LOGIC BLOG 2011

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Iskander Kallimullin and André Nies, *Structures that are computable almost surely*, Logic Blog, March 2010, available at `http://dx.doi.org/2292/9821`.

Contents

Part 1. Randomness, Kolmogorov complexity, and computability

1. Jan-May 2011: Demuth randomness, weak Demuth randomness, and the corresponding lowness notions

This section contains news on Demuth randomness and weak Demuth randomness due to Nies and Kučera. They characterize arithmetical complexity and study the diamond class of no-weakly Demuth random.

The main result of this section is on lowness for (weak) Demuth randomness, due to Bienvenu, Downey, Greenberg, Nies, Turetsky. They characterize lowness for Demuth by a concept called BLR-traceability, which implies being in computably dominated $\cap$ jump traceable (they can now show that it is a proper subclass). Paper submitted Feb. 2012.

1.1. Basic definitions.

For a set $W \subseteq 2^{<\omega}$, we let

$$[W]^{\prec} = \{Z \in 2^{\mathbb{N}} \colon \exists n\, Z\restriction_n \in W\},$$

the corresponding open class in Cantor space.

Definition 1.1. *A* Demuth test *is a sequence of c.e. open sets $(S_m)_{m\in\mathbb{N}}$ such that $\forall m\, \lambda S_m \leq 2^{-m}$, and there is a function $f \leq_{\mathrm{wtt}} \emptyset'$ such that $S_m = [W_{f(m)}]^{\prec}$.*

A set Z passes *the test if $Z \notin S_m$ for almost every m. We say that Z is* Demuth random *if Z passes each Demuth test.*

If we apply the usual passing condition for tests, we obtain the following notion.

Definition 1.2. *We say that Z is* weakly Demuth random *if for each Demuth test $(S_m)_{m\in\mathbb{N}}$ there is an m such that $Z \notin S_m$.*

Remark 1.3. Recall that $f \leq_{\mathrm{wtt}} \emptyset'$ if and only if f is ω-c.e., namely, $f(x) = \lim_t g(x,t)$ for some computable function g such that the number of changes $g(x,t) \neq g(x,t-1)$ is computably bounded in x. Hence the intuition is that we can change the m-component S_m a computably bounded number of times. We will denote by $S_m[t]$ the version of the component S_m that we have at stage t. Thus $S_m[t] = [W_{g(m,t)}]^{\prec}$ where g is understood to be a computable approximation of f as above.

1.2. **Arithmetical Complexity.** Kučera and Nies worked in Prague in May. They proved the following.

Proposition 1.4. *(i) Weak Demuth randomness is Π^0_3.*
(ii) Demuth randomness is not Π^0_3.

In [33] they had shown that both notions are Π^0_4.

Proof. (i) Let $(\Gamma_e, h_e)_{e\in\mathbb{N}}$ be an effective listing of pairs of Turing functionals and use bounds (pc functions), such that if $p = \Gamma^X_e(m)\downarrow$ then $\lambda[W_p]^{\prec} \leq 2^{-m}$. If $\Gamma^{\emptyset'}_e$ is total with use bound h_e (also total) then this determines a Demuth test according to Definition 7.4. Conversely, we obtain all Demuth tests that way. Then Z is not weakly Demuth random iff

$$\exists e\, \forall m \forall s\, \exists t > s\, [p = \Gamma^{\emptyset'}_e(m)[t]\downarrow \text{ with use bound } h_{e,t}(m) \wedge Z \in [W_{p,t}]^{\prec}].$$

For, if the pair $(\Gamma^{\emptyset'}_e, h_e)$ does not determine a Demuth test then the condition for e fails. This shows (i).
(ii) Otherwise, the class of non-Demuth random ML-random sets is a Σ^0_3 null class, so there must be a c.e. noncomputable set A below all of them. This contradicts the fact that there is a high ML-random Δ^0_2 set $Z \not\geq_T A$; this set is not GL_1 and hence not Demuth random. (See [33, Thm 3.2] for a somewhat stronger result that even implies the existence of a weakly Demuth random set Z.)

□

1.3. **The diamond class of the non-weakly Demuth randoms.**

Theorem 1.5 (Kučera and Nies)**.** *Let A be a c.e. set. Then*
A is Turing below all ML-random but non-weak Demuth random sets $\Leftrightarrow$ A is strongly jump traceable.

Proof. ($\Rightarrow$) An ω-c.e. set is not weakly Demuth random. If A is below all ω-c.e. ML-randoms then A is already s.j.t. by a result of [29].
($\Leftarrow$) Suppose $(S_m)_{m\in\mathbb{N}}$ is a Demuth test with $S_m \supseteq S_{m+1}$, . We need to show that $A \leq_{\mathrm{T}} Y$ for any ML-random set Y such that $Y \in \bigcap_m S_m$.

As in Remark 1.3 we have $S_m[t] = [W_{g(m,t)}]^{\prec}$ where g is a computable binary function with a computably bounded number of changes $g(m,t) \neq g(m,t-1)$. We may assume that $S_m[t] \supseteq S_{m+1}[t]$ for each m, t.

For $m < s$ let $r(m,s)$ be the maximum e such that $g(i,t)$ is stable for stages t, $m < t \leq s$, for all $i \leq e$. Let

$$G_m = \bigcup_{s>m} [W_{g(r(m,s),s),s}]^{\prec}$$

Informally, at stage s, the Σ^0_1 set G_m copies the version S_e for e largest that hasn't changed since m. If it does change G_m takes the (larger) version S_{e-1} etc. till this stops.

Clearly $\bigcap_e S_e$ is contained in the Π^0_2 null class $\bigcap_m G_m$. Now consider the usual Hirschfeldt-Miller cost function

$$c(m,s) = \lambda G_{m,s}.$$

Recall that if a set A obeys c then A is below each ML-random set $Y \in \bigcap_m G_m$ by [43, Thm. 5.3.x].

This cost function c is benign. For let $c(x,s) > 2^{-e}$. Then $r(x,s) < e$, so $g(i,t)$ has changed for some $i \leq e$ between stages x and s. So benignity follows from the hypothesis that (S_e) is a Demuth test.

Now, by [30], the c.e. strongly jump traceable set A obeys c. □

Let $\mathcal{C}$ be the class of non-weakly Demuth random sets. Then the theorem says that $\mathcal{C}^\Diamond$ coincides with the strongly jump traceable c.e. sets. We also considered the question whether there is a single weakly Demuth random above all c.e. strongly jump traceables. If not, then this would mean that the non-weakly Demuth random sets form the largest class $\mathcal{C}$ with $\mathcal{C}^\Diamond =$ the c.e. strongly jump traceables, where for a class $\mathcal{C}$ only the intersection with MLR counts.

1.4. **Lowness for weak Demuth randomness.** For details on the following summary, see our forthcoming paper [5]. Z is weakly Demuth *by* an oracle set A if it passes every weak A-Demuth test where the number of version changes is computably bounded.

Proposition 1.6. *The following are equivalent for a set A:*

(i) *Every weak Demuth random is weak Demuth random by A.*
(ii) *A is K-trivial.*

Proof. The implication (i) $\Rightarrow$ (ii) follows from a result of Bienvenu and Miller [6].

Other direction by Claim 5.7 of Nies' paper "c.e. sets below random sets" [44]. □

Theorem 1.7. *The only sets that are low for weak Demuth randomness are the computable sets.*

Since each K-trivial is Δ^0_2, by Proposition 1.6 it suffices to prove the following.

Lemma 1.8. *Let A be a set computing a function f which is not dominated by any computable function. Then there is a weakly Demuth random Z which is not weakly Demuth random relative to A.*

1.5. **BLR-traceability.**

Definition 1.9. *A BLR trace is a sequence $(T_n)_{n\in\mathbb{N}}$ such that $T_n = W_{r(n)}$ where r is an ω-c.e. function. Let h be an order function. We say h is a bound for (T_n) if $|T_n| \leq h(n)$ for each n.*

Definition 1.10. *We say that A is BLR traceable if there is an order function h such that each ω-c.e. by A function f has a BLR trace with bound h.*

As usual, the choice of h doesn't matter. Also, if A is computably dominated then we can take equivalently $T_n = D_{p(n)}$ for some ω-c.e. function p.

Fact 1.11. *Jump traceable & superlow is equivalent to BLR traceable with bound 1.*

Fact 1.12. *Jump traceable & c.e. implies BLR traceable with bound 1.*

These are both by Cole/Simpson, and because having a BLR trace with bound 1 means being ω-c.e.

Proposition 1.13. *BLR traceable with constant bound > 1 does not imply ω-c.e.*

Fact 1.14. *BLR traceable implies jump traceble.*

This is so because the function $f(x) = J^A(x)$ if defined, 0 otherwise is ω-c.e. by A.

Theorem 1.15. *There is an ω-c.e. set which is jump traceable but not BLR traceable.*

Theorem 1.16. *There is jump traceable and computably dominated set that is not BLR traceable.*

Proofs are in the paper.

1.6. **Existence.**

Theorem 1.17. *There is a special Π^0_1 class of BLR traceable sets.*

Proof. Let $\{\langle \Gamma_e, g_e\rangle\}_{e\in\omega}$ be an enumeration of (partial) ω-c.e. by oracle functions. So the following hold:

- g_e is a partial computable function which converges on an initial segment of ω;
- Γ^X_e is total for every oracle X;
- $|\{t \mid \Gamma^X_e(n,t) \neq \Gamma^X_e(n,t+1)\}| < g_e(n)$ for every n such that $g_e(n)\downarrow$ and every oracle X.

We let $f^X_e(n) = \lim_t \Gamma^X_e(n,t)$.

We build a Π^0_1-class $\mathcal{P}$ and computable BLR traces $\{(T^e_n)_{n\in\omega}\})_{e\in\omega}$ with bound 2^n. For all i, we have the requirement:

R_i**:** ϕ_i is not a computable description of a real in $\mathcal{P}$.

For every pair (e,n) with $e < n$, we have the requirement:

$Q_{e,n}$**:** If $g_e(n)\downarrow$, $f^X_e(n) \in T^e_n$ for all reals $X \in \mathcal{P}$.

Clearly these requirements will suffice to prove the result.

Each strategy will receive from the previous strategy some finite collection of strings $\{\alpha_j\}$ all of the same length, and it will create some finite extensions $\{\beta_k\}$ (all of the same length) such that for every j there is at least one k such that $\alpha_j \subseteq \beta_k$, and all reals in $\bigcup_k [\beta_k]$ meet the strategy's requirement.

R_i-strategies will initially define exactly two β_k for every α_j, but may later remove one.

$Q_{e,n}$-strategies will define exactly one β_k for every α_j. They may need to redefine β_k some finite number of times, but each new definition will be an extension of the previous.

At the end of every stage s, we let $\{\hat{\beta}_k\}$ be the outputs of the last strategy to act at stage s, and define the tree P_s to be all strings comparable with one of the $\hat{\beta}_k$. $\mathcal{P}$ will be $\bigcap_s [P_s]$.

Description of R_i-strategy:

This is the standard non-computability requirement on a tree:

(1) Let $\{\alpha_j\}_{j<m}$ be the output of the previous strategy. For every $j < m$, let $\beta_{j,0} = \alpha_j 0$ and $\beta_{j,1} = \alpha_j 1$.
(2) Wait for $\phi_i(|\alpha_0|)$ to converge; while waiting, let the outputs be $\{\beta_{j,0}, \beta_{j,1}\}_{j<m}$.
(3) When $\phi_i(|\alpha_0|)$ converges...
   - ...if $\phi_i(|\alpha_0|) = 0$, let the outputs be $\{\beta_{j,1}\}_{j<m}$.
   - ...if $\phi_i(|\alpha_0|) \neq 0$, let the outputs be $\{\beta_{j,0}\}_{j<m}$.

Description of $Q_{e,n}$-strategy:

Until $g_e(n)$ converges, this strategy takes no action. We ignore for the moment the computable bound on the number of times the index of T_e^n changes.

Let $\{\alpha_j\}_{j<m}$ be the output of the previous strategy. We will keep several values to assist the strategy: ℓ_s will be the number of times the output has been redefined by stage s; $c_s(j)$ will be the current guess for $f_e(n)$ on an extension of α_j. We initially have $\ell_s = 0$ and $c_s(j) = -1$ for all j. Unless otherwise defined, $\ell_{s+1} = \ell_s$ and $c_{s+1}(j) = c_s(j)$.

For every j, let $\beta_j^0 = \alpha_j$. We initially let the outputs be $\{\beta_j^0\}_{j\in\omega}$ and define $T_n^e = \emptyset$. We run the following strategy, where s is the current stage:

(1) Wait for a string $\gamma \in P_s$ with γ extending one of the $\beta_j^{\ell_s}$ and $\Gamma_e^\gamma(n,s) \neq c_s(j)$.
(2) When such a string is found for $\beta_j^{\ell_s}$:
   (a) Define $\beta_j^{\ell_s+1} = \gamma$ and $c_{s+1}(j) = \Gamma_e^\gamma(n,s)$.
   (b) For every $k < m$ with $k \neq j$, choose $\beta_k^{\ell_s+1}$ extending $\beta_k^{\ell_s}$ of the same length as $\beta_j^{\ell_s+1}$.
   (c) Redefine $T_n^e = \{c_{s+1}(k) \mid k < m\}$.
   (d) Define $\ell_{s+1} = \ell_s + 1$.
(3) Return to Step 1.

Full construction:

We make the assumption that for every s, there is precisely one pair (e, n) with $e < n$ and $g_{e,s+1}(n) \downarrow$ but $g_{e,s}(n) \uparrow$. We give the $Q_{e,n}$-strategies priority based on the order in which the $g_e(n)$ converge: if $g_e(n)$ converges before $g_{e'}(n')$, then the $Q_{e,n}$-strategy has stronger priority than the $Q_{e',n'}$-strategy. If $g_e(n)$ never converges, then $Q_{e,n}$ never has a priority, but this is fine because it never acts.

We prioritize the R_i-strategies based on the priorities of the $Q_{e,n}$-strategies: R_i has weaker priority than $R_{i'}$ for any $i' < i$, and also weaker priority than any $Q_{e,n}$-strategy with $n \leq i$. It is given the strongest priority consistent with these restrictions.

Since we only consider $n > e \geq 0$, the R_0-strategy will always have strongest priority. It receives $\alpha_0 = (_i)_{i \in \mathbb{N}}$ as the "output of the previous strategy".

At stage s, let (e, n) be the pair such that $g_e(n)$ has newly converged. We initialise R_n and all strategies which had weaker priority than R_n. The priorities of the various R_i are then redetermined. We then let all Q-strategies with priorities and all R_i-strategies with $i < s$ act, in order of priority.

Whenever a strategy redefines its output, all weaker priority strategies are initialised.

□

1.7. **BLR Traceability is Equivalent to Lowness for Demuth$_{\mathbf{BLR}}$.** We say that a set Z is Demuth random *by* an oracle set A (Demuth$_{\text{BLR}}$ A for short) if it passes every A-Demuth test where the number of version changes is computably bounded. A is low for Demuth$_{\text{BLR}}$ if every Demuth random is already Demuth$_{\text{BLR}}$ A.

Theorem 1.18. *The following are equivalent for a set A.*

(i) *A is BLR traceable.*
(ii) *A is low for* Demuth$_{\text{BLR}}$.

Corollary 1.19. *There exists a perfect class of sets which are all low for Demuth randomness.*

Proof. First observe that any computably dominated set which is low for Demuth$_{\text{BLR}}$ is already low for Demuth randomness. Now by Martin/Miller [39], the Π^0_1-class obtained in Theorem 1.17 contains a perfect subclass of sets which are computably dominated. By the previous theorem, this class is as required. □

2. The collection of weakly-Demuth-random reals is not $\mathbf{\Sigma}^0_3$

By Yu Liang.

This is a result corresponding to Proposition 1.4.

Lemma 2.1. *Given any recursive tree $T \subseteq 2^{<\omega}$ so that $[T]$ only contains Martin-Löf random reals, there is a weakly-Demuth-test $\{W_{f(i)}\}_{i \in \omega}$ so that for any $\sigma \in 2^{<\omega}$, if $[\sigma] \cap [T] \neq \emptyset$, then $[\sigma] \cap [T] \cap (\bigcap_i W_{f(i)}) \neq \emptyset$.*

Proof. The proof is quite similar to the one by Yu for weak-2-randomness.

Given a recursive tree $T \subseteq 2^{<\omega}$ so that $[T]$ only contains Martin-Löf random reals, we try to build a weakly-Demuth-test $\{W_{f(i)}\}_{i \in \omega}$ so that $W_{f(i)}$ densely meets $[T]$ for every i. Suppose $\mu(T) > 2^{-r}$ for some number r.

To build $W_{f(i)}$, for each σ, we try to search some $\tau \succ \sigma$ with $|\tau| = |\sigma| + \ulcorner \sigma \urcorner + i + 1$ so that $[\tau] \cap [T] \neq \emptyset$ where $\ulcorner \sigma \urcorner$ is the Gödel number of σ. We put such τ's into $W_{f(i,0)}$. The searching way is "from left to right". In the most lucky case, we don't make mistake, then $W_{f(i,0)}$ densely meets $[T]$ and $\mu(W_{f(i,0)}) \leq 2^{-i-1}$. But of course we may make mistakes. Note that "making mistakes" is a Σ^0_1-sentence. Once we find the measure of mistakes greater than 2^{-i-1}, we change $W_{f(i,0)}$ to be $W_{f(i,1)}$. By some tricks, we can ensure not to make the same mistakes. This means the times of "making big mistakes" is no more than 2^{i+1} times.

So $f(i)$ change at most 2^i times and $\mu(W_{f(i)}) \leq 2^{-i}$. □

The left part is the exactly same as weak-2-randomness case.

Let $\mathbb{P} = (\mathbf{T}, \leq)$ be partial ordering so that every $T \in \mathbf{T}$ is a recursive tree and only contains Martin-Löf random reals. $T_1 \leq T_2$ if and only if $T_1 \subseteq T_2$. Clearly every $\mathbb{P}$-generic real is weakly-2-random and so weakly-Demuth-random.

