Symmetric Covers of Graphs and Maps

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Abstract

The symmetry of a combinatorial graph can be measured by its automorphism group. Graphs with a high degree of symmetry are interesting from mathematical and aesthetic perspectives, and have also proved useful in practical contexts such as the design of efficient networks (with good broadcast or distribution properties). In particular, the best known graphs for the degree-diameter and cage problems (that is, the largest known connected graphs with given degree and diameter, and the smallest known connected graphs with given degree and girth) are often vertex-transitive or arc-transitive, and are sometimes covers of small and well-known examples such as the 3-cube or the Petersen graph.

In this thesis, we investigate techniques for the construction of covering graphs and regular maps. One particular focus is the construction of arc-transitive abelian regular covering graphs of arc-transitive graphs, in order to help classify the arc-transitive abelian regular covering graphs of small order symmetric graphs, and produce new families of symmetric graphs, some of which may be relevant to the degree-diameter problem.

Until now, covering techniques have been used mainly for the construction of edge- or arc-transitive cyclic and elementary abelian regular covers of small order symmetric cubic and tetravalent graphs. In most cases, the approach has involved voltage graph techniques. But other covers (such as homocyclic regular covers, or more general abelian regular covers) have not received much attention, and voltage graph techniques have a limited range of usefulness. In this thesis, a new approach is introduced that can be used more widely, such as for classifying symmetric abelian regular covers of symmetric graphs or regular maps. This approach use some combinatorial group theory and group

representation theory, and other methods for determining suitable quotients.

As an application, we classify all the symmetric abelian regular covers of the complete graph K_4, the complete bipartite graph $K_{3,3}$, the cube graph Q_3, the Petersen graph and the Heawood graph. We also believe that our methods would open the way for classifying symmetric abelian regular covers of graphs of higher valency.

Also we construct some families of abelian regular covers of orientably-regular maps, in order to show that for at least 83% of all positive integers g, there exists at least one orientably-regular map of genus g with simple underlying graph, and conjecture that there exists at least one such map for every positive integer g.

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Chapter 1

Introduction

This thesis presents some research on a new approach for classifying symmetric abelian regular covering graphs of finite symmetric cubic graphs, and also constructing abelian regular covers of orientable regular maps with simple underlying graphs.

In this Chapter, we give some background theory, in particular, including some details of voltage graph techniques and their applications and limitations, and our new approach for regular covering constructions.

Throughout this thesis, unless explicit exception is made, every graph X is assumed to be finite, undirected, connected and simple — that is, with no loops and no multiple edges. We use $V(X), E(X)$ and $A(X)$ to denote the vertex-set, edge-set and arc-set of X, respectively. Definitions of many of the terms used but not defined in this Chapter can be found in Chapter 2.

1.1 Coverings

Covering techniques are known to be a useful tool in algebraic and topological graph theory. Application of these techniques has resulted in many important examples and classifications of certain families of graphs. Of particular note is the technique of 'voltage graphs', which was developed by Gross and Tucker [36], and is often used. In order to introduce these things, we begin with some background definitions.

A graph *homomorphism* f from a graph $\tilde{X}$ to graph X is a mapping from the vertex set $V(\tilde{X})$ to the vertex set $V(X)$ such that if $\{u,v\} \in E(\tilde{X})$ then $\{f(u), f(v)\} \in E(X)$. When f is surjective, X is called a *quotient* of $\tilde{X}$. For $v \in V(X)$, let $N(v)$ denote the set of neighbours of v in X. A *covering projection* is defined as a graph homomorphism $p : \tilde{X} \to X$ which is surjective and locally bijective, which means that the restriction $p : N(\tilde{v}) \to N(v)$ is a bijection, whenever $\tilde{v}$ is a vertex of $\tilde{X}$ such that $p(\tilde{v}) = v \in V(X)$. We call X the *base graph*, $\tilde{X}$ a *covering graph* (or *derived graph*), and the pre-images $p^{-1}(v), v \in V(X)$ the *fibres*. A covering projection $p : \tilde{X} \to X$ is called *regular* if there exists a semi-regular subgroup N of the group of automorphisms $\mathrm{Aut}(\tilde{X})$ of $\tilde{X}$ such that the quotient graph $\tilde{X}/N$ (with vertices taken as the orbits of N on $V(\tilde{X})$) is isomorphic to X. In that case we call N the *covering* $(transformation)$ *group*, or *voltage group*.

There are two special cases of interest. The regular covering projection is called *cyclic* or *elementary abelian* if N is a cyclic or elementary abelian group. Similarly, we may say that a regular covering projection is *abelian* (or *homocyclic*, or *non-homocyclic abelian*) when the group N is abelian (or homocyclic, or non-homocyclic abelian, respectively).

Two regular covering projections $p : \tilde{X} \to X$ and $p' : \tilde{X}' \to X$ are called *equivalent* if there exists a graph isomorphism $\hat{\alpha} : \tilde{X} \to \tilde{X}'$ such that $p = p'\hat{\alpha}$. Usually, regular covers are studied up to equivalence.

1.2 Voltage techniques and applications

The above properties can be exploited to construct regular covering graphs of a given graph, as follows.

Let X be a connected graph, and let N be a finite group. Suppose $\zeta : A(X) \to N$ is a function assigning a group element to each arc of X, such that $\zeta(v,u) = (\zeta(u,v))^{-1}$ for every arc $(u,v) \in A(X)$. Here ζ is called a *voltage assignment*, N is called the *voltage group* (or *covering group*), and the values of ζ are called *voltages*. In particular, ζ is called *reduced* if the values of ζ on a spanning tree are trivial (equal to the identity element of N). We may construct a larger graph $X \times_\zeta N$, called the (*derived*) *voltage graph* (or *covering graph*), with vertex set $V(X) \times N$ and adjacency defined by $(u, g) \sim (v, h)$ if and only if $u \sim v$ and $h = g\zeta(u, v)$. Examples are given in Figure 1.1 and Figure 1.2.

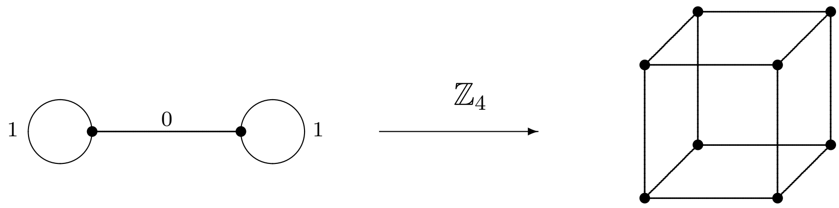


Figure 1.1: The 3-cube graph Q_3 is a $\mathbb{Z}_4$-cover of the dipole graph with voltages as indicated

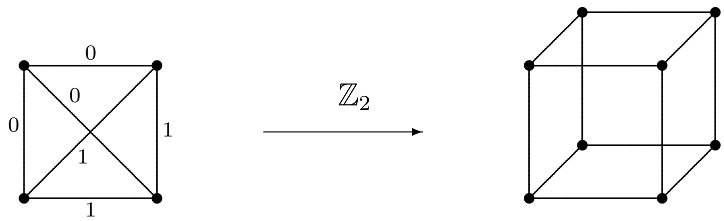


Figure 1.2: Q_3 is a $\mathbb{Z}_2$-cover of the complete graph K_4 with voltages as indicated

The following facts are well known, and can be easily proved.

Lemma 1.2.1 *If the covering graph $\tilde{X}$ is connected, then the covering group can be generated by the voltages.*

Lemma 1.2.2 [46] *Given any voltage assignment $\zeta : A(X) \to N$, there exists a reduced voltage assignment $\eta : A(X) \to N$ such that $X \times_\zeta N$ isomorphic to $X \times_\eta N$.*

The edges of a graph that are not included in a given spanning tree are often called *co-tree edges*, and the number of these is called the *Betti number* of the graph. The *rank* of a group is the smallest cardinality of a generating set. From the above two lemmas, we can see that for a connected covering graph, the rank of the covering group is no larger than the Betti number of the base graph.

Observe that a voltage assignment on arcs can be extended to a voltage assignment on walks in a natural way, such that if C and C' are walks with the last vertex of C being the same as the first vertex of C', then $\zeta(CC') = \zeta(C)\zeta(C')$ in the voltage group.

Let S be the set of voltages of closed walks in X based at a fixed vertex $v \in V(X)$. Given $\alpha \in \mathrm{Aut}(X)$, we define a function $\hat{\alpha} : S \to N$ by

$$(\zeta(C))^{\hat{\alpha}} = \zeta(C^{\alpha})$$

for $C \in S$. Note that if N is abelian, then $\hat{\alpha}$ does not depend on the choice of the base vertex, and the fundamental closed walks at v can be substituted by the fundamental cycles generated by the co-tree arcs of X. See [49] for further details.

Let $p : \tilde{X} \to X$ be a covering projection. Suppose there exists $\alpha \in \mathrm{Aut}(X)$ and $\tilde{\alpha} \in \mathrm{Aut}(\tilde{X})$ such that $\alpha \cdot p = p \cdot \tilde{\alpha}$, that is, such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{p} & X \\ \tilde{\alpha} \downarrow & & \downarrow \alpha \\ \tilde{X} & \xrightarrow[p]{} & X \end{array}$$

x Then we say that $\tilde{\alpha}$ is a *lift* of α, and that α is a *projection* of $\tilde{\alpha}$. Observe that α is uniquely determined by $\tilde{\alpha}$, but it is not true in general that $\tilde{\alpha}$ is uniquely determined by α. The set of all lifts of a given $\alpha \in \mathrm{Aut}(X)$ is denoted by $L(\alpha)$. In particular, when α is the identity automorphism ι, we denote $L(\iota)$ by $\mathrm{CT}(p)$, and call it the group of covering transformations. (That is the group of all self-equivalences of p, that is, of all automorphisms $\tau \in \mathrm{Aut}(\tilde{X})$ such that $p\tau = p$.) Generally, let G be a subgroup of $\mathrm{Aut}(X)$. If for each $\alpha \in G$, the set $L(\alpha)$ of all lifts of α is non-empty, then $\tilde{G} = \bigcup_{\alpha \in G} L(\alpha)$ is a group called the *lift* of G. We also call G a *projection* of $\tilde{G}$.

The use of covering techniques and voltage graph techniques to study symmetric graphs has been a topic of considerable interest, see [27–32, 49–54] for example.

A graph X is called *symmetric* (or *arc-transitive*) if it has the property that given any two pairs of adjacent vertices (u_1, v_1) and (u_2, v_2), there is an automorphism f of X such that $f(u_1) = u_2$ and $f(v_1) = v_2$, — in other words, the automorphism group $\mathrm{Aut}(X)$ of X acts transitively on ordered pairs of adjacent vertices.

An *s-arc* in a graph is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices of the graph such that v_i is adjacent with v_{i+1} for $0 \le i \le s-1$, and $v_{i-1} \ne v_{i+1}$ for $1 \le i \le s-1$; in other words, any two consecutive vertices are adjacent, and any three consecutive vertices are distinct. A graph X is called *s-arc-transitive* if its automorphism group $\mathrm{Aut}(X)$ acts transitively on the set of all s-arcs of X. For a subgroup G of $\mathrm{Aut}(X)$, the graph X is called (G, s)-arc-transitive (respectively, (G, s)-arc-regular) if G is transitive (respectively, sharply–transitive) on the set of s-arcs of X. In particular, if $G = \mathrm{Aut}(X)$, then we call X an $(\mathrm{Aut}(X), s)$-arc-transitive (respectively, $(\mathrm{Aut}(X), s)$-arc-regular) graph, or simply an s-arc-transitive (respectively, s-arc-regular) graph.

Independently in the 1970s, Conway (see [1, Corollary 19.6]) and Djoković [25] used graph covers to construct the first infinite family of finite 5-arc-transitive cubic graphs, as elementary abelian covers of Tutte's 8-cage. Also Djoković [25] used lifts of automorphisms of graphs along covering projections to study s-arc-transitivity of graphs, and showed that if an s-arc-transitive group of automorphisms can be lifted along a regular covering projection, then the covering graph is at least s-arc-transitive. Subsequently Biggs [2] developed a method for constructing certain 5-arc-transitive cubic graphs as covers of cubic graphs that are 4-arc- but not 5-arc-transitive.

Later, Malnič, Marušič and Potočnik [49] took these ideas further in a systematic study of regular covering projections of a given connected graph along which a given group of automorphisms lifts, and used this to give an explicit means of construction of such coverings when the covering group is elementary abelian. Their approach involves taking an appropriate representation of automorphisms of the base graph (by matrices), and then converting the conditions for lifting into a problem of finding invariant subspaces of certain concrete groups of matrices over prime fields.

The approach developed in [49] has been successfully applied to the classification of elementary abelian regular covers of a number of symmetric graphs of small valency. Many examples have been handled by this method.

For example, for cubic graphs, Malnič, Marušič, Miklavič and Potočnik [50] classified non-isomorphic minimal semi-symmetric elementary abelian regular covers of the Möbius-Kantor graph and the generalized Petersen graph $\mathrm{GP}(8,3)$. Malnič and Potočnik [51] classified the vertex-transitive elementary abelian regular covers of the Petersen graph. Also Oh [53] classified all the symmetric cubic graphs of order $16p$ by classifying the arc-transitive cyclic regular covers of the Möbius-Kantor graph. Malnič, Marušič and Potočnik [49] classified the edge-transitive elementary abelian regular covers of prime-dipoles and semi-symmetric elementary abelian regular covers of the Heawood graph. Furthermore, Oh [52] classified all the symmetric cubic graphs of order $14p$ by classifying the arc-transitive cyclic regular covers of the Heawood graph. Oh [54] classified the arc-transitive elementary abelian regular covers of the Pappus graph. For 4-valent graphs, Kuzman [42] classified the arc-transitive elementary abelian regular covers of the complete graph K_5. Kwak and Oh [43] classified the arc-transitive elementary abelian regular covers of the octahedron graph.

Another method was developed by Du, Kwak and Xu [27]. Rather than using matrices to represent the automorphisms, they obtained linear criteria for lifting automorphisms. Using this approach, various arc-transitive cyclic or elementary abelian regular covers of small order cubic graphs were classified.

For instance, Feng and Kwak [30] classified the arc-transitive elementary abelian regular covers of the complete graph K_4, and the complete bipartite graph $K_{3,3}$, and as an application, obtained a complete list of non-isomorphic symmetric cubic graphs of orders $4p$, $6p$, $4p^2$ and $6p^2$, where p is prime. Also Feng and Kwak [28] gave an infinite family of cubic 1-arc-regular graphs, which they constructed by classifying the symmetric cyclic regular covers of the complete bipartite graph $K_{3,3}$. Similarly, Feng and Wang [32] classified the symmetric cyclic covers of the 3-dimensional cube graph Q_3. Also Feng, Kwak and Wang [31] classified the symmetric elementary abelian regular covers of Q_3, and as a further application, classified all symmetric cubic graphs of order $8p$ or $8p^2$ where p is prime. Similarly, Feng and Zhou [59] classified the semi-symmetric elementary abelian regular covers of the Heawood graph; and Feng and Kwak [29] classified the symmetric elementary abelian regular covers of the Petersen graph, and all the symmetric cubic graphs of order $10p$ and $10p^2$ where p is prime are given.

1.3 Limitations of voltage graph techniques

In the construction of symmetric (and semi-symmetric) regular covers of symmetric graphs to date, most attention has been paid to cases where the covering group is either cyclic or elementary abelian. The voltage graph construction works particularly well in these cases, and can also be used to construct regular covers with homocyclic covering groups with non-prime exponent.

Regular covers with more general covering groups have not received much attention. In the method developed by Malnič et al [49], for elementary abelian regular covers, the automorphism matrices are defined over prime fields. For homocyclic regular covers, the corresponding automorphism matrices need to be defined over more general integer rings (such as $\mathbb{Z}_4$ or $\mathbb{Z}_9$) instead of the prime fields. The associated computations for finding invariant subspaces can be carried out easily for small dimensional matrices, but can be much more complicated in larger dimensions. For non-homocyclic abelian regular covers, this method is still possible (with appropriate modifications), but difficult to apply.

Similarly, in the method developed by Du et al [27], the linear criteria for lifting automorphisms are equivalent to solving linear equations over prime fields. (These equations are derived from the action of automorphisms of the voltage group on the fundamental cycles.) Homocyclic regular covers can also be constructed and classified by solving the equations over integer rings, but again, the computations become very difficult if the dimension (which is the number of co-tree edges) is large. Furthermore, it is difficult to apply this method to the classification of non-homocyclic abelian regular covers.

Another limitation of the voltage graph technique is that it does not help much in determining the size of the automorphism group (and in particular, the s-arc-transitivity) of the covering graph. For elementary abelian regular covers, Feng et al [31, 29, 28, 32] used Sylow theorems and other techniques based on a complete list of symmetric cubic graphs of small order given by Conder [14], to find the size of the automorphism group. For homocyclic regular covers and more general abelian regular covers, however, these techniques are limited and difficult to extend.

1.4 New approach for construction of covers

In this section we introduce a new approach for classifying arc-transitive abelian regular covers of symmetric cubic graphs, using some group theory, character theory, and a study of 'layers' of the covering group. As an application, we classify all of the arc-transitive abelian regular covers of some symmetric cubic graphs of small order.

In contrast to the approaches that use voltage graph techniques, our approach is quite general, in that it can also be taken when considering covers of other discrete structures with large automorphism groups, such as regular maps, Hurwitz surfaces (and other compact Riemann surfaces with large automorphism group), and abstract regular polytopes — or indeed whenever there exist universal groups for the kinds of group actions of interest.

Our approach was described briefly by Marston Conder at the AGTAGC 2010 workshop (on algebraic, topological and complexity aspects of graph covers) in Auckland in February 2010. It was largely motivated by work by Leech [44] in determining the structure of certain normal subgroups of the $(2, 3, 7)$ triangle group Δ and the actions by conjugation of the generators of Δ on generators of those subgroups, and the use of these by Cohen [7] to classify abelian covers of Klein's quartic surface (or equivalently, all regular maps of type $\{3, 7\}$ that cover the Klein map of genus 3). The latter covers were critical to a complete determination of all Hurwitz surfaces (or equivalently, all regular maps of type $\{3, 7\}$) of genus 2 to 11905, achieved by Conder in [8, 9].

In order to explain this new approach, we need to describe some further background theory.

Let X be a finite symmetric graph, and suppose G is a group of automorphisms of X acting transitively on $A(X)$. Also let v be any vertex of X, let $H = G_v = \{g \in G : v^g = v\}$, the stabilizer in G of the vertex v, and let $a \in G$ be any automorphism of X that reverses an arc (v, w) incident with v in X. Then the vertices of X can be identified with the right cosets Hx of H in G, and in particular, the arc (v, w) can be identified with the ordered pair (H, Ha). By arc-transitivity, every arc of X is of the form (Hg, Hag) for some $g \in G$, and therefore the vertices Hx and Hy are adjacent if and only if $xy^{-1} \in HaH$.

These observations can be exploited to give a well-known construction for symmetric graphs.

Let G be any group containing a subgroup H of finite index, let a be any element of G such that $a^2 \in H$, and the elements of $H \cup \{a\}$ generate G. Define a graph $X = X(G, H, a)$ by taking the right cosets of H in G as vertices of X, and joining Hx to Hy whenever $xy^{-1} \in HaH$. The group G acts on X (as a group of automorphisms) by right multiplication $g : Hx \mapsto Hxg$ (for $g \in G$) making X a connected arc-transitive graph of order $|G : H|$ and valency $d = |H : H \cap aHa^{-1}|$.

For symmetric cubic graphs, Tutte's theorem [58] shows that every finite symmetric cubic graph is at most 5-arc-transitive. Djoković and Miller [26] showed further that the automorphism group must be a quotient of one of seven finitely-presented 'universal' groups $G_1,\ G_2^1,\ G_2^2,\ G_3,\ G_4^1,\ G_4^2$ and G_5. Indeed every arc-transitive group of automorphisms of a finite symmetric cubic graph is a quotient of one of these seven groups. (More details can be seen in Section 2.4.)

Now suppose X is a symmetric cubic graph, and $\tilde{X}$ is a symmetric regular cover of X. Let G be an arc-transitive group of automorphisms of X that lifts to an arc-transitive group of automorphisms $\tilde{G}$ of $\tilde{X}$. Then G is a quotient $\mathcal{U}/K$, where $\mathcal{U}$ is one of the seven groups $G_1, G_2^1, G_2^2, G_3, G_4^1, G_4^2$ or G_5, and K is a torsion-free normal subgroup of $\mathcal{U}$. Similarly, $\tilde{G}$ is a quotient $\mathcal{U}/L$ where L is a normal subgroup of $\mathcal{U}$ contained in K. The quotient K/L is the voltage group for the regular covering projection of X by $\tilde{X}$.

In order to find and classify all symmetric regular covers of X, for which the subgroup G lifts, we seek to find all possibilities for a normal subgroup L of finite index in $\mathcal{U}$ such that L is contained in K.

To do this, we can first use Reidemeister-Schreier theory (which is implemented as the `Rewrite` command in Magma [3]) to find a presentation for K by generators and relations, using a given presentation for $\mathcal{U}$. Algebraic or computational techniques can be applied to find the actions by conjugation of the generators of $\mathcal{U}$ on the generators of K. Abelianising K allows us to classify abelian regular covers, by investigating the structure of K/K', and finding all subgroups of finite index in K/K' that are invariant under the action of generators of $\mathcal{U}$.

1.5 Regular maps with simple underlying graph

Regular maps are highly symmetric embeddings of graphs or multi-graphs on closed surfaces. The formal study of regular maps was initiated by Brahana [4] in the 1920s and continued by Coxeter (see [24]) and others decades later. Deep connections exist between regular maps and other branches of mathematics, including hyperbolic geometry, Riemann surfaces and, rather surprisingly, number fields and Galois theory. (See some of the references, such as [39], for further background).

Regular maps on the sphere and the torus and other orientable surfaces of small genus are now quite well understood, but until recently, the situation for surfaces of higher genus was something of a mystery. A significant step towards answering some longstanding questions about the genera of orientable surfaces carrying a regular map having no multiple edges, or an 'orientably-regular' map that is chiral (admitting no reflectional symmetry) was taken by Conder, Siráň and Tucker in [23], after the first author noticed patterns in computational data about regular maps of small genus (see [13] and the associated lists of maps available on Conder's website).

One question of interest has been the genus spectrum of orientably-regular maps with simple underlying graph — that is, where the embedded graph has no loops or multiple edges. It is well known that for every $g > 0$ there exists a reflexible regular map of type $\{4g, 4g\}$ on an orientable surface of genus g (with dihedral automorphism group). It follows that there are no 'gaps' in the genus spectrum of orientable surfaces carrying reflexible regular maps. On the other hand, the underlying graphs for these maps are highly degenerate, being bouquets of $2g$ loops based at a single vertex.

A closely-related question concerns the genera of those orientably-regular maps with the property that the underlying graphs of both the map and its dual are simple. From the evidence described in [13], it was discovered that there are gaps in this spectrum: there are no such maps of genus 20, 23, 24, 30, 38, 39, 44, 47, 48, 54, 60, 67, 68, 77, 79, 80, 84, 86, 88 or 95, but there is at least one of genus g for every other g in the range $0 \le g \le 101$.

Two of the main results of [23] were that (a) If M is an orientably-regular but chiral map of genus $p+1$, where p is prime, and $p-1$ is not divisible by 5 or 8, then either M or its topological dual M^* has multiple edges, and (b) if M is a reflexible regular map of genus $p+1$, where p is prime and $p > 13$, then either M or M^* has multiple edges, and if also $p \equiv 1 \bmod 6$, then both M and M^* have multiple edges.

It follows from these that if $g = p+1$ for some prime $p > 13$ such that $p-1$ is not divisible by 5 or 8, then there exists no orientably-regular map of genus g such that the underlying graphs of both the map and its dual are simple. Hence there are infinitely many exceptions, well beyond the brief list given two paragraphs above.

On the other hand, if we are happy for just one of M and M^* to have simple underlying graph, then the situation is intriguing. The exceptions arising from (b) for reflexible regular maps are genera of the form $g = p+1$ where p is a prime congruent to 1 mod 6, but for each of these, there is an orientably-regular but chiral map of type $\{6, 6\}$ of genus g with simple underlying graph. Hence these exceptions for reflexible maps are not exceptions for chiral maps.

In fact, it is easy to see from the Platonic maps, the toroidal regular maps and the lists of all regular maps of small genus (associated with [13]) that for every integer g in the range $0 \le g \le 101$, there exists at least one orientably-regular map of genus g with simple underlying graph.

Hence the obvious question arises: is there any positive integer g for which there exists no orientably-regular map of genus g with simple underlying graph?

We are prepared to conjecture that the answer is 'No', but a proof would be difficult. In this thesis, we provide further evidence in support of it, by proving the existence of several infinite families of examples, covering various pieces of the genus spectrum.

We construct the maps via their automorphism groups (or at least their orientation-preserving groups of automorphisms), using a range of combinatorial group-theoretic techniques. These include semi-direct product constructions (as used in [16] to produce regular maps on non-orientable surfaces of over 77% of all possible genera), and some more general methods similar to those we use for symmetric cubic graphs.

1.6 Layout of remainder of thesis

We give some further background in Chapter 2, on various relevant topics. Then in Chapters 3 to 6, we apply our new approach to find all the symmetric abelian regular covers of K_4, $K_{3,3}$, Q_3 and the Petersen graph, respectively. In Chapter 7, we do the same for the Heawood graph. This case is more difficult than the earlier ones, but is also more interesting, in that every arc-transitive group of automorphisms of the Heawood graph is 1-arc-regular or 4-arc-regular, but some of its regular covers have a 2-arc-regular automorphism group. Some details from this case are given in the Appendix.

In Chapter 8, we switch to the study of regular maps, where we construct families of abelian regular covers of given regular maps in order to prove the existence of regular maps with simple underlying graphs on surfaces of over 5/6 of all possible genera. Finally, we make some concluding remarks and mention some potential topics for future research in Chapter 9.

Before proceeding, we note that very little of this work would have been likely without the benefit of the use of the computational algebra system Magma [3] to produce and analyse examples and to experiment with a number of constructions.

Chapter 2

Preliminaries

In this Chapter, we introduce some further basic definitions and background theory related to permutation groups, finitely-presented groups, graph automorphisms, finite group representations, symmetric cubic graphs, regular maps, and our new approach for finding symmetric abelian regular covers.

2.1 Permutation groups

A *permutation group* G is a group whose elements are permutations of a given set Ω, that is, bijective functions from Ω to Ω, and the group operation is composition of permutations in G. In particular, any subgroup of a symmetric group is a permutation group. A *permutation representation* of a group G on a finite set Ω is a group homomorphism from the group G to $\mathrm{Sym}(\Omega)$, the group of all permutations on Ω.

Let G be a permutation group on a set Ω and let $a \in \Omega$. We denote by G_a the *stabilizer* of a, that is, the subgroup $\{g \in G \mid a^g = a\}$ of G consisting of all elements that fix the point a. We say that G is *semi-regular* on Ω if $G_a = 1$ for all $a \in \Omega$, and *regular* on Ω if G is transitive and semi-regular on Ω. The *orbit* of a point a in Ω is the set of elements of Ω to which a can be moved by elements of G, that is, $\{a^g : \ g \in G\}$. This is denoted by a^G.

The following well known Orbit-Stabilizer Theorem can be easily found in books on permutation groups (such as [6]) and many other texts on group theory:

Theorem 2.1.1 *If G is a permutation group acting on a set Ω, then*

$$|G| = |G_a||a^G|$$

for all $a \in \Omega$.

2.2 Finitely-presented groups

In group theory, there are many different ways of defining and describing particular groups. One way of defining a given group G is by a presentation in terms of generators and relations. Let S be a set of generators for G, so that every element of G can be written as a product of elements of S and their inverses. Also let R be a set of relations involving members of the generating set S. Each relation in R can be written in the form $u = v$ where u and v are words on S, such as $uv^{-1} = 1$, in which case we call uv^{-1} a *relator*. Then we say G has presentation $\langle\ S \mid R\ \rangle$ if every relation satisfied by elements of S is a consequence of the relations in R and the group axioms. Equivalently, $G = \langle\ S \mid R\ \rangle$ if and only if G is isomorphic to the quotient $F(S)/\langle R \rangle^F$ where F is the free group on the alphabets and $\langle R \rangle^F$ is the normal closure in $F = F(S)$ of the relators obtainable from R.

Example 2.2.1 *The dihedral group D_n of order $2n$ has presentation $D_n = \langle\ a, b \mid a^2 = b^n = 1,\ b^a = b^{-1} \rangle$. Hence the elements a and b represent a reflection and a rotation, when D_n is considered as the symmetry group of a regular n-gon.*

2.2.1 The Reidemeister-Schreier process

Given a finitely-presented group $G = \langle\ S \mid R\ \rangle$ where $S = \{x_1, x_2, \cdots, x_m\}$ is the generating set and $R = \{r_1, r_2, \cdots, r_n\}$ is a set of relators, let H be a subgroup of finite index in G. We may form a graph whose vertices are the right cosets of H and whose edges are of the form Hg — Hgx_i for $1 \leq i \leq m$ and $g \in G$. This graph is called the *Schreier coset graph* $\Sigma(G, S, H)$, and gives a diagrammatic representation of the natural action of G on right cosets of H.

In the Schreier coset graph $\Sigma(G, S, H)$, a spanning tree gives a *Schreier transversal* — that is, a set $T = \{t_1, t_2, \cdots, t_k\}$ of coset representatives for the right cosets Hg of H in G, with the property that every left initial sub-word of each t_i also lies in T. The elements of T can be chosen as the words tracing paths from the vertex H of $\Sigma(G, S, H)$ to each other vertex of $\Sigma(G, S, H)$ in the spanning tree. For example, if the edges H — Ha and Ha — Hab^{-1} lie in the spanning tree, then a and ab^{-1} lie in T, with a being a left initial sub-word of ab^{-1} for H in G. Edges of the coset graph not used in the spanning tree (called the co-tree edges) give a Schreier generating set for H in G: the co-tree edge Hu — Hv given by multiplication by x_t gives the Schreier generator ux_tv^{-1} for H.

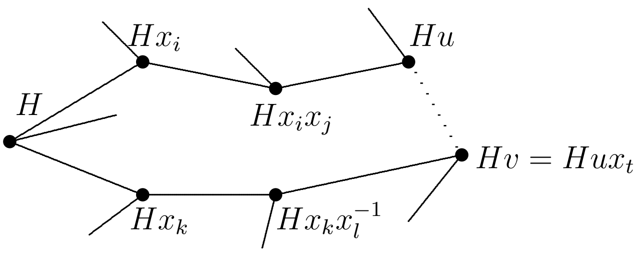


Figure 2.1: Schreier generators given by edges not in spanning tree

Reidemeister-Schreier theory (see [11, 12, 36], for example) provides a method for obtaining a presentation for the subgroup H in terms of generators and relations for the group G. This method consists of four steps, as follows:

1) Construct the Schreier coset graph $\Sigma(G, S, H)$;

2) Take a spanning tree in the Schreier coset graph, giving a Schreier transversal for H in G;

3) Label the co-tree edges with Schreier generators;

4) Apply each of the relators from R to each of the cosets in turn, to obtain the defining relations for H.

An implementation of this process is available in MAGMA [3] via the `Rewrite` command.

Example 2.2.2 *Let* $G = \langle\ x, y \mid x^2 = y^3 = 1\ \rangle$ *be the modular group, let* H *be the stabilizer of* 1 *in the permutation representation* $x \mapsto (2,3)$, $y \mapsto (1,2,3)$. *Taking* $T =$

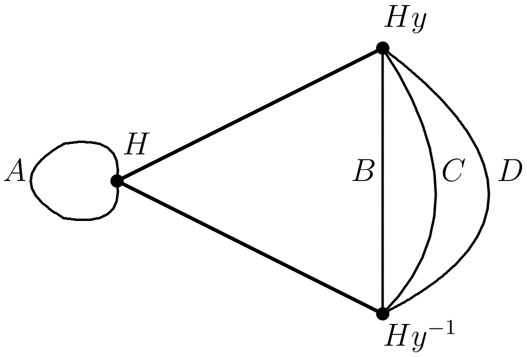


Figure 2.2: The Schreier coset graph of H

$\{1, y, y^{-1}\}$ *as Schreier transversal, we find the following as Schreier generators:* $A = x$, $B = y^3$, $C = yxy$ *and* $D = y^{-1}xy^{-1}$. *Then the relation* $x^2 = 1$ *gives new relations* $A^2 = 1$ *and* $CD = 1$, *while the relation* $y^3 = 1$ *gives new relation* $B = 1$. *After eliminating redundant generators, we find that* H *has presentation* $\langle\ A, C \mid A^2\ \rangle$ *via* $A = x$ *and* $C = yxy$.

Example 2.2.3 *Let* G *be the* $(2, 8, 3)$ *triangle group, with presentation*

$$G = \langle x, y, z \mid x^2 = y^8 = z^3 = xyz = 1 \rangle,$$

and let K *be a normal subgroup of* G *of index* 48. *Then by* MAGMA, *there is only one possibility for* K, *and this has five generators:* $w_1 = y^2xz^{-1}y$, $w_2 = (z^{-1}yx)^2$, $w_3 = xzy^3xy^{-1}$, $w_4 = zy^2xz^{-1}yz^{-1}$ *and* $w_5 = yxzyxz^{-1}xy^{-1}z^{-1}$. *The relations for* G *give the following relations for* K: $w_1^2 = 1$, $w_2^2 = 1$, $w_3^2 = 1$, $w_4^2 = 1$, $(w_1w_5)^2 = 1$ *and* $(w_2w_4w_5^{-1}w_3)^2 = 1$. *Thus* K *has presentation*

$$K = \langle\, w_1, w_2, w_3, w_4, w_5 \mid w_1^2 = w_2^2 = w_3^2 = w_4^2 = (w_1w_5)^2 = (w_2w_4w_5^{-1}w_3)^2 = 1\, \rangle.$$

2.2.2 Low index normal subgroups

In 2006, a method was developed by Firth and Holt, to find all normal subgroups of small index in a finitely-presented group. An implementation of this is available in MAGMA, via the `LowIndexNormalSubgroups` command.

Example 2.2.4 *Let* $G = \langle\, x, y \mid x^2 = y^3 = 1 \,\rangle$ *be the modular group. Using* MAGMA*, it takes only a few minutes to find the* 408 *normal subgroups of* G *with index at most* 2000*.*

2.3 Graph automorphisms

An *automorphism* of a simple graph X is a permutation π of the vertex-set $V(X)$ with the property that $\{u, v\}$ is an edge of X if and only if $\{u^\pi, v^\pi\}$ is an edge of X. The set of all automorphisms of X with the operation of composition is called the *automorphism group* of X, and denoted by $\mathrm{Aut}(X)$. In particular, the automorphism group $\mathrm{Aut}(X)$ is a permutation group on $V(X)$, when X is simple.

Example 2.3.1 *The automorphism group of a complete graph* K_n *is isomorphic to the symmetric group* S_n*. The automorphism group of the cycle graph* C_n *is the dihedral group* D_n *(of order* $2n$*).*

We say a graph X is *vertex-transitive* if $\mathrm{Aut}(X)$ acts transitively on the vertex-set $V(X)$, or in other words, if given any two vertices u and v, there is an automorphism $\pi \in \mathrm{Aut}(X)$ such that $\pi(u) = v$. For example, all Cayley graphs (see [1] for their definition) are vertex-transitive. The action of $\mathrm{Aut}(X)$ on $V(X)$ induces an action on $E(X)$, by the rule $\{x, y\}^\pi = \{x^\pi, y^\pi\}$ for $\{x, y\} \in E(X)$, and we say X is *edge-transitive* if the latter action is transitive. For example, the complete bipartite graph $K_{m,n}$ is edge-transitive, but not vertex-transitive when $m \neq n$.

Next, a *semi-symmetric graph* is a graph which is edge-transitive, but not vertex-transitive. The smallest regular semi-symmetric graph is the Folkman graph (see [33]) which has 20 vertices, 40 edges, diameter 4 and girth 4. Another example is the Ljubljana graph (see [21]) which has 112 vertices and 168 edges.

We say that X is *arc-transitive* if $\mathrm{Aut}(X)$ acts transitively on the arc-set $A(X) = \{(x, y) : \{x, y\} \in E(X)\}$. For example, the simple cycle graph C_n and the complete graph K_n are arc-transitive graphs. A graph is called *half-transitive* if it is vertex-transitive and edge-transitive but not arc-transitive. The smallest half-transitive graph is the Holt graph, which is 4-valent and has 27 vertices (see [5] for more details).

2.4 Symmetric cubic graphs

As we mentioned in Chapter 1, a graph X is called s-arc-transitive if $\mathrm{Aut}(X)$ acts transitively on the s-arcs of X, and s-arc-regular if this action is sharply-transitive. In particular, 0-arc-transitive means vertex-transitive, while 1-arc-transitive means arc-transitive (or symmetric). Note that under the connectedness assumption, s-arc-transitive implies $(s-1)$-arc-transitive, for all $s \geq 1$.

A lot is known about symmetric graphs that are 3-valent, or *cubic*, thanks mostly to two seminal theorems of Tutte [57, 58]. Tutte proved that a finite symmetric cubic graph can be at most 5-arc-transitive (and more details can be found in [58]). He also proved that the automorphism group of any symmetric cubic graph X acts regularly on s-arcs of X for some $s \leq 5$, in which case X is s-arc-regular. For example, K_4 and Q_3 are 2-arc-regular, while $K_{3,3}$ and the Petersen graph are 3-arc-regular, the Heawood graph is 4-arc-regular, and Tutte's 8-cage (on 30 vertices) is 5-arc-regular.

Tutte's work was taken further by Goldschmidt, Sims, Djoković and others (see [35, 56, 25]). We now know that the stabilizer of a vertex in any group acting regularly on the s-arcs of a finite connected cubic graph is isomorphic to either the cyclic group $\mathbb{Z}_3$, the symmetric group S_3, the direct product $S_3 \times \mathbb{Z}_2$, the symmetric group S_4 or the direct product $S_4 \times \mathbb{Z}_2$, depending on whether $s = 1, 2, 3, 4$ or 5 respectively. In the cases $s = 2$ and $s = 4$, there are two different possibilities for the edge-stabilizers, depending on whether or not the group contains an involution that reverses an arc, while for $s = 1, 3$ and 5 there is just one possibility (see [22, 26] for more details).

Taking into account the isomorphism type of the pair consisting of a vertex-stabilizer and edge-stabilizer, Djoković and Miller found seven types of universal groups that act arc-transitvely on the infinite cubic tree with finite vertex-stabilizer [26]. These seven groups were named $G_1, G_2^1, G_2^2, G_3, G_4^1, G_4^2$ and G_5, and can be presented as follows (see [17]):

$G_1 = \langle\, h, a \mid h^3 = a^2 = 1 \,\rangle$ (the modular group);

$G_2^1 = \langle\, h, p, a \mid h^3 = p^2 = a^2 = 1,\ php = h^{-1},\ a^{-1}pa = p \,\rangle$;

$G_2^2 = \langle\, h, p, a \mid h^3 = p^2 = 1,\ a^2 = p,\ php = h^{-1},\ a^{-1}pa = p \,\rangle$;

$G_3 = \langle\, h, p, q, a \mid h^3 = p^2 = q^2 = a^2 = 1,\ pq = qp,\ php = h,\ qhq = h^{-1},\ a^{-1}pa = q \,\rangle$;

$$G_4^1 = \langle\, h, p, q, r, a \mid h^3 = p^2 = q^2 = r^2 = a^2 = 1,\ pq = qp,\ pr = rp,\ (qr)^2 = p,$$
$$h^{-1}ph = q,\ h^{-1}qh = pq,\ rhr = h^{-1},\ a^{-1}pa = p,\ a^{-1}qa = r \,\rangle;$$

$$G_4^2 = \langle\, h, p, q, r, a \mid h^3 = p^2 = q^2 = r^2 = 1,\ a^2 = p,\ pq = qp,\ pr = rp,\ (qr)^2 = p,$$
$$h^{-1}ph = q,\ h^{-1}qh = pq,\ rhr = h^{-1},\ a^{-1}pa = p,\ a^{-1}qa = r \,\rangle;$$

$$G_5 = \langle\, h, p, q, r, s, a \mid h^3 = p^2 = q^2 = r^2 = s^2 = a^2 = 1,\ pq = qp,\ pr = rp,\ ps = sp,$$
$$qr = rq,\ qs = sq,\ (rs)^2 = pq,\ h^{-1}ph = p,\ h^{-1}qh = r,$$
$$h^{-1}rh = pqr,\ shs = h^{-1},\ a^{-1}pa = q,\ a^{-1}ra = s \,\rangle.$$

Now suppose G is a smooth quotient of one of the seven groups above, where 'smooth' means that the orders of the generators are preserved in the quotient. (This is equivalent to supposing that G is a quotient via some torsion-free normal subgroup, since every element of finite order in any of the above seven groups induces an automorphism of the (infinite) cubic tree of finite order and therefore stabilizes a vertex or an edge.) Then we may construct an arc-transitive graph X on which G acts as an arc-transitive group of automorphisms in the following way. Let S be the generating set for G consisting of images of the specified generators (from $\{h, a, p, q, r, s\}$). Let H be the subgroup generated by $S \backslash \{a\}$. Take the coset space $V = \{Hg : g \in G\}$ as the vertex-set, and join two vertices Hx and Hy by an edge whenever $xy^{-1} \in HaH$. This adjacency relation is symmetric, since $HaH = Ha^{-1}H$ in each of the seven cases. The group G acts on the right cosets by multiplication, preserving the adjacency relation. Since $HaH = Ha \cup Hah \cup Hah^{-1}$ in each of the seven cases, the graph X is cubic and symmetric. The graph X itself may be called a *double coset graph*. (See [17] for more details.)

In some cases, the full automorphism group $\text{Aut}(X)$ may contain more than one subgroup acting transitively on the arcs of X. When G is any such subgroup, G will be the image of one of the seven groups G_1, G_2^1, G_2^2, G_3, G_4^1, G_4^2 and G_5. Any subgroup G will be said to be of *type* $1, 2^1, 2^2, 3, 4^1, 4^2$ or 5, according to which of the seven groups it comes from.

The relationships among the seven types of groups $G_1, G_2^1, G_2^2, G_3, G_4^1, G_4^2$ and G_5 were considered in [26], and taken further in [22].

In order to make this thesis self-contained, some of the theorems for introducing the relations among these seven groups are quoted here.

Proposition 2.4.1 [22]

(a) *In the group G_2^1, the subgroup generated by h and a has index 2 and is isomorphic to G_1;*

(b) *The group G_2^2 contains no subgroup isomorphic to G_1;*

(c) *In the group G_3, the subgroup generated by h, a and pq has index 2 and is isomorphic to G_2^1, while the subgroup generated by h, ap and pq has index 2 and is isomorphic to G_2^2, and the subgroup generated by h and a has index 4 and is isomorphic to G_1;*

(d) *In the group G_4^1, the subgroup generated by h and a has index 8 and is isomorphic to G_1, but there are no subgroups of index 2 isomorphic to G_3, and no subgroups of index 4 isomorphic to G_2^1 or G_2^2;*

(e) *In the group* G_4^2*, there are no subgroups of index up to* 8 *that are isomorphic to* G_1, G_2^1, G_2^2 *or* G_3*;*

(f) *In the group* G_5*, the subgroup generated by* hpq, a *and* pq *has index* 2 *and is isomorphic to* G_4^1*, while the subgroup generated by* hpq, ap *and* pq *has index* 2 *and is isomorphic to* G_4^2*, and the subgroup generated by* h *and* a *has index* 16 *and is isomorphic to* G_1*, but there are no subgroups of index* 4 *or* 8 *isomorphic to* G_2^1, G_2^2 *or* G_3*.*

As a consequence of these relations, we also have the following:

Corollary 2.4.2 [22] *Let* G *be an arc-transitive group of automorphisms of a finite symmetric cubic graph* Γ*. Then*

- *if* G *has type* 2^2*, then* G *contains no subgroup of type* 1*;*
- *if* G *has type* 4^1*, then* G *contains no subgroup of type* $2^1, 2^2$ *or* 3*;*
- *if* G *has type* 4^2*, then* G *contains no subgroup of type* $1, 2^1, 2^2$ *or* 3*;*
- *if* G *has type* 5*, then* G *contains no subgroup of type* $2^1, 2^2$ *or* 3.

