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THE DE SAINT-VENANT EQUATIONS IN CURVED CHANNELS

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A thesis submitted in fulfilment of the requirements for the
Degree of Doctor of Philosophy in Civil Engineering,
The University of Auckland, 1998
ABSTRACT

After introducing the subject of curvilinear flow, particularly in the context of meandering natural channels, this thesis then describes the three conventional models for unsteady flow in open channels, namely kinematic, diffusion and dynamic. These descriptions are in terms of the straight channel de Saint-Venant equations. The discussion also considers some aspects of the diffusion model which raise questions as to the appropriateness of the usual engineering approach to this model.

As to date, these models treat curvature cursorily, if at all, the models are then expanded to incorporate curvature in a more systematic manner. This is done by deriving the de Saint-Venant equations in terms of curvilinear coordinates. The models are then presented in terms of the curvilinear mass-conservation and various forms of the curvilinear momentum equation.

The new models are found to be expressed by equations of the form

'linear model + curvilinear correction'

thus allowing the engineer to estimate the size of any curvature effect.

The derived dynamic model is compared with a laboratory study, and the results indicate that the new curvilinear model is a reasonable description of dam-break flow. Subsequent calculations, based on field data, of the celerity of the dynamic wave illustrate how big the corrections can be.
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Dedication

This thesis is dedicated to my mother Eila Agnes Nalder as an expression of thanks for her help, encouragement and support particularly in her capacity (during the last critical months) as chatelaine of The Eila Nalder Doss-House for Indignant Daughters and Free-Loading Cats.
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NOMENCLATURE

A  ccross-sectional area
A_o cross-sectional area for steady uniform flow
dA  element of cross-sectional area
dA  vector of area element dA
B  top level width
B_o top level width for steady uniform flow
b  width scale
C  Chezy friction coefficient
c  long wave celerity or negative of advection value depending on context
c_o  long wave celerity  \( \frac{gA_o}{\sqrt{B_o}} \)
D  diffusion coefficient
D*  dimensionless diffusion coefficient
D_o  depth steady uniform flow (Ponce et.al. notation)
d  depth scale
E  dispersion coefficient
E*  dimensionless dispersion coefficient
E  percentage error
F  Froude number  \( \frac{Q^2B}{gA^3} \)
F_o  Froude number of steady uniform flow
F  total force acting on fluid in control volume
f  single body force per unit mass
f_s component of f in downstream direction of curved channel
g  acceleration due to gravity
K  cross-sectional shape factor
k  wave number, straight channel
L  length scale
n  transverse axis
n  coordinate in n direction
n  Manning friction coefficient (according to context)
n_l coordinate of water's edge at left bank
n_r coordinate of water's edge at right bank
n_m distance between surface midpoint and s axis
\( \bar{n} \)  distance from centroid of cross-section to reference line
M_z first moment of area about z axis
P  wetted perimeter
P'_o derivative of P with respect to A, evaluated at A_o
p  pressure
Q  discharge
q  lateral discharge per unit length
R  hydraulic radius(=A/P)
r  radius of curvature
r_s  radius scale
\( S_f \) friction slope
\( S_o \) downstream slope of channel bed
\( s \) downstream axis for curved channel
\( T \) wave period
\( T_g \) time beginning of rise to centre of gravity of hydrograph (hr)
\( T_r \) time beginning of rise to peak of hydrograph (hr)
\( t \) time
\( U \) average velocity across cross-section
\( u \) velocity vector through area element \( dA \)
\( u \) downstream component of \( u \)
\( u_o \) downstream component of \( u \) for steady uniform flow \((= Q_o/A_o)\) (by context)
\( u_0 \) depth averaged velocity at reference line (by context)
\( u_q \) velocity of lateral flow
\( V \) volume
\( \Delta V \) element of volume
\( \bar{V} \) average downstream velocity (Fread notation)
\( V_p \) peak velocity (Fread notation)
\( V \) parameter \( = \frac{\delta_1}{\theta_2} - u_o \)
\( X \) weighting factor in Muskingum-Cunge method
\( x \) downstream coordinate, straight channel
\( y \) depth measured vertically downwards
\( y_p \) peak depth
\( z \) vertical axis
\( z \) level of channel bed
\( \alpha \) curvature parameter \((=b/r_h)\)
\( \beta \) dimensionless propagation factor (Ponce and Simon notation)
\( \beta \) friction coefficient
\( \delta \) small parameter for linearisation
\( \delta \) logarithmic decrement (Ponce et.al notation)
\( \delta_d \) logarithmic decrement for diffusion model (Ponce et.al notation)
\( \varepsilon \) wave number
\( \zeta \) shallowness parameter \((=d/L)\)
\( \eta \) water level
\( \eta_o \) water level at reference axis
\( \delta_1 \) parameter \( = g S_o \left( \frac{\mu A_o P_o'}{P_o} - (\mu + \nu) \right) \)
\( \delta_2 \) parameter \( = \frac{g S_o A_o \nu}{Q_o} \)
\( \kappa \) curvature \((=1/r)\)
\( \lambda \) general parameter
\( \mu \) constant for friction law, \( = \frac{4}{3} \) for Manning, = 1 for Chezy

\( \nu \) constant for friction law, = 1 for Manning, = 2 for Chezy

\( \xi \) surface level measured vertically upwards (stage)

\( \rho \) density

\( \delta \) dimensionless wave number (Ponce and Simon notation)

\( \sigma \) narrowness parameter (= b/L)

\( \Phi \) cross-sectional area (de Saint-Venant notation)

\( \chi \) wetted perimeter (de Saint-Venant notation)

\[ \int \text{integral over control volume} \]

\[ \int_{ca} \text{integral over control section} \]

\[ \int_{ca} \text{integral over control area} \]
GLOSSARY

As this is a civil engineering thesis, it assumes a knowledge of engineering terminology. The following definitions are included for the benefit of non-engineers.

**Advection:** The velocity of a wave along with dispersive effects

**Backwater:** This is an increase in upstream water level due to downstream obstructions not to increasing discharge

**Celerity:** Wave speed. Although physics defines phase and group velocities, engineers engaged in open channel engineering tend to think in terms of a single wave.

**Channel routing:** The method of predicting the discharge with time at a particular cross-section, when a wave passes down the channel

**Chezy Equation:** This is an empirical equation for uniform flow which is usually expressed in the form

\[ v = C \sqrt{ms} \]

where, \( v \) is water velocity,

\( m \) is hydraulic radius (i.e. are divided by wetted perimeter)

\( C \) is a parameter for roughness with dimensions \( \sqrt{LT^{-2}} \)

The roughness parameter \( C \) is known as **Chezy Constant**, although it is not a true constant.

**Energy Line:** If a length of channel is considered in profile, a line may be drawn above the water surface to represent the energy of the flow. For uniform
flow, **total energy line**, water surface and bed slope are parallel. This is not true for varying flow.

**Friction Slope**: The energy line which represents the amount of energy used to overcome friction.

**Hydrograph**: A graph of discharge against time.

**Long Waves**: A wave where the wavelength is much greater than the amplitude. Waves in open channels are usually of this type. The analysis in this thesis uses the **long wave approximation** by assuming that the wave number is sufficiently small to allow for expansion by the binomial theorem.

**Manning Equation**: An empirical equation for uniform flow, expressed as

\[ v = \frac{1}{n} \left( \frac{m^2}{s} \right)^{\frac{1}{2}} \]

where, \( v \) is velocity

- \( m \) is hydraulic radius
- \( s \) is bed slope
- \( n \) is a roughness parameter ('Mannings \( n \)')

Again the roughness parameter is not a true constant; its dimensions are \( TL^{-1} \).

**Overland Flow**: Water which is unable to enter the soil (e.g. soil saturated or rendered impermeable by paving) and is forced to flow over the surface.

**Routing Coefficient**: The coefficients of the recursive equations used in channel routing.

**Stage**: Water level elevation measured relative to a convenient datum.
Stage-Discharge Rating (or Rating Curve): This is a graph which relates discharge to stage at a given channel cross-section. The rating curve corresponding to the increase to the peak of discharge is the rising curve and that relating to the decrease from the peak is the falling curve. Frequently the falling curve coincides with the rising curve only at the end points and is above it. In this case the cross-section is said to have a looped rating curve.

Talweg: The downstream line connecting the deepest points at each cross-section
I INTRODUCTION

1.1 A FEW CULTURAL POINTS

One of the earliest references to the engineering of meandering rivers is found in Herodotus (Herodotus, c -450 ).

Of the security conscious Queen Nitocris of Babylon he wrote

..she changed the course of the Euphrates, which flows through Babylon. Its course was originally straight, but by cutting channels further upstream, she made it wind about with so many twists and turns that now it actually passes a certain Assyrian village called Ardericca three separate times...

The purpose of both of the excavation and of the diversion of the river was to cause the frequent bends to reduce the speed of the current, and to prevent a direct voyage downstream to the city. A boat would be faced with a devious course and at the end of her trip she would have to make the tedious circuit of the lake.

Years later Cyrus was able to divert the slowed river and capture the city while Nitocris’ son Belshazzar held a particularly unsuccessful party.

Herodotus believed that meandering could be artificially induced. Other writers recognized that rivers meandered without human intervention and regarded them with overtones of the supernatural. Coleridge’s River Alph is probably one of English literature’s best known examples
Five miles meandering with a mazy motion
Through wood and dale the sacred river ran.

Kubla Khan

The term meander comes from the name of a particularly sinuous river in north-west Turkey called in Greek Maiandros (Μαίανδρος) and in Latin Maender. Milton thought of it as a place of enchantment.

Sweet Echo, sweetest nymph that livst unseen
Within thy airy shell
By slow Maender’s margent green
And in the violet embroidered dale

Comus

Ambrose Bierce was less enchanted;

Meander: The word is the ancient name of a river about one hundred and fifty miles south of Troy, which turned and twisted in the effort to get out of hearing when the Greeks and Trojans boasted of their prowess.

The Devil’s Dictionary

An interesting local example arises in the accounts of the various taniwha in New Zealand rivers. These are all said to inhabit bends or meanders, never straight reaches.

A prosaic mind might maintain that helicoidal currents on bends would result in greater losses to drowning. These extra losses would be explained as the predations of a taniwha. On the other hand the fanciful might maintain that taniwha like spa baths!
1.2 THE ENGINEER AND THE RIVER

In one of his lesser known works, Robert Louis Stevenson (1924) provided a rueful description of his father, the civil engineer Thomas Stevenson, 'on vacation' by a river.

On Tweedside, or by the Lyne or Manor, we have spent whole afternoons; to me, at the time, extremely wearisome; to him, as I am now sorry to think bitterly mortifying. The river was to me a pretty and various spectacle; I could not see - I could not be made to see it otherwise. To my father it was a chequer-board of lively forces, which he traced from pool to shallow with minute appreciation and enduring interest. "That bank was being undercut," he might say; "Why? Suppose you were to put a groin out here, would not the filum fluminis be cast abruptly off across the channel? and where would it impinge upon the other shore? and what would be the result? Or suppose you were to blast that boulder, what would happen? Follow it - use the eyes God has given you - can you not see that a great deal of land can be reclaimed upon this side?" It was to me like school in holidays; but to him, until I had worn him out with my invincible triviality, a delight.

Here Stevenson père sees the river, not as a source of aesthetic relief but as a set of interacting forces, susceptible to modification by man.

While we would expect a civil engineer of the 1990s to be more sympathetic to Robert Louis' viewpoint, the interacting forces are still there and engineers are still
fascinated by the aspects described by Thomas Stevenson and by other aspects as well. All these are influenced by the curvature of the river.

The engineer's first concerns are planimetric; the shape of the river, its position on the flood plain, the shapes of the cross-section. The stability of these factors depends on the degree of erosion which in turn depends on velocity.

Flow through a meander is three-dimensional and it can be resolved into a primary downstream velocity and a transverse secondary velocity. The latter is a rotating cell and is responsible for most of the erosion of the outside bank. The loosened sediment is moved downstream and deposited. Eventually, the whole meander pattern migrates across the plain if unhindered.

On a shorter time frame, the engineer is concerned with flood flows. While the volume flow is a given, the engineer is required to predict water levels, to design suitable flood defences and to estimate the effect if the defences fail.

At the back of all this are the theoretical aspects. Meandering is not limited to rivers. It is a fluid phenomena and rivers and streams are near the midpoint of a continuum which ranges from tiny threads of water on plates to ocean currents. Related to this is the question of why rivers meander. Most explanations relate to riverine situations and ignore other cases.

All of these aspects have been exhaustively studied. However a survey of the literature reveals an interesting gap. There has been very little work on unsteady flow in curved channels. This is rather strange as designing for floods, dam break waves etc. are an important part of Civil Engineering.
1.3 UNSTEADY FLOW IN CURVED CHANNELS

While unsteady flow can in general be a continuum of fluctuations, for civil engineers, unsteady flow in open channels usually means discrete disturbances with long wavelengths. The wave generated by a dam break may have a length which corresponds to a multiple of the length of the released reservoir where the size of the multiple depends on the nature of the failure i.e. the lower the breach the more water is lost, and the longer the wavelength. For a flood wave the wavelength depends on the length of the rainfall period and the resulting inflow to the channel. A substantial groundwater inflow, as opposed to overland (surface) flow can extend the wave period to much longer than the rainfall period. It is possible to have flood waves where the wavelength is longer than the channel. Consequently, the equations describing unsteady flow are frequently referred to as long wave equations.' As a rule of thumb long waves are defined to be those where the depth:wavelength ratio is much less than unity.

Probably the only short waves that the civil engineer will encounter in free surface work are the wind induced waves set up by wind blowing along a reservoir. The size of these waves is a function of wind velocity and fetch (distance that the wind has blown over) and the literature contains various empirical formulae for this function. As these waves are rarely a consideration in open channel work they are not considered further in this thesis.

When considering long waves in curved channels, particularly in natural channels, the standard methods in use at present, tend to be rather ad hoc. There are two approaches. One is to treat the channel as if it were a straight channel of length equal to the talweg (the path traced out by the deepest part of the bed) and to include curvature by adjusting the friction terms. (Cowan,1956). Alternatively correction coefficient are added to the coefficient of the de Saint-Venant equations. This latter method was used in the later version of the dam break prediction computer package DAMBRK. (Fread, 1993b)
As neither of these approaches represents a particularly systematic approach, the object of this thesis is to develop a coherent set of long wave models which explicitly include curvature. There are three long wave models in general use. These are, in order of increasing complexity, the kinematic model, and the diffusion model, and the dynamic model. They are described in detail in Section 2.1

1.4 THE SCOPE OF THIS THESIS

The first step in this thesis is to describe the long wave models for the straight channel case as they are presented in the engineering literature. Accordingly Chapter 2 introduces the de Saint-Venant equations and describes the three long wave models mentioned in Section 1.3. An new analysis of a linearisation of the de Saint-Venant equations (Chapter 3) which is based on some unpublished work by Fenton (1996), provides further insight into long waves in straight channels, particularly the dispersive and diffusive effects.

The first step to obtain the equivalent curvilinear models is to derive the de Saint Venant equations in curvilinear coordinates (Chapter 4). For completeness this is done in terms of three sets of variables, discharge and cross-sectional area, discharge and surface level and velocity and surface level. As the de Saint-Venant equations in terms of discharge and area are the simplest of the three, the three basic models are derived in terms of these variables only and are then compared with the straight channel cases (Chapter 5). The methods of linearisation and analysis described in Chapter 3 are applied to the curvilinear case in Chapter 6.

It was possible to carry out a limited verification of the dynamic model developed in Chapter 5 by comparison with data from laboratory experiments on a physical dam break model. This was a simplified situation with a level channel bed, two
straight channels and a single constant curve. The comparison is described in Chapter 7 and indicates that the model developed in this thesis is a reasonable description of the behaviour of long waves in curved channels. Ideally the model should be tested against a more complex set of meanders.

Although chapter 8 concludes the thesis by providing a summary and some thoughts on future work, there are two appendices which present two special cases. Appendix 1 presents a complete model for the case where flow is assumed to be irrotational. This model utilises the curvilinear de Saint-Venant in terms of velocity and surface level which were derived in Chapter 4. Appendix 2 contains the results of a hand calculation done to estimate the velocity of a dynamic wave when allowance has been made for curvature. This calculation suggests that curvature may have a strong effect on the celerity (wave velocity) especially for curved channels with small radii of curvature or markedly assymetric cross-sections. A third appendix, Appendix 3 contains data from the simulated model study.
II UNSTEADY FLOW IN OPEN CHANNELS

2.1 INTRODUCTION

While the main emphasis of this thesis is on the effects of open channel curvature, it is necessary to establish a baseline by describing unsteady flow in straight channels. As the literature on the theory of unsteady flow in straight channels, and the application of that theory goes back to the mid-nineteenth century, the work described in this chapter is now regarded as standard and is generally accepted. It is anticipated that equations describing long waves in curved channels would consist of the straight channel expression with curvilinear corrections.

Unsteady flow in open channels is controlled by the principles of conservation of mass and momentum. These principles are usually described by the de Saint-Venant equations which are discussed in the following Section 2.2.

Before the advent of high-speed computers, engineers rarely endeavored to solve the de Saint-Venant equations. Instead they developed a range of flow routing methods of varying complexity.

The simplest of these are the lumped flow routing methods. These do not utilise the de Saint-Venant equations directly, but instead estimate flow properties such as discharge, water level etc., as a function of time only at critical points on the
hydrologic/hydraulic network — usually water course junctions. These methods are often referred to as hydrologic routing as they are more suited to hydrologic studies than to design for hydraulic structures such as flood defenses.

Distributed flow (or hydraulic) routing methods were developed for use when it was necessary to know flow properties as a function of both space and time, e.g. stopbank design. As these methods are based on the mass conservation equation and various simplifications of the momentum equation, the methods are sometimes referred to as routing models. For the purpose of this thesis, the term model is used for the pair of equations and any supplementary equations which may be developed from the de Saint-Venant equations. The term routing will apply to the process of successively solving the equations along the channel.

The three main models are: kinematic, where friction and bed slope are the only momentum terms; diffusion, where the momentum equation consists of the pressure friction and bed slope terms; and dynamic model, which is described by the full momentum equation i.e. inertia, pressure, friction and bed slope terms. Ponce and Simons (1977) also define a gravity model where the momentum equation contains inertia and pressure terms, but this model appears to have minimal practical application because of the neglect of the friction and bed slope effects. As the momentum equation does not include the bed slope term, the title gravity model is something of a misnomer.

Sections 2.3, 2.4 and 2.5 will discuss each of these models in turn.
2.2 THE DE SAINT-VENANT EQUATIONS

2.2.1 Introduction

Flow in open channels is conventionally described by one dimensional equations of mass conservation (continuity) and conservation of momentum which were first published by the French engineer de Saint-Venant over a hundred years ago (de Saint-Venant, 1871). Since then the subject has generated an extensive body of literature including monographs and textbook sections. A standard text on the application of the de Saint-Venant equations is 'Practical Aspects of Computational River Hydraulics' by J.A.Cunge, F.M. Holly and A.Verwey, Pitman, London 1980.

In his paper de Saint-Venant uses full derivatives, and 's' for the downstream coordinate. It is more correct to express the equations in terms of partial derivatives since the state variables are functions of both distance and time. Also 's' has come to imply an arc length. The derivations which follow beginning in Section 2.2.3 will use x for the downstream distance in the straight channel case.

In the derivation described in paragraph 6 of his paper, de Saint-Venant assumes a prismatic canal with a rectangular cross-section and varying water depth. His form of the mass-conservation equation is
\[
\frac{d\varpi}{dt} + \frac{d(\varpi U)}{ds} = 0 ,
\]  
(2.2.1-1)

where

- \( \varpi \) is the cross-sectional area,
- \( U \) is the average velocity across the cross-section \( \varpi \),
- \( s \) is the downstream coordinate, positive in the downstream direction, and
- \( t \) is time.

The conservation of momentum equation is,

\[
\frac{d\xi}{ds} = \frac{1}{g} \frac{dU}{dt} + \frac{U}{g} \frac{dU}{ds} + \frac{\chi}{\varpi} \frac{F}{\rho g}
\]  
(2.2.1-2)

where,

- \( \xi \) is surface level measured vertically upwards,
- \( \chi \) is the wetted perimeter of the cross-section,
- \( \rho \) is the density,
- \( g \) is acceleration due to gravity, and
- \( \frac{\chi}{\varpi} \frac{F}{\rho g} \) is a form of friction slope.

De Saint-Venant did not specifically define \( F \). Instead he defined \( \rho g F \) as being the bed friction for shallow beds ('le frottement du fond par unité superficielle').

It may be argued that this first set of equations is unsatisfactory, as they are two equations in three variables \((\varpi, U, \xi)\). However for the case de Saint-Venant considered, a prismatic channel with rectangular cross-section, there is an obvious
relationship between \( \omega \) and \( \xi \), although there is the complication that \( \xi \) is measured vertically upwards. It is necessary to define the datum carefully.

In paragraph 7 of his paper, de Saint-Venant simplifies his equations by re-expressing them in terms of two variables, the depth \( y \) (measured vertically downwards so that \( \frac{d\xi}{ds} = -\frac{dy}{ds} \)) and average velocity \( U \), and ignores friction effects. The equations become respectively:

\[
\frac{dy}{dt} + \frac{d(Uy)}{ds} = 0 \tag{2.2.1-3}
\]

and

\[
\frac{dy}{ds} + \frac{dU}{dt} + \frac{UdU}{ds} = 0. \tag{2.2.1-4}
\]

These are the simplest form of the de Saint-Venant equations in the literature but the neglect of friction severely limits their application.

### 2.2.2 Underlying Assumptions of the Conventional Derivation

As the de Saint-Venant equations have become part of the standard open channel lore, it is easy for engineers to forget that their derivation was based on certain assumptions. There remains the danger that the equations may be applied in circumstances where the underlying assumptions are not valid.
Ven Te Chow et al., (1988, Section 9.1.) indicate that derivations which are based on de Saint-Venant’s approach imply the following assumptions:

1. Flow is one-dimensional; depth and velocity vary only in the longitudinal direction of the channel. This implies that the velocity is constant and the water surface is horizontal across any section perpendicular to the longitudinal axis.

2. Flow is assumed to vary gradually along the channel so that hydrostatic pressure prevails and vertical accelerations can be neglected.

3. The longitudinal axis of the channel is approximated as a straight line.

4. The bottom slope of the channel is small and the channel bed is fixed; that is, the effects of scour and deposition are negligible.

5. Resistance coefficients for steady uniform turbulent flow are applicable to unsteady flow, so that relationships such as the Manning and Chézy equations which relate velocity, hydraulic radius (area divided by wetted perimeter), slope and friction coefficient can be used to describe resistance effects.

6. The fluid is incompressible throughout the flow.
For a natural channel, assumptions 2, 5 and 6 are reasonable. Assumption 4 is probably justified within the context of long waves. For other than a very short reach, assumption 3 is not justified and assumption 1 is not justified at all.

However, as the next section shows, it is possible to derive the de Saint-Venant equations without making any assumptions about the velocity distribution. Using discharge (Q) and cross-sectional area (A) as variables and beginning with the principles of conservation of mass and momentum in integral form does this.

2.2.3 Derivations From Integral Forms

Fenton (1996) first set out the following derivation in an unpublished paper on the straight-line case. The same method is use in Chapter 4, to derive the de Saint-Venant equations for curved channels using curvilinear coordinates.

This derivation uses the control volume and notation of Figure 2.2.3-1. Extending the control volume into the air above the water surface obviates the need to establish a boundary condition on the free surface.

**Mass-Conservation Equation:** The principle of mass conservation in integral form (Streeter and Wylie, 1985, Section 3.3) may be written as

\[
\frac{d}{dt} \int_{cv} \rho \, dV + \int_{cs} \rho \, u \, dA = 0,
\]  

\[ (2.2.3-1) \]
where \( \mathit{t} \) is time, \( \rho \) is the fluid density, \( \mathbf{u} \) is the velocity vector through the area element \( dA \) and \( dA \) is a vector representing an area element of the control surface, with direction normal to and directed outwards from the control surface. Hence \( \mathbf{u} \cdot dA \) is equal to the component of velocity normal to the surface at any point multiplied by the elemental area of the control surface. The first term is the time rate of change of mass within the control volume and the second term is mass rate of flow through the area \( dA \).

As the density of air is zero, the domain of integration in the first integral is the volume of water in the control volume. As \( dV = dx \, dA \), the first integral becomes
\[
\frac{d}{dt} \int_{cv} \rho \, dv = \rho \frac{\partial}{\partial t} \int_{cv} \rho \, A \, dx
\]

which by Leibniz' theorem (Section 4.1.3.2) becomes

\[
\rho \frac{\partial}{\partial t} \int_{cv} A \, dx = \rho \, dx \frac{\partial A}{\partial t}
\]  
(2.2.3-2)

Considering the second term in Equation (2.2.3-1), the flow entering the upstream face is \(- \rho \, Q\), and that leaving the control volume through the downstream face is

\[
\rho (Q + dQ) = \rho \left( Q + \frac{\partial Q}{\partial x} \right) \, dx .
\]

Hence, the change within the control volume is

\[
\rho \frac{\partial Q}{\partial x} \, dx .
\]

When there is also inflow from rainfall, groundwater or tributaries ('lateral inflow') entering the channel at a volume rate of \(q\) per unit length, the extra mass leaving the control volume is \(- \rho \, q \, dx\), with a negative sign because of the convention that \(q\) positive corresponds to flow entering the volume.

Combining the two terms gives the total mass-conservation equation as,

\[
\rho \, dx \frac{\partial A}{\partial t} + \rho \, dx \frac{\partial Q}{\partial x} - \rho \, dx \, q = 0,
\]

or,

\[
\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial t} = q.
\]  
(2.2.3-3)

This is an exact expression and suggests that discharge \(Q\) and cross-sectional area \(A\) may be more fundamental variables than velocity and depth.
Momentum Equation: This derivation begins with the integral form of Newton's Second Law (Streeter and Wylie, 1955, Section 3.3):

\[ \frac{\partial}{\partial t} \int_{V} \rho \ u \ dV + \int_{S_{e}} \mathbf{u} \cdot \mathbf{dA} = \Sigma \mathbf{F}, \]

where \( \Sigma \mathbf{F} \) is the total force acting on the fluid in the control volume, including both surface and body forces. As the aim is to develop a one-dimensional momentum equation, only the component in the x (downstream) direction is considered.

First Integral: To evaluate the first integral, \( dV \) is set equal to \( dA \ dx \) so the integral becomes

\[ \frac{\partial}{\partial t} \int_{V} \rho \ u \ dA \ dx, \]

which can be rearranged as

\[ \rho \frac{\partial}{\partial x} \int_{A} u \ dA, \]

where \( u \) is the downstream component of velocity. This equals

\[ \rho \frac{\partial}{\partial t} \frac{\partial Q}{\partial x}. \]

Second integral: The two contributions to this term from upstream and downstream faces are,

\[ \rho \int_{A} (u - u) \ dA \] and \[ \rho \int_{A+\frac{\partial A}{\partial x}ds} \left( u + \frac{\partial u}{\partial x} \right)^2 \ dA, \]

where, in an increment \( dx \), \( u \) has increased by \( \frac{\partial u}{\partial x} \ dx \) and the cross-sectional area by \( \frac{\partial A}{\partial x} \ dx \). The net contribution can be written as

\[ \rho \frac{\partial}{\partial x} \frac{\partial}{\partial x} \int_{A} u^2 \ dA, \] and if the lateral inflow has a velocity of \( u_q \) the rate at which this momentum leaves the control volume is

\[ -\rho \frac{\partial}{\partial x} q u_q \] giving the total net contribution of the term as
\[ \rho dx \frac{\partial}{\partial x} \int_A u^2 dA - \rho dx qu_q. \]

It will be demonstrated in Chapter 4 that
\[ \int_A u^2 dA = \frac{Q^2}{A}, \]
so the second term becomes,
\[ \rho dx \left( 2 \frac{Q}{A} \frac{\partial Q}{\partial x} - \frac{Q^2}{A^2} \frac{\partial A}{\partial x} \right) - \rho dx qu_q. \]

**Force term:** Chow *et al.* (1988) identify five forces acting on the control volume, gravity, friction, pressure, contraction and expansion of the channel and wind shear on the water surface. For the work in this thesis it is assumed that the last two factors are negligible.

The contribution of pressure to the surface forces can be written as:
\[ -\int_{cs} p dA, \]
where \( p \) is the pressure and the negative sign indicates that the pressure acts in the direction opposite to the outward normal. This is more easily derived by utilising Gauss' Divergence Theorem (Milne-Thomson, 1968, Section 2.61), which on allowing for different sign convention for the outwards normal gives
\[ \int_{cs} p dA = \int_{cv} \nabla p dV \]
where \( \nabla \) is the gradient operator. Hence the net pressure force on a closed surface equals the volume integral of the pressure gradient throughout the region enclosed by the surface. As interest is restricted to the \( x \) direction \( \nabla p \) becomes \( \frac{\partial p}{\partial x} \). As it
is assumed that the pressure is hydrostatic then the pressure at any level $z$ is given by,

$$p(x, n, z, t) = \rho g (\eta(x, n, t) - z),$$

where $\eta$ is the water level. Hence,

$$\frac{\partial p}{\partial x} = \rho g \frac{\partial \eta}{\partial x}. \quad (2.2.3-5)$$

Substituting this into the pressure integral term leads to an integral over the area, i.e.

$$- \int \nabla p \, dV = - \rho g \int A \frac{\partial \eta}{\partial x} \, dA = - \rho g \int A \frac{\partial \eta}{\partial x}. \quad (2.2.3-6)$$

As the cross-sectional area can be defined in terms of the water depth as

$$A = \int_b^B (\eta - z_b) \, dy,$$

where $B$ is the width at the surface, then the partial derivative of area with respect to downstream distance is

$$\frac{\partial A}{\partial x} = \int_b^B \left( \frac{\partial \eta}{\partial x} - \frac{\partial z_b}{\partial x} \right) \, dy, \quad (2.2.3-7)$$

where $y$ is the transverse coordinate. Performing the integration gives,

$$\frac{\partial A}{\partial x} = B \left( \frac{\partial \eta}{\partial x} + S_o \right), \quad (2.2.3-8)$$

where $S_o$ is the mean downstream slope across the width of the channel with the usual engineering convention that the mean slope is positive when the channel slopes downwards i.e. $\frac{\partial z_b}{\partial x}$ is negative. Substituting for $\frac{\partial \eta}{\partial x}$ in the last term of
Equation (2.2.3-6) gives,
\[ \rho \frac{dx}{B} \frac{g A}{\partial x} - \rho g A S_o. \]

If the gravity and friction forces are regarded as combined into a single body force per unit mass \( f \), then the total body force is \( \int \rho f dV \). The usual practice is to set the downstream component \( f_r \) as \( -g S_f \) where \( S_f \) is the friction slope.

Substituting this to get the downstream body force gives
\[ \int \rho f_r dV = -\rho g S_f \int dA = -\rho dx g A S_f. \tag{2.2.3-9} \]

Assembling the terms and dividing through by \( \rho dx \) gives the momentum equation as
\[ \frac{\partial Q}{\partial t} + \frac{2Q}{A} \frac{\partial Q}{\partial x} + \left( \frac{g A}{B} - \frac{Q^2}{A^2} \right) \frac{\partial A}{\partial x} = g A S_o + g A S_f = qu. \tag{2.2.3-10} \]

Equations (2.2.3-3) and (2.2.3-10) form a pair of one-dimensional de Saint-Venant equations with the added advantage that their form is not dependent on the shape of the channel. The equations can also be derived in two or three dimensions, but these are generally dependent on the shape of the channel. Engineers concerned with long wave problems generally find a one-dimensional model adequate. In the work that follows in this chapter, each model is described by a pair of one-dimensional equations.
2.3 KINEMATIC MODEL

2.3.1 Derivation of the Model

The model is described by the pair of equations:

\[ \frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = q, \]  \hspace{1cm} (2.3.1-1)

and

\[ S_f = S_o. \]  \hspace{1cm} (2.3.1-2)

This model is based on the assumption that the momentum of unsteady flow is the same as that of steady uniform flow, and the wave property follows primarily from the mass-conservation equation. Consequently, an expression for kinematic wave celerity may be obtained from one of the steady flow equations such as Chézy or Manning, (Chow et al 1988).

Manning’s equation (S.I. form) states that

\[ U = \frac{R^{2/3}}{n} S_f^{1/2}, \]  \hspace{1cm} (2.3.1-3)

where \( U \) is average velocity,

\( R \) is hydraulic radius \((= A/P)\),

\( P \) is wetted perimeter and

\( n \) is the Manning constant (dimensional).
As \( Q = AU \) and \( S_o = S_f \), the Manning equation can also be written in the form

\[
A = \left[ \frac{nP^{2/3}}{S_o^{1/2}} \right] Q^{3/5},
\]

which may be written generally as,

\[
A = \alpha Q^\beta.
\]

Hence the mass-conservation equation in \( Q \) and \( A \) becomes,

\[
\frac{\partial Q}{\partial x} + \alpha \beta Q^{\beta-1} \frac{\partial Q}{\partial t} = q.
\]

An increment in flow \((dQ)\) can be written as \( dQ = \frac{\partial Q}{\partial x} dx + \frac{\partial Q}{\partial t} dt \), which on dividing through by \( dx \) becomes

\[
\frac{dQ}{dx} = \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial t} \frac{dt}{dx},
\]

which equals the volume flow per unit length. This equation is equivalent to the mass-conservation equation provided,

\[
\alpha \beta Q^{\beta-1} = \frac{dt}{dx}.
\]

However as \( \alpha \beta Q^{\beta-1} \), is \( \frac{\partial A}{\partial Q} \) then the kinematic wave velocity \( dx/dt \) equals

\[
\frac{\partial Q}{\partial A}.
\]

Expanding the mass conservation equation into the form,
\[
\frac{\partial A}{\partial t} \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x} = q \tag{2.3.1-9}
\]

and using \( c \) to designate the wave velocity, the equation on multiplying through by \( c \) becomes,

\[
\frac{\partial Q}{\partial t} + c \frac{\partial Q}{\partial x} = cq, \tag{2.3.1-10}
\]

the \textbf{kinematic wave equation}.

Perturbation analysis of the complete one-dimensional de Saint-Venant equations by Ponce and Simon (1977) indicated that kinematic waves;

1) Propagate downstream only,
2) Have celerity equal to 1.5 x the mean flow velocity, and
3) Have zero attenuation.

This analysis assumed the Chézy formula. As is discussed in the following chapter, an analysis, which utilised the Manning formula instead, may give a different value for celerity.