Lemma 2.2. *Given any $\mathbf{\Pi^0_2}$ set G only containing weakly-Demuth-random reals, the set $\mathcal{D}_G = \{P \in \mathbf{T} \mid P \cap G = \emptyset\}$ is dense.*

Proof. Immediately from Lemma 2.1. □

So for any $\mathbf{\Sigma^0_3}$ set H only containing weakly-Demuth-random reals, by Lemma 2.2, any sufficiently $\mathbf{P}$-generic real doesn't belong to H.

I don't know whether the collection of Demuth-random reals is $\mathbf{\Sigma}^0_3$

3. Randomness Zoo

by Antoine Taveneaux.
This has been moved to Logic Blog 2012.

Part 2. Traceability

4. Oct 2011: There are $2^{\aleph_0}$-many $n^{1+\epsilon}$-jump traceable reals

By Yu Liang. This is joint work with Denis Hirschfeldt.

Definition 4.1. *Let h be an order function. We say that a real x is h-jump traceable if for each Turing functional Φ, there is an h-bounded c.e. trace $(U_n)_{n\in\mathbb{N}}$ such that Φ^x total implies $\Phi^x(n) \in U_n$ for a.e. n.*

Theorem 4.2. *Let ϵ be an arbitrary small positive rational. Then there are $2^{\aleph_0}$-many $n^{1+\epsilon}$-jump traceable reals.*

We construct a perfect tree T so that there is a uniformly r.e. sequence $\{U_e\}_{e\in\omega}$ such that

- For every e, $|U_e| \leq e^{1+\epsilon}$;
- for any $x \in [T]$ and any e, if Φ_e^x is total, then $\Phi_e^x(n) \in U_n$ for any $n \geq e$.

First we show that for a single index e, there is a uniformly r.e. sequence $\{U_e\}_{e\in\omega}$ such that

- For every e, $|U_e| \leq e^{1+\epsilon}$;
- for any $x \in [T]$, if Φ_e^x is total, then $\Phi_e^x(n) \in U_n$ for any n.

Let $\delta(n)$ be a computable increasing function so that

$$(\delta(n))^{1+\epsilon} > 2^{n+1} \sum_{i\leq n} \delta(i).$$

We build an embedding function $T : 2^{<\omega} \to 2^\omega$ by approximation.

At stage 0, T_0 is identity.

At stage $s+1$, if there is some $m \in [\delta(n), \delta(n+1))$ and some finite string σ, where we may assume that $|\sigma| > n$, so that $\Phi_e^{T_s(\sigma)}(m) \downarrow$ at stage s. Let $T_{s+1}[\sigma \upharpoonright n]$ be $T_s[\sigma]$ and move other values corresponded to remain T_{s+1} to be an embedding function. In other words, we kill all the branching nodes up to $T_s(\sigma)$ extending $T_s(\sigma \upharpoonright n)$ to narrowing the possible values of Φ_e. Put $\Phi_e^{T_s(\sigma)}(m)$ into U_m.

The intuition behind the construction is to make all the values between $[\delta(n), \delta(n+1))$ be "on some single node".

By the finite injury, $\lim_{s\to\infty} T_s$ exists.

Note that for each $m \in [\delta(n), \delta(n+1))$, we put at most $\delta(n) \cdot 2^{n+1}$, which is not greater than $m^{1+\epsilon}$, many values into U_m.

For the general case, there is no essential difference. Just let δ be even faster so that we may narrow the possible branches.

Conjecture 4.3. *There are only countably many id-jump traceable reals.*

5. December 2011: A c.e. K-trivial which is not $o(\log x)$ jump traceable.

By Turetsky. Paper accepted in Information Processing Letters.

Lemma 5.1. *If h is an order with $\sum_{x=0}^{\infty} 2^{-d\cdot h(x)} = \infty$ for all $d > 0$, then for any $c > 0$, there is an $n > 0$ with $h(2^{cn}) \leq n$.*

Proof. Fix a c, and suppose $h(2^{cn}) > n$ for all n. Then $h(x) > \frac{1}{c}\log x$ for all x, and thus $\sum_{x=0}^{\infty} 2^{-2c\cdot h(x)} < \infty$, contrary to hypothesis. □

Theorem 5.2. *There is a c.e. K-trivial set A which is not jump traceable at any order h with $\sum_{x=0}^{\infty} 2^{-d\cdot h(x)} = \infty$ for all $d > 0$. In particular, it is not jump traceable at any order $h \in o(\log x)$.*

Proof. We make A K-trivial in a standard fashion: fix W a c.e. operator such that $\lambda([W^X]) \leq 1/2$ and $[W^X]$ contains all X-randoms for all oracles X (e.g., $[W^X]$ is an element of the universal oracle ML-test); we shall construct a c.e. set V with $[W^A] \subseteq [V]$ and $\lambda([V]) < 1$. Our construction of V is the obvious one: whenever we see a string $\sigma \in W^A[s]$, we enumerate σ into V. We will ensure that $\lambda([V]) < 1$ by appropriate restraint on A.

Let h^e be an enumeration of all (partial) orders, and let $(T_k^e(x)_i)_{i\in\mathbb{N}} k \in \omega$ be an enumeration of all h^e bounded c.e. traces. We construct a partial A-computable function f^A, and (assuming h^e satisfies the hypothesis of the theorem) make A not jump traceable at order h^e by meeting the following requirements:

P_k^e**:** $(\exists x)[f^A(x)\downarrow \ \& \ f^A(x) \notin T_k^e(x)]$.

We partition the domain of f^A into infinitely many sets B^e, and work to meet requirements for h^e on B^e. However, our choice of pairing function matters: each B^i must be an arithmetic progression. So we let $B^e = \{x \cdot 2^{e+1} + 2^e \mid x \in \omega\}$.

The basic P_k^e-strategy is straightforward. Choose an $x \in B^e$, wait until $h^e(x)\downarrow$, and then define $f^A(x)$ to a large value with large use. Wait until $f^A(x) \in T_k^e(x)$. Change A below the use and redefine $f^A(x)$ to a large value. Eventually we win, since $T_k^e(x)$ has bounded size.

The complication comes in the interaction between positive requirements and ensuring that $\lambda([V]) < 1$ — redefining $f^A(x)$ may cause measure to leave $[W^A]$. As long as the total measure which leaves $[W^A]$ over the course of the construction is strictly less than $1/2$, we are fine. P_k^e must be careful to only contribute a small fraction of that.

Suppose that for some constant c, P_k^e has claimed for its future use $2^{c\cdot n}$ many elements of B^e on which h^e takes value less than n, for some n. P_k^e will follow the basic strategy on the first of these elements, but will discard this and choose a new one if the cost of clearing the computation rises too high. What is "too high"? This will depend on how much progress P_k^e has made on the current witness. Initially (if P_k^e has not yet cleared any computations on this box), P_k^e will discard the box if the measure lost by clearing the computation rises beyond $2^{-c\cdot n}$. Each time P_k^e completes a loop of the basic strategy (that is, each time it clears a computation and the current n-box is promoted), its threshold is multiplied by 2^{c-1}. So if the box has been promoted m times, P_k^e will discard it if the measure lost by clearing the

computation rises beyond $2^{-c\cdot n+(c-1)\cdot m}$. When it discards a box and begins working with the next one, P_k^e's threshold returns to $2^{-c\cdot n}$

Now, let us analyze this strategy in the absence of action for other positive requirements. Assume that $c > 1$. Because uses are always chosen large, the measure which restrains a given n-box of P_k^e is always disjoint from the measure which restrains a different n-box. So if $x_1, \ldots, x_l$ are the n-boxes P_k^e is restrained on, and box x_i is restrained after m_i promotions, we know

$$\sum_{i=1}^{l} 2^{-c\cdot n+(c-1)\cdot m_i} \leq 1/2.$$

Since $m_i \geq 0$, we know $l \leq 2^{c\cdot n}$. Actually, since the measure which restrains x_i must be strictly more than the threshold, we know that $l < 2^{c\cdot n}$.

Now, all the promotions of x_i caused the discarding of at most

$$\sum_{j=0}^{m_i-1} 2^{-c\cdot n+(c-1)\cdot j} \leq 2^{-c\cdot n+(c-1)\cdot m_i-c+2}$$

measure. So the total measure discarded by all the x_i is at most

$$\sum_{i=1}^{l} 2^{-c\cdot n+(c-1)\cdot m_i-c+2} = 2^{-c+2}\sum_{i=1}^{l} 2^{-c\cdot n+(c-1)\cdot m_i} \leq 2^{-c+1}.$$

Since $l < 2^{c\cdot n}$, we know there is some x_{l+1} which is never restrained, and thus causes the P_k^e strategy to succeed. Now, let us analyze the total measure discarded by the promotions of x_{l+1}. It is at most

$$\sum_{j=0}^{n} 2^{-c\cdot n+(c-1)\cdot j} \leq 2^{-c\cdot n+(c-1)\cdot n+1} = 2^{-n+1}.$$

So the total discarded measure caused by P_k^e is at most $2^{-n+1} + 2^{-c+1}$.

We now integrate all the positive strategies via finite injury. When each P_k^e strategy is initialised, it chooses a large integer r. P_k^e then searches for $2^{c\cdot n}$ many unclaimed elements of B^e with $h^e \leq n$ on these elements and $2^{-n+1} + 2^{-c+1} < 2^{-(r+3)}$. By the previous lemma, if h^e satisfies the hypothesis of the theorem, since B^e is an arithmetic progression, P_k^e will eventually find such elements. It then begins running the above strategy. Whenever a higher priority strategy enumerates an element into A, P_k^e is initialised, choosing a new r and searching for new elements to work on. By the usual finite injury argument, every strategy eventually succeeds. Further, the lost measure is at most $\sum_{r=0}^{\infty} 2^{-(r+3)} = 1/4$. So $\lambda([V]) \leq 3/4$. $\square$

Part 3. Randomness and computable analysis

6. April 2011: Randomness and Differentiability

by Nies (mainly), Bienvenu, Hoelzl, Turetsky and others.

Brattka, Miller and Nies have submitted their paper entitled Randomness and Differentiability [10]. The main thesis of the paper is that algorithmic randomness of a real is equivalent to differentiability of effective functions at the real. It goes back to earlier work of Demuth, for instance [14], and Pathak [46].

For most major algorithmic randomness notions, on can now provide a class of effective functions on the unit interval so that

$(*)$ a real $z \in [0,1]$ satisfies the randomness notion $\iff$
each function in the class is differentiable at z.

The matching between algorithmic randomness notions and classes of effective functions is summarized in Figure 1.

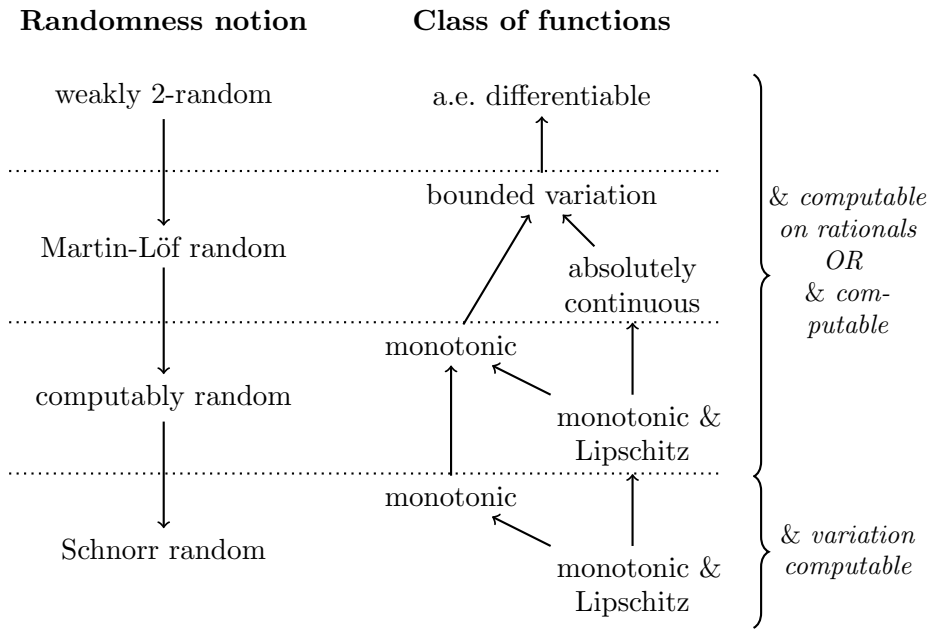

Figure 1. Randomness notions matched with classes of effective functions defined on $[0,1]$ so that $(*)$ holds

Groups currently working in this area include

- Freer, Kjos-Hanssen, Nies: computable Lipschitz function. In prep.
- J. Rute (Carnegie Mellon, student of J. Avigad): higher dimensions, measures. Characterized Schnorr randomness. In preparation.
- Pathak, Rojas and Simpson: Characterized Schnorr randomness. Submitted Nov 2011.
- Kenshi Myabe (Kyoto U): a view via function tests (integral tests). Characterized Schnorr randomness, Kurtz "randomness". Submitted Nov 2011.

6.0.1. *Notation.* For a function f, the *slope* at a pair a, b of distinct reals in its domain is

$$S_f(a,b) = \frac{f(a) - f(b)}{a - b}$$

Recall that if z is in the domain of f then

$$\begin{aligned} \overline{D}f(z) &= \limsup_{h \to 0} S_f(z, z+h) \\ \underline{D}f(z) &= \liminf_{h \to 0} S_f(z, z+h) \end{aligned}$$

$f'(z)$ exists just if those are equal and finite.

6.1. **General thoughts.** Why do algorithmic rdness notions for reals match so well with analytic notions for functions in one variable? OK, the functions have to be computable (or even computable in the variation norm), but then, most functions defined on the unit interval that come up in analysis, such as e^x or $\ln \sqrt{x+1}$, are that. (If the derivative is computable the function is variation computable. I am not sure which kinds of "natural" continuous functions are not differentiable. Who thinks that the Cantor function is natural?)

Analytic notions on functions have been studied systematically by Lebesgue [35] and even earlier. Algorithmic randomness notions have been studied for 45 years, starting with [37] (see [41] for the Russian side of the story).

6.1.1. *How about matching the remaining randomness notions?*

Question 6.1. *Can one match the notions of partial computable randomness and permutation randomness (see* [43, Ch. 7]*) with a class of real functions in the sense of (∗)?*

First of all we would like a proof that these notions are actually about real numbers (not bit sequences). Just like for computable randomness [10, Section 4] one would need to show that they are independent of the choice of a base (which is 2 in the definition).

Question 6.2. *How about matching 2-randomness? Demuth randomness?*

Demuth's paper [20] went a long way towards answering the second part of the question: at a Demuth random real z, the Denjoy (French, 1907, pronounced "dong-jOa") alternative (DA) holds for each Markov computable (=constructive) function f. See Subsection 6.2 for definitions, and Subsection 6.4 for a result stronger than Demuth's.

6.1.2. *Are there function notions from analysis that can't be matched?*

I'm not sure. The analytic notions in Figure 1 seem to be the mayor ones.

6.1.3. *The case of ML-randomness.* Recall that a function $f\colon [0,1] \to \mathbb{R}$ is of *bounded variation* if

$$\infty > \sup \sum_{i=1}^{n} |f(t_{i+1}) - f(t_i)|,$$

where the sup is taken over all partitions $t_1 < t_2 < \ldots < t_n$ in $[0,1]$.

The characterization of ML-randomness is via differentiability of computable functions of bounded variation. A direct proof is probably in Demuth [14]. In [10] we get the harder direction $\Rightarrow$ in $(*)$ above out of the case for computable randomness via the low basis theorem.

A direct proof for $\Rightarrow$ is not available currently. Here is a simple case though:

Fact 6.3. *Let z be ML-random. If f is a computable function of bounded variation, then $\overline{D}f(z) < \infty$.*

Proof. We may assume that the variation $Var(f) \leq 1$. A *dyadic n-interval* inside $[0,1]$ has the form $(i2^{-n}, k2^{-n})$ where $i < k$ are naturals. Let G_r be the union of all intervals (x,y) that are dyadic n intervals for some n such that $S_f(x,y) > 2^r$. We claim that $\lambda G_r \leq 2^{-r}$. For let $G_{r,n}$ be the union of dyadic n-intervals in G_r. Consider the partition consisting of the endpoints of the maximal dyadic n-intervals contained in $G_{r,n}$. Since $V(f) \leq 1$ we can conclude that $\lambda G_{r,n} \leq 2^{-r}$. Since $V(f) \leq 1$ we can conclude that $\lambda G_{r,n} \leq 2^{-r}$.

Now clearly (G_r) is uniformly c.e., hence a ML-test. If $\overline{D}f(z) = \infty$ then (because f is continuous) z must fail this test. □

6.1.4. *A simpler proof of a special case of* [10, Theorem 4.1(iii)→(ii)]. For a nondecreasing function $f\colon [0,1] \to \mathbb{R}$ and a real r let M_r^f be the (dyadic) martingale associated with the slope S_f evaluated at intervals of the form $[r + i2^{-n}, r + (i+1)2^{-n}]$. Solecki, combining ideas from his paper [40] with the middle thirds lemma [10, 2.5], has shown the following (personal communication):

Theorem 6.4. *Suppose f is a nondecreasing function which is not differentiable at $z \in [0,1]$. Suppose that $\underline{D}f(z) = 0$, or $\overline{D}f(z)/\underline{D}f(z) > 72$. Then one of the martingales M^f, $M^f_{1/3}$ does not converge on z.*

Proof. For an interval L with endpoints a, b we write $S(L)$ for $S_f(a,b)$. Note that $S(I \cup J) \leq \max\{S(I), S(J)\}$ for non-disjoint intervals I, J.

For $m \in \mathbb{N}$ let $\mathcal{D}_m$ be the collection of intervals of the form

$$[k2^{-m}, (k+1)2^{-m}]$$

where $k \in \mathbb{Z}$. Let $\mathcal{D}'_m$ be the set of intervals $(1/3) + I$ where $I \in \mathcal{D}_m$.