In particular, we also have the following results:

Proposition 2.4.3 [22, 26] *If the automorphism group of a* 3*-arc-regular finite cubic graph has an arc-transitive subgroup of type* 1 (*and index* 4)*, then it also has arc-transitive subgroups of types* 2^1 *and* 2^2 (*and index* 2)*. Similarly, if the automorphism group of a* 5*-arc-regular finite cubic graph has an arc-transitive subgroup of type* 1 (*and index* 16)*, then it also has arc-transitive subgroups of types* 4^1 *and* 4^2 (*and index* 2)*.*

This eliminates the possibility of the combinations $\{1, 2^1, 3\}$, $\{1, 2^2, 3\}$ and $\{1, 3\}$ for 3-arc-regular cubic graphs, and $\{1, 4^1, 5\}$, $\{1, 4^2, 5\}$ and $\{1, 5\}$ for 5-arc-regular cubic graphs, leaving just 17 possibilities.

The types of arc-transitive action that a finite connected symmetric cubic graph can admit are as follows [22]:

1-arc-regular: $\{1\}$ only;

2-arc-regular: $\{1, 2^1\}$, $\{2^1\}$, $\{2^2\}$;

3-arc-regular: $\{1, 2^1, 2^2, 3\}$, $\{2^1, 2^2, 3\}$, $\{2^1, 3\}$, $\{2^2, 3\}$, $\{3\}$;

4-arc-regular: $\{1, 4^1\}$, $\{4^1\}$, $\{4^2\}$;

5-arc-regular: $\{1, 4^1, 4^2, 5\}$, $\{4^1, 4^2, 5\}$, $\{4^1, 5\}$, $\{4^2, 5\}$, $\{5\}$;

Proposition 2.4.4 [22, 26] *Every symmetric cubic graph admitting actions of types* 1 *and* 4^1 *is a cover of the Heawood graph* (*the incidence graph of a projective plane of order* 2)*, and in particular, is bipartite.*

Proposition 2.4.5 [22, 26] *Every symmetric cubic graph admitting actions of types* 1 *and* 5 *is a cover of the Biggs-Conway graph* (*of order* 2352)*, and in particular, is bipartite.*

Using the above theorems, Conder and Nedela [22] exhibited the types of arc-transitive actions admitted by symmetric cubic graphs of order up to 768.

Example 2.4.6 *The cubic graph* $F112B$ *in* [22] *is* 3*-arc-regular of order* 112*, and admits arc-transitive actions of types* $1,\ 2^1,\ 2^2,\ 3$.

Example 2.4.7 *The cubic graph* $F448C$ *in* [22] *is* 2*-arc-regular of order* 448*, and admits an arc-transitive action of type* 2^2 *only.*

Example 2.4.8 *The cubic graph* $F468$ *in* [22] *is* 5*-arc-regular of order* 468*, and admits arc-transitive actions of types* $4^1,\ 4^2,\ 5$.

2.5 Group representations and characters

A *group homomorphism* from group G to group H (each with multiplication as the binary operation) is a function $f : G \to H$ such that $f(uv) = f(u)f(v)$ for all u, v in G. A *representation* of a group G on a vector space V over a field F is a group homomorphism from G to $GL(V)$, the general linear group on V. In other words, a representation is a function

$$\rho : G \to GL(V)$$

such that $\rho(g_1g_2) = \rho(g_1)\rho(g_2)$, for all $g_1, g_2 \in G$. Here V is called the *representation space*, and the dimension of V is called the *dimension* of the representation.

A subspace W of V which is invariant under the representation is called a *G-invariant subspace*. If V has only two G-invariant subspaces, namely the zero-dimensional subspace and V itself, then the representation is called *irreducible*; otherwise the representation is called *reducible*.

The *character* χ of a group representation $\rho : G \to GL(V)$ is a function on the group which associates to each group element the trace of the corresponding matrix: $\chi(g) = \text{Trace}(\rho(g))$ for all $g \in G$. Each character is constant on every conjugacy class of G. The character χ is called *irreducible* (respectively, *reducible*) if the representation ρ is irreducible (respectively, reducible).

We will make use of characters of representations over the complex field (of characteristic zero), but will also consider what happens to some of these in prime characteristic when necessary.

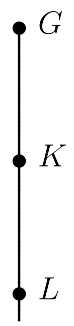
Over the complex field $\mathbb{C}$, the number of irreducible characters of a finite group G is equal to the number of its conjugacy classes. A *character table* of a group G is a two-dimensional table whose rows correspond to its irreducible group representations over $\mathbb{C}$, and whose columns correspond to conjugacy classes of group elements. The entries in each row are the values of the corresponding character on representatives of the respective conjugacy classes of G. (See [37] for more details.) For example, the character table of the alternating group A_4 can be seen in Table 3.1.

2.6 Symmetric regular covers of cubic graphs

Let X be an arc-transitive cubic graph, and let $\tilde{X}$ be an arc-transitive regular covering graph of X, and consider $\mathrm{Aut}(X)$ and $\mathrm{Aut}(\tilde{X})$, the automorphism groups of X and $\tilde{X}$, respectively. Suppose A is a subgroup of $\mathrm{Aut}(X)$ such that A is arc-transitive on X, and that A can be lifted to a subgroup $\tilde{A}$ of automorphisms of $\tilde{X}$ that is arc-transitive on $\tilde{X}$. We have the following well-known fact:

Lemma 2.6.1 *If A is an s-arc-transitive group of automorphisms of the graph X, then the action of $\tilde{A}$ on $\tilde{X}$ is at least s-arc-transitive.*

Let G be one of the seven universal groups $G_1, G_2^1, G_2^2, G_3, G_4^1, G_4^2$ or G_5 introduced in Section 2.4, and let $K,\ L$ be torsion-free normal subgroups of G such that L is a subgroup of K, and $G/K \cong A$ and $G/L \cong \tilde{A}$, with K/L being the covering group (voltage group) of the regular covering projection of X by $\tilde{X}$. This situation is illustrated in Figure 2.3.



$G/K \cong A$ $\qquad$ $G/L \cong \tilde{A}$

$\tilde{X}$ is a regular cover of X

K/L is the covering group

Figure 2.3: A pictorial description of regular covers

Classifying all the symmetric regular covering graphs of X, for which the subgroup A lifts, is equivalent to finding all possibilities for a normal subgroup L of finite index in G such that L is contained in K. The symmetric abelian regular covers are those for which K/L is abelian.

Theoretically, we can find all symmetric abelian regular covers of X if we know the structure of the abelianisation $K/K' = K/[K, K]$, and especially the subgroups of K/K'

that are invariant under conjugation by G. Usually the action of G by conjugation on K/K' can be given by matrices representing the effects of the generators of G.

To find all finite regular covers with abelian covering groups of exponent m, we may consider the action of G by conjugation on the generators of $K/K'K^{(m)}$, where $K^{(m)}$ is the characteristic subgroup of K generated by the mth powers of all elements of K. The problem then reduces to finding all possibilities for a subgroup L of finite index in K such that L contains $K'K^{(m)}$ and L is normal in G.

If $m = p_1^{e_1} p_2^{e_2} \dots p_t^{e_t}$ is the prime-power factorisation of m (with p_i distinct primes), then the factor group K/L is a direct product of its Sylow subgroups, each of which is of the form K/Q_i where Q_i is a G-invariant subgroup of K containing $K'K^{(m)}$ with index $|K:Q_i|$ dividing $p_i^{e_i}$. It follows that we need only find the G-invariant subgroups of prime-power index in K/K', in order to find all finite abelian regular covers.

The key to this step of our approach is to take m as a prime-power, say $m = p^e$, and then view the group $K/K'K^{(m)}$ as being made up of e consecutive 'layers'. For $0 \le j \le e$ we define $K_j = K'K^{(p^j)}$, which is the subgroup generated by K' and the (p^j)th powers of all elements of K. This is a characteristic subgroup of K and therefore normal in G. The quotients $K_0/K_1, K_1/K_2, \dots, K_{e-1}/K_e$ are what we view as the layers of $K/K'K^{(m)}$, from the top down. If K has rank d, and is generated by $w_1, w_2, \cdots, w_d$, say, then each layer K_{j-1}/K_j is an elementary abelian p-group of rank d, generated by the cosets $K_j {w_i}^{p^{j-1}}$ for $1 \le i \le d$. These layers are illustrated in Figure 2.4.

Similarly, if L is any G-invariant subgroup of K containing $K'K^{(m)}$ and having index $|K:L|$ a power of p, then we can define $L_j = L \cap K_j$ for $0 \le j \le e$, and view L as being made up of the layers $L_0/L_1, L_1/L_2, \dots, L_{e-1}/L_e$, again from the top down. Each layer L_{j-1}/L_j of $L/K'K^{(m)}$ is a G-invariant subgroup of the corresponding layer K_{j-1}/K_j of K. Since K_{j-1}/K_j is an elementary abelian p-group, possibilities for the layers L_{j-1}/L_j are relatively easy to find, in the same way that covers with elementary abelian regular covers are found.

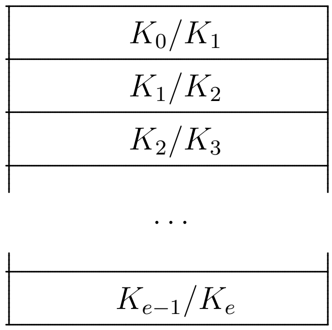


Figure 2.4: The layers of K

Here it is helpful to consider the character table of the group G/K, which gives the degrees (and other details) of irreducible representations of G/K in characteristic 0. If the prime p does not divide the order of the group G/K, then these are also the degrees of the irreducible representations of G in characteristic p, and the other character values are helpful for finding all possibilities. (This can also be achieved using tools in Magma [3] for dealing with modules and submodules.) More details can be seen in [37].

Once we know the G-invariant subgroups of the top layer K_0/K_1, including expressions for their generators in terms of the (images of the) generators w_i of K/K', we immediately have the same for other layers K_{j-1}/K_j for all $j > 1$. For example, if one such subgroup of the top layer is generated by the cosets K_1w_1 and $K_1w_2{w_3}^{-1}$, then each subsequent layer K_{j-1}/K_j has a G-invariant subgroup generated by the cosets $K_jw_1^{p^{j-1}}$ and $K_jw_2^{p^{j-1}}({w_3}^{p^{j-1}})^{-1}$, since for any integer $q > 1$, conjugation of powers w_i^q of the w_i by generators of G is represented by the same matrices (which are induced by the conjugation action of the generators of G on the K/K') as conjugation of the w_i themselves.

What is more challenging is to find all possibilities for L by piecing together the possibilities for its layers. This, however, can be done by considering what happens for small values of e (and hence small values of $p^e = m$). For example, the G-invariant subgroups of the top 'double-layer' group K_0/K_2 (of exponent p^2) tell us the possibilities for every double-layer. Also inspection of the generating sets for the G-invariant subgroups can tell us possibilities for 'triple-layers', and so on.

Our main procedures for classifying symmetric abelian regular covering graphs of symmetric cubic graphs can be summarized as follows:

a) Fix an arc-transitive group A of automorphisms of the base graph, and the universal group G from which the action of A is determined;

b) Find a torsion-free normal subgroup K of G such that $G/K \cong A$, and find a presentation for K (using Reidemeister-Schreier theory) and abelianise K if it is not abelian;

c) Find all the G-invariant subgroups L of finite index in K such that K/L has prime exponent;

d) Use the layer technique to find out all G-invariant subgroups L of finite index in K such that K/L is an abelian p-group (for some prime p);

e) Distinguish these subgroups up to isomorphism of the corresponding covering graphs.

After classifying all these covering graphs, another important question is to determine the level of s-arc-transitivity of each covering graph (and in particular, the size of its automorphism group). By Lemma 2.6.1, we know a lower bound for the largest s for which the cover is s-arc-transitive. Then with the help of the known results in Section 2.4 and other techniques, we can find all the possibilities for s.

In particular, if the G-invariant subgroup L of the kernel K is also normal in some larger universal group F (containing G as a subgroup of finite index), then F/L is a larger group of automorphisms of the covering graph. The size of the automorphism group of the covering graph can be found by determining the largest possibilities for F/L.

2.7 Regular maps

A *map* is a 2-cell embedding of a connected graph (or multigraph) into a closed surface without boundary. The term '2-cell' means that there are no edge-crossings, and each component (or *face*) of the complement $S \setminus X$ of the graph in the surface is simply connected — that is, homeomorphic to an open disk in $\mathbb{R}^2$. The map M is called *orientable* or *non-orientable* according to whether the carrier surface is orientable or non-orientable, and the *genus* and the *Euler characteristic* of the map M are defined as the genus and the Euler characteristic of that surface. The topological *dual* of an orientable map M is obtained from M by interchanging the roles of vertices and faces in the usual way, and is denoted by M^*.

Any map M is composed of a vertex-set, an edge-set, and the set of its faces, denoted by $V = V(M)$, $E = E(M)$ and $F = F(M)$, respectively. The Euler characteristic χ of M is given by the Euler-Poincaré formula $\chi = |V| - |E| + |F|$, and then the genus g of M is given by $\chi = 2 - 2g$ when M is orientable, or $\chi = 2 - g$ when M is non-orientable.

Associated also with any map M is a set of *darts*, or *arcs*, which are the incident vertex-edge pairs $(v, e) \in V \times E$; these can also be viewed as ordered pairs of adjacent vertices when the underlying graph is simple. Each dart is associated with two *blades*, which consist of the dart (v, e) and a chosen side along the edge e; in the non-degenerate cases where every edge lies in two faces, these are the incident vertex-edge-face triples $(v, e, f) \in V \times E \times F$.

An *automorphism* of a map M is any permutation of the edges of the underlying graph that preserves incidence (and hence preserves the embedding), or equivalently, any automorphism of the graph induced by a homeomorphism of the carrier surface to itself. It is important to observe that by connectedness, *every automorphism of a map is uniquely determined by its effect on any blade.*

The set of all automorphisms of a map M forms a group under composition, called the *automorphism group* of a map, and denoted by $\mathrm{Aut}(M)$. If M is orientable, then the subgroup of all orientation-preserving automorphisms has index 1 or 2 in $\mathrm{Aut}(M)$, and is denoted by $\mathrm{Aut}^o(M)$, and sometimes also called the *rotation group* of M. If the orientable map M admits an orientation-reversing automorphism (so that $\mathrm{Aut}^o(M)$ has index 2 in $\mathrm{Aut}(M)$), then M is said to be *reflexible*, and otherwise M is *chiral*. On the other hand, if M is non-orientable, there is no such distinction.

A map M is called *orientably-regular* if it is orientable and $\mathrm{Aut}^o(M)$ acts regularly on the set of all darts of M. If such a map M is reflexible, then $\mathrm{Aut}(M)$ acts regularly on the set of all blades of M. Similarly, a non-orientable map M is called *regular* if $\mathrm{Aut}(M)$ acts regularly on the set of all blades of M. In general, a map is called *regular* if it is either orientably-regular, or non-orientable and regular. Just to make it clear: regular maps fall into three classes: maps that are orientably-regular and reflexible, maps that are orientably-regular but chiral, and maps that are non-orientable and regular.

For any regular map M, the action of $\mathrm{Aut}(M)$ is transitive on the darts of M, and hence on the vertices, on the edges, and on the faces of M. It follows that every face of a regular map M has the same size, say m, and every vertex has the same valence, say k, and then M is said to have *type* $\{m, k\}$. The Platonic solids give the most famous examples, of types $\{3, 3\}$ (tetrahedron), $\{3, 4\}$ (octahedron), $\{4, 3\}$ (cube), $\{3, 5\}$ (icosahedron) and $\{5, 3\}$ (dodecahedron), while every regular map on the torus has type $\{3, 6\}$, $\{4, 4\}$ or $\{6, 3\}$. Note that if the regular map M has type $\{m, k\}$, then its dual (obtained by interchanging the roles of vertices and faces) has type $\{k, m\}$.

Now suppose M is a regular map of type $\{m, k\}$, let (v, e) be any dart of M, and let f be a face incident with e. Then by transitivity, there exists an automorphism r of M that preserves f and induces locally a single-step rotation about the centre of f, and this has order m. Similarly, there exists an automorphism s of M that fixes v and induces a single-step rotation around v, and this has order k. Moreover, we can choose each of r and s (either locally 'clockwise' or 'anti-clockwise') so that their product rs is an automorphism of order 2 that preserves e and acts locally like a rotation about the mid-point of e; in particular, r and s satisfy the relations $r^m = s^k = (rs)^2 = 1$. By connectedness, r and s generate a dart-transitive group of automorphisms of M, which must be either $\mathrm{Aut}(M)$ itself, or $\mathrm{Aut}^o(M)$ in the case where M is orientable and both r and s preserve orientation.

(The existence of such automorphisms is key to the definition of an alternative term for regular map, namely *rotary map*, as coined by Steve Wilson. This has the advantage of allowing the term 'regular' to be reserved for those rotary maps M with the property that $\mathrm{Aut}(M)$ acts regularly on blades, but recent usage has extended this term to cover the orientably-regular but chiral maps as well.)

It follows from the above observations that $\mathrm{Aut}(M)$ or $\mathrm{Aut}^o(M)$ is a quotient of the ordinary $(2, k, m)$ *triangle group*

$$\Delta^o(2, k, m) = \langle\, x, y \mid x^2 = y^k = (xy)^m = 1 \,\rangle,$$

under an epimorphism taking x to rs and y to s^{-1}. Note that the dual M^* is also regular, with the roles of r and s (and hence the roles of xy and y^{-1}) interchanged.

If M admits also an (involutory) automorphism a which reverses the edge e but (unlike rs) preserves each of the two blades associated with (v, e), then the action of $\mathrm{Aut}(M)$ is transitive on blades, and so M is either reflexible or non-orientable. Also in this case $r^a = r^{-1}$ and $(rs)^a = (rs)^{-1} = rs$, and if we define $b = ar$ and $c = bs$, then a, b and c generate $\mathrm{Aut}(M)$ and satisfy the relations

$$a^2 = b^2 = c^2 = (ab)^m = (bc)^k = (ac)^2 = 1,$$

which are the defining relations for the full (or extended) triangle group $\Delta(2, k, m)$.

Note that the automorphism a may be considered geometrically as a reflection, about an axis passing through the midpoints of the edge e and the face f. Similarly, the automorphisms b and c may be considered as a reflection about an axis through v and the midpoint of f (with $r^b = r^{-1}$ and $s^b = s^{-1}$), and a reflection about an axis through v and the midpoint of e (with $(rs)^c = (rs)^{-1} = rs$ and $s^c = s^{-1}$).

Conversely, given any epimorphism $\psi\colon \Delta^o \to G$ from the ordinary $(2, k, m)$ triangle group $\Delta^o = \Delta^o(2, k, m)$ onto a finite group G, in which the orders 2, k and m of the generators x, y and xy are preserved, a map M can be constructed using right cosets of the images of $\langle y\rangle$, $\langle x\rangle$ and $\langle xy\rangle$ as the vertices, edges and faces of M, respectively, with incidence given by non-empty intersection of cosets. (For example, the ordered pair $(v, e) = (\langle y^\psi\rangle, \langle x^\psi\rangle)$ is a dart of M, incident with the face $f = \langle (xy)^\psi\rangle$.) Also the group G acts naturally and transitively by right multiplication on each of $V(M)$, $E(M)$ and

$F(M)$, preserving incidence, and transitively on the darts of M. It follows that M is a regular map of type $\{m, k\}$, with $G = \mathrm{Aut}^o(M)$ or $\mathrm{Aut}(M)$.

This map M admits also the automorphisms a, b and c (described above) if and only if the epimorphism ψ extends to an epimorphism $\widetilde{\psi}\colon \Delta \to \widetilde{G}$ from the full $(2, k, m)$ triangle group $\Delta = \Delta(2, k, m)$ onto a group $\widetilde{G}$ containing G as a subgroup of index 1 or 2. If G has index 2 in $\widetilde{G}$ then M is orientable and reflexible, while if $G = \widetilde{G}$ then M is non-orientable, and vice versa. In both cases, the kernel $K = \ker \psi$ is normal in Δ. On the other hand, if $K = \ker \psi$ is not normal in Δ, then M is orientable but chiral, and the conjugate of K by any element of $\Delta \setminus \Delta^o$ is the kernel of the epimorphism corresponding to the 'mirror image' of M.

In practice, we can tell whether or not an orientably-regular map M of type $\{m, k\}$ is reflexible, either by testing for an automorphism of $\mathrm{Aut}^o(M)$ that inverts the generating pair (r, s) (or the generating pair (rs, s)), or by testing whether the kernel K of the epimorphism $\psi\colon \Delta^o(2, k, m) \to \mathrm{Aut}^o(M)$ is invariant under conjugacy by an element of $\Delta \setminus \Delta^o$.

From this point of view, the study of regular maps can be reduced to the study of non-degenerate quotients of triangle groups.

As is well known (and shown in [23]), the simplicity of the underlying graphs can also be reduced to some easy group theory. If $\langle s \rangle$ stabilises the vertex v, then $\langle s^r \rangle$ stabilises the neighbouring vertex v^r, and their intersection stabilises both vertices. It follows that the existence of multiple edges between v and its neighbour v^{rs} is equivalent to the intersection $\langle s \rangle \cap \langle s^{rs} \rangle$ being non-trivial, and since the latter is normalised by both s and and the involution rs, it is normal in $\langle s, rs \rangle = \langle r, s \rangle = \mathrm{Aut}^o(M)$ or $\mathrm{Aut}(M)$.

Hence if M is an orientably-regular map, then M has simple underlying graph if and only if no non-trivial subgroup of the vertex-stabiliser is normal in $\mathrm{Aut}^o(M)$.

Note that if $\mathrm{Aut}^o(M)$ is a non-abelian simple group, or 'almost simple' (or more generally, if every minimal normal subgroup of $\mathrm{Aut}^o(M)$ is a non-abelian simple group), then every cyclic subgroup must be core-free in $\mathrm{Aut}^o(M)$, and in that case, both M and

M^* have simple underlying graph (see [23] for further details). The genera of such maps, however, are somewhat sparse, and so this observation will be of little use to us when we construct families of maps with simple underlying graphs in Chapter 8.

Chapter 3

Symmetric abelian regular covers of the complete graph K_4

In 2002, Feng and Kwak [30] classified all the symmetric cyclic and elementary abelian regular covers of the complete graph K_4, by using the method of linear criteria for lifting automorphisms, as introduced in [27]. In particular, they showed that all these covering graphs are 2-arc-regular. The same method can be used to classify all the symmetric homocyclic regular covers of K_4, but for more general symmetric abelian regular covers, this method is very difficult to apply.

In this Chapter, by using the new approach we introduced in Section 2.6, we determine all of the symmetric abelian regular covering graphs of the complete graph K_4. The numbers of covers and the largest value of s for which each cover is s-arc-transitive are also given.

3.1 Preliminaries

As we know, the complete graph K_4 is the only symmetric cubic graph on four vertices. Also K_4 is 2-arc-transitive, and has type 2^1 (as described in Section 2.4). Its automorphism group is the symmetric group S_4 of order 24, and the only other arc-transitive group of automorphisms of K_4 is the alternating group A_4, of order 12, which acts regularly on the arcs of K_4.

Consider the finitely-presented group $G_1 = \langle\, h, a \mid h^3 = a^2 = 1 \,\rangle$, and let N be the unique normal subgroup of G_1 of index 12, for which $G_1/N \cong A_4$. Then N is also normal in the group $G_2^1 = \langle\, h, a, p \mid h^3 = a^2 = p^2 = (hp)^2 = (ap)^2 = 1 \,\rangle$, with quotient $G_2^1/N \cong S_4$. By Reidemeister-Schreier theory, or by use of the `Rewrite` command in Magma, we find that the subgroup N is free of rank 3, on generators

$$w_1 = (ha)^3, \quad w_2 = (ah)^3 \quad \text{and} \quad w_3 = h^{-1}(ah)^3 h.$$

The image of w_1 by the conjugation action of h is equal to $h^{-1}w_1h = (ah)^3 = w_2$. Similarly, easy calculations show that the generators h, a and p act by conjugation as below:

$$\begin{array}{lllllllll} h^{-1}w_1h & = & w_2 & a^{-1}w_1a & = & w_2 & p^{-1}w_1p & = & w_2^{-1} \\ h^{-1}w_2h & = & w_3 & a^{-1}w_2a & = & w_1 & p^{-1}w_2p & = & w_1^{-1} \\ h^{-1}w_3h & = & w_1 & a^{-1}w_3a & = & w_1^{-1}w_3^{-1}w_2^{-1} & p^{-1}w_3p & = & w_3^{-1}. \end{array}$$

Now take the quotient G_2^1/N', which is an extension of the free abelian group $N/N' \cong \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$ by the group $G_2^1/N \cong S_4$, and replace the generators h, a, p and all w_i by their images in this group. Also let K denote the subgroup N/N', and let G be G_1/N'. Then, in particular, G is an extension of $\mathbb{Z}^3$ by A_4.

By the above observations, we see that the generators h, a and p induce linear transformations of the free abelian group $K \cong \mathbb{Z}^3$ as follows:

$$h \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad a \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix} \quad \text{and} \quad p \mapsto \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

These matrices generate a group isomorphic to S_4, with the first two generating a subgroup isomorphic to A_4. Note that the matrices of orders 3 and 2 representing h and a have traces 0 and -1, respectively.

3.2 Elementary abelian regular covers

In this section, we find all of the symmetric elementary abelian regular covers of K_4 that can be obtained by lifting the 1-arc-regular subgroup A_4.

The character table of the group A_4 is given below in Table 3.1. By inspecting traces of the matrices representing h and a, we see that the character of the 3-dimensional representation of A_4 over $\mathbb{Q}$ associated with the above action of $G = \langle h, a \rangle$ on K is the character χ_4, which is irreducible. It follows that when we reduce by any prime k that does not divide $|A_4| = 12$, the corresponding action of A_4 on $K/K^{(k)} \cong \mathbb{Z}_k \oplus \mathbb{Z}_k \oplus \mathbb{Z}_k$ is also irreducible, and hence the only non-trivial subgroup of $K/K^{(k)}$ that is normal in $G/K^{(k)}$ is $K/K^{(k)}$ itself.

Element order	1	2	3	3
Class size	1	3	4	4
χ_1	1	1	1	1
χ_2	1	1	λ	λ^2
χ_3	1	1	λ^2	λ
χ_4	3	-1	0	0

Table 3.1: The character table of group A_4 where λ is a primitive cube root of 1

The same argument holds for each 'layer' $K_i/K_{i+1} = K^{(k^i)}/K^{(k^{i+1})}$ of K, since this layer is generated by the cosets of $K^{(k^{i+1})}$ containing $w_1^{k^i}$, $w_2^{k^i}$ and $w_3^{k^i}$, and the effect of conjugation by each of h and a on these generators is given by the same matrices as above.

Similarly, when $k = 3$ we find that the action of G on $K/K^{(k)}$ is irreducible, because there are no subgroups of $K/K^{(3)} \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ of order 3 or 9 that are normalized by both h and a. This is an easy exercise, verifiable with the help of Magma. (Note also that the mod 3 reductions of the characters χ_2 and χ_3 are both trivial.) Hence in particular, the same holds for the action of G on each layer K_i/K_{i+1}.

On the other hand, when $k = 2$ we find that the action of G on $K/K^{(k)}$ is reducible — indeed the subgroup of $K/K^{(2)}$ of order 4 generated by the cosets $K^{(2)}w_1w_2$ and $K^{(2)}w_1w_3$ normalized by both h and a — but there is no subgroup of order 2 with that property.

In fact, because the rank of K is small, we can also prove these (and more) things directly, as follows:

For any prime k, suppose $L/K^{(k)}$ is a non-trivial cyclic subgroup of $K/K^{(k)}$ that is normalized by both h and a, and suppose $L/K^{(k)}$ is generated by the coset $K^{(k)}x$ where $x = w_1^{\alpha}w_2^{\beta}w_3^{\gamma} \in K$. Then L contains $x^h = w_1^{\gamma}w_2^{\alpha}w_3^{\beta}$, so $(\gamma, \alpha, \beta) \equiv \lambda(\alpha, \beta, \gamma)$ mod k for some $\lambda \in \mathbb{Z}_k$, and then since $\alpha \equiv \lambda\beta \equiv \lambda^2\gamma \equiv \lambda^3\alpha$ mod k, each of α, β and γ must be non-zero mod k, and $\lambda^3 \equiv 1$ mod k. Next, L contains x^a, and so contains $w_1^{\beta-\gamma}w_2^{\alpha-\gamma}w_3^{-\gamma}$ and hence also $w_1^{\gamma-\beta}w_2^{\gamma-\alpha}w_3^{\gamma}$, and then because $\gamma \not\equiv 0$ mod k, it follows that $\gamma \equiv \alpha + \beta$ mod k. Similarly, $\alpha \equiv \beta + \gamma$ and $\beta \equiv \alpha + \gamma$ mod k, and then adding these congruences gives $\alpha + \beta \equiv \alpha + \beta + 2\gamma$ mod k, so $2\gamma \equiv 0$ mod k, which implies that $k = 2$. But then $\lambda = 1$, in which case $\alpha \equiv \beta \equiv \gamma \equiv 1$ mod k, and therefore $\gamma \not\equiv \alpha + \beta$ mod k, a contradiction. Hence no such cyclic subgroup exists.

On the other hand, let $L/K^{(k)}$ be a subgroup of $K/K^{(k)}$ isomorphic to $\mathbb{Z}_k \oplus \mathbb{Z}_k$ that is normalized by both h and a. Then L must contain all elements of the form $w_1^{\sigma}w_2^{\tau}w_3^{\mu}$, where (σ, τ, μ) lies in a 2-dimensional subspace of $\mathbb{Z}_k \oplus \mathbb{Z}_k \oplus \mathbb{Z}_k$ (as a vector space over $\mathbb{Z}_k$). The latter is the orthogonal complement of a unique 1-dimensional subspace, generated say by (α, β, γ). Since $L/K^{(k)}$ is normalized by h, we find that every such (σ, τ, μ) is orthogonal also to (γ, α, β) and (β, γ, α), and so as above, we find that $\alpha \equiv \lambda\beta \equiv \lambda^2\gamma$ mod k for some cube root λ of 1 in $\mathbb{Z}_k$. Similarly, because $L/K^{(k)}$ is normalized by a, we find that for any such (σ, τ, μ), also $(\tau-\mu, \sigma-\mu, -\mu)$ is orthogonal to (α, β, γ), and hence so is $(\tau-\mu, \sigma-\mu, -\mu) + (\sigma, \tau, \mu) = (\sigma+\tau-\mu, \sigma+\tau-\mu, 0)$, giving $0 \equiv (\sigma+\tau-\mu)(\alpha + \beta) = (\sigma+\tau-\mu)(1 + \lambda)\beta$ mod k.

Now if $\lambda \not\equiv -1$ mod k, then $\sigma + \tau - \mu \equiv 0$ mod k for all such (σ, τ, μ), in which case we can take $(\alpha, \beta, \gamma) \equiv (1, 1, -1)$ mod k, but that is impossible since we need $\alpha \equiv \lambda\beta \equiv \lambda^2\gamma$ mod k. Thus $\lambda \equiv -1$ mod k. Moreover, since $-1 \equiv \lambda^3 \equiv 1$ mod k, again we get $k = 2$, and $\lambda = 1$ and $\alpha \equiv \beta \equiv \gamma \equiv 1 \mod k$. Hence there is only one subgroup of rank 2 in $K/K^{(2)}$, namely the subgroup generated by $K^{(2)}w_1w_2$ and $K^{(2)}w_1w_3$, and there are no subgroups of rank 2 in $K/K^{(k)}$ when k is an odd prime.

Now we have all the elementary abelian regular covering groups. Note that the above arguments hold also for each layer K_i/K_{i+1} of K.

3.3 Abelian regular covers

In this section, we classify all the abelian covering groups by using our 'layer' technique introduced in Section 2.6.

Suppose $m = k^e$ is any prime-power greater than 1, and suppose $L/K^{(m)}$ is any non-trivial normal subgroup of $G/K^{(m)}$ contained in $K/K^{(m)}$.

If k is odd, then by the arguments in previous section, we know that each layer $L_i/L_{i+1} = (L \cap K_i)/(L \cap K_{i+1})$ of L (for $1 \leq i \leq e$) is either trivial or isomorphic to $\mathbb{Z}_k \oplus \mathbb{Z}_k \oplus \mathbb{Z}_k$. Let t be the smallest non-negative integer for which L_t/L_{t+1} is non-trivial. Then L_t must contain the cosets of $K^{(k^{t+1})}$ represented by each of $w_1^{k^t}, w_2^{k^t}$ and $w_3^{k^t}$, and hence also L_j must contain the cosets represented by $w_1^{k^j}, w_2^{k^j}$ and $w_3^{k^j}$, for $t \leq j < e$, and it follows by an easy induction that $L/K^{(m)} \cong \mathbb{Z}_d \oplus \mathbb{Z}_d \oplus \mathbb{Z}_d$, where $d = k^{e-t}$.

On the other hand, if $k = 2$, then it is possible that the first non-trivial layer L_t/L_{t+1} of L is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (rather than $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$). To explain what happens in that case, we will assume that $t = 0$, but note that the general case (where t may be greater than 0) is similar.

In the case $t = 0$, the group $L/K^{(k)}$ is generated by the cosets $K^{(k)}w_1w_2$ and $K^{(k)}w_1w_3$, and so L must contain the elements $x = w_1w_2^{\beta}w_3^{\gamma}$ and $y = w_1w_2^{\tau}w_3^{\mu}$ for some odd integers β and μ and even integers γ and τ. Then since L is normalized by h and a, it contains each of x^h, x^a and x^{ha}, and therefore contains $w_1^{\gamma}w_2w_3^{\beta}$, $w_1^{\beta-\gamma}w_2^{1-\gamma}w_3^{-\gamma}$ and $w_1^{1-\beta}w_2^{\gamma-\beta}w_3^{-\beta}$. Hence in particular, L contains the product $xx^hx^ax^{ha} = w_1^{1+\gamma+\beta-\gamma+1-\beta}w_2^{\beta+1+1-\gamma+\gamma-\beta} = w_1^2w_2^2$. Conjugation by h then shows that L contains $w_1^2w_3^2$ and $w_2^2w_3^2$, and by multiplying $w_1^2w_2^2$

by the inverse of one of these two, we find that L contains also $w_2^2 w_3^{-2}$ and $w_1^2 w_3^{-2}$, and hence L contains each of w_1^4, w_2^4 and w_3^4. Thus $L/K^{(m)} \cong \mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \mathbb{Z}_{m/2}$ or $\mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \mathbb{Z}_{m/4}$. In particular, if L contains one (and hence all) of w_1^2, w_2^2 and w_3^2, then $L/K^{(m)} \cong \mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \mathbb{Z}_{m/2}$ and is generated by the cosets of $K^{(m)}$ represented by w_1w_2, w_1w_3 and w_3^2, while otherwise $L/K^{(m)} \cong \mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \mathbb{Z}_{m/4}$ and is generated by the cosets of $K^{(m)}$ represented by $w_1w_2^{-1}$, $w_1w_3^{-1}$ and $w_1^2w_2^2$.

In the general case, if L contains one (and hence all) of $w_1^{2^{t+1}}$, $w_2^{2^{t+1}}$ and $w_3^{2^{t+1}}$, then $L/K^{(m)} \cong \mathbb{Z}_d \oplus \mathbb{Z}_d \oplus \mathbb{Z}_{d/2}$ where $d = 2^{e-t}$, and $L/K^{(m)}$ is generated by the cosets of $K^{(m)}$ represented by $w_1^{2^t} w_2^{2^t}$, $w_1^{2^t} w_3^{2^t}$ and $w_3^{2^{t+1}}$; otherwise $L/K^{(m)} \cong \mathbb{Z}_d \oplus \mathbb{Z}_d \oplus \mathbb{Z}_{d/4}$ and is generated by the cosets of $K^{(m)}$ represented by $w_1^{2^t} w_2^{-2^t}$, $w_1^{2^t} w_3^{-2^t}$ and $w_1^{2^{t+1}} w_2^{2^{t+1}}$. The covering group K/L is then isomorphic to $\mathbb{Z}_{2^t} \oplus \mathbb{Z}_{2^t} \oplus \mathbb{Z}_{2^{t+1}}$ or $\mathbb{Z}_{2^t} \oplus \mathbb{Z}_{2^t} \oplus \mathbb{Z}_{2^{t+2}}$, respectively. These two possibilities are illustrated in Figure 3.1 below, for the case $t = 3$.

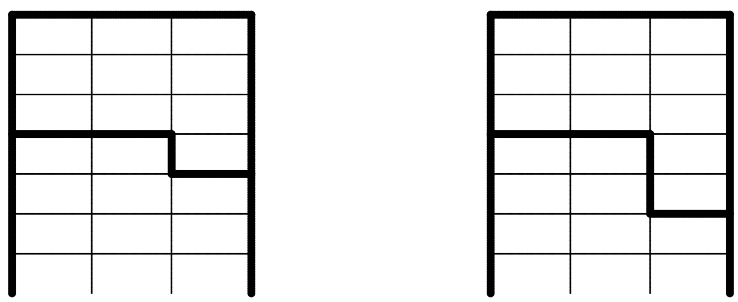


Figure 3.1: The two possibilities for non-homocyclic abelian covers of 2-power exponent

In summary, Table 3.2 provides all possibilities for a normal subgroup L of G contained with finite prime-power index in K.

Index $\|K:L\|$	Generating set for L	Quotient K/L
$\ell^3 = k^{3t}$, any prime k	$\{w_1^\ell,\ w_2^\ell,\ w_3^\ell\}$	$\mathbb{Z}_\ell \oplus Z_\ell \oplus Z_\ell$
$2\ell^3 = 2^{3t+1}$	$\{w_1^\ell w_2^\ell,\ w_1^\ell w_3^\ell,\ w_3^{2\ell}\}$	$\mathbb{Z}_\ell \oplus Z_\ell \oplus Z_{2\ell}$
$4\ell^3 = 2^{3t+2}$	$\{w_1^\ell w_2^{-\ell},\ w_1^\ell w_3^{-\ell},\ w_1^{2\ell} w_2^{2\ell}\}$	$\mathbb{Z}_\ell \oplus Z_\ell \oplus Z_{4\ell}$

Table 3.2: Possibilities for $\langle h, a\rangle$-invariant subgroup L of K when $\langle h, a\rangle/K \cong A_4$

3.4 Automorphism groups of the regular covers

In this section, after classifying all the symmetric abelian regular covers that can be obtained by lifting the 1-arc-regular group of automorphisms A_4, we give the largest value of s for all s-arc-transitive covering graphs by determine the size of their automorphism groups.

It is easy to see that in all of the cases described in Table 3.2, the subgroup L is normal not only in $G = G_1/N'$ but also in the larger group G_2^1/N', since conjugation by the additional generator p has the following effects on the relevant generators:

$$\begin{array}{llll} w_1^i w_2^i & \mapsto\ w_2^{-i} w_1^{-i} = (w_1^i w_2^i)^{-1}, & w_1^i w_2^{-i} & \mapsto\ w_2^{-i} w_1^i = w_1^i w_2^{-i}, \\ w_1^i w_3^i & \mapsto\ w_2^{-i} w_3^{-i} = (w_1^i w_2^i)(w_1^i w_3^i)^{-1}, & w_1^i w_3^{-i} & \mapsto\ w_2^{-i} w_3^i = (w_1^i w_2^{-i})(w_1^i w_3^{-i})^{-1}, \\ w_3^j & \mapsto\ (w_3^j)^{-1}, & w_1^j w_2^j & \mapsto\ w_2^{-j} w_1^{-j} = (w_1^j w_2^j)^{-1}. \end{array}$$

Hence in particular, each of the resulting covers is at least 2-arc-transitive.

In fact, in each case the subgroup K/L is characteristic in the quotient G_2^1/L. This is clear for $k \notin \{2,3\}$ by Sylow theory, while for $k = 3$ it is not difficult to see that K/L is the largest abelian normal 3-subgroup of G_2^1/L, and for $k = 2$, there is no other abelian normal 2-subgroup of G_2^1/L that is isomorphic to K/L (even when $|K/L|$ is small). It follows that each L is not G_3-invariant, for if it were, then G_2^1/L would be invariant under the outer automorphism of G_2^1 that takes h, a and p to h, ap and p (respectively), but then that automorphism would have to preserve the characteristic subgroup K/L and hence preserve K, which we know is not the case (since K_4 is 2-arc-regular). Hence none of these covers can be 3-arc-regular.

Finally, by Corollary 2.4.2 (or [26, Theorem 3]), none of these covers can be 4-arc- or 5-arc-transitive, since each admits a 2-arc-regular group of automorphisms.

3.5 Summary

We have the following:

Theorem 3.5.1 *Let $m = k^e$ be any power of a prime k with $e > 0$. Then the arc-transitive abelian regular covers of the complete graph K_4 with abelian covering groups of exponent m are as follows:*

(a) *If m is odd, there is just one such cover, namely a 2-arc-regular cover of type 2^1 with covering group $\mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m$.*

(b) *If $m = 2^e$ for some $e \geq 2$, then there are exactly three such covers, which are 2-arc-regular covers of type 2^1 with covering groups $\mathbb{Z}_{m/4} \oplus \mathbb{Z}_{m/4} \oplus \mathbb{Z}_m$, $\mathbb{Z}_{m/2} \oplus \mathbb{Z}_{m/2} \oplus \mathbb{Z}_m$ and $\mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m$.*

(c) *If $m = 2$ then there are exactly two such covers, which are 2-arc-regular covers of type 2^1 with covering groups $\mathbb{Z}_2$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.*

Hence in particular, every arc-transitive abelian regular cover of K_4 is 2-arc-transitive.

Chapter 4

Symmetric abelian regular covers of the complete bipartite graph $K_{3,3}$

In 2000, Feng and Kwak [30, 28] classified all the symmetric cyclic and elementary abelian regular covers of the complete bipartite graph $K_{3,3}$, by using the method of linear criteria for lifting automorphisms, as introduced in [27]. As for K_4, the same method can be used to classify all the symmetric homocyclic regular covers of $K_{3,3}$, but for more general symmetric abelian regular covers, this method is very difficult to apply.

In this Chapter, we use our new approach to determine all the symmetric abelian regular covering graphs of the complete bipartite graph $K_{3,3}$. The numbers of covers and the largest value of s for which each cover is s-arc-transitive are also given.

4.1 Preliminaries

We know that the complete bipartite graph $K_{3,3}$ is the only symmetric cubic graph of order 6. Also $K_{3,3}$ is 3-arc-transitive, and is of the type 3 described in Section 2.4. Its automorphism group is the wreath product $S_3 \wr S_2$ of the symmetric group S_3 by the symmetric group S_2, of order 72. This contains unique arc-transitive subgroups of type 1, 2^1 and 2^2, having orders 18, 36 and 36 respectively (see [22, §4.5]). In particular, the minimal groups $A_3 \wr S_2$ (of order 18) and $(A_3 \times A_3) \rtimes C_4$ (of order 36) act regularly on the arcs and 2-arcs of $K_{3,3}$, with type 1 and 2^2 respectively.

We will investigate abelian regular covers of $K_{3,3}$ using the former subgroup, and then later consider what happens with the latter subgroup.