As the kinematic model is effectively controlled by the mass-conservation equation which is exact, the kinematic model is the only one of the three models to have an exact solution. A typical analytical example based on the method of characteristics is as follows. Beginning with Equation (2.3.1-10)

\[
\frac{\partial Q}{\partial t} + c \frac{\partial Q}{\partial x} = cq,
\]
and introducing new characteristic coordinates,

\[ \xi = x + ct \]
\[ \eta = x - ct \]

then from the chain rule,

\[ \frac{\partial Q}{\partial t} = \frac{\partial Q}{\partial \xi} c - \frac{\partial Q}{\partial \eta} c, \quad (2.3.1-11) \]

and

\[ \frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial \xi} + \frac{\partial Q}{\partial \eta} \quad (2.3.1-12) \]

Substituting into Equation (2.3.1-10) gives

\[ \frac{\partial Q}{\partial \xi} = \frac{q}{2} \quad (2.3.1-13) \]

Integrating by separating the variables results in,

\[ Q = \frac{q}{2} \xi + f(\eta) = \frac{q}{2} (x + ct) + (x - ct), \quad (2.3.1-14) \]

where \( f(x-ct) \) is an arbitrary function depending on the boundary conditions, e.g.

\[ Q_{\text{imp}} = F(x). \]

Substituting into Equation (2.3.1-14) gives

\[ \frac{q}{2} x + f(x) = F(x), \quad (2.3.1-15) \]

so that

\[ f(x) = F(x) - \frac{q}{2} x. \quad (2.3.1-16) \]

Therefore,

\[ Q(x,t) = \frac{q}{2} (x + ct) + F(x - ct) - \frac{q}{2} (x - ct) = qct + F(x - ct). \quad (2.3.1-17) \]
2.3.2 Applications

The general kinematic model was first fully described by Lighthill and Whitham (1955). They introduced it as a functional relationship between:

(1) The flow \( q \) (quantity per unit time),
(2) The concentration (quantity per unit distance), and
(3) The position \( x \),

and applied it to a range of physical situations including traffic flow on long crowded roads. In Section 2 of their paper they pointed out that 'several writers independently have given the theory of continuous kinematic waves ..., as applying to flood movement', including Boussinesq (1877), Kleitz (1858, unpublished), and Seddon (1900).

Lighthill and Whitham emphasised that 'we do not claim that kinematic wave theory gives a really exact model of flood movement 'but could act 'as a first approximation' to the dynamic model. Subsequent work has shown that the kinematic model can be applied to a limited range of flood problems. Its use is restricted to cases with single value stage discharge rating curves (i.e. no loop ratings), insignificant backwater effects and channel slopes (i.e tangent of angle of inclination) greater than 0.001, (W.M.O., 1981). Instead of using discharge as a function of stage (water level), the rating curves are handled by using the form
\[ Q = \alpha A^m, \]

where \( \alpha \) and \( m \) are parameters depending on the shape of the cross-section, e.g. for a wide rectangular channel of width \( B \),

\[ \alpha = \left[ \frac{1}{n B^{2/3}} \right] S_0^{0.5} \]

and \( m \) is 1.67 (Lettenmeir & Woods, 1993).

As the waves propagate only downstream, reverse (negative) waves cannot be predicted.

In addition to the analytical solutions using characteristics, direct solutions by finite difference approximation of either explicit or implicit types are also used. Sometimes these solutions of the kinematic model do show attenuation even though the kinematic wave equation does not predict attenuation. This is caused by the numerical errors introduced by the finite difference solution. (Fread 1993a)

The model may be used for overland flow routing in hydrological basin models (such as HEC-1) but should be used for channel routing only when the hydrograph rises slowly, the channel slope is moderate to steep, and the hydrograph attenuation is quite small (Fread, 1993b).
2.4 DIFFUSION MODEL

2.4.1 The Model

In this case, the important force is the pressure, so the model can be described mathematically as

\[
\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = q, \tag{2.4.1-1}
\]

and

\[
\left( \frac{g A - Q^2}{B} \right) \frac{\partial A}{\partial x} = g A \left( S_x - S_o \right) = q u_q. \tag{2.4.1-2}
\]

The terms ‘diffusion’ and ‘dispersion’ have quite clear meanings in physics. They usually imply the evening out of some form of concentration, be it solid, liquid, gas or energy. This is shown mathematically by an expression where the time rate of change of the concentration is proportional to second derivative with respect to distance (diffusion) and to the third derivative with respect to distance (dispersion). (See Section 2.4.2)

For long waves, diffusion means the smoothing out of regions of curvature. Dispersion occurs when the wave speed depends on wavelength so an arbitrary disturbance may disperse out into different waves travelling at different speeds. The terminology seems to have been carried over into analysis of long waves as
the phenomena can be described (mathematically) by the same form of equations -
time rate of change proportional to both the second and third derivative with
respect to distance.

Ponce and Simons (1977) demonstrated by a perturbation analysis that diffusion
waves would

1) Only travel downstream.

2) Have celerity equal to 1.5 x the mean water velocity (i.e. equal to that of
the kinematic wave)

3) Attenuate at a rate that depended on a dimensionless wave number
given by \( \frac{2L_o}{L} \) where, \( L \) is the wavelength of the disturbance and \( L_o \) is the
horizontal length over which a steady uniform flow of the same velocity
drops a head equal to its depth.

As in their analysis of the kinematic model, friction was allowed for by use of the
Chézy equation.

The diffusion model can be used for routing in rivers without significant
backwater effects and where the channel slope is greater than 0.05. If there is
some doubt as to the wisdom of using the diffusion model, the engineer should do
a sample calculation of the size of the deleted inertia factors to check that they are
negligible compared to the pressure, gravitational and friction terms.
2.4.2 Convective-Diffusion Equation

Since the advent of high-speed computers most attention has been focused on solutions for the equations which describe the kinematic and dynamic models. The literature suggests that there has been minimal interest in obtaining solutions for the equations describing the diffusion model. Instead engineers have continued to use what may be called 'the conventional method'. This has been to approach the diffusion model through what has been called a convective-diffusion equation. Equations of similar form can be found in telecommunications (Telegrapher's Equation) and in thermodynamics (Heat-Conduction Equation). These methods can also allow for an estimation of dispersive effects.

The convective-diffusion equation is obtained by combining the mass-conservation and momentum equations into a single equation in one variable. Higher order time derivatives are eliminated in favour of space derivatives to given an equation of the form

\[ \frac{\partial Q}{\partial t} = c \frac{\partial Q}{\partial x} + D \frac{\partial^2 Q}{\partial x^2} + E \frac{\partial^3 Q}{\partial x^3}, \]

where \( c \) is the negative of the celerity value, \( D \) is the diffusion coefficient and \( E \) is the dispersion coefficient.

This method is described in detail in Chapter 3, and a simple case drawn from the literature is described below.
Weinmann and Laurenson (1984) describe the coefficient of the \( \frac{\partial Q}{\partial x} \) as 'responsible for the convective (translational) characteristics', whereas Ferrick \textit{et al.} (1984) define it as the kinematic wave velocity. This is not strictly correct, as the coefficient is the celerity of the diffusive wave. It is numerically equal to the kinematic wave velocity only because Ferrick \textit{et al.}'s derivation forces it to equal the kinematic wave velocity. (See Section 3.3.2.1).

There is general agreement that the coefficient of \( \frac{\partial^2 Q}{\partial x^2} \) is the diffusion coefficient, which represents the attenuation of the flood wave. If the two coefficients are fitted to observed hydrographs, the diffusion coefficient can account for the effects of channel irregularities and flood plain storage, (Weimann and Laurenson, 1984). The coefficient of \( \frac{\partial^3 Q}{\partial x^3} \) is the dispersion coefficient.

This method is demonstrated by the following example, of a diffusion model in a wide prismatic channel with no lateral inflow (Ferrick \textit{et al}, 1984).

The method begins with the de Saint-Venant equations in terms of discharge \( Q \) and flow depth \( y \) i.e.
In this version of the momentum equation, the friction term has been replaced by the Chézy equation.

Ferrick et al. then describe the next step by

'If the coefficients of (1) and (2) (the de Saint-Venant equations) are assumed constant at appropriate reference values, the equations can be combined and expressed in terms of a single dependent variable.'

This is most easily done as follows. The equations are linearised by expanding about a steady uniform flow and in terms of a small parameter \( \delta \) to give,

\[
Q(x,t) = Q_0 + \delta Q_1 + ...
\]

and

\[
y(x,t) = Y_0 + \delta Y_1 + ...
\]

Substituting into the de Saint-Venant equations gives the following equations of zero and first orders:
The mass conservation at first order is,
\[ B \frac{\partial Y_i}{\partial t} + \frac{\partial Q_i}{\partial x} = 0. \quad (2.4.2-3) \]

The momentum equation at zero order is,
\[ \frac{g Q_o^2}{B Y_o^2 C^2} - g BY_o S_o = 0, \]
so that
\[ C^2 = \frac{Q_o^2}{B^2 Y_o^3 S_o}. \quad (2.4.2-4) \]

This shows that, as expected, \( Q_o \) and \( Y_o \) are not independent, and \( Q \) is proportional to \( Y^{3/2} \).

The momentum equation at first order, after substituting for \( C^2 \) from Equation (2.4.2-4), is
\[ \frac{\partial Q_i}{\partial t} + \left( g BY_o - \frac{Q_o^2}{B Y_o} \right) \frac{\partial Y_i}{\partial x} - 3g BY_i + \frac{2g BY_o S_o}{Q_o} Q_i + \frac{2Q_o}{BY_o} \frac{\partial Q_i}{\partial x} = 0. \quad (2.4.2-5) \]

A single equation is obtained by differentiating Equation (2.4.2-3) with respect to \( x \), differentiating Equation (2.4.2-5) with respect to \( t \) and then substituting for
\[ B \frac{\partial^2 Y_i}{\partial x \partial t} \text{ and } B \frac{\partial Y_i}{\partial t}. \]
This gives,
\[ \frac{2g BY_o S_o}{Q_o} \frac{\partial Q_i}{\partial t} + 3g S_o \frac{\partial Q_i}{\partial x} + \left( \frac{Q_o^2}{B^2 Y_o^2} - g Y_o \right) \frac{\partial^2 Q_i}{\partial x^2} + \frac{2Q_o}{BY_o} \frac{\partial^2 Q_i}{\partial x \partial t} + \frac{\partial^2 Q_i}{\partial t^2} = 0, \quad (2.4.2-6) \]

which illustrates the combination of constant coefficients and variables. However
the conventional method is to now substitute \( Q \) and \( y \) for \( Q_o \) and \( Y_o \), to substitute \( Q \) and \( y \) for \( Q_i \) and \( Y_i \) and to try to remember which \( Q \) and \( y \) are variables and which are constants! (This cavalier approach to mathematical analysis is unlikely to be met with favour by an applied mathematician.) This point is taken up in Section 2.4.3.) If this is done and the equation rearranged to give unit coefficient for \( Q/t \), the resulting equation is

\[
\frac{\partial Q}{\partial t} + \frac{3Q}{2By} \frac{\partial Q}{\partial x} + \left( \frac{Q^2 - gB^2y^3Q}{2By^3gS_o} \right) \frac{\partial^2 Q}{\partial x^2} + \frac{Q^2}{2By^2gS_o} \frac{\partial^2 Q}{\partial x \partial t} + \frac{Q}{2BygS_o} \frac{\partial^2 Q}{\partial t^2} = 0, \tag{2.4.2-7}
\]

which is Ferrick et.al's Equation (3). Weinmann and Laurenson (1979) call this the **convective diffusion** equation.

The next step is to remove the \( \frac{\partial^2}{\partial x \partial t} \) and \( \frac{\partial^2}{\partial t^2} \) factors. Equation (2.4.2-7) is rewritten as

\[
\frac{\partial Q}{\partial t} + \frac{3Q}{2By} \frac{\partial Q}{\partial x} + \frac{Q}{2S_oB} \left[ \frac{1}{gy} \left( \frac{\partial}{\partial t} + \frac{Q}{By} \frac{\partial}{\partial x} \right) \right] Q - \frac{\partial^2 Q}{\partial x^2} = 0. \tag{2.4.2-8}
\]

It is then assumed that

\[
\frac{\partial Q}{\partial t} = -\frac{3Q}{2By} \frac{\partial Q}{\partial x},
\]

so that the resulting equation is

\[
\frac{\partial Q}{\partial t} + \frac{3Q}{2By} \frac{\partial Q}{\partial x} + \frac{Q}{2S_oB} \left[ \frac{Q^2}{4B^2y^3g} - 1 \right] \frac{\partial^2 Q}{\partial x^2} = 0. \tag{2.4.2-9}
\]
The actual values agree with Ponce and Simons' values.

The diffusion coefficient derived above can be expressed in terms of the Froude number \( F = \frac{V}{\sqrt{g y}} \) as \( \frac{Q}{2BS_o} \left( \frac{F^2}{4} - 1 \right) \), which is the expression quoted by Ferrick et al. who also present a dispersion coefficient as function of the diffusion coefficient \( D \), i.e.

\[
E = F^2 \frac{y}{2S_o} D = \frac{Qy}{4BS_o} F^2 \left( \frac{F^4}{4} - 1 \right),
\]

(2.4.2-10)

The paper also contains a dimensionless diffusion coefficient

\[
D^* = \frac{Q}{2BS_o c \Delta x} \left( \frac{F^2}{4} - 1 \right),
\]

(2.4.2-11)

and a dimensionless dispersion coefficient,

\[
E^* = F^2 \left( \frac{y}{2S_o \Delta x} \right)
\]

(2.4.2-12)

where \( \Delta x \) is a distance increment down the channel. The further the value of \( D^* \) is from 1, the more diffusion dominates over celerity and the further the value of \( E^* \) is from 1, the more dispersion dominates over diffusion. For natural rivers, \( E^* \) is generally much less than 1, showing that the flow is controlled more by frictional than inertial forces.

2.4.3 Limitations of the Conventional Approach

While the simultaneous use of \( Q \) and \( y \) as both constants and variables in the same equation is questionable, the blithe assumption that \( \frac{\partial Q}{\partial t} = -\frac{3Q}{2By} \frac{\partial Q}{\partial x} \) brings in an
even more serious limitation. This assumption implies that there is a single wave travelling downstream at a speed of 1.5 times the water velocity. Any possibility of a wave travelling upstream is specifically precluded and the value of the celerity of the downstream wave is specifically defined. The sole justification for this seems to be is that it works and gives the required form of equation. An alternative analysis, presented in Chapter 3, indicates that this assumption is suspect.

2.4.4 Routing Methods Based on the Diffusion Model

The Muskingum-Cunge method (Cunge, 1969) is one of the most frequently used diffusion routing methods. In this, the kinematic Muskingum method is changed into a diffusion method which predicts hydrograph attenuation.

Muskingun-Cunge method uses the recursive equations

\[ Q_{i+1}^j = C_1 Q_i^j + C_2 Q_i^{j+1} + C_3 Q_{i+1}^j + C_a \]  \hspace{1cm} (2.4.4-1)

where the coefficients are defined as
\[ C_1 = \frac{\Delta t - 2KX}{2K(1-X) + \Delta t}, \]
\[ C_2 = \frac{\Delta t + 2KX}{2K(1-X) + \Delta t}, \]
\[ C_3 = \frac{2K(1-X) - \Delta t}{2K(1-X) + \Delta t}, \]
\[ C_a = \frac{q_i \Delta x \Delta t}{2K(1-X) + \Delta t}. \]

where, \( K \) is a storage constant with dimensions of time \( t \) and \( X \) is a weighting factor showing the relative importance of inflow and outflow to the storage. They are calculated by

\[ K = \frac{\Delta x}{c} \quad (2.4.4-2) \]

and

\[ X = \frac{1}{2} - \frac{Q_a}{2c B_a S_e \Delta x} \quad (2.4.4-3) \]

where,

- \( Q_a \) is the mean discharge,
- \( c \) is the kinematic wave speed,
- \( B_a \) is the water top width equivalent to \( Q_a \),
- \( S_e \) is the energy grade line,
- \( \Delta x \) is the reach length.

The recursive equation can be solved by either a linear or non-linear method. In the linear solution, \( K \) and \( X \) are assumed to be constant for all time steps in each
reach or are computed from known flow properties. The non-linear solution is more accurate. An estimated value of $Q_i^{/\ast}$ and its corresponding water surface elevation ('h') are used to calculate $K$ and $X$. This iterative solution is continued until it converges to a suitably small values of $h$ such as 0.003 m.

Other diffusion routing schemes include PAB (Parabolic and Backwater) (Todini and Bossi, 1985) and Zero-Inertial (Strelkoff and Katopodes, 1977).

2.4.5 Applicability Criteria for the Kinematic and Diffusion Models

Previous paragraphs have qualitatively stated when the kinematic and diffusion models should be applied. An examination of the literature also shows methods of obtaining quantitative criteria as to when the two models are applicable. These include:

**Ponce, Li and Simons:** In their 1977 paper, referred to previously, Ponce and Simons calculated the propagation celerities and logarithmic decrements for the kinematic and diffusion wave models. Ponce, Li and Simons (1978) used these variables to estimate the limits of applicability of the kinematic and diffusion models. Although the kinematic and diffusion models were treated as having the same celerity, only the diffusion wave shows attenuation. Hence the engineer can judge which of the kinematic or diffusion models best describes the wave in question, by estimating the size of the attenuation. The general logarithmic decrement ($\delta$) is defined as
\[ \delta = \ln(a_i) - \ln(a_o) \]  

(2.4.5-1)

where \( a_o \) and \( a_i \) are respectively the wave amplitudes at the beginning and end of one wave period. The logarithmic decrement for the diffusion model \( (\delta_d) \) was shown by Ponce and Simon to be

\[ \delta_d = \frac{-8\pi^2}{9\tau} \]

where \( \tau = T \left( \frac{u_o}{L_o} \right) \) is the dimensionless wave period of the unsteady component of the linearised analysis with \( T \) the wave period, \( u_o \) the steady uniform flow mean velocity and \( L_o \) is as previously defined (Section 2.4.1). The kinematic model will be valid when \( \delta_d \) is close to 0, or alternatively when the attenuation factor of the diffusion model \( \exp(\delta_d) \) is close to 1. For choice of the model to be 95% accurate, must be greater than \( \frac{1}{71} \) (Table 1, Ponce, Li and Simon). Hence for channels with mild slopes, the kinematic model applies only when the period is very long. This is typical of slow-rising flood waves.

Ponce Li and Simons also established an applicability boundary for the diffusion and dynamic models. This is commented on in Section 2.5.3

**Fread:** In his 1983 paper, Fread makes a quantitative assessment by estimating the size of the terms in the momentum equation, which are neglected. Firstly the terms which are candidates for omission are normalized with respect to the channel bed slope \( S_o \). This ratio is expressed in hydraulic variables which allow
for a range of channel slopes and inflow hydrograph shapes. The momentum equation is obtained in the form

\[
\frac{S}{S_o} = 1 + \frac{\partial y}{\partial t} \frac{\partial}{\partial x} + \frac{1}{K V S_o} \left[ \frac{V^2}{B} \frac{\partial B}{\partial x} + \left( 1 - \frac{1}{K} \right) \frac{V \partial y}{D} \frac{\partial V}{\partial t} \right],
\]  

(2.4.5-2)

where,

- \( B \) is channel width,
- \( K \) is a cross-sectional shape factor, (kinematic wave speed \( c = K V \)),
- \( S \) is the friction slope,
- \( V \) is average downstream velocity, and
- \( y \) is water depth.

This is then re-expressed in terms of the hydrograph parameters and in US Customary units (\( g = 32 \text{ ft/s} \)) to give

\[
\frac{S}{S_o} = 1 + \frac{0.000417 M}{T, S_o K V_p} \frac{0.00000863 M V_p}{T, S_o} \left[ \beta + \left( 1 - \frac{1}{K} \left( \frac{2 y_p}{3 D} \right) \right) - 1 \right]
\]  

(2.4.5-3)

where

\[
\beta = \frac{1600 T, V_p \Delta B_a}{M B} \frac{\Delta B_a}{\Delta x}
\]  

(2.4.5-4)

\( \frac{\Delta B_a}{\Delta x} \) is a measure of the non-prismatic characteristic of the routing reach,

- \( y_p \) is the peak depth,
- \( V_p \) is the peak velocity,
\( T_s \) is the time in hr from the beginning of rise to the centre of gravity of the hydrograph, 

\( M \) is a multiplier, which adjusts the straight line approximation for the rising limb of the hydrograph to that maximum slope associated with the rising limb of a hydrograph having a gamma distribution. It is calculated from, 

\[
M = \frac{\alpha \left( \frac{2}{3} \right)^{\alpha} e^{\frac{\alpha}{3}}}{\frac{T_s}{T_r} - 1}
\]

\( T_r \) is the time in hr from beginning of rise to the peak of the hydrograph. The minimum value of \( M \) is 1 when the rising limb is a straight line, and the maximum value is 2.5 which corresponds to \( \frac{T_s}{T_r} \) of 1.09 and \( t \) is \( \frac{2T_r}{3} \).

The channel geometry is approximated by

\[
B = ky^m,
\]

\[
A = \frac{ky^{m+1}}{m+1},
\]

and

\[
D = \frac{A}{B} = \frac{y}{m+1},
\]

where \( k \) is the scaling factor and \( m \) is the shape factor e.g. \( m = 0 \) for rectangular cross section, \( = 0.5 \) for parabolic, \( = 1.0 \) for triangular. Hence the non-prismatic term \( \beta \) can be expressed as

\[
\beta = \frac{1600 T_r 1.5^m (m+1) \alpha^{0.6(m+1)} \Delta B_1}{M k q_p^{0.6m-0.4} \Delta x}
\]

For the Kinematic Model, the second and third terms of Equation (2.4.5-2) are omitted. Hence if the relative error of the conservation of momentum is to be less
than $E, \%$, then

$$E_r \geq \frac{0.0777M q_p^{0.2} n^{1.2} \Phi (I + 1)}{T, s_n^{1.5}}$$

with $\Phi = \frac{(m+1)^2}{3m+5}$, $\Phi' = \frac{\beta - (m+3)}{3m+5}$, and

$$I = \frac{0.014 S_o^{0.9} q_p^{0.2} \Phi'}{\Phi n^{1.8}}.$$

The parameter $I$ allows for the third term in Equation (2.4.5-2). Fread estimates that for flow with Froude number less than 0.5, $I$ represents less than 17% of the second term. When the Froude number drops to 0.25, $I$ is equivalent to 4% of the second term. This illustrates that the kinematic model is the better approximation for low Froude Number flows.

2.5 DYNAMIC MODEL

2.5.1 The Model

The dynamic model assumes that inertial factors are important as well as pressure, friction and gravitational factors. This requires the full de Saint-Venant Equations, (Equation (2.2.3-3) and Equation (2.2.3-10),
\[ \frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = q , \quad (2.5.1-1) \]

\[ \frac{\partial Q}{\partial t} + \frac{2Q}{A} \frac{\partial Q}{\partial x} + \left( \frac{gA}{B} - \frac{Q^2}{A^2} \right) \frac{\partial A}{\partial x} - gAS_o + gAS_f = q u_q \quad (2.5.1-2) \]

A linear analysis of the dynamic model by Ponce and Simons (1977) results in a second order characteristic equation in terms of \( F_o \), the Froude number of the steady uniform flow, \( \beta \), a dimensionless propagation factor, and \( \sigma \), the dimensionless wave number of the postulated small perturbation in the depth of flow.

This analysis demonstrates that there are two waves ('primary' and 'secondary') propagating along two characteristic paths. These can be either one downstream (primary) and one upstream (secondary) as in the case of a dam-break, or both downstream where the primary wave is the faster.

The actual wave celerities and attenuations as functions of \( F_o \) and \( \sigma \) are as follows;

**Celerities:**

\[ c_{1,2} = \frac{1}{2} \left[ \left( \frac{1}{F_o^2} - \xi^2 \right)^{1/2} + \frac{1}{F_o^2} - \xi^2 \right]^{1/2} \], \quad (2.5.1-3)

where, \( \xi = \frac{1}{\sigma F^2} \).
Logarithmic Decrement

Primary wave

$$\xi_t = \frac{1}{2} \left[ \left( \frac{1}{F_o^2} - \xi^2 \right)^2 + \xi^2 \right]^{1/2} - \frac{1}{F_o^2} + \xi^2 \right]^{1/2}$$

$$\delta_t = -2\pi \frac{1}{1 - \frac{1}{2} \left[ \left( \frac{1}{F_o^2} - \xi^2 \right)^2 + \xi^2 \right]^{1/2} + \frac{1}{F_o^2} - \xi^2 \right]^{1/2}}$$

(2.5.1-4)

Secondary Wave

$$\xi_t = \frac{1}{2} \left[ \left( \frac{1}{F_o^2} - \xi^2 \right)^2 + \xi^2 \right]^{1/2} - \frac{1}{F_o^2} + \xi^2 \right]^{1/2}$$

$$\delta_t = -2\pi \frac{1}{1 - \frac{1}{2} \left[ \left( \frac{1}{F_o^2} - \xi^2 \right)^2 + \xi^2 \right]^{1/2} + \frac{1}{F_o^2} - \xi^2 \right]^{1/2}}$$

(2.5.1-5)

Analysis of these four definitions shows that for $F_o < 2$, primary waves propagate downstream and attenuate; for $F_o = 2$, they again propagate downstream but do not amplify or attenuate and for $F_o > 2$ also propagate downstream and amplify.

When $F_o < 1$ the secondary waves can move either upstream or downstream, at $F_o = 1$ they either remain stationary or propagate downstream, and for $F_o > 1$, they propagate downstream. Secondary waves always attenuate, irrespective of the value of the Froude number.
There is a parallel pattern of dynamic wave celerity compared to kinematic wave celerity. For \( F_o \geq 2 \), the celerity of the primary wave is larger than the kinematic wave, at \( F_o = 2 \) the two celerities are equal, and when \( F_o > 2 \), the primary wave celerity is less than the kinematic.

### 2.5.2 Dynamic Wave Celerity from the Characteristic Equations

As is mentioned in Section 2.5.4 the method of characteristics replaces two partial differential equations by four ordinary differential equations. Two of these define the characteristic equations and are of the form

\[
\frac{dx}{dt} = \text{velocity term} \pm \text{wave celerity}.
\]

Deriving the characteristic equations to obtain an expression for the celerity of the dynamic wave begins with the model,

\[
\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial t} = q, \quad (2.5.2-1)
\]

\[
\frac{\partial Q}{\partial t} + \frac{2Q}{A} \frac{\partial Q}{\partial x} + \left( \frac{gA}{B} - \frac{Q^2}{A^2} \right) \frac{\partial A}{\partial x} - gAS_o + gAS_f = qu_q. \quad (2.5.2-2)
\]

The first step is to multiply Equation (2.5.2-1) by \( \lambda \) and add to Equation (2.5.2-2) and rearrange. This gives
\[
\lambda \frac{\partial A}{\partial t} + \left( gA - \frac{Q^2}{A^2} \right) \frac{\partial A}{\partial x} + \frac{\partial Q}{\partial t} + \left( \lambda + \frac{2Q}{A} \right) \frac{\partial Q}{\partial x} = q \frac{u_s}{u_s} - gA \left( S_f - S_a \right) + \lambda q
\]

(2.5.2-3)

Comparing this with the general difference equation,
\[
\frac{\partial A}{\partial t} + \frac{\partial A}{\partial x} \frac{dx}{dt} + \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x} \frac{dx}{dt} = F(x,t)
\]

shows that the equation of the characteristic, \( \frac{dx}{dt} = f(x,t) \), is equivalent to
\[
\frac{dx}{dt} = \left( \frac{gA - \frac{Q^2}{A^2}}{\lambda} \right) = \left( \lambda + \frac{2Q}{A} \right)
\]

(2.5.2-4)

Cross-multiplying gives a quadratic equation in \( \lambda \),
\[
\lambda^2 + \frac{2Q}{A} \lambda - \left( \frac{gA}{B} - \frac{Q^2}{A^2} \right) = 0
\]

(2.5.2-5)

The solutions of this are,
\[
\lambda = -\frac{Q}{A} \pm \sqrt{\frac{gA}{B}}
\]

(2.5.2-6)

and consequently the equation defining the characteristic is,
\[
\frac{dx}{dt} = \frac{Q}{A} \pm \sqrt{\frac{gA}{B}}
\]

(2.5.2-6)

showing that the celerity of the dynamic wave is \( \sqrt{\frac{gA}{B}} \).

This method is demonstrated in detail in Chapter 4 to obtain an expression for the characteristic equation and wave celerity for the curvilinear dynamic equation. As expected, the expression for the curvilinear case, is that for the straight channel with corrections for curvature.
2.5.3 Applications of the Dynamic Model

The dynamic model can be used for the following examples of unsteady flow,

1. Upstream movement of waves such as tides and storm surges,
2. Backwater effects caused by downstream reservoirs or tributary flows,
3. Unsteady flow in a river with extremely flat bed slopes, i.e. less than 0.0005,
4. Abrupt waves due to rapid releases or dam break.

Routing of spring snowmelt floods of large river systems such as the Mississippi-Ohio-Cumberland-Tennessee requires time steps of the order of 24 hours, while floods due to hurricanes on smaller systems such as the Connecticut rivers would need routing steps of several hours. At the other extreme, a dam break calculation would require time steps of the order of minutes (Lettenmier and Wood, 1993).

2.5.4 Routing Methods with the Dynamic Model

Advances in computer technology have led to a renaissance of the dynamic model, as it is now possible to solve the complete De Saint-Venant equations numerically. One of the earliest attempts was that of Stoker, who in 1953 attempted to model floods in the Ohio River, (Fread 1993b).
Current numerical methods for solving the complete de Saint-Venant equations fall into two categories, characteristic methods and direct methods. The characteristic methods transform the two partial differential equations into a set of four ordinary differential equations, which are then solved using finite differences.

Direct methods are either implicit or explicit. Explicit methods transform the differential equations into a set of algebraic equations, which are solved in sequence at each cross-section at a given time. Implicit methods solve the equations simultaneously for all computational points at a given time. If the set is non-linear, an iterative solution will be needed.

Explicit methods are restricted by considerations of numerical stability. Rounding errors and error introduced by approximating the partial differential equations as finite differences can accumulate sufficiently to cause oscillations of length approximately $2\Delta x$, in the solution. To prevent this, it is necessary to satisfy the Courant condition i.e. the time step must be less than or equal to the ratio of the reach length to the minimum dynamic wave celerity; \( \Delta t \left( \frac{\Delta x}{u} \right) \). Such small steps may be very expensive in computer time.

Implicit methods avoid the problem of numerical stability, but the size of their time steps may be restricted by considerations of numerical convergence. A popular implicit solution technique for the one-dimensional de Saint-Venant equations is the weighted four-point scheme published by Preissmann (1961).
In this scheme the time derivative for a general dependent variable $\Phi$ is approximated by a forward difference ratio at a point centred between the $i$th and $(i+1)$th points along the distance axis and $j$th and $(j+1)$th points along the time axis so that

For the derivative with respect to distance, the point is located between two adjacent time lines according to the weightings, (the ratio $\frac{\Delta t'}{\Delta t}$ as shown in Figure 2.5.4-1) and $(1-\theta)$. This gives

$$\frac{\partial \Phi}{\partial x} = \frac{\theta(\Phi_{i+1}^{j+1} - \Phi_{i}^{j+1})}{\Delta x_i} + \frac{(1-\theta)(\Phi_{i+1}^{j} - \Phi_{i}^{j})}{\Delta x_i}$$

The non-derivative term is approximated at the same point (i.e. with the same coefficients) as the derivative with respect to $x$ and so is,

$$\Phi = \frac{\theta(\Phi_{i+1}^{j+1} + \Phi_{i}^{j+1})}{2} + \frac{(1-\theta)(\Phi_{i+1}^{j} + \Phi_{i}^{j})}{2}$$

Scheme stability requires that $\theta$ be greater than 0.5, but there is some loss of accuracy as it changes from 0.5 to 1.0. The usual practice is to set $\theta$ equal to 0.6.

Substituting these operators into the de Saint-Venant equations leads to a set of weighted four point implicit finite-difference equations. If there are $N$ cross-
sections, the finite difference equations are applied to each of the N-1 rectangular grids; giving 2N-2 equations in 2N unknowns. Boundary conditions at the upstream and downstream boundaries provide the required extra equations. Solution of the equations is usually by an iterative quadratic solution technique such as Newton-Raphson.


Computer packages based on the dynamic model include, DAMBRK, which was developed by the US National Weather Service, and FLUVIAL, which was developed by the University of Illinois.

2.5.4 Boundary of Diffusion and Dynamic Models

In their 1978 paper Ponce, Li and Simons also postulate a boundary between diffusion and dynamic models. They show that as \( \frac{\tau}{F} \) (\( \tau \) is the dimensionless wave period defined in Section 2.4.5 and \( F \) the Froude Number) increases, the celerity of the primary dynamic wave (i.e. downstream) tends to the celerity of the diffusion wave. The celerity error for the diffusion model will be less than 5% provided \( \frac{\tau}{F} \geq 8 \). They postulate that for the amplitude error of the diffusion
model to be less than 5% then \( \frac{t}{F} \geq 30 \). This boundary condition can also be written as

\[
TS_o \left( \frac{g}{d_o} \right)^{0.5} \geq 30,
\]

where \( T \) is the wave period, and \( d_o \) is the steady uniform flow depth. When \( \frac{t}{F} \ltx{30} \), only the dynamic model is suitable.

Fread also developed an error expression for the diffusion/dynamic boundary. If the diffusion model is used instead of the dynamic model the error expression is

\[
E_r \geq \frac{0.000011M \Phi'}{T_oS_o^{0.7}n^{0.6}}.
\]
III GENERAL SOLUTION OF THE LINEARISED DE SAINT-VENANT EQUATIONS

3.1 INTRODUCTION

An alternative examination of unsteady flow in a straight channel to that illustrated in the previous chapter can be made by linearising the de Saint-Venant equations. The linearised pair can be treated as another pair of partial differential equations. However an easier option is to eliminate on variable and combine the equations to give a linear partial differential equation with both diffusive and dispersive properties. If a general solution is applied to the combined equation, it is possible to draw some inference about the diffusion and dispersion of long waves.

This analysis parallels that used to derive the diffusion model in Chapter 2. It is based partially on some unpublished work by Fenton (1994).

3.2 LINEARISATION OF THE DE SAINT-VENANT EQUATIONS

3.2.1 Equations and Expansions

Beginning with the de Saint-Venant equations in terms of $Q$ and $A$;

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = q$$

(3.2.1-1)

and
\[
\frac{\partial Q}{\partial t} + \frac{2Q}{A} \frac{\partial Q}{\partial x} + \left( \frac{g A}{B} - \frac{Q^2}{A^2} \right) \frac{\partial A}{\partial x} + g AS_f - g AS_o = q u_q
\]  \hspace{1cm} (3.2.1-2)

Each variable in the equations is linearised by expanding about a uniform reference flow in terms of a small parameter \( \delta \). These expansions are,

\[
Q = Q_o + \delta Q + ..., \\
A = A_o + \delta A + ..., \\
B = B_o + \delta B + ..., \\
P = P_o + \delta P + ...
\]

where, \( B \) is the width of the channel and \( P \) is the wetted parameter.