The underlying geometric fact is simple:

Fact 6.5. *Let $m \geq 1$. If $I \in \mathcal{D}_m$ and $J \in \mathcal{D}'_m$, then the distance between an endpoint of I and an endpoint of J is at least $1/(3 \cdot 2^m)$.*

To see this: assume that $(k2^{-m} - (p2^{-m} + 1/3) < 1/(3 \cdot 2^m)$. This yields $3k - 3p - 2^m)/(3 \cdot 2^m) < 1/(3 \cdot 2^m)$, and hence $3|2^m$, a contradiction.

Claim 6.6. *Let $z \in I \cap J$ for intervals $I \in \mathcal{D}_m, J \in \mathcal{D}'_m$. Suppose $z \in L$ for some interval L of length d, where $2^{-m}/3 \geq d \geq 2^{-m-1}/3$. Then*

$$S(L)/12 \leq \max\{S(I), S(J)\}. \tag{1}$$

Clearly by Fact 6.5 we have $L \subseteq I \cup J$. Let $I \cup J = [a,b]$. Since f is nondecreasing and $b - a \leq 12|L|$, we have

$S(L) \leq (f(b)-f(a))/|L| \leq 12S(I \cup J) \leq 12 \max\{S(I), S(J)\}$. This shows the claim.

We now give a dual claim where we need the middle thirds.

Claim 6.7. *Let $z \in I \cap J$ for intervals $I \in \mathcal{D}_m, J \in \mathcal{D}'_m$. Suppose z is in the middle third of some interval L of length d, where $2^{-m+1} \geq d/3 \geq 2^{-m}$. Then*

$$6S(L) \geq \max\{S(I), S(J)\}. \tag{2}$$

We only prove it for I. Note that $I \subseteq L$ because z is in the middle third of L. Since $d \leq 6|I|$, and f is nondecreasing, we obtain $6S(L) \geq S(I)$ as required. This shows the claim.

Now we argue similar to [10]. By the hypothesis that $f'(z)$ does not exist and the middle thirds lemma [10, 2.5], we can choose rationals $\widetilde{\beta} < \widetilde{\gamma}$ such that $\widetilde{\gamma}/\widetilde{\beta} > 72$ and

$$\begin{aligned} \widetilde{\gamma} &< \lim_{h \to 0} \sup\{S_f(x,y) \colon 0 \leq y-x \leq h \wedge z \in (x,y)\}, \\ \widetilde{\beta} &> \lim_{h \to 0} \inf\{S_f(x,y) \colon 0 \leq y-x \leq h \wedge z \in \text{ middle third of } (x,y)\} \end{aligned}$$

By definition we can choose arbitrarily short intervals L containing z such that $S(L) \geq \widetilde{\gamma}$, and arbitrarily short intervals L containing z in their middle thirdy such that $S(L) \leq \widetilde{\beta}$. Then, by Claim 6.6, for one of the $\mathcal{D}$ type or the $\mathcal{D}'$ type intervals, there are arbitrarily short intervals K of this type such that $z \in K$ and $S(K) \geq \widetilde{\gamma}/12$. By Claim 6.7, for *both* types of intervals, there are arbitrarily short intervals K of this type such that $z \in K$ and $S(K) \leq \widetilde{6}\beta$. Since $\widetilde{\gamma}/\widetilde{\beta} > 72$, this means that one of the martingales M^f, $M^f_{1/3}$ does not converge on z. □

6.2. **Denjoy alternative, even for partial functions.** Our version of the Denjoy alternative for a function f defined on the unit interval says that

$$\text{either } f'(z) \text{ exists, or } \overline{D}f(z) = \infty \text{ and } \underline{D}f(z) = -\infty. \tag{3}$$

It is a consequence of the classical Denjoy (1907) -Young- (1912) - Saks (1937) theorem that for *any* function defined on the unit interval, the DA holds at almost every z. The actual result is in terms of right and left upper and lower Dini derivatives denoted $D^+f(z)$ (right upper) etc. See Bogachev's proof [7, p. 371]. Denjoy proved it for continuous, Young for measurable and Saks for all functions.

The result is used for instance to show that f' is always Borel (as a partial function). A paper by Alberti, Csornyei, Laczkovich, and Preiss (Real Anal. Exchange Vol. 26(1), 2000/2001, pp. 485-488) revisits the DA.

Definition 6.8. *A* computable real *z is given by a computable Cauchy name, i.e., a sequence $(q_n)_{n \in \mathbb{N}}$ of rationals converging to z such that $|q_{n+1} - q_n| \leq 2^{-n-1}$. Then $|z - q_n| \leq 2^{-n}$. If the Cauchy name is understood we sometimes write $(z)_n$ for q_n.*

Recall the following.

Definition 6.9. *A function g defined on the computable reals is called* Markov computable *if from a computable Cauchy name for x one can compute a computable Cauchy name for $g(x)$.*

Most relevant work of Demuth on the Denjoy alternative for effective functions is in this preprint, simply called *Demuth preprint* below. This later turned into the paper Remarks on Denjoy sets; unfortunately a lot of things from the preprint are missing there.

Since the functions aren't total any more, we have to introduce "pseudo-derivatives" at z, taking the limit of slopes close to z where the function is defined. This is presumably what Demuth did. Consider a function g defined on $I_{\mathbb{Q}}$, the rationals in $[0,1]$. For $z \in [0,1]$ let

$$\underset{\sim}{D}g(z) = \liminf_{h\to 0^+}\{S_g(a,b)\colon a,b \in I_{\mathbb{Q}} \wedge\ a \le x \le b \wedge\ 0 < b-a \le h\}.$$
$$\widetilde{D}g(z) = \limsup_{h\to 0^+}\{S_g(a,b)\colon a,b \in I_{\mathbb{Q}} \wedge\ a \le x \le b \wedge\ 0 < b-a \le h\}.$$

Note that Markov computable functions are continuous on the computable reals, so it does not matter which computable dense set of computable reals we take in the definition of these pseudo-derivatives. For a total continuous function g, we have $\underset{\sim}{D}g(z) = \underline{D}g(z)$ and $\widetilde{D}g(z) = \overline{D}g(z)$. See last section of [10].

Suppose more generally we have a function f with domain containing $I_{\mathbb{Q}}$ we say that the *Denjoy alternative* holds if

(4) either $\widetilde{D}f(z) = \underset{\sim}{D}f(z) < \infty$, or $\widetilde{D}f(z) = \infty$ and $\underset{\sim}{D}f(z) = -\infty$.

This is equivalent to (3) if the function is total and continuous.

6.3. Denjoy randomness coincides with computable randomness.

Definition 6.10 (Demuth preprint, page 4)**.** *A real $z \in [0,1]$ is called Denjoy random (or a Denjoy set) if for no Markov computable function g we have $\underset{\sim}{D}g(z) = \infty$.*

In the Demuth preprint, page 6, it is shown that if $z \in [0,1]$ is Denjoy random, then for every computable $f\colon [0,1] \to \mathbb{R}$ the Denjoy alternative (3) holds at z. Combining this with the results in [10] we can now figure out what Denjoy randomness is, and also obtain a pleasing new characterization of computable randomness through differentiability of computable functions.

Theorem 6.11 (Demuth, Miller, Nies, Kučera)**.** *The following are equivalent for a real $z \in [0,1]$*

(i) *z is Denjoy random.*
(ii) *z is computably random*
(iii) *for every computable $f\colon [0,1] \to \mathbb{R}$ the Denjoy alternative (3) holds at z.*

Proof. (i)$\to$ (iii) is the result of Demuth.

(iii)$\to$(ii) Let f be a nondecreasing computable function. Then f satisfies the Denjoy alternative at z. Since $\underline{D}f(z) \ge 0$, this means that $f'(z)$ exists.

This implies that z is computably random by [10, Thm. 4.1].

(ii)$\to$ (i). Given a binary string σ, we will denote the open interval $(0.\sigma, 0.\sigma + 2^{-|\sigma|})$ by (σ). We also write $S_f(\sigma)$ to mean $S_f(a,b)$ where $(a,b) = (\sigma)$.

By hypothesis z is incomputable, and in particular not a rational. Suppose that the function g is Markov computable and $\underset{\sim}{D}g(z) = +\infty$. Choose dyadic rationals a, b such that $(a, b) = (\sigma)$ for some string σ, $z \in (a, b) \subseteq [0, 1]$ and $S_g(r, s) > 4$ for each r, s such that $z \in (r, s) \subseteq (a, b)$.

Define a computable martingale M on extensions $\tau \succeq \sigma$ that succeeds on (the binary expansion of) z. In the following τ ranges over such extensions.

Firstly, note that $S_g(\tau)$ is a computable real uniformly in τ. Furthermore, the function $\tau \to S_g(\tau)$ satisfies the martingale equality, and succeeds on z in the sense that its values are unbounded (even converge to ∞) along z. However, this function may have negative values; J. Miller has called this "betting with debt" because we can increase our capital at a string $\sigma 0$ beyond $2S_g(\sigma)$ by incurring a debt, i.e. negative value, at $S_g(\sigma 1)$. We now define a computable martingale M that succeeds on z and does not use betting with debt.

Let $M(\emptyset) = S_g(\emptyset)$. Suppose now that $M(\tau)$ has been defined and is positive.

Case 1. There is $u \in \{0, 1\}$ such that, where $v = 1 - u$, we have $S(\tau v)_1 < 1$ (this is the second term in the Cauchy name for the computable real $S(\tau v)$, which is at most $1/2$ away from that real). Then $S(\tau v) < 2$. By choice of a, b we now know that z is not an extension of τv. Thus, we let M double its capital along τu, let $M(\rho) = 0$ for all $\rho \succeq \tau v$. (The martingale M stops betting on these extensions.)

Case 2. Otherwise. Then $S(\tau v) > 0$ for $v = 0, 1$. We let M bet with the same betting factors as S_g:

$$M(\tau u) = M(\tau)\frac{S_g(\tau u)}{S_g(\tau)}$$

for $u = 0, 1$. Note that $M(\tau u) > 0$.

If Case 1 applies to infinitely many initial segments of the binary expansion of z, then M doubles its capital along z infinitely often. Since M has only positive values along z, this means that $\lim_n M(z \restriction_n)$ fails to exist, whence z is not computably random by the effective version of the Doob martingale convergence theorem [].

Otherwise, along z, M is eventually in Case 2. So M succeeds on z because S_g does. □

Note that all we needed for the last implication was that $g(q)$ is a computable real uniformly in a rational $q \in I_{\mathbb{Q}}$. Thus, in Definition 6.10 we can replace Markov computability of g by this weaker hypothesis.

6.4. **The Denjoy alternative for functions satisfying effectiveness notions weaker than computable.**

This is work of Bienvenu, Hölzl and Nies who met at the LIAFA in Paris May-June.

Demuth [20] proved that the DA holds at what is now called Demuth random reals, for each Markov computable function. We show that in fact the much weaker notion of difference randomness is enough! Difference randomness was introduced by Franklin and Ng [26]. They showed that it is equivalent to being ML-random and Turing incomplete.

6.4.1. *Weak 2-randomness yields the DA for functions computable on $I_{\mathbb{Q}}$.* First we review some things from the last section of [10]. Let $I_{\mathbb{Q}} = [0,1] \cap \mathbb{Q}$. A function $f\colon \subseteq [0,1] \to \mathbb{R}$ is called *computable on* $I_{\mathbb{Q}}$ if $f(q)$ is defined and a computable real uniformly in the rational q.

For any rational $p > 0$, let

$$\underset{\sim}{C}(p) = \{z\colon \forall t > 0\, \exists a, b[a \le z \le b \wedge 0 < b - a \le t \wedge S_f(a,b) < p\},$$

where t, a, b range over rationals. Since f is computable on $I_{\mathbb{Q}}$, the set

$$\{z\colon \exists a, b\, [a \le z \le b \wedge 0 < b - a \le t \wedge S_f(a,b) < p\}$$

is a Σ^0_1 set uniformly in t. Then $\underset{\sim}{C}(p)$ is Π^0_2 uniformly in p. Furthermore,

$$\underset{\sim}{D} f(z) < p \Rightarrow z \in \underset{\sim}{C}(p) \Rightarrow \underset{\sim}{D} f(z) \le p. \tag{5}$$

Analogously we define

$$\widetilde{C}(q) = \{z\colon \forall t > 0\, \exists a, b[a \le z \le b \wedge 0 < b - a \le t \wedge S_f(a,b) > q\}.$$

Similar observations hold for these sets.

Theorem 6.12. *Let $f\colon \subseteq [0,1] \to \mathbb{R}$ be computable on $I_{\mathbb{Q}}$. Then f satisfies the Denjoy alternative at every weakly 2-random real z.*

Proof. We adapt the classical proof in [7, p. 371] to the case of pseudo-derivatives. Thereafter we analyse the arithmetical complexity of exception sets to conclude that weak 2-randomness is enough for the DA to hold.

We let a, b, p, q range over $I_{\mathbb{Q}}$. Recall Definition 6.8. For each $r < s$, $r, s \in I_{\mathbb{Q}}$, and for each $n \in \mathbb{N}$, let

$$E_{n,r,s} = \{x \in [0,1]\colon \forall a, b[r \le a \le x \le b \le s \to S_f(a,b)_0 > -n+1]\}. \tag{6}$$

Note that $E_{n,r,s}$ is a Π^0_1 class. For every n we have the implications

$$\underset{\sim}{D} f(z) > -n+2 \to \exists r, s\, [z \in E_{n,r,s}] \to \underset{\sim}{D} f(z) > -n.$$

To show the DA (4) at z, we may assume that $\underset{\sim}{D} f(z) > -\infty$ or $\widetilde{D} f(z) < \infty$. If the second condition holds we replace f by $-f$, so we may assume the first condition holds. Then $z \in E_{n,r,s}$ for some r, s, n as above. Write $E = E_{n,r,s}$.

For $p < q$, the class $E \cap \underset{\sim}{C}(p) \cap \widetilde{C}(q)$ is Π^0_2. By (5) it suffices to show that each such class is null. For this, we show that for a.e. $x \in E$, we have $\underset{\sim}{D} f(z) = \widetilde{D} f(x)$. This remaining part of the argument is entirely within classical analysis. Replacing f by $f(x) + nx$, we may assume that for $x \in E$, we have

$$\forall a, b[r \le a \le x \le b \le s \to S_f(a,b)_0 > 1].$$

Let $f_*(x) = \sup_{a \le x} f(a)$. Then f_* is nondecreasing on E. Let g be an arbitrary nondecreasing function defined on $[0,1]$ that extends f_*. Then by a classic theorem of Lebesgue, $L(x) := g'(x)$ exists for a.e. $x \in [0,1]$.

The following is a definition from classic analysis due to E.P. Dolzhenko, 1967 (see for instance [7, 5.8.124] but note the typo there).

Definition 6.13. *We say that E is* porous at x *via $\epsilon > 0$ if for each $\alpha > 0$ there exists β with $0 < \beta \le \alpha$ such that $(x - \beta, x + \beta)$ constains an open*

interval of length $\epsilon\beta$ that is disjoint from E. We say that E is porous at x if it is porous at x via some ϵ.

By the Lebesgue density theorem, the points in E at which E is porous form a null set.

Claim 6.14. *For each $x \in E$ such that $L(x)$ is defined and E is not porous at x, we have $\widetilde{D}f(x) \leq L(x) \leq \underset{\sim}{D}f(x)$.*

Since $\underset{\sim}{D}f(x) \leq \widetilde{D}f(x)$, this establishes the theorem.

To prove the claim, we show $\widetilde{D}f(x) \leq L(x)$, the other inequality being symmetric. Fix $\epsilon > 0$. Choose $\alpha > 0$ such that

$$\forall u, v \in E\,[(u \leq x \leq v \wedge 0 < v - u \leq \alpha) \to S_{f_*}(u, v) \leq L(x)(1 + \epsilon)]; \tag{7}$$

furthermore, since E is not porous at x, for each $\beta \leq \alpha$, the interval $(x - \beta, x + \beta)$ contains no open subinterval of length $\epsilon\beta$ that is disjoint from E. Now suppose that $a, b \in I_{\mathbb{Q}}$, $a < x < b$ and $\beta = 2(b - a) \leq \alpha$. There are $u, v \in E$ such that $0 \leq a - u \leq \epsilon\beta$ and $0 \leq v - b \leq \epsilon\beta$. Since $u, v \in E$ we have $f_*(u) \leq f(a)$ and $f(b) \leq f_*(v)$. Therefore $v - u \leq b - a + 2\epsilon\beta = (b - a)(1 + 4\epsilon)$. It follows that

$$S_f(a, b) \leq \frac{f_*(v) - f_*(u)}{b - a} \leq S_{f_*}(u, v)(1 + 4\epsilon) \leq L(x)(1 + 4\epsilon)(1 + \epsilon). \qquad \square$$

6.4.2. *Difference randomness yields the Denjoy alternative for Markov computable functions.* We slightly reformulate the definition of difference randomness by Franklin and Ng [26].

Definition 6.15. A *difference test* is given by a Π^0_1 class $P \subseteq [0, 1]$, together with a uniformly Σ^0_1 sequence of classes $(U_n)_{n\in\mathbb{N}}$ where $U_n \subseteq [0, 1]$, such that $\lambda(P \cap U_n) \leq 2^{-n}$ for each n. A real z fails the test if $z \in P \cap \bigcap_n U_n$, otherwise z passes the test. We say z is difference random if it passes each difference test.

Franklin and Ng [26] show that

$$z \text{ difference random} \Leftrightarrow z \text{ is ML-random} \wedge \text{Turing incomplete}.$$

The following direct proof that no left-c.e. real α (such as Chaitin's Ω) is difference random might be helpful to understand the concept of difference tests. Let $P = [\alpha, 1]$. Let $U_n = [0, (i + 1)2^{-n})$ where $i \in \mathbb{N}$ is largest such that $i2^{-n} < \alpha$. Then $P, (U_n)_{n\in\mathbb{N}}$ is a difference test and $\alpha \in P \cap \bigcap_n U_n$.