Take $G_3 = \langle\, h, a, p, q \mid h^3 = a^2 = p^2 = q^2 = [p, q] = [h, p] = (hq)^2 = apaq = 1 \,\rangle$, take $G_1 = \langle h, a\rangle$, and let N be the unique normal subgroup of index 72 in G_3 (and index 18 in G_1), with quotients $G_3/N \cong S_3 \wr S_2$ and $G_1/N \cong A_3 \wr S_2$. Note that the 2-arc-regular group generated by the cosets Nh and Nap of N in the quotient G_3/N is isomorphic to $(A_3 \times A_3) \rtimes C_4$.

Using Reidemeister-Schreier theory or the `Rewrite` command in Magma, we find that the subgroup N is free of rank 4, on generators

$$w_1 = hahah^{-1}ah^{-1}a, \quad w_2 = h^{-1}ahahah^{-1}a,$$
$$w_3 = hah^{-1}ah^{-1}aha, \quad w_4 = h^{-1}ah^{-1}ahaha.$$

Easy calculations show that the generators h, a, p and q act by conjugation as below:

$$\begin{array}{lllllllllll}
h^{-1}w_1h &=& w_2^{-1} & a^{-1}w_1a &=& w_1^{-1} & p^{-1}w_1p &=& w_3 & q^{-1}w_1q &=& w_2 \\
h^{-1}w_2h &=& w_1w_2^{-1} & a^{-1}w_2a &=& w_3^{-1} & p^{-1}w_2p &=& w_4 & q^{-1}w_2q &=& w_1 \\
h^{-1}w_3h &=& w_4^{-1} & a^{-1}w_3a &=& w_2^{-1} & p^{-1}w_3p &=& w_1 & q^{-1}w_3q &=& w_4 \\
h^{-1}w_4h &=& w_3w_4^{-1} & a^{-1}w_4a &=& w_4^{-1} & p^{-1}w_4p &=& w_2 & q^{-1}w_4q &=& w_3.
\end{array}$$

By the above observations, the elements h, a, ap, p and q induce linear transformations of the free abelian group $K \cong \mathbb{Z}^4$ as follows, giving a 4-dimensional representation of G_3:

$$h \mapsto \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad a \mapsto \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad ap \mapsto \begin{pmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$p \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad q \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

These matrices generate a group isomorphic to $S_3 \wr S_2$, with the first two matrices generating a subgroup isomorphic to $A_3 \wr S_2$, and the first and the third generating a subgroup isomorphic to $(A_3 \times A_3) \rtimes C_4$. Note that the traces of the five matrices are $-2, -2, 0, 0$ and 0, respectively.

4.2 Lifting the automorphism subgroup $A_3 \wr S_2$

In this section, we find all of the symmetric abelian regular covers of $K_{3,3}$ that can be obtained by lifting the 1-arc-regular subgroup $A_3 \wr S_2$.

4.2.1 Abelian regular covers

First, take the quotient G_3/N', which is an extension of the free abelian group $N/N' \cong \mathbb{Z}^4$ by the group $G_3/N \cong S_3 \wr S_2$, and replace the generators h, a, p, q and all w_i by their images in G_3/N'. Also let K denote the subgroup N/N', and let $G = G_1/N'$, so that G is an extension of K by $G/K \cong A_3 \wr S_2$, where K is free abelian of rank 4 with generators w_i for $1 \leq i \leq 4$.

Note that the matrices of orders 3 and 2 representing h and a both have trace -2, while the matrices of orders 6, 3 and 3 representing ha, $[h, a]$ and $(ha)^2$ (which is central in the subgroup generated by matrices representing h and a) all have trace 1.

Next, the character table of the group $A_3 \wr S_2$ is as in Table 4.1. By inspecting traces of the above matrices, we see that the character of the 4-dimensional representation of $A_3 \wr S_2$ over $\mathbb{Q}$ associated with the above action of $G = \langle h, a \rangle$ on K is the character $\chi_3 + \chi_4 + \chi_7$, which is reducible to the sum of $\chi_3 + \chi_4$ and χ_7, which are characters of two irreducible

Element order	1	2	3	3	3	3	3	6	6
Class size	1	3	1	1	2	2	2	3	3
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	1	1	1	-1	-1
χ_3	1	-1	λ	λ^2	1	λ	λ^2	$-\lambda$	$-\lambda^2$
χ_4	1	-1	λ^2	λ	1	λ^2	λ	$-\lambda^2$	$-\lambda$
χ_5	1	1	λ	λ^2	1	λ	λ^2	λ	λ^2
χ_6	1	1	λ^2	λ	1	λ^2	λ	λ^2	λ
χ_7	2	0	2	2	-1	-1	-1	0	0
χ_8	2	0	2λ	2λ	-1	$-\lambda$	$-\lambda^2$	0	0
χ_9	2	0	$2\lambda^2$	$2\lambda^2$	-1	$-\lambda^2$	$-\lambda$	0	0

Table 4.1: The character table of the group $A_3 \wr S_2$ where λ is a primitive cube root of 1

2-dimensional representations over the rational field $\mathbb{Q}$.

It follows that for every prime $k \notin \{2,3\}$, the group $K/K^{(k)} \cong (\mathbb{Z}_k)^4$ is the direct sum of two G-invariant subgroups of rank 2, and if $\mathbb{Z}_k$ contains a non-trivial cube root of 1, then one of these is the direct sum of two G-invariant cyclic subgroups and the other is irreducible, while otherwise both of them are irreducible.

In fact, the rank 2 subgroup of K generated by $x = w_1w_3^{-1}w_4$ and $y = w_2w_3^{-1}$ is normal in G, with $x^h = y^{-1}$, $y^h = xy^{-1}$, $x^a = x^{-1}y$ and $y^a = y^{-1}$. Similarly, the rank 2 subgroup generated by $u = w_1w_4^{-1}$ and $v = w_2w_3w_4^{-1}$ is normal in G, with $u^h = v^{-1}$, $v^h = uv^{-1}$, $u^a = u^{-1}$ and $v^a = v^{-1}$. In the quotient $G/K^{(k)}$, the image of the rank 2 subgroup $\langle x, y\rangle$ has no non-trivial G-invariant cyclic subgroup, and hence is irreducible. If $k \equiv 2 \bmod 3$, then the image of $\langle u, v\rangle$ is also irreducible, while if $k \equiv 1 \bmod 3$ and λ is a primitive cube root of 1 mod k, then the image of $\langle u, v\rangle$ is the direct sum of G-invariant subgroups generated by the images of each of $z_\lambda = w_1w_2^\lambda w_3^\lambda w_4^{\lambda^2}$ and $z_{\lambda^2} = w_1w_2^{\lambda^2}w_3^{\lambda^2}w_4^\lambda$, while if $k \equiv 2 \bmod 3$ then it is irreducible.

For $k = 3$, the group $K/K^{(k)}$ still has two G-invariant subgroups of rank 2, one generated by the images of $x = w_1 w_3^{-1} w_4$ and $y = w_2 w_3^{-1}$ and the other generated by the images of $u = w_1 w_4^{-1}$ and $v = w_2 w_3 w_4^{-1}$; but in this case they have non-trivial intersection, namely the cyclic (and G-invariant) subgroup generated by $K^{(k)} w_1 w_2 w_3 w_4$, which equals both $K^{(k)} xy$ and $K^{(k)} uv$. In fact the only non-trivial proper G-invariant subgroups of $K/K^{(3)}$ are these two subgroups of rank 2, their intersection (of rank 1), and the rank 3 subgroup that they generate together.

For $k = 2$, the group $K/K^{(k)}$ is again the direct sum of two G-invariant subgroups of rank 2, just as in the case for any larger prime $k \equiv 2 \bmod 3$.

The above observations were made for the 'top' layer $K_0/K_1 = K/K^{(k)}$ of K, and give all the elementary abelian regular covers. But also the analogous observations hold for each other layer K_i/K_{i+1} of K (with $i \geq 1$).

So now suppose $m = k^e$ is any prime-power greater than 1, and suppose $L/K^{(m)}$ is any non-trivial normal subgroup of $G/K^{(m)}$ contained in $K/K^{(m)}$.

If $k \equiv 2 \bmod 3$, then each layer L_i/L_{i+1} of L is either trivial or isomorphic to $\mathbb{Z}_k \oplus \mathbb{Z}_k$ or $\mathbb{Z}_k \oplus \mathbb{Z}_k \oplus \mathbb{Z}_k \oplus \mathbb{Z}_k$. It then follows from the above observations that there exist divisors c and d of $m = k^e$ such that $L/K^{(m)} \cong \mathbb{Z}_c \oplus \mathbb{Z}_c \oplus \mathbb{Z}_d \oplus \mathbb{Z}_d$, with L being generated by $(w_1 w_4^{-1})^{m/c}$, $(w_2 w_3 w_4^{-1})^{m/c}$, $(w_1 w_3^{-1} w_4)^{m/d}$ and $(w_2 w_3^{-1})^{m/d}$.

On the other hand, if $k \equiv 1 \bmod 3$, and λ is a primitive cube root of 1 mod k, then there exist divisors b, c and d of $m = k^e$ such that $L/K^{(m)} \cong \mathbb{Z}_b \oplus \mathbb{Z}_c \oplus \mathbb{Z}_d \oplus \mathbb{Z}_d$, with L generated by $(w_1 w_2^{\lambda} w_3^{\lambda} w_4^{\lambda^2})^{m/b}$, $(w_1 w_2^{\lambda^2} w_3^{\lambda^2} w_4^{\lambda})^{m/c}$, $(w_1 w_3^{-1} w_4)^{m/d}$ and $(w_2 w_3^{-1})^{m/d}$.

The case $k = 3$ is not quite so straightforward. In this case, each layer can have rank 0, 1, 2, 3 or 4, depending on the layers above it. To see exactly what happens, it is helpful to consider the case $m = 3^2 = 9$. For notational convenience, let $x = w_1 w_3^{-1} w_4$, $y = w_2 w_3^{-1}$, $u = w_1 w_4^{-1}$, $v = w_2 w_3 w_4^{-1}$ and $z = w_1 w_2 w_3 w_4$, and let $\overline{g}$ denote the coset $K^{(9)} g$, when $g \in K$. Then an easy exercise (using MAGMA if necessary) shows that $K/K^{(9)}$ contains exactly 24 subgroups that are normal in $G/K^{(9)}$, and these may be summarised as follows:

- the group $K/K^{(9)} \cong \mathbb{Z}_9 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_9$ itself, generated by all the $\overline{w_i}$;

- one subgroup isomorphic to $\mathbb{Z}_9 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3$, generated by $\overline{x}, \overline{y}, \overline{u}$ (or $\overline{v}$) and $\overline{w_4}^3$;
- two subgroups isomorphic to $\mathbb{Z}_9 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$, one generated by $\overline{x}, \overline{y}$ and all $\overline{w_i}^3$, and the other by $\overline{u}, \overline{v}$ and all $\overline{w_i}^3$;
- one subgroup isomorphic to $\mathbb{Z}_9 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$, generated by $\overline{z}$ and all $\overline{w_i}^3$;
- four subgroups isomorphic to $\mathbb{Z}_9 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3$, generated by $\{\overline{x}, \overline{y}, \overline{u}^3\}$, or $\{\overline{u}, \overline{v}, \overline{y}^3\}$, or $\{\overline{u}\,\overline{w_1}^3, \overline{v}\,\overline{w_2}^3, \overline{y}^3\}$, or $\{\overline{u}\,\overline{w_2}^3, \overline{v}\,\overline{w_1}^3, \overline{y}^3\}$;
- three subgroups isomorphic to $\mathbb{Z}_9 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$, generated by $\overline{x}^3$ and $\overline{y}^3$ plus one of $\overline{z}$, $\overline{z}\,\overline{w_4}^3$ or $\overline{z}\,\overline{w_4}^6$, respectively;
- two subgroups isomorphic to $\mathbb{Z}_9 \oplus \mathbb{Z}_9$, generated by $\{\overline{x}, \overline{y}\}$ and $\{\overline{u}, \overline{v}\}$;
- four subgroups isomorphic to $\mathbb{Z}_9 \oplus \mathbb{Z}_3$, generated by $\{\overline{z}\,\overline{w_3}^6, \overline{y}^3\}$, $\{\overline{z}\,\overline{w_4}^3, \overline{y}^3\}$, $\{\overline{z}\,\overline{w_3}^3\,\overline{w_4}^6, \overline{y}^3\}$ and $\{\overline{z}\,\overline{w_4}^6, \overline{u}^3\}$;
- six G-invariant subgroups of exponent 1 or 3 lying in the second layer $K^{(3)}/K^{(9)}$, isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$, $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$, $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, $\mathbb{Z}_3$ and $\mathbb{Z}_1$.

Note that if we let T_0, T_1, T_2, T_3, T_4 and T_5 denote the six possibilities for a G-invariant subgroup of $K/K^{(3)}$ (of ranks 0, 1, 2, 2, 3 and 4 respectively), with T_2 generated by the images of x and y, and T_3 generated by the images of u and v, then we can represent each of the above subgroup types as a pair (T_i, T_j), where T_i indicates the first layer L_0/L_1 of the subgroup L, and T_j denotes the second layer L_1/L_2. Then in order, the pairs that occur are as follows:

- (T_5, T_5) once, • (T_4, T_5) once, • (T_2, T_5) and (T_3, T_5) once each, • (T_1, T_5) once,
- (T_2, T_4) once and (T_3, T_4) three times, • (T_1, T_4) three times,
- (T_2, T_2) and (T_3, T_3) once each, • (T_1, T_2) three times and (T_1, T_3) once,
- (T_0, T_5), (T_0, T_4), (T_0, T_3), (T_0, T_2), (T_0, T_1) and (T_0, T_0), once each.

The same argument shows that each 'double-layer' section K_i/K_{i+2} of K has exactly 24 G-invariant subgroups, analogous to those in the summary lists above.

As a consequence of these observations, we can draw three important conclusions.

First, for $m = 3^e$ the only G-invariant cyclic subgroups of $K/K^{(m)}$ have orders 1 and 3, because there is no subgroup in the above list isomorphic to $\mathbb{Z}_9$. In fact, if any layer L_i/L_{i+1} of a G-invariant subgroup L of K is non-trivial, then it contains the coset $L_{i+1}(w_1w_2w_3w_4)^{3^i}$, and so contains its image under conjugation by h, namely the coset $L_{i+1}(w_1w_2^{-2}w_3w_4^{-2})^{3^i}$, and hence contains also $L_{i+1}(w_2w_4)^{3^{i+1}}$, in which case the layer L_{i+1}/L_{i+2} has rank 2 or more. In other words, if one layer is T_1, then the next must be T_2, T_3, T_4 or T_5.

Second, the only G-invariant proper subgroup L of K with cyclic quotient K/L of 3-power order is the subgroup generated by $\overline{x}, \overline{y}, \overline{u}$ and $\overline{w_4}^3$, with quotient $K/L \cong C_3$. Hence if the top layer L_0/L_1 of a G-invariant subgroup L of K is T_4 (of rank 3), then the next layer L_1/L_2 is T_5 (of rank 4). Furthermore, the same argument applied to deeper layers shows that if any layer L_i/L_{i+1} of a G-invariant subgroup L of K is T_4 (of rank 3), then the next layer L_{i+1}/L_{i+2} must be T_5 (of rank 4).

Third, if any layer L_i/L_{i+1} of a G-invariant subgroup L of K has rank 2, then the next layer L_{i+1}/L_{i+2} can have rank 2, 3 or 4 (and if it has rank 3, then the subsequent layer L_{i+2}/L_{i+3} must have rank 4). In fact, the possible pairs for two successive layers in this case are (T_2, T_2), (T_2, T_4), (T_2, T_5), (T_3, T_3), (T_3, T_4) and (T_3, T_5), and then the possible triples for three successive layers are (T_2, T_2, T_2), (T_2, T_2, T_4), (T_2, T_2, T_5), (T_2, T_4, T_5), (T_2, T_5, T_5), (T_3, T_3, T_3), (T_3, T_3, T_4), (T_3, T_3, T_5), (T_3, T_4, T_5) and (T_3, T_5, T_5).

We can now put these observations together to find all possibilities for a normal subgroup L of $G = \langle h, a \rangle$ contained in K with index $|K : L|$ being a power $m = k^e$ of a prime k.

When $k = 3$, the layers of any such L must consist of (say) e_0 copies of T_0 (where $e_0 \geq 0$), followed by e_1 copies of T_1 (where $e_1 = 0$ or 1), followed by e_2 copies of either T_2 or T_3, followed by e_3 copies of T_4 (where $e_3 = 0$ or 1), followed by e_4 copies of T_5 (where $e_4 \geq 0$), with $e_0 + e_1 + e_2 + e_3 + e_4 = e$. Moreover, any such combination determines a unique L, except in the cases where a pair of successive layers is of the form (T_1, T_2), (T_1, T_4) or (T_3, T_4), where there are exactly three such L.

All the possibilities for L are listed in the summary Table 4.2.

Index $\lvert K:L \rvert$	Generating set for L	Quotient K/L
$d^4 = k^{4t}$, for any k	$\{w_1^d,\ w_2^d,\ w_3^d,\ w_4^d\}$	$\mathbb{Z}_d \oplus Z_d \oplus Z_d \oplus Z_d$
$c^2d^2 = k^{2s}k^{2t}$, with $s < t$, for any $k \neq 3$	$\{u^c,\ v^c,\ x^d,\ y^d\}$ or $\{x^c,\ y^c,\ u^d,\ v^d\}$	$\mathbb{Z}_c \oplus Z_c \oplus Z_d \oplus Z_d$
$bcd^2 = k^r k^s k^{2t}$, with $r < s$, for any $k \equiv 1 \bmod 3$	$\{{z_\lambda}^b,\ z_{\lambda^2}^c,\ x^d,\ y^d\}$ or $\{z_{\lambda^2}^b,\ {z_\lambda}^c,\ x^d,\ y^d\}$	$\mathbb{Z}_b \oplus Z_c \oplus Z_d \oplus Z_d$
$3d^4 = 3^{4t+1}$	$\{x^d,\ y^d,\ u^d,\ w_4^{3d}\}$	$\mathbb{Z}_d \oplus Z_d \oplus Z_d \oplus Z_{3d}$
$27d^4 = 3^{4t+3}$	$\{z^d,\ w_2^{3d},\ w_3^{3d},\ w_4^{3d}\}$	$\mathbb{Z}_d \oplus Z_{3d} \oplus Z_{3d} \oplus Z_{3d}$
$81d^4 = 3^{4t+3}$	$\{z^d,\ y^{3d},\ u^{3d},\ w_4^{9d}\}$, or $\{z^d w_4^{3d},\ y^{3d},\ u^{3d},\ w_4^{9d}\}$ or $\{z^d w_4^{-3d},\ y^{3d},\ u^{3d},\ w_4^{9d}\}$	$\mathbb{Z}_d \oplus Z_{3d} \oplus Z_{3d} \oplus Z_{9d}$
$c^2d^2 = 3^{2s}3^{2t}$, with $s < t$	$\{x^c,\ y^c,\ w_3^d,\ w_4^d\}$ or $\{u^c,\ v^c,\ w_3^d,\ w_4^d\}$	$\mathbb{Z}_c \oplus Z_c \oplus Z_d \oplus Z_d$
$3c^2d^2 = 3^{2s+1}3^{2t}$, with $s+1 < t$	$\{z^c w_4^{-3c},\ u^{3c},\ y^d,\ w_4^d\}$ or $\{z^c w_3^{-3c},\ y^{3c},\ u^d,\ w_4^d\}$ or $\{z^c w_3^{\frac{d}{3}-3c} w_4^{\frac{d}{3}},\ y^{3c},\ u^d,\ w_4^d\}$ or $\{z^c w_3^{-\frac{d}{3}-3c} w_4^{-\frac{d}{3}},\ y^{3c},\ u^d,\ w_4^d\}$	$\mathbb{Z}_c \oplus Z_{3c} \oplus Z_d \oplus Z_d$
$3c^2d^2 = 3^{2s}3^{2t+1}$, with $s+1 < t$	$\{x^c,\ y^c,\ u^d,\ w_4^{3d}\}$ or $\{u^c,\ v^c,\ y^d,\ w_4^{3d}\}$ or $\{u^c w_1^d,\ v^c w_2^d,\ y^d,\ w_4^{3d}\}$ or $\{u^c w_2^d,\ v^c w_1^d,\ y^d,\ w_4^{3d}\}$	$\mathbb{Z}_c \oplus Z_c \oplus Z_d \oplus Z_{3d}$
$9c^2d^2 = 3^{2s+1}3^{2t+1}$, with $s+2 < t$	$\{z^c w_3^{-3c},\ y^{3c},\ u^d,\ w_4^{3d}\}$ or $\{z^c w_3^{d-3c},\ y^{3c},\ u^d,\ w_4^{3d}\}$ or $\{z^c w_3^{-d-3c},\ y^{3c},\ u^d,\ w_4^{3d}\}$ or $\{z^c w_4^{-3c},\ u^{3c},\ y^d,\ w_4^{3d}\}$ or $\{z^c w_4^{d-3c},\ u^{3c},\ y^d,\ w_4^{3d}\}$ or $\{z^c w_4^{-d-3c},\ u^{3c},\ y^d,\ w_4^{3d}\}$	$\mathbb{Z}_c \oplus Z_{3c} \oplus Z_d \oplus Z_{3d}$

Table 4.2: Possibilities for $\langle h, a \rangle$-invariant subgroup L of K when $\langle h, a \rangle / K \cong A_3 \wr S_2$

4.2.2 Automorphism groups of the regular covers

To determine the size of the automorphism group of the covering graph, we consider which of the G-invariant subgroups of $K/K^{(m)}$ are normalized by the additional generators p, q and pq of the larger groups G_2^1, G_2^2 and G_3.

Note that conjugation by pq (which lies in G_2^1/N but not G_1/N) takes

$$x = w_1 w_3^{-1} w_4 \ \mapsto\ w_4 w_2^{-1} w_1 = xy^{-1}, \qquad y = w_2 w_3^{-1} \ \mapsto\ w_3 w_2^{-1} = y^{-1},$$
$$u = w_1 w_4^{-1} \ \mapsto\ w_4 w_1^{-1} = u^{-1}, \qquad v = w_2 w_3 w_4^{-1} \ \mapsto\ w_3 w_2 w_1^{-1} = u^{-1} v,$$

while if λ is a cube root of 1 in $\mathbb{Z}_k$, then also conjugation by pq takes

$$z_\lambda = w_1 w_2^\lambda w_3^\lambda w_4^{\lambda^2} \ \mapsto\ w_4 w_3^\lambda w_2^\lambda w_1^{\lambda^2} = (w_1 w_2^{\lambda^2} w_3^{\lambda^2} w_4^\lambda)^{\lambda^2} = z_{\lambda^2}^{\lambda^2}.$$

It follows that in each layer of $K/K^{(m)}$, the G_1-invariant subgroups of rank 2 that we encountered above are invariant under conjugation by pq, while those of rank 1 are not, except when $k = 3$. Hence in particular, for $k \neq 3$, every G_1-invariant subgroup of $K/K^{(m)}$ for which each layer has rank 0, 2 or 4 is G_2^1-invariant, but those for which some layer has rank 1 or 3 are not. In other words, those in rows 1 and 2 of the Table 4.2 are G_2^1-invariant, while those in row 3 are not.

In the exceptional case ($k = 3$), again some more careful attention is needed, but in fact it is easy to check that all G_1-invariant subgroups of $K/K^{(m)}$ are G_2^1-invariant apart from the 3rd and 4th subgroups in each of rows 8 and 9 of Table 4.2.

Similarly, conjugation by p takes

$$x = w_1 w_3^{-1} w_4 \ \mapsto\ w_3 w_1^{-1} w_2 = u^{-1} v, \qquad y = w_2 w_3^{-1} \ \mapsto\ w_4 w_1^{-1} = u^{-1},$$
$$u = w_1 w_4^{-1} \ \mapsto\ w_3 w_2^{-1} = y^{-1}, \qquad v = w_2 w_3 w_4^{-1} \ \mapsto\ w_4 w_1 w_2^{-1} = xy^{-1},$$

and hence interchanges the two rank 2 subgroups $\langle x, y\rangle$ and $\langle u, v\rangle$ of K.

It follows that when $k \neq 3$, the only G_3-invariant subgroups of $K/K^{(m)}$ are the homocyclic subgroups $\mathbb{Z}_d \oplus \mathbb{Z}_d \oplus \mathbb{Z}_d \oplus \mathbb{Z}_d$ (for d dividing m). When $k = 3$, the only G_2^1-invariant subgroups of $K/K^{(m)}$ that are also G_3-invariant are those in rows 1, 4 and 5 and the first of the subgroups in row 6 of Table 4.2; in all other cases, conjugation by p takes the subgroup to another of the G_2^1-invariant subgroups from the same row.

By Corollary 2.4.2 (or [26, Theorem 3]), none of the covers obtained from the subgroups that are G_2^1- or G_3-invariant can be 4-arc- or 5-arc-transitive, since each admits a 2-arc- or 3-arc-regular group of automorphisms. Similarly, if the subgroup L is G_1-invariant but not G_2^1-invariant, then the cover admits a 1-arc-regular group of automorphisms, and cannot be 3-arc-regular by Proposition 2.4.4 (or [26, Proposition 26], or [22, Proposition 2.3]).

Next, suppose the subgroup L is G_1-invariant but not G_2^1-invariant, and the cover is 4-arc-transitive. In that case, the cover has both a 1-arc-regular and a 4-arc-regular group of automorphisms, and so by Proposition 2.4.5 (or [26, Proposition 29], or [22, Proposition 3.2]), the cover must be a regular cover of the Heawood graph. Hence the quotient G_1/L must have a normal subgroup J/L of index $336/8 = 42$ in G_1/L, with $G_1/J \cong (G_1/L)/(J/L)$ isomorphic to an extension of C_7 by C_6, and J normal in the larger group G_4^1. In particular, $|G_1/L|$ is divisible by 7, and then since $|G_1/K| = |A_3 \wr S_2| = 18$, we find that $|K|$ is divisible by 7, so $m = |K/L| = 7^e$ for some e.

Now let $H = (G_1)'$, the derived group of G_1, which is the unique normal subgroup of index 6 in G_1 for which the corresponding quotient is cyclic of order 6. Then H contains both J and K, since both $G_1/K \cong A_3 \wr S_2$ and $G_1/J \cong C_7 \rtimes C_6$ have a cyclic quotient of order 6. Moreover, $H = JK$ since J is maximal in H and $K \not\subseteq J$, and so $J \cap K$ has index $|JK/K| = |H/K| = 3$ in J and index $|JK/J| = |H/J| = 7$ in K. It follows that $(J \cap K)/L$ has order $|K/L|/7 = m/7 = 7^{e-1}$, and index 3 in J/L, and is therefore a normal Sylow 7-subgroup of J/L.

In particular, $(J \cap K)/L$ is characteristic in J/L, and therefore normal in G_4^1/L. Factoring it out gives a quotient $G_4^1/(J \cap K) \cong (G_4^1/L)/((J \cap K)/L)$ of G_4^1 of order $(8 \cdot 18m)/(m/7) = 1008$, which is then a 4-arc-transitive group of automorphisms of a cubic graph of order $1008/24 = 42$. (In fact this will be a 3-fold cover of the Heawood graph.) There is, however, only one arc-transitive cubic graph of order 42, namely the graph F042 listed in the appendix of [22], but that graph is 1-arc-regular. Hence this possibility can be eliminated.

Similarly, if L is G_1-invariant but not G_2^1-invariant and the cover is 5-arc-transitive, then by Proposition 2.4.5 (or [26, Proposition 30], or [22, Proposition 3.4]), the cover is a regular cover of the Biggs-Conway graph. In particular, its 1-arc-regular group of automorphisms must have $\mathrm{PSL}(2,7)$ as a composition factor, and hence is insoluble. But on the other hand, G_1/L is a normal extension of an abelian group by $A_3 \wr S_2$ and is therefore soluble, so this possibility can also be eliminated.

Thus all of the regular covers obtained above are 1-arc-, 2-arc- or 3-arc-regular.

Next, we determine isomorphisms between the covering graphs that arise from the G_1-invariant subgroups in the Table 4.2. When the subgroup is G_3-invariant, the regular cover is 3-arc-regular, and is then unique up to isomorphism. When the subgroup is G_2^1-invariant but not G_3-invariant, the regular cover is 2-arc-regular of type 2^1, but isomorphic to the regular cover that arises from (exactly) one other subgroup in the same row of the table. Similarly, when the subgroup is G_1-invariant but not G_2^1-invariant, the regular cover is 1-arc-regular, and isomorphic to the regular cover that arises from (exactly) one other subgroup in the same row of the table.

4.3 Lifting the automorphism subgroup $(A_3 \times A_3) \rtimes C_4$

In the previous section we determined all symmetric abelian regular covers of $K_{3,3}$ via its 1-arc-regular group of automorphisms, $A_3 \wr S_2$. To complete the analysis, we consider what happens with the other minimal arc-transitive group of automorphisms, which is the 2-arc-regular group generated by the cosets Nh and Nap of N in the quotient G_3/N, and isomorphic to the automorphism subgroup $(A_3 \times A_3) \rtimes C_4$.

In this case, we work inside the quotient G_2^2/N', using the linear transformations of K induced by h and ap. The character table of group $(A_3 \times A_3) \rtimes C_4$ is as in Table 4.3.

Since the trace of the matrix induced by h is -2, it is immediately obvious that the

Element order	1	2	3	3	4	4
Class size	1	9	4	4	9	9
ψ_1	1	1	1	1	1	1
ψ_2	1	1	1	1	-1	-1
ψ_3	1	-1	1	1	ξ	$-\xi$
ψ_4	1	-1	1	1	$-\xi$	ξ
ψ_5	4	0	-2	1	0	0
ψ_6	4	0	1	-2	0	0

Table 4.3: The character table of the group $(A_3 \times A_3) \rtimes C_4$ where ξ is a primitive 4th root of 1

character of the given representation is ψ_5 , which is irreducible over $\mathbb{Q}$. It follows that for every prime $k \notin \{2, 3\}$, the group $K/K^{(k)} \cong (\mathbb{Z}_k)^4$ has no non-trivial proper G_2^2-invariant subgroup. The same holds also for $k = 2$, since the mod 2 reductions of each of the characters ψ_1 to ψ_4 are all trivial.

Hence for any prime $k \neq 3$, if L is a G_2^2-invariant subgroup of K with index $|K : L|$ a power of k, then every layer of L has rank 0 or 4, and so $|K : L| = d^4$ and $K/L \cong \mathbb{Z}_d \oplus \mathbb{Z}_d \oplus \mathbb{Z}_d \oplus \mathbb{Z}_d$ for some d. Every such subgroup L, however, is also preserved by conjugation by p (and by a) and hence is G_3-invariant, and so gives one of the 3-arc-regular covers found earlier.

For $k = 3$, the group $K/K^{(k)}$ has just two non-trivial proper G_2^2-invariant subgroup, namely the cyclic subgroup generated by the image of $z = w_1w_2w_3w_4$ (which is centralized by h and inverted under conjugation by ap), and the subgroup generated by any three of $x = w_1w_3^{-1}w_4$, $y = w_2w_3^{-1}$, $u = w_1w_4^{-1}$ and $v = w_2w_3w_4^{-1}$, with h conjugating x, y, u and v to y^{-1}, xy^{-1}, v^{-1} and uv^{-1}, and ap conjugating x, y, u and v to v^{-1}, u^{-1}, y and $x^{-1}y$, respectively.

Next, an easy exercise shows that $K/K^{(9)}$ has no G_2^2-invariant cyclic subgroup of order 9; equivalently, if the top layer of a G_2^2-invariant subgroup L is cyclic, then its next layer must have rank at least 3. Similarly, $K/K^{(9)}$ has no G_2^2-invariant rank 3 subgroup isomorphic to $\mathbb{Z}_9 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_9$; equivalently, if the top layer of a G_2^2-invariant subgroup L has rank 3, then its next layer must have (full) rank 4.

Hence the only G_2^2-invariant subgroups of K with 3-power index are those given in rows 1, 4, 5 and 6 of Table 4.2. On the other hand, if L is one of the second or third kinds of subgroup in row 6 of Table 4.2, generated by $z^d w_4^{\pm 3d}$, y^{3d}, u^{3d} and w_4^{9d}, then L is not G_2^2-invariant, since conjugation by ap takes $z^d w_4^{\pm 3d}$ to $z^{-d} w_2^{\mp 3d}$, which does not lie in L. In particular, every G_2^2-invariant subgroup of K with 3-power index is also G_3-invariant, and again gives one of the 3-arc-regular covers found earlier.

In other words, every symmetric abelian regular cover of $K_{3,3}$ obtainable via the 2-arc-regular subgroup $(A_3 \times A_3) \rtimes C_4$ of $S_3 \wr S_2$ is 3-arc-regular, and hence is obtainable via the 1-arc-regular subgroup $A_3 \wr S_2$.

4.4 Summary

Our findings can be summarised as follows:

Theorem 4.4.1 *Let $m = k^e$ be any power of a prime k, with $e > 0$. Then the arc-transitive abelian regular covers of the complete bipartite graph $K_{3,3}$ with abelian covering group of exponent m are as follows:*

(a) *If $k \equiv 2 \bmod 3$, there are exactly $e+1$ such covers, namely a 3-arc-regular cover with covering group $\mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m$, and one 2-arc-regular cover of type 2^1 with covering group $\mathbb{Z}_\ell \oplus \mathbb{Z}_\ell \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m$ for each proper divisor ℓ of m.*

(b) *If $k \equiv 1 \bmod 3$, then there are exactly $\frac{3}{2}e(e+1)+1$ such covers, namely a 3-arc-regular cover with covering group $\mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m$, and one 2-arc-regular cover of type 2^1 with covering group $\mathbb{Z}_\ell \oplus \mathbb{Z}_\ell \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m$ for each proper divisor ℓ of m, plus one 1-arc-regular cover with covering group $\mathbb{Z}_j \oplus \mathbb{Z}_j \oplus \mathbb{Z}_\ell \oplus \mathbb{Z}_m$ for each divisor j of m and each proper divisor ℓ of m, plus one 1-arc-regular cover with covering group $\mathbb{Z}_j \oplus \mathbb{Z}_\ell \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m$ for each pair $\{j, \ell\}$ of distinct proper divisors of m.*

(c) *If* $k = 3$ *and* $e \geq 3$*, then there are exactly* $8e - 5$ *such covers, namely four* 3*-arc-regular covers with covering groups* $\mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m$, $\mathbb{Z}_{m/3} \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m$, $\mathbb{Z}_{m/3} \oplus \mathbb{Z}_{m/3} \oplus \mathbb{Z}_{m/3} \oplus \mathbb{Z}_m$ *and* $\mathbb{Z}_{m/9} \oplus \mathbb{Z}_{m/3} \oplus \mathbb{Z}_{m/3} \oplus \mathbb{Z}_m$*, plus one* 2*-arc-regular cover of type* 2^1 *with covering group* $\mathbb{Z}_{m/9} \oplus \mathbb{Z}_{m/3} \oplus \mathbb{Z}_{m/3} \oplus \mathbb{Z}_m$*, plus one* 2*-arc-regular cover of type* 2^1 *with covering group* $\mathbb{Z}_\ell \oplus \mathbb{Z}_\ell \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m$ *for each proper divisor* ℓ *of* m*, plus two* 2*-arc-regular covers of type* 2^1 *with covering groups* $\mathbb{Z}_\ell \oplus \mathbb{Z}_{3\ell} \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m$ *and* $\mathbb{Z}_\ell \oplus \mathbb{Z}_\ell \oplus \mathbb{Z}_{m/3} \oplus \mathbb{Z}_m$ *for each proper divisor* ℓ *of* $m/3$*, plus three pairwise non-isomorphic* 2*-arc-regular covers of type* 2^1 *with covering groups* $\mathbb{Z}_\ell \oplus \mathbb{Z}_{3\ell} \oplus \mathbb{Z}_{m/3} \oplus \mathbb{Z}_m$ *for each proper divisor* ℓ *of* $m/9$*, plus two* 1*-arc-regular covers with covering groups* $\mathbb{Z}_\ell \oplus \mathbb{Z}_{3\ell} \oplus \mathbb{Z}_m \oplus \mathbb{Z}_m$ *and* $\mathbb{Z}_\ell \oplus \mathbb{Z}_\ell \oplus \mathbb{Z}_{m/3} \oplus \mathbb{Z}_m$ *for each proper divisor* ℓ *of* $m/3$*.*

(c) *If* $k = 3$ *and* $e = 2$*, then there are exactly* 11 *such covers, namely four* 3*-arc-regular covers with covering groups* $\mathbb{Z}_9 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_9$, $\mathbb{Z}_3 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_9$, $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_9$ *and* $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_9$*, plus five* 2*-arc-regular covers of type* 2^1 *with covering groups* $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_9$, $\mathbb{Z}_3 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_9$, $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_9$, $\mathbb{Z}_9 \oplus \mathbb{Z}_9$ *and* $\mathbb{Z}_3 \oplus \mathbb{Z}_9$*, and two* 1*-arc-regular covers with covering groups* $\mathbb{Z}_3 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_9$ *and* $\mathbb{Z}_3 \oplus \mathbb{Z}_9$*.*

(d) *If* $k = 3$ *and* $e = 1$*, then there are exactly* 4 *such covers, namely three* 3*-arc-regular covers with covering groups* $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$, $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ *and* $\mathbb{Z}_3$*, and one* 2*-arc-regular covers of type* 2^1 *with covering group* $\mathbb{Z}_3 \oplus \mathbb{Z}_3$*.*

Chapter 5

Symmetric abelian regular covers of the cube graph Q_3

By using the method of linear criteria for lifting automorphisms [27], Feng and Wang [32] classified all the symmetric cyclic regular covers of the 3-dimensional cube graph Q_3 in 2002, and Feng, Kwak and Wang [31] classified all the symmetric elementary abelian regular covers in 2004. The same method can be used to classify all the symmetric homocyclic regular covers of Q_3, but again for more general symmetric abelian regular covers, this method is very difficult to apply.

In this Chapter, we use our new approach to determine all the symmetric abelian regular covering graphs of Q_3. The numbers of covers and the largest value of s for which each cover is s-arc-transitive are also given.

5.1 Preliminaries

We know that the 3-dimensional cube graph Q_3 is the only symmetric cubic graph of order 8. Also Q_3 is 2-arc-transitive, and has type 2^1 (as described in Section 2.4). Its automorphism group is the direct product $S_4 \times C_2$, of order 48, and the only other arc-transitive subgroups of automorphisms of Q_3 are the subgroups $A_4 \times C_2$ and S_4, each of which acts regularly on the arcs of Q_3, and has order 24.

Take $G_2^1 = \langle\, h, a, p \mid h^3 = a^2 = p^2 = [a, p] = (hp)^2 = 1\,\rangle$. By MAGMA, we know that this group has two normal subgroups of index 48, both with quotient $S_4 \times C_2$, but these are interchanged by the outer automorphism of G_2^1 that takes the generators h, a and p to h, ap and p, and so without loss of generality we can take either one of them. We will take N to be the torsion free normal subgroup of G_2^1 of index 48 that is freely generated by

$$w_1 = (ha)^4, \quad w_2 = (ah)^4, \quad w_3 = h^{-1}ahahahah^{-1},$$
$$w_4 = ah^{-1}ahahahah^{-1}a \quad \text{and} \quad w_5 = (hah^{-1}a)^3.$$

Easy calculations show that the generators h, a and p act by conjugation as below:

$$\begin{array}{lllllllll}
h^{-1}w_1h & = & w_2 & a^{-1}w_1a & = & w_2 & p^{-1}w_1p & = & w_2^{-1} \\
h^{-1}w_2h & = & w_3 & a^{-1}w_2a & = & w_1 & p^{-1}w_2p & = & w_1^{-1} \\
h^{-1}w_3h & = & w_1 & a^{-1}w_3a & = & w_4 & p^{-1}w_3p & = & w_3^{-1} \\
h^{-1}w_4h & = & w_2^{-1}w_3^{-1}w_5^{-1} & a^{-1}w_4a & = & w_3 & p^{-1}w_4p & = & w_4^{-1} \\
h^{-1}w_5h & = & w_3^{-1}w_4w_5^{-1} & a^{-1}w_5a & = & w_5^{-1} & p^{-1}w_5p & = & w_3w_4^{-1}w_5\ .
\end{array}$$

Now take the quotient G_2^1/N', which is an extension of the free abelian group $N/N' \cong \mathbb{Z}^5$ by the group $G_2^1/N \cong S_4 \times C_2$.

The two subgroups of the latter group that act regularly on the arcs of Q_3 are the quotients (mod N) of the subgroups $\langle h, a\rangle$ and $\langle h, ap\rangle$, isomorphic to S_4 and $A_4 \times C_2$, respectively. Also replace the generators h, a, p and all w_i by their images in G_2^1/N', and let K denote the subgroup N/N'.

By the above observations, the generators h, a, p and ap induce linear transformations of the free abelian group $K \cong \mathbb{Z}^5$ as follows:

$$a \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad h \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 & -1 \end{pmatrix},$$

$$p \mapsto \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix} \quad \text{and} \quad ap \mapsto \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 \end{pmatrix}.$$

The matrices representing h and a generate a subgroup isomorphic to S_4, and while those for h and ap generate a subgroup isomorphic to $A_4 \times C_2$.

5.2 Lifting the automorphism subgroup S_4

In this section, we find all of the symmetric abelian regular covers of Q_3 that can be obtained by lifting the 1-arc-regular subgroup S_4.

The matrices of orders 3 and 2 representing h and a both have trace -1, while the matrices of orders 4, 3 and 2 representing ha, $[h,a]$ and $(ha)^2$ have trace $1, -1$ and 1.

Next, the character table of the group S_4 is as in Table 5.1. By inspecting traces, we see that the character of the 5-dimensional representation of S_4 over $\mathbb{Q}$ associated with the above action of $G = \langle h, a \rangle$ on K is the character $\chi_3 + \chi_4$, which is the sum of the characters of irreducible 2-dimensional and 3-dimensional representations over the rational field $\mathbb{Q}$.

It follows that for every prime $k \notin \{2, 3\}$, the group $K/K^{(k)} \cong (\mathbb{Z}_k)^5$ is direct sum of two G-invariant irreducible subgroups of rank 2 and rank 3, and these are its only non-trivial proper G-invariant subgroups.

In fact, the rank 2 subgroup of K generated by $u = w_1 w_2^{-1} w_4 w_5^{-1}$ and $v = w_3 w_4$ is normal in $\langle h, a \rangle$, with $u^h = v^{-1}$, $v^h = uv^{-1}$, $u^a = u^{-1}v$ and $v^a = v$, while the rank 3 subgroup generated by $x = w_1 w_2 w_4$, $y = w_3 w_4^{-1}$ and $z = w_5$ is normal in $\langle h, a \rangle$, with $x^h = z^{-1}$, $y^h = xyz$, $z^h = (yz)^{-1}$, $x^a = xy$, $y^a = y^{-1}$ and $z^a = z^{-1}$.

In the quotient $G/K^{(k)}$ for prime $k \notin \{2, 3\}$, the image of the rank 2 subgroup $\langle u, v \rangle$ has no non-trivial G-invariant cyclic subgroup, and hence is irreducible, and similarly, the

Element order	1	2	2	3	4
Class size	1	3	6	8	6
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	2	2	0	-1	0
χ_4	3	-1	-1	0	1
χ_5	3	-1	1	0	-1

Table 5.1: The character table of the group S_4

image of the rank 3 subgroup $\langle x, y, z\rangle$ has no non-trivial proper G-invariant subgroup, and hence is irreducible.

