated with $W(\cdot)$ is a unique time invariant control rule $v_t = h(x_t)$ which solves the maximization problem.

3.2 *How to Use These Results:*

The solution techniques given in the deterministic infinite horizon problem still work in the stochastic infinite horizon problem, and there is no need to repeat them. Perhaps the best way to illustrate these results is by using an example.

3.3 *Example 5*: A Stochastic Optimal Growth Model with a Labour-Leisure Trade-off.

This example is chosen not only to illustrate the solution techniques given above, but also to introduce the reader to a particular type of model. This modelling approach is used in the "Real Business Cycle" literature of authors such as Kydland and Prescott [1982] and Long and Plosser [1983].

Under the appropriate assumptions made about preferences and technology in an economy, the following optimal growth model can be interpreted as a competitive equilibrium model.[5] That is, this model mimics a simple dynamic stochastic general equilibrium economy. Because of this, the solution paths for the endogenous variables generated by this model can be compared with actual paths of these variables observed in real-world macroeconomies.

---

[4] The same conditions as in Note 2, with the added assumption of homogeneous first order Markov shocks.

[5] See Stokey *et al.*, (1989), chapter 15 for further discussion on this topic.

Consider the following problem:

$$\underset{\{C_t, k_t, n_t\}}{Max} \quad E_0 \sum_{t=0}^{\infty} \beta^t [\ell n\, C_t + \delta\, \ell n(1 - n_t)] \tag{1}$$

subject to:

i) $C_t + k_t = y_t$ (2)

ii) $y_{t+1} = A_{t+1} k_t^{\alpha} n_t^{1-\alpha}$ (3)

iii) $\ell n\, A_{t+1} = \rho \ell n\, A_t + \xi_{t+1}$ (4)

where n_t represents units of labour chosen in period t, δ is a positive parameter, ρ is a parameter which lies in the interval (-1, 1), and ξ_t represents a white noise error process.[6] The endogenous variable y_t represents output in period *t*. The rest of the notation is the same as in example 2.

These are only two substantive differences between this example and example 2. First, a labour-leisure decision is added to the model, represented by the inclusion of the term $(1 - n_t)$ in the payoff function (1) and the inclusion of n_t in the production function (3). Second, the production technology parameter A_t in equation 3 is assumed to evolve over time according to the Markov process (4).

Notice that in this example $C_t, k_t,$ and n_t are the control variables chosen every period by the decision maker, whereas y_t and A_t are the state variables. To solve this problem, we first set up the Bellman equation:

$$W(y_t, A_t) = \underset{\{C_t, k_t, n_t\}}{Max} \quad \{[\ell\text{n}\, C_t + \delta \ell n(1 - n_t)\,] + \beta E_t(y_{t+1}, A_{t+1})\} \tag{5}$$

Since this is a logarithmic example, we can use the guess-verify method of solution. The obvious first *guess* for the form of the value function is:

$$W(y_t, A_t) = D + G\ell n\; y_t + H\ell n\; A_t \tag{6}$$

---

[6] For example, ξ_t is normally distributed with mean 0 and variance σ_{ξ}^{2}.

where *D, G* and *H* are coefficients to be determined.

$$\Rightarrow \quad W(y_{t+1}, A_{t+1}) = D + G\ell n\ y_{t+1} + H\ell n\ A_{t+1} \qquad (6')$$

To verify this guess, first substitute (3) into (6') to get

$$W(y_{t+1}, A_{t+1}) = D + G[\ell n\ A_{t+1} + \alpha\ell n\ k_t + (1-\alpha)\ell n\ n_t] + H\ell n\ A_{t+1}$$

Now substitute this into equation 5:

$$W[y_t, A_t] = \underset{\{C_t, k_t, n_t\}}{Max} \ \{[\ell n\ C_t + \delta\ell n(1-n_t)] + \beta E_t[D + (G+H)\ell n\ A_{t+1} + G\alpha\ell n\ k_t + G(1-\alpha)\ell n\ n_t]\}$$