Theorem 6.16. *Let $f\colon \subseteq [0, 1] \to \mathbb{R}$ be Markov computable. Then f satisfies the Denjoy alternative at every difference random real z.*

Proof. Note that each Markov computable function is computable on $I_{\mathbb{Q}}$. We will show that under the stronger condition of Markov computability, the relevant null sets in the proof of the foregoing Theorem 6.12 are effective null sets in the sense of difference randomness. The proof relies on Lemmas 6.18 and 6.19, which will be established in Subsection 6.4.3 below.

Given $n \in \mathbb{N}$ and $r < s$ in $I_{\mathbb{Q}}$, define the set E as above. As before, we may assume that for $x \in E$, we have $\forall a, b[r \leq a \leq x \leq b \leq s \to S_f(a, b)_0 > 1]$, and hence the function $f_*(x) = \sup_{a\leq x} f(a)$ is nondecreasing on E.

Firstly, we show that some total nondecreasing extension g of $h = f_* \restriction_E$ can be chosen to be computable. Since f is Markov computable, we obtain the following.

Claim 6.17. *The function $f_* \restriction_E$ is computable.*

To see this, recall that p, q range over $I_{\mathbb{Q}}$, and let $f^*(x) = \inf_{q \geq x} f(q)$. If $x \in E$ and $f_*(x) < f^*(x)$ then x is computable: fix a rational d in between these two values. Then $p < x \leftrightarrow f(p) < d$, and $q > x \leftrightarrow f(q) > d$. Hence x is both left-c.e. and right-c.e., and therefore computable. Now a Markov computable function is continuous at every computable x. Thus $f_*(x) = f^*(x)$ for each x in E.

To compute $f(x)$ for $x \in E$ up to precision 2^{-n}, we can now simply search for rationals $p < x < q$ such that $0 < f(q)_{n+2} - f(p)_{n+2} < 2^{-n-1}$, and output $f(p)_{n+2}$. If during this search we detect that $x \notin E$, we stop. This shows the claim.

Lemma 6.18. *Let $h\colon \subseteq [0,1] \to \mathbb{R}_0^+$ be a computable function that is defined and non-decreasing on a Π^0_1 class E . Then $h \restriction_E$ can be extended to a function $g\colon [0,1] \to \mathbb{R}_0^+$ that is computable and non-decreasing on $[0,1]$.*

By [10, Thm. 5.1] we know that $L(x) := g'(x)$ exists for each computably random (and hence certainly each ML-random) real x.

To show that in fact $\underset{\sim}{D} f(z) = \widetilde{D} f(z) = L(z)$ for each difference random real z, we need the following.

Lemma 6.19. *Let $E \subseteq [0,1]$ be a Π^0_1 class. If $z \in E$ is difference random, then E is not porous at z.*

Given the lemma, we can conclude the proof of the theorem by invoking Claim 6.14. □

6.4.3. *Proving the two lemmas.*

Lemma 6.20. *Let E be a nonempty Π^0_1 class. Let $h\colon \subseteq [0,1] \to \mathbb{R}_0^+$ be a computable function with domain containing E. Then $\sup_{x \in E}\{h(x)\}$ is right-c.e. and $\inf_{x \in E}\{h(x)\}$ is left-c.e. uniformly in an index for E.*

Proof. We proof the statement for the supremum; the proof for the infimum is analogous. We use the signed digit representation of reals, that is every real is represented by an infinite sequence in $\{-1, 0, 1\}^\infty$.

We run in parallel an enumeration of E^c and for all $x \in [0,1]$ (given by a Cauchy name $(x_n)_{n \in \mathbb{N}}$) the computations of $h(x)$ up to precision 2^{-n}. That is, we want to compute $(h(x))_n$, the n-th entry of a Cauchy name for $h(x)$.

Due to uniform continuity, for each n there is a number n' such that for all x in order to accomplish the computation it suffices to have access to the initial segment $(x_0, \ldots, x_{n'})$ of the Cauchy name of x. When the computation of $(h(x))_n$ halts for some x it also halts for all other x' which have a Cauchy name that begins with $(x_0, \ldots, x_{n'})$, since the computation is clearly the same. We do not know n', so we build a tree of computations that branches into three directions (-1, 0 and 1) whenever we access a new entry of the Cauchy name of the input. We remove a branch of the tree when it gets covered by E^c. Due to the existence of n' the tree will remain finite.

Write $\sup_{x\in E}\{h(x)\}[n]$ for the approximation to the value $\sup_{x\in E}\{h(x)\}$ that we achieve when we proceed as described with precision level 2^{-n}. If we increase the precision, more of E^c may get enumerated before halting has occured everywhere on the tree; so we see that the sequence $(\sup_{x\in E}\{h(x)\}[n]+2^{-n})_{n\to\infty}$ is a right-c.e. approximation to $\sup_{x\in E}\{h(x)\}$. □

Proof of Lemma 6.18. Since E is compact and closed, h is uniformly continuous on E, that is, for every $\varepsilon > 0$ there exists a *single* $\delta(\varepsilon) > 0$ such that for any point $x \in E$ the continuity condition

$$|y - x| < \delta(\varepsilon) \Rightarrow |h(y) - h(x)| < \varepsilon$$

is satisfied.

Idea. We do not know $\delta(\varepsilon)$ for a given ε, but we can search for it using in parallel the following construction for different candidate δ's:

We split the unit interval into intervals $(I_k)_k$ of length δ and write l_k for the left border point of I_k. Write i_k for $\inf\{h(x) \mid x \in I_k \cap E\}$ and s_k for $\sup\{h(x) \mid x \in I_k \cap E\}$ if these values exist. We use lemma 6.20 to approximate i_k and s_k for all intervals, and at the same time we enumerate E^c.

We do this until we have found a δ (called *ε-fit*) such that every interval has been dealt with; by this we mean that for every interval I_k we have either covered I_k with E^c, or we have found that $s_k - i_k < \varepsilon$, that is we already know h up to precision ε. In the latter case we set our approximation to h to be the line from point (l_k, i_k) to point (l_{k+1}, s_k); on the remaining intervals (the former case) we interpolate linearly. Call the new function g_0. We can then output $g_0(x)$ up to precision ε at any point $x \in [0, 1]$.

A problem with this construction. The following problem can occur with g_0: Assume we have for some ε found a δ that is ε-fit. We construct g_0 as described and interpolate linearly on, say, the maximal connected sequence $I_k, \ldots, I_{k+n}$, all contained in E^c. But if we look at the same construction for g_0 at a better precision $\varepsilon' < \varepsilon$, we might actually enumerate more of E^c until we find an ε'-fit δ', and this might extend the sequence $I_k, \ldots, I_{k+n}$ to, say, $I_{k-i}, \ldots, I_{k+n+j}$, all contained in E^c. The linear interpolation on this sequence of intervals would then be *significantly flatter* than at level ε. So for some $x \in I_{k-i} \cup \cdots \cup I_{k+n+j}$ we might have that the approximation to g_0 with precision ε differs by more than ε from the approximation to g_0 with precision ε', which is not allowed.

To fix this problem we need to define g inductively over all precision levels, and "commit" to all linear interpolations that have happened at earlier precision levels, as will be described now.

Formal construction. Assume we want to compute the n-th entry of a Cauchy name for $g(x)$, that is we want to compute $g(x)$ up to precision 2^{-n}. We say that *we are at precision level n*. We do not know $\delta(2^{-n})$ so we do the following with $\delta = 2^{-p}$ in parallel for all p until we find a δ that is *n-fit*, defined as follows:

Split the interval $[0, 1]$ into intervals of length δ and write

$$I_k = [(k - 1) \cdot 2^{-p}, k \cdot 2^{-p})$$

for the k-th interval and $l_k = (k-1)\cdot 2^{-p}$ for the left border point of I_k. For mathematical precision set $I_p := [1-2^{-p}, 1]$. Call an interval I_k n-treated if there exists a smaller precision level $n' < n$ where I_k has been covered by E^c and therefore a linear interpolation on I_k has been defined. We say that δ is *n-fit* if

- for every interval I_k we have that
  (1) I_k is n-treated or
  (2) $I_k \cap E = \emptyset$ or
  (3) $s_k - i_k < 2^{-n}$
- and if for all k, where both I_k and I_{k+1} fulfill condition 3, we have $i_{k+1} < s_k$; that is, we have that intervals that directly follow each other have a "vertical overlap" in their approximations.

The following linear interpolation is an 2^{-n}-close approximation to g:

First, inductively replay the construction for all precision levels $n' < n$ to find all n-treated intervals. For the remaining intervals, run in parallel the right-c.e. and left-c.e. approximations to s_k and i_k, respectively, and the enumeration of E^c, until for every interval either condition 2 or 3 are satisfied.

Build the following piecewise linear function: For all intervals that are already n-treated, keep the linear interpolations from the earlier precision level $n-1$. In all remaining intervals I_k that fulfill condition 3 we set g to be the line from point (l_k, i_k) to (l_{k+1}, s_k), that is, for $x \in I_k$ we let

$$g(x) = i_k + (x - l_k)\cdot\frac{(s_k - i_k)}{l_{k+1} - l_k} = i_k + (x - l_k)\frac{(s_k - i_k)}{\delta}.$$

Now look at the remaining intervals that have not yet been assigned a linear interpolation. In every maximal connected sequence $I_k, \ldots, I_{k+n}$ of such intervals every interval must fulfill condition 2. We interpolate linearly over $I_k, \ldots, I_{k+n}$ in the straightforward way, that is we draw a line from point $(s_k, g(l_k))$ to point $(s_{k+n+1}, g(l_{k+n+1}))$ (strictly speaking $g(l_k)$ is not yet defined, so use $\lim_{x\to l_k} g(x)$ instead).

Verification. It is clear that g is everywhere defined and non-decreasing. Whenever h was defined on a point $x \in I_k$ inside E, g gets assigned a value between i_k and s_k and since $s_k > h(x) > i_k$ and $s_k - i_k < 2^{-n}$ we have $|h(x) - g(x)| < 2^{-n}$. To see that g is computable note that s_k and i_k are defined on any interval that is not entirely contained in E^c and that these values can be approximated in a right-c.e. and left-c.e. way, respectively, by using lemma 6.20. □

Proof of Lemma 6.19. In this proof, we say that a string σ meets $\mathcal{C}$ if $[\![\sigma]\!] \cap \mathcal{C} \neq \emptyset$.

Fix $c \in \mathbb{N}$ such that $\mathcal{C}$ is porous at z via 2^{-c+2}. For each string σ consider the set of minimal "porous" extensions at stage t,

$$N_t(\sigma) = \left\{ \rho \succeq \sigma \;\middle|\; \exists \tau \succeq \sigma \left[\begin{array}{c} |\tau| = |\rho| \;\wedge\; |0.\tau - 0.\rho| \leq 2^{-|\tau|+c} \wedge \\ [\![\tau]\!] \cap \mathcal{C}_t = \emptyset \;\wedge\; \rho \text{ is minimal with this property} \end{array} \right] \right\}.$$

We claim that

$$\sum_{\substack{\rho\in N_t(\sigma)\\ \rho \text{ meets } \mathcal{C}}} 2^{-|\rho|} \le (1-2^{-c-2})2^{-|\sigma|}. \tag{8}$$

To see this, let R be the set of strings ρ in (8). Let V be the set of prefix-minimal strings that occur as witnesses τ in (8). Then the open sets generated by R and by V are disjoint. Thus, if r and v denote their measures, respectively, we have $r+v \le 2^{-|\sigma|}$. By definition of $N_t(\sigma)$, for each $\rho \in R$ there is $\tau \in V$ such that $|0.\tau - 0.\rho| \le 2^{-|\tau|+c}$. This implies $r \le 2^{c+1}v$. The two inequalities together imply (8) because $r \le 2^{c+1}(1-r)$ implies $r \le 1 - 1/(2^c+1) + 1$.

Note that by the formal details of this definition even the "holes" τ are ρ's, and therefore contained in the sets $N_t(\sigma)$. This will be essential for the proof of the first of the following two claims. At each stage t of the construction we define recursively a sequence of anti-chains as follows.

$$B_{0,t} = \{\emptyset\}, \text{ and for } n>0\colon B_{n,t} = \bigcup\{N_t(\sigma)\colon \sigma \in B_{n-1,t}\}$$

Claim. If a string ρ is in $B_{n,t}$ then it has a prefix ρ' in $B_{n,t+1}$.
This is clear for $n=0$. Suppose inductively that it holds for $n-1$. Suppose further that ρ is in $B_{n,t}$ via a string $\sigma \in B_{n-1,t}$. By the inductive hypothesis there is $\sigma' \in B_{n-1,t+1}$ such that $\sigma' \preceq \sigma$. Since $\rho \in N_t(\sigma)$, ρ is a viable extension of σ' at stage $t+1$ in the definition of $N_{t+1}(\sigma')$, except maybe for the minimality. Thus there is $\rho' \preceq \rho$ in $N_{t+1}(\sigma')$. $\Diamond$

Claim. For each n,t, we have $\sum\{2^{-|\rho|}\colon \rho \in B_{n,t} \wedge \rho \text{ meets } \mathcal{C}\} \le (1-2^{-c-2})^n$.
This is again clear for $n=0$. Suppose inductively it holds for $n-1$. Then, by (8),

$$\sum_{\substack{\rho\in B_{n,t}\\ \rho \text{ meets } \mathcal{C}}} 2^{-|\rho|} = \sum_{\substack{\sigma\in B_{n-1,t}\\ \sigma \text{ meets } \mathcal{C}}} \sum_{\substack{\rho\in N_t(\sigma)\\ \rho \text{ meets } \mathcal{C}}} 2^{-|\rho|} \le \sum_{\substack{\sigma\in B_{n-1,t}\\ \sigma \text{ meets } \mathcal{C}}} 2^{-|\sigma|}(1-2^{-c-2}) \le (1-2^{-c-2})^n.$$

This establishes the claim. $\Diamond$

Now let $U_n = \bigcup_t [\![B_{n,t}]\!]$. Clearly the sequence $(U_n)_{n\in\mathbb{N}}$ is uniformly effectively open. By the first claim, $U_n = \bigcup_t [\![B_{n,t}]\!]$ is a nested union, so the second claim implies that $\lambda(\mathcal{C}\cap U_n) \le (1-2^{-c-2})^n$. We show $z \in \bigcap U_n$ by induction on n. Clearly $z \in U_0$. If $n>0$ suppose inductively $\sigma \prec z$ where $\sigma \in \bigcup_t B_{n-1,t}$. Since z is random there is η such that $\sigma \prec \eta \prec z$ and η ends in 0^c1^c. Every interval $(a,b) \subseteq [0,1]$ contains an interval of the form $[\![\rho]\!]$ for a dyadic string ρ such that the length of $[\![\rho]\!]$ is no less than $(b-a)/4$. Thus, since $\mathcal{C}$ is porous at z via 2^{-c+2}, there is t, $\rho \succeq \eta$ and τ satisfying the condition in the definition of $N_t(\sigma)$. By the choice of η one verifies that $\tau \succeq \sigma$. Thus $z \in U_n$.

Now take a computable subsequence $(U_{g(n)})_{n\in\mathbb{N}}$ such that $\lambda(\mathcal{C}\cap U_{g(n)}) \le 2^{-n}$ to obtain a difference test that z fails. $\square$

6.5. **Sets of non-differentiability for single computable functions $f\colon [0,1] \to \mathbb{R}$.** The plan is to characterize the sets $\{z\colon f'(z) \text{ is undefined}\}$ for computable functions on the unit interval. We also want to consider the

case when the functions have additional analytical properties, such as being of bounded variation, or being Lipschitz.

6.5.1. *The classical case.* Zahorski [52] showed the following (also see Fowler and Preiss [25]).

Theorem 6.21. *The sets of non-differentiability of a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ are exactly of the form $L \cup M$, where L is G_δ (ie boldface Π^0_2) and M is a $G_{\delta\sigma}$ (ie boldface Σ^0_3) null set.*

Proof. Given f, the set

$$L = \{z\colon \overline{D}f(z) = \infty \wedge \underline{D}f(z) = -\infty\}$$

is G_δ. Let

$$M = \{z\colon z \notin L \wedge \exists p < q \left[p, q \in \mathbb{Q} \wedge \underline{D}f(z) < p \wedge q < \overline{D}f(z)\right]\}.$$

Then M is $G_{\delta\sigma}$.

Note that $f'(z)$ is undefined iff $z \in L \cup M$, and M is null by the Denjoy alternative.

For the converse direction, we are given L, M and have to build f. We may assume that L, M are disjoint, else replace M by $M - L$. Zahorski builds a continuous function g which is non-differentiable exactly in L (this is hard). Fowler and Preiss build a Lipschitz function h nondifferentiable exactly in M. Now let $f = g + h$.

6.5.2. *The algorithmic case.* If f is computable, then the set L defined above is lightface Π^0_2, and M is lightface Σ^0_3.

For the other direction, it's not clear what happens now. Suppose L is lightface Π^0_2 and M is lightface Σ^0_3. Fowler and Preiss don't say what Zahorski did, and the original paper is probably awful to read. However, the proof [10, Thm 3.1] is likely to come close. There, a Π^0_2 set L is given, and one builds a computable function g that is nondifferentiable in L and differentiable outside a Π^0_2 set $\widehat{L} \supseteq L$, where after modifying that construction a bit, one can ensure that $\widehat{L} - L$ is null.

Since the Fowler-Preiss function for M is Lipschitz, we can't hope this part of the construction works algorithmically, because a computable Lipschitz function is differentiable at all computably randoms by [10, Section 5]. □

6.6. **Extending the results of [10] to higher dimensions.** So far, most of the interaction between randomness and computable analysis is restricted to the 1-dimensional case. Anyone who has done calculus will confirm that differentiation becomes more interesting (and challenging) in higher dimensions. When randomness notions are defined for infinite sequences of bits, that is points in Cantor space $2^{\mathbb{N}}$, it is irrelevant to procede to higher dimensions, because $(2^{\mathbb{N}})^n$ is in all aspects (metric, measure) equivalent to $2^{\mathbb{N}}$. This is clearly no longer true when randomness for n-tuples of reals (yes, reals) in the unit interval $[0, 1]$ is studied, because the n-cube $[0, 1]^n$ for $n > 1$ is much more complex than the unit interval.