For $k = 2$, the image of the rank 2 subgroup $\langle u, v\rangle$ in $G/K^{(k)}$ is irreducible as an G-invariant subgroup, but in this case, it is also contained in the image of the rank 3 subgroup $\langle x, y, z\rangle$ (since $K^{(2)}xz = K^{(2)}w_1w_2w_4w_5 = K^{(2)}u$ and $K^{(2)}y = K^{(2)}w_3w_4^{-1} = K^{(2)}v$), as well as in a third non-trivial proper G-invariant subgroup, namely the rank 4 subgroup generated by the images of the products w_iw_j for all i, j (which contains the image of $\langle u, v\rangle$ but not the image of $\langle x, y, z\rangle$).

In contrast, for $k = 3$ the image of the rank 3 subgroup $\langle x, y, z\rangle$ in $G/K^{(k)}$ is irreducible as an G-invariant subgroup, while the image of the rank 2 subgroup $\langle u, v\rangle$ contains a non-trivial cyclic G-invariant subgroup, namely the subgroup generated by the image of $uv = w_1w_2^{-1}w_3w_4^2w_5^{-1}$, since $(uv)^h = uv$ and $(uv)^a = (uv)^{-1}$. Hence in this case, we have also a fourth non-trivial proper G-invariant subgroup, namely the rank 4 subgroup generated by the images of x, y, z and uv.

As before, analogous observations hold also for each other layer K_i/K_{i+1} of K (with $i \geq 1$).

So now suppose $m = k^e$ is any prime-power greater than 1, and suppose $L/K^{(m)}$ is any non-trivial normal subgroup of $G/K^{(m)}$ contained in $K/K^{(m)}$.

If $k \notin \{2, 3\}$, then it is easy to see that there exist divisors c and d of $m = k^e$ such that L is generated by $u^{m/c}, v^{m/c}, x^{m/d}, y^{m/d}$ and $z^{m/d}$, and $L/K^{(m)} \cong (\mathbb{Z}_c)^2 \oplus (\mathbb{Z}_d)^3$.

When $k = 3$, we note that conjugation by h and a behave well, in that they take the triple $(x^{k^t}, y^{k^t}, z^{k^t})$ to $(z^{-k^t}, (xyz)^{k^t}, (yz)^{-k^t}) = ((z^{k^t})^{-1}, x^{k^t}y^{k^t}z^{k^t}, (y^{k^t})^{-1}(z^{k^t})^{-1})$ and to $((xy)^{k^t}, y^{-k^t}, z^{-k^t}) = (x^{k^t}y^{k^t}, (y^{k^t})^{-1}, (z^{k^t})^{-1})$ respectively, for every positive integer t. The same kind of thing happens also with the pair (u, v). On the other hand, conjugation by h and a take uv to $uv^{-2} = (uv)v^{-3}$ and $u^{-1}v^2 = (uv)^{-1}v^3$, and hence if any layer L_i/L_{i+1} of a G-invariant subgroup L of $K/K^{(m)}$ has rank 1 or 4 (generated by images of $(uv)^i$ or of $(uv)^i$, x^i, y^i and z^i) then its next layer must have rank 2 or 5 (generated by images of $(uv)^{3i}$ and v^{3i} or of $(uv)^{3i}$, v^{3i}, x^{3i}, y^{3i} and z^{3i}).

The case $k = 2$ is not quite so straightforward. In this case, each layer can have rank 0, 1, 2, 3, 4 or 5, depending on the layers above it. To see exactly what happens, it is helpful to consider the case $m = 2^3 = 8$. Let $\overline{g}$ denote the coset $K^{(8)}g$, when $g \in K$. Then an easy exercise (using MAGMA if necessary) shows that $K/K^{(8)}$ contains exactly 35 subgroups that are normal in $G/K^{(8)}$, and these may be summarised as follows:

- the group $K/K^{(8)} \cong (\mathbb{Z}_8)^5$ itself, generated by all the $\overline{w_i}$;
- three other homocyclic subgroups $K^{2^i}/K^{(8)} \cong (\mathbb{Z}_{2^{3-i}})^5$, generated by all the $\overline{w_i}^{2^i}$;
- one subgroup isomorphic to $(\mathbb{Z}_8)^4 \oplus \mathbb{Z}_4$, generated by $\{\overline{u}, \overline{v}, \overline{w_1w_4}, \overline{w_2w_4}, \overline{w_5}^2\}$;
- one subgroup isomorphic to $(\mathbb{Z}_4)^4 \oplus \mathbb{Z}_2$, generated by $\{\overline{u}^2, \overline{v}^2, (\overline{w_1w_4})^2, (\overline{w_2w_4})^2, \overline{w_5}^4\}$;
- one subgroup isomorphic to $(\mathbb{Z}_2)^4$, generated by $\{\overline{u}^4, \overline{v}^4, (\overline{w_1w_4})^4, (\overline{w_2w_4})^4\}$;
- one subgroup isomorphic to $(\mathbb{Z}_8)^3 \oplus (\mathbb{Z}_4)^2$, generated by $\{\overline{x}, \overline{y}, \overline{z}, (\overline{w_2})^2, \overline{w_4}^2\}$;
- one subgroup isomorphic to $(\mathbb{Z}_8)^3 \oplus (\mathbb{Z}_2)^2$, generated by $\{\overline{x}, \overline{y}, \overline{z}, (\overline{w_2})^4, \overline{w_4}^4\}$;
- one subgroup isomorphic to $(\mathbb{Z}_4)^3 \oplus (\mathbb{Z}_2)^2$, generated by $\{(\overline{x})^2, (\overline{y})^2, (\overline{z})^2, (\overline{w_2})^4, \overline{w_4}^4\}$;
- one subgroup isomorphic to $(\mathbb{Z}_8)^2 \oplus (\mathbb{Z}_4)^3$, generated by $\{\overline{u}, \overline{v}, (\overline{w_2})^2, (\overline{w_4})^2, \overline{w_5}^2\}$;
- two subgroups isomorphic to $(\mathbb{Z}_8)^2 \oplus (\mathbb{Z}_2)^3$, generated by $\{\overline{u}, \overline{v}, (\overline{w_2})^4, (\overline{w_4})^4, \overline{w_5}^4\}$, and $\{\overline{uw_2}^2\overline{w_5}^2, \overline{vw_4}^2\overline{w_5}^2, (\overline{w_2})^4, (\overline{w_4})^4, \overline{w_5}^4\}$;
- one subgroup isomorphic to $(\mathbb{Z}_4)^2 \oplus (\mathbb{Z}_2)^3$, generated by $\{(\overline{u})^2, (\overline{v})^2, (\overline{w_2})^2, (\overline{w_4})^4, \overline{w_5}^4\}$;
- one subgroup isomorphic to $(\mathbb{Z}_8)^2 \oplus (\mathbb{Z}_4)^2 \oplus \mathbb{Z}_2$, generated by $\{\overline{u}, \overline{v}, (\overline{w_1w_4})^2, (\overline{w_2w_4})^2, \overline{w_5}^4\}$;
- two subgroups isomorphic to $(\mathbb{Z}_8)^2 \oplus \mathbb{Z}_4 \oplus (\mathbb{Z}_2)^2$, generated by $\{\overline{u}, \overline{v}, (\overline{w_5})^2, (\overline{w_2})^4, \overline{w_4}^4\}$,

and $\{\overline{uw_2}^2, \overline{vw_4}^{-2}, (\overline{w_5})^2, (\overline{w_2})^4, \overline{w_4}^4\}$;

- one subgroup isomorphic to $(\mathbb{Z}_8)^2 \oplus (\mathbb{Z}_2)^2$, generated by $\{\overline{u}, \overline{v}, (\overline{w_1w_4})^4, (\overline{w_2w_4})^4\}$;
- one subgroup isomorphic to $(\mathbb{Z}_4)^2 \oplus (\mathbb{Z}_2)^2$, generated by $\{\overline{u}^2, \overline{v}^2, (\overline{w_1w_4})^4, (\overline{w_2w_4})^4\}$;
- one subgroup isomorphic to $(\mathbb{Z}_8)^3$, generated by $\{\overline{x}, \overline{y}, \overline{z}\}$, plus two other subgroups isomorphic to $(\mathbb{Z}_{2^{3-i}})$, generated by $\{\overline{x}^{2^i}, \overline{y}^{2^i}, \overline{z}^{2^i}\}$ for $i = 1, 2$;
- two subgroups isomorphic to $(\mathbb{Z}_8)^2 \oplus \mathbb{Z}_4$, generated by $\{\overline{uw_2}^2, \overline{vw_4}^{-2}, \overline{w_5}^2\}$ and $\{\overline{uw_2}^{-2}, \overline{vw_4}^2, \overline{w_5}^2\}$;
- two subgroups isomorphic to $(\mathbb{Z}_4)^2 \oplus \mathbb{Z}_2$, generated by $\{\overline{u}^2, \overline{v}^2, \overline{w_5}^4\}$ and $\{\overline{u}^2\overline{w_2}^4, \overline{v}^2\overline{w_4}^4, \overline{w_5}^4\}$;
- four subgroups isomorphic to $(\mathbb{Z}_8)^2 \oplus \mathbb{Z}_2$, generated by $\{\overline{u}, \overline{v}, \overline{w_5}^4\}$, $\{\overline{uw_2}^4, \overline{vw_4}^4, \overline{w_5}^4\}$, $\{\overline{uw_2}^2\overline{w_5}^2, \overline{vw_4}^{-2}\overline{w_5}^2, \overline{w_5}^4\}$ and $\{\overline{uw_2}^{-2}\overline{w_5}^2, \overline{vw_4}^2\overline{w_5}^2, \overline{w_5}^4\}$;
- two subgroups isomorphic to $(\mathbb{Z}_8)^2$, generated by $\{\overline{u}, \overline{v}\}$ and $\{\overline{uw_2}^4, \overline{vw_4}^4\}$;
- two subgroups isomorphic to $(\mathbb{Z}_4)^2$, generated by $\{\overline{u}^2, \overline{v}^2\}$ and $\{\overline{u}^2\overline{w_2}^4, \overline{v}^2\overline{w_4}^4\}$;
- one subgroup isomorphic to $(\mathbb{Z}_2)^2$, generated by $\{\overline{u}^4, \overline{v}^4\}$.

If we let T_0, T_1, T_2, T_3 and T_4 denote the five possibilities for a G-invariant subgroup of $K/K^{(2)}$ (of ranks 0, 2, 3, 4 and 5 respectively), then once again we can represent each of the 2-layer combinations as a pair (T_i, T_j). The pairs that arise in this case can be found from looking at G-invariant subgroups of $K/K^{(4)}$ (or equivalently, the top two layers or the bottom two layers of $K/K^{(8)}$), and these are easily found to be the following, counted according to their frequency: (T_4, T_4) once; (T_3, T_4) once; (T_2, T_2) once; (T_2, T_4) once; (T_1, T_1) twice; (T_1, T_2) twice; (T_1, T_3) once; (T_1, T_4) once; and (T_0, T_0), (T_0, T_1), (T_0, T_2), (T_0, T_3) and (T_0, T_4), once each.

The 3-layer combinations that arise from the above list of G-invariant subgroups of $K/K^{(8)}$ can now be summarised with their associated frequencies, as follows:

- (T_4, T_4, T_4), once,
- (T_3, T_4, T_4), once,
- (T_2, T_4, T_4), once,
- (T_1, T_4, T_4), once,
- (T_1, T_3, T_4), once,
- (T_0, T_4, T_4), (T_0, T_0, T_4) and (T_0, T_0, T_0), once each,
- (T_3, T_3, T_4), once,
- (T_2, T_2, T_4), once,
- (T_1, T_1, T_4), twice,
- (T_1, T_2, T_4), twice,
- (T_0, T_0, T_3), once,
- (T_0, T_2, T_4), once,
- (T_0, T_1, T_4), once,
- (T_1, T_1, T_3), once,

- (T_0, T_1, T_3), once,
- (T_2, T_2, T_2), (T_0, T_2, T_2) and (T_0, T_0, T_2), once each,
- (T_1, T_2, T_2), twice,
- (T_0, T_1, T_2), twice,
- (T_1, T_1, T_2), four times,
- (T_1, T_1, T_1), twice,
- (T_0, T_1, T_1), twice,
- (T_0, T_0, T_1), once.

The same argument shows that each 'triple-layer' section K_i/K_{i+3} of K has exactly 35 G-invariant subgroups, analogous to those in the summary list above, when $k = 2$.

We can now put these observations together to find all possibilities for a normal subgroup L of $G = \langle h, a \rangle$ contained in K with index $|K : L|$ being a power $m = k^e$ of a prime k.

When $k = 2$, the layers of any such L must consist of (say) e_0 copies of T_0, followed by e_1 copies of T_1, followed by e_2 copies of T_2, followed by e_3 copies of T_3, followed by e_4 copies of T_4, with $e_0 + e_1 + e_2 + e_3 + e_4 = e$, and $e_i \geq 0$ for all i, but $e_3 \leq 1$, and $e_2 e_3 = 0$. Moreover, any such combination determines a unique L, except in some cases where a pair of successive layers is of the form (T_1, T_1) or (T_1, T_2) and no layer is T_3, namely as follows:

a) If $(e_1, e_2, e_3) = (2, 1, 0)$, so that L is a tower of two copies of T_1 on top of a single copy of T_2, on top of a tower of any number of copies of T_4, then there are four possibilities for L,

b) If $e_3 = 0$ and either $e_1 > 1$ or $e_1 e_2 > 0$, but $(e_1, e_2) \neq (2, 1)$, then there are two possibilities for L.

All the possibilities for L are listed in the summary Table 5.2.

Index $\lvert K:L \rvert$	Generating set for L	Quotient K/L
$d^5 = k^{5t}$, for any k	$\{w_1^d,\ w_2^d,\ w_3^d,\ w_4^d,\ w_5^d\}$	$(\mathbb{Z}_d)^5$
$c^2d^3 = k^{2s}k^{3t}$, with $s \neq t$, for any $k > 2$	$\{u^c,\ v^c,\ x^d,\ y^d,\ z^d\}$	$(\mathbb{Z}_c)^2 \oplus (\mathbb{Z}_d)^3$
$8d^3 = 2^{5t+3}$	$\{u^d,\ v^d,\ x^{2d},\ y^{2d},\ z^{2d}\}$	$(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{2d})^3$
$c^2d^3 = 2^{2s+3t}$, with $s+1 < t$	$\{u^c,\ v^c,\ w_2^d,\ w_4^d,\ w_5^d\}$ or $\{u^c w_2^{\frac{d}{2}} w_5^{\frac{d}{2}},\ v^c w_4^{\frac{d}{2}} w_5^{\frac{d}{2}},\ w_2^d,\ w_4^d,\ w_5^d\}$	$(\mathbb{Z}_c)^2 \oplus (\mathbb{Z}_d)^3$
$c^3d^2 = 2^{3s+2t}$, with $s < t$	$\{x^c,\ y^c,\ z^c,\ w_2^d,\ w_4^d\}$	$(\mathbb{Z}_c)^3 \oplus (\mathbb{Z}_d)^2$
$4c^2d^3 = 2^{2s}2^{3t+2}$, with $s < t$	$\{u^c,\ v^c,\ w_5^d,\ w_2^{2d},\ w_4^{2d}\}$ or $\{u^c w_2^d,\ v^c w_4^d,\ w_5^d,\ w_2^{2d},\ w_4^{2d}\}$	$(\mathbb{Z}_c)^2 \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{2d})^2$
$2c^3d^2 = 2^{3s+1}2^{2t}$, with $s+2 < t$	$\{u^c w_2^{2c},\ v^c w_4^{-2c},\ w_5^{2c},\ w_2^d,\ w_4^d\}$ or $\{u^c w_2^{2c-\frac{d}{2}},\ v^c w_4^{-2c+\frac{d}{2}},\ w_5^{2c},\ w_2^d,\ w_4^d\}$	$(\mathbb{Z}_c)^2 \oplus \mathbb{Z}_{2c} \oplus (\mathbb{Z}_d)^2$
$4c^3d^2 = 2^{3s+2}2^{2t}$, with $s+2 < t$	$\{u^c w_2^{2c} w_5^{2c},\ v^c w_4^{-2c} w_5^{2c},\ w_5^{4c},\ w_2^d,\ w_4^d\}$ or $\{u^c w_2^{2c-\frac{d}{2}} w_5^{2c},\ v^c w_4^{-2c+\frac{d}{2}} w_5^{2c},\ w_5^{4c},\ w_2^d,\ w_4^d\}$	$(\mathbb{Z}_c)^2 \oplus \mathbb{Z}_{4c} \oplus (\mathbb{Z}_d)^2$
$4c^2d^3 = 2^{2s}2^{3t+1}$, with $s < t$	$\{u^c,\ v^c,\ (w_1w_4)^d,\ (w_2w_4)^d,\ w_5^{2d}\}$	$(\mathbb{Z}_c)^2 \oplus (\mathbb{Z}_d)^2 \oplus \mathbb{Z}_{2d}$
$3c^3d^2 = 3^{3s}3^{2t+1}$, any s and t	$\{x^c,\ y^c,\ z^c,\ (uv)^d,\ v^{3d}\}$	$(\mathbb{Z}_c)^3 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_{3d}$

Table 5.2: Possibilities for $\langle h,a\rangle$-invariant subgroup L of K when $\langle h,a\rangle/K \cong S_4$

Next, we consider which of the $\langle h, a\rangle$-invariant subgroups of $K/K^{(m)}$ are normalized by the additional generator p of the larger group G_2^1. Here we will work inside the group G_2^1/N', and adopt the same notation for images of elements in this group.

Note that conjugation by p (which lies in G_2^1/N' but not G_1/N') takes

$$\begin{array}{ccc} u = w_1w_2^{-1}w_4w_5^{-1} & \mapsto & w_1w_2^{-1}w_3^{-1}w_5^{-1} = uv^{-1} \\ v = w_3w_4 & \mapsto & w_3^{-1}w_4^{-1} = v^{-1} \\ x = w_1w_2w_4 & \mapsto & w_1^{-1}w_2^{-1}w_4^{-1} = x^{-1} \\ y = w_3w_4^{-1} & \mapsto & w_3^{-1}w_4 = y^{-1} \\ z = w_5 & \mapsto & w_3w_4^{-1}w_5 = yz \end{array}$$

and also preserves the subgroup generated by the products w_i, w_j for all i, j. Moreover, when $k = 3$ this takes $uv = w_1w_2^{-1}w_3w_4^2w_5^{-1}$ to $w_1w_2^{-1}w_3^{-2}w_4^{-1}w_5^{-1} = uv$.

Hence all of the $\langle h, a\rangle$-invariant subgroups of $K/K^{(k)}$ that we met for each prime k are also $\langle h, a, p\rangle$-invariant. In fact it is not difficult to see that all of the $\langle h, a\rangle$-invariant subgroups summarised in Table 5.2 are G_2^1-invariant.

It follows that all of the resulting covers of Q_3 admit a 2-arc-regular group of automorphisms. In particular, none of these covers can be 4-arc- or 5-arc-transitive, by Proposition 2.4.1. Also by the same arguments as used for K_4 (in Chapter 3), none of them can be 3-arc-transitive, because the subgroup N itself is not normal in G_3. Thus all of the covers resulting from the above possibilities for the subgroup L are 2-arc-regular. Moreover, again since the subgroup N itself is not normal in G_3, each of them is unique up to isomorphism.

This completes the determination of symmetric abelian regular covers of Q_3 obtainable via the 1-arc-regular group of automorphisms isomorphic to S_4.

5.3 Lifting the automorphism subgroup $A_4 \times C_2$

In this section, we consider what happens with the other minimal arc-transitive group of automorphisms, namely the 1-arc-regular group generated by the cosets Nh and Nap of

N in the quotient G_2^1/N, which is isomorphic to the direct product $A_4 \times C_2$.

The character table of the group $A_4 \times C_2$ is as in Table 5.3. Note that the matrices of orders 3 and 2 representing h and ap have trace -1 and -3, while the matrices of order 6, 3, and 2 representing hap, $(hap)^2$ and $(hap)^3$ have trace $1, -1$ and 1.

By inspecting traces, we see that the character of the 5-dimensional representation of $A_4 \times C_2$ over $\mathbb{Q}$ associated with the action of $\langle h, ap \rangle$ on K is the character $\psi_5 + \psi_6 + \psi_7$, which is reducible to the sum of $\psi_5 + \psi_6$ and ψ_7, which are characters of two irreducible 2-dimensional and 3-dimensional representations over the rational field $\mathbb{Q}$.

Element order	1	2	2	2	3	3	6	6
Class size	1	1	3	3	4	4	4	4
ψ_1	1	1	1	1	1	1	1	1
ψ_2	1	-1	1	-1	1	1	-1	-1
ψ_3	1	1	1	1	λ	$-1-\lambda$	λ	$-1-\lambda$
ψ_4	1	1	1	1	$-1-\lambda$	λ	$-1-\lambda$	λ
ψ_5	1	-1	1	-1	$-1-\lambda$	λ	$1+\lambda$	$-\lambda$
χ_6	1	-1	1	-1	λ	$-1-\lambda$	$-\lambda$	$1+\lambda$
ψ_7	3	3	-1	-1	0	0	0	0
ψ_8	3	-3	-1	1	0	0	0	0

Table 5.3: The character table of the group $A_4 \times C_2$ where the λ is a primitive cube root of 1

For every prime $k \notin \{2, 3\}$, the group $K/K^{(k)} \cong (\mathbb{Z}_k)^5$ is direct sum of two G-invariant irreducible subgroups of rank 2 and rank 3 generated by the images of u and v on one hand, and of x, y and z on the other. Both of these subgroups are $\langle h, ap \rangle$-invariant, since $u^{ap} = u^{-1}$, $v^{ap} = v^{-1}$, $x^{ap} = x^{-1}y^{-1}$, $y^{ap} = y$ and $z^{ap} = y^{-1}z^{-1}$. Moreover, these two $\langle h, ap \rangle$-invariant subgroups are irreducible, unless $\mathbb{Z}_k$ contains a cube root of 1, in which case the rank 2 subgroup is reducible.

When $k > 2$ and $k \equiv 2 \mod 3$, it follows that the $\langle h, ap \rangle$-invariant subgroups with index in K being a power of k are precisely the same as the $\langle h, a\rangle$-invariant subgroups, given in the first two rows of Table 5.2.

When $k > 2$ and $k \equiv 1 \mod 3$ (and $m = k^e$ for some $e > 0$), the rank 2 subgroup of the quotient $K/K^{(m)}$ generated by the images of u and v is the direct sum of two $\langle h, ap\rangle$-invariant rank 1 subgroups, generated by the images of v_λ and v_{λ^2}, where λ is a primitive cube root of 1 in $\mathbb{Z}_m$ and $v_t = w_1 w_2^{-1} w_3^t w_4^{1+t} w_5^{-1}$ for $t \in \{\lambda, \lambda^2\}$. Note that since $t^2+t+1 = 0$ in $\mathbb{Z}_m$ and $v_t^h = w_1^t w_2^{-t} w_3^{t^2} w_4^{-1} w_5^{-t} = v_t^t$ and $v_t^{ap} = w_1^{-1} w_2 w_3^{-t} w_4^{-1-t} w_5 = v_t^{-1}$ modulo $K^{(k)}$. On the other hand, the rank 3 subgroup of $K/K^{(m)}$ generated by the images of x, y and z remains irreducible under conjugation by $\langle h, ap\rangle$.

When $k = 3$, the $\langle h, ap\rangle$-invariant subgroup generated by the image of x, y and z remains irreducible, and also the rank 1 subgroup generated by the image of uv is $\langle h, ap\rangle$-invariant, with $(uv)^h = uv$ and $(uv)^{ap} = w_1^{-1} w_2 w_4^{-1} w_3^{-2} w_4^{-1} w_5 w_3 = (uv)^{-1}$. Hence the $\langle h, ap\rangle$-invariant subgroups with index in K being a power of 3 are precisely the same as the $\langle h, a\rangle$-invariant subgroups, given in the first two rows and the last row of Table 5.2.

When $k = 2$ on the other hand, the $\langle h, ap\rangle$-invariant rank 2 subgroup of $K/K^{(k)}$ generated by the images of u and v remains irreducible (since $u^h = v^{-1}$ and $v^h = uv^{-1}$ while $u^{ap} = u^{-1}$ and $v^{ap} = v^{-1}$). In fact, the $\langle h, ap\rangle$-invariant subgroups of $K/K^{(2)}$ are precisely the same as the $\langle h, a\rangle$-invariant subgroups, namely the ones we called T_0, T_1, T_2, T_3 and T_4 above. Other possibilities arise, however, for $\langle h, ap\rangle$-invariant subgroups of $K/K^{(m)}$ when m is a larger power of 2.

Again it is helpful to consider the case $m = 8 = 2^3$, to see what happens. It is an easy exercise (using MAGMA if necessary) to show that $K/K^{(8)}$ contains exactly 16 subgroups that are preserved under conjugation by h and ap but not by a (or p), and these may be summarised as follows:

- two subgroups isomorphic to $(\mathbb{Z}_8)^2 \oplus (\mathbb{Z}_2)^3$, generated by $\{\overline{uw_2}^2\overline{w_4}^2, \overline{vw_2}^2\overline{w_5}^2, \overline{w_2}^4, \overline{w_4}^4, \overline{w_5}^4\}$ and $\{\overline{uw_4}^2\overline{w_5}^2, \overline{vw_2}^2\overline{w_4}^2, \overline{w_2}^4, \overline{w_4}^4, \overline{w_5}^4\}$;
- two subgroups isomorphic to $(\mathbb{Z}_8)^2 \oplus \mathbb{Z}_4 \oplus (\mathbb{Z}_2)^2$, generated by $\{\overline{uw_4}^2, \overline{vw_2}^2\overline{w_4}^2, \overline{w_5}^2, \overline{w_2}^4, \overline{w_4}^4\}$ and $\{\overline{uw_2}^2\overline{w_4}^2, \overline{vw_2}^2, \overline{w_5}^2, \overline{w_2}^4, \overline{w_4}^4\}$;

- two subgroups isomorphic to $(\mathbb{Z}_8)^2 \oplus \mathbb{Z}_4$, generated by $\{\overline{uw_2}^2\overline{w_4}^4, \overline{vw_2}^4\overline{w_4}^2, \overline{w_5}^2\}$ and $\{\overline{uw_2}^6\overline{w_4}^4, \overline{vw_2}^4\overline{w_4}^{-2}, \overline{w_5}^2\}$;
- two subgroups isomorphic to $(\mathbb{Z}_4)^2 \oplus \mathbb{Z}_2$, generated by $\{\overline{u}^2\overline{w_4}^4, \overline{v}^2\overline{w_2}^4\overline{w_4}^4, \overline{w_5}^4\}$ and $\{\overline{u}^2\overline{w_2}^4\overline{w_4}^4, \overline{v}^2\overline{w_2}^4, \overline{w_5}^4\}$;
- four subgroups isomorphic to $(\mathbb{Z}_8)^2 \oplus \mathbb{Z}_2$, generated by $\{\overline{uw_4}^4, \overline{vw_2}^4\overline{w_4}^4, \overline{w_5}^4\}$, $\{\overline{uw_2}^4\overline{w_4}^4, \overline{vw_2}^4, \overline{w_5}^4\}$, $\{\overline{uw_1}^{-2}\overline{w_4}^2, \overline{vw_1}^6\overline{w_2}^2, \overline{w_5}^4\}$ and $\{\overline{uw_1}^2\overline{w_4}^{-2}, \overline{vw_1}^2\overline{w_2}^6, \overline{w_5}^4\}$;
- two subgroups isomorphic to $(\mathbb{Z}_8)^2$, generated by $\{\overline{uw_2}^4\overline{w_4}^4, \overline{vw_2}^4\overline{w_5}^4\}$ and $\{\overline{uw_4}^4\overline{w_5}^4, \overline{vw_2}^4\overline{w_4}^4\}$;
- two subgroups isomorphic to $(\mathbb{Z}_4)^2$, generated by $\{\overline{u}^2\overline{w_2}^4\overline{w_4}^4, \overline{v}^2\overline{w_2}^4\overline{w_5}^4\}$ and $\{\overline{u}^2\overline{w_4}^4\overline{w_5}^4, \overline{v}^2\overline{w_2}^4\overline{w_4}^4\}$.

The triples representing these 16 subgroups are: (T_1, T_1, T_4) twice, (T_1, T_2, T_4) twice, (T_1, T_2, T_2), twice, (T_0, T_1, T_2) twice, (T_1, T_1, T_2) four times, (T_1, T_1, T_1) twice, and (T_0, T_1, T_1) twice, respectively.

Using these observations (and again some more for the case $m = 16$ for clarity), we find that the only possibilities for a subgroup L of prime-power index in K that is $\langle h, ap \rangle$-invariant but not $\langle h, a \rangle$-invariant are those in the summary Table 5.4.

The subgroups in the last four rows of Table 5.4 all come from the case $k = 2$. The 'layer combinations' for a subgroup L from each of these rows are respectively as follows:

- Row 2: a tower of two or more copies of T_1 on top of a tower of copies of T_4;
- Row 3: a tower of two or more copies of T_1 on top of a single copy of T_2 on top of a tower of copies of T_4;
- Row 4: a single copy of T_1 on top of a tower of two or more copies of T_2 on top of a tower of copies of T_4;
- Row 5: a tower of two copies of T_1 on top of a single copy of T_2 on top of a tower of copies of T_4.

Note that there are two possibilities in each case; the triple (T_1, T_1, T_2) which appeared four times in the summary for $m = 8$ occurs twice in each of rows 3 and 5.

The covering graphs of Q_3 corresponding to the subgroups of K in Table 5.4 admit

Index $\|K:L\|$	Generating set for L	Quotient K/L
$b^3cd = k^{3r+s+t}$, with $s < t$, for $k \equiv 1 \bmod 3$	$\{x^b,\ y^b,\ z^b,\ (v_\lambda)^c,\ (v_{\lambda^2})^d\}$ or $\{x^b,\ y^b,\ z^b,\ (v_{\lambda^2})^c,\ (v_\lambda)^d\}$	$(\mathbb{Z}_b)^3 \oplus \mathbb{Z}_c \oplus \mathbb{Z}_d$
$c^2d^3 = 2^{2s+3t}$, with $s+1 < t$	$\{u^c w_2^{\frac{d}{2}} w_4^{\frac{d}{2}},\ v^c w_2^{\frac{d}{2}} w_5^{\frac{d}{2}},\ w_2^d,\ w_4^d,\ w_5^d\}$ or $\{u^c w_4^{\frac{d}{2}} w_5^{\frac{d}{2}},\ v^c w_2^{\frac{d}{2}} w_4^{\frac{d}{2}},\ w_2^d,\ w_4^d,\ w_5^d\}$	$(\mathbb{Z}_c)^2 \oplus (\mathbb{Z}_d)^3$
$4c^2d^3 = 2^{2s}2^{3t+2}$, with $s+1 < t$	$\{u^c w_4^d,\ v^c w_2^d w_4^d,\ w_5^d,\ w_2^{2d},\ w_4^{2d}\}$ or $\{u^c w_2^d w_4^d,\ v^c w_2^d,\ w_5^d,\ w_2^{2d},\ w_4^{2d}\}$	$(\mathbb{Z}_c)^2 \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{2d})^2$
$2c^3d^2 = 2^{3s+1}2^{2t}$, with $s+2 < t$	$\{u^c w_2^{2c} w_4^{\frac{d}{2}},\ v^c w_2^{\frac{d}{2}} w_4^{-2c+\frac{d}{2}},\ w_5^{2c},\ w_2^d,\ w_4^d\}$ or $\{u^c w_2^{2c+\frac{d}{2}} w_4^{\frac{d}{2}},\ v^c w_2^{\frac{d}{2}} w_4^{-2c},\ w_5^{2c},\ w_2^d,\ w_4^d\}$	$(\mathbb{Z}_c)^2 \oplus \mathbb{Z}_{2c} \oplus (\mathbb{Z}_d)^2$
$4c^3d^2 = 2^{3s+2}2^{2t}$, with $s+3 < t$	$\{u^c w_1^{-2c} w_4^{-2c+\frac{d}{2}},\ v^c w_1^{2c+\frac{d}{2}} w_2^{2c},\ w_5^{4c},\ w_2^d,\ w_4^d\}$ or $\{u^c w_1^{-2c+\frac{d}{2}} w_4^{-2c},\ v^c w_1^{2c} w_2^{2c+\frac{d}{2}},\ w_5^{4c},\ w_2^d,\ w_4^d\}$	$(\mathbb{Z}_c)^2 \oplus \mathbb{Z}_{4c} \oplus (\mathbb{Z}_d)^2$

Table 5.4: Additional possibilities for $\langle h, ap\rangle$-invariant subgroup L of K when $\langle h, ap\rangle/K \cong A_4 \times C_2$

a 1-arc-regular but not a 2-arc-regular group of automorphisms (since the subgroup is $\langle h, ap\rangle$-invariant but not $\langle h, a\rangle$-invariant). Hence in particular, by Proposition 2.4.3 (or [26, Proposition 26], or [22, Proposition 2.3]), none of them can be 3-arc-regular.

Next, suppose the subgroup L is G_1-invariant but not G_2^1-invariant, and the cover is 4-arc-transitive. In that case, by the same arguments as for $K_{3,3}$, the cover must be a regular cover of the Heawood graph, and since $G_1/K \cong A_4 \times C_2$ has a cyclic quotient of order 6, this implies the existence of a group of order $336 \cdot 4 = 1344$ acting 4-arc-transitively on a symmetric cubic graph of order $1344/24 = 56$. There are, however, only three symmetric cubic graphs of that order (namely the graphs F056A, F056B and F056C listed in [22]), and none of them is 4-arc-transitive. Hence this possibility can be eliminated.

Similarly, if L is G_1-invariant but not G_2^1-invariant and the cover is 5-arc-transitive, then by the same arguments as for $K_{3,3}$, the covering graph must be a regular cover of the Biggs-Conway graph, and hence the 1-arc-regular group $\langle h, ap \rangle / L$ of automorphisms must be insoluble, which is not the case.

Hence all of the resulting covers of Q_3 are 1-arc-regular.

Moreover, since conjugation by a (or p) interchanges the subgroups in each row of Table 5.4 in pairs, these covering graphs are isomorphic in pairs.

5.4 Summary

We have the following theorem:

Theorem 5.4.1 *Let $m = k^e$ be any power of a prime k, with $e > 0$. Then the arc-transitive abelian regular covers of the 3-dimensional cube graph Q_3 with abelian covering group of exponent m are as follows:*

(a) *If $k \equiv 2 \bmod 3$ and $k > 2$, there are exactly $2e + 1$ such covers, namely a 2-arc-regular cover with covering group $(\mathbb{Z}_m)^5$, plus one 2-arc-regular cover with covering group $(\mathbb{Z}_\ell)^2 \oplus (\mathbb{Z}_m)^3$ and one 2-arc-regular cover with covering group $(\mathbb{Z}_\ell)^3 \oplus (\mathbb{Z}_m)^2$, for each proper divisor ℓ of m.*

(b) *If $k \equiv 1 \bmod 3$, then there are exactly $\frac{1}{2}e(e+1) + e^2 + 2e + 1$ such covers, namely a 2-arc-regular cover with covering group $(\mathbb{Z}_m)^5$, plus one 2-arc-regular cover with covering group $(\mathbb{Z}_\ell)^2 \oplus (\mathbb{Z}_m)^3$ and one 2-arc-regular cover with covering group $(\mathbb{Z}_\ell)^3 \oplus (\mathbb{Z}_m)^2$, for each proper divisor ℓ of m, plus one 1-arc-regular cover with covering group $\mathbb{Z}_j \oplus \mathbb{Z}_\ell \oplus (\mathbb{Z}_m)^3$ for each pair $\{j, \ell\}$ of distinct divisors of m, and one 1-arc-regular cover with covering group $(\mathbb{Z}_j)^3 \oplus \mathbb{Z}_\ell \oplus \mathbb{Z}_m$ for each ordered pair (j, ℓ) of proper divisors of m.*

(c) *If $k = 3$, then there are exactly $4e+1$ such covers, namely a 2-arc-regular cover with covering group $(\mathbb{Z}_m)^5$, plus one 2-arc-regular cover with covering group $(\mathbb{Z}_\ell)^2 \oplus (\mathbb{Z}_m)^3$, one 2-arc-regular cover with covering group $(\mathbb{Z}_\ell)^3 \oplus (\mathbb{Z}_m)^2$, one 2-arc-regular cover* *with covering group* $\mathbb{Z}_\ell \oplus \mathbb{Z}_{3\ell} \oplus (\mathbb{Z}_m)^3$ *and one* 2*-arc-regular cover with covering group* $(\mathbb{Z}_\ell)^3 \oplus \mathbb{Z}_{m/3} \oplus (\mathbb{Z}_m)$*, for each proper divisor* ℓ *of* m*.*

(d) *If* $k = 2$ *and* $e > 2$*, then there are exactly* $14e - 16$ *such covers, namely a* 2*-arc-regular cover with covering group* $(\mathbb{Z}_m)^5$*, plus one* 2*-arc-regular cover with covering group* $(\mathbb{Z}_{m/2})^2 \oplus (\mathbb{Z}_m)^3$*, plus two* 2*-arc-regular covers and one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_\ell)^2 \oplus (\mathbb{Z}_m)^3$*, for each proper divisor* ℓ *of* $m/2$*, plus one* 2*-arc-regular cover with covering group* $(\mathbb{Z}_\ell)^3 \oplus (\mathbb{Z}_m)^2$ *for each proper divisor* ℓ *of* m*, plus two* 2*-arc-regular covers and one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_{m/4})^2 \oplus \mathbb{Z}_{m/2} \oplus (\mathbb{Z}_m)^2$*, plus four* 2*-arc-regular covers and two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_{m/8})^2 \oplus \mathbb{Z}_{m/2} \oplus (\mathbb{Z}_m)^2$*, plus two* 2*-arc-regular covers and one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_\ell)^2 \oplus \mathbb{Z}_{m/2} \oplus (\mathbb{Z}_m)^2$ *for each proper divisor* ℓ *of* $m/8$*, plus two* 2*-arc-regular covers and one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_\ell)^2 \oplus \mathbb{Z}_{2\ell} \oplus (\mathbb{Z}_m)^2$ *for each proper divisor* ℓ *of* $m/4$*, plus two* 2*-arc-regular covers and one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_\ell)^2 \oplus \mathbb{Z}_{4\ell} \oplus (\mathbb{Z}_m)^2$ *for each proper divisor* ℓ *of* $m/8$*, and one* 2*-arc-regular cover with covering group* $(\mathbb{Z}_\ell)^2 \oplus (\mathbb{Z}_{m/2})^2 \oplus \mathbb{Z}_m$ *for each proper divisor* ℓ *of* m*.*

(e) *If* $k = 2$ *and* $e = 3$*, then there are exactly* 26 *such covers, namely one* 2*-arc-regular cover with covering group* $(\mathbb{Z}_4)^r \oplus (\mathbb{Z}_8)^{5-r}$ *for each* $r \in \{0, 2, 3, 4\}$*, plus two* 2*-arc-regular covers and one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_2)^2 \oplus (\mathbb{Z}_8)^3$*, plus one* 2*-arc-regular cover with covering group* $(\mathbb{Z}_2)^2 \oplus (\mathbb{Z}_4)^2 \oplus \mathbb{Z}_8$*, plus two* 2*-arc-regular covers and one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_2)^2 \oplus \mathbb{Z}_4 \oplus (\mathbb{Z}_8)^2$*, plus one* 2*-arc-regular cover with covering group* $(\mathbb{Z}_2)^3 \oplus (\mathbb{Z}_8)^2$*, plus two* 2*-arc-regular covers and one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_8)^3$*, plus four* 2*-arc-regular covers and two* 1*-arc-regular covers with covering group* $\mathbb{Z}_4 \oplus (\mathbb{Z}_8)^2$*, plus one* 2*-arc-regular cover with covering group* $(\mathbb{Z}_4)^2 \oplus \mathbb{Z}_8$*, plus two* 2*-arc-regular covers and one* 1*-arc-regular cover with covering group* $\mathbb{Z}_2 \oplus (\mathbb{Z}_8)^2$*, and one* 2*-arc-regular cover with covering group* $(\mathbb{Z}_8)^2$*.*

(f) *If* $k = 2$ *and* $e = 2$*, then there are exactly* 12 *such covers, namely one* 2*-arc-*

regular cover with covering group $(\mathbb{Z}_2)^r \oplus (\mathbb{Z}_4)^{5-r}$ *for each* $r \in \{0, 2, 3, 4\}$*, plus two* 2*-arc-regular covers and one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_4)^3$*, plus two* 2*-arc-regular covers and one* 1*-arc-regular cover with covering group* $\mathbb{Z}_2 \oplus (\mathbb{Z}_4)^2$*, plus one* 2*-arc-regular cover with covering group* $(\mathbb{Z}_2)^2 \oplus \mathbb{Z}_4$*, and one* 2*-arc-regular cover with covering group* $(\mathbb{Z}_4)^2$*.*

(g) *If* $k = 2$ *and* $e = 1$*, then there are exactly* 4 *such covers, namely namely one* 2*-arc-regular cover with covering group* $(\mathbb{Z}_2)^r$ *for each* $r \in \{2, 3, 4, 5\}$*.*

Chapter 6

Symmetric abelian regular covers of the Petersen graph

In this Chapter, we deal with the Petersen graph, which turns our to be easier than $K_{3,3}$ and Q_3. Using the method of linear criteria for lifting automorphisms [27], Feng and Wang [29] classified all the symmetric cyclic and elementary abelian regular covers of the Petersen graph in 2006, and as an application, they found all of the symmetric cubic graphs of order $10p$ or $10p^2$ for prime p. By using the method of finding invariant subspaces of matrix automorphisms introduced in [49], Malnič and Potočnik [51] classified all the vertex-transitive elementary abelian regular covers of the Petersen graph in 2006 as well. The same methods can be used to classify all the symmetric homocyclic regular covers, but again for more general symmetric abelian regular covers, these methods are difficult to apply.

In this Chapter, we use our new approach to determine all the symmetric abelian regular covering graphs of the Petersen graph. The numbers of covers and the largest value of s for which each cover is s-arc-transitive are also given.

6.1 Preliminaries

We know that the Petersen graph is the only symmetric cubic graph of order 10. It is 3-arc-regular, and its automorphism group is the symmetric group S_5, of order 120. The only other arc-transitive group of automorphisms is the alternating subgroup A_5, which acts regularly on the 2-arcs, with type 2^1.

Take $G_3 = \langle\, h, a, p, q \mid h^3 = a^2 = p^2 = q^2 = [p, q] = [h, p] = (hq)^2 = apaq = 1\,\rangle$, let $G_2^1 = \langle h, a, pq\rangle$, and let N be the unique normal subgroup of index 120 in G_3 (and index 60 in G_2^1), with quotients $G_3/N \cong S_5$ and $G_2^1/N \cong A_5$. Using Reidemeister-Schreier theory or the `Rewrite` command in Magma, we find that the subgroup N is free of rank 6, on generators

$$w_1 = (ha)^5, \quad w_2 = (ah)^5, \quad w_3 = h^{-1}ahahahahah^{-1}, \quad w_4 = pqahah^{-1}ahah^{-1}ah,$$
$$w_5 = pqhahah^{-1}ahah^{-1}ah^{-1} \quad \text{and} \quad w_6 = pqh^{-1}ahah^{-1}ahah^{-1}a.$$

Easy calculations show that the generators h, a, p and q act by conjugation as follows:

$$\begin{array}{lll}
h^{-1}w_1h = w_2 & a^{-1}w_1a = w_2 & p^{-1}w_1p = w_4^{-1} & q^{-1}w_1q = w_6 \\
h^{-1}w_2h = w_3 & a^{-1}w_2a = w_1 & p^{-1}w_2p = w_6^{-1} & q^{-1}w_2q = w_4 \\
h^{-1}w_3h = w_1 & a^{-1}w_3a = w_4w_3w_6 & p^{-1}w_3p = w_5^{-1} & q^{-1}w_3q = w_5 \\
h^{-1}w_4h = w_5 & a^{-1}w_4a = w_4^{-1} & p^{-1}w_4p = w_1^{-1} & q^{-1}w_4q = w_2 \\
h^{-1}w_5h = w_6 & a^{-1}w_5a = w_1^{-1}w_5^{-1}w_2^{-1} & p^{-1}w_5p = w_3^{-1} & q^{-1}w_5q = w_3 \\
h^{-1}w_6h = w_4 & a^{-1}w_6a = w_6^{-1} & p^{-1}w_6p = w_2^{-1} & q^{-1}w_6q = w_1.
\end{array}$$

Now take the quotient G_3/N', which is an extension of the free abelian group $N/N' \cong \mathbb{Z}^6$ by the group $G_3/N \cong S_5$, and replace the generators h, a, p, q and all w_i by their images in G_3/N'. Also let K denote the subgroup N/N', and let $G = G_2^1/N'$, so that G is an extension of K by A_5.