### 3.2.2 Linearising the Mass Conservation Equations

Substituting into the derived equation gives,

\[
\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = \frac{\partial (A_o + \delta A)}{\partial t} + \frac{\partial (Q_o + \delta Q)}{\partial x} + O(\delta^2) = q ,
\]

which at zeroth order is,

\[
\frac{\partial A_o}{\partial t} + \frac{\partial Q_o}{\partial x} = q .
\]  \hspace{1cm} (3.2.2-1)

If the base flow is taken to be uniform (as is assumed in linearising \( S_f \) in Section 3.2.3) then all derivatives of the base flow are zero. This forces the engineer to assume that \( q \) and \( q u_q \) in the momentum equation must also equal zero and this analysis does not apply when there is lateral flow.
The mass conservation equation to first order is

$$\frac{\partial A_i}{\partial t} + \frac{\partial Q_i}{\partial x} = 0$$  \hspace{1cm} (3.2.2-2)

### 3.2.3 Linearising the Momentum Equation

The momentum equations is linearised term by term for clarity,

**Linearising the variables:**

$$\frac{\partial Q}{\partial t} \text{ becomes } \frac{\partial Q_0}{\partial t} + \delta \frac{\partial Q_1}{\partial t},$$

$$\frac{2Q}{A} \frac{\partial Q}{\partial x} \text{ becomes (with } u_o = \frac{Q_o}{A_o})$$

$$2u_o \frac{\partial Q_o}{\partial x} + 2\delta \frac{Q_o}{A_o} \frac{\partial Q_1}{\partial x} - 2\delta u_o \frac{A_i}{A_o} \frac{\partial Q_1}{\partial x} + 2u_o \frac{\partial Q_1}{\partial x},$$

$$\left( \frac{gA - \frac{Q^2}{A^2}}{B} \right) \frac{\partial A}{\partial x} \text{ becomes}$$

$$g \left( \frac{A_o + \delta A_i}{(B_o + \delta B_i) \frac{\partial A}{\partial x}} \right) \left( A_o + \delta A \right) - \left( \frac{A_o^2 + 2\delta A_i}{(B_o^2 + 2\delta B_i) \frac{\partial A}{\partial x}} \right) \left( A_o + \delta A \right)$$

$$\left( c_o^2 - u_o^2 + \delta \frac{gA_i}{B_o} \frac{A_i}{B_o^2} - \frac{gB_i A_o}{B_o^2} - 2\delta \frac{Q_1}{A_o} + 2\delta u_o \frac{A_i}{A_o^2} \right) \frac{\partial A}{\partial x} \left( A_o + \delta A \right),$$

where $c_o = \sqrt{\frac{gA}{B}}$ the wave celerity of the steady flow

**Linearising the friction term:** The Chezy and Manning formulas can be expressed in the general form

$$S_f = \frac{\beta(Q^\nu P^\mu)}{A^{(j+\nu)}},$$
where $S_f$ is the friction slope, $\beta$ is the friction coefficient and $P$ is the wetted perimeter. If the friction coefficient is calculated using Manning’s formula, then

$$\mu = \frac{4}{3}, \quad \nu = 2; \text{ if by Chezy’s formula } \mu = 1, \nu = 2.$$

Substituting into the expression for $S_f$ leads to

$$S_f = \frac{\beta (Q_o + \delta Q_1 + \ldots) (P_0 + \delta P_1 + \ldots)^\nu}{(A_o + \delta A_1 + \ldots)^{\mu+\nu}} \frac{P_o^\mu (1 + \mu \frac{\delta P_1}{P_o})}{A_o^{\nu+\mu} (1 + (\mu + \nu) \delta \frac{A_1}{A_o})}$$

For both steady and unsteady flow, the friction coefficient is $\beta = \frac{S_o A_o^{(\mu+\nu)}}{Q_o^\nu P_o^\mu}$, and the friction expression becomes,

$$S_f = S_o \left( 1 - (\mu + \nu) \delta \frac{A_1}{A_o} + \nu \delta \frac{Q_1}{Q_o} + \mu \delta \frac{P_1}{P_o} \right)$$

Then the linearised version of the term $g A (S_o - S_f)$ becomes,

$$g A_o S_o \delta \mu \frac{P_1}{P_o} + g A_o S_o \delta \nu \frac{Q_1}{Q_o} - g S_o \delta (\mu + \nu) A_1,$$

which is first order.

Gathering the zeroth and first order term and gives the following:

**zeroth order:**

$$\frac{\partial Q_o}{\partial t} + 2 u_o \frac{\partial Q_o}{\partial x} + (c_o^2 - u_o^2) \frac{\partial A_o}{\partial x} = 0,$$

(3.2.3-1)
and,

**first order:**

\[
\frac{\partial Q_1}{\partial t} + 2u_o \frac{\partial Q_1}{\partial x} + 2\left( \frac{Q_1}{A_1} - u_o A_1 \right) \frac{\partial Q_o}{\partial x} - \frac{\partial Q_o}{\partial x} + \left( c_o^2 - u_o^2 \right) \frac{\partial A_1}{\partial x} \\
+ \left( \frac{g A_1}{B_o} - \frac{g B_o A_1}{B_o^2} - 2 \frac{Q_1}{A_o} + 2 \frac{u_o^2 A_1}{A_o^2} \right) \frac{\partial A_o}{\partial x} + g A_o S_o \left( \mu \frac{P_o}{P_o} + v \frac{Q_1}{Q_o} - (\mu + v) \frac{A_1}{A_o} \right) = 0
\]

(3.2.3-2)

As the zeroth order flow is uniform, the space derivatives \( \frac{\partial A_o}{\partial x} \) and \( \frac{\partial Q_o}{\partial x} \) vanish, and the first order momentum equation becomes

\[
\frac{\partial Q_1}{\partial t} + 2u_o \frac{\partial Q_1}{\partial x} + \left( c_o^2 - u_o^2 \right) \frac{\partial A_1}{\partial x} + g A_o S_o \left( \mu \frac{P_o}{P_o} + v \frac{Q_1}{Q_o} - (\mu + v) \frac{A_1}{A_o} \right) = 0
\]

(3.2.3-3)

As the wetted perimeter is a function of area, \( P_1 \) can be set equal to \( A_1 P_o' \), where \( P_o' \) is the derivative of \( P \) with respect to \( A \), evaluated at \( A_o \). This substitution gives

\[
\frac{\partial Q_1}{\partial t} + 2u_o \frac{\partial Q_1}{\partial x} + \left( c_o^2 - u_o^2 \right) \frac{\partial A_1}{\partial x} + g A_o S_o \left( \mu \frac{P_o'}{P_o} A_1 + v \frac{Q_1}{Q_o} - (\mu + v) \frac{A_1}{A_o} \right) = 0
\]

which can be written as,

\[
\frac{\partial Q_1}{\partial t} + 2u_o \frac{\partial Q_1}{\partial x} + \left( c_o^2 - u_o^2 \right) \frac{\partial A_1}{\partial x} + \theta_1 A_1 + \theta_2 Q_1 = 0
\]

(3.2.3-4)

with \( \theta_1 \) equal to \( g S_o \left( \mu \frac{P_o'}{P_o} \right) \), \( \mu + v \), and \( \theta_2 \) is equal to \( \frac{g S_o A_o v}{Q_o} \).
3.2.4 The Combined Equation

The combined equation is obtained by cross-differentiating Equations (3.2.2-2) and (3.2.3-4). The mass-conservation equation becomes, after differentiating with respect to \( x \),

\[
\frac{\partial^2 A}{\partial x \partial t} = -\frac{\partial^2 Q}{\partial x^2}.
\] (3.2.4-1)

The momentum equation is differentiated with respect to \( t \) and becomes

\[
\frac{\partial^2 Q_1}{\partial t^2} + 2u_o \frac{\partial^2 Q_1}{\partial x \partial t} + (c_o^2 - u_o^2) \frac{\partial^2 A_1}{\partial x \partial t} + \theta_1 \frac{\partial A_1}{\partial t} + \theta_2 \frac{\partial Q_1}{\partial t} = 0.
\] (3.2.4-2)

Substituting for \( \frac{\partial^2 A_1}{\partial x \partial t} \) and \( \frac{\partial A_1}{\partial t} \) gives the momentum equation as

\[
\frac{\partial^2 Q_1}{\partial t^2} + 2u_o \frac{\partial^2 Q_1}{\partial x \partial t} - (c_o^2 - u_o^2) \frac{\partial^2 Q_1}{\partial x^2} - \theta_1 \frac{\partial Q_1}{\partial x} + \theta_2 \frac{\partial Q_1}{\partial t} = 0
\] (3.2.4-3)

which coalesces into

\[
\theta_2 \frac{\partial Q_1}{\partial t} - \theta_1 \frac{\partial Q_1}{\partial x} + \left( \frac{\partial}{\partial t} + u_o \frac{\partial}{\partial x} \right)^2 Q_1 - c_o^2 \frac{\partial^2 Q_1}{\partial x^2} = 0
\] (3.2.4-4)

This equation can be further simplified by making the substitution

\[
V = \frac{\theta_1 + \theta_2 u_o}{\theta_2} = \frac{\theta_1}{\theta_2} - u_o
\]

to give,

\[
\theta_2 \left( \frac{\partial Q_1}{\partial t} + (u_o + V) \frac{\partial Q_1}{\partial x} \right) + \left( \frac{\partial}{\partial t} + u_o \frac{\partial}{\partial x} \right)^2 Q_1 - c_o^2 \frac{\partial^2 Q_1}{\partial x^2} = 0
\] (3.2.4-5)
This method can be applied to any pair of equations. A parallel derivation in electrical engineering leads to the Telegrapher's Equation. Although this title has been applied to Equation (3.2.4-2), this thesis will continue to refer to it as the 'Combined Equation, to avoid any confusion with the method's use in electrical engineering.

3.3 COMPARISON WITH PREVIOUS DIFFUSION MODEL

3.3.1 Convective-Diffusion Equation

The previous discussion of the conventional method, (Section 2.4.2) only considered the diffusion coefficient. Here dispersion is incorporated by assuming that the first differential with respect to time is approximately equal to the negative of the sum of the first and second differentials with respect to distance i.e.

\[
\frac{\partial}{\partial t} \approx -(u_o + V) \frac{\partial}{\partial x} + \left( \frac{c_o^2 - u_o^2}{\theta^2} \right) \frac{\partial^2}{\partial x^2}.
\]  

(3.3.2.1-1)

However the criticisms levelled at this method in Section 2.4.3 are still valid here.

Applying the Equation (3.3.1-1) to Equation (3.2.4-5) gives

\[
\frac{\partial Q_i}{\partial t} = -(u_o + V) \frac{\partial Q_i}{\partial x} - \left( \frac{V^3 - c_o^3}{\theta} \right) \frac{\partial^3 Q_i}{\partial x^3} + \left( \frac{2V(c_o^2 - u_o^2)}{\theta^2} \right) \frac{\partial^3 Q_i}{\partial x^3}.
\]  

(3.3.1-2)
The single velocity indicates a single long wave which is travelling downstream at $u_o + V$ with diffusion and dispersion.

3.3.2 Coefficient Values

3.3.2.1 Celerity

Substituting for $V$ and using the values for $\mu$ and $\nu$ corresponding to the Chezy equation gives a velocity of $u_o \left(\frac{3}{2} - \frac{A_o P_o'}{2 P_o}\right)$. For the case of a wide prismatic rectangular channel of width $B$, Ferrick et.al. (1984) quote a value of $\frac{3Q}{2By}$. As $y$ is the water depth corresponding to $A_o$ and $Q_o$ this expression is equivalent to $\frac{3}{2} u_o$, i.e. only the first term.

The second factor in the expression for velocity found in this analysis becomes $\frac{2By^2}{B + 2y}$ for the prismatic rectangular channel. This factor does not appear in the derivations by Ponce and Simons (1977) or Ferrick et.al. (1984) because in both these derivations the friction factor was subsumed into the Chezy coefficient and lost. Consequently, the derivations produced the misleading result that the diffusion wave celerity was identical to the kinematic wave celerity. When friction is specifically included, the celerity is shown to be less than the kinematic celerity.
3.3.2.2 Diffusion Coefficient

The diffusion coefficient is

\[ D = \frac{c_o^2 - v^2}{\theta_2} \]

When substitutions are made for \( V \) and \( \theta_2 \), the coefficient becomes,

\[ D = \frac{c_o^2 - \frac{1}{4} u_o^2 \left( 1 - \frac{2 A_o P'_o}{P_o} + \frac{A_o^2 (P'_o)^2}{P_o^2} \right)}{2 g S_o} \]  \tag{3.3.2.1-1}

For the case of a wide rectangular channel with wave speed \( c_o = \sqrt{g y} \), the diffusion coefficient becomes:

\[ D = \frac{Q}{2 S_o B} \left[ 1 - \frac{Q^2}{4 B^2 g^2 y^3} \left( 1 - \frac{4 B y^2}{B + 2 y} + \frac{4 B^2 y^4}{(B + 2 y)^2} \right) \right] \]  \tag{3.3.2.2-2}

which when simplified and written in terms of the Froude number becomes,

\[ D = \frac{Q}{2 S_o B} \left[ 1 - \frac{F^2}{4} \left( 1 - \frac{4 B y^2}{B + 2 y} + \frac{4 B^2 y^4}{(B + 2 y)^2} \right) \right] \]  \tag{3.3.2.2-3}

which is the coefficient developed by Ponce and Simons with a correction factor for friction effects

3.3.2.3 Dispersion Coefficient

The dispersion coefficient is

\[ E = \frac{2 V (c_o^2 - u_o^2)}{\theta_2} \]

which on substitution for \( V \) and \( \theta_2 \) becomes,

\[ E = \frac{u_o^3}{4 g S_o} (c_o^2 - u_o^2) \left( 1 - \frac{A_o P'_o}{P_o} \right) \]  \tag{3.3.2.3-1}
For unsteady flow in a wide rectangular channel the dispersion coefficient is, in terms of \( Q, B \) and \( y \),

\[
E = \frac{Q^3}{4 g S_o B^2 y^2} \left( 1 - \frac{4 B y^2}{B + 2 y} \right),
\]

(3.3.2.3-2)

which again is the Ponce and Simons coefficient with a correction for friction effects.

### 3.3.3 Coefficients With the Manning Equation

The values of the coefficients derived in Sections 3.3.2.1, 3.3.2.2, and 3.3.2.3 depend on the choice of friction equation. If the Manning equation had been used instead of Chézy (i.e. \( \mu = \frac{4}{3}, \nu = 2 \)), then the following values would have been obtained for the case of a rectangular channel.

**Celerity:** Here the value is

\[
c = \frac{Q}{B y} \left[ \frac{5}{3} - \frac{8}{3} \left( \frac{B y^2}{B + 2 y} \right) \right]
\]

(3.3.3-1)

If the Manning equation had been incorporated into the momentum equation as was done with the Chézy equation then the celerity would have been \( \frac{5}{3} \frac{Q}{B y} \), i.e. faster than that predicted by the Chézy equation. The friction correction may change the celerity to greater than or less than the Chézy value depending on the sizes of \( B \) (a constant) and \( y \) (a function of \( Q \)).
Diffusion Coefficient: This coefficient becomes,

\[
D = \frac{Q}{2 B S_o} \left[ 1 - \frac{4 F^2}{9} \left( 1 - \frac{4 B \, y^2}{B + 2 \, y} + \frac{4 B^2 \, y^4}{(B + 2 \, y)^2} \right) \right],
\]

so that when the Manning equation is used, the estimated diffusion coefficient is bigger.

Dispersion Coefficient: With the Manning equation the dispersion coefficient is

\[
E = \frac{Q^3}{4 g S_o^2 B^3 y^2} \frac{4}{3} \left( 1 - F^2 \right) \left( 1 - \frac{4 B \, y^2}{B + 2 \, y} \right),
\]

which is higher than the coefficient obtained using the Chézy equation.

We have thus obtained two equally valid and differing sets of coefficients. While this may be unsatisfactory from a purely mathematical perspective, civil engineers recognize that it reflects a basic quality of the Chezy and Manning equations. Both the equations are approximate descriptions of steady uniform flow and so it is to be expected that results derived from them would be only approximate. Which to use is a matter of engineering judgement.
3.4 GENERAL SOLUTION OF THE COMBINED EQUATION

3.4.1 Introduction

The diffusive and dispersive characteristics of a wave can be examined without recourse to the questionable assumptions of the conventional method, by finding a general solution of the combined equation.

3.4.2 The General Solution

A general solution can be obtained by assuming a solution of the form

(Coulson, 1958)

\[ Q = Q_0 e^{(ikx + jyt)} \]

and substituting this into Equation (3.2.4-5) where \( Q_1 \) has been replaced by \( Q \), i.e.

\[ \theta_x \left( \frac{\partial Q}{\partial t} + (u_o + V) \frac{\partial Q}{\partial x} \right) + \left( \frac{\partial}{\partial t} + u_x \frac{\partial}{\partial x} \right) \]

(3.4.2-1)

The derivatives of the given solution are,
\[
\frac{\partial Q}{\partial t} = \mu Q_0 e^{ikx + \mu t} \quad \frac{\partial Q}{\partial x} = ik Q_0 e^{ikx + \mu t}
\]
\[
\frac{\partial^2 Q}{\partial t^2} = \mu^2 Q_0 e^{ikx + \mu t} \quad \frac{\partial Q}{\partial t} = -k^2 Q_0 e^{ikx + \mu t}
\]
\[
\frac{\partial^2 Q}{\partial x \partial t} = i k Q_0 e^{ikx + \mu t}
\]

Substituting into Equation (3.4.2-1) and dividing through by \(Q_0 e^{ikx + \mu t}\) results in a quadratic in \(\mu\)
\[
\mu^2 + (2u_0 ik + \theta_2)\mu + \left[(c_o^2 - u_o^2)k^2 + \theta_2 (u_o + V)i k\right] = 0.
\]

This can be solved by the quadratic formula i.e.,
\[
\mu = \frac{-2u_0 ik + \theta_2}{2} \pm \frac{\sqrt{(2u_0 ik + \theta_2)^2 - 4\left(c_o^2 - u_o^2\right)k^2 + \theta_2 (u_o + V)i k}}{2}
\]
\[
= \frac{-2u_0 ik + \theta_2}{2} \pm \frac{\sqrt{\theta_2^2 \left(1 - \frac{4}{\theta_2} \left(c_o^2 k^2 + \theta_2 V i k\right)\right)}}{2}
\]

(3.4.2-3)

An expression for the square root is obtained by expanding by the binomial theorem. This is permissible because \(k\) is small for a long wave. The infinite binomial series is truncated at \(\theta_2^6\) as this is deemed sufficient to demonstrate the wave nature of the diffusion and dispersion coefficients.

The solution for \(\mu\) is,
\[ \mu = -\left( u_o i k + \frac{\theta_2}{2} \right) \]
\[ \frac{\theta_2}{2} \left( \frac{c_o^2 - V^2}{2} \right) k^2 + \frac{\theta_3^2}{2} \left( c_o^4 - 6 c_o^2 V^2 + 5 V^4 \right) k^4 \]
\[ - \frac{\theta_4^2}{2} \left( 2 c_o^6 - 15 c_o^4 V + 35 c_o^2 V^4 - 21 V^6 \right) k^6 \]

\[ \pm \left[ \frac{V}{2} \left( \frac{c_o^2 - V^2}{2} \right) k^2 + \frac{\theta_3^2}{2} \left( 2 c_o^4 V - 10 c_o^2 V^3 + 7 V^5 \right) k^4 \right] \]
\[ - \frac{i k}{2} \left( \frac{\theta_2^2}{2} \left( 4 c_o^6 V - 35 c_o^2 V^3 + 63 c_o^2 V^5 + 33 V^7 \right) k^6 \right) \]

which can then be substituted into the general solution to give,

\[ Q = Q_o \exp \left[ i k x - \left( u_o i k + \frac{\theta_2}{2} \right) t \right] \]
\[ + \frac{\theta_2}{2} \left( \frac{c_o^2 - V^2}{2} \right) k^2 + \frac{\theta_3^2}{2} \left( c_o^4 - 6 c_o^2 V^2 + 5 V^4 \right) k^4 \]
\[ - \frac{\theta_4^2}{2} \left( 2 c_o^6 - 15 c_o^4 V + 35 c_o^2 V^4 - 21 V^6 \right) k^6 \right] \]

\[ \pm \left[ \frac{V}{2} \left( \frac{c_o^2 - V^2}{2} \right) k^2 + \frac{\theta_3^2}{2} \left( 2 c_o^4 V - 10 c_o^2 V^3 + 7 V^5 \right) k^4 \right] \]
\[ - \frac{i k}{2} \left( \frac{\theta_2^2}{2} \left( 4 c_o^6 V - 35 c_o^2 V^3 + 63 c_o^2 V^5 + 33 V^7 \right) k^6 \right) \]

\[ (3.4.2-4) \]

\[ (3.4.2-5) \]

### 3.4.3 General Advection and Diffusion Coefficients

The general diffusion and dispersion coefficients can be derived by expressing the solution as

\[ Q = Q_o \exp \left[ i k (x - ct) - D k^2 t \right] \]

(3.4.3-1)

where \( c \) is the advection i.e. an expression for the wave speed which incorporates the dispersive effect and \( D \) is the diffusion coefficient (Coulson, 1958).
Advection of Long Wave:

From Equation (3.4.2-5), the expression for the advection is

$$c = u_o \pm \left[ V - \frac{2(c_o^3 V - V^3)k^2}{\theta_2^2} - \frac{2(3c_o^4 V - 10c_o^2 V^3 + 7V^5)k^4}{\theta_2^4} \right]$$

\[\frac{4(5c_o^6 V - 35c_o^4 V^3 + 63c_o^2 V^5 + 33V^7)k^6}{\theta_2^6}\]

(3.4.3-2)

The general solution indicates two associated waves corresponding to the $\pm$ options. This shows that the conventional method assumption that there is only a single downstream wave is not justified. The positive option gives the same downstream wave as that derived by the conventional approach to the diffusion model, but with extra factors. In contrast to the conventional approach, this analysis does not result in a dispersion coefficient, but it does demonstrate the dispersive effects by showing that the advection is a function of wavelength and friction.

As yet we do not have a physical explanation for the negative wave but for the present work, the existence of both the negative and positive diffusion waves will be accepted.

Diffusion Coefficient.

The presence of the $\pm$ leads to two values of diffusion coefficient corresponding to the two waves.
Positive Square Root:

When the positive square root is chosen the diffusion coefficient is

\[ D = \frac{(c_o^2 - V^2)k^2}{\theta_2} + \frac{(c_o^4 - 6c_o^2V^2 + 5V^4)k^4}{\theta_2^3} + \frac{2(c_o^6 - 15c_o^4V + 35c_o^2V^4 - 21V^6)k^6}{\theta_2^5} + \ldots \]

(3.4.3-3)

The first term corresponds to the diffusion coefficient derived by the conventional method of deriving the diffusion model (Section 3.3.2.2). The remaining terms provide more information on the effects of friction and wavelength on the diffusion of the wave.

Negative Square Root:

When the negative square root is chosen, Equation (3.4.2-5) becomes,

\[ Q = Q_o \exp \left\{ \left( \frac{ikx - u_o ik t}{\theta_2} \right) - \left( \frac{c_o^2 - V^2 k^2}{\theta_2^3} - \frac{c_o^4 - 6c_o^2V^2 + 5V^4 k^4}{\theta_2^5} - \frac{2(c_o^6 - 15c_o^4V + 35c_o^2V^4 - 21V^6)k^6}{\theta_2^7} \right) \right\} \]

(3.4.3-4)

Comparison with Equation (3.4.3-1) is complicated by the existence of the extra factor \( Q_o \exp(-\theta_2^3) \). This may represent some extra frictional effects on the
second wave. Alternatively the $\theta_1 t$ may be treated as equivalent to $\theta_2 t k^\circ$ and incorporated into the diffusion coefficient. In this case

$$D = \theta_2 k^2 - \left( \frac{c_o^2 - V^2}{\theta_2} \right) k^2 - \left( \frac{c_o^4 - 6 c_o^2 V^2 + 5 V^4}{\theta_2^3} \right) k^4 - 2 \left( \frac{c_o^6 - 15 c_o^4 V + 35 c_o^2 V^4 - 21 V^6}{\theta_2^5} \right) k^6 + ...$$

(3.4.3-5)

3.5 UNSTEADY FLOW IN CURVED OPEN CHANNELS

3.5.1 Introduction

The discussion in Chapter 2 and Chapter 3.1 - 3.4 has established the straight channel case. As was pointed out in Chapter 1, a common engineering practice when dealing with a curved channel, such as a meandering river, is to apply the straight channel model. The channel is treated as straight with length equal to the talweg, and curvature is allowed for by varying the friction values. This 'straightening out' ensures conformity with assumption 3 in Section 2.2.2 above, but it does not ensure conformity with the other assumptions.

Another method of allowing for curvature, is to apply correction coefficient to the de Saint-Venant equations. The correction relates the actual channel length (measured along the talweg) to the flow path distance along the flood plain. This method is used in NWS DAMBRK-88 model. (Fread 1988). Both methods are basically ad hoc.
In flow around a curve, velocities form a three dimensional pattern with a primary downstream component and a secondary transverse circulation. The primary flow shows a transverse velocity gradient. Flow around a bend sets superelevation with higher water level on the concave (outer) bank. Also, the channel cross-section is rarely prismatic; at bends it can be markedly asymmetric even if it is symmetric in straight reaches.

The remainder of this thesis is given over to the derivation of long wave models in curved channels. As the interest is primarily in the streamwise movement of long waves, the models developed are one-dimensional. The question of allowing for curvature is dealt with by returning to first principles and developing the models using a curvilinear coordinate system.

3.5.2 The Curvilinear de Saint-Venant Equations

Pairs of de Saint-Venant equations in curvilinear coordinates are derived from the integral forms of the mass-conservation and momentum conservation principles. By defining a particular control volume and using an integral approach it is possible to derive a model that avoids the complications of velocity distributions. The coordinate system chosen (Figure 4.1.2(a)) is orthogonal with a downstream axis (s), a transverse axis (n) and a vertical axis (z). This enables a single set of axes to define the entire length of the river.
In this scheme the time derivative for a general dependent variable \( \Phi \) is approximated by a forward difference ratio at a point centred between the \( i \)th and \((i+1)\)th points along the distance axis and \( j \)th and \((j+1)\)th points along the time axis so that

\[
\frac{\partial \Phi}{\partial x} = \frac{\theta (\Phi_{i+1}^{j+1} - \Phi_{i}^{j+1})}{\Delta x} + \frac{(1 - \theta)(\Phi_{i+1}^{j} - \Phi_{i}^{j})}{\Delta x}.
\]

The non-derivative term is approximated at the same point (i.e. with the same coefficients) as the derivative with respect to \( x \) and so is,

\[
\Phi = \frac{\theta (\Phi_{i+1}^{j+1} + \Phi_{i}^{j+1})}{2} + \frac{(1 - \theta)(\Phi_{i+1}^{j} + \Phi_{i}^{j})}{2}.
\]

Scheme stability requires that \( \theta \) be greater than 0.5, but there is some loss of accuracy as it changes from 0.5 to 1.0. The usual practice is to set \( \theta \) equal to 0.6.

Substituting these operators into the de Saint-Venant equations leads to a set of weighted four point implicit finite-difference equations. If there are \( N \) cross-
IV CURVILINEAR DE SAINT-VENANT EQUATIONS

4.1 PRELIMINARY

4.1.1 Introduction

The work in this chapter lays the basis for the three curvilinear unsteady flow models described in Chapter 5 by deriving the de Saint-Venant equations in curvilinear coordinates from integral first principles. This is initially done in terms of discharge $Q$ and cross-sectional area $A$, followed by derivations in variables discharge $Q$ and surface level, and finally in variables velocity $u_o$ and surface level $\eta$ at a reference line.

Although it may be argued that a model in terms of discharge and area avoids the need to describe a complex three-dimensional velocity distribution, in the course of the derivation some assumptions are required about the distribution. Velocities are assumed to be depth averaged and a general transverse velocity distribution is defined. Both the $(Q, A)$ and $(Q, \eta_o)$ forms are shown to be independent of the velocity distribution to first order, while the $(u_o, \eta_o)$ form depends on the velocity distribution.

4.1.2 Curvilinear Coordinates and the Control Section

This thesis assumes the coordinate system and control section illustrated in Figures 4.1.2(a), (b), (c) where $s$ is the downstream coordinate, $n$ is the transverse coordinate and $z$ the vertical coordinate.
Figure 4.1.2(a) Curvilinear Coordinate System

Figure 4.1.2(b) Curvilinear Control Section
Centre of curvature $n = r = 1/\kappa$

Surface mid-point, $n = n_m$

Centroid

$\bar{n}$

Figure 4.1.2(c) Meander Channel Cross-Section
Figure 4.1.2(c) shows the reference line intersecting the talweg. It will be shown later in Section (4.2.4) that at the level of the approximation of this analysis, the placement of the axis is arbitrary. The author places the reference line at the talweg for the purely utilitarian reason of keeping it within the water.

The derivations in Sections 4.2, 4.3 and 4.4, are based upon an integration of the principles of conservation of mass and momentum over the control section described in Figures 4.1.2 (b) and (c). The sides of the control section lie on the radii of curvature and so are consistent with the curvilinear coordinate system. The control volume extends into the air above the water surface. This obviates the need for water surface boundary conditions.

4.1.3 Geometric Derivatives

The derivations of this chapter require derivatives of certain geometric factors of the cross-section. These factors are, surface level, cross-sectional area and moment of area. The derivatives are established initially for clarity in the subsequent derivations.

4.1.3.1 Surface Slope and Superelevation

The transverse surface curve is approximated by a straight line given by

\[ \eta = \eta_0 - \left( \frac{u^2 \kappa}{g} \right) n. \]

This is derived as follows. An element of fluid of unit mass travelling at speed \( u \) around a path of radius \( r \) has an apparent centrifugal force of
\[ \frac{u^2}{r} \] and hence has a transverse slope of \[ \frac{u^2 \kappa}{g} n \]. Given that \( \eta_o \) is the surface level at the s axis and \( u_o \) is the depth averaged velocity at the s axis, then the surface level at a distance \( n \) from the s axis is \( \eta = \eta_o - \left( \frac{u^2 \kappa}{g} \right) n \) as the water level decreases with increasing positive \( n \).

The derivatives with respect to \( s \) and \( t \) are,

\[ \frac{\partial \eta}{\partial s} = \frac{\partial \eta_o}{\partial s} - \frac{u_o^2 \partial \kappa}{g} n - \frac{2u_o \kappa n \partial u_o}{g} \partial s, \quad (4.1.3.1-1) \]

and

\[ \frac{\partial \eta}{\partial t} = \frac{\partial \eta_o}{\partial t} - \frac{2u_o \kappa n \partial u_o}{g} \partial t. \quad (4.1.3.1-2) \]

4.1.3.2 Cross-Sectional Area (Figure 4.1.2 (b))

The cross-sectional area is defined as,

\[ A = \int \int_{-n_i}^{n} d\eta \, d\eta = \int_{-n_i}^{n} (\eta - z_b) \, d\eta. \quad (4.1.3.2-1) \]

where \( z_b \) is the elevation of the bed and \( n_i \) and \( n_r \) are the coordinates of the water's edge at the left and right banks.
Differentiating with respect to \( s \) and applying Leibniz’ theorem gives

\[
\frac{\partial A}{\partial s} = \int_{n_r}^{n_i} \left( \frac{\partial \eta}{\partial s} - \frac{\partial z_s}{\partial s} \right)dn.
\]

(4.1.3.2-1)

There are no end contributions as the integrand is zero there. Substituting for \( \frac{\partial \eta}{\partial s} \) and integrating leads to

\[
\frac{\partial A}{\partial s} = B \left( \frac{\partial \eta}{\partial s} + S_o - \frac{u_0^2 n_m}{g} \frac{\partial \kappa}{\partial s} - \frac{2 u_0 \kappa n_m}{g} \frac{\partial u_o}{\partial s} \right),
\]

(4.1.3.2-2)

where \( B \) is the width of the water surface, \( S_o \) is the slope of the bed (given by \(- \frac{1}{B} \int_{n_r}^{n_i} \frac{\partial z_s}{\partial s} dn \)), and \( n_m \) is the distance between surface midpoint (i.e. midpoint between \( n_i \) and \( n_r \)) and the \( s \) axis (Figure 4.1.2(c)).

Differentiating the area with respect to time \( t \) and again applying Leibniz’ theorem gives

\[
\frac{\partial A}{\partial t} = \int_{n_r}^{n_i} \frac{\partial \eta}{\partial t} dn \quad \text{as} \quad z_s \quad \text{is assumed to be constant over the time scale considered in this work. Substituting for} \quad \frac{\partial \eta}{\partial t} \quad \text{and integrating gives}
\]

\[
\frac{\partial A}{\partial t} = B \left( \frac{\partial \eta}{\partial t} - \frac{2 u_0 \kappa n_m}{g} \frac{\partial u_o}{\partial t} \right).
\]

(4.1.3.2-3)

\(^1\) Leibniz’ theorem states that \( \frac{d}{dc} \int f(x,c)dx + f(b,c) \frac{db}{dc} - f(a,c) \frac{da}{dc} \). In Equation (4.1.3.2-1), \( f(b,c) \) and \( f(a,c) \) are both equal to 0 as \( (\eta - z_s) = 0 \) when \( n = n_i \) and \( n = n_r \). (see ‘Handbook of Mathematical Functions’, ed Abramowitz & Stegun)
4.1.3.3 Moment of Area About z Axis

The first moment of area about the z axis is defined as

\[ M_z = \int_{z_a}^{z_b} n dx \, dz = \int_{z_a}^{z_b} (\eta - z_b) n \, dz. \]

\( M_z \) is also equal to \( A \bar{h} \) where \( \bar{h} \) is the distance from the centroid of the cross-section to the reference line.

Differentiating with respect to \( t \) gives,

\[ \frac{\partial M_z}{\partial t} = \int_{n_s}^{n} \frac{\partial \eta}{\partial t} n \, dz. \] (4.1.3.3-1)

As in this derivation, we neglect term of order \( \kappa^2 \) and above, (See Section 4.2.2) and as \( M_z \) is always multiplied by \( \kappa \) in the derivations which follow, \( \eta \) can be approximated by \( \eta_0 \)

\[ \frac{\partial M_z}{\partial t} = \int_{n_s}^{n} \frac{\partial \eta_0}{\partial t} n \, dz = B n_m \frac{\partial \eta_0}{\partial t}. \] (4.1.3.3-2)

Differentiating \( M_z \) with respect to \( s \) gives

\[ \frac{\partial M_z}{\partial s} = \int_{n_s}^{n} \frac{\partial \eta}{\partial s} (\eta - z_b) n \, dz = \int_{n_s}^{n} \frac{\partial \eta_0}{\partial s} n \, dz - \int_{n_s}^{n} \frac{\partial z_b}{\partial s} n \, dz \] (4.1.3.3-3)

to first order. When \( S_1 \) is defined as the weighted mean slope, \( -\int_{n_s}^{n} \frac{\partial z_b}{\partial s} n \, dz \), the derivative of \( M_z \) with respect to \( s \) becomes,
4.2 DERIVATION OF EQUATIONS FROM INTEGRAL PRINCIPLES IN TERMS OF DISCHARGE AND CROSS-SECTIONAL AREA \((Q, A)\)

4.2.1 Introduction

The discussion in Chapter 2 indicates that the straight channel de Saint-Venant equations have some implied assumptions. The most restrictive is the assumption of one-dimensional flow with depth and velocity varying only in the longitudinal direction. It also implies that the transverse surface is level.