What I am trying to say is that the identification of Cantor space and $[0, 1]$ doesn't always make sense any longer for higher dimensions, because methods typical for n-space aren't there in the setting of $(2^{\mathbb{N}})^n \cong 2^{\mathbb{N}}$. It

makes sense in the setting of L_p computability, though. Also we can still use the identification to *define* notions such as computable randomness in $[0,1]^n$, if pure measure theory won't do it (it does for Martin-Löf and Schnorr randomness).

Currently several researchers investigate extensions of the results in [10] to higher dimensions. Already Pathak [46] showed that a weak form of the Lebesgue differentiation theorem holds for Martin-Löf random points in the n-cube $[0,1]^n$. Work in progress of Rute, and Pathak, Simpson, and Rojas might strengthen this to Schnorr random points in the n-cube. On the other hand, functions of bounded variation can be defined in higher dimension [7, p. 378], and one might try to characterize Martin-Löf randomness in higher dimensions via their differentiability.

By work submitted May 2012 of [27], computable randomness can be characterized via differentiability of computable Lipschitz functions. The obvious analog of computable randomness in higher dimensions has been introduced in [10, 48].

Definition 6.22. *Call a point $x = (x_1, \dots, x_n)$ in the n-cube computably random if no computable martingale succeeds on the binary expansions of $x_1, \dots, x_n$ joined in the canonical way (alternating between the sequences).*

Rute studies it via measures. Differentiability of Lipschitz functions in higher dimensions has been studied to great depth (see for instance [1]). Effective aspects of this theory could be used as an approach to such a randomness notion. One could investigate whether this is equivalent to differentiability at x of all computable Lipschitz functions. In [27] it is shown that if $z \in [0,1]^n$ is not computably random, some computable Lipschitz function is not differentiable at z (in fact some partial fails to exist). This is done by a straightforward reduction to the 1-dimensional case.

In any computable probability space, to be weakly 2-random means to be in no null Π^0_2 class. For weak 2-randomness in n-cube, new work of Nies and Turetsky yields the analog of [10, Thm 3.1]. The following writeup is due to Turetsky.

Theorem 6.23 (almost)**.** *Let $z \in [0,1]^n$. Then the following are equivalent:*

(1) *z is weakly 2-random;*
(2) *each a.e. differentiable computable function $f\colon [0,1]^n \to \mathbb{R}$ is differentiable at z;*
(3) *each a.e. differentiable computable function $f\colon [0,1]^n \to \mathbb{R}$ is Gâteaux differentiable at z; and*
(4) *each a.e. differentiable computable function $f\colon [0,1]^n \to \mathbb{R}$ has all partial derivatives at z.*

Proof. (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are facts from classical analysis. We show (1) $\Rightarrow$ (4), (1) + (4) $\Rightarrow$ (2) and (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (4). Suppose z is weakly 2-random and f is an a.e. differentiable computable function. Fix coordinate i. For $\vec{a} \in [0,1]^n$, $h \in \mathbb{R}$, let

$$S^i_f(\vec{a}, h) = \frac{f(a_1, \dots, a_i + h, \dots, a_n) - f(a_1, \dots, a_i, \dots, a_n)}{h}.$$

Recall that $\partial f/\partial x_i$ exists at $\vec{a}$ precisely if the upper derivative $\overline{D}^i f(\vec{a}) = \limsup_{|h|\to 0} S^i_f(\vec{a},h)$ and the lower derivative $\underline{D}^i f(\vec{a}) = \liminf_{|h|\to 0} S^i_f(\vec{a},h)$ are finite and equal.

For q a rational, $\overline{D}^i f(\vec{a}) \geq q$ is equivalent to the formula.

$$(\forall p < q)(\forall \delta > 0)(\exists |h| < \delta)[S^i_f(\vec{a},h) > p]$$

By density, we can take p and δ to range over the rationals. Since f is computable, and thus continuous, we can take h to range over the rationals. Thus $\{\vec{a} \mid \overline{D}^i f(\vec{a}) \geq q\}$ is a Π^0_2-set uniformly in q. Symmetrically, so is $\{\vec{a} \mid \underline{D}^i f(\vec{a}) \leq q\}$. Then the $\vec{a}$ such that $\partial f/\partial x_i$ does not exist are precisely those $\vec{a}$ satisfying

$$(\forall q)[\overline{D}^i f(\vec{a}) \geq q] \vee (\forall q)[\underline{D}^i f(\vec{a}) \leq q] \vee (\exists q,p)[\underline{D}^i f(\vec{a}) \leq q < p \leq \overline{D}^i f(\vec{a})].$$

This is a Σ^0_3-set contained in the set of all points at which f is not differentiable, and thus has measure 0. Thus it cannot contain z, and so $\partial f/\partial x_i$ exists at z.

$(1) + (4) \Rightarrow (2)$. Suppose z is weakly 2-random, f is an a.e. differentiable computable function and $\frac{\partial f}{\partial x_i}(z)$ exists for every i. Let $S^i_f(\vec{a},h)$ be as defined above.

For $\vec{a} \in [0,1]^n$ and $h \in \mathbb{R}$, let

$$J_f(\vec{a},h) = \left[\ S^1_f(\vec{a},h) \quad \ldots \quad S^n_f(\vec{a},h)\ \right].$$

By definition, $\lim_{|h|\to 0} J_f(\vec{a},h) = J_f(\vec{a})$, the Jacobian of f at $\vec{a}$ (when this exists).

Again by definition, the derivative of f exists at $\vec{a}$ if $J_f(\vec{a})$ exists and

$$\lim_{||\vec{h}||\to 0} \frac{f(\vec{a}+\vec{h}) - f(\vec{a}) - J_f(\vec{a})\vec{h}}{||\vec{h}||} = 0.$$

By continuity of f, we can take $\vec{h}$ to have rational coordinates.

Let X be the Π^0_3-set consisting of those a satisfying

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall ||\vec{h}|| < \delta)(\forall |b| < \delta)\frac{\left|f(\vec{a}+\vec{h}) - f(\vec{a}) - J_f(\vec{a},b)\vec{h}\right|}{||\vec{h}||} < \epsilon.$$

Here ϵ, δ and b are rationals, and $\vec{h}$ has rational coordinates. We argue first that X contains every point at which f is differentiable.

Suppose $\vec{a}$ is a point at which f is differentiable. Fix $\epsilon > 0$. Let δ be sufficiently small that

$$(\forall ||\vec{h}|| < \delta)\frac{\left|f(\vec{a}+\vec{h}) - f(\vec{a}) - J_f(\vec{a})\vec{h}\right|}{||\vec{h}||} < \epsilon/2,$$

and also

$$(\forall |b| < \delta)\left|\left|(J_f(\vec{a},b) - J_f(\vec{a}))^T\right|\right| < \epsilon/2.$$

Here we treat $J_f(\vec{a},b) - J_f(\vec{a})$ as a row vector. Then for any $||\vec{h}|| < \delta$ and $|b| < \delta$,

$$\begin{aligned}
\frac{\left|f(\vec{a}+\vec{h}) - f(\vec{a}) - J_f(\vec{a},b)\vec{h}\right|}{||\vec{h}||} &= \frac{\left|f(\vec{a}+\vec{h}) - f(\vec{a}) - J_f(\vec{a})\vec{h} + J_f(\vec{a})\vec{h} - J_f(\vec{a},b)\vec{h}\right|}{||\vec{h}||} \\
&\leq \frac{\left|f(\vec{a}+\vec{h}) - f(\vec{a}) - J_f(\vec{a})\vec{h}\right|}{||\vec{h}||} + \frac{\left|(J_f(\vec{a}) - J_f(\vec{a},b))\vec{h}\right|}{||\vec{h}||} \\
&< \epsilon/2 + \frac{\left|\left|(J_f(\vec{a}) - J_f(\vec{a},b))^T\right|\right| ||\vec{h}||}{||\vec{h}||} \\
&< \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

Thus X contains every point at which f is differentiable, and so X has full measure. So $z \in X$.

Next we show that f is differentiable at z. Fix $\epsilon > 0$. Let δ be sufficiently small that

$$(\forall ||\vec{h}|| < \delta)(\forall |b| < \delta)\frac{\left|f(z+\vec{h}) - f(z) - J_f(z,b)\vec{h}\right|}{||\vec{h}||} < \epsilon/2,$$

and fix some $|b| < \delta$ such that

$$\left|\left|(J_f(z,b) - J_f(z))^T\right|\right| < \epsilon/2.$$

Then, similar to the above, for any $||\vec{h}|| < \delta$,

$$\begin{aligned}
\frac{\left|f(z+\vec{h}) - f(z) - J_f(z)\vec{h}\right|}{||\vec{h}||} &= \frac{\left|f(z+\vec{h}) - f(z) - J_f(z,b)\vec{h} + J_f(z,b)\vec{h} - J_f(z)\vec{h}\right|}{||\vec{h}||} \\
&\leq \frac{\left|f(z+\vec{h}) - f(z) - J_f(z,b)\vec{h}\right|}{||\vec{h}||} + \frac{\left|(J_f(z,b) - J_f(z))\vec{h}\right|}{||\vec{h}||} \\
&< \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

Thus f is differentiable at z.

(4) $\Rightarrow$ (1). □

6.7. **The Lebesgue differentiation theorem.** A version of the Lebesgue diferentiation theorem in one dimension can be found, for instance, in [7, Section 5.4].

Theorem 6.24. *Let $g \in L_1([0,1])$. Then λ-almost surely,*

$$g(z) = \lim_{r\to 0} \frac{1}{r}\int_z^{z+r} g\, d\lambda.$$

In other words, if $G(z) = \int_0^z g(x)dx$, then for λ-almost every z, $G'(z)$ exists and equals $g(z)$. (This explains the name given to the theorem.) In one dimension, the usual version becomes

$$g(z) = \lim_{r,s\to +0} \frac{1}{r+s}\int_{z-r}^{z+s} g d\lambda.$$

This is equivalent to the formulation above that $G'(z) = g(z)$ for a.e. z. For if f is a function on $[0,1]$, then

$$f'(z) \text{ exists} \Leftrightarrow \lim_{r,s\to 0+} S_f(z-s, z+r) \text{ exists},$$

in which case they are equal. To see this, note that if the limit on the right exists, it clearly equals $f'(z)$. Now suppose conversely that $c = f'(z)$ exists. Given $\varepsilon > 0$, choose $\delta > 0$ such that for $r, s > 0$,

$$\max(r,s) < \delta \text{ implies } |c - S_f(z-s,z)| \leq \varepsilon \wedge |c - S_f(z, z+r)| \leq \varepsilon.$$

Then

$$\begin{aligned} |(s+r)c - (f(z+r) - f(z-s))| &= |sc - sS_f(z-s,z) + rc - rS_f(z,z+r)| \\ &\leq |sc - sS_f(z-s,z)| + |rc - rS_f(z,z+r)| \\ &\leq \varepsilon(s+r) \end{aligned}$$

Hence $|c - S_f(z-s, z+r)| \leq \varepsilon$.

7. Demuth's path to algorithmic randomness

Kučera and Nies worked in Prague in May. Their main project was unearthing Demuth's results. Eventually they want to put the outcome of this into a BSL paper. A preliminary version appears in an LNCS volume for Calude's 60th birthday, Demuth's path to randomness (extended abstract) [34].

Osvald Demuth was a constructivist and a finitist. His primary interest was constructive analysis in the Russian style, which was initiated by Markov, Šanin, Ceĭtin, and others.

Demuth was born in 1936. In 1959 he graduated from the Faculty of Mathematics and Physics at Charles University, Prague with the equivalent of a masters degree. Thereafter he studied constructive mathematics in Moscow under the supervision of A. A. Markov jr., where he successfully defended his doctoral thesis (equivalent to a PhD thesis) in 1964. After that he returned to Charles University, where he worked until the end of his life in 1988.

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7.1. Demuth's world.

Demuth used the Russian style terminology of constructive mathematics, adding some of his own terms and notions.

From the beginning through the 1970s, in line with Russian style constructivism, he only believed in computable reals, which he called constructive real numbers, and sometimes, simply, numbers. He still accepted talking about Δ^0_2 reals, which he called pseudonumbers. They were given by approximations in the sense of the limit lemma, so it was not necessary to view them as entities of their own. Later on, in the 1980s, he relaxed his standpoint somewhat, also admitting arithmetical reals.

For Markov computable functions see Definition 6.9. They were discussed in Subsection 6.2 (April entry), in particular their pseudo-differentiability. Demuth called them *constructive* functions. By a *c-function* he meant a constructive function that is constant on $(-\infty, 0]$ and on $[1, \infty)$. By a result of Ceĭtin, and also a similar result of Kreisel, Shoenfield and Lacombe, each c-function is continuous on the computable reals. However, since such a function only needs to be defined on the computable reals, it is not necessarily uniformly continuous.

A *modulus of uniform continuity* for a function f is a function θ on positive rationals such that $|x - y| \leq \theta(\epsilon)$ implies $|f(x) - f(y)| \leq \epsilon$ for each rational $\epsilon > 0$. If a c-function is uniformly continuous (or equivalently, if it can be extended to a continous function on $[0, 1]$) then its has a modulus of uniform continuity that is computable in $\emptyset'$. Demuth also considered $\emptyset$-uniformly continuous c-functions, i.e. c-functions which even have a computable modulus of uniform continuity; this is the same as computable functions on the unit interval in the usual sense of computable analysis (see [9]).

7.2. Martin-Löf randomness and differentiability.

Demuth introduced a randomness notion equivalent to Martin-Löf randomness in the paper [13]. He was not aware of Martin-Löf's earlier definition in [37]. Among other things, Demuth gave his own proof that there is a universal Martin-Löf test.

The notion was originally only considered for pseudonumbers (i.e., Δ^0_2 reals). As a constructivist, Demuth found it more natural to first define the *non*-Martin-Löf random pseudonumbers. He called them Π_1 numbers. Pseudonumbers that are not Π_1 numbers were called Π_2 numbers. Thus, in modern language, the Π_2 numbers are exactly the Martin-Löf random Δ^0_2-reals.

As already noted, from the 1980s on Demuth also admitted arithmetical reals (in parallel with the decline of communism, and thereby its background of philosophical materialism). In [17] he called the arithmetical *non*-ML-random reals $\mathcal{A}_1$ numbers,

and the arithmetical ML-random reals $\mathcal{A}_2$ numbers. For instance, the definition of $\mathcal{A}_1$ can be found in [17, page 457]. By then, Demuth knew of Martin-Löf's work: he defined $\mathcal{A}_1$ to be $\bigcap_k [W_{\langle g \rangle(k)}]$, where g is a computable function determining a universal ML-test, and $[X]$ is the set of arithmetical reals extending a string in X. In the English language papers such as [20], the non-ML random reals were called AP (for approximable, or approximable in measure), and the ML-random reals were called NAP (for non-approximable).

Demuth needed Martin-Löf randomness for his study of differentiability of Markov computable functions (Definition 6.9), which he called constructive. The abstract of the paper [14], translated literally, is as follows:

> It is shown that every constructive function f which cannot fail to be a function of weakly bounded variation is finitely pseudo-differentiable on each Π_2 number.
>
> For almost every pseudo number ξ there is a pseudo number which is a value of pseudo derivative of function f on ξ, where the differentiation is almost uniform.

Converted into modern language, the first statement says that each Markov computable function of bounded variation is (pseudo-)differentiable at each ML-random real. We do not know how Demuth proved this. However, his result has been recently reproved in [10] in an indirect way, relying on a similar result on computable randomness in the same paper [10].

The second statement says that for almost every Δ^0_2 real z, the derivative $f'(z)$ is also Δ^0_2, and can be computed from z. The notion that a property $\mathcal{S}$ holds for "almost all" pseudonumbers (i.e., Δ^0_2 reals) is defined in [14, page 584]; see Figure 2.

<u>Определние</u>. Пусть $\mathcal{S}$ свойство псевдочисел, а $x \triangle y$ сегмент. Мы скажем, что $\mathcal{S}(\xi)$ выполнено для почти всех псевдочисел ξ (соотв. ξ из $x \triangle y$), если для всякого НЧ m существуют последовательность последовательностей рациональных сегментов $\{\{Q^n_k\}_k\}_n$ и последовательность неинфинитных рекурсивно перечислимых (р.п.) множеств НЧ $\{D_n\}_n$ такие, что $\forall n \ell \left(\sum_{1 \le k \le \ell \,\&\, \neg(k \in D_n)} |Q^n_k| < \frac{1}{2^{m+n}}\right)$ и для всякого ПЧ ξ (соотв. ПЧ ξ из $x \triangle y$) верно $(\neg \exists n k (\neg(k \in D_n) \,\&\, \xi \in Q^n_k) \supset \mathcal{S}(\xi))$ (см. лемму 7 из [2]).

FIGURE 2. [14, page 584]: Definition of Y-tests

We rephrase this definition in modern (classical) language. Demuth introduces a notion of tests; let us call them Y-tests. A Y-test uniformly in a number $m \in \mathbb{N}$ provides a computable double sequence of rational intervals $(Q^m_r(k))_{r,k \in \mathbb{N}}$, and a uniformly c.e. sequence of finite sets $(E^m_r)_{r \in \mathbb{N}}$, such that

$$\lambda(\bigcup\{Q^m_r(k) \colon k \notin E^m_r\}) \le 2^{-(m+r)}$$

(where λ denotes Lebesgue measure). A real z *fails* the test if for each m there is r such that for some $k \notin E^m_r$ we have $z \in Q^m_r(k)$, i.e., $z \in \bigcup\{Q^m_r(k) \colon k \notin E^m_r\}$. Otherwise z *passes* the test. Thus, z passes the test if there is an m such that for all r, we have $z \notin \bigcup\{Q^m_r(k) \colon k \notin E^m_r\}$. Demuth says that a property $\mathcal{S}$ holds for *almost all* z if there is a Y-test such that $\mathcal{S}$ holds for all z passing the test.