By the above observations, the generators h, a, p and q induce linear transformations of the free abelian group $K \cong \mathbb{Z}^6$ as follows:

$$h \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad a \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$p \mapsto \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad q \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

These matrices generate a subgroup isomorphic to S_5, and the matrices representing h, a and pq generate a subgroup isomorphic to A_5. Note that the traces of the above four matrices are 0, -2, 0 and 0, respectively.

6.2 Abelian regular covers

In this section, we find all the symmetric abelian regular covers of the Petersen graph that can be obtained by lifting the 2-arc-regular subgroup A_5.

The character table of the group A_5 is given in Table 6.1, with α and β being the zeroes of the polynomial $t^2 - t - 1$ (or in other words, the golden ratio $\frac{1+\sqrt{5}}{2}$ and its conjugate $\frac{1-\sqrt{5}}{2}$).

By inspecting traces of the matrices induced by each of a and h, we see that the character of the 6-dimensional representation of A_5 over $\mathbb{Q}$ associated with the above action of $\langle h, a, pq\rangle$ on K is the rational character $\chi_2 + \chi_3$, which is reducible over fields containing zeroes of the polynomial $t^2 - t - 1$.

Element order	1	2	3	5	5
Class size	1	15	20	12	12
χ_1	1	1	1	1	1
χ_2	3	-1	0	α	β
χ_3	3	-1	0	β	α
χ_4	4	0	1	-1	1
χ_5	5	1	-1	0	0

Table 6.1: The character table of the group A_5

So now let k be any odd prime. Then $\alpha^2 - \alpha - 1 = 0$ for some $\alpha \in \mathbb{Z}_k$ if and only if $(2\alpha - 1)^2 = 4\alpha^2 - 4\alpha + 1 = 5$ for some $\alpha \in \mathbb{Z}_k$, or equivalently, if and only if 5 is a quadratic residue mod k.

Hence if $k \equiv \pm 1$ mod 5, then the group $K/K^{(k)} \cong (\mathbb{Z}_k)^6$ is direct sum of two G-invariant subgroups of rank 3. In fact, these rank 3 subgroups are the images in $K/K^{(k)}$ of the two subgroups generated by $\{x_\alpha, y_\alpha, z_\alpha\}$ and $\{x_\beta, y_\beta, z_\beta\}$ where

$$x_\lambda = w_1 w_4 w_5 w_6^\lambda, \quad y_\lambda = w_2 w_4^\lambda w_5 w_6 \quad \text{and} \quad z_\lambda = w_3 w_4 w_5^\lambda w_6$$

for $\lambda \in \{\alpha, \beta\}$, the set of zeroes of $t^2 - t - 1$ in $\mathbb{Z}_k$. Note that conjugation by h cyclically permutes x_λ, y_λ and z_λ, and conjugation by a inverts each of x_λ and y_λ and takes z_λ to $x_\lambda^{-\lambda} y_\lambda^{-\lambda} z_\lambda$, while conjugation by pq interchanges x_λ with y_λ^{-1} and inverts z_λ, for each $\lambda \in \{\alpha, \beta\}$. Moreover, these are the only non-trivial proper G-invariant subgroups of $K/K^{(k)}$.

If $k \equiv \pm 2$ mod 5 (and k is odd), then no such zeroes of $t^2 - t - 1$ exist in $\mathbb{Z}_k$, and the corresponding 6-dimensional representation of A_5 is irreducible over $\mathbb{Z}_k$, and it follows that $K/K^{(k)}$ has no non-trivial proper G-invariant subgroups. Note that this holds just as well when $k = 3$, since the representations χ_2 and χ_3 are distinct when defined over GF(9).

When $k = 5$, the mod k reductions of the characters χ_2 and χ_3 coincide, and we have just one non-trivial proper G-invariant subgroup of $K/K^{(k)}$, namely the rank 3 subgroup generated by the images of $x_\lambda = w_1w_4w_5w_6^2$, $y_\lambda = w_2w_4^2w_5w_6$ and $z_\lambda = w_3w_4w_5^2w_6$, where $\lambda = 2$ (the unique zero of $t^2 - t - 1$ in $\mathbb{Z}_5$).

For $k = 2$, with the help of Magma, if necessary, it is easy to show that the group $K/K^{(k)}$ has four non-trivial proper G-invariant subgroups, namely one of rank 4 generated by the images of w_1w_3, w_2w_3, w_4w_5 and w_5w_6, plus three of rank 5 containing the latter, with additional generators w_1, w_4 and w_1w_4, respectively.

As before, analogous observations hold also for each other layer K_i/K_{i+1} of K.

Next, suppose $m = k^e$ is any prime-power greater than 1, and suppose $L/K^{(m)}$ is any non-trivial normal subgroup of $G/K^{(m)}$ contained in $K/K^{(m)}$.

If k is odd and $k \equiv \pm 2 \bmod 5$, then every layer L_i/L_{i+1} has rank 0 or 6, and therefore $L \cong (\mathbb{Z}_\ell)^6$ for some ℓ dividing m.

On the other hand, if $k \equiv \pm 1 \bmod 5$, then every layer L_i/L_{i+1} has rank 0, 3 or 6. Moreover, the polynomial $t^2 - t - 1$ always has two zeroes α and β in $\mathbb{Z}_m$, and if $\lambda \in \{\alpha, \beta\}$, then the elements $x_\lambda = w_1w_4w_5w_6^\lambda$, $y_\lambda = w_2w_4^\lambda w_5w_6$ and $z_\lambda = w_3w_4w_5^\lambda w_6$ generate a G-invariant subgroup of rank 3. It follows that $L \cong (\mathbb{Z}_j)^3 \oplus (\mathbb{Z}_\ell)^3$ for some j and ℓ dividing m, with two possibilities for L for each pair (j, ℓ) such that $j > \ell$: one generated by the images of the (m/j)th powers of x_α, y_α and z_α and the (m/ℓ)th powers of x_β, y_β and z_β, and the other with the roles of α and β reversed.

Similarly, when $k = 5$, every layer L_i/L_{i+1} has rank 0, 3 or 6, but in this case the polynomial $t^2 - t - 1$ has a zero only in $\mathbb{Z}_m$ only when $m = k = 5$, and it follows that if some layer L_i/L_{i+1} has rank 3, then the next layer L_{i+1}/L_{i+2} must have rank 6. Hence for $e > 1$, we have $L \cong (\mathbb{Z}_\ell)^6$ or $L \cong (\mathbb{Z}_{5\ell})^3 \oplus (\mathbb{Z}_\ell)^3$ for some ℓ dividing $m/5$. Any subgroup of the latter form is generated by the images of the (m/ℓ)th powers of the elements x_λ, y_λ and z_λ (given above) and the $(m/5\ell)$th powers of all the w_i.

Finally, when $k = 2$, an easy analysis of the situation for the case $m = 2^3 = 8$ shows that $L/K^{(m)}$ can have at most one non-trivial layer of rank less than 6. More specifically, the smallest non-trivial layer (which we might call the 'top' layer) must have rank 4, 5 or

6, and all other non-trivial layers have rank 6. This top layer can be specified in terms of any one of the four non-trivial proper G-invariant subgroups of $K/K^{(2)}$ found above, or have rank 6, and it then determines $L/K^{(m)}$ uniquely.

Putting these observations together, we find that the only possibilities for a normal subgroup L of G contained in K with index $|K:L|$ being a power of a prime k are those included in the summary Table 6.2.

Index $\|K:L\|$	Generating set for L	Quotient K/L
$d^6 = k^{6t}$, for any k	w_i^d for all i	$(\mathbb{Z}_d)^6$
$c^3d^3 = k^{3(s+t)}$, with $s < t$, for $k \equiv \pm 1 \bmod 5$	$x_\alpha^c,\ y_\alpha^c,\ z_\alpha^c,\ x_\beta^d,\ y_\beta^d,\ z_\beta^d$, or $x_\beta^c,\ y_\beta^c,\ z_\beta^c,\ x_\alpha^d,\ y_\alpha^d,\ z_\alpha^d$	$(\mathbb{Z}_c)^3 \oplus (\mathbb{Z}_d)^3$
$125d^6 = 5^{6s+3}$	$x_\lambda^d,\ y_\lambda^d,\ z_\lambda^d$, and all w_i^{5d}	$(\mathbb{Z}_d)^3 \oplus (\mathbb{Z}_{5d})^3$
$2d^6 = 2^{6s+1}$	$(w_1w_3)^d,\ (w_2w_3)^d,\ (w_4w_5)^d,\ (w_5w_6)^d$, all w_i^{2d}, and one of $w_1^d,\ w_4^d$ or $(w_1w_4)^d$	$(\mathbb{Z}_d)^5 \oplus \mathbb{Z}_{2d}$
$4d^6 = 2^{6s+2}$	$(w_1w_3)^d,\ (w_2w_3)^d,\ (w_4w_5)^d,\ (w_5w_6)^d$, and all w_i^{2d}	$(\mathbb{Z}_d)^4 \oplus (\mathbb{Z}_{2d})^2$

Table 6.2: Possibilities for $\langle h, a, pq\rangle$-invariant subgroup L of K when $\langle h, a, pq\rangle/K \cong A_5$

6.3 Automorphism groups of the regular covers

In this section, we consider which of the $\langle h, a, pq\rangle$-invariant subgroups of $K/K^{(m)}$ are normalized by the additional generator p of the larger group G_3. Here we will work inside the group G_3/N', and adopt the same notation for images of elements in this group.

Note first that if λ is a zero of $t^2 - t - 1$ in $\mathbb{Z}_m$, then so is $-1 - \lambda$, and conjugation by p (which lies in G_3/N' but not G_2^1/N') takes

$$
\begin{aligned}
x_\lambda = w_1w_4w_5w_6^\lambda &\quad\mapsto\quad w_4^{-1}w_1^{-1}w_3^{-1}w_2^{-\lambda} = (x_{-1-\lambda})^{-1}(y_{-1-\lambda})^{-\lambda}(z_{-1-\lambda})^{-1},\\
y_\lambda = w_2w_4^\lambda w_5w_6 &\quad\mapsto\quad w_6^{-1}w_1^{-\lambda}w_3^{-1}w_2^{-1} = (x_{-1-\lambda})^{-\lambda}(y_{-1-\lambda})^{-1}(z_{-1-\lambda})^{-1},\\
z_\lambda = w_3w_4w_5^\lambda w_6 &\quad\mapsto\quad w_5^{-1}w_1^{-1}w_3^{-\lambda}w_2^{-1} = (x_{-1-\lambda})^{-1}(y_{-1-\lambda})^{-1}(z_{-1-\lambda})^{-\lambda}.
\end{aligned}
$$

Hence in particular, if $t^2 - t - 1$ has two distinct zeroes α and β in $\mathbb{Z}_m$, then conjugation by p interchanges the rank 3 subgroups generated by $\{x_\alpha^\ell, y_\alpha^\ell, z_\alpha^\ell\}$ and $\{x_\beta^\ell, y_\beta^\ell, z_\beta^\ell\}$ for each divisor ℓ of m, while if there is just one zero (namely in the case $m = k = 5$), then the rank 3 subgroup generated by $\{x_\lambda, y_\lambda, z_\lambda\}$ is preserved.

On the other hand, in the case $k = 2$, we note that conjugation by p interchanges w_1w_3 with w_4w_5, and w_2w_3 with w_5w_6, and w_1 with w_4, and so fixes w_1w_4. Hence p preserves the rank 4 subgroup generated by $S = \{w_1w_3, w_2w_3, w_4w_5, w_5w_6\}$ and the rank 5 subgroup generated by $S \cup \{w_1w_4\}$, but interchanges the rank 5 subgroups generated by $S \cup \{w_1\}$ and $S \cup \{w_4\}$.

It follows that all of the covers of the Petersen graph that arise in the cases described by rows 1, 3 and 5 of Table 6.2, and one of the three cases in row 4, admit a 3-arc-regular group of automorphisms, while the others do not. Also none of these covers can be 4-arc- or 5-arc-transitive, by Corollary 2.4.2 (or [26, Theorem 3]), and each of them is unique up to isomorphism.

6.4 Summary

Thus we have the following theorem:

Theorem 6.4.1 *Let $m = k^e$ be any power of a prime k, with $e > 0$. Then the arc-transitive regular covers of the Petersen graph with abelian covering group of exponent m are as follows:*

(a) *If $k \equiv \pm 2$ mod 5 and $k > 2$, there is exactly one such cover, namely a 3-arc-regular cover with covering group $(\mathbb{Z}_m)^6$.*

(b) *If* $k \equiv \pm 1$ *mod* 5*, then there are exactly* $e+1$ *such covers, namely a* 3*-arc-regular cover with covering group* $(\mathbb{Z}_m)^6$ *plus one* 2*-arc-regular cover with covering group* $(\mathbb{Z}_\ell)^3 \oplus (\mathbb{Z}_m)^3$ *for each proper divisor* ℓ *of* m*.*

(c) *If* $k = 5$*, then there are exactly two such covers, namely one* 3*-arc-regular cover with covering group* $(\mathbb{Z}_m)^6$ *and one* 3*-arc-regular cover with covering group* $(\mathbb{Z}_{m/5})^3 \oplus (\mathbb{Z}_m)^3$*.*

(d) *If* $k = 2$*, then there are exactly* 4 *such covers, namely three* 3*-arc-regular covers with covering groups* $(\mathbb{Z}_m)^6$*,* $(\mathbb{Z}_{m/2})^4 \oplus (\mathbb{Z}_m)^2$*, and* $(\mathbb{Z}_{m/2})^5 \oplus \mathbb{Z}_m$*, plus one* 2*-arc-regular cover with covering group* $(\mathbb{Z}_{m/2})^5 \oplus \mathbb{Z}_m$*.*

Remark 6.4.2 *In case (c) of the above theorem, taking* $m = 5$ *gives a* 3*-arc-regular cubic graph of order* 1250*, which turns out to have diameter* 10 *and girth* 16*. In fact, at the time of writing, this is the largest known connected* 3*-valent graph of diameter* 10*. It was found almost by accident in* 2006 *by Conder, as a result of a computation to find all symmetric cubic graphs of order up to* 2048*, which he has since extended to find all symmetric cubic graphs of order up to* 10000*. The existence of this graph as a cover of the Petersen graph was one of the motivation for this thesis project.*

Chapter 7

Symmetric abelian regular covers of the Heawood graph

In 2004, Malnič, Marušič and Potočnik [49] classified all the semi-symmetric elementary abelian regular covers of the Heawood graph, using the method of finding invariant subspaces. The same method may be used to classify all the symmetric homocyclic regular covers, but is difficult to apply for other kinds of abelian regular covers.

In this Chapter, we determine all the symmetric abelian regular covering graphs of the Heawood graph. The numbers of covers and the largest value of s for which each cover is s-arc-transitive are also given.

One remarkable finding is that although every arc-transitive group of automorphisms of the Heawood graph $\mathcal{H}$ is either 1-arc-regular or 4-arc-regular, there exist two families of abelian regular covers of $\mathcal{H}$ that are 2-arc-regular. Another interesting outcome is the appearance of examples of a regular covering graph of $\mathcal{H}$ with two non-isomorphic possibilities for the covering group.

7.1 Preliminaries

We know that the Heawood graph (the incidence graph of a projective plane of order 7) is the only symmetric cubic graph of order 14. This graph is 4-arc-regular, of type 4^1. Its automorphism group is $\mathrm{PGL}(2,7)$, of order 336. Every other arc-transitive group of automorphisms of the Heawood graph lies in a conjugacy class of eight arc-regular subgroups of $\mathrm{PGL}(2,7)$, each of which is isomorphic to a semi-direct product $C_7 \rtimes_3 C_6$ (where the 3 indicates that a generator of C_6 conjugates each element of C_7 to its 3rd power), of order 42.

Take the group G_4^1, with presentation $\langle\, h, a, p, q, r \mid h^3 = a^2 = p^2 = q^2 = r^2 = (pq)^2 = (pr)^2 = p(qr)^2 = h^{-1}phq = h^{-1}qhpq = (hr)^2 = (ap)^2 = aqar = 1 \,\rangle$, and observe that the three elements h, a and p suffice as generators (because $q = h^{-1}ph$ and $r = aqa$). This group G_4^1 has two normal subgroups of index 336, both with quotient $\mathrm{PGL}(2,7)$, but these are interchanged by the outer automorphism (induced by conjugation by an element of the larger group G_5) that takes the three generators h, a and p to h, ap and p respectively, and so without loss of generality we can take either one of them.

We will take the one that is contained in the subgroup $G_1 = \langle h, a\rangle$; this is a normal subgroup N of index 42 in G_1 with $G_1/N \cong C_7 \rtimes_3 C_6$.

Using Reidemeister-Schreier theory or the `Rewrite` command in Magma, we find that the subgroup N is free of rank 8, on generators

$$w_1 = (ha)^6, \qquad w_2 = hah^{-1}ah^{-1}ahahah^{-1}a,$$
$$w_3 = (h^{-1}a)^6, \qquad w_4 = h^{-1}ahah^{-1}ah^{-1}ahaha,$$
$$w_5 = hah^{-1}ahah^{-1}ah^{-1}aha, \qquad w_6 = hahah^{-1}ahah^{-1}ah^{-1}a,$$
$$w_7 = h^{-1}ahahah^{-1}ahah^{-1}a, \qquad w_8 = h^{-1}ah^{-1}ahahah^{-1}aha.$$

Easy calculations show that the generators h, a and p act by conjugation particularly nicely, as below:

$$\begin{array}{rclrclrcl}
h^{-1}w_1h &=& w_3^{-1} & a^{-1}w_1a &=& w_3^{-1} & p^{-1}w_1p &=& w_5 \\
h^{-1}w_2h &=& w_4^{-1} & a^{-1}w_2a &=& w_2^{-1} & p^{-1}w_2p &=& w_6 \\
h^{-1}w_3h &=& w_2w_7^{-1} & a^{-1}w_3a &=& w_1^{-1} & p^{-1}w_3p &=& w_7 \\
h^{-1}w_4h &=& w_3^{-1}w_6 & a^{-1}w_4a &=& w_8^{-1} & p^{-1}w_4p &=& w_8 \\
h^{-1}w_5h &=& w_8^{-1} & a^{-1}w_5a &=& w_7^{-1} & p^{-1}w_5p &=& w_1 \\
h^{-1}w_6h &=& w_7^{-1} & a^{-1}w_6a &=& w_6^{-1} & p^{-1}w_6p &=& w_2 \\
h^{-1}w_7h &=& w_1w_4^{-1} & a^{-1}w_7a &=& w_5^{-1} & p^{-1}w_7p &=& w_3 \\
h^{-1}w_8h &=& w_5w_8^{-1} & a^{-1}w_8a &=& w_4^{-1} & p^{-1}w_8p &=& w_4.
\end{array}$$

Now take the quotient G_4^1/N', which is an extension of the free abelian group $N/N' \cong \mathbb{Z}^8$ by the group $G_4^1/N \cong \mathrm{PGL}(2,7)$, and replace the generators h, a, p and all w_i by their images in this group. Also let K denote the subgroup N/N', and let G be G_1/N'. Then, in particular, G is an extension of $\mathbb{Z}^8$ by $C_7 \rtimes_3 C_6$.

By the above observations, we see that the generators h, a and p induce linear transformations of the free abelian group $K \cong \mathbb{Z}^8$ as follows:

$$h \mapsto \begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1
\end{pmatrix},$$

$$a \mapsto \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$p \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

These matrices generate a group isomorphic to $\mathrm{PGL}(2,7)$, with the first two generating a subgroup isomorphic to $C_7 \rtimes_3 C_6$.

Next, the character table of the group $C_7 \rtimes_3 C_6$ is as in Table 7.1.

By inspecting traces of the matrices of orders 2, 3, 6 and 7 induced by each of a, $h^{\pm 1}$, $(ah)^{\pm 1}$ and $[a,h]$, we see that the character of the 8-dimensional representation of $C_7 \rtimes_3 C_6$ over $\mathbb{Q}$ associated with the above action of $G = \langle h, a \rangle$ on K is $\chi_5 + \chi_6 + \chi_7$, which is expressible as the sum of $\chi_5 + \chi_6$ and χ_7, the characters of two irreducible representations over $\mathbb{Q}$ of dimensions 2 and 6.

In the next two sections, for every positive integer m we let $K^{(m)}$ denote the subgroup of K generated by the mth powers of all its elements, and if m is a prime-power, say $m = k^e$, then we will consider G_1-invariant subgroups of each layer K_{j-1}/K_j of $K/K^{(m)}$,

Element order	1	2	3	3	6	6	7
Class size	1	7	7	7	7	7	6
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	-1	1
χ_3	1	1	λ	λ^2	λ^2	λ	1
χ_4	1	1	λ^2	λ	λ	λ^2	1
χ_5	1	-1	λ	λ^2	$-\lambda^2$	$-\lambda$	1
χ_6	1	-1	λ^2	λ	$-\lambda$	$-\lambda^2$	1
χ_7	6	0	0	0	0	0	-1

Table 7.1: The character table of subgroup $C_7 \rtimes_3 C_6$ where λ is a primitive cube root of 1

where $K_j = K^{(k^j)}$ for every non-negative integer j, in order to find G_1-invariant subgroups of $K/K^{(m)}$.

7.2 Characteristic other than 7

When we reduce by any prime k, the quotient $K/K^{(k)} \cong (\mathbb{Z}_k)^8$ is the direct sum of two G_1-invariant subgroups of ranks 2 and 6, and the latter is irreducible when $k \neq 7$. In fact, these two subgroups are the images of the normal subgroups U and V of ranks 2 and 6 in G generated by

$$u_1 = w_1 w_3 w_5^{-1} w_6^{-1} w_7^{-1} \quad \text{and} \quad u_2 = w_2 w_4 w_5^{-1} w_6^{-1} w_7^{-1} w_8$$

and

$$v_1 = w_1,\ v_2 = w_2 w_7^{-1},\ v_3 = w_3,\ v_4 = w_4 w_8^{-1},\ v_5 = w_5 w_7^{-1} \ \text{ and } \ v_6 = w_6 w_7^{-1} w_8;$$

with conjugation of the respective generators by h and a given as follows:

$$u_1^h = u_1^{-1} u_2, \quad u_2^h = u_1^{-1}, \quad u_1^a = u_1^{-1} \quad \text{and} \quad u_2^a = u_2^{-1},$$

$$v_1^h = v_3^{-1}, \quad v_2^h = v_1^{-1}, \quad v_3^h = v_2, \quad v_4^h = v_3^{-1}v_5^{-1}v_6, \quad v_5^h = v_1^{-1}v_4, \quad v_6^h = v_1^{-1}v_4v_5,$$

$$v_1^a = v_3^{-1}, \quad v_2^a = v_2^{-1}v_5, \quad v_3^a = v_1^{-1}, \quad v_4^a = v_4, \quad v_5^a = v_5, \quad v_6^a = v_4^{-1}v_5v_6^{-1}.$$

Hence for every prime $k \neq 7$ and every pair (c, d) of integer powers of k, there exists a G_1-invariant subgroup L of K with index $|K : L| = c^2d^6$ and with quotient $K/L \cong (\mathbb{Z}_c)^2 \oplus (\mathbb{Z}_d)^6$, generated by the images of the elements u_i^c for $1 \leq i \leq 2$ and v_j^d for $1 \leq j \leq 6$.

When $k \equiv 2 \bmod 3$ and $k > 2$, the corresponding subgroups of $K/K^{(k)}$ are both irreducible as G_1-invariant subgroups, since the mod k reductions of the characters $\chi_5 + \chi_6$ and χ_7 are irreducible over $\mathbb{Z}_k$. The same holds also when $k = 2$, since there is no G_1-invariant cyclic subgroup of the rank 2 subgroup in that case. Hence for every prime $k \equiv 2 \bmod 3$, the only G_1-invariant subgroups of K with index a power of k are the subgroups with quotients $K/L \cong (\mathbb{Z}_c)^2 \oplus (\mathbb{Z}_d)^6$ described above.

When $k \equiv 1 \bmod 3$, however, the rank 2 subgroup of $K/K^{(k)}$ splits into the direct sum of two G_1-invariant subgroups of rank 1, generated by the images of

$$z_t = w_1 w_2^t w_3 w_4^t w_5^{t^2} w_6^{t^2} w_7^{t^2} w_8^t$$

for $t \in \{\lambda, \lambda^2\}$, where λ is a primitive cube root of 1 in $\mathbb{Z}_k$.

Here

$$z_t^a = w_1^{-1} w_2^{-t} w_3^{-1} w_4^{-t} w_5^{-t^2} w_6^{-t^2} w_7^{-t^2} w_8^{-t} = z_t^{-1},$$

while

$$z_t^h = w_1^{-t^2} w_2^{-1} w_3^{1+t} w_4^{t+t^2} w_5^{-t} w_6^{-t} w_7^{1+t^2} w_8^{t+t^2},$$

the image of which in $K/K^{(k)}$ is $z_t^{-t^2}$, since $t^2 + t + 1 \equiv 0 \bmod k$ in each case. The same holds when k is replaced by a higher power of K, say $m = k^e$: if λ is a primitive cube root of 1 in $\mathbb{Z}_m$, and $z_t = w_1 w_2^t w_3 w_4^t w_5^{t^2} w_6^{t^2} w_7^{t^2} w_8^t$ for $t \in \{\lambda, \lambda^2\}$, then the image of each of z_λ and z_{λ^2} generates a G_1-invariant subgroup of rank 1 in $K/K^{(m)}$, and their direct sum is a G_1-invariant subgroup of rank 2. Moreover, the latter is complementary to the image of V (of rank 6) when $k \neq 7$.

It follows that for every prime $k \equiv 1 \bmod 3$, and for every triple (b, c, d) of powers of k with $b \neq c$, there is also a G_1-invariant subgroup L of K with index $|K:L| = bcd^6$ and quotient $K/L \cong \mathbb{Z}_b \oplus \mathbb{Z}_c \oplus (\mathbb{Z}_d)^6$, generated by the images of the elements $z_\lambda^{\,b}$, $z_{\lambda^2}^{\,c}$ and $v_j^{\,d}$ for $1 \leq j \leq 6$. Moreover, when $k \neq 7$, each layer of any G_1-invariant subgroup L of K with index a power of k must have rank 1, 2, 6, 7 or 8, and it is easy to see that there are no other possibilities for L (when $k \equiv 1 \bmod 3$ and $k \neq 7$).

When $k = 3$, the quotient $K/K^{(k)} \cong (\mathbb{Z}_3)^8$ has six G_1-invariant subgroups. These include the subgroups of ranks 0, 2, 6 and 8 that occur for every other prime k, plus the cyclic subgroup generated by the image of $z_1 = w_1w_2w_3w_4w_5w_6w_7w_8$ (which coincides with the image of $u_1u_2 = w_1w_2w_3w_4w_5^{-2}w_6^{-2}w_7^{-2}w_8$), and the subgroup of rank 7 generated by the images of z_1 and the elements v_j for $1 \leq j \leq 6$.

In $K/K^{(9)}$, however, there is no G_1-invariant cyclic subgroup of order 9; the only non-trivial G_1-invariant subgroups of $K/K^{(9)}$ of rank 1 or 2 are unique subgroups isomorphic to $\mathbb{Z}_3$, $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, $\mathbb{Z}_9 \oplus \mathbb{Z}_3$ and $\mathbb{Z}_9 \oplus \mathbb{Z}_9$, generated by the images of $\{(u_1u_2)^3\}$, $\{(u_1u_2)^3, u_2^3\}$ (or $\{u_1^3, u_2^3\}$), $\{u_1u_2, u_2^3\}$ and $\{u_1u_2, u_2\}$ (or $\{u_1, u_2\}$), respectively. It follows that every G_1-invariant subgroup L of K with index a power of 3 generated by the images of the elements $(u_1u_2)^{\,b}$, u_2^c and $v_j^{\,d}$ for $1 \leq j \leq 6$, where b, c and d are powers of 3 with $c = b$ or $3b$, in which case $|K:L| = bcd^6$ and $K/L \cong \mathbb{Z}_b \oplus \mathbb{Z}_c \oplus (\mathbb{Z}_d)^6$.

This completes the analysis of G_1-invariant subgroups of $K/K^{(m)}$ when m is a power of some prime $k \neq 7$. These will be summarised in Table 7.3 in Section 7.4.

7.3 Characteristic 7

The case $k = 7$ is not quite so straightforward. In this case, each layer can have rank 0, 1, 2, 3, 4, 5, 6, 7 or 8, depending on the layers above it. Here, as we will see, in $K/K^{(7)}$ the images of the subgroups U and V of ranks 2 and 6 considered in the previous section intersect non-trivially in a subgroup of rank 1.

To describe the possibilities for a G_1-invariant subgroup of each layer, again it helps to let λ be a primitive cube root of 1 in $\mathbb{Z}_m$ (when $m = 7^e$), and define $z_t = w_1 w_2^t w_3 w_4^t w_5^{t^2} w_6^{t^2} w_7^{t^2} w_8^t$ for $t \in \{\lambda, \lambda^2\}$. This time, however, we choose λ so that $\lambda \equiv 2 \bmod 7$ (while $\lambda^2 \equiv 4 \bmod 7$). Also, take $v_1 = w_1$, $v_2 = w_2 w_7^{-1}$, $v_3 = w_3$, $v_4 = w_4 w_8^{-1}$, $v_5 = w_5 w_7^{-1}$ and $v_6 = w_6 w_7^{-1} w_8$ (as before), and define $y_t = w_7 w_8^t$ for each $t \in \{\lambda, \lambda^2\}$. Then an alternative basis for the group $K/K^{(m)}$ is formed by the images of the following eight elements:

$$x_1 = z_\lambda = w_1 w_2^\lambda w_3 w_4^\lambda w_5^{\lambda^2} w_6^{\lambda^2} w_7^{\lambda^2} w_8^\lambda, \qquad x_2 = z_{\lambda^2} = w_1 w_2^{\lambda^2} w_3 w_4^{\lambda^2} w_5^\lambda w_6^\lambda w_7^\lambda w_8^{\lambda^2},$$

$$x_3 = v_2 v_3 v_4^2 v_5 v_6^2 = w_2 w_3 w_4^2 w_5 w_6^2 w_7^{-4}, \qquad x_4 = v_3 v_6^{-2} = w_3 w_6^{-2} w_7^2 w_8^{-2},$$

$$x_5 = v_4 v_5^3 = w_4 w_5^3 w_7^{-3} w_8^{-1}, \qquad x_6 = v_6 y_\lambda^2 = w_6 w_7 w_8^{1+2\lambda},$$

$$x_7 = v_6 = w_6 w_7^{-1} w_8, \qquad x_8 = y_{\lambda^2} = w_7 w_8^{\lambda^2}.$$

With help from Magma [3], we find that the group $K/K^{(7)}$ has exactly 22 G_1-invariant subgroups. We will denote the trivial subgroup by T_0 and the group $K/K^{(7)}$ itself by T_{21}, and then the 20 non-trivial proper G_1-invariant subgroups can be labelled T_1 to T_{20} and summarised in Table 7.2.

	Rank	Generated by images of		Rank	Generated by images of
T_1	1	x_1	T_2	1	x_2
T_3	2	$x_1,\ x_2$	T_4	2	$x_1,\ x_3$
T_5	3	$x_1,\ x_2,\ x_3$	T_6	3	$x_1,\ x_3,\ x_4$
T_7	4	$x_1,\ x_2,\ x_3,\ x_4$	T_8	4	$x_1,\ x_3,\ x_4,\ x_5$
T_9	5	$x_1,\ x_2,\ x_3,\ x_4,\ x_5$	T_{10}	5	$x_1,\ x_3,\ x_4,\ x_5,\ x_6$
T_{11}	5	$x_1,\ x_3,\ x_4,\ x_5,\ x_2 x_6$	T_{12}	5	$x_1,\ x_3,\ x_4,\ x_5,\ x_2^2 x_6$
T_{13}	5	$x_1,\ x_3,\ x_4,\ x_5,\ x_2^3 x_6$	T_{14}	5	$x_1,\ x_3,\ x_4,\ x_5,\ x_2^4 x_6$
T_{15}	5	$x_1,\ x_3,\ x_4,\ x_5,\ x_2^5 x_6$	T_{16}	5	$x_1,\ x_3,\ x_4,\ x_5,\ x_2^6 x_6$
T_{17}	6	$x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6$	T_{18}	6	$x_1,\ x_3,\ x_4,\ x_5,\ x_2^4 x_6,\ x_7$
T_{19}	7	$x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ x_7$	T_{20}	7	$x_1,\ x_3,\ x_4,\ x_5,\ x_2^4 x_6,\ x_7,\ x_8$

Table 7.2: The non-trivial proper G_1-invariant subgroups of $K/K^{(7)}$

When the exponent m of K/L is a higher power of 7, say $m = 7^e$ with $e > 1$, finding the G_1-invariant subgroups of $K/K^{(m)}$ is much more challenging than in earlier cases (namely in the previous Section and Chapters 3 to 6).

For all $j > 0$, the G_1-invariant subgroups of the jth layer $K_{j-1}/K_j = K^{(7^{j-1})}/K^{(7^j)}$ of K are isomorphic to the G_1-invariant subgroups of $K/K^{(7)}$, and are generated by the images of the (7^{j-1})th powers of the corresponding sets of x_i in each case. In some sense, what we have to do is see how the possibilities at each layer can fit together.

For each $t \in \{\lambda, \lambda^2\}$ the image of z_t generates a G_1-invariant subgroup of rank 1 in $K/K^{(m)}$, and these two subgroups may be viewed as a tower of copies of T_1 and a tower of copies of T_2 (from Table 7.2). The images of z_λ and z_{λ^2} together generate a G_1-invariant subgroup of rank 2, coinciding with the image of the subgroup U defined earlier, since in $K/K^{(m)}$ the image of z_t is the same as the image of $u_1u_2^t$ for each t (because $-1-t \equiv t^2$ mod m). (Also conversely, $z_\lambda^\lambda z_{\lambda^2}^{-1} = u_1^{\lambda-1}$ and $z_\lambda^{-1}z_{\lambda^2} = u_2^{\lambda^2-\lambda}$.) This subgroup is a tower of copies of the subgroup T_3 from Table 7.2.

Also, and again as before, the subgroup V generated by $v_1 = w_1$, $v_2 = w_2w_7^{-1}$, $v_3 = w_3$, $v_4 = w_4w_8^{-1}$, $v_5 = w_5w_7^{-1}$ and $v_6 = w_6w_7^{-1}w_8$ is G_1-invariant, and so this gives a G_1-invariant homocyclic subgroup of rank 6 in $K/K^{(m)}$, which can be viewed as a tower of copies of T_{18}. (It is an easy to show that in $K/K^{(7)}$, the images of T_{18} is isomorphic to the images of V.)

Note that the intersection of the images of the rank 6 subgroup V and the rank 2 subgroup U (or equivalently, the intersection of the T_3 and T_{18} towers) is neither trivial nor one of the rank 1 towers generated by z_λ and z_{λ^2}, except in the case $m = 7$: in fact, it is the cyclic subgroup of order 7 generated by the image of $z_\lambda^{m/7}$ $(= x_1^{m/7})$.

Next, for each $t \in \{\lambda, \lambda^2\}$, the image of the subgroup generated by $V \cup \{y_t\}$ is a G_1-invariant subgroup of rank 7 in $K/K^{(m)}$, since

$$y_t^h = w_1w_4^{-1}w_5^tw_8^{-t} = w_1(w_4w_8^{-1})^{-1}(w_5w_7^{-1})^t(w_7w_8^t)^tw_8^{-(1+t+t^2)} = v_1v_4^{-1}v_5^ty_t^tw_8^{-(1+t+t^2)}$$

(with $1+t+t^2 \equiv 0$ mod m), while

$$y_t^a = w_5^{-1}w_4^{-t} = (w_4w_8^{-1})^{-t}(w_5w_7^{-1})^{-1}(w_7w_8^t)^{-1} = v_4^{-t}v_5^{-1}y_t^{-1}.$$

These two homocyclic subgroups of rank 7 may be viewed as a tower of copies of T_{19} (when $t = \lambda$) and a tower of copies of T_{20} (when $t = \lambda^2$), since in $K/K^{(7)}$ the images of x_2, x_6 and x_8 coincide with those of $v_1v_2^4v_3v_4^4v_5^2v_6^2y_\lambda^3$, $v_6y_\lambda^2$ and y_{λ^2} respectively. Just for the sake of interest, the other rank 6 subgroup of $K/K^{(7)}$, namely T_{17}, is generated by the images of $v_1v_6^2$, $v_2v_6^5$, $v_3v_6^5$, $v_4v_6^5$, $v_5v_6^3$ and $v_6y_\lambda^2$.

It turns out that the above towers of copies of T_1, T_2, T_3, T_{18}, T_{19} or T_{20} account for all of the homocyclic G_1-invariant subgroups of exponent m in $K/K^{(m)}$, but that will not become clear until we have found all the G_1-invariant subgroups of $K/K^{(m)}$, below.

To see exactly what happens, it is helpful to consider the case $m = 7^2 = 49$. Subgroups of $K/K^{(49)}$ that have rank 8 must all have second layer equal to K_1/K_2 (and a subgroup of $K/K^{(7)}$ as first layer), and are not so interesting for us. Similarly, subgroups of exponent 7 have trivial first layer, and we will ignore those for now.

There are exactly 101 non-trivial subgroups of $K/K^{(49)}$ of exponent 49 and rank at most 7 that are normal in $G/K^{(49)}$, and these can be summarised as follows, with $V^{(j)}$ denoting the set $\{v_1{}^j, v_2{}^j, v_3{}^j, v_4{}^j, v_5{}^j, v_6{}^j\}$ of jth powers of the generators of V:

Rank 1:

- two subgroups isomorphic to $\mathbb{Z}_{49}$, generated by the images of x_1 and x_2;

Rank 2:

- three subgroups isomorphic to $\mathbb{Z}_{49} \oplus \mathbb{Z}_7$, generated by the images of $\{x_1, x_2{}^7\}, \{x_1, x_3{}^7\}$ and $\{x_2, x_1{}^7\}$;
- one subgroup isomorphic to $(\mathbb{Z}_{49})^2$, generated by the image of $\{x_1, x_2\}$;

Rank 3:

- three subgroups isomorphic to $\mathbb{Z}_{49} \oplus (\mathbb{Z}_7)^2$, generated by the images of $\{x_1, x_2{}^7, x_3{}^7\}$, $\{x_1, x_3{}^7, x_4{}^7\}$ and $\{x_2, x_1{}^7, x_3{}^7\}$;
- one subgroup isomorphic to $(\mathbb{Z}_{49})^2 \oplus \mathbb{Z}_7$, generated by the image of $\{x_1, x_2, x_3{}^7\}$;

Rank 4:

- three subgroups isomorphic to $\mathbb{Z}_{49} \oplus (\mathbb{Z}_7)^3$, generated by the images of $\{x_1, x_2{}^7, x_3{}^7, x_4{}^7\}$, $\{x_1, x_3{}^7, x_4{}^7, x_5{}^7\}$ and $\{x_2, x_1{}^7, x_3{}^7, x_4{}^7\}$;

- one subgroup isomorphic to $(\mathbb{Z}_{49})^2 \oplus (\mathbb{Z}_7)^2$, generated by the image of $\{x_1, x_2, {x_3}^7, {x_4}^7\}$;

Rank 5:

- 15 subgroups isomorphic to $\mathbb{Z}_{49} \oplus (\mathbb{Z}_7)^4$, generated by the images of $\{x_1, {x_2}^7, {x_3}^7, {x_4}^7, {x_5}^7\}$, $\{x_1, {x_3}^7, {x_4}^7, {x_5}^7, ({x_2}^i x_6)^7\}$ for $0 \leq i \leq 6$, and $\{x_2\, {x_6}^{7i}, {x_1}^7, {x_3}^7, {x_4}^7, {x_5}^7\}$ for $0 \leq i \leq 6$;
- 7 subgroups isomorphic to $(\mathbb{Z}_{49})^2 \oplus (\mathbb{Z}_7)^3$, generated by the images of $\{x_1, x_2\, {x_6}^{7i}, {x_3}^7, {x_4}^7, {x_5}^7\}$ for $0 \leq i \leq 6$;

Rank 6:

- 9 subgroups isomorphic to $\mathbb{Z}_{49} \oplus (\mathbb{Z}_7)^5$, generated by the images of $\{x_1, {x_2}^7, {x_3}^7, {x_4}^7, {x_5}^7, {x_6}^7\}$, $\{x_2, {x_1}^7, {x_3}^7, {x_4}^7, {x_5}^7, {x_6}^7\}$ and $\{x_1 {x_8}^{7i}\} \cup V^{(7)}$ for $0 \leq i \leq 6$;
- two subgroups isomorphic to $(\mathbb{Z}_{49})^2 \oplus (\mathbb{Z}_7)^4$, generated by the images of $\{x_1, x_2, {x_3}^7, {x_4}^7, {x_5}^7, {x_6}^7\}$ and $\{x_1 {x_8}^{14}, x_3\} \cup V^{(7)}$;
- one subgroup isomorphic to $(\mathbb{Z}_{49})^3 \oplus (\mathbb{Z}_7)^3$, generated by the image of $\{x_1 {x_8}^{14}, x_3, x_4\} \cup V^{(7)}$;
- one subgroup isomorphic to $(\mathbb{Z}_{49})^4 \oplus (\mathbb{Z}_7)^2$, generated by the image of $\{x_1 {x_8}^{14}, x_3, x_4, x_5\} \cup V^{(7)}$;
- 7 subgroups isomorphic to $(\mathbb{Z}_{49})^5 \oplus \mathbb{Z}_7$, generated by the images of $\{x_1 {x_8}^{14}, x_3, x_4, x_5, ({x_2}^4 x_6) {x_6}^{7i}\} \cup V^{(7)}$ for $0 \leq i \leq 6$;
- one subgroup isomorphic to $(\mathbb{Z}_{49})^6$, generated by the image of $V^{(1)}$;

Rank 7:

- 9 subgroups isomorphic to $\mathbb{Z}_{49} \oplus (\mathbb{Z}_7)^6$, generated by the images of $\{x_1, {x_8}^7\} \cup V^{(7)}$, $\{x_2, (x_6 x_7^{-1})^7\} \cup V^{(7)}$ and $\{x_1 {x_8}^{7i}, (x_6 x_7^{-1})^7\} \cup V^{(7)}$ for $0 \leq i \leq 6$;
- 9 subgroups isomorphic to $(\mathbb{Z}_{49})^2 \oplus (\mathbb{Z}_7)^5$, generated by the images of $\{x_1 {x_8}^{14}, x_3, (x_6 x_7^{-1})^7\} \cup V^{(7)}$, $\{x_1, x_3, {x_8}^7\} \cup V^{(7)}$, and $\{x_1 {x_8}^{7i}, x_2, (x_6 x_7^{-1})^7\} \cup V^{(7)}$ for $0 \leq i \leq 6$;
- three subgroups isomorphic to $(\mathbb{Z}_{49})^3 \oplus (\mathbb{Z}_7)^4$, generated by the images of $\{x_1 {x_8}^{14}, x_2, x_3, (x_6 x_7^{-1})^7\} \cup V^{(7)}$, $\{x_1 {x_8}^{14}, x_3, x_4, (x_6 x_7^{-1})^7\} \cup V^{(7)}$,

and $\{x_1, x_3, x_4, {x_8}^7\} \cup V^{(7)}$;

- three subgroups isomorphic to $(\mathbb{Z}_{49})^4 \oplus (\mathbb{Z}_7)^3$, generated by the images of $\{x_1{x_8}^{14}, x_2, x_3, x_4, (x_6x_7^{-1})^7\} \cup V^{(7)}$, $\{x_1{x_8}^{14}, x_3, x_4, x_5, (x_6x_7^{-1})^7\} \cup V^{(7)}$, and $\{x_1, x_3, x_4, x_5, {x_8}^7\} \cup V^{(7)}$;
- 15 subgroups isomorphic to $(\mathbb{Z}_{49})^5 \oplus (\mathbb{Z}_7)^2$, generated by the images of $\{x_1, x_3, x_4, x_5, ({x_2}^4x_6){x_6}^{7i}, {x_8}^7\} \cup V^{(7)}$ for $0 \le i \le 6$, $\{x_1{x_8}^{14}, x_2, x_3, x_4, x_5\} \cup V^{(7)}$, and $\{x_1{x_8}^{14}, x_3, x_4, x_5, {x_2}^ix_6, (x_6x_7^{-1})^7\} \cup V^{(7)}$ for $0 \le i \le 6$;
- three subgroups isomorphic to $(\mathbb{Z}_{49})^6 \oplus \mathbb{Z}_7$, generated by the images of $\{x_1{x_8}^{14}, x_2, x_3, x_4, x_5, x_6\} \cup V^{(7)}$, $V^{(1)} \cup \{{x_6}^7\}$ and $V^{(1)} \cup \{{x_8}^7\}$;
- two subgroups isomorphic to $(\mathbb{Z}_{49})^7$, generated by the images of $V^{(1)} \cup \{x_6\}$ and $V^{(1)} \cup \{x_8\}$.