This is not true in the case of flow in curved channels. The velocity is a complicated three-dimensional structure and there is superelevation on the outside of the bend.

In the derivation in Section 4.2 these problems are resolved by using discharge \((Q)\) and cross-sectional area \((A)\) as variables. As the control volume extends into the air above the water surface it is not necessary to allow for the shape of the surface. In engineering practice some estimation of the superelevation will need to be made for the design of stopbanks.

4.2.2 Mass-Conservation Equation

The general mass-conservation principle in integral form (Streeter and Wylie,

\[
\frac{\partial M}{\partial s} = B n_m \frac{\partial n_o}{\partial s} + S_i
\]
1981 Section 3.3) has already been quoted as,

\[ \frac{\partial}{\partial t} \int_{c_v} \rho \, dV + \int_{c_s} \rho \mathbf{u} \cdot d\mathbf{A} = 0 \]  \hspace{1cm} (4.2.2-1)

where \( t \) is time, \( d\mathbf{A} \) is vector representing an area element \( dA \) of the control surface, with direction normal to and directed outwards from the control surface, \( \mathbf{u} \cdot d\mathbf{A} \) is the component of velocity normal to the surface at any point times area such that it is the local volume flux, \( dV \) is \((1 - \kappa \, n) \, ds \, dn \, dz\), the volume of the control volume, (Figures 4.1.2(a), (b)).

**First term:** Although the control volume extends above the water surface, the domain of integration is the volume of water in the control section, as the density of air is negligible compared to water. The first term becomes

\[ \rho \, ds \, \frac{\partial}{\partial t} \int_{n, z_b}^{n, \eta} (1 - \kappa \, n) \, dz \, dn = \rho \, ds \, \frac{\partial}{\partial t} \int_{n}^{\eta} (1 - \kappa \, n)(\eta - z_b) \, dn \]

\hspace{1cm} (4.2.2-2)

where, \( \eta, \, z_b, \, n, \) and \( n_r \) have been defined in Section 4.1.3. Applying Leibniz' theorem for the derivative of an integral, and recognising that again there is no contribution from the limits as the integrand there is zero, results in

\[ \rho \, ds \, \frac{\partial}{\partial t} \int_{n_r}^{n} (1 - \kappa \, n) \, \frac{\partial \eta}{\partial t} \, dn = \rho \, ds \left[ \int_{n_r}^{n} \frac{\partial \eta}{\partial t} \, dn - \kappa \int_{n_r}^{n} n \, \frac{\partial \eta}{\partial t} \, dn \right] \]  \hspace{1cm} (4.2.2-3)

Using the definition in Section 4.1.3.2, this reduces to,
\[ \rho \, ds \left[ \frac{\partial A}{\partial t} - \kappa \int_n^{n'} n \frac{\partial n}{\partial t} \, dn \right]. \]

Again only the first order term need be substituted for \( n \) as the integral is multiplied by \( \kappa \). This gives,

\[ \int_n^{n'} n \frac{\partial n}{\partial t} \, dn = B n_m \frac{\partial n}{\partial t}. \]

(4.2.2-4)

Similarly, as the integral is multiplied by \( \kappa \), \( \frac{\partial n}{\partial t} \) may be set equal to \( n_m \frac{\partial A}{\partial t} \), and the first term becomes,

\[ \rho \, ds \left[ \frac{\partial A}{\partial t} - \kappa n_m \frac{\partial A}{\partial t} \right] = \frac{\partial A}{\partial t} \left( 1 - \kappa n_m \right) \rho \, ds. \]

(4.2.2-5)

**Second term:** This is the net outflow from the control volume. When mass inflow is \( \rho Q \), and outflow is \( \rho (Q + \Delta Q) \), the net outflow is \( \rho \left( Q + \frac{\partial Q}{\partial s} \, ds \right) - \rho Q \), which is \( \rho \frac{\partial Q}{\partial s} \, ds \). When lateral inflow \( q \) per unit length is included, then the extra mass exiting the control volume is \( \rho q \, ds \).

**Mass Conservation Equation:**

The mass conservation equation is therefore

\[ \frac{\partial A}{\partial t} (1 - \kappa n_m) + \frac{\partial Q}{\partial s} = q + O \left( (\kappa n)^2, (\kappa n)^2 \right) \]

(4.2.2-6)

This derivation has treated as negligible, all terms to the order of \( \kappa^2 \) and above.
For the theory to be accurate, \((\kappa n_l)^2 = \left(\frac{n_l}{r}\right)^2 \ll 1\), and \((\kappa n_r)^2 = \left(\frac{n_r}{r}\right)^2 \ll 1\).

Consequently the theory is more accurate for streams where the width is small compared to radius of curvature.

**4.2.3 Momentum Equation**

Again the derivation begins with the momentum principle in terms of integrals (Streeter and Wylie, 1985, Section 3.3),

\[
\frac{\partial}{\partial t} \int_{V} \rho \, u \, dV + \int_{S_s} \rho \, u u \, dA = \Sigma F
\]  

(4.2.3-1)

where \(\Sigma F\) is the force acting on the fluid in the control volume. Although Chow et al. (1988) identify five different forces, this derivation is only concerned with pressure, gravitational and frictional forces. Pressure is regarded as hydrostatic, and the derivation concentrated on the downstream components of the gravitational and frictional forces.

**First integral:**

Substituting for \(dV\) in the first term and using \(u\) for the s-component of depth averaged velocity, leads to,

\[
\rho \, ds \, \frac{\partial}{\partial t} \int_{n_r, z_0}^{n_r, \eta} (1 - \kappa n) \, u \, dz \, dn = \rho \, ds \, \frac{\partial}{\partial t} \int_{n_r}^{\eta} (1 - \kappa n)(\eta - z_n) \, u \, dA,
\]  

(4.2.3-2)

where \(dA = dn \, dz\).
Second integral:

Fluid passing through the upstream face of the control section contributes

\[ \rho \int \limits_{\Delta} u(-u) dA \]

to the second integral and the downstream face contributes

\[ \rho \int \limits_{\Delta+\Delta} (u + \Delta u)^2 dA . \]

If the inflow from groundwater etc. is \( q \) per unit length and has a velocity of \( u_q \), then the second integral becomes

\[ \rho \left( \int \limits_{\Delta+\Delta} (u + \Delta u)^2 dA - \int \limits_{\Delta} u(-u) dA \right) ds - \rho u_q q ds . \]

This can be shown to equal,

\[ \rho ds \left( \frac{\partial}{\partial s} \int \limits_{A} u^2 dA - u_q q \right) . \]

Pressure Force:

The net pressure force is \(- \int p d\mathbf{S} \), which by Gauss’ divergence theorem becomes \(- \int \nabla p \ dV \). As the streamwise component of \( \nabla p \) is \( \frac{1}{(1-\kappa n)} \frac{\partial p}{\partial s} \) (Batchelor, Appendix 2, 1967) the pressure force expression becomes,

\[ \int \frac{\partial p}{\partial s} \frac{1}{(1-\kappa n)} dV . \]

If the pressure is assumed to be hydrostatic then \( \frac{\rho}{\rho g} = \eta(s,n,t) - z \) and

\[ \frac{\partial p}{\partial s} = \rho g \frac{\partial \eta}{\partial s} . \]

This shows that the pressure gradient at any point is due only to the slope of the surface above. The net pressure force is,
\[- \rho g \int_{v_i} \frac{1}{\partial_s (1 - \kappa n)} dV \, ,\]

which on substituting for \(dV\) becomes

\[- \rho g \int_{v_i} \frac{\partial \eta}{\partial_s (\eta - z_b)} dn \, .\]

**Friction Force:**

The friction force is defined as \(- \rho g \int S_f dV\) where \(S_f\) is the friction slope.

Substituting for \(dV\) leads to

\[- \rho g S_f \int_{v_i} (1 - \kappa n) (\eta - z_b) dn \, .\]

**Total Force:**

Combining pressure and friction forces gives the total force as

\[- \rho g \int_{v_i} \frac{\partial \eta}{\partial_s (\eta - z_b)} dn - \rho g S_f \int_{v_i} (1 - \kappa n) (\eta - z_b) dn \, .\]

Assembling the derived terms, dividing through by \(ds\) and rearranging gives the momentum equation,

\[\frac{\partial}{\partial t} \int_{v_i} u (1 - \kappa n) (\eta - z_b) dn + \frac{\partial}{\partial A} \int_{A} u^2 dA - g \int_{v_i} \frac{\partial \eta}{\partial_s (\eta - z_b)} dn \]

\[- g S_f \int_{v_i} (1 - \kappa n) (\eta - z_b) dn = u_q q \quad (4.2.3-3)\]

Expressing this equation in terms of \(Q\) and \(A\) initially requires the specification of a velocity distribution. This derivation assumes a general linear velocity distribution in the transverse direction,

\[u = u_o (1 + \beta n)\]
where again \( u_o \) is the depth averaged velocity at the reference line and \( u \) is the depth averaged velocity distance \( n \) from the reference line, and \( \beta \) is of order \( \kappa \).

The terms of Equation (4.2.3-3) are simplified term by term for clarity.

**First term:** Substituting the linear velocity distribution, changes the first term to

\[
\frac{\partial}{\partial t} \int u = u_o (1 + \beta n) (1 - \kappa n) (\eta - z_o) \, dn.
\]

After expanding within the integral sign and removing terms of order 2 (including \( \beta \kappa \)) the integral becomes,

\[
\frac{\partial}{\partial t} [u_o (A + (\beta - \kappa) M_z)]
\]

where \( M_z \) is the first moment of area about the z axis (as defined in Section 4.1.4.3).

Substituting for \( u_o \) requires, \( Q = \int u \, dA = \int u_o (1 + \beta n) \, dA \). Therefore \( Q \) is equal to \( u_o (A + \beta M_z) \) which in turn equals \( u_o A (1 + \beta \bar{n}) \) and accordingly, the expression for \( u_o \) is \( \frac{Q(1 - \beta \bar{n})}{A} \). Substituting for \( u_o \) and \( M_z \) leads to an expression which can be simplified to

\[
\frac{\partial}{\partial t} [Q(1 - \kappa \bar{n})].
\]

The factor \( \beta \) has been eliminated, which shows that, at this order, the details of the transverse velocity distribution are not critical.
Differentiating gives, \( \frac{\partial Q}{\partial t} (1 - \kappa \eta) - Q \kappa \frac{\partial \tilde{n}}{\partial t} \). However, as \( \tilde{n} \) is defined as

\[
\frac{1}{A} \int n (\eta - z_b) \, dn, \quad \frac{\partial \tilde{n}}{\partial t}
\]

becomes,

\[
- \frac{1}{A} \frac{\partial A}{\partial t} A \tilde{n} + \int \frac{1}{A} n \frac{\partial \eta}{\partial t} \, dn.
\]

At this order, \( \frac{\partial \eta}{\partial t} = \frac{1}{B} \frac{\partial A}{\partial t} \) so the first term becomes,

\[
\frac{\partial}{\partial t} [Q(1 - \kappa \tilde{n})] = \frac{\partial Q}{\partial t} (1 - \kappa \eta) - \frac{Q}{A} \kappa (n_m - \tilde{n}) \frac{\partial A}{\partial t}. \tag{4.2.3-4}
\]

**Second Term:** As \( u \) is the downstream velocity component of the element of area \( dA \), the term \( \int u^2 \, dA \) is equivalent to the square of the average velocity across the cross-section multiplied by the area. In terms of \( Q \) and \( A \) this is \( \frac{Q^2}{A} \). Accordingly the partial derivative with respect to \( s \) is

\[
\int u^2 \, dA = \frac{\partial}{\partial s} \frac{Q^2}{A} = \frac{2Q}{A} \frac{\partial Q}{\partial s} - \frac{Q^2}{A} \frac{\partial A}{\partial s} \tag{4.2.3-5}
\]

**Third Term:** The third term, \( g \int n \frac{\partial \eta}{\partial s} (\eta - z_b) \, dn \), can be written as \( g \int \frac{\partial \eta}{\partial s} \, dA \)

Describing the transverse surface gradient in terms of a new variable \( \eta_i \) as,

\[
\eta = \eta_o - \left( \frac{\nu^2 \kappa}{g} \right) \eta_i = \eta_o + \kappa \eta_i \eta,
\]

the required integral then becomes,

\[
g A \frac{\partial \eta}{\partial s} + g A \tilde{n} \frac{\partial \kappa}{\partial s} \eta_i + g A \tilde{n} \kappa \frac{\partial \eta_i}{\partial s}.
\]
To find an expression for $\frac{\partial \eta}{\partial s}$, $\eta$ is substituted into the integral equation for the area to give,

$$A = \int_{n_o}^{n} \left( \eta + \kappa \eta n - z_b \right) \, dn$$  \hspace{1cm} (4.2.3-7)

and

$$\frac{\partial A}{\partial s} = \int_{n_o}^{n} \left( \frac{\partial \eta}{\partial s} + \frac{\partial \kappa}{\partial s} \eta n + \kappa \frac{\partial \eta}{\partial s} - \frac{\partial z_b}{\partial s} \right) \, dn$$  \hspace{1cm} (4.2.3-8)

$$= B \frac{\partial \eta}{\partial s} + BS_o \kappa n_m + \kappa \frac{\partial \eta}{\partial s} B n_m$$

where $S_o$ is the mean bed slope defined as $-\frac{1}{B} \frac{\partial z_b}{\partial s} \, dn$. Rearranging gives

$$\frac{\partial \eta}{\partial s} = \frac{1}{B} \frac{\partial A}{\partial s} - S_o \kappa n_m - \kappa \frac{\partial \eta}{\partial s} n_m$$  \hspace{1cm} (4.2.3-9)

As $\eta_i$ has been defined as equal to $-\frac{u^2}{g}$ and it has been demonstrated above that

$$(u = \frac{Q}{A})$$

may be substituted for $u_o$ to first order, then $\eta_i$ becomes $-\frac{Q^2}{A^2 g}$, and,

$$\frac{\partial \eta_i}{\partial s} = 2 \frac{Q^2}{g A^2} \frac{\partial A}{\partial s} - 2 \frac{Q}{g A^2} \frac{\partial Q}{\partial s}$$  \hspace{1cm} (4.2.3-10)

Substituting into the required integral gives,

$$g \int_{n_o}^{n} \frac{\partial \eta}{\partial s} \, dA$$

$$= \frac{\partial A}{\partial s} \left( \frac{g A}{B} + 2 \frac{Q^2}{A^2} \kappa (\bar{n} - n_m) \right) - 2 \frac{Q}{A} \kappa (\bar{n} - n_m) \frac{\partial Q}{\partial s} - g A S_o - \frac{\partial \kappa Q^2}{\partial s} (\bar{n} - n_m)$$  \hspace{1cm} (4.2.3-11)

At this stage, the gravity factor $g A S_o$ has appeared.
**Fourth term:** The fourth term \(- g S_f \int_{n_r} (1 - \kappa \eta) (\eta - z_b) \, \text{d}n\) can be integrated to give \( g A S_f (1 - \kappa \bar{n}) \).

**Momentum Equation**

Assembling the terms gives the momentum equation in a curvilinear coordinate system as

\[
(1 - \kappa \bar{n}) \frac{\partial Q}{\partial t} + \frac{Q}{A} \kappa (\bar{n} - n_m) \frac{\partial A}{\partial t} + 2 \frac{Q}{A} (1 - \kappa (\bar{n} - n_m)) \frac{\partial Q}{\partial s} \\
\left( \frac{g A}{B} + \frac{Q^2}{A^2} (2\kappa (\bar{n} - n_m) - 1) \right) \frac{\partial A}{\partial s} - \frac{Q^2}{A} (\bar{n} - n_m) \frac{\partial \kappa}{\partial s} \\
- g A S_o + g A S_f (1 - \kappa \bar{n}) = u_q q
\]

(4.2.3-12)

The momentum equation can be simplified by replacing \(\frac{\partial A}{\partial t}\) by \(q - \frac{\partial Q}{\partial s}\), as in the mass conservation equation to first order \(\frac{\partial A}{\partial t} = q - \frac{\partial Q}{\partial s}\).

This simplification gives:

\[
(1 - \kappa \bar{n}) \frac{\partial Q}{\partial t} + \frac{Q}{A} (2 - 3\kappa (\bar{n} - n_m)) \frac{\partial Q}{\partial s} + \left( \frac{g A}{B} + \frac{Q^2}{A^2} (2\kappa (\bar{n} - n_m) - 1) \right) \frac{\partial A}{\partial s} \\
- g A S_o - \frac{Q^2}{A} (\bar{n} - n_m) \frac{\partial \kappa}{\partial s} + g A S_f (1 - \kappa \bar{n}) + q \left( \frac{Q}{A} \kappa (\bar{n} - n_m) - u_q \right) = 0
\]

(4.2.3-13)

**4.2.4 Comparison With Straight Channel Equations**

The curvilinear equations derived above follow a similar pattern to the straight channel equations but with an extra term for rate of change of curvature in the momentum equation and extra factors in the coefficients of all but two of the other
terms.

It has already been indicated (Section 3.5.1) that to date there are two methods for incorporating curvature into a study of long waves: one is to vary the friction factor, the other is to include a factor in the momentum equation that relates the actual length of the channel to the distance along the floodplain. Although calibrating models has a long pedigree in water engineering, the method of varying friction factors is more an example of tweaking than a systematic incorporation of curvature. The second method is a legacy of the 'flood routing as a substitute for solving inconvenient equations' approach and is very broad brush.

The equations derived in this chapter allow a more detailed consideration of the influence of curvature as they include both local curvature and rate of change of curvature at each cross-section.

They also incorporate details of the channel cross-section, an aspect of unsteady flow which the literature suggests does not seem to have invited much interest. However the equations suggest that the channel cross-sectional shape is important. As it is usual for curved natural channels to have asymmetric cross-sections it is not surprising that channel shape is an important factor in long wave behaviour.

It will be recalled that $n_m$ is the coordinate of the midpoint of the surface width and $\bar{n}$ is the coordinate of the centroid of the cross-section. They are both indices of the asymmetry of the bed and the fluid cross-section. They will decrease in
value as the cross-section becomes more symmetrical, but will never both equal 0 at the same time. While the channel may be symmetrical as in a laboratory flume or an artificial channel, the wetted area will always be asymmetrical due to superelevation. It may be possible to place the reference line, so that either \( \bar{n} \) or \( n_m \) is equal to 0, but never both.

### 4.2.5 Wave Celerity by the Method of Characteristics

Section 2.5.2 demonstrates using the method of characteristics to find an expression for the celerity of a long wave which can be described by the full de Saint-Venant equations. Although discussion of curvilinear long wave models is deferred until Chapter 5, the expression for the celerity of the dynamic wave is derived here so as to allow some comments to be made on the positioning of the reference line.

Rearranging the equations developed in the two previous section gives,

\[
\frac{\partial A}{\partial t} (1 - \kappa n_m) + \frac{\partial Q}{\partial s} = q \tag{4.2.5-1}
\]

\[
(1 - \kappa \bar{n}) \frac{\partial Q}{\partial t} + \frac{Q}{A} (2 - 3 \kappa (\bar{n} - n_m)) \frac{\partial Q}{\partial s} + \left( \frac{g}{B} + \frac{Q^2}{A^2} (2 \kappa (\bar{n} - n_m) - 1) \right) \frac{\partial A}{\partial s} = g A S_o + \frac{Q^2}{A} (\bar{n} - n_m) \frac{\partial \kappa}{\partial s} - g A S_f \left( 1 - \kappa \bar{n} \right) - q \left( \frac{Q}{A} \kappa (\bar{n} - n_m) - u_q \right) \tag{4.2.5-2}
\]

Equation (4.2.5-1) is multiplied by a general parameter \( \lambda \) and added to Equation
(4.2.5-2) to give on rearranging

\[
\lambda \left(1 - \kappa n_m \right) \frac{\partial A}{\partial t} + \left( \frac{gA}{B} + \frac{Q^2}{A^2} \left(2\kappa (\bar{n} - n_m) - 1 \right) \right) \frac{\partial A}{\partial s} \\
+ (1 - \kappa \bar{n}) \frac{\partial Q}{\partial t} + \left( \lambda + \frac{Q}{A} \left(2 - 3\kappa (\bar{n} - n_m) \right) \right) \frac{\partial Q}{\partial s} \\
= g AS_o + \frac{Q^2}{A} (\bar{n} - n_m) \frac{\partial \kappa}{\partial s} - g AS_f (1 - \kappa \bar{n}) - q \left( \frac{Q}{A} \kappa (\bar{n} - n_m) - u_q - \lambda \right)
\]

(4.2.5-3)

This equation is compared with the general difference equation for two variables,

\[
\frac{\partial A}{\partial t} + \frac{\partial A}{\partial s} \frac{ds}{dt} + \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial s} \frac{ds}{dt} = F(s,t)
\]

Hence the equations for the characteristics give

\[
\frac{ds}{dt} = \frac{\left( \frac{gA}{B} + \frac{Q^2}{A^2} \left(2\kappa (\bar{n} - n_m) - 1 \right) \right)}{\lambda \left(1 - \kappa n_m \right)} = \frac{\left( \lambda + \frac{Q}{A} \left(2 - 3\kappa (\bar{n} - n_m) \right) \right)}{(1 - \kappa \bar{n})}
\]

(4.2.5-4)

which on cross-multiplying and simplifying gives

\[
\lambda^2 + \frac{Q}{A} \left(2 - 3\kappa (\bar{n} - n_m) \right) \lambda - \left( \frac{gA}{B} \left(1 - \kappa (\bar{n} - n_m) \right) + \frac{Q^2}{A^2} \left(3\kappa (\bar{n} - n_m) - 1 \right) \right) = 0
\]

(4.2.5-5)

The solutions of this quadratic are

\[
\lambda = -\frac{Q}{A} \left(1 - \frac{3}{2} \kappa (\bar{n} - n_m) \right) \pm \sqrt{\frac{gA}{B} \left(1 - \frac{1}{2} \kappa (\bar{n} - n_m) \right)}
\]

(4.2.5-6)

Substituting these into definition of \( \frac{ds}{dt} \) (Equation (4.2.5-3)) and multiplying by

\[(1 - \kappa \bar{n})^{-1} (\equiv (1 + \kappa \bar{n})) \]

leads to
\[
\frac{ds}{dt} = \frac{Q}{A} \left(1 + \frac{3}{2} \kappa n_m - \frac{1}{2} \kappa \tilde{n}\right) \pm \sqrt{\frac{gA}{B} \left(1 + \frac{1}{2} \kappa n_m + \frac{1}{2} \kappa \tilde{n}\right)}
\]  
(4.2.5-7)

This expression parallels that developed for the linear case, \(\frac{ds}{dt} = u \pm c\), with

\[
c = \sqrt{\frac{gA}{B} \left(1 + \frac{1}{2} \kappa n_m + \frac{1}{2} \kappa \tilde{n}\right)}
\]  
(4.2.5-8).

The equation for long wave celerity is now that of a straight channel multiplied by a curvilinear correction which combines the effects of curvature and cross-sectional asymmetry.

4.3 DERIVATION OF EQUATIONS FROM INTEGRAL PRINCIPLES IN TERMS OF DISCHARGE AND WATER LEVEL \((Q, \eta)\)

4.3.1 Introduction

In some engineering solutions, it is desirable to obtain a value for a typical water level. Section 4.3 contains a derivation of the curvilinear de Saint-Venant equations in terms of discharge \((Q)\) and the water level at the reference line \(\eta_o\). However these equations still contain the variable \((A)\) and a new variable \((B)\), the top water width giving two equations in four unknowns. It is necessary to find two additional equations to obtain a unique solution.

It may be argued that the design engineer could use the \((Q,A)\) form of the equations and work out the elevation from the geometry of the cross-section. However this second step may not be straightforward and the total work involved may in the end lead to a less efficient method than that using the de Saint-Venant
equations in terms of $Q$ and $\eta_o$.

The resulting equation for the celerity of the dynamic wave is the same as that obtained in Section 4.2.3.

4.3.2 Mass-Conservation Equation

The mass conservation equation for the control volume of Figure 4.1.2(b) can be written as,

$$
\int_{n_r}^{n} \frac{\partial \eta}{\partial t} (1 - \kappa n) dn + \frac{\partial Q}{\partial s} = q.
$$

(4.3.2-1)

It has previously been shown (in Section 4.1.3.1) that $\frac{\partial \eta}{\partial t}$ can be approximated by

$$
\frac{\partial \eta}{\partial t} = \frac{\partial \eta_o}{\partial t} - \frac{2 u_o \kappa n}{g} \frac{\partial u_o}{\partial t},
$$

so the first term becomes,

$$
\int_{n_r}^{n} \frac{\partial \eta_o}{\partial t} (1 - \kappa n) - \frac{2 u_o \kappa n}{g} \frac{\partial u_o}{\partial t} dn
$$

which is equivalent to

$$
B \left( \frac{\partial \eta_o}{\partial t} (1 - \kappa n_m) - \frac{2 u_o \kappa n_m}{g} \frac{\partial u_o}{\partial t} \right).
$$

This then requires substitution for $u_o$ and $\frac{\partial u_o}{\partial t}$, to give the equation in terms of discharge.

Again assuming a general linear velocity distribution $u = u_o (1 + \beta n)$, it has been
shown in Section 4.2.3 that

\[ u_o = \frac{Q}{A} (1 - \beta \bar{n}) \]

and hence

\[ \frac{\partial u_o}{\partial t} = (1 - \beta \bar{n}) \left( \frac{1}{A} \frac{\partial Q}{\partial t} - \frac{Q}{A^2} \frac{\partial A}{\partial t} \right). \]  

(4.3.2-2)

Using the result of Section 4.1.3.2, \( \frac{\partial A}{\partial t} = B \left( \frac{\partial \eta_o}{\partial t} - \frac{2 u_o \kappa n_m \partial u_o}{g} \right) \), the required derivative becomes

\[ \frac{\partial u_o}{\partial t} = (1 - \beta \bar{n}) \left( \frac{1}{A} \frac{\partial Q}{\partial t} - \frac{QB}{A^2} \frac{\partial \eta_o}{\partial t} + \frac{QB}{A^2} \frac{2 u_o \kappa n_m \partial u_o}{g} \right). \]

Multiplying both sides of this equation by \( \kappa n_m \frac{2u_o}{g} \) gives,

\[ \kappa n_m \frac{2u_o}{g} \frac{\partial u_o}{\partial t} = \left( \frac{1}{A} \frac{\partial Q}{\partial t} - \frac{QB}{A^2} \frac{\partial \eta_o}{\partial t} \right) \kappa n_m \frac{2u_o}{g}. \]  

(4.3.2-3)

Substituting for \( u_o \) and \( \frac{\partial u_o}{\partial t} \) leads to the mass conservation equation in terms of discharge and water level:

\[ B \left[ 1 + \left( \frac{2Q^2B}{g A^3} - 1 \right) \kappa n_m \right] \frac{\partial \eta_o}{\partial t} - \frac{2QB}{g A^3} \kappa n_m \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial s} = q \]  

(4.3.2-4)

which can also be written as

\[ B \left[ 1 + (2 F^2 - 1) \kappa n_m \right] \frac{\partial \eta_o}{\partial t} - \frac{2QB}{g A^3} \kappa n_m \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial s} = q \]  

(4.3.2-5)
where, \( F \) is the Froude number (= \( \sqrt{\frac{Q^2 B}{g A^3}} \))

### 4.3.3 Momentum Equation

The momentum equation has already been obtained in the form,

\[
\frac{\partial}{\partial t} \int_{n_i}^n u (1 - \kappa n)(\eta - z_b) \, dn + \frac{\partial}{\partial s} \int u_z dA - g \int \frac{\partial}{\partial s} (\eta - z_b) \, dn - g S_f \int (1 - \kappa n)(\eta - z_b) \, dn = u_q q \tag{4.3.3-1}
\]

**First term:** Assuming the velocity distribution

\[ u = u_0 (1 + \beta n) = \left( \frac{Q}{A} (1 - \beta \bar{n} + \beta n) \right), \]

the first term becomes

\[
\frac{\partial}{\partial t} \left[ \frac{Q}{A} (1 - \beta \bar{n})(A - \kappa \bar{M}_z) + \frac{Q}{A} \beta \bar{M}_z \right] = \frac{\partial}{\partial t} [Q (1 - \kappa \bar{n})] \tag{4.3.3-2}
\]

However as \( \bar{n} \) is defined by \( \bar{n} = \frac{1}{A} \int n (\eta - z_b) \, dn \), its derivative is, (using the product rule),

\[
\frac{\partial \bar{n}}{\partial t} = - \frac{1}{A^2} \frac{\partial A}{\partial t} A \bar{n} + \frac{1}{A} \int_{n_i}^n \frac{\partial n}{\partial t} \, dn \tag{4.3.3-3}
\]

As \( \frac{\partial \bar{n}}{\partial t} \) will be multiplied by \( \kappa \) in the first term, \( \frac{\partial n}{\partial t} \) can be approximated by \( \frac{\partial n}{\partial t} \). It is also true to first order that \( \frac{\partial n}{\partial t} = \frac{1}{B} \frac{\partial A}{\partial t} \). Consequently,
\[-\frac{\partial \tilde{n}}{\partial t} = \frac{1}{A^2} \frac{\partial A}{\partial t} A \tilde{n} - \frac{1}{\eta_0} n \frac{\partial \eta}{\partial t} \, dn, \quad (4.3.3-4)\]

and the first term becomes

\[(1 - \kappa \eta) \frac{\partial Q}{\partial t} + \frac{BQ}{A^2} \kappa (\tilde{n} - n_m) \frac{\partial \eta}{\partial t}.\]

**Second term:** To expand the second term, use is made of the previous results,

\[\int u^2 dA = \frac{\partial}{\partial S} \left( \frac{Q^2}{A} \right) = \frac{2Q}{A} \frac{\partial Q}{\partial s} - \frac{Q^2}{A^2} \frac{\partial A}{\partial s}, \quad (4.2.3-5)\]

and

\[\frac{\partial A}{\partial s} = B \left( \frac{\partial \eta_0}{\partial s} + \frac{u^2 n_m \partial \kappa}{g} \frac{\partial}{\partial s} - \frac{2u \kappa n_m \partial u}{g} \frac{\partial}{\partial s} \right). \quad (4.1.3.2-2)\]

From the form of the velocity distribution used in Section 4.3.2

\[u^2_o = \frac{Q^2}{A^2} \left( 1 - 2 \beta \tilde{n} \right), \]

so that

\[\frac{\partial u_o}{\partial s} = (1 - \beta \tilde{n}) \left( \frac{1}{A} \frac{\partial Q}{\partial s} - \frac{1}{A^2} \frac{\partial Q}{\partial s} \right),\]

and the second term is thus,

\[\frac{2Q}{A} \left( 1 + \kappa n_m \frac{Q^2 B}{g A^3} \right) \frac{\partial Q}{\partial s} - \frac{Q^2 B}{A^2} \left( 1 + \kappa n_m \frac{2Q^2 B}{g A^3} \right) \frac{\partial \eta_0}{\partial s} + \frac{\partial \kappa}{\partial s} \frac{n_m}{g A^3} \frac{Q^4 B}{A^3} - \frac{Q^2 B}{A^2} \left( 1 + \kappa n_m \frac{2Q^2 B}{g A^3} \right) S_o.\]

**Third and fourth terms:** The third term, \(-g \int_{\eta_0}^{\eta_0} \frac{\partial \eta}{\partial s} (\eta - z_v) \, dn\) requires \(\eta\) in
terms of $Q$ and $A$, i.e. $\eta = \eta_o - \left( \frac{Q^2 \kappa}{g A^2} \right)$

which gives

$$\frac{\partial \eta}{\partial s} = \frac{\partial \eta_o}{\partial s} - \frac{Q^2 \partial \kappa}{g A^2} n - \frac{2Q\kappa n \partial Q}{g A^2} + \frac{2Q^2 \kappa n \partial A}{g A^3}.$$

It is only necessary to substitute the zero order part of the expression for $\frac{\partial A}{\partial s}$

developed above, which gives $\frac{\partial \eta}{\partial s}$ as

$$\frac{\partial \eta}{\partial s} = \frac{\partial \eta_o}{\partial s} - \frac{Q^2 \partial \kappa}{g A^2} n - \frac{2Q\kappa n \partial Q}{g A^2} + \frac{2Q^2 \kappa n \partial A}{g A^3}$$

$$\left(1 + \kappa n \frac{2Q^2 B}{g A^3}\right) \frac{\partial \eta_o}{\partial s} - \frac{Q^2 \partial \kappa}{g A^2} n - \frac{2Q\kappa n \partial Q}{g A^2} + \frac{2Q^2 B \kappa n}{g A^3} S_o + S_f (1 - \kappa n).$$

With this expression for $\frac{\partial \eta}{\partial s}$ the combined integrand of the and fourth term becomes

$$\left(1 + \kappa n \frac{2Q^2 B}{g A^3}\right) \frac{\partial \eta_o}{\partial s} - \frac{Q^2 \partial \kappa}{g A^2} n - \frac{2Q\kappa n \partial Q}{g A^2} + \frac{2Q^2 B \kappa n}{g A^3} S_o + S_f (1 - \kappa n).$$

Integrating and multiplying by $g$ gives,

$$Ag \left(1 + \kappa \bar{n} \frac{2Q^2 B}{g A^3}\right) \frac{\partial \eta_o}{\partial s} - \frac{Q^2 \partial \kappa}{A \partial s} - \frac{2Q\kappa \bar{n} \partial Q}{A^2} + \frac{2Q^2 B \kappa \bar{n}}{A^2} S_o + S_f (1 - \kappa \bar{n}).$$
The momentum equation is therefore

\[
\frac{B Q}{A} \kappa \left( \tilde{n} - n_m \right) \frac{\partial \eta_0}{\partial t} - \left[ A g \left( 1 + \kappa \tilde{n} \frac{2 Q^2 B}{g A^3} \right) + \frac{Q^2 B}{A^2} \left( 1 + \kappa n_m \frac{Q^2 B}{g A^3} \right) \right] \frac{\partial \eta_0}{\partial s} + (1 - \kappa n) \frac{\partial Q}{\partial t} + \frac{2 Q}{A} \left[ 1 + \kappa n_m \frac{Q^2 B}{g A^3} + \kappa \tilde{n} \right] \frac{\partial Q}{\partial s} + \frac{Q^2}{A} \left[ n_m \frac{Q^2 B}{g A^3} + \tilde{n} \right] \frac{\partial \kappa}{\partial s} - \frac{Q^2 B}{A^2} \left( 1 + \kappa n_m \frac{Q^2 B}{g A^3} - 2 \kappa \tilde{n} \right) S_o - S_f (1 - \kappa \tilde{n}) = u_q q
\]

(4.3.3-5)

This equation can be simplified slightly by expressing it in terms of the Froude number F. It is

\[
\frac{B Q}{A} \kappa \left( \tilde{n} - n_m \right) \frac{\partial \eta_0}{\partial t} - \left[ A g \left( 1 + \kappa \tilde{n} 2 F^2 \right) + \frac{Q^2 B}{A^2} \left( 1 + \kappa n_m F^2 \right) \right] \frac{\partial \eta_0}{\partial s} + (1 - \kappa n) \frac{\partial Q}{\partial t} + \frac{2 Q}{A} \left[ 1 + \kappa n_m F^2 + \kappa \tilde{n} \right] \frac{\partial Q}{\partial s} + \frac{Q^2}{A} \left[ n_m F^2 + \tilde{n} \right] \frac{\partial \kappa}{\partial s} - \frac{Q^2 B}{A^2} \left( 1 + \kappa n_m F^2 - 2 \kappa \tilde{n} \right) S_o - S_f (1 - \kappa \tilde{n}) = u_q q
\]

(4.3.3-6).