We can make the notion of Y-tests look more like a modern-day test notion by using Σ^0_1 sets of reals and parameterizing by $p = m + r$. Thus, for $p \geq m$ we write

$$\begin{aligned} U^m_p &= \bigcup\{Q^m_{p-m}(k) \colon k \in \mathbb{N}\} \\ V^m_p &= \bigcup\{Q^m_{p-m}(k) \colon k \in E^m_{m-p}\} \end{aligned}$$

These sets of reals are uniformly Σ^0_1, and $\lambda(U^m_p - V^m_p) \leq 2^{-p}$ for each p. A real z fails the test if $\forall m \exists p \geq m\, [z \in U^m_p - V^m_p]$. This shows that Y-tests are closely related to the difference tests of Franklin and Ng [26], further discussed in 7.7 below. However, unlike [26] we now have a passing condition more like the one for Solovay-tests.

7.3. **Denjoy alternative and Denjoy sets.** It is a classic and well-known fact that for any real-valued function $g \colon [0,1] \to \mathbb{R}$, the reals z such that $g'(z) = \infty$ (i.e., $\underline{D}g(z) = \infty$) form a null set. This is usually proved via covering theorems such as Vitali's or Sierpinskis's. Cater [11] has given an elementary proof of a stronger fact: the reals where the right derivative is infinite form a null set.

Demuth knew results of this kind. The following definition from Subsection 6.3 originates in [16]. Demuth tries to determine which type of null classes is needed to make an analog of the classic fact hold for Markov computable functions. The paper [16] is entitled "The constructive analogue of a theorem by Garg on derived numbers". Garg's Theorem is a variant of the Denjoy-Young-Saks theorem, which has the somewhat obscure reference [28]. As usual, for functions not defined everywhere we have to work with pseudo-derivatives (see Subsection 6.2).

Definition 6.10 *A real $z \in [0,1]$ is called Denjoy random (or a Denjoy set) if for no Markov computable function g we have $\underset{\sim}{D}g(z) = \infty$.*

The Denjoy alternative, which was discussed in Subsection 6.2 of the April 2011 entry, motivated a lot of Demuth's work on algorithmic randomness. The work of Demuth on the Denjoy alternative for effective functions is surveyed in his preprint, called *preprint survey* below. This is based on a talk Demuth gave at the Logic Colloquium 1988 in Padova, Italy (close to the end of communist era in 1989, it became easier to travel to the "West"). He later turned the preprint survey into the paper Remarks on Denjoy sets [21], but it contains only part of the preprint survey. Also recall Theorem 6.11 of Subsection 6.3 due to Demuth, Miller, Nies, Kučera, which shows that Denjoy randomness coincides with computable randomness:

The following are equivalent for a real $z \in [0,1]$

(i) *z is Denjoy random.*
(ii) *z is computably random*
(iii) *for every computable $f \colon [0,1] \to \mathbb{R}$ the Denjoy alternative (3) holds at z.*

We do not know at present how Demuth obtained (i)→(iii) of the theorem, namely, that every Denjoy real z makes the Denjoy alternative hold for computable functions. A proof of this using classical language would be useful.

However, a direct proof of (i)→(ii) is in [10, Thm. 4.6]: if a martingale M with the so-called "savings property" succeeds on (the binary expansion of) a real z, then the authors build in fact a computable function g such that $\underline{D}g(z) = \underset{\sim}{D}g(z) = \infty$. Together with the remark after the proof of Theorem 6.11 we obtain:

Corollary 7.1. *The following are equivalent for a real z:*

(i) *For no $I_{\mathbb{Q}}$ computable function g we have $\underset{\sim}{D}g(z) = \infty$.*
(ii) *z is Denjoy random, i.e., for no Markov computable function g we have $\underset{\sim}{D}g(z) = \infty$.*

(iii) *For no computable function g we have $\underline{D}g(z) = \infty$.*

This implies that the particular choice of Markov computable functions in Definition 6.10 is irrelevant. Similar equivalences stating that the exact level of effectivity of functions does not matter have been obtained in the article [10]. For instance, the authors characterize computable randomness via differentiability of nondecreasing effective functions; this works with any of the three particular effectiveness properties above.

One can say that Demuth studied computable randomness indirectly via his Denjoy sets. Presumably he didn't know the notion of computable randomness, which was introduced by Schnorr in [49], a monograph in German. Demuth also proved in [20, Thm. 2] that every Denjoy set that is AP (i.e., non ML-random) must be high. The analogous result for computable randomness was later obtained in [45]. There the authors also show a kind of converse: each high degree contains a computably random set that is not ML-random. This fact was apparently not known to Demuth.

7.4. **Demuth randomness and weak Demuth randomness.** As told above, Demuth knew that Denjoy randomness of a real z implies the Denjoy alternative at z for all computable functions. The next question for Demuth to ask was the following:

Question 7.2. *How much randomness for a real z is needed to ensure the Denjoy alternative at z for all Markov computable functions?*

Demuth showed the following (see preprint survey, page 7, Thm 5, item 4), which refers to [15].

Theorem 7.3. *There is a Markov computable function f that is extendable to a continuous function on $[0,1]$ such that the Denjoy alternative fails at some ML-random real z.*

This has been reproved by Bienvenu, Hölzl, Miller and Nies [**?**]. In fact one can make z left-c.e.

It was now clear to Demuth that a randomness notion stronger than ML was needed. Weak 2-randomness may have seemed ignoble to him because a Δ^0_2 real cannot be weakly 2-random. He needed a notion compatible with being Δ^0_2. Such a notion was introduced in the paper "Some classes of arithmetical reals" [17, page 458]. The definition is reproduced in the preprint survey on page 4.

In modern (classical) language the definitions are as follows (also see Subsection 1.1 in the January entry).

Definition 7.4. *A* Demuth test *is a sequence of c.e. open sets $(S_m)_{m\in\mathbb{N}}$ such that $\forall m\, \lambda S_m \le 2^{-m}$, and there is a function $f \le_{\mathrm{wtt}} \emptyset'$ such that $S_m = [W_{f(m)}]^{\prec}$.*

A set Z passes *the test if $Z \notin S_m$ for almost every m. We say that Z is* Demuth random *if Z passes each Demuth test.*

Recall that $f \le_{\mathrm{wtt}} \emptyset'$ if and only if f is ω-c.e., namely, $f(x) = \lim_t g(x,t)$ for some computable function g such that the number of changes $\#\{t\colon\ g(x,t) \ne g(x,t-1)\}$ is computably bounded in x. Hence the intuition is that we can change the m-component S_m for a computably bounded number of times.

Fig. 3 shows what the definition of Demuth randomness looks like in the 1982 paper [17, p. 458]. Demuth first defines tests via certain conditions γ_q, where q is an index for a binary computable function $\phi_q(x,k)$. The condition γ_q holds for a real z if

$$\forall m \exists k \ge m\ z \text{ is in } [W_{\lim(s^1_1(q,k))}].$$

$$\bar{S}_0(q) \rightleftharpoons (S_0(q) \,\&\, \forall k\,(\mu_0(\mathcal{L}im(s_1^1(q,k+1))) \leq 2^{-k-1})),$$

$$\mathcal{K}(p,q) \rightleftharpoons (\mathcal{K}_0(q) \,\&\, \forall k\,(!\langle p\rangle(k) \,\&\, \mathcal{M}is(s_1^1(q,k)) \leq \langle p\rangle(k))),$$

где $\mathcal{K}$ одно из выражений S, $\hat{S}$ и $\bar{S}$,

б) если верно $S_0(q)$, то

$$\mathcal{T}_q \rightleftharpoons \bigcap_m \bigcup_{n \geq m} [W_{\mathcal{L}im(s_1^1(q,n))}],$$

$$\mathcal{T}_q^* \rightleftharpoons \bigcup_m \bigcap_{n \geq m} [W_{\mathcal{L}im(s_1^1(q,n))}],$$

$$\mathcal{Y}_q \rightleftharpoons \bigcup_k [\mathcal{B}_{\mathcal{L}im(s_1^1(q,k))}]_c,$$

в) $\mathcal{A}_\alpha \rightleftharpoons \bigwedge X(\neg\neg \exists m\,(\hat{S}(p,m) \,\&\, X \in \mathcal{T}_m))$;

г) $\mathcal{A}_\alpha \rightleftharpoons \bigwedge X(\neg\neg \exists pq\,(\hat{S}(p,q) \,\&\, X \in \mathcal{T}_q))$,

$\mathcal{A}_\alpha^* \rightleftharpoons \bigwedge X(\neg\neg \exists p\,q\,(\hat{S}(p,q) \,\&\, X \in \mathcal{T}_q^*))$,

$\mathcal{A}_\beta \rightleftharpoons \mathcal{A} \setminus \mathcal{A}_\alpha$.

FIGURE 3. [17, page 458]: $\mathcal{A}_\beta$ is the definition of Demuth randomness

The expression in the subscript simply means $\lim_x \phi_q(x,k)$, which is the final version r of the test. A further condition, involving an index p for a computable unary function, yields the bound $\phi_p(k)$ on the number of changes. The bound 2^{-k} on measures of the k-th component can also be found.

If we apply the usual passing condition for tests, we obtain the following notion which only occurs in [17, page 458].

Definition 7.5. *We say that Z is* weakly Demuth random *if for each Demuth test $(S_m)_{m\in\mathbb{N}}$ there is an m such that $Z \notin S_m$.*

In the context of the Definition 7.4, if we also have $S_m \supseteq S_{m+1}$, for each m, we say that $(S_m)_{m\in\mathbb{N}}$ is a monotonic Demuth test. Thus, a set Z is weakly Demuth random if and only if Z passes all monotonic Demuth tests.

In [17] this is given by conditions γ_q^*, where the quantifiers are switched compared to γ_q:

$$\exists m \forall k \geq m \ z \text{ is in } [W_{\lim(s_1^1(q,k))}].$$

The class of arithmetical non-Demuth randoms is called $\mathcal{A}_\alpha$, and the class of arithmetical non-weakly Demuth randoms is called $\mathcal{A}_\alpha^*$. The complement of $\mathcal{A}_\alpha$ within the arithmetical reals is called $\mathcal{A}_\beta$ and, similarly, the complement of $\mathcal{A}_\alpha^*$ within the arithmetical reals is called $\mathcal{A}_\beta^*$. Later on, in the preprint survey, Demuth used the terms WAP sets (weakly approximable) for the non-Demuth randoms, and NWAP for the Demuth randoms and analogously, in an obviuous sense, the terms WAP* sets and NWAP* sets.

The notation $Mis(s_1^1(q,k))$ in Fig. 3 refers to the number of "mistakes", i.e. changes, and Demuth requires it be bounded by $\langle p\rangle(k)$.

In the preprint survey, page 7, Thm 5, item 5) Demuth states that Demuth randomness is sufficient to get the Denjoy alternative for Markov computable functions. This refers to the paper [18].

Theorem 7.6. *Let z be a Demuth random real. Then the Denjoy Alternative holds at z for every Markov computable function.*

This result is actually hard to pin down in [18]. Theorem 2 on page 399 comes close but has some extra conditions not present in the original Denjoy alternative.

Remark 7.7. Franklin and Ng [26] introduced difference randomness, a concept much weaker than even weak Demuth randomness, but still stronger than ML-randomness. In recent work, Bienvenu, Hölzl and Nies have shown that difference randomness is sufficient as a hypothesis on the real z in Theorem 7.6. No converse holds. It turns out that the randomness notion to make the Denjoy Alternative hold for each Markov computable function is incomparable with ML-randomness! See April entry, Subsection 6.4.

Demuth randomness is way too strong for its original purpose, ensuring the Denjoy alternative for Markov computable functions. However, it has recently turned out to be very interesting a notion on its own, because it interacts nicely with computability theoretic notions such as strong jump traceability. For instance, every c.e. set Turing below a Demuth random is strongly jump traceable [33]. Greenberg and Turetsky have provided a converse of this: given a c.e. strongly jump traceable set, there is a Demuth random above.

7.5. **Randomness and computational complexity.** Near the end of his life Demuth became more interested in the purely recursion theoretic aspects of randomness notions. The mathematics department at Prague University had a seminar in the 1980s on recursion theory, which was based on Rogers' book [47] and some draft of Soare's book [50].

Demuth proved the following.

Theorem 7.8. *(i) Each Demuth random set is GL_1.*
(ii) Each Demuth random set is of hyperimmune T-degree.

(i). [20], Remark 10, [part 3b] gives a sketch of a proof. A full proof can be found in [43, Section 5.5].
(ii). Only a sketch of a proof is given in Remark 2 and Remark 11 of the preprint survey. An alternative proof can be derived from (i) and the result of Miller and Nies [43, Thm. 8.1.19] that no GL_1 set of hyperimmune-free degree is d.n.c.

It is of interest that Kučera and Demuth ([22], Theorem 18) proved a result which plays the same role as a later result of Miller and Yu (see, [43], 5.1.14), namely, if A is ML-random then there is a constant c such that $\forall n\, \lambda S^A_{\Phi,n} \leq 2^{-n+c}$, where $S^A_{\Phi,n}$ means (for a Turing functional Φ, $n > 0$)

$$S^A_{\Phi,n} = [\{\sigma\colon\ A\restriction_n \preceq \Phi^\sigma\}]^{\prec}.$$

7.6. **Work on semigenericity.** The following direction of Demuth's late work is only loosely connected to randomness. A set Z is called *semigeneric* [19] if every Π^0_1 class containing Z has a computable path. Any ML-random set is contained in a whole Π^0_1 class of ML-randoms, and therefore is not semigeneric. Intuitively, to be semigeneric means to be close to computable in the sense that the set cannot be separated from the computable sets by a Π^0_1 class.

Demuth proved in [19, Thm. 9] that if a set Z is semigeneric then any set B such that $\emptyset <_{tt} B \leq_{tt} Z$ is also semigeneric. In particular, its tt-degree contains only semigeneric sets.

Kučera and Demuth [22] studied semigenericity and its relationship with other types of genericity. We review some of their results.

Ceĭtin's notion of strong undecidability. Ceĭtin ([12]) called a set Z *strongly undecidable* if there is a computable function p such that for any computable set M and any index v of its characteristic function, $p(v)$ is defined and $Z\restriction_{p(v)} \neq M\restriction_{p(v)}$.

By [22, Cor. 2], a set Z is semigeneric if and only if Z is neither computable nor strongly undecidable. Furthermore, strong undecidability can be characterized by some kind of "uniform non-hyperimmunity": by [22, Thm. 5], a set Z is strongly undecidable if and only if there is a computable function f such that for each computable set M and any index v of its characteristic function, the symmetric difference $M \triangle Z$ is infinite, and its listing in order of magnitude dominated by the computable function with index $f(v)$.

[22, Thm. 14] characterizes the sets Z such that the Turing-degree of Z contains a strongly undecidable sets: this happens precisely when there is a Π^0_2 class containing Z but no computable sets. So we have a weaker form of separation from the computable sets than for non-semigenericity.

[22, Thm. 14] was actually proved in terms of so-called V-coverings (where V stands for Vitali). A set Z is V-covered by a c.e. set of strings A if for all k there is a string $\sigma \in A$ such that $|\sigma| \geq k$ and $\sigma \prec Z$. It is easy to see that a class of sets $\mathcal{A}$ is a Π^0_2 class if and only if there is a c.e. set of strings B such that $\mathcal{A}$ is equal to the class of sets V-covered by B (see [43, 1.8.60]).

Connection to weak 1-genericity and hyperimmunity. Recall that a set Z is weakly 1-generic if Z is in each dense Σ^0_1 class (see [43, 1.8.47]). Clearly any weakly 1-generic set is semigeneric. The converse fails.

Demuth [19, Thm. 16] showed that a set Z is weakly 1-generic if and only if for any computable set M the symmetric difference $M \triangle Z$ is hyperimmune. Kurtz ([31], [32]) proved that a Turing-degree contains a weakly 1-generic set if and only if it is hyperimmune. It follows from Kurtz's results, using a fact of Martin-Miller [36], that the weakly 1-generic T-degrees are closed upwards. As a corollary we have that there are weakly 1-generic Turing degrees which do contain ML-random sets and, thus, they can compute d.n.c. functions. On the other hand, Kučera and Demuth showed that the classes of 1-generic Turing degrees and of Turing degrees of d.n.c. functions are disjoint. In fact, they proved in [22, Cor. 2] that no d.n.c. function (and, thus, no ML-random set) is computable in a 1-generic set (also see [43, 4.1.6]).

Demuth [19, Cor. 12] proved that any hyperimmune or co-hyperimmune set is semigeneric. Furthermore, by [19, Thm. 21] there is a semigeneric set E (even hypersimple) such that no set $A \leq_{tt} E$ is weakly 1-generic.

8. K-triviality and incompressibility in computable metric spaces

Nies and PhD student Melnikov (co-supervised with Khoussainov) worked in Auckland and on Rakiura/Stewart Island (December 2010).

For more detail than this, see the their paper "K-triviality in computable metric spaces" [38], although arguments and some definitions are a bit different in that improved version. Also see Nies' Talk "K -triviality in computable metric spaces" at the Cape Town 2011 CCR. (Both are available on Nies' web page.)

8.1. **Background on computable metric spaces.**

Definition 8.1. Let (M, d) be a Polish space, and let $(q_i)_{i\in\mathbb{N}}$ be a dense sequence in M without repetitions. We say that $\mathcal{M} = (M, d, (q_i)_{i\in\mathbb{N}})$ is a *computable metric space* if $d(q_i, q_k)$ is a computable real uniformly in i, k. We say that $(q_i)_{i\in\mathbb{N}}$ a *computable structure* on (M, d), and refer to the elements of the sequence $(q_i)_{i\in\mathbb{N}}$ as the *special points*.

Definition 8.2. A sequence $(p_s)_{s\in\mathbb{N}}$ of special points is called a *Cauchy name* if $d(p_s, p_t) \leq 2^{-s}$ for each $s \in \mathbb{N}, t \geq s$. Since the metric space is complete, $x = \lim_s p_s$ exists; we say that $(p_s)_{s\in\mathbb{N}}$ is a Cauchy name for x. Note that $d(x, p_s) \leq 2^{-s}$ for each s.

Definition 8.3. We say that a point in a computable metric space is *computable* if it has a computable Cauchy name.