Using equation 2 to substitute out C_t in the above yields:

$$W[y_t, A_t] = \underset{\{k_t, n_t\}}{Max} \ \{[\ell n\ (y_t - k_t) + \delta\ell n(1-n_t)] + \beta D + \beta G\alpha\ell n\ k_t + \beta G(1-\alpha)\ell n\ n_t \beta(G+H)E_t\ell n\ A_{t+1}\}$$

(7)

The FOCs are:

$$n_t : \frac{-\delta}{1-n_t} + \frac{\beta G(1-\alpha)}{n_t} = 0 \quad \Rightarrow \quad n_t = \frac{\beta G(1-\alpha)}{\delta + \beta G(1-\alpha)} \qquad (8)$$

$$k_t : -\frac{1}{y_t - k_t} + \frac{\beta G\alpha}{k_t} = 0 \quad \Rightarrow \quad k_t = \frac{\alpha\beta G}{1+\alpha\beta G} y_t \qquad (9)$$

Now we can substitute equations 8 and 9 back into 7 to get:

$$W(y_t, A_t) = \ell n\left[\frac{1}{1+\alpha\beta G} y_t\right] + \delta\ell n\left[\frac{\delta}{\delta + \beta G(1-\alpha)}\right] + \beta D + \alpha\beta\ell n\left[\frac{\alpha\beta G}{1+\alpha\beta G} y_t\right]$$

$$+ \beta G(1-\alpha)\ell n\left[\frac{\beta G(1-\alpha)}{\delta + \beta G(1-\alpha)}\right] + \beta(G+H)E_t\ell n\ A_{t+1} \qquad (7')$$

Recall from equation 4 that $\ell n\ A_{t+1} = \rho\ell n\ A_t + \xi_{t+1}$ where ξ_{t+1} is a white noise error term with mean zero. Hence:

$$E_t\ell n\ A_{t+1} = \rho\ell n\ A_t.$$

Collecting terms in equation 7' yields:

$$W(y_t, A_t) = (1 + \alpha\beta G)\ell n\, y_t + \beta(G + H)\rho \ell n\, A_t + \text{constants} \tag{10}$$

Comparing equations 6 and 10, it is clear that

$$G = 1 + \alpha\beta \Rightarrow G = \frac{1}{1 - \alpha\beta} \tag{11}$$

$$H = \rho\beta(G + H) \Rightarrow \frac{\rho\beta}{(1 - \alpha\beta)^2} \tag{12}$$

Similarly, the constant D can be determined from equations 6 and 10, knowing the values of *G* and *H* given in equations 11 and 12.

Since all the coefficients have been determined, the initial guess of the form of the value function (equation 6) has been *verified*. We can now use equations 11 and 12 to solve for the optimal paths of $n_t, k_{t,}$ and C_t as functions of the state y_t.

From equation 9: $k_t = \alpha\beta y_t$ (13)

From equation 8: $n_t = \dfrac{\beta(1 - \alpha)}{\delta(1 - \alpha\beta) + \beta(1 - \alpha)}$ (14)

From equations 13 and 2: $C_t = (1 - \alpha\beta) y_t$ (15)

Notice that while the optimal k_t and C_t are functions of the state y_t, the optimal n_t is not. That is, n_t is *constant*. This is a result that is peculiar to the functional forms chosen for the preferences and technology, and it provides some justification for the absence of labour-leisure decisions in growth models with logarithmic functional forms.

REFERENCES

Blanchard, O., and Fischer, S., *Lectures on Macroeconomics*, MIT Press, (1989).

Chaing, A.C., *Fundamental Methods of Mathematical Economics*, Third Edition, McGraw-Hill, New York (1984).

Kydland, F., and Prescott, E. "Time to Build and Aggregate Fluctuations", *Econometrica* 50, pp. 1345-1370 (1982).

Long, J., and Plosser, C., "Real Business Cycles", *Journal of Political Economy* 90, pp. 39-69 (1983).

Manuelli, R., and Sargent, T., *Exercises in Dynamic Macroeconomic Theory*, Harvard University Press, (1987).

Sargent, T., *Dynamic Macroeconomic Theory*, Harvard University Press, (1987).

Stokey, N., Lucas, R.E., with Prescott, E., *Recursive Methods for Economic Dynamics*, Harvard University Press (1989).