These equations are more complicated than the corresponding Q,A forms, Equation (4.2.2-6) and Equation (4.2.3-13).

4.3.4 Long Wave Celerity by the Method of Characteristics

For clarity the calculation for characteristics uses the two de Saint-Venant Equations in terms of the Froude Number and with the forcing terms set equal to \( q_1 \) and \( q_2 \).
The mass conservation equation becomes

\[
B \left[ 1 + (2,F^2 - 1)\kappa n_m \right] \frac{\partial \eta_o}{\partial t} - \frac{2QB}{gA^3} \kappa n_m \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial s} = q_1, \quad (4.3.4-1)
\]

where \( q_1 = q \), and the momentum equation becomes

\[
\frac{BQ}{A} \kappa (\bar{n} - n_m) \frac{\partial \eta_o}{\partial t} - \left[ A g \left( 1 + \kappa \bar{n} 2F^2 \right) + \frac{Q^2 B}{A^2} \left( 1 + \kappa n_m F^2 \right) \right] \frac{\partial \eta_o}{\partial s} \\
(1 - \kappa n) \frac{\partial Q}{\partial t} + \frac{2Q}{A} \left[ (1 + \kappa n_m F^2 + \kappa \bar{n}) \right] \frac{\partial Q}{\partial s} = q_2
\]

where,

\[
q_2 = u_q q - \frac{Q^2}{A} \left[ n_m \frac{Q^4 B}{gA^3} + \bar{n} \right] \frac{\partial \kappa}{\partial s} + \frac{Q^2 B}{A^2} \left( 1 + \kappa n_m \frac{Q^2 B}{gA^3} - 2 \kappa \bar{n} \right) S_o + S_f (1 - \kappa \bar{n}).
\]

(4.3.4-2)

Equation (4.3.4-2) + \( \lambda \) Equation (4.3.4-1) is

\[
\left[ \frac{BQ}{A} \kappa (\bar{n} - n_m) + \lambda B \left[ 1 + (2,F^2 - 1)\kappa n_m \right] \right] \frac{\partial \eta_o}{\partial t} - \left[ A g \left( 1 + \kappa \bar{n} 2F^2 \right) + \frac{Q^2 B}{A^2} \left( 1 + \kappa n_m F^2 \right) \right] \frac{\partial \eta_o}{\partial s} \\
\left[ (1 - \kappa n) - \lambda \frac{2QB}{gA^3} \kappa n_m \right] \frac{\partial Q}{\partial t} + \left[ \frac{2Q}{A} \left( 1 + \kappa n_m F^2 + \kappa \bar{n} \right) + \lambda \right] \frac{\partial Q}{\partial s} = q_2 + \lambda q_1
\]

(4.3.4-3)

Hence along the characteristic given by \( \frac{ds}{dt} = f(Q, A) \),

\[
\frac{2Q}{A} \left( 1 + \kappa n_m F^2 + \kappa \bar{n} \right) + \lambda = \left[ A g \left( 1 + \kappa \bar{n} 2F^2 \right) + \frac{Q^2 B}{A^2} \left( 1 + \kappa n_m F^2 \right) \right] \\
\left[ (1 - \kappa n) - \lambda \frac{2QB}{gA^3} \kappa n_m \right]
\]

(4.3.4-4)

and cross-multiplying and rearranging leads to a quadratic equation in \( \lambda \),
\[
\left[ 1 + (2 F^2 - 1) \kappa n_m \right] \lambda^2 + \frac{Q}{A} \left( 2 + 4 \kappa n_m F^2 - 3 \kappa \bar{n} + \kappa n_m \right) \lambda \\
+ \frac{Q^2}{A^2} \left( 1 + \kappa n_m 2 F^2 - 3 \kappa \bar{n} + 2 \kappa n_m \right) - \frac{A g}{B} \left[ 1 + (2 F^2 - 1) \kappa \bar{n} \right] = 0
\]

(4.3.4-5)

Dividing through by the coefficient of \( \lambda^2 \), substituting for \( F \) and simplifying brings the quadratic equation to

\[
\lambda^2 + \frac{Q}{A} \left( 2 - 3 \kappa \bar{n} + 3 \kappa n_m \right) \lambda \\
+ \frac{Q^2}{A^2} \left( 1 - 3 \kappa \bar{n} + 3 \kappa n_m \right) - \frac{A g}{B} \left( 1 - \kappa \bar{n} + \kappa n_m \right) = 0
\]

(4.3.4-6)

Applying the quadratic formula again gives,

\[
\lambda = -\frac{Q}{A} \left( 1 - \frac{3}{2} \kappa \bar{n} + \frac{3}{2} \kappa n_m \right) \pm \sqrt{\frac{A g}{B} \left( 1 - \frac{1}{2} \kappa \bar{n} + \frac{1}{2} \kappa n_m \right)},
\]

(4.3.4-7)

which on substituting back into the characteristic equation gives,

\[
\frac{ds}{dt} = \frac{Q}{A} \left( 1 + \frac{3}{2} \kappa \bar{n} + \frac{3}{2} \kappa n_m \right) \pm \sqrt{\frac{A g}{B} \left( 1 + \frac{1}{2} \kappa \bar{n} + \frac{1}{2} \kappa n_m \right)}
\]

(4.3.4-8)

This is the same as the corresponding equation for the derivation in terms of \( Q \) and \( A \) (Equation (4.2.5-7)).
4.4 DERIVATION OF EQUATIONS FROM INTEGRAL PRINCIPLES IN TERMS OF VELOCITY AND SURFACE LEVEL \((u_o, \eta_o)\)

### 4.4.1 Introduction

It is possible to obtain the curvilinear equations in terms of velocity and water level at the reference line. However these are more complicated than the previous cases and are thus of limited practical value, but are included in this thesis for completeness.

Although in the course of the derivations described in Sections 4.2 and 4.3, it was necessary to assume a velocity distribution, this function was eliminated in the later part of each derivation and do not occur in the final equations. However in the derivation of the curvilinear equations in terms of velocity and water level, it forms part of the final equations.

### 4.4.2 Mass-Conservation Equation

The derivation starts with the general integral mass conservation equation

\[ \int_{n} \frac{\partial \eta}{\partial t} (1 - \kappa \eta) \, dn + \frac{\partial Q}{\partial s} = q \]  \hspace{1cm} (4.4.2-1)

**First term:** The first term has already been shown (Section 4.3.2) to be equal to

\[ B \left[ (1 - \kappa n_m) \frac{\partial \eta_o}{\partial t} - \kappa n_m \frac{2u_o}{g} \frac{\partial u_o}{\partial t} \right]. \]
Second term: As \( Q = \int u \, dA \), then assuming a general linear distribution gives the second term as \( \frac{\partial}{\partial s} \int u_o (1 + \beta n)(\eta - z_o) \, dn \), which is the same as,

\[
\frac{\partial u_o}{\partial s} (A + \beta M_z) + u_o \left( \frac{\partial A}{\partial s} + \frac{\partial \beta}{\partial s} M_z - \beta \frac{\partial M_z}{\partial s} \right).
\]

\( M_z \) has been defined as \( A \tilde{n} \) and \( \frac{\partial A}{\partial s} \) was shown in Section 4.1.3.2 to be

\[
B \left( \frac{\partial \eta_o}{\partial s} + S_o - \frac{u_o^2 n_m}{g} \frac{\partial \kappa}{\partial s} - \frac{2 \kappa n_m u_o}{g} \frac{\partial u_o}{\partial s} \right).
\]

Section 4.1.4.3 calculated \( \frac{\partial M_z}{\partial s} \) as \( B n_m \left( \frac{\partial \eta_o}{\partial s} + S_1 \right) \) although in this case the approximation \( B n_m \left( \frac{\partial \eta_o}{\partial s} + S_1 \right) \) is used as this factor is multiplied by \( \beta \). Now the second term is

\[
\left( A + \beta A \tilde{n} - B \kappa n_m \frac{2u_o^2}{g} \right) \frac{\partial u_o}{\partial s} + u_o B \left( 1 + \beta n_m \right) \frac{\partial \eta_o}{\partial s} + u_o B S_o + u_o B \beta n_m S_1
\]

\[
- \frac{\partial \kappa}{\partial s} B n_m u_o^3 + u_o \frac{\partial \beta}{\partial s} A \tilde{n}.
\]

Combining the two terms and dividing through by \( B \) gives

\[
(1 - \kappa n_m) \frac{\partial \eta_o}{\partial t} - \kappa n_m \frac{2u_o^2}{g} \frac{\partial u_o}{\partial t} + u_o \left( 1 + \beta n_m \right) \frac{\partial \eta_o}{\partial s} + \left( \frac{A}{B} (1 + \beta \tilde{n}) - \kappa n_m \frac{2u_o^2}{g} \right) \frac{\partial u_o}{\partial s}
\]

\[
+ u_o \left( S_o + \beta n_m S_1 \right) - \frac{\partial \kappa}{\partial s} n_m u_o^3 + u_o \frac{\partial \beta}{\partial s} A \tilde{n} = \frac{q}{B}
\]

\[\text{(4.4.2-2)}\]
This version of the mass-conservation equation is considerably more complex than the previous two. It includes the extra terms $S_o$ and $S_i$ and the dependence on the choice of transverse velocity distribution is obvious.

### 4.4.3 Momentum Equation

Beginning with the integral momentum equation as a modified form of Equation (4.2.4-1),

$$\frac{\partial}{\partial t} \int_{n_{sw}}^{n_{j}} u (1 - \kappa n)(\eta - z_b) \, dn + \frac{\partial}{\partial s_c} \int_{n_{sw}}^{n_{j}} u^2 \, dA + g \int_{n_{sw}}^{n_{j}} \left( \frac{\partial \eta}{\partial s} + S_f (1 - \kappa n) \right)(\eta - z_b) \, dn = u_q \, q$$

(4.4.3-1)

**First term:** Assuming a general linear velocity gradient, the first term becomes,

$$\frac{\partial}{\partial t} \int_{n_{sw}}^{n_{j}} u_o (1 + \beta n - \kappa n)(\eta - z_b) \, dn.$$  

Integrating gives the derivative, $\frac{\partial}{\partial t} u_o (A + (\beta - \kappa)M_z)$, which can be expanded to give,

$$\frac{\partial u_o}{\partial t} [A + (\beta - \kappa)M_z] + u_o \left[ \frac{\partial A}{\partial t} + (\beta - \kappa) \frac{\partial M_z}{\partial t} \right],$$

as $\beta$ and $\kappa$ are both assumed to be constant with time.

Substituting for $M_z$, $\frac{\partial A}{\partial t}$ (as defined in Section 4.1.3.2), and $\frac{\partial M_z}{\partial t}$ (as defined in Section 4.1.3.3.) gives the first term as
\[
\left( A \left[ 1 + (\beta - \kappa) \tilde{\eta} \right] - B \kappa n_m \frac{2u_o^2}{g} \right) \frac{\partial u_o}{\partial t} + u_o B \left[ 1 + (\beta - \kappa n_m) \right] \frac{\partial \eta_o}{\partial t}.
\]

**Second term:** On substituting for a general linear velocity distribution, the second term becomes \( \frac{\partial}{\partial s} \int u_o^2 (1 + 2 \beta n)(\eta - z_b) \, dn \). After integration, the derivative \( \frac{\partial}{\partial s} \left( u_o^2 [A + 2 \beta M_z] \right) \) is obtained. Substituting for \( M_z, \frac{\partial A}{\partial s} \), and \( \frac{\partial M_z}{\partial t} \) leads to

\[
2u_o \left[ A(1 + 2 \beta \tilde{n}) - B \kappa n_m \frac{u_o^2}{g} \right] \frac{\partial u_o}{\partial s} + u_o^2 B (1 + 2 \beta n_m) \frac{\partial \eta_o}{\partial s} + u_o^2 (B S_o + 2 \beta S_z)
\]

\[
- \frac{u_o^4}{g} n_m B \frac{\partial \kappa}{\partial s} + 2 A \tilde{n} u_o \frac{\partial \beta}{\partial s}.
\]

**Third term:** Substituting into the third term \( g \int \left( \frac{\partial \eta}{\partial s} + S_f (1 - \kappa n) \right)(\eta - z_b) \, dn \), and integrating gives,

\[
g A \frac{\partial \eta_o}{\partial s} - u_o^2 A \tilde{n} \frac{\partial \kappa}{\partial s} - 2 \kappa A \tilde{n} u_o \frac{\partial u_o}{\partial s} + g A S_f (1 - \kappa \tilde{n})
\]

Hence gathering up the constituent terms and dividing through by \( B \) gives the momentum equation as,
\[
\left( \frac{A}{B} \left[ 1 + (\beta - \kappa) \tilde{n} \right] - \kappa n_m \frac{2u_o^2}{g} \right) \frac{\partial u_o}{\partial t} + 2u_o \left[ \frac{A}{B} \left( 1 + 2\beta \tilde{n} - \kappa \tilde{n} \right) - \kappa n_m \frac{u_o^2}{g} \right] \frac{\partial u_o}{\partial s} \\
+ u_o \left[ 1 + (\beta - \kappa n_m) \right] \frac{\partial \eta_o}{\partial t} + \left[ u_o \left( 1 + 2\beta n_m \right) + \frac{g A}{B} \right] \frac{\partial \eta_o}{\partial s} \\
+ u_o^2 \left( S_o + 2\frac{\beta}{B} S_1 \right) - u_o \left( \frac{\eta_o}{B} + \frac{u_o^2}{g} n_m \right) \frac{\partial \kappa}{\partial s} + 2\frac{\beta}{B} \frac{\partial \kappa}{\partial s} + \frac{g A}{B} S_f (1 - \kappa \tilde{n}) = u_q q
\]

Like the mass-conservation equation this version of the momentum equation is more complex than the previous two and also relies on the choice of transverse velocity distribution.

**4.4.4 Long Wave Celerity From the Method of Characteristics**

The curvilinear de Saint-Venant Equations are each rewritten in two parts as

\[
(1 - \kappa n_m) \frac{\partial \eta_o}{\partial t} + u_o \left( 1 + \beta n_m \right) \frac{\partial \eta_o}{\partial s} + \left( \frac{A}{B} \left( 1 + \beta \tilde{n} \right) - \kappa n_m \frac{2u_o^2}{g} \right) \frac{\partial u_o}{\partial s} - \kappa n_m \frac{2u_o}{g} \frac{\partial u_o}{\partial t} = q_i
\]

where,

\[
q_i = \frac{q}{B} - u_o \left( S_o + \beta n_m S_1 \right) + \frac{\partial \kappa}{\partial s} n_m u_o^3 - u_o \frac{\partial \beta}{\partial s} \frac{A}{B} \tilde{n}
\]

and
\[
\left( \frac{A}{B} \left[ 1 + (\beta - \kappa) \tilde{n} \right] - \kappa n_m \frac{2u_o^2}{g} \right) \frac{\partial u_o}{\partial t} + 2u_o \left[ \frac{A}{B} \left( 1 + 2 \beta \tilde{n} - \kappa \tilde{n} \right) - \kappa n_m \frac{u_o^2}{g} \right] \frac{\partial u_o}{\partial s} \\
+ u_o \left[ 1 + (\beta - \kappa n_m) \right] \frac{\partial \eta_o}{\partial t} + \left[ u_o^2 \left( 1 + 2 \beta n_m \right) + \frac{gA}{B} \right] \frac{\partial \eta_o}{\partial s} = q_2
\]

\[\text{(4.4.4-2)}\]

where,

\[q_2 = u_o q - u_o^3 \left( S_o + 2 \frac{B}{A} S_1 \right) + u_o^2 \left( \frac{A}{B} \tilde{n} + \frac{u_o^2}{g} n_m \right) \frac{\partial \kappa}{\partial s} - 2 \tilde{A} u_o^2 \frac{\partial \beta}{\partial s} \frac{\partial \kappa}{\partial s} - \frac{gA}{B} S_f (1 - \kappa \tilde{n})\]

Equation \((4.4.3-2) + \lambda \ (4.4.3-1)\) eventually gives the quadratic

\[-\frac{A}{B} (1 + \beta \tilde{n} - \kappa n_m) \lambda^2 + \frac{A}{B} u_o (\kappa \tilde{n} + \kappa n_m - 4 \beta \tilde{n} - 2) \lambda \]
\[+ \frac{A}{B} u_o^2 (\kappa \tilde{n} - 3 \beta \tilde{n} - 1) - \frac{gA}{B} \left[ 1 + (\beta - \kappa) \tilde{n} \right] = 0\]

\[\text{(4.4.3-3)}\]

Solving by the quadratic formula gives the values,

\[\lambda = \frac{u_o}{2} (\kappa \tilde{n} - \kappa n_m - 2 - 2 \beta \tilde{n}) \pm \sqrt{\frac{gA}{B} (1 - \kappa \tilde{n} + \kappa n_m)}\]

\[\text{(4.4.3-4)}\]

which on substituting into one of the expressions for the characteristics gives,

\[\frac{ds}{dt} = u_o \left( 1 + \beta \tilde{n} - \frac{1}{2} \kappa \tilde{n} + \frac{3}{2} \kappa n_m \right) \pm \sqrt{\frac{gA}{B} \left( 1 + \frac{1}{2} \kappa \tilde{n} + \frac{1}{2} \kappa n_m \right)}\]

\[\text{(4.4.3-5)}\]

which gives the same expression for celerity as the previous two cases.

### 4.4.5 Irrotational Flow

The special case of irrotational flow (i.e. \(\beta = \kappa\)) has the corresponding characteristic equation
\[
\frac{ds}{dt} = u_o \left( 1 + \frac{1}{2} \kappa \tilde{n} + \frac{3}{2} \kappa n_m \right) \pm \sqrt{\frac{g A}{B} \left( 1 + \frac{1}{2} \kappa \tilde{n} + \frac{1}{2} \kappa n_m \right) }.
\] (4.4.5-1)

Substitution for \( u_o = \frac{Q}{A} (1 - \kappa \tilde{n}) \) leads to an equation identical to Equations (4.2.5-7) and (4.3.4-7) even though they were derived by taking a general linear velocity distribution.

**Irrotational Model:** The version of the curvilinear de Saint-Venant equations derived in Section 4.4, along with expressions for celerity, transverse velocity gradient and transverse surface gradient for the irrotational flow case are briefly described in Section 5.4.3. Appendix 2 contains a discussion of the background of this model and describes the derivations of the expressions for transverse velocity and surface gradients that complete the model.

**4.5 POSITION OF REFERENCE LINE**

It has already been indicated that the position of the reference line is, within reason, arbitrary. The following derivation supports that.

Consider two possible downstream axes \( s_1 \) and \( s_2 \) a distance \( \Delta \) apart (Figure 4.5).

The reference lines on the axes have corresponding radii of curvature \( r_1 \) and \( r_2 \) where

\[
r_2 = r_1 + \Delta.
\] (4.5-1)

Inverting this relation gives,
Expanding the inverse by the binomial theorem gives to first order,

\[ \kappa_2 \approx \kappa_1 \left( 1 - \frac{\Delta}{r_1} \right)^{-1} = \kappa_1 (1 - \kappa_1 \Delta), \]

and

\[ \frac{\kappa_2}{\kappa_1} = (1 - \kappa_1 \Delta). \quad (4.5-3) \]

The wave speeds along, \( s_1 \) and \( s_2 \) respectively are given by

\[ c_1^2 = \frac{gA}{B} (1 + \kappa_1 \bar{n}_1 + \kappa n_{m1}). \quad (4.5-4) \]

and

\[ c_2^2 = \frac{gA}{B} (1 + \kappa_2 \bar{n}_2 + \kappa_2 n_{m2}) \quad (4.5-5) \]

where

\[ \bar{n}_1, n_{m1} \] are relative to the axis with radius of curvature \( r_1 \)

\[ \bar{n}_2, n_{m2} \] are relative to the axis with radius of curvature \( r_2 \), and

\[ \bar{n}_2 - \bar{n}_1 = n_{m2} - m_{m1} = \Delta, \]
If \( t_1 \) and \( t_2 \) are the times taken for a long wave to travel from cross-section I to cross-section II measured relative to reference lines at \( s_1 \) and \( s_2 \), then
\[
t_1 = \frac{r_1 \delta \theta}{c_1} \tag{4.5-6}
\]
and
\[
t_2 = \frac{r_2 \delta \theta}{c_2} \tag{4.5-7}
\]
where \( \delta \theta \) is the angle formed when the lines of the cross-sections are extended to meet at the centre of curvature. (Figure 4.5). Dividing \( t_1 \) by \( t_2 \) gives,
\[
\frac{t_1}{t_2} = \frac{r_1}{r_2} \frac{c_2}{c_1} = \frac{\kappa_2 c_2}{\kappa_1 c_1} \tag{4.5-8}
\]
which on substituting for \( \kappa_1, \kappa_2, c_1, c_2 \) leads to
\[
\frac{t_1}{t_2} = (1 - \kappa_1 \Delta) \sqrt{\frac{gA}{B} \left( 1 + \kappa_2 (\bar{n}_2 + n_{m_2}) \right)} \tag{4.5-9}
\]
\frac{gA}{B} \left( 1 + \kappa_1 (\bar{n}_1 + n_{m_1}) \right)
\]
If the square roots are expanded by the binomial theorem to the first order this ratio becomes,
\[
\frac{t_1}{t_2} = (1 - \kappa_1 \Delta) \left( 1 + \frac{1}{2} \kappa_2 (\bar{n}_2 + n_{m_2}) \right) \left( 1 + \frac{1}{2} \kappa_1 (\bar{n}_1 + n_{m_1}) \right) \tag{4.5-10}
\]
which is equivalent to,
\[
\frac{t_1}{t_2} = (1 - \kappa_1 \Delta) \left( 1 + \frac{1}{2} \kappa_1 (\bar{n}_2 - \bar{n}_1 + n_{m2} - n_{m1}) \right)
\]

(4.5-11)

This gives the required ratio as

\[
\frac{t_1}{t_2} = (1 - \kappa_1 \Delta) \left( 1 + \frac{1}{2} \kappa_1 (\bar{n}_2 - \bar{n}_1 + n_{m2} - n_{m1}) \right)
\]

\[
\frac{t_1}{t_2} = (1 - \kappa_1 \Delta)(1 + \kappa_1 \Delta) = \left(1 - (\kappa \Delta)^2\right),
\]

(4.5-12)

i.e. independent of curvature to first order.

Although the expressions for celerity used in this analysis are functions of \( \kappa \) to first order, design engineers are more concerned with time of travel of the long wave than celerity. Hence the critical function is the ratio of time and so the placement of the axis can be regarded as arbitrary.
V CURVILINEAR LONG WAVE MODELS

5.1 INTRODUCTION

The derivations in Chapter 4 resulted in curvilinear de Saint-Venant equations in three different sets of variables. This and the next two chapters will use the equations in terms of discharge and area, as they are the simplest of the three.

The curvilinear kinematic model is straightforward and is described in the following Section 5.2. As it has been demonstrated that the conventional method of deriving the diffusion model cannot be used when there is lateral inflow (Section 3.2.2), the basic two equation diffusion model is presented. Although the limitations of the conventional method have been referred to, it is anticipated that it will remain part of engineering practice in the near future. It will therefore be applied to the curvilinear case. The general solution of the combined equation will be discussed in Chapter 6.

The dynamic model will be briefly presented as it has already been derived in the previous chapter. The special case of irrotational flow will be outlined. The dynamic model derived in this thesis is, in Chapter 7, compared with published laboratory data from model dam break studies.

5.2 CURVILINEAR KINEMATIC MODEL

5.2.1 Kinematic Wave Equation

It has already been noted (Section 2.3.1) that for the kinematic model, the momentum equation is given by

\[ S_f = S_o. \]  

(5.2.1-1)

This shows that there is a unique relationship between \( Q \) and \( A \), given here by the curvilinear mass conservation equation,
It is possible to derive the kinematic wave equation in terms of either Q or A. (The linear case used only Q.)

**In terms of Q:** Taking $A = f(Q)$, then \( \frac{\partial A}{\partial t} = \frac{dA}{dQ} \frac{\partial Q}{\partial t} \), so that substituting into the mass-conservation equation gives

\[
(1 - \kappa n_m) \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial s} = q .
\]

and comparing with the identity from calculus,

\[
dQ = \frac{\partial Q}{\partial t} \frac{dt}{ds} + \frac{\partial Q}{\partial s}
\]

indicates that,

\[
\frac{dt}{ds} = (1 - \kappa n_m) \frac{dA}{dQ} .
\]

Hence the curvilinear kinematic wave speed \( c_c = \frac{ds}{dt} \) is to first order is

\[
(1 + \kappa n_m) \frac{dQ}{dA}
\]

and the kinematic wave equation is

\[
\frac{\partial Q}{\partial t} + c \frac{\partial Q}{\partial s} = cq .
\]

**In terms of A:** If we take $Q = h(A)$, (where \( h = f^{-1} \)), then \( \frac{\partial Q}{\partial s} = \frac{dQ}{dA} \frac{\partial A}{\partial s} \), and the mass-conservation equation becomes,

\[
(1 - \kappa n_m) \frac{\partial A}{\partial t} + \frac{dQ}{dA} \frac{\partial A}{\partial s} = q .
\]

which on multiplying by \( (1 + \kappa n_m) \) becomes to first order
\[ \frac{\partial A}{\partial t} + \frac{dQ}{dA} \frac{\partial A}{\partial s} (1 + \kappa n_m) = q (1 + \kappa n_m). \] (5.2.1-8)

Comparing with the identity,
\[ dA = \frac{\partial A}{\partial t} dt + \frac{\partial A}{\partial s} ds, \]
again gives a curvilinear kinematic wave speed of \( c = (1 + \kappa n_m) \frac{dQ}{dA} \). The kinematic wave equation in terms of \( A \) is
\[ \frac{\partial A}{\partial t} + c \frac{\partial A}{\partial s} = q (1 + \kappa n_m). \] (5.2.1-9)

In this case a curvilinear factor has appeared on the right hand side of the equation. As \( q \) is inflow per unit length and imperfectly known, the curvilinear correction in unlikely to be of any impact. Consequently a more reasonable form of the kinematic wave equation may be
\[ \frac{\partial A}{\partial s} + c \frac{\partial A}{\partial s} = q. \] (5.2.1-10)

### 5.2.2 Comparison With Straight Channel Case

Comparison with Section 2.3.1 shows that the kinematic wave equation has the same form in the curvilinear and straight channel case. The only difference is the \( (1 + \kappa n_m) \) factor, which corrects the kinematic wave celerity for curvature.
5.3 CURVILINEAR DIFFUSION MODEL

5.3.1 Curvilinear Diffusion Model with Inflow

It has been demonstrated that derivations of the diffusion model that rely on linearisations cannot be used when there is lateral inflow. In that case it is necessary to go to first principles and solve the basic equations.

This curvilinear diffusion model is obtained from the curvilinear de Saint Venant equations (i.e. Equation (4.2.2-1) and Equation (4.2.3-6)) by retaining the friction, gravity and pressure terms in the momentum equation. The model is thus,

\[ \frac{\partial A}{\partial t}(1-\kappa n_m) + \frac{\partial Q}{\partial s} = q \]  

(5.3.1-1)

and,

\[ \left( \frac{gA}{B} + \frac{Q^2}{A^2} \left[ 2\kappa (\bar{n} - n_m) - 1 \right] \right) \frac{\partial A}{\partial s} - \frac{Q^2}{A} \frac{\partial \kappa}{\partial s} (\bar{n} - n_m) \frac{\partial A}{\partial s} + gAS_o + gAS_f (1-\kappa \bar{n}) \]

\[ = \quad q \left[ u_q - \frac{Q}{A} \kappa (\bar{n} - n_m) \right]. \]

(5.3.1-2)

5.3.2 Conventional Derivation of the Curvilinear Diffusion Equation

It is shown in Chapter 6, that the curvilinear de Saint-Venant equations can be linearised and combined to give,

\[ \frac{\partial^2 Q}{\partial t^2} + 2u_o \left( 1 - \frac{1}{2} \kappa \bar{n}_o + \frac{3}{2} \kappa n_{mo} \right) \frac{\partial^2 Q}{\partial t \partial s} + u_o^2 \left( 1 - \kappa \bar{n}_o + 3 \kappa n_{mo} \right) \frac{\partial^2 Q}{\partial s^2} \]

\[-c_o^2 \left( 1 + \kappa \bar{n}_o + \kappa n_{mo} \right) \frac{\partial^2 Q}{\partial s^2} - \theta_1 \left( 1 + \kappa \bar{n}_o + \kappa n_{mo} \right) \frac{\partial Q}{\partial s} + \theta_2 \left( 1 + \kappa \bar{n}_o \right) \frac{\partial Q}{\partial t} = 0. \]

(5.3.1-1)
Which can coalesce into
\[
\left( \frac{\partial}{\partial t} + u_0 \left( 1 - \frac{1}{2} \kappa \tilde{n}_0 \right) \frac{\partial}{\partial s} \right)^2 Q - c_o^2 \left( 1 + \kappa \tilde{n}_0 + \kappa n_{mo} \right) \frac{\partial^2 Q}{\partial s^2} \\
- \theta_1 \left( 1 + \kappa \tilde{n}_0 + \kappa n_{mo} \right) \frac{\partial Q}{\partial s} + \theta_2 \left( 1 + \kappa \tilde{n}_0 \right) \frac{\partial Q}{\partial t} = 0.
\]

(5.3.2-2).

Here the subscript \( o \) indicates that the term is relative to a uniform base flow, \( \kappa \) is assumed constant and \( c_o \) is the celerity of the long wave \( \left( = \sqrt{\frac{g A_o}{B_o}} \right) \).

As the usual convection-diffusion equation has unit coefficient for \( \frac{\partial Q}{\partial t} \), Equation (5.3.2-2) is rewritten as
\[
\frac{\partial Q}{\partial t} - \theta_1 \left( 1 + \kappa n_{mo} \right) \frac{\partial Q}{\partial s} + \frac{\left( 1 - \kappa \tilde{n}_0 \right)}{\theta_2} \left( \frac{\partial}{\partial t} + u_0 \left( 1 - \frac{1}{2} \kappa \tilde{n}_0 + \frac{3}{2} \kappa n_{mo} \right) \frac{\partial}{\partial s} \right)^2 Q \\
- c_o^2 \frac{\left( 1 + \kappa \tilde{n}_0 + \kappa n_{mo} \right)}{\theta_2} \frac{\partial^2 Q}{\partial s^2} = 0.
\]

(5.3.2-3)

The squared term is simplified by assuming that
\[
\frac{\partial}{\partial t} = \frac{\theta_1}{\theta_2} \left( 1 + \kappa n_{mo} \right) \frac{\partial}{\partial s} - \frac{u_0^2}{\theta_2} \left( 1 - 2 \kappa \tilde{n} + 3 \kappa n_{mo} \right) \frac{\partial^2}{\partial s^2} + \frac{C_o^2}{\theta_2} \left( 1 + \kappa n_{mo} \right) \frac{\partial^2}{\partial s^2}.
\]

(5.3.2-4)

This is the author's curvilinear equivalent of the approximation used for the straight channel case described in Section 2.4.2.

Substituting into Equation (5.3.2-3) gives, when the Chézy law describes friction,
\[
\frac{\partial Q}{\partial t} - \frac{\theta_1}{\theta_2}(1 + \kappa n_{mo}) \frac{\partial Q}{\partial s} + \\
+ \left[ \frac{u_o^2}{\theta_2} \left( \frac{A_o P_o'}{2 P_o} \right)^2 \left( 1 - \kappa \tilde{n}_o + 2\kappa n_{mo} \right) - \frac{A_o P_o'}{2 P_o} \left( 1 + \kappa n_{mo} \right) + \frac{3}{4} \left( 1 + \kappa \tilde{n} \right) \right] \frac{c_o^2}{\theta_2} \left( 1 + \kappa n_{mo} \right) \right] \frac{\partial^2 Q}{\partial s^2} \\
+ \frac{2u_o}{\theta_2^2} \left( \frac{A_o P_o'}{2 P_o} \right) \left( 1 - \kappa \tilde{n}_o + \kappa n_{mo} \right) - \frac{1}{2} \left( c_o^2 \left( 1 + \kappa n_{mo} \right) - u_o^2 \left( 1 - 2\kappa \tilde{n}_o + 3\kappa n_{mo} \right) \right) \frac{\partial^3 Q}{\partial s^3} \\
+ \frac{1 - \kappa \tilde{n}_o}{\theta_2} \left[ \frac{c_o^2}{\theta_2} \left( 1 + \kappa n_{mo} \right) - \frac{u_o^2}{\theta_2} \left( 1 - 2\kappa \tilde{n}_o + 3\kappa n_{mo} \right) \right]^2 \frac{\partial^4 Q}{\partial s^4}.
\]

For clarity, the coefficients of \( \frac{\partial Q}{\partial s} \), \( \frac{\partial^2 Q}{\partial s^2} \), and \( \frac{\partial^3 Q}{\partial s^3} \) are considered separately.

**Celerity:** The negative coefficient of \( \frac{\partial Q}{\partial s} \) becomes on substitution for \( \theta_1 \) and \( \theta_2 \),

\[
u_o^2 \left( \frac{3}{2} - \frac{A_o P_o'}{2 P_o} \right) \left( 1 + \kappa n_{mo} \right).
\]

This is the expression for the celerity of the straight channel, which was, derived in Section 3.3.2.1 plus a curvilinear correction.

**Diffusion Coefficient:** Expanding, substituting for \( \theta_2 \), and simplifying gives the diffusion coefficient as,
\[
\left( \frac{u_o}{4} \left[ 1 - \frac{2A_o P'_o}{P_o} + \left( \frac{A_o P'_o}{P_o} \right)^2 - c_o^2 \right] u_o \right) \frac{2g S_o}{2g S_o} \\
\left( \frac{u_o}{4} \left[ \kappa \tilde{n} - \frac{2A_o P_o}{P_o} \kappa n_{\text{mo}} + \left( \frac{A_o P'_o}{P_o} \right)^2 \left( 2\kappa n_{\text{mo}} - \kappa \tilde{n} \right) - c_o^2 \kappa n_{\text{mo}} \right] u_o \right) \frac{2g S_o}{2g S_o} 
\]

This once again demonstrates the expected pattern of the straight channel case along with curvilinear corrections. Here the correction requires both cross-section factors.