If a computable metric space $\mathcal{M} = (M, d, (q_i)_{i\in\mathbb{N}})$ is fixed in the background, we will use the letters p, q etc. for special points. We may identify the special point q_i with $i \in \mathbb{N}$. Thus, we may view a Cauchy name as a function $\alpha\colon \mathbb{N} \to \mathbb{N}$. We also write $\lim_n \alpha(n) = x$ meaning that $\lim_n q_{\alpha(n)} = x$.

Example 8.4. The following are computable metric spaces.

(i) The unit interval $[0, 1]$ with the usual distance function and the computable structure given by some effective listing without repetition of the rationals in $[0, 1]$, i.e., by fixing a computable bijection τ between $\mathbb{N}$ and $\mathbb{Q} \cap [0, 1]$, and letting $q_n = \tau(n)$.

(ii) The Baire space $\mathbb{N}^{\mathbb{N}}$ consisting of the functions $f\colon \mathbb{N} \to \mathbb{N}$ with the usual ultrametric distance function

$$d(f, g) = \max\{2^{-n}\colon\ f(n) \neq g(n)\},$$

(where $\max \emptyset = 0$). The computable structure is given by fixing some effective listing without repetition of the functions that are eventually 0. (Note that such functions can be described by strings in $\mathbb{N}^{<\omega}$ that don't end in 0.)

(iii) Cantor space $2^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}}$, with the inherited distance function and computable structure.

8.2. K**-triviality.**

We will generalize the usual definition of K-triviality in Cantor space

$$\exists b\, \forall n\, K(x\restriction_n) \leq K(0^n) + b$$

to points in general computable metric spaces.

We first provide some preliminary material. Thereafter, we introduce our main concept.

K-triviality for functions. Fix some effective encoding of tuples x over $\mathbb{N}$ by binary strings, so that $K(x)$ is defined for any such tuple.

Definition 8.5. *We say that a function $\alpha\colon \mathbb{N} \to \mathbb{N}$ is K-*trivial *if*

$$\exists b\, \forall n\, K(\alpha\restriction_n) \leq K(0^n) + b.$$

Proposition 8.6. *A function α is K-trivial iff its graph $\Gamma_\alpha \leftrightharpoons \{\langle n, \alpha(n)\rangle\colon n \in \mathbb{N}\}$ is K-trivial in the usual sense of sets.*

Proof. One uses that K-triviality implies lowness for K (see [43, Section 5.4]). □

Solovay functions.
Recall that a computable function $h\colon \mathbb{N} \to \mathbb{N}$ is called a *Solovay function* [4] if $\forall r\, [K(r) \leq^+ h(r)]$, and $\exists^\infty r\, [K(r) =^+ h(r)]$.

Solovay [51] constructed an example of such a function. The following simpler recent example is due to Merkle. We include the short proof for completeness' sake.

Fact 8.7. *There is a Solovay function h.*

Proof. Let $\mathbb{U}$ denote the optimal prefix-free machine. Given $r = \langle \sigma, n, t\rangle$, if t is least such that $\mathbb{U}_t(\sigma) = n$, define $h(r) = |\sigma|$. Otherwise let $h(r) = r$.

We have $K(r) \leq^+ h(r)$ because there is a prefix-free machine M which on input σ outputs $r = \langle \sigma, \mathbb{U}(\sigma), t\rangle$ if t is least such that $\mathbb{U}_t(\sigma)$ halts. If σ is also a shortest string such that $\mathbb{U}(\sigma) = n$, then we have $h(r) = |\sigma| = K(N) \leq^+ K(r)$. □

The following generalizes the corresponding fact for sets also due to Merkle.

Fact 8.8. *Let $\alpha\colon \mathbb{N} \to \mathbb{N}$ be a function such that $\forall r\, K(\alpha\restriction_r) \leq h(r) + b$. Then α is K-trivial via a constant $b + O(1)$.*

Proof. Given n, let σ be a shortest $\mathbb{U}$-description of n, and let t be least such that $\mathbb{U}_t(\sigma) = n$. Let $r = \langle \sigma, n, t\rangle$. Then

$$K(\alpha\restriction_n) \leq^+ K(\alpha\restriction_r) \leq h(r) + b = |\sigma| + b = K(n) + b.$$

□

8.3. **K-trivial points in computable metric space.** Our principal notion is K-triviality in computable metric spaces. Recall from Definition 8.3 that a point x in a computable metric space $\mathcal{M}$ is called computable if it has a computable Cauchy name. We define K-triviality in a similar fashion.

Definition 8.9. We say that a point x in a computable metric space is *K-trivial* if it has a Cauchy name that is K-trivial as a function.

Proposition 8.10. *Let $\mathcal{M}, \mathcal{N}$ be computable metric spaces, and let the map $F\colon \mathcal{M} \to \mathcal{N}$ be computable. If x is K-trivial in $\mathcal{M}$, then $F(x)$ is K-trivial in $\mathcal{N}$.*

Proof. Let α be a K-trivial Cauchy name for x. Since F is computable, there is a Cauchy name $\beta \leq_T \alpha$ for $F(x)$. Then β is K-trivial by Proposition 8.6 and the result of [42] that K-triviality for sets is closed downward under $\leq_{\mathrm{T}}$. □

If F is effectively uniformly continuous (say Lipschitz), then one can also give a direct proof which avoids the hard result from [42]. Moreover, from a K-triviality constant for x one can effectively obtain a K-triviality constant for $F(x)$, which is not true in the general case.

Since the identity map is Lipschitz, we obtain:

Corollary 8.11. *K-triviality in a computable metric space is invariant under changing of the computable structure to an equivalent one. More specifically, if x is K-trivial for b with respect to a computable structure, then x is K-trivial for $b + O(1)$ with respect to an equivalent structure.*

8.4. **Existence of non-computable K-trivial points.**

Example 8.12. *There is a computable metric space $\mathcal{M}$ with a noncomputable point such that the only K-trivial points are the computable points.*

Proof. Let $\mathcal{M}$ be the computable metric space with domain $\{\Omega_s\colon\ s \in \mathbb{N}\} \cup \{\Omega\}$, the metric inherited from the unit interval and with the computable structure given by $q_s = \Omega_s$.

Suppose g is a Cauchy name for Ω in $\mathcal{M}$. Then, for a rational p, we have $p > \Omega \leftrightarrow \exists n[g(n) + 2^{-n} < p]$. Thus $\{p \in \mathbb{Q}\colon p < \Omega\} \leq_{\mathrm{T}} g$. This set is Turing complete, so g is not K-trivial. □

A metric space is said to be *perfect* if it has no isolated points. In the following we take Cantor space $2^{\mathbb{N}}$ as the computable metric space with the usual computable structure of Example 8.4.

Proposition 8.13. [8, Prop. 6.2]*] Suppose the computable metric space $\mathcal{M}$ is perfect. Then there is a computable injective map $F\colon\ 2^{\mathbb{N}} \to \mathcal{M}$ which is Lipschitz with constant 1.*

Given $\delta > 0$, we denote by $B(x, \delta)$ the open ball of radius δ and with center in x, that is, $B(x, \delta) = \{y \in M : d(x, y) < \delta\}$.

Theorem 8.14. *Let $\mathcal{M}$ be a computable perfect Polish space. Then for every special point $p \in M$ and every rational $\delta > 0$ there exists a non-computable K-trivial point $x \in B(p, \epsilon)$.*

Proof. Note that the closure in $\mathcal{M}$ of $B(p, \delta)$ is again a perfect Polish space $\mathcal{N}$ which has an inherited computable structure. By the result of Brattka and Gherardi there is a computable injective Lipschitz map $F\colon\ 2^{\mathbb{N}} \to \mathcal{N}$.

Let A be a non-computable K-trivial point in Cantor space. Then $x = F(A)$ is K-trivial in $\mathcal{N}$, and hence in $\mathcal{M}$, by Proposition 8.10; actually only the easier case of Lipschitz functions discussed after 8.10 is needed.

As Brattka and Gherardi point out before Prop. 6.2, the inverse of F is computable (on its domain). Thus, if x is computable then so is A, which is not the case. □

8.5. **An equivalent local definition of K-triviality.** We fix a computable metric space $\mathcal{M} = (M, d, (q_n)_{n\in\mathbb{N}})$. When it comes to K-triviality for points in $\mathcal{M}$, one's first thought would be to directly adapt the definition of K-triviality of some point x in Cantor Space,

$$\exists b\, \forall n\, K(x\restriction_n) \leq K(0^n) + b.$$

Recall from Example 8.4 that in Cantor space $2^{\mathbb{N}}$ we chose as the special points the sequences of bits that are eventually 0. Since we identify a special point $p = \alpha(n)$ with n, we get the following tentative definition:

$$\exists b\, \forall n\, \exists p[d(x, p) \leq 2^{-n} \wedge K(p) \leq K(n) + b]. \tag{9}$$

This isn't right, though:

Proposition 8.15. *There is a Turing complete Π^0_1 set $A \in 2^{\mathbb{N}}$ satisfying condition (9)*

Proof. For a string α, let $g(\alpha)$ be the longest prefix of α that ends in 1, and $g(\alpha) = \emptyset$ if there is no such prefix. We say that a set A is *weakly K-trivial* if

$$\forall n\, [K(g(A\restriction_n)) \leq^+ K(n)].$$

This is equivalent to (9). (Clearly, every K-trivial set is weakly K-trivial. Every *c.e.* weakly K-trivial set is already K-trivial.)

We now build a Turing complete Π^0_1 set A that is weakly K-trivial. We maintain the condition that

$$\forall i\, \forall w\, [\gamma_i < w \to K(w) > i], \tag{10}$$

where γ_i is the i-th element of A. This implies that A is Turing complete, as follows. We build a prefix-free machine N. When i enters $\emptyset'$ at stage s, we declare that $N(0^i 1) = s$. This implies $K(s) \leq i + d$ for some fixed coding constant d. Now $i \in \emptyset' \leftrightarrow i \in \emptyset'_{\gamma_{i+d+1}}$, which implies $\emptyset' \leq_{\mathrm{T}} A$.

We let $A = \bigcap A_s$, where A_s is a cofinite set effectively computed from s, $A_0 = \mathbb{N}$, $[s, \infty) \subseteq A_s$, and $A_{s+1} \subseteq A_s$ for each s. We view γ_i as a movable marker; γ_i^s denotes its position at stage s, which is the i-th element of A_s.

Construction of A and a prefix-free machine M.
Stage 0. Let $A_0 = \mathbb{N}$.

Stage $s > 0$. Suppose that there is w such that $i := K_s(w) < K_{s-1}(w)$. By convention, w is unique and $w < s$. Thus, there is a new computation $\mathbb{U}_s(\sigma) = w$ with $|\sigma| = i$ at stage s.

If $w \leq \gamma_i^{s-1}$ then let $A_s = A_{s-1}$. If $w > \gamma_i^{s-1}$ then, to maintain (10) at stage s, we *move* the marker γ_i: we let $A_s = A_{s-1} - [\gamma_i^{s-1}, s)$, which results in $\gamma_{i+k}^s = s + k$ for $k \geq i$, while $\gamma_j^s = \gamma_j^{s-1}$ for $j < i$.

In any case, declare $M(\sigma) = g(A_s \restriction_w)$.

Verification. Clearly, each marker γ_i moves at most 2^{i+1} times, so $A = \bigcap_s A_s$ is an infinite co-c.e. set. Furthermore, condition (10) holds because it is maintained at each stage of the construction.

We show by induction on s that

$$\forall n\, [K(g(A_s \restriction_n)) \leq^+ K_s(n))]. \tag{11}$$

For $s = 0$ the condition is vacuous. Now suppose $s > 0$. We may suppose that w as in stage s of the construction exists, otherwise (11) holds at stage s by inductive hypothesis.

As in the construction let $i = K_s(w)$, and let σ be the string of length i such that $\mathbb{U}_s(\sigma) = w$. If $w \leq \gamma_i^s$ then $A_s = A_{s-1}$, so setting $M(\sigma) = g(A_s \restriction_w)$ maintains (11).

Now suppose that $w > \gamma_i^s$. Let $n < s$. We verify (11) at stage s for n.

If $n \leq \gamma_i^{s-1}$ then $A_s \restriction_n = A_{s-1} \restriction_n$ and $K_s(n) = K_{s-1}(n)$, so the condition holds at stage s for n by inductive hypothesis. Now suppose that $n > \gamma_i^{s-1}$. By (10) at stage $s - 1$ we have $K_{s-1}(n) > i$, and hence $K_s(n) \geq i$ (equality holds if $n = w$). Because we move the marker γ_i at stage s, we have $g(A_s \restriction_n) = g(A_s \restriction_w)$. Thus setting $M(\sigma) = g(A_s \restriction_w)$ ensures that the condition (11) holds at stage s for n. $\square$

8.6. **Local definition of K-triviality.** The analog of the definition in Cantor space $\exists b\, \forall n\, K(x \restriction_n) \leq K(0^n) + b$ should be a stronger property: From the string $x \restriction_n$ we can compute the maximum distance 2^{-n} we want the highly compressible special point p to have from x. Thus, we should actually require that $K(p, n) \leq K(n) + b$ (where $K(p, n)$ is the complexity of the pair $\langle p, n\rangle$).

Definition 8.16. Let $b \in \mathbb{N}$. We say that $x \in M$ is *locally K-trivial via b*, or locally K-trivial(b) for short, if

$$\forall n\, \exists p\, [d(x, p) < 2^{-n} \wedge K(p, n) \leq K(n) + b]. \tag{12}$$

K-trivial implies locally K-trivial:

Fact 8.17. *Suppose f is a K-trivial via v Cauchy name for x. Then x is locally K-trivial via $v + O(1)$.*

Proof. Note that $d(f(n), n) \leq 2^{-n}$ for each n. Clearly,

$$K(f(n), n) \leq^+ K(f\restriction_{n+1}) + O(1) \leq K(n+1) + v + O(1) \leq K(n) + v + O(1).$$

Hence, for each n, the point $p = f(n)$ is a special point at a distance of at most 2^{-n} from x such that $K(p, n) \leq K(n) + v + O(1)$. □

Showing the definition is natural. We show that the notion of locally K-trivial point in (12) is actually independent of the fact that we chose the bounds on distances to be of the form 2^{-n}. We list the positive rationals as $(r_i)_{i\in\mathbb{N}}$ in an effective way without repetitions. Note that for $\epsilon = r_i$ we have $K(\epsilon) = K(i)$ be definition. Given that, we could also define local K-triviality(b) like this:

$$\forall\epsilon\, \exists p\, [d(x, p) < \epsilon \wedge K(p, \epsilon) \leq K(\epsilon) + b], \tag{13}$$

where ϵ ranges over positive rationals. This is apparently stronger than the definition above. However, if we encode positive rationals as $(r_i)_{i\in\mathbb{N}}$ in a reasonable way then $2^{-i} \leq r_i$ for all $i \geq$ some i_0. In this case, Fact 8.17 still holds: if n is least such that $2^{-n} \leq r_i < 2^{-n+1}$, then for $r_i = \epsilon$ we can take $p = f(n)$ as a witness for local K-triviality in the sense of (13): we have $i \geq n$, and hence

$$K(f(n), i) \leq K(f\restriction_{i+1}) + O(1) \leq K(i) + v + O(1).$$

In Theorem 8.21 below we will show the converse of the foregoing Fact 8.17. Thus, being locally K-trivial, either in the original or the strong sense, is all equivalent to having a K-trivial Cauchy name.

However, this equivalence holds only for points themselves, not for their approximating sequences. The following example shows that a point x can have continuum many Cauchy names $(p_n)_{n\in\mathbb{N}}$ of special points so that (12) in Definition 8.16 holds for each n via p_n.

Example 8.18. Let the special points of the computable metric space $\mathcal{M}$ be all the pairs $\langle r, n\rangle$, where $r \in \{0, 1\}$ and $n \in \mathbb{N}$. We declare

$$d(\langle 0, n\rangle, \langle 1, n\rangle) = 2^{-n} \text{ and } d(\langle r, n\rangle, \langle k, n+1\rangle) = 2^{-n-1}.$$

All the special points are isolated. There is only one non-isolated point, namely $x = \lim_n \langle 0, n\rangle$. This point x is computable. Each sequence $(p_n)_{n\in\mathbb{N}}$ of the form $(\langle r_n, n\rangle)_{n\in\mathbb{N}}$ is a Cauchy name for x. For an appropriate b, (12) holds for each n via p_n.

Fact 8.19. $\#\{p \in \mathbb{N}\colon\ K(p, n) \leq K(n) + b\} = O(2^b)$.

Proof. This is like [43, 2.2.26], with the change that we let $M(\sigma) = n$ if $\mathbb{U}(\sigma) = \langle i, n\rangle$. □

We next derive a fact on the number and distribution of the points that are locally K-trivial(b). Suppose that distinct points $x_1, \ldots, x_k \in M$ are locally K-trivial(b). Pick $n^* \geq 2$ so large that $2^{-n^*+1} < d(x_i, x_k)$ for any $i \neq j$, and choose p_i for x_i, n^* according to (12). Then all the p_i are distinct. By Fact 8.19, this implies that $k = O(2^b)$. Furthermore, x_i is the only locally K-trivial(b) point x such that $d(x, p_i) \leq 2^{-n^*}$.

We summarize the preceding remarks.

Lemma 8.20.

(a) For each $b \in \mathbb{N}$, at most $O(2^b)$ many $x \in M$ are locally K-trivial(b).

(b) There is $n^* \in \mathbb{N}$, $n^* \geq 2$ as follows: for each point x that is locally K-trivial via b, there is a special point $\widetilde{p}$ with $K(\widetilde{p}, n^*) \leq K(n^*) + b$ such that x is the only locally K-trivial(b) point within a distance of 2^{-n^*} of $\widetilde{p}$.

We are now ready to prove the converse of Fact 8.17: local K-triviality in the sense of Definition 8.16 implies K-triviality in the sense of Definition 8.9.

Theorem 8.21. *Suppose that $x \in M$ is locally K-trivial via b. Then x has a Cauchy name h that is K-trivial via $b + O(1)$.*

Proof. After adding a constant to the Solovay function h from Fact 8.7 if necessary, we may suppose that $\forall r\, K(r) \leq h(r)$ and $\exists^\infty r\, K(r) = h(r)$. Thus, the inequalities and equalities hold literally, not merely up to a constant.