Now just as we did for the examples considered in previous Chapters, we may represent each of the above subgroups as a pair $(T_i,\ T_j)$ indicating the first layer L_0/L_1 and second layer L_1/L_2 of the subgroup L, respectively, where $L_j = L \cap K_j = L \cap K^{(7^j)}$ for all j. In order, the pairs that occur are as follows:

Rank 1: (T_1, T_1) and (T_2, T_2) once each;

Rank 2: (T_1, T_3), (T_1, T_4) and (T_2, T_3) once each; (T_3, T_3) once;

Rank 3: (T_1, T_5), (T_1, T_6) and (T_2, T_5) once each; (T_3, T_5) once;

Rank 4: (T_1, T_7), (T_1, T_8) and (T_2, T_7) once each; (T_3, T_7) once;

Rank 5: (T_1, T_j) for $9 \le j \le 16$ once each, and (T_2, T_9) seven times;
(T_3, T_9) seven times;

Rank 6: (T_1, T_{17}) and (T_2, T_{17}) once each, and (T_1, T_{18}) seven times;
(T_3, T_{17}) and (T_4, T_{18}) once each; (T_6, T_{18}) once; (T_8, T_{18}) once;
(T_{14}, T_{18}) seven times; (T_{18}, T_{18}) once;

Rank 7: (T_1, T_{20}) and (T_2, T_{19}) once each, and (T_1, T_{19}) seven times;
(T_4, T_{19}) and (T_4, T_{20}) once each, and (T_3, T_{19}) seven times;
(T_5, T_{19}), (T_6, T_{19}) and (T_6, T_{20}) once each;
(T_7, T_{19}), (T_8, T_{19}) and (T_8, T_{20}) once each;

(T_{14}, T_{20}) seven times, and (T_j, T_{19}) for $9 \leq j \leq 16$ once each;

(T_{17}, T_{19}), (T_{18}, T_{19}) and (T_{18}, T_{20}) once each;

(T_{19}, T_{19}) and (T_{20}, T_{20}) once each.

Note that these pairs are also exactly the same as the pairs that occur as G_1-invariant subgroups of any given 'double-layer' section K_{j-1}/K_{j+1} of K.

One thing that is immediately clear from them is that each allowable pair occurs either once only, or exactly seven times. Those that occur seven times are the following: (T_1, T_{18}), (T_1, T_{19}), (T_2, T_9), (T_3, T_9), (T_3, T_{19}), (T_{14}, T_{18}) and (T_{14}, T_{20}). These are the cases involving an extra parameter i, with $0 \leq i \leq 6$.

Moreover, the generating sets for the subgroups that arise in the case of the pair (T_3, T_9) are easily obtained from those for the pair (T_2, T_9), simply by adjoining $x_1 = z_\lambda$, the generator of a rank 1 tower. Similarly, those for the pair (T_1, T_{19}) are easily obtained from those for the pair (T_1, T_{18}), by adjoining $(x_6 x_7^{-1})^7 = y_\lambda^7$, while those for the pair (T_3, T_{19}) can be obtained from those for the pair (T_1, T_{19}) by adjoining $x_2 = z_{\lambda^2}$ (or from the pair (T_2, T_9) by adjoining z_{λ^2} and y_λ^7), and those for the pair (T_{14}, T_{20}) can be obtained from those for the pair (T_{14}, T_{18}) by adjoining ${x_8}^7 = {y_{\lambda^2}}^7$. Adjoining these extra generators does not create any particular complications, and so for larger values of m, we need only pay close attention to the cases involving the pairs (T_1, T_{18}), (T_2, T_9) and (T_{14}, T_{18}).

When $m = 343$, there are 216 G_1-invariant subgroups with exponent m and rank at most 7, and we find the following triples occur for the subgroups in the first three layers of these subgroups:

Rank 1: (T_1, T_1, T_1) and (T_2, T_2, T_2) once each;

Rank 2: (T_1, T_1, T_3), (T_2, T_2, T_3), (T_1, T_3, T_3), (T_2, T_3, T_3), (T_3, T_3, T_3) and (T_1, T_1, T_4) once each;

Rank 3: (T_1, T_1, T_5), (T_2, T_2, T_5), (T_1, T_3, T_5), (T_2, T_3, T_5), (T_3, T_3, T_5) and (T_1, T_1, T_6) once each;

Rank 4: (T_1, T_1, T_7), (T_2, T_2, T_7), (T_1, T_3, T_7), (T_2, T_3, T_7), (T_3, T_3, T_7) and (T_1, T_1, T_8) once each;

Rank 5: (T_1, T_1, T_j) for $9 \le j \le 16$ once each,
and (T_2, T_2, T_9), (T_1, T_3, T_9), (T_2, T_3, T_9) and (T_3, T_3, T_9) seven times each;

Rank 6: (T_1, T_1, T_{17}), (T_2, T_2, T_{17}), (T_1, T_3, T_{17}), (T_2, T_3, T_{17}), (T_3, T_3, T_{17}),
(T_4, T_{18}, T_{18}), (T_6, T_{18}, T_{18}), (T_8, T_{18}, T_{18}) and (T_{18}, T_{18}, T_{18}) once each,
and (T_1, T_1, T_{18}), (T_1, T_{18}, T_{18}), (T_{14}, T_{18}, T_{18}) seven times each;

Rank 7: (T_2, T_2, T_{19}), (T_2, T_5, T_{19}), (T_2, T_7, T_{19}), (T_2, T_{17}, T_{19}), (T_4, T_{18}, T_{19}), (T_6, T_{18}, T_{19}),
(T_8, T_{18}, T_{19}), (T_{18}, T_{18}, T_{19}), (T_2, T_{19}, T_{19}), (T_j, T_{19}, T_{19}) for $4 \le j \le 19$,
(T_1, T_1, T_{20}), (T_1, T_4, T_{20}), (T_1, T_6, T_{20}), (T_1, T_8, T_{20}),(T_4, T_{18}, T_{20}), (T_6, T_{18}, T_{20}),
(T_8, T_{18}, T_{20}), (T_{18}, T_{18}, T_{20}), (T_1, T_{20}, T_{20}), (T_4, T_{20}, T_{20}), (T_6, T_{20}, T_{20}),
(T_8, T_{20}, T_{20}), (T_{18}, T_{20}, T_{20}), (T_{20}, T_{20}, T_{20}) once each;
and (T_1, T_1, T_{19}), (T_1, T_3, T_{19}), (T_2, T_3, T_{19}), (T_3, T_3, T_{19}), (T_2, T_9, T_{19}),
(T_1, T_{18}, T_{19}), (T_{14}, T_{18}, T_{19}), (T_1, T_{19}, T_{19}), (T_3, T_{19}, T_{19}), (T_1, T_{14}, T_{20}),
(T_1, T_{18}, T_{20}), (T_{14}, T_{18}, T_{20}) and (T_{14}, T_{20}, T_{20}), seven times each.

(The above observations can be confirmed with the help of MAGMA.)

7.4 Summary

Putting the results of Sections 7.2 and 7.3 together, we find that the only possibilities for a normal subgroup L of G contained in K with index $|K:L|$ being a power of a prime k are those included in the summary Table 7.3.

Each row of this table describes a class of such subgroups, and for ease of reference, the jth class is denoted in the left-most column by the symbol of the form 'j_S' where S is a single parameter or sequence of parameters, sometimes with an asterisk added. The parameters b, c and d are powers of k, and unless otherwise indicated, we will take $b = k^t$, $c = k^u$, $d = k^v$, and $e = k^w$. If the asterisk appears, then there are exactly seven subgroups of that type with the given parameters, while if it does not, then there is just one such subgroup. The second column gives conditions on the prime k and the other parameters. The third column gives a description of the subgroup(s) in the class; when

$k \neq 7$ this is an explicit generating set for L, but when $k = 7$, we indicate the layers of L from the top down, by a sequence of T_j's (for various j) followed by a term K_w (for $K^{(7^w)} = K^{(e)}$), where $e = 7^w$ is the exponent of K/L. Again we use $V^{(j)}$ to denote the set $\{{v_1}^j, {v_2}^j, {v_3}^j, {v_4}^j, {v_5}^j, {v_6}^j\}$ of jth powers of the generators of V. Finally, the fourth column gives the structure of the quotient K/L.

For notational convenience, we use the symbol ${}^fT_j$ to indicate a subsequence $T_j, \overset{f}{\ldots}, T_j$ of f successive copies of the subgroup T_j. Hence, for example, the sequence $({}^2T_2, T_{19}, K_3)$ denotes a subgroup L such that K/L has exponent $7^3 = 343$, with $L_3 = K_3 = K^{(7^3)}$, and for this subgroup, L/L_3 is a a copy of T_{19} extended by a tower of two copies of T_2 (as in the first of the rank 7 subgroups listed for the case $m = 343$ in the previous section). Since T_2 and T_{19} have ranks 1 and 7, we have $K/L \cong (\mathbb{Z}_{343/343})^1 \oplus (\mathbb{Z}_{343/7})^6 \oplus \mathbb{Z}_{343/1} \cong (\mathbb{Z}_{49})^6 \oplus \mathbb{Z}_{343}$ for this example .

Explicit generating sets for all these cases can be found in the Appendix.

Type	Conditions	Description of L	Quotient K/L
$1_{(c,d)}$	$k \neq 7$	$\langle u_1^c, u_2^c, V^{(d)} \rangle$	$(\mathbb{Z}_c)^2 \oplus (\mathbb{Z}_d)^6$
$2_{(b,c,d)}$	$k \equiv 1 \bmod 3$; $k \neq 7$; $b \neq c$	$\langle z_\lambda^b, z_{\lambda^2}^c, V^{(e)} \rangle$	$\mathbb{Z}_b \oplus \mathbb{Z}_c \oplus (\mathbb{Z}_d)^6$
$3_{(c,d)}$	$k = 3$	$\langle (u_1u_2)^c, u_2^{3c}, V^{(d)} \rangle$	$\mathbb{Z}_c \oplus \mathbb{Z}_{3c} \oplus (\mathbb{Z}_d)^6$
4_e	$k = 7$	$({}^wT_0, K_w)$	$(\mathbb{Z}_e)^8$
$5_{(d,e)}$	$k = 7$; $d < e$	$({}^vT_0, {}^{w-v}T_1, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_e)^7$
$6_{(d,e)}$	$k = 7$; $d < e$	$({}^vT_0, {}^{w-v}T_2, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_e)^7$
$7_{(c,d,e)}$	$k = 7$; $c < d < e$	$({}^uT_0, {}^{v-u}T_1, {}^{w-v}T_3, K_w)$	$\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_e)^6$
$8_{(c,d,e)}$	$k = 7$; $c < d < e$	$({}^uT_0, {}^{v-u}T_2, {}^{w-v}T_3, K_w)$	$\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_e)^6$
$9_{(d,e)}$	$k = 7$; $d < \frac{e}{7}$	$({}^vT_0, {}^{w-v-1}T_1, T_4, K_w)$	$\mathbb{Z}_d \oplus \mathbb{Z}_{\frac{e}{7}} \oplus (\mathbb{Z}_e)^6$

$10_{(d,e)}$	$k=7;\ d<e$	$({}^{v}T_0, {}^{w-v}T_3, K_w)$	$(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_e)^6$
11_e	$k=7;\ e>1$	$({}^{w-1}T_0, T_4, K_w)$	$(\mathbb{Z}_{\frac{e}{7}})^2 \oplus (\mathbb{Z}_e)^6$
$12_{(c,d,e)}$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_1, {}^{w-v-1}T_3, T_5, K_w)$	$\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus \mathbb{Z}_{\frac{e}{7}} \oplus (\mathbb{Z}_e)^5$
$13_{(c,d,e)}$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_2, {}^{w-v-1}T_3, T_5, K_w)$	$\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus \mathbb{Z}_{\frac{e}{7}} \oplus (\mathbb{Z}_e)^5$
$14_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_5, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^2 \oplus (\mathbb{Z}_e)^5$
$15_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_2, T_5, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^2 \oplus (\mathbb{Z}_e)^5$
$16_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_6, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^2 \oplus (\mathbb{Z}_e)^5$
$17_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_3, T_5, K_w)$	$(\mathbb{Z}_d)^2 \oplus \mathbb{Z}_{\frac{e}{7}} \oplus (\mathbb{Z}_e)^5$
18_e	$k=7;\ e>1$	$({}^{w-1}T_0, T_5, K_w)$	$(\mathbb{Z}_{\frac{e}{7}})^3 \oplus (\mathbb{Z}_e)^5$
19_e	$k=7;\ e>1$	$({}^{w-1}T_0, T_6, K_w)$	$(\mathbb{Z}_{\frac{e}{7}})^3 \oplus (\mathbb{Z}_e)^5$
$20_{(c,d,e)}$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_1, {}^{w-v-1}T_3, T_7, K_w)$	$\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^2 \oplus (\mathbb{Z}_e)^4$
$21_{(c,d,e)}$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_2, {}^{w-v-1}T_3, T_7, K_w)$	$\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^2 \oplus (\mathbb{Z}_e)^4$
$22_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_7, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^3 \oplus (\mathbb{Z}_e)^4$
$23_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_2, T_7, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^3 \oplus (\mathbb{Z}_e)^4$
$24_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_8, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^3 \oplus (\mathbb{Z}_e)^4$
$25_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_3, T_7, K_w)$	$(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{\frac{e}{7}})^2 \oplus (\mathbb{Z}_e)^4$
26_e	$k=7;\ e>1$	$({}^{w-1}T_0, T_7, K_w)$	$(\mathbb{Z}_{\frac{e}{7}})^4 \oplus (\mathbb{Z}_e)^4$
27_e	$k=7;\ e>1$	$({}^{w-1}T_0, T_8, K_w)$	$(\mathbb{Z}_{\frac{e}{7}})^4 \oplus (\mathbb{Z}_e)^4$
${28_{(c,d,e)}}^*$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_1, {}^{w-v-1}T_3, T_9, K_w)$	$\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^3 \oplus (\mathbb{Z}_e)^3$
${29_{(c,d,e)}}^*$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_2, {}^{w-v-1}T_3, T_9, K_w)$	$\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^3 \oplus (\mathbb{Z}_e)^3$
$30_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_9, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^4 \oplus (\mathbb{Z}_e)^3$

$31_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_{10}, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^4 \oplus (\mathbb{Z}_e)^3$
$32_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_{11}, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^4 \oplus (\mathbb{Z}_e)^3$
$33_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_{12}, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^4 \oplus (\mathbb{Z}_e)^3$
$34_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_{13}, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^4 \oplus (\mathbb{Z}_e)^3$
$35_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_{14}, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^4 \oplus (\mathbb{Z}_e)^3$
$36_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_{15}, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^4 \oplus (\mathbb{Z}_e)^3$
$37_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_{16}, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^4 \oplus (\mathbb{Z}_e)^3$
${38_{(d,e)}}^*$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_2, T_9, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^4 \oplus (\mathbb{Z}_e)^3$
${39_{(d,e)}}^*$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_3, T_9, K_w)$	$(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{\frac{e}{7}})^3 \oplus (\mathbb{Z}_e)^3$
40_e	$k=7;\ e>1$	$({}^{w-1}T_0, T_9, K_w)$	$(\mathbb{Z}_{\frac{e}{7}})^5 \oplus (\mathbb{Z}_e)^3$
41_e	$k=7;\ e>1$	$({}^{w-1}T_0, T_{10}, K_w)$	$(\mathbb{Z}_{\frac{e}{7}})^5 \oplus (\mathbb{Z}_e)^3$
42_e	$k=7;\ e>1$	$({}^{w-1}T_0, T_{11}, K_w)$	$(\mathbb{Z}_{\frac{e}{7}})^5 \oplus (\mathbb{Z}_e)^3$
43_e	$k=7;\ e>1$	$({}^{w-1}T_0, T_{12}, K_w)$	$(\mathbb{Z}_{\frac{e}{7}})^5 \oplus (\mathbb{Z}_e)^3$
44_e	$k=7;\ e>1$	$({}^{w-1}T_0, T_{13}, K_w)$	$(\mathbb{Z}_{\frac{e}{7}})^5 \oplus (\mathbb{Z}_e)^3$
45_e	$k=7;\ e>1$	$({}^{w-1}T_0, T_{14}, K_w)$	$(\mathbb{Z}_{\frac{e}{7}})^5 \oplus (\mathbb{Z}_e)^3$
46_e	$k=7;\ e>1$	$({}^{w-1}T_0, T_{15}, K_w)$	$(\mathbb{Z}_{\frac{e}{7}})^5 \oplus (\mathbb{Z}_e)^3$
47_e	$k=7;\ e>1$	$({}^{w-1}T_0, T_{16}, K_w)$	$(\mathbb{Z}_{\frac{e}{7}})^5 \oplus (\mathbb{Z}_e)^3$
$48_{(c,d,e)}$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_1, {}^{w-v-1}T_3, T_{17}, K_w)$	$\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^4 \oplus (\mathbb{Z}_e)^2$
$49_{(c,d,e)}$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_2, {}^{w-v-1}T_3, T_{17}, K_w)$	$\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^4 \oplus (\mathbb{Z}_e)^2$
$50_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_{17}, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^5 \oplus (\mathbb{Z}_e)^2$
$51_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_2, T_{17}, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^5 \oplus (\mathbb{Z}_e)^2$

$52_{(d,e)}{}^*$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_{18}, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^5 \oplus (\mathbb{Z}_e)^2$
$53_{(d,e)}{}^*$	$k=7;\ d<\frac{e}{49}$	$({}^{v}T_0, T_1, {}^{w-v-1}T_{18}, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{7d})^5 \oplus (\mathbb{Z}_e)^2$
$54_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_3, T_{17}, K_w)$	$(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{\frac{e}{7}})^4 \oplus (\mathbb{Z}_e)^2$
$55_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_4, {}^{w-v-1}T_{18}, K_w)$	$(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{7d})^4 \oplus (\mathbb{Z}_e)^2$
$56_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_6, {}^{w-v-1}T_{18}, K_w)$	$(\mathbb{Z}_d)^3 \oplus (\mathbb{Z}_{7d})^3 \oplus (\mathbb{Z}_e)^2$
$57_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_8, {}^{w-v-1}T_{18}, K_w)$	$(\mathbb{Z}_d)^4 \oplus (\mathbb{Z}_{7d})^2 \oplus (\mathbb{Z}_e)^2$
$58_{(d,e)}{}^*$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_{14}, {}^{w-v-1}T_{18}, K_w)$	$(\mathbb{Z}_d)^5 \oplus \mathbb{Z}_{7d} \oplus (\mathbb{Z}_e)^2$
59_e	$k=7;\ e>1$	$({}^{w-1}T_0, T_{17}, K_w)$	$(\mathbb{Z}_{\frac{e}{7}})^6 \oplus (\mathbb{Z}_e)^2$
$60_{(d,e)}$	$k=7;\ d<e$	$({}^{v}T_0, {}^{w-v}T_{18}, K_w)$	$(\mathbb{Z}_d)^6 \oplus (\mathbb{Z}_e)^2$
$61_{(c,d,e)}{}^*$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_1, {}^{w-v-1}T_3, T_{19}, K_w)$	$\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^5 \oplus \mathbb{Z}_e$
$62_{(c,d,e)}{}^*$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_2, {}^{w-v-1}T_3, T_{19}, K_w)$	$\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^5 \oplus \mathbb{Z}_e$
$63_{(c,d,e)}$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_1, T_4, {}^{w-v-1}T_{20}, K_w)$	$\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{7d})^5 \oplus \mathbb{Z}_e$
$64_{(c,d,e)}{}^*$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_2, T_3, {}^{w-v-1}T_{19}, K_w)$	$\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{7d})^5 \oplus \mathbb{Z}_e$
$65_{(c,d,e)}$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_1, T_6, {}^{w-v-1}T_{20}, K_w)$	$\mathbb{Z}_c \oplus (\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{7d})^4 \oplus \mathbb{Z}_e$
$66_{(c,d,e)}$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_2, T_5, {}^{w-v-1}T_{19}, K_w)$	$\mathbb{Z}_c \oplus (\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{7d})^4 \oplus \mathbb{Z}_e$
$67_{(c,d,e)}$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_1, T_8, {}^{w-v-1}T_{20}, K_w)$	$\mathbb{Z}_c \oplus (\mathbb{Z}_d)^3 \oplus (\mathbb{Z}_{7d})^3 \oplus \mathbb{Z}_e$
$68_{(c,d,e)}$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_2, T_7, {}^{w-v-1}T_{19}, K_w)$	$\mathbb{Z}_c \oplus (\mathbb{Z}_d)^3 \oplus (\mathbb{Z}_{7d})^3 \oplus \mathbb{Z}_e$
$69_{(c,d,e)}{}^*$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_1, T_{14}, {}^{w-v-1}T_{20}, K_w)$	$\mathbb{Z}_c \oplus (\mathbb{Z}_d)^4 \oplus (\mathbb{Z}_{7d})^2 \oplus \mathbb{Z}_e$
$70_{(c,d,e)}{}^*$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_2, T_9, {}^{w-v-1}T_{19}, K_w)$	$\mathbb{Z}_c \oplus (\mathbb{Z}_d)^4 \oplus (\mathbb{Z}_{7d})^2 \oplus \mathbb{Z}_e$
$71_{(c,d,e)}{}^*$	$k=7;\ 7c<d<e$	$({}^{u}T_0, T_1, {}^{v-u-1}T_{18}, {}^{w-v}T_{19}, K_w)$	$\mathbb{Z}_c \oplus (\mathbb{Z}_{7c})^5 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$
$72_{(c,d,e)}{}^*$	$k=7;\ 7c<d<e$	$({}^{u}T_0, T_1, {}^{v-u-1}T_{18}, {}^{w-v}T_{20}, K_w)$	$\mathbb{Z}_c \oplus (\mathbb{Z}_{7c})^5 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$
$73_{(c,d,e)}{}^*$	$k=7;\ 7c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_1, T_{18}, {}^{w-v-1}T_{20}, K_w)$	$\mathbb{Z}_c \oplus (\mathbb{Z}_d)^5 \oplus \mathbb{Z}_{7d} \oplus \mathbb{Z}_e$

$74_{(c,d,e)}$	$k=7;\ c<d<\frac{e}{7}$	$({}^{u}T_0, {}^{v-u}T_2, T_{17}, {}^{w-v-1}T_{19}, K_w)$	$\mathbb{Z}_c \oplus (\mathbb{Z}_d)^5 \oplus \mathbb{Z}_{7d} \oplus \mathbb{Z}_e$
${75_{(d,e)}}^*$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_1, T_{19}, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{e}{7}})^6 \oplus \mathbb{Z}_e$
$76_{(c,d,e)}$	$k=7;\ c<d<e$	$({}^{u}T_0, {}^{v-u}T_1, {}^{w-v}T_{20}, K_w)$	$\mathbb{Z}_c \oplus (\mathbb{Z}_d)^6 \oplus \mathbb{Z}_e$
$77_{(c,d,e)}$	$k=7;\ c<d<e$	$({}^{u}T_0, {}^{v-u}T_2, {}^{w-v}T_{19}, K_w)$	$\mathbb{Z}_c \oplus (\mathbb{Z}_d)^6 \oplus \mathbb{Z}_e$
${78_{(d,e)}}^*$	$k=7;\ d<\frac{e}{49}$	$({}^{v}T_0, T_1, {}^{w-v-1}T_{19}, K_w)$	$\mathbb{Z}_d \oplus (\mathbb{Z}_{7d})^6 \oplus \mathbb{Z}_e$
${79_{(d,e)}}^*$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, {}^{w-v-1}T_3, T_{19}, K_w)$	$(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{\frac{e}{7}})^5 \oplus \mathbb{Z}_e$
${80_{(d,e)}}^*$	$k=7;\ d<\frac{e}{49}$	$({}^{v}T_0, T_3, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{7d})^5 \oplus \mathbb{Z}_e$
$81_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_4, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{7d})^5 \oplus \mathbb{Z}_e$
$82_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_4, {}^{w-v-1}T_{20}, K_w)$	$(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{7d})^5 \oplus \mathbb{Z}_e$
$83_{(c,d,e)}$	$k=7;\ 7c<d<e$	$({}^{u}T_0, T_4, {}^{v-u-1}T_{18}, {}^{w-v}T_{19}, K_w)$	$(\mathbb{Z}_c)^2 \oplus (\mathbb{Z}_{7c})^4 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$
$84_{(c,d,e)}$	$k=7;\ 7c<d<e$	$({}^{u}T_0, T_4, {}^{v-u-1}T_{18}, {}^{w-v}T_{20}, K_w)$	$(\mathbb{Z}_c)^2 \oplus (\mathbb{Z}_{7c})^4 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$
$85_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_5, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^3 \oplus (\mathbb{Z}_{7d})^4 \oplus \mathbb{Z}_e$
$86_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_6, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^3 \oplus (\mathbb{Z}_{7d})^4 \oplus \mathbb{Z}_e$
$87_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_6, {}^{w-v-1}T_{20}, K_w)$	$(\mathbb{Z}_d)^3 \oplus (\mathbb{Z}_{7d})^4 \oplus \mathbb{Z}_e$
$88_{(c,d,e)}$	$k=7;\ 7c<d<e$	$({}^{u}T_0, T_6, {}^{v-u-1}T_{18}, {}^{w-v}T_{19}, K_w)$	$(\mathbb{Z}_c)^3 \oplus (\mathbb{Z}_{7c})^3 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$
$89_{(c,d,e)}$	$k=7;\ 7c<d<e$	$({}^{u}T_0, T_6, {}^{v-u-1}T_{18}, {}^{w-v}T_{20}, K_w)$	$(\mathbb{Z}_c)^3 \oplus (\mathbb{Z}_{7c})^3 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$
$90_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_7, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^4 \oplus (\mathbb{Z}_{7d})^3 \oplus \mathbb{Z}_e$
$91_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_8, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^4 \oplus (\mathbb{Z}_{7d})^3 \oplus \mathbb{Z}_e$
$92_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_8, {}^{w-v-1}T_{20}, K_w)$	$(\mathbb{Z}_d)^4 \oplus (\mathbb{Z}_{7d})^3 \oplus \mathbb{Z}_e$
$93_{(c,d,e)}$	$k=7;\ 7c<d<e$	$({}^{u}T_0, T_8, {}^{v-u-1}T_{18}, {}^{w-v}T_{19}, K_w)$	$(\mathbb{Z}_c)^4 \oplus (\mathbb{Z}_{7c})^2 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$
$94_{(c,d,e)}$	$k=7;\ 7c<d<e$	$({}^{u}T_0, T_8, {}^{v-u-1}T_{18}, {}^{w-v}T_{20}, K_w)$	$(\mathbb{Z}_c)^4 \oplus (\mathbb{Z}_{7c})^2 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$

$95_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_9, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^5 \oplus (\mathbb{Z}_{7d})^2 \oplus \mathbb{Z}_e$
$96_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_{10}, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^5 \oplus (\mathbb{Z}_{7d})^2 \oplus \mathbb{Z}_e$
$97_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_{11}, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^5 \oplus (\mathbb{Z}_{7d})^2 \oplus \mathbb{Z}_e$
$98_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_{12}, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^5 \oplus (\mathbb{Z}_{7d})^2 \oplus \mathbb{Z}_e$
$99_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_{13}, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^5 \oplus (\mathbb{Z}_{7d})^2 \oplus \mathbb{Z}_e$
$100_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_{14}, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^5 \oplus (\mathbb{Z}_{7d})^2 \oplus \mathbb{Z}_e$
$101_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_{15}, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^5 \oplus (\mathbb{Z}_{7d})^2 \oplus \mathbb{Z}_e$
$102_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_{16}, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^5 \oplus (\mathbb{Z}_{7d})^2 \oplus \mathbb{Z}_e$
$103_{(d,e)}{}^*$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_{14}, {}^{w-v-1}T_{20}, K_w)$	$(\mathbb{Z}_d)^5 \oplus (\mathbb{Z}_{7d})^2 \oplus \mathbb{Z}_e$
$104_{(c,d,e)}{}^*$	$k=7;\ 7c<d<e$	$({}^{u}T_0, T_{14}, {}^{v-u-1}T_{18}, {}^{w-v}T_{19}, K_w)$	$(\mathbb{Z}_c)^5 \oplus \mathbb{Z}_{7c} \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$
$105_{(c,d,e)}{}^*$	$k=7;\ 7c<d<e$	$({}^{u}T_0, T_{14}, {}^{v-u-1}T_{18}, {}^{w-v}T_{20}, K_w)$	$(\mathbb{Z}_c)^5 \oplus \mathbb{Z}_{7c} \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$
$106_{(d,e)}$	$k=7;\ d<\frac{e}{7}$	$({}^{v}T_0, T_{17}, {}^{w-v-1}T_{19}, K_w)$	$(\mathbb{Z}_d)^6 \oplus \mathbb{Z}_{7d} \oplus \mathbb{Z}_e$
$107_{(c,d,e)}$	$k=7;\ c<d<e$	$({}^{u}T_0, {}^{v-u}T_{18}, {}^{w-v}T_{19}, K_w)$	$(\mathbb{Z}_c)^6 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$
$108_{(c,d,e)}$	$k=7;\ c<d<e$	$({}^{u}T_0, {}^{v-u}T_{18}, {}^{w-v}T_{20}, K_w)$	$(\mathbb{Z}_c)^6 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$
$109_{(d,e)}$	$k=7;\ d<e$	$({}^{v}T_0, {}^{w-v}T_{19}, K_w)$	$(\mathbb{Z}_d)^7 \oplus \mathbb{Z}_e$
$110_{(d,e)}$	$k=7;\ d<e$	$({}^{v}T_0, {}^{w-v}T_{20}, K_w)$	$(\mathbb{Z}_d)^7 \oplus \mathbb{Z}_e$

Table 7.3: Possibilities for G_1-invariant subgroup L of K when $G_1/K \cong C_7 \rtimes_3 C_6$ [Note: $b=k^t,\ c=k^u,\ d=k^v$ and $e=k^w$ (with $t,u,v,w \geq 0$) in all relevant cases]

7.5 Additional automorphisms

In this section, we find out which of the abelian regular covers obtainable from G_1-invariant subgroups of finite prime-power index in $K = N/N'$ admit a larger group of automorphisms than the lift of the group $G_1/N \cong C_7 \rtimes_3 C_6$.

First, we note that none of these regular covers can be 5-arc-transitive, since the Heawood graph itself is not 5-arc-transitive (and in particular, the subgroup N is not normal in the group G_5).

The next possibility we check is that the cover is 4-arc-transitive. To do this, we consider whether or not the G_1-invariant subgroup L is G_4^1-invariant, which we can do by checking whether L is normalised by the additional generator p of G_4^1. If it is, then each layer of L must also be normalised by p, since the subgroups $K_j = K^{(k^j)}$ of K are characteristic in K. For this reason, we begin by determining which of the G_1-invariant subgroups of $K/K^{(k)}$ are normalised by p. Recall that p conjugates w_i to w_j whenever $j \equiv i + 4 \bmod 8$.

Now for every prime k, it is easy to see that the rank 2 subgroup U generated by $u_1 = w_1 w_3 w_5^{-1} w_6^{-1} w_7^{-1}$ and $u_2 = w_2 w_4 w_5^{-1} w_6^{-1} w_7^{-1} w_8$ is not G_4^1-invariant, since $u_1{}^p = w_1^{-1} w_2^{-1} w_3^{-1} w_5 w_7$, which does not lie in U. Also the rank 6 subgroup V generated by $v_1 = w_1$, $v_2 = w_2 w_7^{-1}$, $v_3 = w_3$, $v_4 = w_4 w_8^{-1}$, $v_5 = w_5 w_7^{-1}$ and $v_6 = w_6 w_7^{-1} w_8$ is not G_4^1-invariant, since $v_1{}^p = w_5$, which does not lie in V. Similarly, when $k \equiv 1 \bmod 3$ and t is a primitive cube root of 1 mod k, the rank 1 subgroup of $K/K^{(k)}$ generated by $z_t = w_1 w_2^t w_3 w_4^t w_5^{t^2} w_6^{t^2} w_7^{t^2} w_8^t$ is not G_4^1-invariant, because $z_t{}^p$ is not expressible as a power of z_t. On the other hand, when $k = 3$, the rank 1 subgroup generated by $z_1 = w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_8$ is G_4^1-invariant, since z_1 is centralized by p. This one, however, does not extend to a rank 1 subgroup of $K/K^{(m)}$ when m is a higher power of 3, since $K/K^{(9)}$ has no cyclic G_1-invariant subgroup of order greater than 3.

It follows that for $k \neq 7$, the only G_1-invariant subgroups of k-power index in K that are also G_4^1-invariant are the rank 8 subgroups $K^{(m)}$ themselves, with covering group $K/K^{(m)} \cong (\mathbb{Z}_m)^8$, for $m = k^\ell$ (for any such k), and the subgroups generated by the images of $z_1^{\frac{m}{3}}$ and all w_i^m, with covering group $\mathbb{Z}_{\frac{m}{3}} \oplus (\mathbb{Z}_m)^7$, when $m = 3^\ell$ for some $\ell > 0$. These are the subgroups of types $1_{(m,m)}$ and $3_{(\frac{m}{3},m)}$ in Table 7.3.

In the case $k = 7$, again we let λ be a primitive cube root of 1 mod m, where $m = 7^\ell$ is the exponent of the covering group K/L, chosen such that $\lambda \equiv 2 \bmod 7$ and $\lambda^2 \equiv 4 \bmod 7$. For notational convenience, we will write $x \simeq y$ when the elements x and y have the same image in the top layer $K/K^{(7)}$ of K, so that (for example) $z_\lambda \simeq w_1w_2^2w_3w_4^2w_5^4w_6^4w_7^4w_8^2$. The effect of conjugation by p on the generators x_1 to x_8 defined in Section 7.3 can now be given as follows:

$$\begin{array}{lll}
x_1 \simeq w_1w_2^2w_3w_4^2w_5^4w_6^4w_7^4w_8^2 & \mapsto & w_1^4w_2^4w_3^4w_4^2w_5w_6^2w_7w_8^2 \simeq x_1^2x_2^2x_3^6x_4x_5^6x_6 \\
x_2 \simeq w_1w_2^4w_3w_4^4w_5^2w_6^2w_7^2w_8^4 & \mapsto & w_1^2w_2^2w_3^2w_4^4w_5w_6^4w_7w_8^4 \simeq x_2^2x_3x_4^6x_5x_6^4x_7^6x_8 \\
x_3 \simeq w_2w_3w_4^2w_5w_6^2w_7^3 & \mapsto & w_1w_2^2w_3^3w_6w_7w_8^2 \simeq x_2x_3^5x_4^4x_6^4 \\
x_4 \simeq w_3w_6^5w_7^2w_8^5 & \mapsto & w_2^5w_3^2w_4^5w_7 \simeq x_1^2x_2^5x_3^2x_5^5x_6^6 \\
x_5 \simeq w_4w_5^3w_7^4w_8^6 & \mapsto & w_1^3w_3^4w_4^6w_8 \simeq x_1^3x_3x_5^5 \\
x_6 \simeq w_6w_7w_8^5 & \mapsto & w_2w_3w_4^5 \simeq x_1^5x_2^2x_3^4x_4^4x_6^2x_7^2x_8^5 \\
x_7 \simeq w_6w_7^6w_8 & \mapsto & w_2w_3^6w_4 \simeq x_1^6x_2x_3^6x_5x_6x_7^3x_8^3 \\
x_8 \simeq w_7w_8^4 & \mapsto & w_3w_4^4 \simeq x_1^6x_2x_3^5x_4^3x_5^6x_6x_7^4x_8^2.
\end{array}$$

Note that the images of x_1^p and x_2^p both lie outside the image of the subgroup generated by x_1, x_2, x_3, x_4 and x_5, and so it follows from the definition of the G_1-invariant subgroups of $K/K^{(7)}$ (in Table 7.2) that none of the subgroups T_1 to T_9 of $K/K^{(7)}$ is normalised by p. Similarly, none of the subgroups $T_{10}, T_{11}, T_{13}, T_{14}, T_{15}, T_{16}, T_{18}$ and T_{20} is normalised by p, since each contains the image of x_3 but not the image of x_3^p, and the subgroups T_{17} and T_{19} are not normalised by p, since they contain the image of x_2 but not the image of x_2^p (and contain the image of x_6 but not the image of x_6^p).

On the other hand, the subgroup T_{12} is normalised by p, because

$$\begin{aligned}
x_1^p &\simeq x_1^2x_2^2x_3^6x_4x_5^6x_6 \simeq x_1^2x_3^6x_4x_5^6(x_2^2x_6), \\
x_3^p &\simeq x_2x_3^5x_4^4x_6^4 \simeq x_3^5x_4^4(x_2^2x_6)^4, \\
x_4^p &\simeq x_1^2x_2^5x_3^2x_5^5x_6^6 \simeq x_1^2x_3^2x_5^5(x_2^2x_6)^6, \\
x_5^p &\simeq x_1^3x_3x_5^5, \quad \text{and} \\
(x_2^2x_6)^p &\simeq x_1^5x_2^6x_3^6x_4^2x_5^2x_6^3 \simeq x_1^5x_3^6x_4^2x_5^2(x_2^2x_6)^3.
\end{aligned}$$

Thus T_{12} is the only non-trivial proper G_1-invariant subgroup of $K/K^{(7)}$ normalised by p. Furthermore, since there are no G_1-invariant subgroups of $K/K^{(49)}$ with T_{12} as both layers, this subgroup can occur in at most one layer of L.

Hence we find that the only G_1-invariant subgroups of 7-power index in K that are also G_4^1-invariant are the subgroups $K^{(m)}$, with covering group $K/K^{(m)} \cong (\mathbb{Z}_m)^8$, with $m = 7^\ell$ for $\ell \geq 0$, plus one subgroup with covering group $(\mathbb{Z}_{\frac{m}{7}})^5 \oplus (\mathbb{Z}_m)^3$ where $m = 7^\ell$, for each $\ell > 0$. These are the subgroups of types 4_m and 43_m in Table 7.3.

Next, we consider the possibility that the G_1-invariant subgroup L of K is also G_2^1-invariant. Of course this is not very likely to happen, since the Heawood graph has no 2-arc-regular group of automorphisms (and in particular, the subgroup K of G/N' itself is not G_2^1-invariant), but remarkably, it does happen.

The group G_2^1 can be obtained as an extension of G_1 by adjoining the involutory automorphism θ of G_1 that takes h and a to h^{-1} and a^{-1} $(= a)$, respectively. This is like a reflection, and takes $w_1 = (ha)^6$ to $(h^{-1}a)^6 = w_3$, and vice versa, but takes each of $w_2, w_4, w_5, w_6, w_7, w_8$ to an element outside of K.

(For example, $w_8^\theta = (h^{-1}ah^{-1}ahahah^{-1}aha)^\theta = hahah^{-1}ah^{-1}ahah^{-1}a = w_6w_3^{-1}h^{-1}aha$, which does not lie in K, for otherwise K would contain $[h, a] = h^{-1}aha$.)