**Dispersion Coefficient:** The coefficient of \( \frac{\partial^3 Q}{\partial s^3} \) becomes on substitution \( g_2 \) and simplification,

\[
\frac{Q^3}{4g^2 S_o^2} \left( u_o^3 - c_o^3 \right) \left( 1 - \frac{A_o P'_o}{P_o} \right) \\
+ \frac{Q^3}{4g^2 S_o^2} \left[ \frac{A_o P'_o}{P_o} c_o^2 \left( 2\kappa n_{\text{mo}} - \kappa \tilde{n}_o \right) - \frac{A_o P'_o}{P_o} u_o \left( 3\kappa n_{\text{mo}} - 3\kappa \tilde{n}_o \right) + \frac{u_o}{2} \left( 3\kappa n_{\text{mo}} - 2\kappa \tilde{n}_o \right) - \frac{c_o^2}{2} \kappa n_{\text{mo}} \right]
\]

This shows the now familiar same pattern of straight channel case and curvilinear corrections in terms of \( \kappa \tilde{n}_o \) and \( \kappa n_{\text{mo}} \).
5.4 DYNAMIC MODEL

5.4.1 Dynamic Model Equations

The curvilinear dynamic model to first order is:

\[
\frac{\partial A}{\partial t}(1 - \kappa n_m) + \frac{\partial Q}{\partial s} = q \tag{5.4.1-2}
\]

and

\[
(1 - \kappa \bar{n}) \frac{\partial Q}{\partial t} + \left[ \frac{Q}{A} (2 - 3\kappa(\bar{n} - n_m)) \right] \frac{\partial Q}{\partial s} + \left( \frac{gA}{B} + \frac{Q^2}{A^2} [2\kappa(\bar{n} - n_m) - 1] \right) \frac{\partial A}{\partial s} - \frac{Q^2}{A} (\bar{n} - n_m) \frac{\partial K}{\partial s} = 0. \tag{5.4.1-1}
\]

These are of course, the full de Saint-Venant equations derived in Chapter 4.

5.4.2 Celerity of the Dynamic Wave

The celerity of the dynamic wave has already been shown to be (Section 4.2.5-8),

\[
c = \sqrt{\frac{gA}{B} \left( 1 + \frac{1}{2} \kappa n_m + \frac{1}{2} \kappa \bar{n} \right)} \tag{5.4.2-1}
\]

Appendix 1 describes a hand calculation of the size of dynamic model celerity for three floods of and several cross-sections along the Wairoa River in New Zealand. The calculations demonstrate the effects of cross-section asymmetry and changing curvature.
5.4.3 Irrotational Model

In the course of the work described in this thesis a complete model was developed for the dynamic model for irrotational flow (Nalder, 1995). This is set out in Table 5.4.3. Mass conservation and conservation of momentum are described by the curvilinear de Saint Venant equations in terms of $u_o$ and $\eta_o$ with $\beta$ set equal to $\kappa$. The expression for the celerity of the long wave is that derived previously in Chapter 4.

Expressions for the transverse velocity gradient and the transverse surface gradient are obtained by analysis of the irrotational condition. These derivations are described in detail in Appendix 2.

5.4.4 Model Study

In Chapter 7 the dynamic model is applied to the case of a laboratory scale study of dam breaks. This is not intended to be an exhaustive comparison. It is an initial comparison to see whether the derived equations are reasonable. The laboratory model is a simplification of an actual engineering dam break, but does contain sufficient features to enable an estimate to be made of the model's accuracy. It is found that the dynamic model derived in this thesis provides a reasonable prediction of long wave behaviour.
Mass-Conservation Equation

\[
(1 - \kappa n_m) \frac{\partial \eta}{\partial t} - \kappa n_m \frac{2u_o}{g} \frac{\partial u_o}{\partial t} + u_o (1 + \kappa n_m) \frac{\partial \eta}{\partial s} + \left[ \frac{A}{B} (1 + \kappa \bar{n}) - \kappa n_m \frac{2u_o^2}{g} \right] \frac{\partial u_o}{\partial s} + u_o (S_o + \kappa n_m S_1) + \left( \frac{u_o}{B} \bar{n} - n_m \frac{u_o^3}{g} \right) \frac{\partial \kappa}{\partial s} = q / B.
\]  

(5.4.3-1)

Momentum Equation

\[
\left( \frac{A}{B} - 2\kappa n_m \frac{u_o^2}{g} \right) \frac{\partial u_o}{\partial t} + u_o \frac{\partial \eta}{\partial t} \left[ 2u_o \frac{A}{B} (1 + \kappa \bar{n}) - 2\kappa n_m \frac{u_o^3}{g} \right] \frac{\partial u_o}{\partial s} + u_o^2 (1 + 2\kappa n_m) + \frac{g A}{B} \frac{\partial \eta}{\partial s} + 2u_o^2 \frac{A}{B} \frac{\partial \kappa}{\partial s} - u_o^3 \left( \frac{A}{B} \bar{n} + n_m \frac{u_o^3}{g} \right) \frac{\partial \kappa}{\partial s} + u_o (S_o + 2n_m S_1) + S_f \frac{g A}{B} (1 - \kappa \bar{n}) = u_q q.
\]

(5.4.3-2)

Transverse Velocity Gradient

\[u = \frac{u_o}{1 - \kappa \bar{n}}.\]

(5.4.3-3)

Transverse Surface Gradient

\[\eta = \eta_o + \frac{u_o^2}{2g} - \frac{u^2}{2g}.\]

(5.4.3-4)

Long Wave Celerity

\[c = \sqrt{\left( \frac{Ag}{B} \right) \left( 1 + \frac{1}{2} \kappa \bar{n} + \frac{1}{2} \kappa n_m \right)}.\]

(5.4.3-5)

Table 5.4.3 Irrotational Flow Model
VI LINEARISED CURVILINEAR DE SAINT-VENANT EQUATIONS

6.1 INTRODUCTION

Chapter 3 considered unsteady flow in open channels by linearising the de Saint-Venant equations and considering a general solution of the single equation obtained by combining the mass-conservation and momentum equations. This analysis is repeated here for the curvilinear case.

6.2 LINEARISATION OF CURVILINEAR EQUATIONS

6.2.1 Linearisation of Mass-Conservation Equation

The curvilinear mass-conservation equation without lateral inflow is

\[ \frac{\partial A}{\partial t} (1 - \kappa n_m) + \frac{\partial Q}{\partial s} = 0. \]  \hspace{1cm} (6.2.1-1)

The individual variables are expanded about a steady base flow in terms of a small parameter \( \delta \). The following expansions are used:

\[ A = A_o + \delta A_1 + ..., \]
\[ Q = Q_o + \delta Q_1 + ..., \]
\[ n_m = n_{m0} + \delta n_{m1} + ..., \text{where } n_{m1} = A_1 n'_{m0}, \]
\[ \tilde{n} = \tilde{n}_o + \delta \tilde{n}_1 + ..., \text{where } \tilde{n}_1 = A_1 n'_{o}. \]

(In the expansions for \( \tilde{n} \), and \( n_m \), the prime denoted differentiation with respect to \( A_o \).) As \( \kappa \) is constant for a given cross-section being fixed by topography, it is not linearised.

Substituting into Equation (6.2.1-1) gives

\[ [1 - \kappa (n_{m0} + \delta n_{m1})] \frac{\partial}{\partial t} (A_o + \delta A_1) + \frac{\partial}{\partial s} (Q_o + \delta Q_1) = 0. \]  \hspace{1cm} (6.2.1-2)
Zeroth order

\[(1 - \kappa n_{m0}) \frac{\partial A_o}{\partial t} + \frac{\partial Q_o}{\partial s} = 0. \quad (6.2.1-3)\]

Hence \(\frac{\partial A_o}{\partial t}\), and \(\frac{\partial Q_o}{\partial s}\) must both equal zero which is consistent with a steady base flow.

First order:

\[(1 - \kappa n_{m0}) \frac{\partial A_1}{\partial t} - \kappa n_{m1} \frac{\partial A_m}{\partial t} + \frac{\partial Q_1}{\partial s} = 0. \quad (6.2.1-4)\]

As it has been shown that \(\frac{\partial A_o}{\partial t} = 0\), the mass conservation equation becomes

\[(1 - \kappa n_{m0}) \frac{\partial A_1}{\partial t} + \frac{\partial Q_1}{\partial s} = 0 \quad (6.2.1-5)\]

6.2.2 Linearisation of the Momentum Equation

With no inflow, the momentum equation is,

\[
(1 - \kappa \bar{n}) \frac{\partial Q}{\partial t} + \frac{Q}{A} [2 - 3\kappa (\bar{n} - n_m)] \frac{\partial Q}{\partial s} + \left( \frac{gA}{B} + \frac{Q^2}{A^2} \left[ 2\kappa (\bar{n} - n_m) - 1 \right] \right) \frac{\partial A}{\partial s}
- \frac{Q^2}{A} (\bar{n} - n_m) \frac{\partial \kappa}{\partial s} + S_f gA (1 - \kappa \bar{n}) - gAS_o = 0. \quad (6.2.2-1)
\]

For clarity, each term is linearised separately.
First term:

\[(1-\kappa \bar{n}) \frac{\partial Q}{\partial t} \text{ becomes } (1-\kappa \bar{n}_o) \frac{\partial Q_o}{\partial t} + \delta \left\{ -\kappa \bar{n}_1 \frac{\partial Q_o}{\partial t} + (1-\kappa \bar{n}_o) \frac{\partial Q_1}{\partial t} \right\}.\]

Second term:

\[\frac{Q}{A} \left[ 2 - 3\kappa (\bar{n} - n_m) \right] \frac{\partial Q}{\partial s} \text{ on linearisation becomes,}\]

\[u_o \left[ 2 - 3\kappa (\bar{n}_o - n_{mo}) \right] \frac{\partial Q_o}{\partial s} + \delta \left[ \left( -3\kappa (\bar{n}_1 - n_{m1}) + \left( \frac{Q_1}{Q_o} - \frac{A_1}{A_o} \right) (2 - 3\kappa (\bar{n}_o - n_{mo})) \right) \frac{\partial Q_o}{\partial s} \right] + \left( 2 - 3\kappa (\bar{n}_o - n_{mo}) \right) \frac{\partial Q_1}{\partial s}.\]

Third term:

\[\left( \frac{gA_o}{B_o} + \frac{Q^2}{A^2} [2\kappa(\bar{n} - n_m) - 1] \right) \frac{\partial A}{\partial s} \] is linearised term by term to give,

\[\left( \frac{gA_o}{B_o} + u_o^2 [2\kappa(\bar{n}_o - n_{mo}) - 1] \right) \frac{\partial A_o}{\partial s} + \delta \left( \frac{gA_o}{B_o} + u_o^2 [2\kappa(\bar{n}_o - n_{mo}) - 1] \right) \frac{\partial A_1}{\partial s} + \delta \left( \frac{A_1}{A_o} \right) + 2u_o^2 \left( \kappa (\bar{n}_1 - n_{m1}) + \left( \frac{Q_1}{Q_o} - \frac{A_1}{A_o} \right) (2\kappa (\bar{n}_o - n_{m1}) - 1) \right) \frac{\partial A_o}{\partial s}.\]

Fourth term:

The term \[\frac{Q^2}{A} (\bar{n} - n_m) \frac{\partial \kappa}{\partial s}, \] becomes

\[u_o^2 A_o \left( \bar{n}_o - n_{mo} \right) + \delta \left( \bar{n}_1 - n_{m1} \right) + \left( 2 \frac{Q_1}{Q_o} - \frac{A_1}{A_o} \right) (\bar{n}_o - n_{mo}) \right\} \frac{\partial \kappa}{\partial s}.\]
**Fifth term:**

The friction and bed slope terms are combined to give the term

\[ gA [S_f (1 - \kappa \tilde{n}) - S_o] \]

However, it may be argued that as the friction slope \( S_f \) is at best an estimate, multiplying it by a curvature correction is a specious improvement. For the work considered here, the curvature factor will be ignored.

The friction slope has already been linearised in Section 3.2.3 as,

\[ S_f = S_o \left( 1 + (\mu + \nu) \delta \frac{A_1}{A_o} + \mu \delta \frac{P_1}{P_o} + \nu \delta \frac{Q_1}{Q_o} \right) \]  \hspace{1cm} (6.2.2-2)

where \( \mu = 1 \) and \( \nu = 2 \), when the friction is described by the Chézy equation and \( \mu = \frac{4}{3} \) and \( \nu = 2 \), when it is described by the Manning equation.

This gives the forcing term in the form

\[ gS_o \left( (\mu + \nu) \delta A_1 + \mu \delta \frac{P_1 A_o}{P_o} + \nu \delta \frac{Q_1}{Q_o} \right) \]

which can be further simplified as \( \theta_1 A_1 + \theta_2 Q_1 \), where, as in Section 3.2.3,

\[ \theta_1 = gS_o \left( \mu A_o \frac{P'_o}{P_o} - (\mu + \nu) \right) \]

\[ \theta_2 = gS_o \nu \frac{A_o}{Q_o} \]

\[ P_1 = A_o P'_o \]

After these substitutions the momentum equation is obtained to zeroth order and then to first order.
Zeroth order:

The zero order momentum equation is:

\[
(1 - \kappa \bar{n}_o) \frac{\partial Q_0}{\partial t} + u_o \left[ 2 - 3\kappa (\bar{n}_o - n_{mo}) \right] \frac{\partial Q_o}{\partial s} + \left( \frac{g A_o}{B_o} + u_o^2 [2\kappa (\bar{n}_o - n_{mo}) - 1] \right) \frac{\partial A_o}{\partial s} + u_o^2 A_o (\bar{n}_o - n_{mo}) \frac{\partial \kappa}{\partial s} = 0.
\]

(6.2.2-3)

It has already been shown that \( \frac{\partial Q_0}{\partial s} = 0 \). However, \( \frac{\partial Q_o}{\partial t} \) is also zero because the base flow is assumed to be steady. Substituting for these values gives the zeroth order momentum equation as:

\[
\left( \frac{g A_o}{B_o} + u_o^2 [2\kappa (\bar{n}_o - n_{mo}) - 1] \right) \frac{\partial A_o}{\partial s} + u_o^2 A_o (\bar{n}_o - n_{mo}) \frac{\partial \kappa}{\partial s} = 0.
\]

(6.2.2-4)

(The assumption of steady flow does not imply that \( \frac{\partial A_o}{\partial s} = 0 \).)

First order:

Assembling the first order components of the momentum equation and substituting for \( \frac{\partial Q_0}{\partial s} \), \( \frac{\partial Q_o}{\partial t} \), and \( \frac{\partial A_o}{\partial t} \) gives:

\[
(1 - \kappa \bar{n}_o) \frac{\partial Q_1}{\partial t} + u_o \left[ 2 - 3\kappa (\bar{n}_o - n_{mo}) \right] \frac{\partial Q_1}{\partial s} + \left( \frac{g A_o}{B_o} + u_o^2 [2\kappa (\bar{n}_o - n_{mo}) - 1] \right) \frac{\partial A_1}{\partial s} + \left[ \frac{A_i}{A_o} \right] \frac{\partial Q_i}{\partial s} + \left( \frac{Q_i}{Q_o} - \frac{A_i}{A_o} \right) \left( 2\kappa (\bar{n}_1 - n_{m1}) - 1 \right) \frac{\partial A_i}{\partial s} + u_o^2 A_o \left( \bar{n}_1 - n_{m1} \right) + \left( 2 \frac{Q_i}{Q_o} - \frac{A_i}{A_o} \right) (\bar{n}_o - n_{mo}) \frac{\partial \kappa}{\partial s} + \theta_1 A_i + \theta_2 Q_i = 0.
\]

(6.2.2-5)
Rearranging Equation (6.2.2-3) gives the following expression for $\partial A_1/\partial s$ as a function of $\partial \kappa/\partial s$:

$$\frac{\partial A_1}{\partial s} = -\frac{u_o^2 A_o^2 (\bar{n}_o - n_{mo})}{\frac{g A_o}{B_o} + u_o^2 [2 \kappa (\bar{n}_o - n_{mo}) - 1]} \frac{\partial \kappa}{\partial s}. \tag{6.2.2-6}$$

When this substitution is made, a complicated first order momentum equation appears i.e.,

$$\left(1 - \kappa \bar{n}_o \right) \frac{\partial Q_1}{\partial t} + u_o \left(2 - 3 \kappa (\bar{n}_o - n_{mo}) \right) \frac{\partial Q_1}{\partial s}$$

$$+ \left(\frac{g A_o}{B_o} + u_o^2 [2 \kappa (\bar{n}_o - n_{mo}) - 1] \right) \frac{\partial A_1}{\partial s} + C \frac{\partial \kappa}{\partial s} + \theta_1 A_1 + \theta_2 Q_1 = 0,$$

where

$$C =$$

$$\left[ \left( \frac{A_1}{A_o} - \frac{B_1}{B_o} \right) + 2 u_o^3 \left[ \kappa (\bar{n}_1 - n_{m1}) + \left( \frac{Q_1}{Q_o} - \frac{A_1}{A_o} \right) (2 \kappa (\bar{n}_1 - n_{m1}) - 1) \right] \right]$$

$$- \frac{u_o^2 A_o^2 (\bar{n}_o - n_{mo})}{\frac{g A_o}{B_o} + u_o^2 [2 \kappa (\bar{n}_o - n_{mo}) - 1]}$$

$$+ u_o^2 A_o \left[ (\bar{n}_1 - n_{m1}) + \left( 2 \frac{Q_1}{Q_o} - \frac{A_1}{A_o} \right) (\bar{n}_o - n_{mo}) \right]. \tag{6.2.2-7}$$

However as is shown below (Section 6.3), the remainder of the analysis can be carried out using the momentum equation with the simple coefficient $C$.

**6.3 COMBINED EQUATION.**

As the analysis in Section 6.2 has removed the derivatives of $Q_o$ and $A_o$ from the first order equation, mass-conservation and momentum equations, they can be
rewritten in terms of \( Q (=Q_1) \) and \( A (=A_1) \). The momentum equation can also be simplified by setting \( c_o^2 = \frac{gA_0}{B_0} \). The equations are now:

**Mass-Conservation Equation:**

\[
(1 - \kappa n_{mo}) \frac{\partial A}{\partial t} + \frac{\partial Q}{\partial s} = 0 \quad (6.3-1)
\]

**Momentum Equation:**

\[
(1 - \kappa \bar{n}_o) \frac{\partial Q}{\partial t} + u_o (2 - 3\kappa(\bar{n}_o - n_{mo})) \frac{\partial Q}{\partial s} \\
+ \left(c_o^2 + u_o^2 [2\kappa(\bar{n}_o - n_{mo}) - 1]\right) \frac{\partial^2 A}{\partial s \partial t} + C \frac{\partial \kappa}{\partial s} + \theta_1 A + \theta_2 Q = 0. \quad (6.3-2)
\]

Again the equations are combined by cross-differentiating and substituting.

Differentiating Equation (6.3-1) with respect to \( s \) gives:

\[
(1 - \kappa n_{mo}) \frac{\partial^2 A}{\partial s \partial t} + \frac{\partial^2 Q}{\partial s^2} = 0, \quad (6.3-3)
\]

while differentiating Equation (6.3-2) with respect to \( t \) gives:

\[
(1 - \kappa \bar{n}_o) \frac{\partial^2 Q}{\partial t^2} + u_o (2 - 3\kappa(\bar{n}_o - n_{mo})) \frac{\partial^2 Q}{\partial t \partial s} \\
+ \left(c_o^2 + u_o^2 [2\kappa(\bar{n}_o - n_{mo}) - 1]\right) \frac{\partial^2 A}{\partial t \partial s} + \theta_1 \frac{\partial A}{\partial t} + \theta_2 \frac{\partial Q}{\partial t} = 0. \quad (6.3-4)
\]

The \( C \frac{\partial \kappa}{\partial s} \) factor has vanished from the momentum equation. Within the time scale of the work in this thesis, the pattern of meanders is not expected to change. Hence a derivative with respect to \( t \) of \( \frac{\partial \kappa}{\partial s} \) is zero.
Substituting for $\partial^2 A/\partial s \partial t$ from Equation (6.3-3) and $\partial A/\partial t$ from Equation (6.3-1) gives the momentum equation in terms of $Q$ only, i.e.

$$(1 - \kappa \bar{n}_o) \frac{\partial^2 Q}{\partial t^2} + u_o \left(2 - 3\kappa(\bar{n}_o - n_{mo})\right) \frac{\partial^2 Q}{\partial t \partial s} - u_o^2 \left(2\kappa \bar{n}_o - 3\kappa n_{mo} - 1\right) \frac{\partial^2 Q}{\partial s^2}$$

$$- c_o^2 \left(1 + \kappa n_{mo}\right) \frac{\partial^2 Q}{\partial s^2} - \theta_1 (1 + \kappa n_{mo}) \frac{\partial Q}{\partial s} + \theta_2 \frac{\partial Q}{\partial t} = 0.$$  

(6.3-5)

Rearranging the equation to give a unit coefficient for $\frac{\partial^2 Q}{\partial t^2}$ gives the combined equation as:

$$\frac{\partial^2 Q}{\partial t^2} + 2u_o \left(1 - \frac{1}{2} \kappa \bar{n}_o + \frac{3}{2} \kappa n_{mo}\right) \frac{\partial^2 Q}{\partial t \partial s} + u_o^2 \left(1 - \kappa \bar{n}_o + 3\kappa n_{mo}\right) \frac{\partial^2 Q}{\partial s^2}$$

$$- c_o^2 \left(1 + \kappa \bar{n}_o + \kappa n_{mo}\right) \frac{\partial^2 Q}{\partial s^2} - \theta_1 (1 + \kappa \bar{n}_o + \kappa n_{mo}) \frac{\partial Q}{\partial s} + \theta_2 (1 + \kappa \bar{n}_o) \frac{\partial Q}{\partial t} = 0.$$  

(6.3-6)

This equation coalesces into

$$\left(\frac{\partial}{\partial t} + u_o \left(1 - \frac{1}{2} \kappa \bar{n}_o + \frac{3}{2} \kappa n_{mo}\right) \frac{\partial}{\partial s}\right)^2 Q - c_o^2 \left(1 + \kappa \bar{n}_o + \kappa n_{mo}\right) \frac{\partial^2 Q}{\partial s^2}$$

$$- \theta_1 (1 + \kappa \bar{n}_o + \kappa n_{mo}) \frac{\partial Q}{\partial s} + \theta_2 (1 + \kappa \bar{n}_o) \frac{\partial Q}{\partial t} = 0.$$  

(6.3-7)

which is the curvilinear equivalent of Equation (3.2.4-4).
6.4 GENERAL SOLUTION OF THE COMBINED EQUATION

6.4.1 General Solution

The general solution is again assumed to be of the form

\[ Q(s,t) = Q_0 \exp(i \varepsilon s + \mu t), \tag{6.4.1-1} \]

where \( \varepsilon \) is the wave number, used instead of \( k \) to avoid confusion with \( \kappa \), the curvature. To substitute into the linearised momentum equation the following derivatives are required:

\[ \frac{\partial Q}{\partial t} = \mu Q_0 e^{i(\varepsilon s + \mu t)}, \quad \text{and} \quad \frac{\partial^2 Q}{\partial t^2} = \mu^2 Q_0 e^{i(\varepsilon s + \mu t)}, \]

\[ \dot{Q}/\dot{s} = i \varepsilon Q_0 e^{i(\varepsilon s + \mu t)}, \quad \text{and} \quad \ddot{Q}/\ddot{s} = -\varepsilon^2 Q_0 e^{i(\varepsilon s + \mu t)}, \]

and

\[ \frac{\partial^2 Q}{\partial s \partial t} = i \varepsilon \mu Q_0 e^{i(\varepsilon s + \mu t)}. \]

Substituting these expressions into Equation (6.3-6) and dividing through by \( Q_0 e^{i(\varepsilon s + \mu t)} \) results in a quadratic in \( \mu \):

\[
\mu^2 + \left[ 2u_o i \varepsilon \left( 1 - \frac{1}{2} \kappa \bar{n}_o + \frac{3}{2} \kappa n_{mo} \right) + \theta_2 \left( 1 + \kappa \bar{n}_o \right) \right] \mu + e_o^2 \left( 1 + \kappa \bar{n}_o + \kappa n_{mo} \right) + \e^2 = 0. \tag{6.4.1-2}
\]

For ease of calculation the quadratic is rewritten with simplified coefficients, i.e.

\[
\mu^2 + \left[ d i \varepsilon + f \right] \mu + g e^2 + j i e = 0, \tag{6.4.1-3}
\]

where,
\[ d = 2u_o \left(1 - \frac{1}{2} \kappa n_o + \frac{3}{2} \kappa n_{mo}\right), \]
\[ f = \theta_2(1 + \kappa n_o), \]
\[ g = c_o^2(1 + \kappa n_o + \kappa n_{mo}) - u_o^2(1 - \kappa n_o + 3\kappa n_{mo}) \text{ and} \]
\[ j = -\theta_1(1 + \kappa n_o + \kappa n_{mo}). \]

This quadratic is solved using the standard formula with the square root evaluated using the binomial theorem. As the discussion in Chapter 3 has already demonstrated that the wave has dispersive effects, the binomial expansion is truncated at \( e^3 \). This is sufficient to enable comparison with the expressions for celerity and diffusion coefficient derived in Section 3.4.3.

Solving Equation (6.4.1-3) for \( \mu \) gives

\[ \mu = -\frac{d}{2} i \epsilon - \frac{f}{2} \]

\[ \pm \left[ \frac{f}{2} + \frac{(2df - 4j)}{4f} i \epsilon + \left[ \frac{(df - 2j)^3}{4f^3} - \frac{(d^2 + 4g)}{4f} \right] \epsilon^2 \right] \]

\[ + \left[ \frac{(df - 2j) (d^2 + 4g)}{4f^3} - \frac{(df - 2j)^3}{4f^5} \right] \epsilon^3 \]

(6.4.1-4)

Substituting back into the general solution and rearranging gives,

\[ Q = Q_o \exp \left[ \left( \frac{f}{2} t + \epsilon^2 \left[ \frac{(df - 2j)^3}{4f^3} - \frac{(d^2 + 4g)}{4f^5} \right] t \right) \pm \left[ \frac{d - 2j}{2f} + \epsilon^3 \left[ \frac{(df - 2j)(d^2 + 4g)}{4f^3} - \frac{(df - 2j)^3}{4f^5} \right] t \right) \right]. \]

(6.4.1-5)
6.4.2 Advection and Diffusion Coefficient.

It was shown in Chapter 2 that the general solution can also be written in form, which displays dispersive and diffusive behaviour i.e.,

\[ Q = Q_0 \exp \left\{ i \epsilon (s - ct) - De^2 r \right\}, \quad (6.4.2-1) \]

where,

- \( c \) is the advection velocity, and
- \( D \) is the diffusion coefficient.

**Celerity:**

Comparing Equations (6.4.1-5) and (6.4.2-1) shows that the advection is

\[ c = \frac{d}{2} \pm \frac{df - 2j}{4f} + e^2 \left( \frac{(df - 2j)(a^2 + 4g)}{4f^3} - \left( \frac{df - 2j}{4f^5} \right)^3 \right). \]

This again shows that there are two waves associated with this long wave model.

Substituting for \( d, f \) and \( j \) gives the curvilinear advection as

\[ c = u_o \left( 1 - \frac{1}{2} \kappa \bar{n}_o + \frac{3}{2} \kappa n_m \right) \]

\[ \pm u_o \left[ \begin{array}{c}
\frac{2c_o^2 u_o}{\theta_2^2} \left( 1 - \frac{3}{2} \kappa \bar{n}_o + \frac{5}{2} \kappa n_m \right) + \frac{\theta_1}{\theta_2} (1 - \kappa n_m) \\
- \frac{2c_o^2 u_o}{\theta_2^2} \left( 1 + \frac{3}{2} \kappa \bar{n}_o + \frac{9}{2} \kappa n_m \right) - \frac{6u_o}{\theta_2^3} (1 + 2 \kappa \bar{n}_o + 4 \kappa n_m) \\
- \frac{6u_o}{\theta_2^3} (1 + \frac{5}{2} \kappa \bar{n}_o + \frac{7}{2} \kappa n_m) - \frac{2\theta_1^3}{\theta_2^5} (1 + 3 \kappa \bar{n}_o + 3 \kappa n_m) \\
\end{array} \right] \quad (6.4.2-2). \]
This can be compared with the straight channel case in \( u_o, \theta_1 \) and \( \theta_2 \) by substituting for \( V \) in Equation (3.4.3-3) and truncating at \( k^2 \). The truncated Equation (3.4.3-2), is

\[
c = u_o \pm \left[ V - \frac{2(c_o^2 V - V^3)}{\theta_1^2} \right],
\]

which on substituting becomes,

\[
c = u_o \pm \left[ u_o + \frac{\theta_1}{\theta_2} - k^2 \left( \frac{2c_o^2 u_o}{\theta_2^2} - \frac{2c_o^2 \theta_1}{\theta_2^3} - \frac{2\theta_1^3}{\theta_2^4} - \frac{6\theta_1^2}{\theta_2^5} + \frac{6\theta_1}{\theta_2^6} + \frac{2u_o^3}{\theta_2} \right) \right].
\]

Comparing Equations (6.4.2-2) and (6.4.2-3) demonstrates that the former has the now familiar pattern of straight channel case plus curvilinear correction.

**Diffusion Coefficient**

Again there are two coefficients corresponding to each of the wave. When the positive square root is chosen, the value for \( D \) is straightforward,

\[
D = \frac{(df - 2j)^2}{4f^3} - \frac{(d^2 + 4g)}{4f}
\]

which on substitution becomes;

\[
D = \frac{u_o^2}{\theta_2^2} \left( 1 - 2\kappa \tilde{n}_o + 3\kappa n_{mo} \right) + \frac{2u_o \theta_1}{\theta_2^3} \left( 1 - \frac{3}{2} \kappa \tilde{n}_o + \frac{5}{2} \kappa n_{mo} \right) + \frac{\theta_1^2}{\theta_2^4} \left( 1 - \kappa \tilde{n}_o + 2\kappa n_{mo} \right)
\]

\[
- \frac{c_o^2}{\theta_2} \left( 1 - 4\kappa \tilde{n}_o + \kappa n_{mo} \right).
\]

(6.4.2-3)
The equivalent expression for the straight channel case is $\frac{(c_o^2 - V^2)}{\theta_2}$, which on substituting for $V$ gives,

$$D = \frac{u_o^2}{\theta_2} + \frac{2u_o \theta_1}{\theta_2} + \frac{\theta_2^2 - c_o^2}{\theta_2}. \quad (6.4.2-4)$$

Both Equations (6.4.2-3) and (6.4.2-4) are of the same form.

When the negative square root is chosen the resulting version of Equation (6.4.1-5) contains the factor 'f'. This is equivalent to the $\theta_2$ of the straight channel case, but in the spirit of this work it is no surprise to discover that it has the form of the straight channel expression along with a correction for curvature i.e. $f(= \theta_2 (1 + \kappa \eta_o))$.

If $f$ is regarded as some frictional effect on the wave, then the diffusion coefficient is,

$$D = -\frac{(df - 2j)^2}{4f^3} + \frac{(d^2 + 4g)}{4f}. \quad (6.4.2-5)$$

If on the other hand $f$ is regarded as $s$ function of $e^2$ then the diffusion coefficient is

$$D = -fe^{-2} - \frac{(df - 2j)^2}{4f^3} + \frac{(d^2 + 4g)}{4f}. \quad (6.4.2-6)$$
6.5 CONCLUSION

Chapters 5 and 6 detail a complete description of long wave models in curved open channels. As the model equations have been presented in terms of discharge and cross-sectional area they are applicable to all types of curved channels including natural channels with their changing shapes and curvatures.

In each case the model can be described in terms of the straight channel case plus a correction for curvilinearity. The engineer is then able to make an estimate of the size of the correction for a particular channel or watercourse and decide whether that correction is sufficiently large to warrant inclusion in the design calculations.

The shortcomings of the conventional derivation of the diffusion model are still apparent in the curvilinear version and the engineer is advised to use the diffusion model by solving the basic equations.
VII SOLUTION OF DYNAMIC MODEL FOR RECTANGULAR CHANNEL

7.1 INTRODUCTION.

In this chapter the dynamic model developed in Chapter 3 is applied to an open channel, with prismatic rectangular cross-section, constant curvature and no lateral inflow. The solutions are presented in the form of 'Matlab' routines.

This simple case is reported in the paper Experimental Results of Two-Dimensional Dam-Break Flow by Bell, Elliot and Chaudhry, published in Journal of Hydraulics Research Volume 30, 1992, No.2, pages 225-252. This experiment was carried out to validate the two-dimensional model of Dammuller et al., (1989) and thus can only be applied to a one-dimensional model to a limited extent.

It should also be emphasised that this is a very simple case. The dynamic model in a natural channel must allow for variations in curvature and variations in cross-sectional shape usually expressed in terms of $\bar{n}$ and $n$. These in turn will vary for a given cross-section when the water level rises and falls. Also in a natural channel, the cross-sections of interest are unlikely to be evenly spaced.

7.2 THE LABORATORY STUDY

The channel-reservoir arrangement is shown in Figure 7.2.1-1, which reproduces the figure from the paper. The curved channel was 0.3m wide and 0.41m deep with a centreline radius of 1.07m.
Figure 7.2.1-1 Laboratory Dam-Break Model (Ref. Bell et al.)
Positive waves down the channel and negative waves in the upstream reservoir were generated by a dam-break mechanism, which instantaneously lifted a plate separating the reservoir from the channel. Details of this are given in the paper.

Tests were conducted with smooth and rough bed. The smooth steel plate channel was roughened with a layer of steel wire mesh held down by wooden battens attached at 80 mm intervals (centreline distance). The Manning roughness coefficient $n$ was determined by trial and error. A steady state flow was set up and the measured water surface profile compared with that predicted by an equation describing a quasi-steady-state flow i.e.,

$$\frac{dy}{dx} = \frac{S_o - S_f}{1 - \frac{Q^2B}{gA^3}}$$

where $\frac{dy}{dx}$ is the rate of change of depth along the channel length and the other terms are as previously defined. The calculated values were $n = 0.0165$ for the smooth channel and 0.040 for the rough channel. In practice, there was a small variation in $n$ with respect to discharge over the range of flows considered. For the work described in this chapter, the value of $n$ was assumed to be fixed at the quoted values.

Capacitance probes in the reservoir monitored the negative wave and water profiles were recorded in the straight and curved sections of the channel, by video camera. Again details are to be found in the paper.

For both roughnesses, the reservoir water level was varied from 0.35 m to 0.2 m in 0.05 m steps. For each reservoir level, tests were conducted with stationary water
channel depths of 0.0, 0.13, 0.025, 0.051, and 0.076 m for the smooth channel case, and 0.0, 0.025, 0.051, and 0.076 m for the rough channel case. The paper contains results for the two pairs of tests are shown in Table 7.2.2-1.

Although the paper presented the results for four studies, this study was confined to the two smooth channels, Test 2 and Test 21.