The c.e. tree T. Choose a number n^* and a special point $\widetilde{p}$ for the given point x and the constant b according to Lemma 8.20. We define a c.e. tree $T \subseteq \mathbb{N}^{<\omega}$. Since special points are identified with natural numbers, we can think of the nodes of the tree T as being labelled by special points. The root is labelled by $\widetilde{p}$. Of course, the same label p may be used at many nodes. We think of a node at level n labelled p as the first $n+1$ items of a Cauchy name.

As usual, $[B]$ denotes the set of (infinite) paths of a tree $B \subseteq \mathbb{N}^{<\omega}$. Each $\alpha \in [T]$ will be a Cauchy name for some point y. The special points $\alpha(i)$ will be witnesses for the local K-triviality(b) of the point y at n , where $n = n^* + i$.

Formally, we let $T = \bigcup_s T_s$, where

$$T_s = \{(p_{n^*}, \ldots, p_v) \colon p_{n^*} = \widetilde{p} \wedge \forall i . n^* \leq i < v\, [K_s(p_i, i) \leq h(i) + b \wedge d(p_i, p_{i+1}) \leq 2^{-i-1}]\} \tag{14}$$

Note that $[T]$ contains a Cauchy name for x by the hypothesis that x is locally K-trivial(b), the choice of $\widetilde{p}$ and since $n^* \geq 2$. Actually, any path of T does this:

Claim 8.22. *Each $\alpha \in [T]$ is a Cauchy name of x.*

To see this, let $p_0, \ldots, p_{n^*-1}$ witness local K-triviality(b) of x for $n < n^*$. Since the metric space (M, d) is complete, α is a Cauchy name of some point $y \in M$, and the special points

$$p_0, \ldots, p_{n^*-1}, \alpha(n^*), \alpha(n^*+1), \ldots$$

show that y is locally K-trivial(b). Also $d(y, \widetilde{p}) \leq 2^{-n^*}$. Thus in fact $y = x$ by Lemma 8.20. This proves the claim.

For instance, in the case of the computable metric space of Example 8.18, once again let x be the only limit point. We may let $n^* = 2$ and $\widetilde{p} = \langle 0, 2 \rangle$. Then T is a full binary tree, consisting of all the tuples of the form $(\widetilde{p}, \langle r_3, 3 \rangle, \ldots, \langle r_v, v \rangle)$ where $r_i \in \{0, 1\}$.

A very thin c.e. subtree G of T. The tree T for the computable metric space in Example 8.18 shows that there may be lots of Cauchy names with witnesses for local K-triviality; we cannot expect that each such Cauchy name is K-trivial. We will prune T to a c.e. subtree G that is so thin that all of its nodes τ are strongly compressible in the sense that $K(\tau) \leq h(|\tau|) + b + O(1)$; hence each infinite path is K-trivial by Fact 8.8.

We say that a label $p \in \mathbb{N}$ is *present at level* n of a tree $B \subseteq \mathbb{N}^{<\omega}$ if there is $\eta \in B$ such that η has length n and ends in p. While G is only a thin subtree of T, we will ensure that each label present at a level n of T is also present at level n of G. This will show that G still contains a Cauchy name for x.

To continue Example 8.18, there are only two labels at each level of T, so for $G \subseteq T$ we can simply take the tuples where each r_i is 0, except possibly the last. Then the only infinite path of G is a computable Cauchy name of the limit point x.

We will build a computable enumeration $(G_s)_{s\in\mathbb{N}}$ of the tree G where $G_s \subseteq T_s$ for each s. To help with the definition of this computable enumeration, we first define a slower computable enumeration $(\widetilde{T}_s)_{s\in\mathbb{N}}$ of T that grows "one leaf at a time". The $\widetilde{T}_s$ are subtrees of the T_s.

Why can each node of G be compressed? Suppose a new leaf labelled p appears at level n of T, but is not yet at level n of G. Suppose also that p is a successor on T of a node labelled q. Inductively, q is already on level $n-1$ of G; that is, there already is a node $\overline{\eta}$ of length $n-1$ on G that ends in q. Since p is on level n of T, there is a $\mathbb{U}$-*description* showing that $K(p,n) \leq h(n)+b$ (that is, a string $|w|$ with $|w| \leq h(n)+b$ such that $\mathbb{U}(\omega) = \langle p,n\rangle$). Since p is not at level n of G, this $\mathbb{U}$-description is "unused". Hence we can use it as a description of a new node $\eta = \overline{\eta}\,\hat{}\,p$ on G. This ensures that $K(\eta) \leq h(n)+b+O(1)$.

Note that we make use of the fact that a node η on G, once strongly compressible at a stage, remains so at later stages. This is why we need the Solovay function h. If we tried to satisfy the condition $K(\eta) \leq K(n)+b+O(1)$, we might fail, because $K(n)$ on the right side could decrease later on. We also needed the Solovay function to ensure that T is c.e.

In the formal construction, we build a prefix-free machine L (see [43, Ch. 2]) to give short descriptions of these nodes. The argument above is implemented via maintaining the conditions (16,17) below.

A slower computable enumeration $(\widetilde{T}_s)_{s\in\mathbb{N}}$ of T. Let $\widetilde{T}_0$ consist only of the empty string. If $s>0$ and $\widetilde{T}_{s-1}$ has been defined, see whether there is $\tau \in T_s - \widetilde{T}_{s-1}$. If so, choose τ least in some effective numbering of $\mathbb{N}^{<\omega}$. Pick v maximal such that $\tau\restriction_v \in \widetilde{T}_{s-1}$, and put $\tau\restriction_{v+1}$ into $\widetilde{T}_s$. Clearly we have $T = \bigcup_s \widetilde{T}_s$.

Three conditions that need to be maintained at each stage. For strings $\tau,\eta \in \mathbb{N}^{<\omega}$ we write $\tau \sim \eta$ if they have the same length and end in the same elements. Recall that each label present at a level n of T needs to be also present at level n of G. Actually, in the construction we ensure that for each stage s, each label p that is present at a level n of $\widetilde{T}_s$ is also present at level n of G_s:

$$\forall \tau \in \widetilde{T}_s\, \exists \eta \in G_s\, [\tau \sim \eta]. \tag{15}$$

To make sure that each $\eta \in G$ satisfies $K(\eta) \leq h(|\eta|)+b+O(1)$, we construct, along with $(G_s)_{s\in\mathbb{N}}$, a computable enumeration $(L_s)_{s\in\mathbb{N}}$ of (the graph of) a prefix-free machine L. Let m,n range over natural numbers, and v,w over strings. We maintain at each stage s the conditions

$$\forall \eta \in G_s\, \forall m \left[0 < m \leq |\eta| \to \exists v\ [\,|v| \leq h(m)+b \wedge L_s(v) = \eta\restriction_m]\right]; \tag{16}$$

$$\text{if } \mathbb{U}_s(w) = \langle p,n\rangle, \text{ then } \left[w \in \mathrm{dom}(L_s) \ \to\ p \text{ is at level } n \text{ of } G_s\right]. \tag{17}$$

For full construction and verification see the paper. □

Note that all this also works for plain descriptive complexity C instead of K. We say that $x \in M$ is *locally C-trivial via b* if

$$\forall n\, \exists p\, [d(x,p) < 2^{-n} \wedge C(p,n) \leq C(n)+b]. \tag{18}$$

Then by the method in the proof above but for plain machines, we have a local, "AIT" characterization of computable points in computable metric spaces:

Theorem 8.23. *A point is locally C-trivial iff it is computable.*

For details see the upcoming paper.

8.7. **Incompressibility and randomness.** This material is not in the paper [38].

Martin-Löf randomness is a central randomness notion. The usual definition (see [43, 3.2.1]) via passing all Martin-Löf tests works in all computable probability spaces [], In particular, it can be applied both in Cantor space with the product measure, and in the unit interval with the usual uniform measure. Note that a real $r \in [0,1]$ is Martin-Löf random iff its binary expansion is when viewed as an element of $2^{\mathbb{N}}$.

The Schnorr-Levin Theorem (see [43, 3.2.9]) states that a set $A \in 2^{\mathbb{N}}$ is Martin-Löf random if and only if there is a constant $b \in \mathbb{N}$ such that $K(A\restriction_n) > n - b$ for all n.

We let

$$\begin{aligned} K(z\,;n) &= \min\{K(p,n)\colon\ d(z,p) < 2^{-n}\}, \\ K_*(z\,;n) &= \min\{K(p)\colon\ d(z,p) < 2^{-n}\}. \end{aligned}$$

Definition 8.24. We say that a point z in a computable metric space $\mathcal{M}$ is *incompressible in approximation (i.a.)* if $\exists b \in \mathbb{N}\,\forall n\ [K(z;\,n) > n - b]$. We say that z is *strongly i.a.* if $\exists b \in \mathbb{N}\,\forall n\ [K_*(z;\,n) > n - b]$.

By the following fact, the second notion above expresses that the closer a special point is to z, the less it can be compressed.

Fact 8.25. *z is strongly i.a. via $b \Leftrightarrow \forall p\ [d(z,p) \geq 2^{-K(p)-b}]$.*

Proof. Note that the condition $\forall p\ [d(z,p) \geq 2^{-K(p)-b}]$ is equivalent to

$$\forall n\ [d(z,p) < 2^{-n} \to K(p) + b > n],$$

i.e., $\forall n\, K_*(z\,;n) > n - b$. □

Fact 8.26. *Let $\mathcal{M}$ be either Cantor space or the unit interval, with the computable structures introduced in Example 8.4. Let x be an element of $\mathcal{M}$. Then the following are equivalent.*

(i) *x is incompressible in approximation.*
(ii) *x is strongly incompressible in approximation.*
(iii) *x is Martin-Löf random.*

Proof. (i) → (iii). We first consider the case that x is a bit sequence A in Cantor space. Suppose x is incompressible in approximation via $b \in \mathbb{N}$.

Given $n \in \mathbb{N}$, let p be the special point $A\restriction_n 0^\infty$, that is, p consists of the first n bits of A, followed by 0s. Then, since $d(A,p) < 2^{-n}$, by our hypothesis we have $K(A\restriction_n) \geq^+ K(p,n) > n - b)$. This shows that A is Martin-Löf random.

Now suppose $x \in [0,1]$ is incompressible in approximation. Then $x < 1$. Let A be the binary expansion of x with infinitely many 0s. If p is a special point in $2^{\mathbb{N}}$ with $d(A,p) < 2^{-n}$, then the dyadic rational q corresponding to p is a special point in $[0,1]$ with $d(x,q) < 2^{-n}$. Since q can be computed from p, this shows that A is incompressible in approximation within Cantor space. Hence x is Martin-Löf random.

(iii) → (ii). The following argument works in both Cantor space and the unit interval. Let

$$U_b = \bigcup_p B(p, 2^{-K(p)-b-1}).$$

Then the sequence $(U_b)_{b\in\mathbb{N}}$ of open sets is uniformly c.e. Furthermore, the measure of U_b is bounded by

$$\sum_p 2^{-K(p)-b} \leq 2^{-b}\Omega\,.$$

Thus, $\{U_b\}_{b\in\mathbb{N}}$ is a Martin-Löf test. Since z passes this test, z is incompressible in approximation. $\square$

This only works because we have dimension 1. In the unit *square*, for instance, a ML-random x will satisfy $K_*(x;n) \geq^+ 2n$, by a proof similar as above. In the k-cube .... you guessed it, $\geq^+ kn$. In these spaces there is a computable upper bound on $K_*(x;n)$. This fails for instance in $\mathcal{C}[0,1]$

We give an example of a computable metric spaces such that no incompressible point exists.

Example 8.27. Let $\mathcal{M}$ be the Cantor space $2^{\mathbb{N}}$ with the distance function squared, i.e., with $\widetilde{d}(f,g) = d(f,g)^2$, and the same computable structure as in Example 8.4(iii). Then no point in M is incompressible in approximation.

To see this, suppose that $A \in M$ is incompressible in approximation via b. Let p_n be the special point $(A\restriction_n)0^\infty$. Then $\widetilde{d}(A,p_n) < 2^{-2n}$, whence $K(p_n,n) \geq K(A;2n) > 2n-b$. For large n, this contradicts the fact that $K(p_n,n) \leq^+ K(A\restriction_n) \leq^+ n + 2\log n$.

We show that being i.a. is preserved under any computable map $F\colon \mathcal{M}\to\mathcal{N}$ such that F^{-1} is Lipschitz. For instance, via a computable embedding of the unit interval we can find points that are i.a. in any computable Banach space.

Proposition 8.28. *Let $F\colon \mathcal{M}\to\mathcal{N}$ be a computable map such that F^{-1} is Lipschitz, namely, there is $v\in\mathbb{N}$ such that $d(x,y)\leq 2^v d(F(x),F(y))$. Then for each z in $\mathcal{M}$ and each $n\geq v+1$ we have*

$$K(F(z);n) \geq^+ K(z;n-v-1).$$

Proof. Define a prefix-free machine L as follows. On input τ, if $\mathbb{U}(\tau) = \langle q,n\rangle$ for a special point q of $\mathcal{N}$, then L searches for a special point p of $\mathcal{M}$ such that $d(F(p),q) < 2^{-n}$. If p is found, it outputs $\langle p, n-v-1\rangle$.

Suppose now that $\mathbb{U}(\tau) = \langle q,n\rangle$ for some τ of least length, where

$$d(F(z),q) < 2^{-n}.$$

Then on input τ the machine L finds p because F is continous. This shows that $K(p,n-v-1) \leq^+ |\tau| = K(q,n)$. Furthermore, we have

$$d(p,z) \leq 2^v d(f(p),F(z)) \leq 2^v[d(F(p),q) + d(q,F(z))] < 2^v 2^{-n+1}.$$

Since $K(F(z);n)$ is the minimum of all such $K(q,n)$ such that $d(f(z),q) < 2^{-n}$, this establishes the required inequality. $\square$

In particular, F preserves being i.a. To preserve being strongly i.a. we also need that the range of F is dense:

Proposition 8.29. *Let $F\colon \mathcal{M}\to\mathcal{N}$ be a computable map with range dense in $\mathcal{N}$ such that F^{-1} is Lipschitz with constant 2^v. If z in $\mathcal{M}$ is strongly i.a. then so is $F(z)$. In particular, being strongly i.a. is preserved under changing the computable structure to an equivalent one.*

Proof. Suppose z is strongly i.a. via b. Then, by Fact 8.25, for each special point p of $\mathcal{M}$ and each σ, we have

$$\mathbb{U}(\sigma) = p \to d(z,p) \geq 2^{-|\sigma|-b}. \tag{19}$$

We define a prefix-free machine L. By the recursion theorem, we may assume that a coding constant d_L for L is given in advance. On input τ, if $\mathbb{U}(\tau) = q$ for a special point q of $\mathcal{N}$, then L outputs a special point p of $\mathcal{M}$ such that

$$d(F(p),q) < 2^{-|\tau|-v-b-d_L-1}.$$

Note that p exists because the range of F is dense in $\mathcal{N}$.

Since d_L is the coding constant for L, we have $\mathbb{U}(\sigma) = p$ for some σ such that $|\sigma| \leq |\tau| + d_L$. Thus, by (19) $d(z,p) \geq 2^{-|\sigma|-b} \geq 2^{-|\tau|-d_L-b}$. By the Lipschitz condition on F^{-1} we obtain

$$\begin{aligned} d(F(z),q) &\geq d(F(z),F(p)) - d(F(p),q) \\ &\geq 2^{-v} d(z,p) - d(F(p),q) \\ &\geq 2^{-|\tau|-v-d_L-b} - 2^{-|\tau|-v-d_L-b-1} \\ &= 2^{-|\tau|-v-d_L-b-1}. \end{aligned}$$

If $|\tau| = K(q)$ then by Fact 8.25, this shows that $F(z)$ is strongly i.a. in $\mathcal{N}$ via the constant $v + d_L + b + 1$. $\square$

Part 4. Various

9. Some new exercises on computability and randomness

These will go into next edition of Nies' book. Solutions next page.

Stephan showed that every wtt-incomplete c.e. set B is "i.o. K-trivial" in the sense that $\exists^\infty n\, K(B\restriction_n) \leq^+ K(n)$. See [43, 5.2.8]. The following exercise shows that the weakly K-trivials exist in every wtt degree.

Fact 9.1. *For each set A there is a set $B \equiv_{wtt} A$ such that*

$$\exists^\infty n\, K(B\restriction_n) \leq^+ K(n).$$

Moreover the set of n where this happens is computably bounded.

We consider the smallest cost function that makes sense at all. Ω is random and hence does not obey even that.

Fact 9.2. *Let $c(x,s) = 2^{-x}$. Show that Ω does not obey this cost function.*

Fact 9.3. *(Yu Liang) Use Arslanov's completeness criterion to give a direct proof that Ω_0 is low.*

Solutions:

9.1 Define sets $(B_k)_{k\geq -1}$ of size $k+1$ inductively (but not computably), as follows. Let $B_{-1} = \emptyset$. If B_k has been defined, let $n = n_k$ be the number such that $D_n = B_k$ (strong index). Note that $n_k > \max B_k$. If $k \in A$ let $B_{k+1} = B_k \cup \{2n\}$. Otherwise, let $B_{k+1} = B_k \cup \{2n+1\}$.

Verification. Clearly $B \leq_{\mathrm{wtt}} A$ by the recursive definition of B. We also have $A \leq_{\mathrm{wtt}} B$ because the sequence (n_k) is computably bounded.

To show that B is i.o. K-trivial, note that for $m = 2n_k$, we have that $K(B\restriction_m) \leq^+ K(m)$ because $D_{n_k} = B \cap [0, m)$.

9.2 We view Ω_s as a binary string. At stage $s > 0$, if there is i least such that $\Omega_s(i) = 1$ and $\Omega_{s-1}(i) = 0$, enumerate the interval $[\Omega_s\restriction_{i+1}]$ into a set $\mathcal{S}$.

If Ω obeys c then $\mathcal{S}$ is a Solovay test capturing Ω.

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