In particular, θ does not preserve K, but takes K to another subgroup of index 42 in G_1, with intersection $J = K \cap K^\theta$ having index 7 in K (and index 294 in G_1). In fact $J = K \cap K^\theta$ is generated by the eight elements $v_1 = w_1$, $v_2 = w_2w_7^{-1}$, $v_3 = w_3$, $v_4 = w_4w_8^{-1}$, $v_5 = w_5w_7^{-1}$, $v_6 = w_6w_7^{-1}w_8$, $y_2 = w_7w_8^2$ and w_8^7, with:

$$\begin{aligned}
v_1^\theta &= w_1^\theta = ((ha)^6)^\theta = (h^{-1}a)^6 = w_3 = v_3, \\
v_3^\theta &= w_3^\theta = w_1 = v_1, \\
v_2^\theta &= (w_2w_7^{-1})^\theta = (h^{-1}w_3h)^\theta = hw_1h^{-1} = w_7w_2^{-1} = (w_2w_7^{-1})^{-1} = v_2^{-1}, \\
v_4^\theta &= (w_4w_8^{-1})^\theta(h^{-1}ah^{-1}w_3hah)^\theta = hahw_1h^{-1}ah^{-1} = w_5w_8^{-1}w_3w_6^{-1} \\
&= w_3(w_5w_7^{-1})(w_6w_7^{-1}w_8)^{-1} = v_3v_5v_6^{-1},
\end{aligned}$$

$$
\begin{aligned}
v_5^{\theta} &= (w_5w_7^{-1})^{\theta} = (hah^{-1}ahah^{-1}ah^{-1}ah^{-1}ah^{-1}ahah^{-1}ah^{-1}ah)^{\theta} \\
&= h^{-1}ahah^{-1}ahahahahah^{-1}ahahah^{-1} = w_4w_1^{-1}w_6w_2^{-1} \\
&= w_1^{-1}(w_2w_7^{-1})^{-1}(w_4w_8^{-1})(w_6w_7^{-1}w_8) = v_1^{-1}v_2^{-1}v_4v_6, \\
v_6^{\theta} &= (w_6w_7^{-1}w_8)^{\theta} = (hahah^{-1}ahahahah^{-1}ahahahah^{-1}aha)^{\theta} \\
&= h^{-1}ah^{-1}ahah^{-1}ah^{-1}ah^{-1}ahah^{-1}ah^{-1}ah^{-1}ahah^{-1}a \\
&= w_8w_5^{-1}w_1w_4^{-1}w_7 = w_1(w_4w_8^{-1})^{-1}(w_5w_7^{-1})^{-1} = v_1v_4^{-1}v_5^{-1}, \\
y_2^{\theta} &= (w_7w_8^2)^{\theta} \\
&= (h^{-1}ahahah^{-1}ahah^{-1}ah^{-1}ah^{-1}ahahah^{-1}ahah^{-1}ah^{-1}ahahah^{-1}aha)^{\theta} \\
&= hah^{-1}ah^{-1}ahah^{-1}ahahahah^{-1}ah^{-1}ahah^{-1}ahahah^{-1}ah^{-1}ahah^{-1}a \\
&= w_2w_3^{-1}w_5w_4^{-1}w_7 = (w_2w_7^{-1})\,w_3^{-1}(w_4w_8^{-1})^{-1}(w_5w_7^{-1})(w_7w_8^2)^3\,w_8^{-7} \\
&= v_2v_3^{-1}v_4^{-1}v_5\,y_2^3\,w_8^{-7}, \\
(w_7^7)^{\theta} &= ((h^{-1}ahahah^{-1}ahah^{-1}a)^7)^{\theta} = (hah^{-1}ah^{-1}ahah^{-1}aha)^7 \\
&= w_2w_3^{-1}w_4w_1^{-1}w_5w_7w_3^{-1}w_8w_1^{-1}w_2w_4w_1^{-1}w_6w_3^{-1}w_8 \\
&= w_1^{-3}w_2^2w_3^{-3}w_4^2w_5w_6w_7w_8^2 \\
&= w_1^{-3}(w_2w_7^{-1})^2w_3^{-3}(w_4w_8^{-1})^2(w_5w_7^{-1})(w_6w_7^{-1}w_8)(w_7w_8^2)^5w_8^{-7} \\
&= v_1^{-3}v_2^2v_3^{-3}v_4^2v_5v_6\,y_2^5\,(w_8^7)^{-1}, \\
(w_8^7)^{\theta} &= ((h^{-1}ah^{-1}ahahah^{-1}aha)^7)^{\theta} = (hahah^{-1}ah^{-1}ahah^{-1}a)^7 \\
&= w_6w_3^{-1}w_7w_2^{-1}w_5w_1^{-1}w_6w_5w_8^{-1}w_7w_1^{-1}w_6w_3^{-1}w_5w_4^{-1}w_7 \\
&= w_1^{-2}w_2^{-1}w_3^{-2}w_4^{-1}w_5^3w_6^3w_7^3w_8^{-1} \\
&= w_1^{-2}(w_2w_7^{-1})^{-1}w_3^{-2}(w_4w_8^{-1})^{-1}(w_5w_7^{-1})^3(w_6w_7^{-1}w_8)^3(w_7w_8^2)^8w_8^{-21} \\
&= v_1^{-2}v_2^{-1}v_3^{-2}v_4^{-1}v_5^3v_6^3\,y_2^8\,(w_8^7)^{-3}.
\end{aligned}
$$

It follows that $J = K \cap K^{\theta}$ contains V, and w_i^7 for all i, as well as $y_{\lambda} = w_7w_8^{\lambda}$ (which is the product of $y_2 = w_7w_8^2$ and a power of w_8^7), whenever m is a power of 7 and λ is a primitive cube root of 1 mod m with $\lambda \equiv 2$ mod 7. On the other hand, J contains neither $u_1 = w_1w_3w_5^{-1}w_6^{-1}w_7^{-1}$ nor $u_2 = w_2w_4w_5^{-1}w_6^{-1}w_7^{-1}w_8$ (from Section 7.2), but J does contain each of u_1^7, u_2^7 and $u_1u_2^2 = y_2$.

It is also easy to see that this subgroup J is G_1-invariant, by checking the images of V, y_2 and w_8^7 under conjugation by h and a. But in fact K^θ itself is G_1-invariant, because $(K^\theta)^h = K^{\theta h} = K^{h^{-1}\theta} = K^\theta$ and $(K^\theta)^a = K^{\theta a} = K^{a\theta} = K^\theta$, and it follows directly from this that $J = K \cap K^\theta$ is G_1-invariant. Similarly, J is θ-invariant, since $(K \cap K^\theta)^\theta = K^\theta \cap K = K \cap K^\theta$.

We may view the 'top layer' of J as a copy of the rank 7 subgroup T_{19} of $K/K^{(7)}$, with every subsequent layer of J being isomorphic to T_{21} (generated by the images of the appropriate powers of all the w_i).

Now let L be any G_1-invariant subgroup of finite prime-power index in K, such that L^θ lies in K. Then also L^θ is G_1-invariant, by the same argument as used for K^θ a few lines above. Also L^θ lies in K^θ, so lies in $K \cap K^\theta = J$ as well. In particular, the index $|K:L|$ must be a multiple of $|K:J| = 7$. Hence we may restrict our attention to the case of characteristic 7, and the subgroups we found in Section 7.3.

Next, consider the commutator $c_{ij} = [w_i, w_j] = w_i^{-1}w_j^{-1}w_iw_j$ of any two of the generators w_i and w_j of K. Since these two elements commute in K, and L^θ lies in K, we know that L^θ (trivially) contains c_{ij}, and it follows that L must contain the θ-image c_{ij}^θ, for all such i and j.

These commutators are easily computed. For example,

$$\begin{aligned}
c_{12}^\theta &= (w_1^\theta)^{-1}(w_2^\theta)^{-1}w_1^\theta w_2^\theta \\
&= (ah)^6(ah^{-1}ahahah^{-1}ah^{-1}ah)(h^{-1}a)^6(h^{-1}ahahah^{-1}ah^{-1}aha) \\
&= (ah)^6ah^{-1}ahahah^{-1}ahah^{-1}ah^{-1}ah^{-1}ah^{-1}ah^{-1}ahahah^{-1}ah^{-1}aha \\
&= (ah)^6(ah^{-1}ahahah^{-1}ahah^{-1})(ah^{-1})^6hah^{-1}ahah^{-1}ah^{-1}aha \\
&= w_3^{-1}w_5^{-1}w_1^{-1}w_5 = w_1^{-1}w_3^{-1} = v_1^{-1}v_3^{-1}.
\end{aligned}$$

All such θ-images c_{ij}^θ are given below:

$$\begin{aligned}
c_{12}^\theta &= w_1^{-1}w_3^{-1} = v_1^{-1}v_3^{-1}, \\
c_{13}^\theta &= w_3^{-1}w_1^{-1}w_3w_1 = 1, \\
c_{14}^\theta &= w_3^{-1}w_6^{-1}w_1w_2^{-1}w_5w_1^{-1}w_6 = v_2^{-1}v_3^{-1}v_5,
\end{aligned}$$

$$
\begin{aligned}
c_{15}^{\theta} &= w_3^{-1}w_2^{-1}w_1^{-1}w_2 = v_1^{-1}v_3^{-1},\\
c_{16}^{\theta} &= w_3^{-1}w_4^{-1}w_2w_7^{-1}w_4 = v_2v_3^{-1},\\
c_{17}^{\theta} &= w_3^{-1}w_8^{-1}w_3w_6^{-1}w_1^{-1}w_6w_3^{-1}w_8 = v_1^{-1}v_3^{-1},\\
c_{18}^{\theta} &= w_3^{-1}w_7^{-1}w_4w_2^{-1}w_5w_4^{-1}w_7 = v_2^{-1}v_3^{-1},\\
c_{23}^{\theta} &= w_5^{-1}w_1w_4^{-1}w_7w_6^{-1}w_5w_1 = v_1^{2}v_4^{-1}v_6^{-1},\\
c_{24}^{\theta} &= w_5^{-1}w_1w_7^{-1}w_4w_1^{-1}w_6w_3^{-1}w_8w_1^{-1}w_6 = v_1^{-1}v_3^{-1}v_4v_5^{-1}v_6^{2},\\
c_{25}^{\theta} &= w_5^{-1}w_1w_8^{-1}w_4w_1^{-1}w_2 = v_2v_4v_5^{-1},\\
c_{26}^{\theta} &= w_5^{-1}w_1w_4^{-1}w_7w_2^{-1}w_5w_1^{-1}w_2w_7^{-1}w_4 = 1,\\
c_{27}^{\theta} &= w_5^{-1}w_1w_4^{-1}w_3^{-1}w_4w_1^{-1}w_6w_3^{-1}w_8 = v_3^{-2}v_5^{-1}v_6,\\
c_{28}^{\theta} &= w_5^{-1}w_1w_5^{-1}w_2w_7^{-1}w_4w_1^{-1}w_6w_3^{-1}w_8w_4^{-1}w_7 = v_2v_3^{-1}v_5^{-2}v_6,\\
c_{34}^{\theta} &= w_1^{-1}w_6^{-1}w_1w_4^{-1}w_8w_1^{-1}w_6 = v_1^{-1}v_4^{-1},\\
c_{35}^{\theta} &= w_1^{-1}w_2^{-1}w_6w_7^{-1}w_4w_1^{-1}w_2 = v_1^{-2}v_4v_6,\\
c_{36}^{\theta} &= w_1^{-1}w_4^{-1}w_7w_2^{-1}w_5w_8^{-1}w_3w_6^{-1}w_2w_7^{-1}w_4 = v_1^{-1}v_3v_5v_6^{-1},\\
c_{37}^{\theta} &= w_1^{-1}w_8^{-1}w_3w_7^{-1}w_4w_1^{-1}w_6w_3^{-1}w_8 = v_1^{-2}v_4v_6,\\
c_{38}^{\theta} &= w_1^{-1}w_7^{-1}w_8w_4^{-1}w_7 = v_1^{-1}v_4^{-1},\\
c_{45}^{\theta} &= w_6^{-1}w_1w_5^{-1}w_1w_4^{-1}w_7w_1^{-1}w_2 = v_1v_2v_4^{-1}v_5^{-1}v_6^{-1},\\
c_{46}^{\theta} &= w_6^{-1}w_1w_8^{-1}w_6w_3^{-1}w_2w_7^{-1}w_4 = v_1v_2v_3^{-1}v_4,\\
c_{47}^{\theta} &= w_6^{-1}w_1w_8^{-1}w_3w_1^{-1}w_6w_3^{-1}w_8 = 1,\\
c_{48}^{\theta} &= w_6^{-1}w_1w_5^{-1}w_7w_2^{-1}w_5w_4^{-1}w_7 = v_1v_2^{-1}v_4^{-1}v_6^{-1},\\
c_{56}^{\theta} &= w_2^{-1}w_1w_4^{-1}w_7w_2^{-1}w_5w_2w_7^{-1}w_4 = v_1v_2^{-1}v_5,\\
c_{57}^{\theta} &= w_2^{-1}w_1w_4^{-1}w_3^{-1}w_8w_1^{-1}w_6w_3^{-1}w_8 = v_2^{-1}v_3^{-2}v_4^{-1}v_6,\\
c_{58}^{\theta} &= w_2^{-1}w_1w_5^{-1}w_2w_7^{-1}w_4w_1^{-1}w_5w_4^{-1}w_7 = 1,
\end{aligned}
$$

$$
\begin{aligned}
c_{67}^{\theta} &= w_4^{-1}w_7w_2^{-1}w_1w_4^{-1}w_8w_5^{-1}w_2w_7^{-1}w_4w_1^{-1}w_6w_3^{-1}w_8 = v_3^{-1}v_4^{-1}v_5^{-1}v_6,\\
c_{68}^{\theta} &= w_4^{-1}w_7w_2^{-1}w_6^{-1}w_8w_4^{-1}w_7 = v_2^{-1}v_4^{-2}v_6^{-1},\\
c_{78}^{\theta} &= w_8^{-1}w_3w_6^{-1}w_1w_5^{-1}w_2w_3^{-1}w_8w_4^{-1}w_7 = v_1v_2v_4^{-1}v_5^{-1}v_6^{-1}.
\end{aligned}
$$

Note that every element in the list above is expressible in terms of the generators v_1 to v_6 of the rank 6 subgroup V of K. In fact, each of them is expressible as a word in the following 'base' elements: v_1v_3, $v_2v_3^{-1}$, v_1v_4, $v_1v_2^{-1}v_5$, $v_2v_4^2v_6$ and v_4^7, or perhaps better still, the elements v_1v_4, $v_2v_4^{-1}$, $v_3v_4^{-1}$, $v_5v_4^{-2}$, $v_6v_4^3$ and v_4^7.

So now let F be the subgroup generated by the six elements v_1v_4, $v_2v_4^{-1}$, $v_3v_4^{-1}$, $v_5v_4^{-2}$, $v_6v_4^3$ and v_4^7. Then F contains $v_1^7 = (v_1v_4)^7v_4^{-7}$, and similarly contains v_2^7, v_3^7, v_5^7 and v_6^7, so F has index 7 in V, with $K/F \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_7$. Also it is easy to check using the conjugacy details given at the beginning of Section 7.2 that F is G_1-invariant. Similarly, using the θ-images of the elements v_i, we can see that F is preserved by θ.

The first layer of F is a copy of T_{14}, since the images of the five elements v_1v_4, $v_2v_4^{-1}$, $v_3v_4^{-1}$, $v_5v_4^{-2}$ and $v_6v_4^3$ in $K/K^{(7)}$ generate the same subgroup as $\{x_1,\ x_3,\ x_4,\ x_5,\ x_2^4x_6\}$, while all subsequent layers are copies of T_{18}. It follows that F is one of the seven subgroups of type $58_{(1,7)}{}^*$ from Table 7.3, and in fact F can be generated by $\{z_2{w_7}^{-14},\ x_3,\ x_4,\ x_5,\ v_5{v_6}^3\} \cup V^{(7)}$, and is one of the seven subgroups of type $58_{(1,7)}{}^*$ (see Appendix) — a fact which can be easily checked with the help of MAGMA.

Now once again, let L be any G_1-invariant subgroup of finite 7-power index in K, such that L^θ lies in K. Then we know that L^θ is G_1-invariant, and $F \subseteq L \subseteq J$. It follows that the top layer of L is isomorphic to a subgroup of $K/K^{(7)}$ containing T_{14} and contained in T_{19}, and so must be a copy of one of T_{14}, T_{17}, T_{18} or T_{19}, while every subsequent layer of L contains a copy of T_{18} and hence is a copy of T_{18}, T_{19}, T_{20} or T_{21} itself.

Conversely, if L is any G_1-invariant subgroup of K such that $F \subseteq L \subseteq J$, then $F = F^\theta \subseteq L^\theta \subseteq J^\theta = J$, and in particular, also L^θ is a G_1-invariant subgroup of K. Moreover, L^θ has the same index in G_1 as L, and hence the same index in K as L.

The relevant subgroup types from Table 7.3 are given in Table 7.4 below, with the asterisks dropped from types $58_{(1,e)}{}^*$, $103_{(1,e)}{}^*$, $104_{(1,d,e)}{}^*$ and $105_{(1,d,e)}{}^*$ since there is just one subgroup containing F in each of those cases.

Type	Conditions	Description of L	Quotient K/L
45_7	$k = 7$	(T_{14}, K_7)	$(\mathbb{Z}_7)^3$
$58_{(1,e)}$	$k = 7$; $e > 7$	$(T_{14}, {}^{w-1}T_{18}, K_w)$	$\mathbb{Z}_7 \oplus (\mathbb{Z}_e)^2$
59_7	$k = 7$	(T_{17}, K_7)	$(\mathbb{Z}_7)^2$
$60_{(1,e)}$	$k = 7$; $e > 1$	$({}^wT_{18}, K_w)$	$(\mathbb{Z}_e)^2$
$100_{(1,e)}$	$k = 7$; $e > 7$	$(T_{14}, {}^{w-1}T_{19}, K_w)$	$(\mathbb{Z}_7)^2 \oplus \mathbb{Z}_e$
$103_{(1,e)}$	$k = 7$; $e > 7$	$(T_{14}, {}^{w-1}T_{20}, K_w)$	$(\mathbb{Z}_7)^2 \oplus \mathbb{Z}_e$
$104_{(1,d,e)}$	$k = 7$; $7 < d < e$	$(T_{14}, {}^{v-1}T_{18}, {}^{w-v}T_{19}, K_w)$	$\mathbb{Z}_7 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$
$105_{(1,d,e)}$	$k = 7$; $7 < d < e$	$(T_{14}, {}^{v-1}T_{18}, {}^{w-v}T_{20}, K_w)$	$\mathbb{Z}_7 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_e$
$106_{(1,e)}$	$k = 7$; $e > 7$	$(T_{17}, {}^{w-1}T_{19}, K_w)$	$\mathbb{Z}_7 \oplus \mathbb{Z}_e$
$107_{(1,d,e)}$	$k = 7$; $1 < d < e$	$({}^vT_{18}, {}^{w-v}T_{19}, K_w)$	$\mathbb{Z}_d \oplus \mathbb{Z}_e$
$108_{(1,d,e)}$	$k = 7$; $1 < d < e$	$({}^vT_{18}, {}^{w-v}T_{20}, K_w)$	$\mathbb{Z}_d \oplus \mathbb{Z}_e$
$109_{(1,e)}$	$k = 7$; $e > 1$	$({}^wT_{19}, K_w)$	$\mathbb{Z}_e$

Table 7.4: Possibilities for G_1-invariant subgroup L of K lying between F and J

[Note: $d = k^v$ and $e = k^w$ in all relevant cases]

Next, the following is helpful in considering the effect of θ on these subgroups.

Proposition 7.5.1 *Let L be any G_1-invariant subgroup of K such that $F \subseteq L \subseteq J$.*

(a) *If K/L has exponent $m = 7^\ell$ where $\ell > 1$, then the ℓth layer of L^θ contains the image of $x_6^{\frac{m}{7}} = (v_6 y_\lambda^2)^{\frac{m}{7}}$, and hence is a copy of T_{19} or T_{21}.*

(b) *If the top two layers of L are copies of T_{14} and T_{18}, then the top two layers of L^θ are copies of T_{14} and T_{19}, or T_{14} and T_{18}, according to whether or not the third layer of L contains a copy of T_{20}.*

(c) *If the top two layers of L are copies of T_{14} and T_{19}, then the top two layers of L^θ are copies of T_{14} and T_{18}, or T_{14} and T_{19}, according to whether the third layer of L has rank 7 or 8.*

(d) *If the top two layers of L are copies of T_{14} and T_{20}, then the top two layers of L^θ are copies of T_{17} and T_{19}.*

(e) *If the top two layers of L are copies of T_{17} and T_{19}, then the top two layers of L^θ are copies of T_{14} and T_{20}, or T_{14} and T_{21}, according to whether the third layer of L has rank 7 or 8.*

(f) *If j successive layers of L form a tower of j copies of T_{18}, where $j \geq 2$, then the corresponding j layers of L^θ are either a tower of $j-1$ copies of T_{18} on top of a single copy of T_{19}, or a a tower of j copies of T_{18}, depending on whether or not the next layer of L contains a copy of T_{20}.*

(g) *If two successive layers of L are copies of T_{18} and T_{19}, then the corresponding layers of L^θ are two copies of T_{18}.*

(h) *If two successive layers of L are copies of T_{18} and T_{20}, then the corresponding layers of L^θ are two copies of T_{19}.*

(i) *If j is the largest non-negative integer for which j successive layers of L form a tower of copies of T_{19}, and $j \geq 2$, then the corresponding j layers of L^θ are a copy of T_{18}, followed by a tower of $j-2$ copies of T_{20}, and then a copy of T_{21}, unless the first layer of L is a copy of T_{17}, in which case the top layer of L^θ is a copy of T_{14}, and the next j layers of L^θ consist of a tower of $j-1$ copies of T_{20} followed by a copy of T_{21}.*

(j) *If j successive layers of L form a tower of j copies of T_{20}, where $j \geq 2$, then the corresponding j layers of L^θ are a tower of j copies of T_{19}.*

Proof. We will prove just some of this, and the rest can be similarly proved. Most of it follows from observations about the θ-images of particular elements considered earlier. We can use those (and the θ-images of y_λ and $(y_{\lambda^2})^7$) to help us see what happens to layers of G_1-invariant subgroups of K under the action of θ.

First, suppose K/L has exponent $m = 7^\ell$, where $\ell \geq 2$. Then L^θ contains the elements v_i^7 and hence also the elements $v_i^{\frac{m}{7}}$, for $1 \leq i \leq 6$, since these lie in F. But also L contains w_j^m for $1 \leq j \leq 8$, and therefore L^θ must also contain $(w_j^m)^\theta = (w_j^\theta)^m$ for all such j. Now we know that $(w_8^7)^\theta = v_1^{-2}v_2^{-1}v_3^{-2}v_4^{-1}v_5^3v_6^3\, y_2^8\, (w_8^7)^{-3}$, and it follows that L^θ contains $(w_8^m)^\theta = ((w_8^7)^\theta)^{\frac{m}{7}} = (v_1^{-2}v_2^{-1}v_3^{-2}v_4^{-1}v_5^3v_6^3)^{\frac{m}{7}}\, y_2^{\frac{8m}{7}}\, (w_8^m)^{-3}$.

Hence the ℓth layer $(L^\theta)_{\ell-1}/(L^\theta)_\ell$ of L^θ contains the image of the subgroup generated by $V^{(\frac{m}{7})} \cup \{y_2^{\frac{8m}{7}}\}$, or equivalently, by $V^{(\frac{m}{7})} \cup \{y_2^{\frac{m}{7}}\}$. This is the same as the image of the subgroup generated by $V^{(\frac{m}{7})} \cup \{x_6^{\frac{m}{7}}\}$, by observations made a few paragraphs after Table 4.1, and so is a copy of T_{19}. Thus the ℓth layer of L^θ contains a copy of T_{19}, which proves part (a).

Now recall that we chose λ as a primitive root of 1 mod m, with $\lambda \equiv 2$ mod 7 (and $\lambda^2 \equiv 4$ mod 7). For m divisible by 49 this means $\lambda \equiv 30$ mod 49, while for m divisible by 343 it means $\lambda \equiv 324$ mod 343, so that $\lambda = 2 + 7d$ for some integer d, with $d \equiv 4$ mod 7 when $\ell > 1$, and $d \equiv 46$ mod 49 when $\ell > 2$. Also $\lambda^2 = 4 + 7e$, where $e = 4d + 7d^2 \equiv 2$ mod 7 when $\ell > 1$, and $e \equiv 2$ mod 49 when $\ell > 2$.

By definition, we know that $y_\lambda = w_7 w_8^\lambda = w_7 w_8^{2+7d} = y_2 (w_8^7)^d$, and then similarly, we have $y_{\lambda^2} = w_7 w_8^{\lambda^2} = w_7 w_8^{4+7e} = y_2 w_8^{2+7e}$.

Using the θ-images of y_2 and w_8^7 we calculated earlier, we find that

$$\begin{aligned} y_\lambda^\theta &= (y_2(w_8^7)^d)^\theta = y_2^\theta((w_8^7)^\theta)^d \\ &= (v_2 v_3^{-1} v_4^{-1} v_5\, y_2^3 w_8^{-7})\,(v_1^{-2} v_2^{-1} v_3^{-2} v_4^{-1} v_5^3 v_6^3\, y_2^8 (w_8^7)^{-3})^d \\ &= v_1^{-2d} v_2^{1-d} v_3^{-1-2d} v_4^{-1-d} v_5^{1+3d} v_6^{3d}\, y_2^{3+8d} w_8^{-7-21d}. \end{aligned}$$

Note that $3 + 8d \equiv 0 \bmod 7$ (and also $-7 - 21d \equiv 0 \bmod 7$), and so the image of y_λ^θ in $K/K^{(7)}$ lies in the image of the subgroup V (generated by v_1 to v_6).

The analogous property holds for higher powers of these elements, and so if some layer L_i/L_{i+1} of L is a copy of T_{19} (of rank 7), then the corresponding layer of L^θ can be a copy of T_{18} (of rank 6), depending on what happens with the layers above and below it.

On the other hand, $3 + 8d \equiv 28 \bmod 49$ while $-7 - 21d \equiv 7 \equiv 56 \bmod 49$, and so the image of $y_2^{3+8d} w_8^{-7-21d}$ in $K/K_2 = K/K^{(49)}$ is the same as the image of $(y_2 w_8^2)^{28}$, and then since $y_{\lambda^2} = w_7 w_8^{2+7e}$, this is the same as the image of $y_{\lambda^2}^{28}$. Hence if a layer of L is a copy of T_{19}, then the next layer of L^θ contains not only a copy of T_{18} but also the non-trivial image of a power of $x_8 = y_{\lambda^2}$, and therefore contains a copy of T_{20}, so must be a copy of T_{20} or T_{21}.

In fact we have more than that, because

$$\begin{aligned} x_6^\theta &= (v_6 y_\lambda^2)^\theta = v_6^\theta (y_\lambda^\theta)^2 \\ &= (v_1 v_4^{-1} v_5^{-1})\,(v_1^{-2d} v_2^{1-d} v_3^{-1-2d} v_4^{-1-d} v_5^{1+3d} v_6^{3d}\, y_2^{3+8d} w_8^{-7-21d})^2 \\ &= v_1^{1-4d} v_2^{2-2d} v_3^{-2-4d} v_4^{-3-2d} v_5^{1+6d} v_6^{6d}\, y_2^{6+16d} w_8^{-14-42d} \\ &= (v_1 v_4)^{1-4d} (v_2 v_4^{-1})^{2-2d} (v_3 v_4^{-2-4d})^{-3-2d} (v_5 v_4^{-2})^{1+6d} (v_6 v_4^3)^{6d}\, v_4^{-2-10d} \\ &\qquad y_2^{6+16d} w_8^{-14-42d}. \end{aligned}$$

Noting that $-2 - 10d$, $6 + 16d$ and $-14 - 42d$ are all divisible by 7, we see from this that the image of x_6^θ in $K/K^{(7)}$ lies in the image of the subgroup generated by $v_1 v_4$, $v_2 v_4^{-1}$, $v_3 v_4^{-1}$, $v_5 v_4^{-2}$ and $v_6 v_4^3$, namely T_{14}.

Hence if the top layer of L is a copy of T_{17} (which is generated by T_{14} and the image of x_6), then the top layer of L^θ can be a copy of T_{14}. On the other hand, the second layer of L^θ contains a copy of T_{18} and the image of of $(y_2w_8^2)^{28}$, and hence contains a copy of T_{20}, so must be a copy of T_{20} or T_{21}.

It follows, for example, that if the top two layers of L are copies of T_{17} and T_{19}, then the top layer of L^θ contains a copy of T_{14} and the second layer contains a copy of T_{20}. In fact, since we are assuming that K/L has exponent $m = 7^\ell$, and T_{19} has rank 7, all of the next $\ell{-}1$ layers of L after the first one will be copies of T_{19}, and so all of the corresponding layers of L^θ must contain copies of T_{20}. Also by (a), the ℓth layer of L^θ contains a copy of T_{19} as well, and hence must have rank 8. In particular, $|K\!:\!L| = |T_{21}\!:\!T_{17}||T_{21}\!:\!T_{19}|^{\ell-1} = 7^{\ell+1}$, while $|K\!:\!L^\theta| \le |T_{21}\!:\!T_{14}||T_{21}\!:\!T_{20}|^{\ell-2} = 7^{\ell+1}$, and since we know that $|K\!:\!L| = |K\!:\!L^\theta|$, this forces the top layer of L^θ to be T_{14} and all of the next $\ell-2$ layers to be T_{20}. In particular, this proves part (e). The proof of part (i) is similar.

Next, $x_8 = y_{\lambda^2} = w_7w_8^{\lambda^2}$ $(= w_7w_8^{4+7e} = y_2w_8^{2+7e})$, and therefore

$$\begin{aligned} x_8^\theta &= ((y_{\lambda^2})^7)^\theta = ((w_7w_8^{\lambda^2})^7)^\theta = (w_7^7)^\theta\,((w_8^7)^\theta)^{\lambda^2} \\ &= (v_1^{-3}v_2^2v_3^{-3}v_4^2v_5v_6\,y_2^5\,(w_8^7)^{-1})\,(v_1^{-2}v_2^{-1}v_3^{-2}v_4^{-1}v_5^3v_6^3\,y_2^8\,(w_8^7)^{-3})^{\lambda^2} \\ &= v_1^{-3-2\lambda^2}\,v_2^{2-\lambda^2}\,v_3^{-3-2\lambda^2}\,v_4^{2-\lambda^2}\,v_5^{1+3\lambda^2}\,v_6^{1+3\lambda^2}\,y_2^{5+8\lambda^2}(w_8^7)^{-1-3\lambda^2} \\ &= (v_1v_4)^{-3-2\lambda^2}(v_2v_4^{-1})^{2-\lambda^2}(v_3v_4^{-1})^{-3-2\lambda^2}(v_5v_4^{-2})^{1+3\lambda^2}(v_6v_4^3)^{1+3\lambda^2}\,v_4^{3-5\lambda^2} \\ &\qquad y_2^{5+8\lambda^2}(w_8^7)^{-1-3\lambda^2}. \end{aligned}$$

In this case $3-5\lambda^2 \equiv -7 \equiv 0$ mod 7 while $5+8\lambda^2 \equiv 37 \not\equiv 0$ mod 7, and so the image of x_8^θ in $K/K^{(7)}$ lies in the subgroup generated by the images of v_1v_4, $v_2v_4^{-1}$, $v_3v_4^{-1}$, $v_5v_4^{-2}$, $v_6v_4^3$ and y_2, which is T_{17}.

Hence if some layer of L is a copy of T_{20} (generated by the images of V and x_8), then the next layer up in L^θ contains a copy of T_{17} and so must be T_{17} or T_{19}. This cannot be a copy of T_{21}, by (a), and moreover, it is a copy of T_{17} only if those layers are the second layer of L and the top layer of L^θ. In all other cases it is a copy of T_{19}.

Proofs of parts (d), (h) and (j) follow easily from this, and proofs of the remaining parts are similar to these and the ones completed above. □

The observations in Proposition 7.5.1 now make it easy to determine all of the G_2^1-invariant subgroups of finite prime-power index in K. For example, if L has type $100_{(1,49)}$, with the first two layers being copies of T_{14} and T_{19} and all subsequent layers having rank 8, then it follows from part (c) that L^θ has the same type, and hence L is preserved by θ. On the other hand, if L has type $100_{(1,343)}$, with the first three layers being copies of T_{14}, T_{19} and T_{19}, and all subsequent layers having rank 8, then it follows from part (i) that the first three layers of L^θ are copies of T_{14}, T_{18} and T_{21}, and so L is not preserved by θ.

Thus we obtain the following, which will also be used shortly when we consider isomorphisms between the covers:

Corollary 7.5.2 *The effect of θ on the G_1-invariant subgroups of K lying between F and J is as described in Table* 7.5.

Type of L	Type of L^θ
45_7	$106_{(1,49)}$
$58_{(1,49)}$	$100_{(1,343)}$
$58_{(1,7^w)}$, where $w \geq 3$	$104_{(1,7^{w-1},7^{w+1})}$
59_7	59_7
$60_{(1,7)}$	$109_{(1,49)}$
$60_{(1,7^w)}$, where $w \geq 2$	$107_{(1,7^{w-1},7^{w+1})}$
$100_{(1,49)}$	$100_{(1,49)}$
$100_{(1,343)}$	$58_{(1,49)}$

Type of L	Type of L^θ
$100_{(1,7^w)}$, where $w \geq 4$	$105_{(1,49,7^{w-1})}$
$103_{(1,7^w)}$, where $w \geq 2$	$106_{(1,7^{w+1})}$
$104_{(1,7^{w-1},7^w)}$, where $w \geq 3$	$104_{(1,7^{w-1},7^w)}$
$104_{(1,7^{w-2},7^w)}$, where $w \geq 4$	$58_{(1,7^{w-1})}$
$104_{(1,7^v,7^w)}$, where $w-3 \geq v \geq 2$	$105_{(1,7^{v+1},7^{w-1})}$
$105_{(1,49,7^w)}$, where $w \geq 3$	$100_{(1,7^{w+1})}$
$105_{(1,7^v,7^w)}$, where $w > v > 2$	$104_{(1,7^{v-1},7^{w+1})}$
$106_{(1,49)}$	45_7
$106_{(1,7^w)}$, where $w \geq 3$	$103_{(1,7^{w-1})}$
$107_{(1,7^{w-1},7^w)}$, where $w \geq 2$	$107_{(1,7^{w-1},7^w)}$
$107_{(1,7^{w-2},7^w)}$, where $w \geq 3$	$60_{(1,7^{w-1})}$
$107_{(1,7^v,7^w)}$, where $w-3 \geq v \geq 1$	$108_{(1,7^{v+1},7^{w-1})}$
$108_{(1,7,7^w)}$, where $w \geq 2$	$109_{(1,7^{w+1})}$
$108_{(1,7^v,7^w)}$, where $w > v > 1$	$107_{(1,7^{v-1},7^{w+1})}$
$109_{(1,7)}$	$109_{(1,7)}$
$109_{(1,49)}$	$60_{(1,7)}$
$109_{(1,7^w)}$, where $w \geq 3$	$108_{(1,7,7^{w-1})}$

Table 7.5: Effect of θ on the G_1-invariant subgroups from Table 7.4

In particular, this gives us all of the G_1-invariant subgroups of prime-power index in K that are also G_2^1-invariant:

Corollary 7.5.3 *The G_2^1-invariant subgroups of finite prime-power index in K are the following, from Table* 7.5*:*

- *the subgroup of type* $109_{(1,7)}$*, which is J, with quotient* $K/L \cong \mathbb{Z}_7$*,*
- *the subgroup of type* 59_7*, generated by* $F \cup \{z_{\lambda^2}\}$*, with quotient* $K/L \cong (\mathbb{Z}_7)^2$*,*
- *the subgroup of type* $100_{(1,49)}$*, with quotient* $K/L \cong (\mathbb{Z}_7)^2 \oplus \mathbb{Z}_{49}$*,*
- *the subgroup of type* $107_{(1,7^{\ell-1},7^\ell)}$*, with quotient* $K/L \cong \mathbb{Z}_{7^{\ell-1}} \oplus \mathbb{Z}_{7^\ell}$*, for each* $\ell \geq 2$*,*
- *one of the subgroups of type* ${104_{(1,7^{\ell-1},7^\ell)}}^*$*, with quotient* $K/L \cong \mathbb{Z}_7 \oplus \mathbb{Z}_{7^{\ell-1}} \oplus \mathbb{Z}_{7^\ell}$*, for each* $\ell \geq 3$*.*

Note that none of these subgroups has top layer isomorphic to T_{12} or T_{21}, and so none of them can be G_4^1-invariant, but actually that follows also from the fact that no finite symmetric cubic graph admits both a 2-arc-regular and a 4-arc-regular group of automorphisms (see Corollary 2.4.2, or [26, Theorem 3]).

We still need to check for G_3-invariance, but this is easy:

By Proposition 2.4.3 (or [26, Proposition 26], or [22, Proposition 2.3]), if the regular cover resulting from a G_1-invariant subgroup L has a 3-arc-regular group of automorphisms, then it must also admit a 2-arc-regular group of automorphisms, and so L must come from the restricted set of G_2^1-invariant possibilities that we found above. On the other hand, the group G_3 can be obtained from G_2^1 by adjoining the involutory automorphism τ that interchanges h, a and θ with h, $a\theta$ and θ (respectively). This automorphism τ interchanges $(ha)^2$ with $hah^{-1}a$, and $(h^{-1}a)^2$ with $h^{-1}aha$, and hence takes the element $v_1 = w_1 = (ha)^6$ to $(hah^{-1}a)^3 = w_5w_1^{-1}hahah$, which does not lie in K, let alone in any subgroup L contained in K. Similarly, τ takes $v_1v_4 = w_1w_4w_8^{-1}$ to $w_5w_1^{-1}w_7^{-1} = v_1^{-1}v_5$, but the image of this in $K/K^{(7)}$ does not lie in the subgroup T_{14}, so τ does not preserve any G_2^1-invariant subgroup L with T_{14} as its top layer. Hence τ preserves no G_2^1-invariant subgroup of finite index, and therefore we have no 3-arc-regular cover.

Finally, we determine isomorphisms between the covering graphs that arise from the G_1-invariant subgroups we have found.

When the subgroup L is G_4^1-invariant, the cover is 4-arc-regular, and unique up to isomorphism, since the subgroup K is normal in G_4^1 but not in G_5. Similarly, when the subgroup is G_2^1-invariant, the cover is 2-arc-regular, and unique up to isomorphism, since K is normal in G_1 but not in G_2^1.

So now suppose L is G_1-invariant, but not G_2^1- or G_4^1-invariant. Then the cover obtained from L will be unique up to isomorphism unless there exists an outer automorphism of G_1 taking L to another G_1-invariant subgroup of K. Let us suppose that happens.

The group G_1 is the modular group $\mathrm{PSL}(2,\mathbb{Z})$, and isomorphic to the free product $C_2 \star C_3$, so (as is well known) the automorphism group of G_1 is the group G_2^1, generated by G_1 and the involutory automorphism θ that inverts the two standard generators of G_1, and in particular, $G_2^1 \cong \mathrm{PGL}(2,\mathbb{Z})$. Hence we may suppose the outer automorphism takes L to L^θ. In particular, since L^θ lies in K, we find that L must be one of the subgroups described in Table 7.5, but not one of those that are preserved by θ.

It follows that if L is a G_1-invariant subgroup of J containing F (in which case L will certainly not be G_4^1-invariant), then either $L = L^\theta$ and the cover is 2-arc-regular, or $L^\theta \neq L$ but L and L^θ define the same 1-arc-regular cover of the Heawood graph. Note, however, that in the latter case, the exponents of K/L and K/L^θ are always different — in fact one of them is always 7 times the other — so we do not have to take much account of them when enumerating all possibilities for L such that the covering group K/L has given exponent.

In all other cases, where L does not contain F or is not contained in J, the cover is unique up to isomorphism.