7.3 THE MATHEMATICAL MODEL

7.3.1 Basic Mathematical Model

The design of the experimental setup creates certain requirements for the basic mathematical model. The salient factors are

- two separate sections with constant curvature so that curvature has to be specified at each computational point,
- two points where curvature changes, so that \( \frac{\partial k}{\partial s} \) has to be specified at all points, even if the value is 0,
- there is no lateral inflow so that \( q=0 \),
- the reference line is placed at the mid-point of a symmetric cross-section, thus setting \( n_m =0 \),
- the channel width \( B \) is constant as the channel is prismatic,
- as the rig was set horizontally, \( S_o =0 \).
Incorporating these factors gives the following pair of equations;

\[
\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial s} = 0, \quad (7.3.1-1)
\]

\[
(1 - \kappa \bar{n}) \frac{\partial Q}{\partial t} + \frac{Q}{A} (2 - 3 \kappa \bar{n}) \frac{\partial Q}{\partial s} + \left[ \frac{gA}{B} + \frac{Q^2}{A^2} (2\kappa \bar{n} - 1) \right] \frac{\partial A}{\partial s} - \frac{Q^2}{A} \bar{n} \frac{\partial \kappa}{\partial s} + gAS_f (1 - \kappa \bar{n}) = 0. \quad (7.3.1-2)
\]

As discharge changes, so do both cross-sectional area and velocity, i.e. the water level rises and falls and its slope changes. For a general channel, the shape of the cross-section and the superelevation governs the value of \( \bar{n} \).

However, for a symmetric channel, \( \bar{n} \) is controlled only by the super-elevation and \( \kappa \bar{n} \) is thus, in effect, a second order effect. As the preceding analysis has dispensed with second order terms, this suggests that for a symmetric channel the curvature effects can be ignored. This is not necessarily so. The following section uses the data reported by Bell et.al. to demonstrate that although \( \kappa \bar{n} \) is of second order, its value is sufficiently greater than those of the deleted second order terms that it is more accurate to keep it in the equation than to leave it out.

<table>
<thead>
<tr>
<th>Test No.</th>
<th>Reservoir Level</th>
<th>Initial Water Channel Depth</th>
<th>Value ( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.305 m</td>
<td>0.0</td>
<td>0.0165</td>
</tr>
<tr>
<td>2n</td>
<td>0.305 m</td>
<td>0.0</td>
<td>0.040</td>
</tr>
<tr>
<td>21</td>
<td>0.305 m</td>
<td>0.0762 m</td>
<td>0.0165</td>
</tr>
<tr>
<td>21n</td>
<td>0.305 m</td>
<td>0.0762 m</td>
<td>0.040</td>
</tr>
</tbody>
</table>

Table 7.2.1-1 Details of Laboratory Study
7.3.2 Second Order Terms for the Laboratory Study

In the course of deriving the curvilinear de Saint Venant equations the following factors were assumed to be negligible; \( \kappa^2 \), \( \beta \kappa \) (from the definition of the general linear velocity distribution), and \( 2 \frac{u_o \kappa^2 n}{g} \) (in Section 3.1.3.1). As \( \beta \) is typically of the same order as \( \kappa \), then the relevant factors are \( \kappa^2 \) and \( 2 \frac{u_o \kappa^2 n}{g} \).

Calculation of \( \bar{n} \) for prismatic channel.

The moment of area for the cross-section is obtained by summing those for each of the four subsections of Figure 7.3.2-1. The total is divided by area to given an expression for \( \bar{n} \).

- Moment of area (a) \( \frac{(d_1B^2)}{8} \)
- Moment of area (b) \( -(d_1 + y)B^2/8 \)
- Moment of area (c) \( yB^2/24 \)
- Moment of area (d) \( -\frac{yB^2}{12} \)

Hence the total moment of area is \( -4yB^2/24 \), so that \( \bar{n} \) is \( -4yB^2/24A \). (The negative sign indicates that the centre of mass is to the right of the reference line.)

A photo in the paper by Bell et.al. shows an example with the following data

\[
d_1 = 102 \text{ mm}, \\
d_1 + 2y = 138 \text{ mm}, \\
B = 305 \text{ mm}, \\
\text{Radius of curvature} = 1070 \text{ mm}.
\]
This corresponds to a curvature of $9.3 \times 10^{-4}$ mm$^{-1}$ and a $\kappa^2$ value of $8.7 \times 10^{-7}$ mm$^{-2}$.

As the tangent of the angle of superelevation has been shown in section 3.1.3.1 to be equal to $\frac{u_0^2 \kappa}{g}$, the value of $u_0$ is 1115.7 mm s$^{-1}$. The maximum value of $2 \frac{u_0 \kappa^2 n}{g}$ occurs when $n$ is 152.5 mm. It is $3.0 \times 10^5$. Substituting into the derived formula for $n$ gives a value of 13.07 mm so that $\kappa n$ is $1215.6 \times 10^{-5}$.

For this particular experiment, the value for $\kappa n$ has been shown to be 400 times the value of the largest negligible term. It is also sufficiently large to require the retention of the $\kappa n$ term in the mathematical model.

7.3.3 Discretising the Equations

Solving the equations using finite differences is easier if the momentum equation has unit coefficient for $\frac{\partial Q}{\partial t}$. This form of the momentum equation is

$$\frac{\partial Q}{\partial t} + \frac{Q}{A} (2 - \kappa \bar{n}) \frac{\partial Q}{\partial s} + \left[ \frac{gA}{B} (1 + \kappa \bar{n}) - \frac{Q^2}{A^2} (1 - \kappa \bar{n}) \right] \frac{\partial A}{\partial s} - \frac{Q^2}{A} \bar{n} \frac{\partial \kappa}{\partial s} + gAS_f = 0.$$  

(7.3.3-1)

As it has been shown that $\bar{n} = -\frac{4yB^2}{24A}$, then as $y = \frac{u_0^2 \kappa B \kappa}{2g}$, $\bar{n}$ can be set equal to $-\kappa B^3 Q^2 \frac{Q^2}{12g A^3}$, or $-\kappa R_i \frac{Q^2}{A^3}$. where, $R_i = \frac{B^3}{12g}$. The equations then become,
\[
\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial s} = 0, \quad (7.3.3-2)
\]
\[
\frac{\partial Q}{\partial t} + \frac{Q}{A} \left( 2 + \kappa^2 R_1 \frac{Q^2}{A^3} \right) \frac{\partial Q}{\partial s} + \left[ \frac{gA}{B} \left( 1 - \kappa^2 R_1 \frac{Q^2}{A^3} \right) - \frac{Q^2}{A^2} \left( 1 + \kappa^2 R_1 \frac{Q^2}{A^3} \right) \right] \frac{\partial A}{\partial s} - \kappa R_4 \frac{Q^4}{A^4} \frac{\partial \kappa}{\partial s} + gA S_f = 0.
\]
\[
(7.3.3-3)
\]

The model is further simplified to
\[
\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial s} = 0, \quad (7.3.3-4)
\]
\[
\frac{\partial Q}{\partial t} + \frac{Q}{A} \left( 2 + R_4 \frac{Q^2}{A^3} \right) \frac{\partial Q}{\partial s} + \left[ R_4 A \left( 1 - R_4 \frac{Q^2}{A^3} \right) - \frac{Q^2}{A^2} \left( 1 + R_4 \frac{Q^2}{A^3} \right) \right] \frac{\partial A}{\partial s} - R_5 \frac{Q^4}{A^4} + R_3 A = 0.
\]
\[
(7.3.3-5)
\]

\[
R_2 = \frac{g}{B},
\]
\[
R_3 = g S_f,
\]
where,
\[
R_4 = \kappa^2 R_1,
\]
\[
R_5 = \kappa \frac{\partial \kappa}{\partial s} R_1.
\]

As \( \frac{\partial \kappa}{\partial s} \) is known for the channel (and will be obtained by surveying for a natural channel) its value can be drawn into a constant. (Its value will be zero except where it abruptly changes.)

These equations are solved using finite differences. Points in time are spaced ‘k’ apart and points in distance are spaced ‘h’ apart. The momentum equation has the added complications of \( A \) and \( (Q/A) \) outside the partial derivative. This is dealt
with by using a right differencing scheme (i.e. $\Delta Q$ is $Q_i - Q_{i-1}$, where $i$ is the step in space) and then moving the whole scheme one step backwards. The step size was adjusted until sufficient accuracy was obtained.

When the dynamic model was presented in Chapter 2, a section was devoted to describing the Preissmann scheme to solve the dynamic model. This scheme is not being used in this case, as there is no downstream boundary condition.

**Mass Conservation:** Discretising the momentum equation gives,

$$\frac{A_{i,j+1} - A_{i,j}}{k} + \frac{Q_{i,j} - Q_{i-1,j}}{h} = 0$$

so that,

$$h[A_{i,j+1} - A_{i,j}] + k[Q_{i,j} - Q_{i-1,j}] = 0,$$

and

$$A_{i,j+1} = \frac{k}{h}[Q_{i-1,j} - Q_{i,j}] + A_{i,j}.$$

If $j$ becomes $(j-1)$, the discretisation becomes

$$A_{i,j} = \frac{k}{h}[Q_{i-1,j-1} - Q_{i,j-1}] + A_{i,j-1}. \quad (7.3.3-1)$$

**Momentum Equation:** This becomes

$$\left[ \frac{Q_{i,j+1} - Q_{i,j}}{k} \right] + \frac{Q_{i,j}}{A_{i,j}} \left( 2 + R_4 \frac{Q_{i,j}^2}{A_{i,j}^3} \right) \left[ \frac{Q_{i,j} - Q_{i-1,j}}{h} \right]$$

$$+ \left[ R_2 A_{i,j} \left( 1 - R_4 \frac{Q_{i,j}^2}{A_{i,j}^3} \right) - R_4 \frac{Q_{i,j}^2}{A_{i,j}^3} \left( 1 + R_4 \frac{Q_{i,j}^2}{A_{i,j}^3} \right) \right] - R_3 \frac{Q_{i,j}^3}{A_{i,j}^5} + R_3 A_{i,j} = 0$$
which can be rearranged to give,

\[ Q_{i,j+1} = Q_{i,j} - k \frac{Q_{i,j}}{A_{i,j}} \left( 2 + R_4 \frac{Q_{i,j}^2}{A_{i,j}^3} \right) \frac{Q_{i,j} - Q_{i-1,j}}{h} \]

\[ -k \left[ R_2 A_{i,j} \left( 1 - R_4 \frac{Q_{i,j}^2}{A_{i,j}^3} \right) - \frac{Q_{i,j}^2}{A_{i,j}^3} \left( 1 + R_4 \frac{Q_{i,j}^2}{A_{i,j}^3} \right) \right] \frac{A_{i,j} - A_{i-1,j}}{h} \]

\[ + k R_3 A_{i,j}^4 - k R_3 A_{i,j} = 0 \]

Then allowing j to be replaced by (j-1) gives;

\[ Q_{i,j} = Q_{i,j-1} - k \frac{Q_{i,j-1}}{h} \frac{Q_{i,j-1}}{A_{i,j-1}} \left( 2 + R_4 \frac{Q_{i,j-1}^2}{A_{i,j-1}^3} \right) \frac{Q_{i,j-1} - Q_{i-1,j-1}}{h} \]

\[ -k \left[ R_2 A_{i,j-1} \left( 1 - R_4 \frac{Q_{i,j-1}^2}{A_{i,j-1}^3} \right) - \frac{Q_{i,j-1}^2}{A_{i,j-1}^3} \left( 1 + R_4 \frac{Q_{i,j-1}^2}{A_{i,j-1}^3} \right) \right] \frac{A_{i,j-1} - A_{i-1,j-1}}{h} \]

\[ + k R_3 A_{i,j-1}^4 - k R_3 A_{i,j-1} = 0 \]

(7.3.3-2)

7.3.4 Friction Slope Calculation

Bell et al. describe the roughness of the channel in terms of Manning’s n whilst the model requires the friction slope Sf. Figure 7 of the paper contains two pair of measured and computed water surface profiles along with the corresponding steady flow and the calculated n. Taking the water depth at mid-distance along the channel as typical, the friction slope is calculated for each set of profiles by substituting into the Manning equation in the form
\[ Q = \frac{1}{A} \left( \frac{A}{n} \right)^{\frac{2}{3}} S_f \]  

(7.3.4-1)

The higher flow case where the discharge is 0.0165 m³ s⁻¹, the depth 0.156 m and \( n = 0.017 \) gives an \( S_f \) value of 0.0011. The other measurement was for a lower flow of 0.0095 m³ s⁻¹ with depth 0.09 m and \( n = 0.016 \) which gave an \( S_f = 0.0008 \). An approximate value of 0.0009 was used in the simulation.

Initially it had been hoped that the two cases for the rough channel could be modelled, but it was found that the mathematical scheme chosen is very sensitive to the \( S_f \) factor. When the higher value for friction slope was used, the calculation became unstable and consequently was abandoned. This problem is referred to in Chapter 8.

7.4 TEST 2.

7.4.1 Defining the Input Function

Bell et al.'s Figure 16 contains plots of depth vs time for Stations 1, 2, 4, 6 and 8 for Test 2. Ideally the input would describe the wave as it came from the model dam break. Unfortunately this is very difficult to establish, so for this study, the wave front measured at Station 1 is taken to be the input wave. (Figure 7.4.1-1). The simulation requires an expression for discharge as a function of time (to establish the boundary condition \( Q(0,t) \)) and for cross-sectional area as a function of time (to establish the boundary condition \( A(0,t) \)).

7.4.2 Mathematical Expressions for the Input Function

This is established as follows. A table of values of depth vs time is obtained from the Station 1 plot. As the width of the channel is known, a table of areas vs time is then calculated. The speed of the wave in the straight channel is
Figure 7.4.1-1 Input Wave Station 1, TEST 2
estimated by comparing the times at which the wave is detected at Station 1 and then at Station 2. Multiplying each cross-sectional area by velocity gives a table of discharge against time. This data is reproduced in Appendix 3.

**Discharge Input:** The input data are plotted on Figure 7.4.2-1. The shape indicates that they probably represent a flow of the form

\[ Q = Q_o - b \exp(ct), \]

where

- \( Q \) is the discharge,
- \( Q_o \) is the plateau discharge,
- \( b, c \) are constants with \( c \) assumed to be negative, and
- \( t \) is time.

Initially a \( \log Q v \log t \) linear regression was carried out to provide estimates of the model parameters. The final form of the function was obtained by using the computer package SAS to perform a nonlinear regression. This gave the function

\[ Q = 0.066 - 0.063 \exp(-1.920t). \]  

(7.4.2-1)

The SAS output is shown in Table 7.4.2-1. The resulting input function is plotted in Figure 7.4.2-2.

**Area Input:** The same methods are used for the data relating cross-sectional area to time. The resulting function (with \( A = \text{area} \) is;

\[ A = 0.038 - 0.036 \exp(-1.927t). \]  

(7.4.2-2)
### Non-Linear Least Squares Summary Statistics

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<th>Source</th>
<th>DF</th>
<th>Sum of Squares</th>
<th>Mean Square</th>
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<td>(Corrected Total)</td>
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<td>0.01177151369</td>
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</table>

<table>
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<th>Parameter</th>
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<th>Asymptotic Std. Error</th>
<th>Asymptotic 95% Confidence Interval</th>
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</thead>
<tbody>
<tr>
<td>A</td>
<td>0.065854606</td>
<td>0.00142568711</td>
<td>0.0629929615 to 0.0687111986</td>
</tr>
<tr>
<td>B</td>
<td>0.062956089</td>
<td>0.00279243350</td>
<td>0.0573599317 to 0.0685522463</td>
</tr>
<tr>
<td>C</td>
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<td>0.1976957613</td>
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Asymptotic Correlation Matrix

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<td>C</td>
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Table 7.4.2-1 SAS Output: Non-Linear Regression, Q (0,t) Boundary Condition, TEST 2
Figure 7.4.2-2: Input Discharge Data from Regression Equation TEST 2
The initial data are plotted on Figure 7.4.2-3 and the resulting regression expression is plotted on Figure 7.4.2-4. Table 7.4.2-2 contains the SAS output.

### 7.4.3 Matlab Routines and Boundary Conditions

The matlab routines developed for this calculation are attached and form Tables 7.4.3-1,-2,-3,-4,-5,and -6. Tables 1, 2, 3, and 4 are the boundary conditions. The input functions derived in Section 7.4.2 form two of them and initially discharge is zero along the channel. Since the wave enters a dry channel, the initial area is also zero along the channel.

However as the mathematics requires division by the area it is necessary to provide a dummy boundary condition for initial area along the channel. Assuming a small positive value (0.001), which was considered to be sufficiently small as to not disturb the computation, did this.

The computations are carried out for positions along the channel approximately equivalent to Bell et.al's Stations 2, 4, 6 and 8 and the results are compared with the plots published in the paper.
### Non-Linear Least Squares Summary Statistics

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>Sum of Squares</th>
<th>Mean Square</th>
</tr>
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<td>Regression</td>
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<td>Residual</td>
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<tr>
<td>Uncorrected Total</td>
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(Corrected Total) 57 0.01177151369

### Parameter Estimates

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<th>Asymptotic 95% Confidence Interval</th>
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<th>Upper</th>
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<td>0.0372229728</td>
<td>0.00140941127</td>
<td>0.0400843715</td>
</tr>
<tr>
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<td>0.00140941127</td>
<td>0.035161970</td>
<td>0.0391652391</td>
<td></td>
</tr>
<tr>
<td>F</td>
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<td>0.1727492779</td>
<td>-2.2726984877</td>
<td>-1.5803044619</td>
<td></td>
</tr>
</tbody>
</table>

### Asymptotic Correlation Matrix

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<th>F</th>
<th>G</th>
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<td>0.8536814838</td>
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<tr>
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<tr>
<td>G</td>
<td>0.8536814838</td>
<td>-0.52765608</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 7.4.2-2 SAS Output: Non-Linear Regression, A (0,t) Boundary Condition, TEST 2
% BA1 is the boundary condition of \( A(s, 0) \); INPUT
function \( z = BA1(s) \);
\( z = 0.001 \);

Table 7.4.3-1 Matlab Routine BA1.m for Boundary Condition \( A(s,0) \), TEST 2

% BQ1 is the boundary condition of \( Q(s, 0) \); INPUT
function \( z = BQ1(s) \)
\( z = 0.000 \)

Table 7.4.3-2 Matlab Routine BQ1.m For Boundary Condition \( Q(s,0) \), TEST 2

% BA2 is the boundary condition of \( A(0,t) \); INPUT
function \( z = BA2(t) \)
\( z = 0.38 - 0.36 \exp(-1.927t) \);

Table 7.4.3-3 Matlab Routine BA2.m For Boundary Condition \( A(0,t) \), TEST 2

% BQ2 is the boundary condition of \( Q(0,t) \); INPUT
function \( z = BQ2(t) \)
\( z = 0.66 - 0.063 \exp(-1.92t) \);

Table 7.4.3-4 Matlab Routine BQ2.m For Boundary Condition \( Q(0,t) \), TEST 2
function [A, Q] = code (BA1, BA2, BQ1, BQ2, h, k, N, M, l, g, B, S1)

%BA1 is the boundary condition of A (s,0); INPUT
%BA2 is the boundary condition of A (0,t); INPUT
%BQ1 is the boundary condition of Q (s, 0); INPUT
%BQ2 is the boundary condition of Q (0,t); INPUT
%\( s \) is the length of [0 x] along S direction; INPUT
%N is the number of mesh points in the S direction; INPUT
%\( l, \ m, g, b, S1 \) are the constants; INPUTS
FR=(k/h);
R1=(B^3)/(12*g); R2=g*B; R3=g^*S1;
for i=1:N
A (i, 1)=feval (BA1, h*(i-1));
Q (i, 1)=feval (BQ1, h*(i-1));
end

for j=2:M
A (1,j)=feavl (BA2, k*(j-1));
Q (1,j)=feavl (BQ2, k*(j-1));
end
for j=2:M
for i=2:N
R4=(1(i)^2)*R1
A (i, j)=A (i, j-1)+FR*(Q (i-1, j-1)-Q (i, j-1));
F=(Q (i, j-1)/A (i, j-1));
G=R4*(Q (i, j-1)^2)/(A (i, j-1)^3);
Q (i, j)=Q (i, j-1)-FR*F*(2+G)*(Q (i, j-1)-Q (i-1, j-1))-FR*(R2*A (i, j-1)*(1-G)-F^2*(1+G))*(A (i, j-1)-A (i-1, j-1))^k*R3*A (i, j-1);
end
end

Table 7.4.3-5 Matlab Routine 'Code.m', for TEST 2
% Set values corresponding to code.m
% clear
s=9.5; N=20; M=8001;
h=s/(N-1);
k=0.001;
y=k*(M-1);
B=0.305; g=9.8; S1=0.0009;
l=zeros(1,N);

% for i=9:15
l(i)=0.93;
end

% This is the calling sequence
[A,Q]=code ('BA1','BA2','BQ1','BQ2';h,k,N,M,l,g,B,S1);
t=0:k: y;

% ZA=A';
ZQ=Q';

% The plot required at distance s=0.6 (for example)
DIS=0.6;
NP=round (DIS/h)+1;
AD=ZA (:, NP);
QD=ZQ (:, NP);
plot(t, AD)
QD=Q (:, NP);
plot(t, AD)

**Table 7.4.3-6 Matlab Routine 'Dymic.m', for TEST 2**
7.4.4 Results of Simulations

In simulations of long waves in open channels the engineer is not primarily concerned with getting an exact reproduction of the experimental data. (The existence of experimental error makes an exact copy impossible.) Instead the concern is to accurately model the speed of the wave and its maximum dimension. The former would affect the time available for emergency services to operate, the latter is need to define engineering structures such as flood banks. In comparing the predictions of the curvilinear dynamic model with the laboratory data, these are the two factors, which will be considered. As the Matlab routines produce a list of the area values plotted it is easy to see that each simulation plateaus out and the corresponding depth can be obtained by dividing by 0.305 m, the channel width.

Figures 7.4.4-1, -2, -3, and -4 show the graphs of cross-sectional area vs time and discharge vs time for each of the four Stations. These are compared with Bell et.al's graphs for depth vs time, which are reproduced as Figure 7.4.4-5. For Stations 2 and 8 a single plot is given corresponding to flow in a straight channel. The graphs for Station 4 and 6 show two plots corresponding to the inner and outer sides of the wave as it moves around the curve. The reference line is placed at the midpoint of the channel cross-section, but the depth at the reference line is not expected to be the mean of the wall heights. While the transverse surface has been treated as a straight line, it is in fact hyperbolic. We would therefore expect that the depth at the reference to be less than the mean.

Comparison of the Figure 7.4.4 plots shows that the velocity of the wave is predicted reasonably correctly. However comparison of depths shows a major anomaly. The maximum depths with i/s representing depth at the inside wall and o/s representing depths at the outside wall are:
Figure 7.4.4–1: TEST2 Station 2 $s=4.1m$
Figure 7.4.4–2: TEST2 Station 4 s=5.97m
Figure 7.4.4-3 TEST2 Station 6 s=7.32m
Figure 7.4.4-4: TEST2 Station 8 $s=9.37\text{m}$
The simulated values are much higher than the values reported in Bell et al. However comparing the graphs for Stations 1 and 2 in Figure 7.4.4-5, it can be seen that there appears to have been substantial attenuation between the two. Over a distance of approximately 4 metres and a time span of two seconds, the depth falls from 0.12 m to 0.04 m, which is highly unlikely. Furthermore, examining the corresponding graphs for Test 2n where the channel is much rougher, ($S_r=0.0048$ compared with Test 2's $S_r=0.009$) shows a change from 0.14 m to 0.10 m over the same distance. It appears that the values given for Station 1 for Test 2 have (probably inadvertently) been inflated. The author is of the opinion that the error was typographical - incorrect figures were placed on the vertical axis during typeseting.

### 7.5 TEST 21

#### 7.5.1 Input Data

The input data are obtained from the plot of depth vs time for Station 1 found in Figure 22 of Bell et al. and are tabulated in Appendix 3. It had been hoped that input expressions could be developed for Test 21 as for Test 2. However as can
Figure 7.4.4-5(a) TEST 2, Experimental data, Station 2 $s = 4.1$ m

Figure 7.4.4-5(b) TEST 2, Experimental data, Station 4 $s = 5.97$ m

Figure 7.4.4-5(c) TEST 2, Experimental data, Station 6 $s = 7.32$ m

Figure 7.4.4-5(c) TEST 2, Experimental data, Station 8 $s = 9.37$ m
be seen in the plot (Figure 7.5.1-1), the graph rises very steeply and most of the points form a straight line with a slight downward slope. When a non-linear regression is performed on these data, the straight section overshadows the steep rise and the wave front is not defined. It is necessary to read the input data directly.

7.5.2 Matlab Routines and Boundary Conditions

For Test 21, the previous Matlab routines were used with the following modifications.

The file 'code' was modified to accept matrix data. The submitted data were smoothed out using a spline routine and then data points were read off at constant intervals. Files 'BQ1.m' and 'BA1.m' were retained to provide boundary conditions for Q (s, 0) and A (s, 0).

Files 'BQ2.m' and 'BA2.m' were replaced by a single file 'data.m' containing the input data which had to be extrapolated to allow simulation of the long wave at further down the channel and hence at a later period. Although the plot of depth v time for Station 1 only extended to 2.0 seconds the data from the depth probe immediately upstream of the 'dam' indicated that the water level was roughly level during the period 1-5 second. It is therefore reasonable to extrapolate the plateau at least a further 4 seconds. However to ensure that calculations could be carried out at Station 8 it was necessary to have input data stretching to about 8 seconds.

In Test 21 the dam break wave was released into a channel containing still water of depth of 0.0762m. It was initially thought that this could form the (s, 0) boundary condition, but when this was tried, the simulation became unstable and the previous boundary condition was reinstated. As the Station 1 data begin at approximately time = 0.027s, area = 0.0269 m² and discharge = 0.048 m³s⁻¹, it was decided to extrapolate the data linearly backwards so that area went from 0.0269
0.0269 m² down to 0.0232 m² (i.e. 0.0762 m x 0.305m) and discharge from 0.048 m³s⁻¹ to 0.0 m³s⁻¹. This was over a period of 1.6 seconds. This extrapolation along with the plateau extrapolation gave the required 8 seconds of data.

The original input data included an initial spike. The early work showed that the simulation did not reproduce this spike, but as these points were in the data set, they distorted the rising limb of the plots for both area and discharge. As this initial spike is not significant from an engineering point of view, the data were removed and the remaining points closed up. The routines are found in Table 7.5.2-1,-2,-3, and -4. The routine 'data.m' is not included, as it is simply a repeat of the data in Appendix 3.

7.5.3 Results of Simulation

The results of the simulation for Stations 2, 4, 6 and 8 are shown as Figures 7.5.3-1, -2, -3, -4 and the original data from Bell et.al. are shown on Figure 7.5.3-5.

**Wave Celerity:** Comparison of the corresponding plots shows that the simulation predicts the wave celerity quite accurately. The rising limbs of the simulated plots are not quite as steep as those of the laboratory data, but from an engineering viewpoint this is not a major problem. Estimating the time of arrival of the wave is more critical.

**Maximum Height:** As the data has been extrapolated to enable the calculation to proceed, it has in effect distorted the time scale. We therefore do not expect that at numerically identical points on the time scale of model and simulated plots for a given Station would result in an identical value for depth. As the variable of
% BA1 is the boundary condition of A (s, 0); INPUT
function z = BA1(s);
z = 0.001;

Table 7.5.2-1 Matlab Routine BA1.m for Boundary Condition A(s,0).

TEST 21

% BQ1 is the boundary condition of Q (s, 0); INPUT
function z = BQ1(s)
z = 0.000

Table 7.5.2-2 Matlab Routine BQ1.m For Boundary Condition Q (s, 0).

TEST 21
% Set values corresponding to code.m
%
clear
s=9.5; N=20; M=8001;
h=s/(N-1);
\( k=0.001; \)
y=k*(M-1);
B=0.305; g=9.8; S1=0.0009;
l=zeros(1,N);
%
for i=9:15
1(i)=0.93;
end
%
% This is the calling sequence
[A,Q]=code('BA1','BQ1','BQ2',h,k,N,M,l,g,B,S1);
t=0:k:y;
%
ZA=A';
ZQ=Q';
% The plot required at distance \( s=9.37 \) (for example)
DIS=9.37;
NP=round(DIS/h)+1;
AD=ZA(:,NP);
QD=ZQ(:,NP);
plot(t,AD)
hold on
plot(t,QD,':-')
hold off

Table 7.5.2-3 Matlab Routine 'Dymic.m', For TEST 21
function [A, Q] = code (BA1, BA2, BQ1, BQ2, h, k, N, M, l, g, B, S1)
% BA1 is the boundary condition of A(s,0); INPUT
% BQ1 is the boundary condition of Q (s, 0); INPUT
% s is the length of [0 x] along S direction; INPUT
% N is the number of mesh points in the S direction; INPUT
% l,d, g, b, S1 are the constants; INPUTS
FR=(k/h);
R1=(B^3)/(12*g); R2=g/B; R3=g*S1;
for i=1:N
A(i,1)=feval(8A1,1';1-1));
Q(i,1)=feval(BQ1, h*(i-1));
end
%for j=1:M
%A(1.j)=feval (BA2, k*(j-1));
%Q (1.j)=feavl(BQ2, k*(j-1));
%end
data
t=0:k:2;
A1=spline(DA(:,1),DA(:,2),t);
Q1=spline(DA(:,1),DA(:,3),t);
if (M>2001)
for j=2:2001
A(1.j)=A1(j);
Q(1.j)=Q1(j);
end
for j=2002:M
A(1.j)=0.055748;
Q(1.j)=0.098674;
end
else
for j=2:M
A(1.j)=A1(j);
Q(1.j)=Q1(j);
end
for j=2:M
for i=2:N
R4=(1(i)^2)*R1
A(i,j)=A(i,j-1)+FR*(Q(i-1,j-1)-Q(i,j-1));
F=(Q(i,j-1)/A(i,j-1));
G=R4*(Q(i,j-1)^2)/(A(i,j-1)^3);
Q(i,j)=Q(i,j-1)-FR*F*(2+G)*(Q(i,j-1)-Q(i-1,j-1))-FR*(R2*A(i,j-1)*(1-G)-F^2*(1+G)))*(A(i,j-1)-A(i-1,j-1))-k*R3*A(i,j-1);
end
end
end

Table 7.5.2-4 Matlab Routine 'Code.m', for TEST 21
Figure 7.5.3-1 TEST21 Station2 s = 4.1m
Figure 7.5.3-2 TEST21 Station 4 s = 5.97 m
Figure 7.5.3–3 TEST21 Station 6 s = 7.32
Figure 7.5.3-4 TEST21 Station 8 s = 9.37m
engineering importance is the maximum water level it is more reasonable to run
the simulation for a longer period and then to compare the values at which each of
the pair of plots plateau out.

It should also be noted that although the transverse surface has been treated as a
straight line it is in fact hyperbolic (see Appendix II), so that dividing the
calculated area by the width will not necessarily give the depth at the reference line.
Instead it can be regarded as a 'typical' depth. The simulation is allowed to run for
12 seconds and the calculated values along with the corresponding values from
Figure 7.5.3-5 are given in Table 7.5.3-1

<table>
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<tr>
<th>Station</th>
<th>Simulated Depth (m)</th>
<th>Laboratory Depth</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.180</td>
<td>0.18</td>
</tr>
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<td>4</td>
<td>0.171</td>
<td>0.17 (i/s)-0.21(o/s)</td>
</tr>
<tr>
<td>6</td>
<td>0.163</td>
<td>0.17(i/s)-0.21(o/s)</td>
</tr>
<tr>
<td>8</td>
<td>0.158</td>
<td>0.17</td>
</tr>
</tbody>
</table>

Table 7.5.3-1 Comparison Model Results and Simulation for TEST 21

Table 7.5.3-1 indicates that the simulation gives depth values which are close to
those measured in the laboratory but show higher attenuation around the curve.
This underestimation of wave height would not in practice be critical, as the
engineer would, as a matter of course, add a factor of safety.
7.6 CONCLUSION

This brief simulation demonstrates that the curvilinear dynamic model derived in this thesis is a reasonable representation of the behaviour of a dynamic wave in a curved channel. However it should be kept in mind that this simulation is for a simple case. A more conclusive demonstration would be to compare it with field data. However such data are not readily available.

Although this is a simple case, the solution of the equations was not straightforward. As the defining equations form a pair of highly non-linear coupled differential equations, developing a solution scheme suitable for a range of conditions such as varying roughnesses is a major undertaking.

It also demonstrates that engineers who use a programme prepared by others should appreciate the underlying mathematical structure and any assumptions or qualifications that may have been built into the programme.
VIII CONCLUSION

8.1 SUMMARY OF THESIS

The work described in this thesis begins by establishing the three main long wave models for straight open channels - kinematic, diffusion and dynamic (Chapters 2 and 3). It is then demonstrated that curvilinearity can be systematically allowed for by incorporating it from first principles.

After deriving the de Saint-Venant equations in curvilinear coordinates, (Chapter 4) curvilinear versions of the kinematic, diffusion and dynamic models were obtained (Chapters 5 and 6). The equations describing these models take the following form -

'straight channel case + curvilinear correction'.

With equations in this form, the engineer can easily estimate the size of the curvilinear correction and decide whether it is significant. An example of this is provided in Appendix 2, where the celerity of the dynamic wave is calculated for three sets of data from the Wairoa River, (New Zealand). This calculation indicates that for long waves in channels which are either strongly curved or have very asymmetric cross-sections the curvilinear correction can be sufficiently large to warrant attention.

The validity of the curvilinear dynamic model is checked by comparison with a laboratory study of dam break waves (Chapter 7). Although solving the derived equations does not exactly reproduce the laboratory data, the agreement is close enough to ensure that the model is sufficiently accurate to warrant use in an engineering application.
The superelevation of the curved flow is shown by this analysis to be a function of curvature. Consequently, for the case of a symmetric channel where the values of the shape factors $\tilde{n}$ and $n_m$ depend only on the superelevation, the dimensionless terms $\kappa \tilde{n}$ and $\kappa n_m$ are of order $\kappa^2$. While the analysis is conducted to first order and in general, second order terms are discarded, calculations with data from the paper by Bell et al. (1992) indicate that the superelevation term is sufficiently large to demand the retention of the dimensionless shape factors despite being second order. This reservation is true only for a symmetric channel. If the channel is asymmetric, then the shape factors depend on the channel shape as well as the elevation.

8.2 FURTHER WORK

8.2.1 Diffusion Model

In the course of describing the diffusion model, some limitations have been uncovered. It has been shown that the conventional approach found in engineering literature does not apply when the open channel receives lateral inflow. It also arbitrarily restricts the model to a single downstream wave. When a general derivation of the diffusion model is used (i.e. by substituting a general solution into the differential equation obtained when the linearised de Saint-Venant equations are combined by cross-differentiating and substituting), this analysis indicates that the model implies the existence of two diffusion waves with dispersive effects (Chapters 3 and 6). A next step would be to try to identify the physical equivalent of the second (presumably upstream) wave and of the extra factor $\theta_2$ or $\theta_2(1-\kappa \tilde{n})$ which appears in the general solution.

If the engineer has to apply the diffusion model to an open channel with lateral inflow, (s)he will have no recourse but to solve the basic equations. Although some computer packages based on diffusion routing exist, the general literature
indicates a paucity of interest in attempts to solve these equations. Further work needs to be done to develop numerical schemes to solve these equations particularly over a range of roughnesses. This comment applies to both the curved and straight channel cases.

8.2.2 Kinematic and Dynamic Models

The curvilinear kinematic model is fairly straightforward, but like the diffusion model it has not to date been compared with existing field or laboratory data. Such a comparison would be of interest.

While a successful attempt has been made to verify the curvilinear dynamic model by comparing with a simple model study, the limitations of the solution scheme employed restricted the application to a case of long waves in a smooth channel. It would be useful to establish a solution scheme that could be applied over the range of friction conditions found in engineering practice, i.e. Manning’s n from 0.009 (smooth concrete line channel) to 0.10 (rivers with irregular alignment).

A further challenge is to apply the derived curvilinear dynamic model to a long wave such as a dam-break in a natural open channel. This would be more complex than the analysis described in Chapter 7. Now both shape factors \((\bar{h}, \bar{n}_{mo})\) and both slope measures \((S_o, S_f)\) appear. The curvilinear de Saint-Venant equations are a pair of highly non-linear coupled differential equations, which are not easily solved.
APPENDIX I - CALCULATION OF CURVILINEAR DYNAMIC WAYE Celerity

1. INTRODUCTION

This appendix contains the results of three hand calculations of the curvilinear dynamic wave celerity and compares them with the traditional straight channel formula. These calculations were done to provide an estimate of how big the curvilinear correction would be in practice, particularly in extreme cases.

These calculations are based upon data from the Wairoa River, (New Zealand) which were supplied by the Hawkes Bay Regional Council. The data consist of a plan with the positions of the bench marks and line of cross-sections, the surveyed cross-sections and bank water levels for floods which occurred in October 1990, April 1991, and November 1991.

Radii of curvature were scaled off the typographical map of the river supplied by HBRC. For each cross-section, the reference line was placed at the talweg, the bank levels plotted and values of $\bar{H}$ and $n_m$ calculated for each available cross-section/flood level combination. These plots are attached as Figures 1-15. Figure 16 shows the positions of the cross-sections.

2 WAIROA RIVER FLOOD OF OCTOBER 1990

Table AI-1 contains a summary of the analysis of this data where $c$ is the celerity calculated using the straight channel formula, $C$ is the celerity using the new formula (Equation 4.2.5-8), and $\% \ inc$ is the percentage difference between $c$ and $C$. The other symbols are as previously defined.
There is some doubt as to the reliability of the measurements for cross-sections 2, 3 and 4 as here the superelevation is opposite to what would be expected. This is probably due to a surveying error. The usual practice is to estimate the position of the flood level after the flood has subsided by looking for evidence such as debris levels and tops of slips. It is easy to mistake the position of the top water level.

<table>
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<th>$n_m$</th>
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<th>$B$</th>
<th>$c$</th>
<th>$C$</th>
<th>% inc.</th>
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<td>37.0</td>
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<td>9.48</td>
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<td>11.67</td>
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<td>-24.0</td>
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<td>9.65</td>
<td>11.11</td>
<td>15.09</td>
</tr>
<tr>
<td>26</td>
<td>0.003</td>
<td>26.9</td>
<td>32.0</td>
<td>1106.3</td>
<td>122.5</td>
<td>9.41</td>
<td>10.24</td>
<td>8.84</td>
</tr>
</tbody>
</table>

Table AI-1 Dynamic Wave Celerity for Wairoa River, October 1990

The smallest increases occur on cross-sections 6 and 21. Here the channel is incised and roughly symmetrical so that the reference line is near the centre. The correction reflects the small values of $\bar{n}$ and $n_m$. 
Cross-section 18 has the highest correction but the result is specious. As can be seen on the plot, the water level stretches beyond the centre of curvature, which violates the assumptions of the derivation. The value of $(\kappa \eta_i)^2$ is 0.6.

The next highest corrections are those of cross-section 16 and 8. In these two cases, the intersection of the water level with the bank is just within the centre of curvature. Cross-section 16 has a $(\kappa \eta_i)^2$ value of 0.2 and so the correction can be regarded as reasonable. Cross-section 8 on the other hand has a $(\kappa \eta_i)^2$ value of 0.5, which puts this cross-section out of the range of application of the derived equations.

An interesting comparison can be made between cross-sections 12 and 13. Both have the same sized curvature but considerably different corrections. This can be ascribed mainly to the difference in cross-section shape. Cross-section 12 approximates to what may be called the classic meander cross section asymmetric with the talweg to one side. Cross-section 13 is much more symmetric. It should also be noted that it also has a high value of $(\kappa \eta_i)^2 = 0.82$, (Cross-section 12 has $(\kappa \eta_i)^2 = 0.66$) even though the surface water level is within the radius of curvature.

3 WAIROA RIVER FLOOD OF APRIL 1991

Fewer data are available for this flood. The analysis is summarised in Table AI-2. This set of data shows a similar pattern to that of the October 1990 flood. The smallest correction is again cross-section 6, the largest is for cross-section 12 but again it is a questionable result. Cross-section 18 has the next highest correction and in this case $(\kappa \eta_i)^2$ is 0.3, which is just on the border of applicability. The same contrast occurs between cross-sections 12 and 13 as shown previously, although in
<table>
<thead>
<tr>
<th>Section No.</th>
<th>Curvature ( \kappa )</th>
<th>( \bar{h} ) (m)</th>
<th>( n_m ) (m)</th>
<th>A ( (m^2) )</th>
<th>B (m)</th>
<th>c (m)</th>
<th>C (m)</th>
<th>% inc.</th>
</tr>
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Table AI-2 Dynamic Wave Speed for Wairoa River Flood, April 1991

<table>
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<tr>
<th>Section No.</th>
<th>Curvature ( \kappa )</th>
<th>( \bar{h} ) (m)</th>
<th>( n_m ) (m)</th>
<th>A ( (m^2) )</th>
<th>B (m)</th>
<th>c (m)</th>
<th>C (m)</th>
<th>% inc.</th>
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<td>8.73</td>
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<td>1017.9</td>
<td>105.1</td>
<td>9.75</td>
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<td>9.37</td>
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<tr>
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<td>47.0</td>
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<td>128.0</td>
<td>7.78</td>
<td>8.55</td>
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<td>112.9</td>
<td>9.40</td>
<td>10.33</td>
<td>9.9</td>
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</tbody>
</table>

Table AI-3 Dynamic Wave Celerity for Wairoa River Flood, November 1991
this case the \((\kappa n_i)^2\) value for cross-section is now down to 0.4 which may still be sufficiently high to render application of this analysis questionable.


Table A1-3 summarises this case. This time cross-sections 2, 3 and 4 have superelevation in the expected direction. Overall there is the same pattern of relative increases in celerity. Cross-section 8 has the third highest correction, but also has the highest value of \((\kappa n_i)^2 - 0.53\). All the other cross-sections have \((\kappa n_i)^2\) values of less than 0.1. Cross-section 10 is on a reach, which is approximately straight.
Fig. 3: Wairoa River Site No.3
Fig. 4: Wairoa River Site No.5
Fig. 5: Wairoa River Site No.6

<table>
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<th>Reduced level (m)</th>
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<tr>
<td>15.08</td>
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<td>14.00</td>
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</table>

Datum 3.00 m

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<tr>
<td>11.90</td>
<td>135.60</td>
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<td>16.89</td>
<td>141.00</td>
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</table>

Scales - H: 1:1500, V: 1:150

Benchmark

CL
PL

13.70
13.40
13.80

13.70(Co ordinates: 13.76(Az: 80))
13.80(Co ordinates: 13.80(Az: 80))
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<th>Datum 3.00 m</th>
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</thead>
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<tr>
<td>11.09</td>
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<td></td>
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<td>10.81</td>
<td></td>
</tr>
<tr>
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<td>16.01</td>
<td></td>
</tr>
<tr>
<td>11.76</td>
<td>21.91</td>
<td></td>
</tr>
</tbody>
</table>

Scales: H 1:1750, V 1:150

Fig. 8: Waipoo River Site No.10

1. Peg
2. RL
3. CL
4. 14.000 m
5. 14.000 m
6. Benchmark
Fig. 9: Wairoa River Site No. 12
Fig. 15: Wairoa River Site No. 21

<table>
<thead>
<tr>
<th>Reduced level (m)</th>
<th>Distance (m)</th>
</tr>
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<tbody>
<tr>
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<td>-16.44</td>
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<td>17.35</td>
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Scales: H 1:5000 V 1:200

RL CL

Benchmark

16.80
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</thead>
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<td>14.61</td>
<td>113.10</td>
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<tr>
<td>16.88</td>
<td>122.40</td>
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</tbody>
</table>

Fig 16: Waioara River Site No.23

Scales: H: 1:5000 V: 1:200

Datum 3.00 m

Benchmark

RL

CL

Peg

17.22

15.80

15.80 (Oct '90)

15.66 (Apr '91)
APPENDIX II IRROTATIONAL FLOW AROUND A CURVE

1. INTRODUCTION

1.1 Fluid Flow In a Curved Channel

Measurements in curved channels have shown that when flow enters a curve the core of maximum velocity moves to the inner bank. Once into the curve the secondary (transverse) flow may gradually skew the velocity distribution and the line of fastest flow gradually moves across the channel. This effect has been measured in such rivers as the River South Esk, Scotland (Bridge and Jarvis, 1976), Squamish River, British Columbia (Hickin, 1978), Muddy Creek, western Wyoming (Dietrich et.al., 1979, Nelson and Smith, 1989) and Dommel River, the Netherlands (de Vriend and Geldorf, 1983). This may be the phenomenon referred to by Thomas Stevenson quoted in Chapter 1.

A similar pattern of velocity shift has been measured in curved rectangular or trapezoidal flumes although here the shift usually occurs near the bend exit. (Ippen and Drinker, 1962, Rozovski quoted in Leschziner and Rodi, 1984, Kikkawa et.al., 1979).

Since these studies indicate that for most of the flow through a curved rectangular or trapezoidal laboratory flume, and for the start of flow into a curved natural channel, flow can be regarded as part of a free vortex, it is of interest to consider the special case of irrotational flow in a curved channel.

In the early stages of the work described in this thesis, a partial model was developed for irrotational flow in a curved channel i.e., the derivation of expressions for transverse velocity distribution and the shape of the transverse water level. The subsequent derivation of the curvilinear de Saint-Venant
Equations in $u_0$ and $\eta_o$ (Section 4.4) allowed a complete model for irrotational flow (Nalder, 1995).

The derivation of the equations for transverse velocity and surface shape are presented in this appendix. It begins with the conservation of momentum equation in differential form and expresses it in curvilinear coordinates using standard expressions. The irrotational condition is then used to simplify the equations and to develop the required expressions. The other parts of the model, the curvilinear de Saint-Venant equations and an expression for dynamic wave velocity have already been derived in Section 4.4.

1.2 Conservation of Momentum in Differential Form

The momentum equation in vector form is

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P + \mathbf{g},$$  

(A1.2-1)

where $\mathbf{u}$ is the velocity vector and $\mathbf{g}$ is the acceleration due to gravity.

The equation is expressed in orthogonal curvilinear coordinates by using the standard expressions for common differential quantities found in Batchelor (1967).

1.3 Change of Coordinates Using Standard Expressions

Given a system of orthogonal curvilinear coordinates, $\xi_1, \xi_2, \xi_3$ with unit vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ (parallel to the coordinate lines in the direction of increasing $\xi$ ) and scale factors $h_1, h_2, h_3$, then the divergence operator $\nabla$ is defined by [Appendix II Batchelor, (1967)] as:
\[ \nabla = \frac{a}{h_1} \frac{\partial}{\partial \xi_1} + \frac{b}{h_2} \frac{\partial}{\partial \xi_2} + \frac{c}{h_3} \frac{\partial}{\partial \xi_3}. \]  
(A1.3-1)

With the general vector, \( \mathbf{F} = F_1 \mathbf{a} + F_2 \mathbf{b} + F_3 \mathbf{c} \),

grad \( \mathbf{F}(= \nabla \cdot \mathbf{F}) \) becomes

\[
\frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial (h_2 h_3 F_1)}{\partial \xi_1} + \frac{\partial (h_1 h_3 F_2)}{\partial \xi_2} + \frac{\partial (h_1 h_2 F_3)}{\partial \xi_3} \right\},
\]

(A1.3-2)

curl \( \mathbf{F}(= \nabla \times \mathbf{F}) \) becomes

\[
\frac{1}{h_1 h_2 h_3} \begin{vmatrix}
    h_1 & h_2 & h_3 \\
    \frac{\partial}{\partial \xi_1} & \frac{\partial}{\partial \xi_2} & \frac{\partial}{\partial \xi_3} \\
    h_1 F_1 & h_2 F_2 & h_3 F_3
\end{vmatrix}
\]

(A1.3-3)

In the curvilinear system described in Chapter 3, \( \mathbf{F} \) is replaced by the velocity vector

\[ \mathbf{u} = u_s \mathbf{s} + u_n \mathbf{n} + u_z \mathbf{z}, \]

which means that,

\[ (\xi_1, \xi_2, \xi_3) = (s, n, z), \]

and

\[ (a, b, c) = (s, n, z). \]
The scale factors \( h_2 \) and \( h_3 \) are both equal to unity. The scale factor \( h_1 \) needs to be derived by considering an incremental movement of the radius of curvature.

Consider Fig. A.1 where positive curvature is anticlockwise and the radius of curvature sweeps out an angle \( d\theta \). Therefore,

\[
ds = R d\theta,
\]

and

\[
ds_1 = (R-n) d\theta.
\]

As both \( ds \) and \( ds_1 \) are along arcs parallel to the \( s \) axis, \( h_1 \) can be defined as

\[
h_1 = \frac{ds_1}{ds} = \frac{(R-n) d\theta}{R d\theta} = \left( 1 - \frac{n}{R} \right) = (1 - \kappa n),
\]

where \( \kappa \) is curvature.

### 1.4 Irrotational Condition

The irrotational condition is \( \nabla \times \mathbf{u} = 0 \), which in terms of Batchelor's formula is

\[
\begin{vmatrix}
1 & (1 - \kappa n) s & n & z \\
1 & \frac{\partial}{\partial s} & \frac{\partial}{\partial n} & \frac{\partial}{\partial z} \\
(1 - \kappa n) u_s & u_n & u_z & 0
\end{vmatrix} = 0. \tag{A1.4-1}
\]

If the components are considered separately, three scalar equations are obtained:
\[ \frac{\partial u_z}{\partial n} - \frac{\partial u_n}{\partial z} = 0, \quad (A1.4-2) \]

\[ \frac{\partial (1 - \kappa n)u_z}{\partial z} - \frac{\partial u_z}{\partial s} = 0, \quad (A1.4-3) \]

\[ \frac{\partial u_n}{\partial s} - \frac{\partial (1 - \kappa n)u_z}{\partial n} = 0. \quad (A1.4-4) \]

2. CURVILINEAR MOMENTUM EQUATION

Applying the results of Section 1.3 to the momentum equation gives the first term on the left hand side as,

\[ \frac{\partial u}{\partial t} = \frac{\partial u_z}{\partial t} s + \frac{\partial u_n}{\partial t} n + \frac{\partial u_z}{\partial t} z. \quad (A2-1) \]

The second term becomes,

\[ (u \cdot \nabla)u = \frac{u_z}{(1 - \kappa n)} \frac{\partial u}{\partial s} + u_n \frac{\partial u}{\partial n} + u_z \frac{\partial u}{\partial z}. \quad (A2-2) \]

After substituting for \( u \) the equation is simplified by applying the Frenêt formulæ (ignoring the binormal),

\[ \frac{\partial n}{\partial s} = \kappa s, \]

\[ \frac{\partial s}{\partial s} = \kappa n, \]

and removing the trivial cases,
\[
\frac{\partial u}{\partial s} = \frac{\partial u}{\partial n} = \frac{\partial u}{\partial z} = 0,
\]
\[
\frac{\partial n}{\partial n} = \frac{\partial n}{\partial z} = 0,
\]
\[
\frac{\partial s}{\partial n} = \frac{\partial s}{\partial z} = 0.
\]

Rearranging the resulting expression gives

\[
(u \cdot \nabla)u = \left( \frac{u_s}{1 - \kappa n} \frac{\partial u_s}{\partial s} + u_n \frac{\partial u_n}{\partial n} + u_z \frac{\partial u_z}{\partial z} - \kappa \frac{u_s u_n}{1 - \kappa n} \right) s + \]
\[
\left( \frac{u_s}{1 - \kappa n} \frac{\partial u_s}{\partial s} + u_n \frac{\partial u_n}{\partial n} + u_z \frac{\partial u_z}{\partial z} + \kappa u_s^2 \right) n + \]
\[
\left( \frac{u_s}{1 - \kappa n} \frac{\partial u_s}{\partial s} + u_n \frac{\partial u_n}{\partial n} + u_z \frac{\partial u_z}{\partial z} \right) z.
\]

The two factors on the right hand side of the momentum equation become

\[
g = g(z) = -gz, \quad (A2.4)
\]
as \( z \) is positive upwards, and

\[
-\frac{1}{\rho} \nabla p = -\frac{1}{\rho} \left( \frac{1}{1 - \kappa n} \frac{\partial p}{\partial s} s + \frac{\partial p}{\partial n} n + \frac{\partial p}{\partial z} z \right). \quad (A2.5)
\]

The momentum equation can then be split into components.

The \( s \) component gives,
\[
\frac{\partial u_x}{\partial t} + \frac{u_x}{1-\kappa n} \frac{\partial u_x}{\partial s} + u_n \frac{\partial u_x}{\partial n} + u_z \frac{\partial u_x}{\partial z} - \kappa \frac{u_x u_n}{1-\kappa n} = -\frac{1}{\rho} \frac{1}{1-\kappa n} \frac{\partial p}{\partial s}, \tag{A2-6}
\]

the \( n \) component gives,

\[
\frac{\partial u_n}{\partial t} + \frac{u_x}{1-\kappa n} \frac{\partial u_n}{\partial s} + u_n \frac{\partial u_n}{\partial n} + u_z \frac{\partial u_n}{\partial z} + \kappa u_x^2 = -\frac{1}{\rho} \frac{\partial p}{\partial n}, \tag{A2-7}
\]

and the \( z \) component gives,

\[
\frac{\partial u_z}{\partial t} + \frac{u_x}{1-\kappa n} \frac{\partial u_z}{\partial s} + u_n \frac{\partial u_z}{\partial n} + u_z \frac{\partial u_z}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g. \tag{A2-8}
\]

### 3 SCALING COORDINATES

The non-dimensional factors used in the model development are as follows.

**Variables:**

Defining \( L \) as a length scale (typically a meander length), \( b \) as a width scale and \( d \) as a depth scale, the variables and their derivatives are scaled as follows;

\[
S = \frac{s}{L}, \quad \frac{\partial}{\partial s} = \frac{1}{L} \frac{\partial}{\partial S},
\]

\[
N = \frac{n}{b}, \quad \frac{\partial}{\partial n} = \frac{1}{b} \frac{\partial}{\partial N},
\]

\[
Z = \frac{z}{d}, \quad \frac{\partial}{\partial z} = \frac{1}{d} \frac{\partial}{\partial Z},
\]

Also,

\[
T = \frac{\sqrt{g d}}{L} \frac{t}{T}, \quad \frac{\partial}{\partial t} = \frac{\sqrt{g d}}{L} \frac{\partial}{\partial T},
\]

and
pressure \( P = \frac{p}{\rho g d} \)

where \( g \) is acceleration due to gravity, and \( \sqrt{gd} \) is the long wave celerity.

**Velocities:**

Velocities are scaled using the method of Fredrichs (1948). Here the streamwise velocity \( u_s \) is scaled by the long wave celerity. The radial and transverse velocities are scaled in terms of long wave celerity and 'shallowness' parameters leading to

...new and independent variables
which will stretch the domain to
keep it from vanishing as
shallowness parameter(s) approach
their limit.

(Dressler, 1978)

The downstream component is scaled such that

\[
\frac{u_s}{\sqrt{gd}} = U,
\]

which on rearranging becomes

\[
u_s = \sqrt{gd} U.
\]

The transverse component becomes

\[
\frac{u_n}{\sqrt{gd}} = \frac{b}{L} V,
\]

which can be set equal to,

\[
u_n = \sqrt{gd} \sigma V , \text{ where } \sigma \text{ is the narrowness parameter } \left( = \frac{b}{L} \right).
\]

Like wise the vertical component is scaled as,
\[
\frac{u_z}{\sqrt{gd}} = \frac{d}{L} W,
\]
which can be rearranged to give,
\[
u_z = \sqrt{gd} \zeta W, \text{where } \zeta \text{ is the streamwise shallowness parameter } \left(= \frac{d}{L}\right).
\]

**Radius of Curvature**

If \( r_a \) is defined as the radius scale, the non-dimensional radius \( R \) is
\[
R = \frac{\text{radius}}{r_a},
\]
and the curvature factor \( 1 - \kappa n = 1 - \frac{n}{r} \) can be non-dimensionalised as
\[
1 - \kappa n = 1 - \frac{bN}{r_a R} = 1 - \alpha \frac{N}{R},
\]
which is non-zero as \( \kappa n \) is defined to be less than unity.

**4. TRANSVERSE VELOCITY GRADIENT**

Scaling the three components of the irrotational condition gives two useful results and an expression for the transverse gradient of the downstream velocity.

Scaling Equation (1.3-1), s component of the irrotational condition gives
\[
\frac{1}{b} \frac{\partial}{\partial N} \sqrt{gd} \frac{d}{L} W - \frac{1}{d} \frac{\partial}{\partial Z} \sqrt{gd} \frac{b}{L} V = 0,
\]
which simplifies to
\[
\varepsilon^3 \frac{\partial W}{\partial N} - \sigma^2 \frac{\partial V}{\partial Z} = 0,
\]
As \( \zeta^2 \) is negligible, this indicates that \( \frac{\partial V}{\partial Z} = 0 \).

(A4.3)
The \( n \) component, Equation (1.3-2), becomes

\[
\frac{1}{d} \frac{\partial}{\partial Z} \left[ \left( 1 - \alpha \frac{N}{R} \right) \sqrt{g d U} \right] - \frac{1}{L} \frac{\partial}{\partial S} \left( \sqrt{g d} \frac{d}{L} W \right) = 0, \quad (A4.4)
\]

and can be simplified to give,

\[
\frac{\partial}{\partial Z} \left[ \left( 1 - \alpha \frac{N}{R} \right) U \right] - \zeta^2 \frac{\partial W}{\partial S} = 0. \quad (A4.5)
\]

As \( \zeta^2 \) is negligible and \( \left( 1 - \alpha \frac{N}{R} \right) \) is defined previously to be non-zero, then

\[
\frac{\partial U}{\partial Z} = 0. \quad (A4.6)
\]

Scaling the \( z \) component leads to,

\[
\frac{1}{B} \frac{\partial}{\partial S} \left( \sqrt{g d} \frac{b}{L} V \right) - \frac{1}{b} \frac{\partial}{\partial N} \left[ \left( 1 - \alpha \frac{N}{R} \right) \sqrt{g d U} \right] = 0, \quad (A4.7)
\]

which on simplification becomes,

\[
\sigma^2 \frac{\partial V}{\partial S} - \frac{\partial}{\partial N} \left[ \left( 1 - \alpha \frac{N}{R} \right) U \right] = 0. \quad (A4.8)
\]

Assuming \( \sigma^2 \ll 1 \) (justified by field data), this equation becomes

\[
\frac{\partial}{\partial N} \left[ \left( 1 - \alpha \frac{N}{R} \right) U \right] = 0, \quad (A4.9)
\]
which gives,

\[
\left(1 - \alpha \frac{N}{R}\right) \frac{\partial U}{\partial N} = \frac{\alpha}{R} U. \tag{A4-10}
\]

As Equation (4-1) can be written as a variables separable ordinary differential equation, it can be integrated to give,

\[
\log U = -\log \left(1 - \alpha \frac{N}{R}\right) + G, \tag{A4-11}
\]

where \(G\) is the coefficient of integrating. \(G\) can be evaluated by substituting the values of \(U\) and \(N\) that correspond to the \(s\) axis, i.e.

\[
U = U_0, \\
n = bN = 0, \text{ hence} \\
N = 0.
\]

This gives,

\[
\log U_0 = \log 1 + G = 0 + G, \tag{A4-12}
\]

which on substituting in Equation (4-2) becomes,

\[
\log U = -\log \left(1 - \alpha \frac{N}{R}\right) + \log U_0 = \log \frac{U_0}{1 - \alpha \frac{N}{R}}. \tag{A4-13}
\]

i.e.

\[
U = \frac{U_0}{\left(1 - \alpha \frac{N}{R}\right)} + O(\sigma^2, \xi^2), \tag{A4-14}
\]
which is, when recast into dimensional variables becomes,

\[ u_s = \frac{u_{so}}{(1 - \kappa n)}. \quad (A4-15) \]

This equation gives the downstream velocity at position \( n \) in terms of the velocity at the reference axis. The convention adopted ensures that \( N \) and \( R \) have the same positive direction. On moving inwards from the reference line towards the centre of curvature, \((1 - \kappa n)\) decreases and \( u \) increases. The reverse occurs in moving away from the centre of curvature. In practice, this simple distribution will be modified by bank friction and secondary flows.

5 TRANSVERSE SURFACE GRADIENT

Scaling the \( n \) and \( z \) components of the momentum equation leads to an expression for the superelevation of the water surface.

Beginning with the scaled version of the \( n \) component, Equation (A2-2)

\[
\frac{\sqrt{\text{gd}}}{L} \frac{\partial}{\partial T} \left[ \sqrt{\text{gd}} \frac{b}{L} V \right] + \sqrt{\text{gd}} \frac{U}{L} \frac{1}{\left(1 - \frac{aN}{R}\right)} \frac{\partial}{\partial S} \left[ \sqrt{\text{gd}} \frac{b}{L} V \right] + \sqrt{\text{gd}} \frac{b}{L} V \left(1 - \frac{\partial}{\partial N} \left[ \sqrt{\text{gd}} \frac{b}{L} V \right] \right)
\]

\[
+ \sqrt{\text{gd}} \frac{d}{L} \frac{1}{N} \frac{\partial}{\partial Z} \left[ \sqrt{\text{gd}} \frac{b}{L} V \right] + \frac{1}{r_a} \frac{\partial}{\partial R} \left( \frac{gdU^2}{\left(1 - \frac{\alpha N}{R}\right)} \right) + \frac{1}{\rho} \frac{\partial}{\partial N} (P \rho gd) = 0,
\]

\((A5-1)\)

this can be simplified to give,
\[
\sigma^2 \frac{\partial V}{\partial T} + \sigma^2 \frac{U}{1 - \alpha \frac{N}{R}} \frac{\partial V}{\partial S} + \sigma^2 V \frac{\partial V}{\partial N} + \sigma^2 W \frac{\partial V}{\partial Z} + \frac{\alpha U^2}{R \left(1 - \alpha \frac{N}{R}\right)} + \frac{\partial P}{\partial N} = 0.
\]

(A5-2)

As \( \sigma^2 \) is negligible,

\[
\frac{\alpha U^2}{R \left(1 - \alpha \frac{N}{R}\right)} + \frac{\partial P}{\partial N} = 0.
\]

(A5.3)

Substituting for \( U \), from Equation 4-3, gives,

\[
\frac{\partial P}{\partial N} = -\frac{\alpha}{R \left(1 - \alpha \frac{N}{R}\right)} \frac{U^2}{\left(1 - \alpha \frac{N}{R}\right)^3}.
\]

(A5.4)

As this expression contains only \( P \) and \( N \) it can be re-expressed as an ordinary differential equation, with separated variables, and can be integrated with the substitution \( \theta = 1 - \alpha \frac{N}{R} \) and the limits \( \theta = 1 \), \( \theta = \left(1 - \alpha \frac{N}{R}\right) \).

Integrating gives

\[
P = -\frac{U^2}{2\theta^2} \left[1 - \frac{1}{\theta^2} \frac{1}{\left(1 - \alpha \frac{N}{R}\right)^2}\right] + F(S,Z,T),
\]

(A5.5)

where \( F(S,Z,T) \) is the coefficient of integration allowing for the dependence on \((s,n,z)\).
Part of this expression can be found by non-dimensionalising the $z$ component (Equation (2-3)) which gives,

$$
\frac{\sqrt{gd}}{L} \frac{\partial}{\partial T} \left[ \sqrt{gd} \frac{dW}{L} \right] + \sqrt{gd} U \frac{1}{L} \frac{\partial}{\partial S} \left[ \sqrt{gd} \frac{dW}{L} \left( 1 - \frac{\alpha N}{R} \right) \right]
$$

$$
+ \sqrt{gd} \frac{b}{L} \sqrt{ \frac{1}{b} \frac{\partial}{\partial N} \left[ \sqrt{gd} \frac{dW}{L} \right] } + \sqrt{gd} \frac{dW}{L} \frac{1}{d} \frac{\partial}{\partial Z} \left[ \sqrt{gd} \frac{dW}{L} \right]
$$

$$
+ \frac{1}{\rho d} \frac{\partial}{\partial Z} (P \rho gd) + g = 0,
$$

(A5-6)

which can be simplified to,

$$
\xi^2 \left[ \frac{\partial W}{\partial T} + \frac{1}{1 - \alpha \frac{N}{R}} \frac{\partial W}{\partial S} + \frac{\partial W}{\partial Z} \right] + \frac{\partial P}{\partial Z} + 1 = 0.
$$

(A5-7)

As $\xi^2$ is negligible, the differential equation becomes the hydrostatic case

$$
\frac{\partial P}{\partial Z} = -1.
$$

(A5-8)

Integration gives

$$
P = -Z + G(S, N, T).
$$

(A5-9)

The pressure equation is now

$$
P = \frac{U_n^2}{2} \left[ 1 - \frac{1}{\left( 1 - \alpha \frac{N}{R} \right)^2} \right] - Z + I(S, T),
$$

(A5-10)

where the function $I(S, T)$ is obtained by substituting the variables which define the water surface at the $s$ axis, i.e.
\[ Z = Z_o = H_o(S,T) \]
\[ P = 0 \]
\[ N = 0. \]

Substituting these values into the total pressure equation gives,
\[ 0 = I(S,T) + \frac{U_o^2}{2} \left( 1 - 1 \right)^2 - H_o(S,T), \]
so that
\[ I(S,T) = H_o(S,T). \]  
(A5-11)

The pressure at the point defined by \( N \) and \( Z \) is therefore;
\[ P = H_o + \frac{U_o^2}{2} \left[ 1 - \frac{1}{\left( 1 - \alpha \frac{N}{R} \right)^2} \right] - Z. \]  
(A5-12)

An equation for the surface profile is obtained by substituting values for a general point on the surface, i.e.
\[ Z = H(S,N,T), \]
\[ P = 0. \]

This gives the equation
\[ 0 = H_o + \frac{U_o^2}{2} \left[ 1 - \frac{1}{\left( 1 - \alpha \frac{N}{R} \right)^2} \right] - Z, \]  
(A5-13)

which can be rearranged to give,
\[ H = H_o + \frac{U_o^2}{2} - \frac{U_o^2}{2 \left(1 - \alpha \frac{N}{R}\right)^2} = H_o + \frac{U_o^2}{2} - \frac{U^2}{2}. \quad (A5-14) \]

In dimensional variables these expressions become.

\[ \eta = \eta_o + \frac{u_o^2}{2g} - \frac{u_o^2}{2g(1 - \kappa n)} \quad (A5-15) \]

and

\[ \eta = \eta_o + \frac{u_o^2}{2g} - \frac{u^2}{2g}. \quad (A5-16) \]
6 COMPLETE MODEL FOR IRROTATIONAL FLOW.

Assembling the expressions derived in this appendix and in Section 4.4 the complete engineering model is:

Mass Conservation Equation

\[
(1 - \kappa n_m) \frac{\partial \eta_o}{\partial t} - \kappa n_m \frac{2u_o}{g} \frac{\partial u_o}{\partial t} + u_o (1 + \kappa n_m) \frac{\partial \eta_o}{\partial s} + \left[ \frac{A}{B} \left(1 + \kappa \bar{n}\right) - \kappa n_m \frac{2u_o^2}{g} \right] \frac{\partial u_o}{\partial s} = \frac{q}{B} - u_o (S_o + \kappa n_m S_l) + \frac{\partial \kappa}{\partial s} n_m \frac{u_o}{g} - u_o \frac{A}{B} \bar{n} \frac{\partial \kappa}{\partial s}.
\]  
\quad (A6.1)

Momentum Equation

\[
\left( \frac{A}{B} - 2\kappa n_m \frac{u_o^2}{g} \right) \frac{\partial u_o}{\partial t} + \left[ 2u_o \frac{A}{B} \left(1 + \kappa \bar{n}\right) - 2\kappa n_m \frac{u_o^2}{g} \right] \frac{\partial u_o}{\partial s} + u_o \frac{\partial \eta_o}{\partial t} + \left[ u_o^2 \left(1 + 2\kappa n_m\right) + \frac{gA}{B} \right] \frac{\partial \eta_o}{\partial s} = u_o q - 2u_o^2 \frac{A}{B} \bar{n} \frac{\partial \kappa}{\partial s} + u_o^2 \left( \frac{A}{B} \bar{n} + n_m \frac{u_o^2}{g} \right) \frac{\partial \kappa}{\partial s} - u_o \left( S_o + 2\kappa n_m S_l \right) - S_f gA \frac{A}{B} (1 - \kappa \bar{n}).
\]  
\quad (A6.2)

Dynamic Wave Celerity

\[
c = \sqrt{\left( \frac{Ag}{B} \right) \left( 1 + \frac{1}{2} \kappa \bar{n} + \frac{1}{2} \kappa n_m \right)}
\]  
\quad (A6.3)

Transverse Velocity Distribution

\[
u_v = \frac{u_{vo}}{1 - \kappa n}
\]  
\quad (A6.4)

Transverse Surface Gradient

\[
\eta = \eta_o + \frac{u_o^2}{2g} - \frac{u_o^2}{2g(1 - \kappa n)}
\]  
\quad (A6.5)
APPENDIX III INPUT DATA FOR CHAPTER 7

The following tables contain the basic data for TEST 2 and TEST 21. These were obtained by reading off the values of time and depth for each of the Station 1 graphs.

TEST 2

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<th>Time (sec)</th>
<th>Depth (m)</th>
<th>Area A (sq.m.)</th>
<th>D/chg (cume)</th>
<th>Time (sec)</th>
<th>Depth (m)</th>
<th>Area A (sq.m.)</th>
<th>D/chg (cume)</th>
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<td>0.111</td>
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TABLE AIII - 1 INPUT DATA FOR TEST 2
**TEST 21**

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<th>Time (sec)</th>
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