7.6 Main theorem

Thus we have the following, with 'for each $d \,|\, m$' and 'for each $d \,||\, m$' meaning 'for each divisor d of m' and 'for each proper divisor d of m', respectively:

Theorem 7.6.1 *Let $m = k^\ell$ be any power of a prime k, with $\ell > 0$. Then the symmetric abelian regular covers of the Heawood graph with covering group of exponent m are as follows:*

(a) *If $k \equiv 2 \bmod 3$, there are exactly $2\ell + 1$ such covers, namely*

- *one 4-arc-regular cover with covering group $(\mathbb{Z}_m)^8$,*
- *one 1-arc-regular cover with covering group $(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_m)^6$ and one 1-arc-regular cover with covering group $(\mathbb{Z}_d)^6 \oplus (\mathbb{Z}_m)^2$, for each $d \,||\, m$.*

(b) *If $k \equiv 1 \bmod 3$ and $k \neq 7$, there are exactly $3\ell^2 + 3\ell + 1$ such covers, namely*

- *one 4-arc-regular cover with covering group $(\mathbb{Z}_m)^8$,*
- *two 1-arc-regular covers with covering group $\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_m)^6$ and one 1-arc-regular cover with covering group $(\mathbb{Z}_c)^6 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_m$, for each ordered pair (c, d) of distinct divisors of m.*

(c) *If $k = 3$, there are exactly $4\ell + 1$ such covers, namely*

- *two 4-arc-regular covers, with covering groups $(\mathbb{Z}_m)^8$ and $\mathbb{Z}_{\frac{m}{3}} \oplus (\mathbb{Z}_m)^7$,*
- *one 1-arc-regular cover with covering group $\mathbb{Z}_d \oplus \mathbb{Z}_{3d} \oplus (\mathbb{Z}_m)^6$ for each $d \,||\, \frac{m}{3}$,*
- *one 1-arc-regular cover with covering group $(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_m)^6$, one 1-arc-regular cover with covering group $(\mathbb{Z}_d)^6 \oplus (\mathbb{Z}_m)^2$, and one 1-arc-regular cover with covering group $(\mathbb{Z}_d)^6 \oplus \mathbb{Z}_{\frac{m}{3}} \oplus \mathbb{Z}_m$, for each $d \,||\, m$.*

(d) *If $k = 7$ and $\ell \geq 3$, there are exactly $54\ell^2 - 54\ell + 14$ such covers, namely*

- *two 4-arc-regular covers, with covering groups $(\mathbb{Z}_m)^8$ and $(\mathbb{Z}_{\frac{m}{7}})^5 \oplus (\mathbb{Z}_m)^3$,*
- *two 2-arc-regular covers, with covering groups $\mathbb{Z}_7 \oplus \mathbb{Z}_{\frac{m}{7}} \oplus \mathbb{Z}_m$ and $\mathbb{Z}_{\frac{m}{7}} \oplus \mathbb{Z}_m$,*
- *two 1-arc-regular covers with covering group $\mathbb{Z}_d \oplus (\mathbb{Z}_m)^7$, for each $d \,||\, m$,*

- *three* 1-*arc-regular covers with covering group* $\mathbb{Z}_d \oplus \mathbb{Z}_{\frac{m}{7}} \oplus (\mathbb{Z}_m)^6$, *for each* $d \,||\, \frac{m}{7}$,
- *two* 1-*arc-regular covers with covering group* $\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_m)^6$, *for each pair* $\{c, d\}$ *of distinct divisors of* $\frac{m}{49}$,
- *two* 1-*arc-regular covers with covering group* $(\mathbb{Z}_{\frac{m}{7}})^2 \oplus (\mathbb{Z}_m)^6$,
- *one* 1-*arc-regular cover with covering group* $(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_m)^6$, *for each* $d \,||\, \frac{m}{7}$,
- *two* 1-*arc-regular covers with covering group* $\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus \mathbb{Z}_{\frac{m}{7}} \oplus (\mathbb{Z}_m)^5$, *for each pair* $\{c, d\}$ *of distinct divisors of* $\frac{m}{49}$,
- *three* 1-*arc-regular covers with covering group* $\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{m}{7}})^2 \oplus (\mathbb{Z}_m)^5$, *for each* $d \,||\, \frac{m}{7}$,
- *one* 1-*arc-regular cover with covering group* $(\mathbb{Z}_d)^2 \oplus \mathbb{Z}_{\frac{m}{7}} \oplus (\mathbb{Z}_m)^5$, *for each* $d \,||\, \frac{m}{7}$,
- *two* 1-*arc-regular covers with covering group* $(\mathbb{Z}_{\frac{m}{7}})^3 \oplus (\mathbb{Z}_m)^5$,
- *two* 1-*arc-regular covers with covering group* $\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{m}{7}})^2 \oplus (\mathbb{Z}_m)^4$, *for each pair* $\{c, d\}$ *of distinct divisors of* $\frac{m}{49}$,
- *three* 1-*arc-regular covers with covering group* $\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{m}{7}})^3 \oplus (\mathbb{Z}_m)^4$, *for each* $d \,||\, \frac{m}{7}$,
- *one* 1-*arc-regular cover with covering group* $(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{\frac{m}{7}})^2 \oplus (\mathbb{Z}_m)^4$, *for each* $d \,||\, \frac{m}{7}$,
- *two* 1-*arc-regular covers with covering group* $(\mathbb{Z}_{\frac{m}{7}})^4 \oplus (\mathbb{Z}_m)^4$,
- *fourteen* 1-*arc-regular covers with covering group* $\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{m}{7}})^3 \oplus (\mathbb{Z}_m)^3$, *for each pair* $\{c, d\}$ *of distinct divisors of* $\frac{m}{49}$,
- *fifteen* 1-*arc-regular covers with covering group* $\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{m}{7}})^4 \oplus (\mathbb{Z}_m)^3$, *for each* $d \,||\, \frac{m}{7}$,
- *seven* 1-*arc-regular covers with covering group* $(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{\frac{m}{7}})^3 \oplus (\mathbb{Z}_m)^3$, *for each* $d \,||\, \frac{m}{7}$,
- *seven* 1-*arc-regular covers with covering group* $(\mathbb{Z}_{\frac{m}{7}})^5 \oplus (\mathbb{Z}_m)^3$,
- *two* 1-*arc-regular covers with covering group* $\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{m}{7}})^4 \oplus (\mathbb{Z}_m)^2$, *for each pair* $\{c, d\}$ *of distinct divisors of* $\frac{m}{49}$,
- *nine* 1-*arc-regular covers with covering group* $\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{m}{7}})^5 \oplus (\mathbb{Z}_m)^2$, *for each* $d \,||\, \frac{m}{7}$,
- *seven* 1-*arc-regular covers with covering group* $\mathbb{Z}_d \oplus (\mathbb{Z}_{7d})^5 \oplus (\mathbb{Z}_m)^2$, *for each* $d \,||\, \frac{m}{49}$,
- *two* 1-*arc-regular covers with covering group* $(\mathbb{Z}_{\frac{m}{49}})^2 \oplus (\mathbb{Z}_{\frac{m}{7}})^4 \oplus (\mathbb{Z}_m)^2$,
- *one* 1-*arc-regular cover with covering group* $(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{\frac{m}{7}})^4 \oplus (\mathbb{Z}_m)^2$, *for each* $d \,||\, \frac{m}{49}$,

- *one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{7d})^4 \oplus (\mathbb{Z}_m)^2$*, for each* $d \,||\, \frac{m}{49}$*,*
- *one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_d)^3 \oplus (\mathbb{Z}_{7d})^3 \oplus (\mathbb{Z}_m)^2$*, for each* $d \,||\, \frac{m}{7}$*,*
- *one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_d)^4 \oplus (\mathbb{Z}_{7d})^2 \oplus (\mathbb{Z}_m)^2$*, for each* $d \,||\, \frac{m}{7}$*,*
- *seven* 1*-arc-regular covers with covering group* $(\mathbb{Z}_d)^5 \oplus \mathbb{Z}_{7d} \oplus (\mathbb{Z}_m)^2$*, for each* $d \,||\, \frac{m}{7}$*, but with one of these for* $d = 1$ *having* $\mathbb{Z}_7 \oplus \mathbb{Z}_{\frac{m}{7}} \oplus \mathbb{Z}_{7m}$ *as an alternative covering group,*
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_{\frac{m}{7}})^6 \oplus (\mathbb{Z}_m)^2$*,*
- *one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_d)^6 \oplus (\mathbb{Z}_m)^2$*, for each* $d \,||\, \frac{m}{7}$*, but with the one for* $d = 1$ *having* $\mathbb{Z}_{\frac{m}{7}} \oplus \mathbb{Z}_{7m}$ *as an alternative covering group,*
- *fifteen* 1*-arc-regular covers with covering group* $\mathbb{Z}_d \oplus \mathbb{Z}_{\frac{m}{49}} \oplus (\mathbb{Z}_{\frac{m}{7}})^5 \oplus \mathbb{Z}_m$*, for each* $d \,||\, \frac{m}{49}$*,*
- *fourteen* 1*-arc-regular covers with covering group* $\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{m}{7}})^5 \oplus \mathbb{Z}_m$*, for each pair* $\{c, d\}$ *of distinct divisors of* $\frac{m}{343}$*,*
- *eight* 1*-arc-regular covers with covering group* $\mathbb{Z}_c \oplus \mathbb{Z}_d \oplus (\mathbb{Z}_{7d})^5 \oplus \mathbb{Z}_m$*, for each pair* $\{c, d\}$ *of distinct divisors of* $\frac{m}{343}$*,*
- *two* 1*-arc-regular covers with covering group* $\mathbb{Z}_c \oplus (\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{7d})^4 \oplus \mathbb{Z}_m$*, for each ordered pair* (c, d) *of distinct divisors of* $\frac{m}{49}$ *with* $c < d$*,*
- *two* 1*-arc-regular covers with covering group* $\mathbb{Z}_c \oplus (\mathbb{Z}_d)^3 \oplus (\mathbb{Z}_{7d})^3 \oplus \mathbb{Z}_m$*, for each ordered pair* (c, d) *of distinct divisors of* $\frac{m}{49}$ *with* $c < d$*,*
- *fourteen* 1*-arc-regular covers with covering group* $\mathbb{Z}_c \oplus (\mathbb{Z}_d)^4 \oplus (\mathbb{Z}_{7d})^2 \oplus \mathbb{Z}_m$*, for each ordered pair* (c, d) *of distinct divisors of* $\frac{m}{49}$ *with* $c < d$*,*
- *fifteen* 1*-arc-regular covers with covering group* $\mathbb{Z}_d \oplus (\mathbb{Z}_{7d})^5 \oplus \mathbb{Z}_{49d} \oplus \mathbb{Z}_m$*, for each* $d \,||\, \frac{m}{49}$*,*
- *fourteen* 1*-arc-regular covers with covering group* $\mathbb{Z}_c \oplus (\mathbb{Z}_{7c})^5 \oplus \mathbb{Z}_{49d} \oplus \mathbb{Z}_m$*, for each ordered pair* (c, d) *of distinct divisors of* $\frac{m}{343}$ *with* $c < d$*,*
- *eight* 1*-arc-regular covers with covering group* $\mathbb{Z}_c \oplus (\mathbb{Z}_{7d})^5 \oplus \mathbb{Z}_{49d} \oplus \mathbb{Z}_m$*, for each ordered pair* (c, d) *of distinct divisors of* $\frac{m}{343}$ *with* $c < d$*,*

- *nine* 1*-arc-regular covers with covering group* $\mathbb{Z}_d \oplus (\mathbb{Z}_{\frac{m}{7}})^6 \oplus \mathbb{Z}_m$*, for each* $d \,||\, \frac{m}{7}$*,*
- *nine* 1*-arc-regular covers with covering group* $\mathbb{Z}_d \oplus (\mathbb{Z}_{7d})^6 \oplus \mathbb{Z}_m$*, for each* $d \,||\, \frac{m}{49}$*,*
- *two* 1*-arc-regular covers with covering group* $\mathbb{Z}_c \oplus (\mathbb{Z}_d)^6 \oplus \mathbb{Z}_m$*, for each ordered pair* (c, d) *of distinct divisors of* $\frac{m}{343}$ *with* $c < d$*,*
- *nine* 1*-arc-regular covers with covering group* $(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{7d})^5 \oplus \mathbb{Z}_m$*, for each* $d \,||\, \frac{m}{7}$*,*
- *seven* 1*-arc-regular covers with covering group* $(\mathbb{Z}_d)^2 \oplus (\mathbb{Z}_{\frac{m}{7}})^5 \oplus \mathbb{Z}_m$*, for each* $d \,||\, \frac{m}{49}$*,*
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_c)^2 \oplus (\mathbb{Z}_{7c})^4 \oplus \mathbb{Z}_{7d} \oplus \mathbb{Z}_m$*, for each ordered pair* (c, d) *of distinct divisors of* $\frac{m}{49}$ *with* $c < d$*,*
- *three* 1*-arc-regular covers with covering group* $(\mathbb{Z}_d)^3 \oplus (\mathbb{Z}_{7d})^4 \oplus \mathbb{Z}_m$*, for each* $d \,||\, \frac{m}{7}$*,*
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_c)^3 \oplus (\mathbb{Z}_{7c})^3 \oplus \mathbb{Z}_{7d} \oplus \mathbb{Z}_m$*, for each ordered pair* (c, d) *of distinct divisors of* $\frac{m}{49}$ *with* $c < d$*,*
- *three* 1*-arc-regular covers with covering group* $(\mathbb{Z}_d)^4 \oplus (\mathbb{Z}_{7d})^3 \oplus \mathbb{Z}_m$*, for each* $d \,||\, \frac{m}{7}$*,*
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_c)^4 \oplus (\mathbb{Z}_{7c})^2 \oplus \mathbb{Z}_{7d} \oplus \mathbb{Z}_m$*, for each ordered pair* (c, d) *of distinct divisors of* $\frac{m}{49}$ *with* $c < d$*,*
- *fifteen* 1*-arc-regular covers with covering group* $(\mathbb{Z}_d)^5 \oplus (\mathbb{Z}_{7d})^2 \oplus \mathbb{Z}_m$*, for each* $d \,||\, \frac{m}{7}$*, but with one of those for* $d = 1$ *having* $\mathbb{Z}_7 \oplus \mathbb{Z}_{49} \oplus \mathbb{Z}_{\frac{m}{49}}$ *as an alternative covering group, and another of those for* $d = 1$ *having* $\mathbb{Z}_7 \oplus \mathbb{Z}_{7m}$ *as an alternative covering group,*
- *thirteen* 1*-arc-regular covers with covering group* $\mathbb{Z}_7 \oplus \mathbb{Z}_{\frac{m}{7}} \oplus \mathbb{Z}_m$*,*
- *fourteen* 1*-arc-regular covers with covering group* $(\mathbb{Z}_c)^5 \oplus \mathbb{Z}_{7c} \oplus \mathbb{Z}_{7d} \oplus \mathbb{Z}_m$*, for each ordered pair* (c, d) *of distinct divisors of* $\frac{m}{49}$ *with* $c < d$ *other than* $(1, \frac{m}{49})$*, but with one of those for each pair* (c, d) *with* $c = 1$ *having* $\mathbb{Z}_7 \oplus \mathbb{Z}_{49d} \oplus \mathbb{Z}_{\frac{m}{7}}$ *as an alternative covering group, and another one of those for each pair* (c, d) *with* $c = 1$ *having* $\mathbb{Z}_7 \oplus \mathbb{Z}_d \oplus \mathbb{Z}_{7m}$ *as an alternative covering group,*
- *three* 1*-arc-regular covers with covering group* $(\mathbb{Z}_d)^6 \oplus \mathbb{Z}_{7d} \oplus \mathbb{Z}_m$*, for each* $d \,||\, \frac{m}{7}$*, but with the three such covers in the case* $d = 1$ *having (in some order) respectively* $(\mathbb{Z}_7)^2 \oplus \mathbb{Z}_{\frac{m}{7}}$*,* $\mathbb{Z}_{49} \oplus \mathbb{Z}_{\frac{m}{7}}$ *and* $\mathbb{Z}_{7m}$ *as an alternative covering group,*

- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_c)^6 \oplus \mathbb{Z}_{7d} \oplus \mathbb{Z}_m$*, for each ordered pair* (c, d) *of distinct divisors of* $\frac{m}{49}$ *with* $c < d$ *other than* $(1, \frac{m}{49})$*, but with the two such covers in each case with* $c = 1$ *having (in some order) respectively* $\mathbb{Z}_{49d} \oplus \mathbb{Z}_{\frac{m}{7}}$ *and* $\mathbb{Z}_d \oplus \mathbb{Z}_{7m}$ *as an alternative covering group,*
- *one* 1*-arc-regular cover with covering group* $\mathbb{Z}_{\frac{m}{7}} \oplus \mathbb{Z}_m$*,*
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_d)^7 \oplus \mathbb{Z}_m$*, for each* $d \,||\, m$*, but with one of those for* $d = 1$ *having* $\mathbb{Z}_7 \oplus \mathbb{Z}_{\frac{m}{7}}$ *as an alternative covering group.*

(e) *If* $k = 7$ *and* $e = 2$ *(so that* $m = 49$*), there are exactly* 122 *such covers, namely*

- *two* 4*-arc-regular covers, with covering groups* $(\mathbb{Z}_{49})^8$ *and* $(\mathbb{Z}_7)^5 \oplus (\mathbb{Z}_{49})^3$*,*
- *two* 2*-arc-regular covers, with covering groups* $(\mathbb{Z}_7)^2 \oplus \mathbb{Z}_{49}$ *and* $\mathbb{Z}_7 \oplus \mathbb{Z}_{49}$*,*
- *two* 1*-arc-regular covers with covering group* $\mathbb{Z}_7 \oplus (\mathbb{Z}_{49})^7$*,*
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_{49})^7$*,*
- *three* 1*-arc-regular covers with covering group* $\mathbb{Z}_7 \oplus (\mathbb{Z}_{49})^6$*,*
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^2 \oplus (\mathbb{Z}_{49})^6$*,*
- *one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_{49})^6$*,*
- *three* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^2 \oplus (\mathbb{Z}_{49})^5$*,*
- *one* 1*-arc-regular cover with covering group* $\mathbb{Z}_7 \oplus (\mathbb{Z}_{49})^5$*,*
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^3 \oplus (\mathbb{Z}_{49})^5$*,*
- *three* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^3 \oplus (\mathbb{Z}_{49})^4$*,*
- *one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_7)^2 \oplus (\mathbb{Z}_{49})^4$*,*
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^4 \oplus (\mathbb{Z}_{49})^4$*,*
- *fifteen* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^4 \oplus (\mathbb{Z}_{49})^3$*,*
- *seven* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^3 \oplus (\mathbb{Z}_{49})^3$*,*
- *seven* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^5 \oplus (\mathbb{Z}_{49})^3$*,*
- *nine* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^5 \oplus (\mathbb{Z}_{49})^2$*,*
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^4 \oplus (\mathbb{Z}_{49})^2$*,*

- *one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_7)^3 \oplus (\mathbb{Z}_{49})^2$,
- *one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_7)^2 \oplus (\mathbb{Z}_{49})^2$,
- *seven* 1*-arc-regular covers with covering group* $\mathbb{Z}_7 \oplus (\mathbb{Z}_{49})^2$*, but with one of these having* $(\mathbb{Z}_7)^2 \oplus \mathbb{Z}_{343}$ *as an alternative covering group,*
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^6 \oplus (\mathbb{Z}_{49})^2$,
- *one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_{49})^2$*, but with one of these having* $\mathbb{Z}_7 \oplus \mathbb{Z}_{343}$ *as an alternative covering group,*
- *nine* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^6 \oplus \mathbb{Z}_{49}$,
- *nine* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^5 \oplus \mathbb{Z}_{49}$,
- *three* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^4 \oplus \mathbb{Z}_{49}$,
- *three* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^3 \oplus \mathbb{Z}_{49}$,
- *fourteen* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^2 \oplus \mathbb{Z}_{49}$*, but with one of these having* $\mathbb{Z}_7 \oplus \mathbb{Z}_{343}$ *as an alternative covering group,*
- *two* 1*-arc-regular covers with covering group* $\mathbb{Z}_7 \oplus \mathbb{Z}_{49}$*, but with one of these having* $(\mathbb{Z}_7)^3$ *as an alternative covering group, and the other having* $\mathbb{Z}_{343}$ *as an alternative covering group,*
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^7 \oplus \mathbb{Z}_{49}$*, and*
- *two* 1*-arc-regular covers with covering group* $\mathbb{Z}_{49}$*, but with one of these having* $(\mathbb{Z}_7)^2$ *as an alternative covering group.*

(f) *If* $k = 7$ *and* $e = 1$ *(so that* $m = 7$*), there are exactly* 21 *such covers, namely*

- *two* 4*-arc-regular covers, with covering groups* $(\mathbb{Z}_7)^8$ *and* $(\mathbb{Z}_7)^3$*,*
- *two* 2*-arc-regular covers, with covering groups* $\mathbb{Z}_7$ *and* $(\mathbb{Z}_7)^2$*,*
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^7$*,*
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^6$*,*
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^5$*,*
- *two* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^4$*,*
- *seven* 1*-arc-regular covers with covering group* $(\mathbb{Z}_7)^3$*, but with one of these having*

$\mathbb{Z}_7 \oplus \mathbb{Z}_{49}$ *as an alternative covering group,*

- *one* 1*-arc-regular cover with covering group* $(\mathbb{Z}_7)^2$*, but also having* $\mathbb{Z}_{49}$ *as an alternative covering group, and*
- *one* 1*-arc-regular cover with covering group* $\mathbb{Z}_7$*.*

Chapter 8

Regular maps with simple underlying graphs

In this Chapter, we show that for over 83% of all positive integer g, there exists an orientably-regular maps of genus g with simple underlying graph. We use a range of constructions, including semi-direct products, and other methods similar to those used for symmetric graphs in earlier Chapters. The maps we construct are all covers of a given base map, with abelian covering groups. Background on regular maps was given in Section 2.7.

8.1 Construction of helpful families of maps

In this Section, we construct several families of orientably-regular maps with simple underlying graph, which will help us prove our main theorem.

8.1.1 Family A: Orientably-regular maps of type $\{3n, 4\}$

It is well known that a regular octahedron can be viewed as a regular embedding of its 1-skeleton (which is a 4-valent graph of order 8) on the sphere, giving a Platonic regular map M, of type $\{3, 4\}$ and genus 0. The automorphism group of this map is $S_4 \times C_2$, with the S_4 preserving orientation.

It is also well-known that an infinite family of regular maps of type $\{3n, 4\}$ can be constructed as cyclic regular coverings of the octahedral map. (see [40] or [48] for example.) These maps can be constructed in a number of ways.

One way is by using semi-direct products, in a way similar to the approach taken in [16]: for any positive integer n, form the semi-direct product $G = NH \cong C_{3n} \rtimes S_4$ of a cyclic group $N = \langle\, w \mid w^{3n} = 1 \,\rangle$ of order $3n$ by the symmetric group $H = S_4 = \langle\, u, v \mid u^2 = v^4 = (uv)^3 = 1 \,\rangle$, with conjugation of N by H given by $w^u = w^{-1}$ and $w^v = w^{-1}$. In this group, define $x = wu$ and $y = v$; then $x^2 = y^4 = 1$ and $(xy)^3 = (wuv)^3 = w^3$, which has order n, so the subgroup of G generated by x and y has order $24n$ and is the rotation group of an orientably-regular map of characteristic $\chi = 24n/(3n) - 24n/2 + 24n/4 = 8 - 6n$ and genus $g = 3n - 3$.

The vertex-stabiliser (in the rotation group $\langle x, y\rangle$) is the cyclic subgroup of order 4 generated by $y = v$, and as this contains no non-trivial normal subgroup of $\langle x, y\rangle$, the underlying graph of the map is simple. (On the other hand, the face-stabiliser is the cyclic subgroup of order $3n$ generated by $xy = wuv$, which contains the cyclic normal subgroup generated by $(xy)^3 = w^3$, and so the underlying graph of the dual map has multiple edges, for $n > 1$.) Also each map is reflexible, since the involutory automorphism of $H = S_4$ inverting each of u and v extends to an involutory automorphism of G that inverts each of $x = wu$ and y (and centralises w).

Another way to construct these maps is to take the group Γ with presentation $\Gamma = \langle\, x, y \mid x^2 = y^4 = (xyxy^2)^2 = 1 \,\rangle$, and consider the normal subgroup N of index 24 in Γ generated by the element $z = (xy)^3$. Now the relation $(xyxy^2)^2 = 1$ can be re-written as $(yxyxy)^2 = 1$, which gives $(yx)^3 = (yxyxy)x = (y^{-1}xy^{-1}xy^{-1})x = (xy)^{-3}$, and it follows that conjugation by each of x and y^{-1} inverts the element $z = (xy)^3$. Hence in particular, z generates a cyclic normal subgroup K in Γ, of index 24, with quotient $\Gamma/K = \langle\, x, y \mid x^2 = y^4 = (xyxy^2)^2 = (xy)^3 = 1 \,\rangle \cong S_4$. Next, by Reidemeister-Schreier theory, or by use of the `Rewrite` command in Magma [3], we find that the subgroup K is free of rank 1, and hence infinite. Thus for each positive integer n, we can factor out the normal subgroup generated by z^n, and get an extension of C_n by S_4, just as above. The resulting orientably-regular map has type $\{3n, 4\}$ and genus $g = 3n - 3$, and is reflexible (because its rotation group admits an involutory automorphism that inverts the images of the two generators x and y), and its underlying graph is simple, because the cyclic subgroup of order 2 generated by the image of y^2 is not normal in the rotation group.

A presentation for the rotation group of the nth map in this family is simply

$$\langle\, r, s \mid (rs)^2 = s^4 = (r^2 s^{-1})^2 = r^{3n} = 1 \,\rangle,$$

which can be obtained by taking $r = xy$ and $s = y^{-1}$. Similarly, a presentation for the full automorphism group is

$$\langle\, a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = (bc)^4 = (ababcb)^2 = (ab)^{3n} = 1 \,\rangle,$$

with the rotation group generated by $r = ab$ and $s = bc$ as usual. The first few members of this family (after the first one, of genus 0) are the duals of the maps named R3.4, R6.3, R9.11, R12.1 and R15.5 in [15].

Thus we have the following:

Proposition 8.1.1 *For every positive integer n, there exists a reflexible regular map of type $\{3n, 4\}$ and genus $3n - 3$, with simple underlying graph (and rotation group an extension of C_n by S_4).*

8.1.2 Family B: Orientably-regular maps of type $\{4n, 4\}$

In this case we can start with the toroidal map of type $\{4, 4\}_4$ (see [24]), with rotation group $H = \langle\, u, v \mid u^2 = v^4 = (uv)^4 = [u, v]^2 = 1 \,\rangle$, which is an extension of $C_2 \times C_2$ by D_4, of order 32. and construct an infinite family of cyclic regular coverings of this.

As in the previous sub-section, for any positive integer n we can form the semi-direct product $G = NH \cong C_{4n} \rtimes H$ of a cyclic group $N = \langle\, w \mid w^{4n} = 1 \,\rangle$ of order $4n$ by the above group H, with conjugation of N by H given by $w^u = w^{-1}$ and $w^v = w^{-1}$. Again we may define $x = wu$ and $y = v$ in this group, and this time we find $x^2 = y^4 = 1$ while $(xy)^4 = (wuv)^4 = w^4$, which has order n, so the subgroup of G generated by x and y has order $32n$ and is the rotation group of an orientably-regular map of characteristic $\chi = 32n/(4n) - 32n/2 + 32n/4 = 8 - 8n$ and genus $g = 4n - 3$.

Again the vertex-stabiliser $\langle y \rangle$ (of order 4) contains no non-trivial normal subgroup of $\langle x, y \rangle$, so the underlying graph of the map is simple, while the face-stabiliser $\langle xy \rangle$ (of order $4n$) contains the cyclic normal subgroup generated by $(xy)^4 = w^4$, and so the underlying graph of the dual map has multiple edges, for $n > 1$. Also each map is reflexible, since the automorphism of H inverting each of u and v extends to an involutory automorphism of G that inverts each of $x = wu$ and y (and centralises w).

For an alternative construction, take the normal subgroup N of index 32 generated by $z = (xy)^4$ in the group $\Phi = \langle\, x, y \mid x^2 = y^4 = xyxy^2xy^{-1}xy^2 = 1 \,\rangle$. In this group, the relation $xyxy^2xy^{-1}xy^2 = 1$ can be re-written as $yxyxy = y^{-1}xyxy^{-1}$, giving

$$\begin{aligned}(yx)^4 &= yx(yxyxy)x = yx(y^{-1}xyxy^{-1})x = (yxy^{-1}xy)xy^{-1}x = (y^{-1}xyxy^{-1})^{-1}xy^{-1}x \\ &= (yxyxy)^{-1}xy^{-1}x = (y^{-1}xy^{-1}xy^{-1})xy^{-1}x = (y^{-1}x)^4 = (xy)^{-4},\end{aligned}$$

from which it follows that $z = (xy)^4$ is inverted under conjugation by x and y. Accordingly, z generates a cyclic normal subgroup K of index 32 in Φ, with quotient $\Phi/K = \langle\, x, y \mid x^2 = y^4 = xyxy^2xy^{-1}xy^2 = (xy)^4 = 1 \,\rangle \cong H$, and by Reidemeister-Schreier theory, or by use of the `Rewrite` command in MAGMA [3], we find that K is infinite.

Again for each positive integer n, we can factor out the normal subgroup generated by z^n, and get an extension of C_n by H. The resulting orientably-regular map has type $\{4n, 4\}$ and genus $g = 4n - 3$, and is reflexible (since $\langle x, y \rangle$ admits an involutory automorphism that inverts x and y and centralises $z = (xy)^4$), and its underlying graph is simple (since $\langle y^2 \rangle$ is not normal in the rotation group).

In fact the group we obtain in this way has the same presentation (in terms of the images of x and y) as the group defined using the semi-direct product construction, since in the former case, the relations $uw = w^{-1}u$ and $vw = w^{-1}v$ imply that

$$xyxy^2xy^{-1}xy^2 = wuvwuv^2wuv^{-1}wuv^2 = w^{1+1-1-1}uvuv^2uv^{-1}uv^2 = uvuv^2uv^{-1}uv^2,$$

which is trivial. We will exploit this fact in the next sub-section.

Meanwhile we have the following:

Proposition 8.1.2 *For every positive integer n, there exists a reflexible regular map of type $\{4n, 4\}$ and genus $4n-3$, with simple underlying graph (and rotation group an extension of C_n by the rotation group of the toroidal map of type $\{4,4\}_4$).*

A presentation for the rotation group of the nth map in this family is simply

$$\langle\, r, s \mid (rs)^2 = s^4 = (rs^{-1})^2(r^{-1}s)^2 = r^{4n} = 1\,\rangle,$$

which again can be obtained by taking $r = xy$ and $s = y^{-1}$.

Note that the relation $xyxy^2xy^{-1}xy^2 = 1$ can be rewritten as $1 = xy^2xy^{-1}xy^2xy$, and when we take $r = xy$ and $s = y^{-1}$, this gives $1 = rs^{-1}rs^{-2}rs^{-1}r = (rs^{-1})^2(s^{-1}r)^2$. On the other hand, it can also be rewritten as $1 = xy^2xyxy^2xy^{-1} = xy^2xyxy^{-2}xy^{-1} = (xy^2)^2(y^{-1}xy^{-1})^2$, which becomes $1 = (rs^{-1})^2(r^{-1}s)^2$. Hence we find that $(r^{-1}s)^2 = (rs^{-1})^2 = (s^{-1}r)^2$, and in particular, $(rs^{-1})^4 = (r^{-1}s)^2(r^{-1}s)^2 = (r^{-1}s)^2(s^{-1}r)^2 = 1$. Again we will use this in the next sub-section.

A presentation for the full automorphism group is

$$\langle\, a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = (bc)^4 = (abcb)^2(babc)^2 = (ab)^{4n} = 1\,\rangle,$$

with the rotation group generated by $r = ab$ and $s = bc$ as usual. The first few members of the resulting family (after the first one, of genus 1) are the duals of the maps named R5.6, R9.10 and R13.4 in [15].

8.1.3 Family C: Orientably-regular maps of type $\{8n, 4\}$

Take the group $\Phi = \langle\, x, y \mid x^2 = y^4 = xyxy^2xy^{-1}xy^2 = 1\,\rangle$ from the previous sub-section, and for any positive integer n, factor out the cyclic normal subgroup K generated by $z^{2n} = (xy)^{8n}$. The resulting quotient is the same as the one obtained using the semi-direct product construction, and in the group $G \cong C_{8n} \rtimes H$ from that, we have

$$[x, y^2] = xy^2xy^2 = wuv^2wuv^2 = w^{1-1}uv^2uv^2 = (uv^2)^2,$$

which is an involution that centralises every power of w. Since the element $(uv^2)^2$ is central in $H = \langle u, v\rangle$, this involution is centralised also by each of x $(= wu)$ and y $(= v)$, and therefore central in G. Also the element $(xy)^{4n}$ is the unique involution in the cyclic normal subgroup generated by $(xy)^4$, and hence is central in G as well.

These two central involutions $[x, y^2] = (uv^2)^2$ and $(xy)^{4n}$ are distinct, so their product $[x, y^2](xy)^{4n}$ is a third central involution. Taking the quotient of $\langle x, y\rangle$ of the central subgroup of order 2 generated by this third involution, we obtain a group of order $32(2n)/2 = 32n$, generated by two elements of orders 2 and 4 with product of order $8n$. This gives an orientably-regular map of type $\{8n, 4\}$, with characteristic $\chi = 32n/(8n) - 32n/2 + 32n/4 = 4 - 8n$, and genus $g = 4n - 1$.

Again the map is reflexible, since the automorphism inverting the two generators of the earlier group centralises both $[x, y^2]$ and $(xy)^{4n}$, and therefore centralises their product as well. Also the underlying graph of the map is simple (for the same reasons as before).

Thus we have the following:

Proposition 8.1.3 *For every positive integer n, there exists a reflexible regular map of type $\{8n, 4\}$ and genus $4n - 1$, with simple underlying graph.*

A presentation for the rotation group of the nth map in this family is simply

$$\langle\, r, s \mid (rs)^2 = s^4 = (rs^{-1})^2(r^{-1}s)^2 = sr^{-1}sr^{-1+4n} = r^{8n} = 1\,\rangle,$$

although the last relation is redundant, since the fourth relation $sr^{-1}sr^{-1+4n} = 1$ gives $r^{4n} = (rs^{-1})^2$ and therefore $r^{8n} = (r^{4n})^2 = (rs^{-1})^4 = 1$, by what we observed in the previous sub-section. A presentation for the full automorphism group is

$$\langle\, a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = (bc)^4 = (abcb)^2(babc)^2 = bcbabc(ab)^{-1+4n} = 1\,\rangle,$$

with the rotation group generated by $r = ab$ and $s = bc$ as usual.

The first few members of the resulting family are the duals of the maps named R3.5, R7.3, R11.2 and R15.6 in [15]. Also the carrier surfaces of these maps can easily be shown to be the same as those considered by Kulkarni in [41]. On the other hand, the regular maps with the same parameters obtainable from Maclachlan's surface actions in [48] (which include the maps named R3.6, R7.4, R11.3 and R15.7 in [15]) do not have simple underlying graphs.

8.1.4 Family D: Orientably-regular maps of type $\{6n, 6\}$

In this sub-section we exhibit three families of orientably-regular maps of type $\{6n, 6\}$ and genus $6n-2$, for $n \equiv 0, 1$ and 2 mod 3 respectively. The maps in the first family are chiral, while those in the second and third families are reflexible.

We begin with the second and third families. For $n \not\equiv 0$ mod 3, we take the group Σ with presentation

$$\Sigma = \langle\, x, y \mid x^2 = y^6 = xyxy^{-2}xy^{-1}xy^2 = 1 \,\rangle,$$

and consider the two elements $u = (xy)^2$ and $v = (xy^2)^2$.

Note that the relation $xyxy^{-2}xy^{-1}xy^2 = 1$ is equivalent (by inversion and conjugation) to $xyxy^2xy^{-1}xy^{-2} = 1$. Also this relation implies that $xyx = y^{-2}xyxy^2$, and hence that $(yx)^2 = y(xyx) = y^{-1}xyxy^2 = (y^{-1}xy^{-1}x)(xy^2xy^2) = u^{-1}v$. Similarly, the relation $xyxy^{-2}xy^{-1}xy^2 = 1$ gives $xy^2x = y^{-1}xy^2xy$, and so $(y^2x)^2 = y^2(xy^2x) = yxy^2xy = y(y^{-1}xy^2xy)y = xy^2xy^2 = v$. From these observations, we deduce that

$$u^x = (yx)^2 = u^{-1}v, \qquad u^y = y^{-1}xyxy^2 = u^{-1}v \text{ (as above)},$$
$$v^x = (y^2x)^2 = xy^2xy^2 = v, \qquad v^y = y^{-1}xy^2xy^3 = (y^{-1}xy^2xy)y^2 = xy^2xy^2 = v.$$

In particular, the subgroup N generated by u and v is normal in Σ. The quotient Σ/N is generated by the (involutory) images of the elements x and xy, and hence is dihedral of order 12, so N has index 12 in Σ. Also v is centralised by both generators of Σ, and hence by u, and therefore N is abelian. Moreover, $u^3 = (xy)^6$, while

$$\begin{aligned} v^3 &= (xy^2)^2(xy^2)^2(xy^2)^2 = (xy^2)^2(y^2x)^2(xy^2)^2 = xy^2xy^{-2}xy^{-2}xy^2 \\ &= y^{-1}xy^2xyy^{-2}xy^{-2}xy^2 = y^{-1}(xy^2xy^{-1}xy^{-2}xy)y = 1. \end{aligned}$$

Thus Σ is isomorphic to an extension of $\mathbb{Z} \oplus Z_3$ by D_6. Also $(u^3)^x = (u^3)^y = (u^{-1}v)^3 =$

$u^{-3}v^3 = u^{-3}$, and therefore the element u^3 generates a cyclic normal subgroup of Σ, with index 9 in N and index 108 in Σ. By Reidemeister-Schreier theory, this subgroup is infinite. (Some of these things can also be verified with the help of Magma.)

Now for any positive integer n, we may factor out the normal subgroup generated by u^{3n}, and get a quotient of order $108n$ that is the rotation group of an orientably-regular map of type $\{6n, 6\}$, characteristic $108n/(6n) - 108n/2 + 108n/6 = 18 - 36n$, and genus $18n-8$. This map is reflexible, since the group Σ admits an automorphism θ which inverts x and y, and takes $u = (xy)^2$ to $(xy^{-1})^2 = x(y^{-1}x)^2x = (u^{-1})^x = (u^{-1}v)^{-1} = uv^{-1}$ and $v = (xy^2)^2$ to $(xy^{-2})^2 = x(y^{-2}x)^2x = (v^{-1})^x = v^{-1}$, and this automorphism takes u^3 to $(uv^{-1})^3 = u^3$, so preserves the quotient $\Sigma/\langle u^{3m}\rangle$.

But also the normal subgroup $N/\langle u^{3n}\rangle$ of this quotient has a characteristic abelian subgroup of order 9 generated by $u^n = (xy)^{2n}$ and v (each of order 3).

If $n \equiv 1 \bmod 3$, say $n = 3d - 2$, then the element $u^n v$ generates a cyclic normal subgroup of Σ, since $(u^n v)^y = u^n v$ and

$$(u^n v)^x = (u^{-1}v)^n v = u^{-n}v^{n+1} = (u^n v)^{-1}v^{n+2} = (u^n v)^{-1}v^{3d} = (u^n v)^{-1}.$$

When we factor out this subgroup of order 3, we get a quotient of order $36n$ that is the rotation group of an orientably-regular map of type $\{6n, 6\}$, characteristic $36n/(6n) - 36n/2 + 36n/6 = 6 - 12n$, and genus $6n - 2$. The underlying graph of the map is simple for all such $n > 1$, since neither $\langle y^2\rangle$ nor $\langle y^3\rangle$ is normal in the rotation group, but that does not happen for the case $n = 1$ (since in that case $v = (xy^2)^2$ becomes trivial, so $(y^2)^x = y^{-2}$, which then makes $\langle y^2\rangle$ normal in $\langle x, y\rangle$). Also the map is reflexible (for all n), since the inverting automorphism θ of Σ takes $u^n v$ to $(uv^{-1})^n v^{-1} = u^n v^{-n-1} = (u^n v)v^{-n-2} = (u^n v)v^{-3d} = u^n v$.

Similarly, if $n \equiv 2 \bmod 3$, say $n = 3d + 2$, then the element $u^n v^{-1}$ generates a cyclic normal subgroup of Σ, since $(u^n v^{-1})^y = u^n v^{-1}$ and

$$(u^n v^{-1})^x = (u^{-1}v)^n v^{-1} = u^{-n}v^{n-1} = (u^n v^{-1})^{-1}v^{n-2} = (u^n v^{-1})^{-1}v^{3d} = (u^n v^{-1})^{-1},$$

and when we factor out this subgroup of order 3, again we get a quotient of order $36n$ that is the rotation group of an orientably-regular map of type $\{6n, 6\}$, characteristic

$36n/(6n) - 36n/2 + 36n/6 = 6 - 12n$, and genus $6n-2$, with simple underlying graph. Again the map is reflexible, since the inverting automorphism θ of Σ takes $u^n v^{-1}$ to $(uv^{-1})^n v = u^n v^{-n+1} = (u^n v^{-1})v^{-n+2} = (u^n v^{-1})v^{-3d} = u^n v^{-1}$.

Thus we get two families of reflexible maps with the desired properties. The first few members of these two families are the duals of the maps named R10.16, R22.9, R28.21, R40.5, R40.5 and R46.23 in Conder's website of orientably-regular maps of genus 2 to 101 (see [13]).

For given $n \not\equiv 0$ mod 3, a presentation for the rotation group of the map is given by

$$\langle\, r, s \mid (rs)^2 = s^6 = r^2 s^3 r s^2 r s^{-1} = r^{2n}(rs^{-3})^{\pm 2} = 1 \,\rangle,$$

with the final superscript in the last relator being $+2$ for all $n \equiv 1$ mod 3, and -2 for all $n \equiv 2$ mod 3. Correspondingly, a presentation for the full automorphism group is

$$\langle\, a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = (bc)^6 = (ab)^2(bc)^3 acbcabcb = (ab)^{2n}(abcbcbcb)^{\pm 2} = 1 \,\rangle,$$

with the rotation group generated by $r = ab$ and $s = bc$ as usual.

For the case $n \equiv 0$ mod 3, the above approach does not work; indeed there is no such map (with the above parameters and with simple underlying graph) when $n = 3, 6, 9, 12$ or 15, for example. Instead, we start with a different group Λ, with presentation

$$\Lambda = \langle\, x, y \mid x^2 = y^6 = (xyxy^2)^2 = (xy^2)^2(xy^{-2})^2 = 1 \,\rangle.$$

The relations in this group imply $yxy^2xy = xy^{-2}x$, and hence $1 = xy^2xy^2xy^{-2}xy^{-2}$ $= xy(yxy^2xy)y^3xy^{-2} = xy(xy^{-2}x)y^3xy^{-2}$, which gives us $xy^{-2}xy^3 = y^{-1}xy^2x$.

Now take $u = xy^{-2}xy^2$ and $v = (xy)^3(xy^{-1})^3$. Then we find $u^x = y^{-2}xy^2x = u^{-1}$ and $v^x = (yx)^3(y^{-1}x)^3 = v^{-1}$, while $u^y = y^{-1}xy^{-2}xy^3 = y^{-1}y^{-1}xy^2x = u^{-1}$, and $v^{y^{-1}} = yvy^{-1} = (yx)^3(y^{-1}x)^3 = v^{-1}$, from which it follows that also $v^y = v^{-1}$. Hence both u and v are inverted under conjugation by each of x and y, and in particular, the subgroup N generated by u and v is normal in Λ. Moreover, since v is centralised by y^2, we find that v is centralised by $xy^{-2}xy^2 = u$, and so N is abelian.

Also $(xy)^6 = (xy)^3(xy^{-1})^3(yx)^3(xy)^3 = vyxyxy^2xyxy = vyx(xy^{-2}x)xy = v$. It follows from this (with the help of Magma [3] if necessary) that the quotient Λ/N has order 36, and then by Reidemeister-Schreier theory, we find that N is isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}$, with the $\mathbb{Z}_3$ generated by u, and with v infinite. (Indeed

$$u^3 = xy^{-2}xy^2xy^{-2}xy^2xy^{-2}xy^2 = xy^{-2}xy^2(y^2xy^{-2}x)xy^{-2}xy^2 = xy^{-2}xy^{-2}xy^2xy^2 = 1.)$$

It follows that the subgroup generated by v itself is normal in N, with index 108, and when we factor out the subgroup generated by v^n for any positive integer n, we get a quotient of order $108n$ in which the image of xy has order $6n$. Accordingly, this quotient is again the rotation group of an orientably-regular map of type $\{6n, 6\}$, characteristic $108n/(6n) - 108n/2 + 108n/6 = 18 - 36n$, and genus $18n - 8$. Also this map is reflexible, since the rotation group admits an automorphism that inverts the images of x and y, and then takes the image of $u = xy^{-2}xy^2$ to the image of $xy^2xy^{-2} = y^2x(xy^{-2}xy^2)xy^{-2} = u^{xy^{-2}} = u^{-1}$, and similarly the image of $v = (xy)^3(xy^{-1})^3$ to the image of $(xy^{-1})^3(xy)^3 = (y^{-1}x)^3((xy)^3(xy^{-1})^3)(xy)^3 = v^{(xy)^3} = v$.

But if n is divisible by 3, say $n = 3m$, and we factor out the normal subgroup generated by uv^m, then we get a different quotient, of order $108m = 36n$, in which the image of xy has order $18m$, since the image of $v = (xy)^6$ has order $3m$ (with the image of v^m coinciding with the image of u^{-1}). This gives an orientably-regular map of type $\{18m, 6\}$, characteristic $108m/(18m) - 108m/2 + 108m/6 = 6 - 36m$, and genus $18m - 2$, but the map is no longer reflexible, since any automorphism that inverts the images of x and y must take the image of uv^m (which is trivial) to $u^{-1}v^m$ (which is not). On the other hand, the underlying graph is simple, since the images of the subgroups generated by y^2 and y^3 are not normal.

Thus we have a family of chiral maps of type $\{6n, 6\}$ and genus $6n - 2$, for n divisible by 3, all with simple underlying graphs.

A presentation for the rotation group (in the case $n = 3m$) is

$$\langle\, r, s \mid (rs)^2 = s^6 = (r^2s^{-1})^2 = (rs^{-1})^2(rs^3)^2 = rs^3rs^{-1}r^{n/3} = 1\,\rangle,$$

again obtainable by taking $r = xy$ and $s = y^{-1}$.

The first few members of the resulting family are the duals of the maps named C16.1, C34.1, C52.1, C70.1 and C88.1 in Conder's website of orientably-regular maps of genus 2 to 101 (see reference [13]).

Thus we have the following:

Proposition 8.1.4 *For every integer $n > 1$, there exists an orientably-regular map of type $\{6n, 6\}$ and genus $6n-2$, with simple underlying graph. In fact if $n \not\equiv 0$ mod 3, then there exists such a map that is reflexible, while if $n \equiv 0$ mod 3, there exists such a map that is chiral.*

Note that the four families of maps we have described so far have genera congruent to 0, 1, 3, 4, 5, 6, 7, 9, 10 and 11 mod 12. For the remaining two congruence classes (namely 2 and 8 mod 12), maps described in the next section will be helpful.

8.2 Orientably-regular maps of type $\{6, 6\}$

A well-known family of regular maps of type $\{6, 6\}$ was introduced by Sherk [55] in the 1960s, using a construction based on the automorphism groups of the toroidal maps of type $\{3, 6\}$. The maps in Sherk's family are indexed by ordered pairs (α, β) of non-negative integers, with rotation group of the form

$$G_{\alpha,\beta} = \langle\, r, s \mid (rs)^2 = r^6 = s^6 = (r^2s^{-1})^2 = (r^{-2}s^{-2})^\alpha(r^2s^2)^\beta = 1 \,\rangle$$

for each such pair $(\alpha, \beta) \neq (0, 0)$. This group has order $12k$ where $k = \alpha^2 + \alpha\beta + \beta^2$, and the corresponding map (which we will denote by $\mathcal{S}_{(\alpha,\beta)}$) has genus $k + 1$, and the map $\mathcal{S}_{(\alpha,\beta)}$ is reflexible if and only if $\alpha\beta(\alpha - \beta) = 0$. For example, $\mathcal{S}_{(0,1)}$, $\mathcal{S}_{(1,0)}$, $\mathcal{S}_{(0,2)}$, $\mathcal{S}_{(1,1)}$, $\mathcal{S}_{(2,0)}$, $\mathcal{S}_{(1,2)}$ and $\mathcal{S}_{(2,1)}$ are respectively the maps R2.5, R2.5, R5.10, the dual of R4.8, R5.10, the dual of C8.1, and the mirror image of the dual of C8.1 in [15].

The underlying graph of $\mathcal{S}_{(\alpha,\beta)}$ is simple except when $\alpha + \beta \leq 2$, for in those cases $\langle s^3\rangle$ or $\langle s^2\rangle$ is normal in $G_{\alpha,\beta}$ (while no such degeneracy occurs when $\alpha + \beta > 2$).

Hence the Sherk family gives orientably-regular maps of genus g (and type $\{6,6\}$) with simple underlying graphs, for all g expressible in the form $\alpha^2+\alpha\beta+\beta^2+1$ where α and β are non-negative integers with $\alpha+\beta>2$. This set of possible genera is 'quadratic' rather than 'linear', and so asymptotically is less dense than the arithmetic progressions of genera provided by the families in the sub-sections above, but nevertheless it covers some genera that the previous families do not, such as 8, 14, 20, 26, 32, 38, 44, 50, 62, 68, 74, 80, 92 and 98 (but not 56 or 86).

On the other hand, the underlying graph of the dual of $\mathcal{S}_{(\alpha,\beta)}$ is never simple, for the relation $(r^2s^{-1})^2=1$ can be rewritten as $(r^3(rs)^{-1})^2=1$, which implies that conjugation by the involution rs inverts r^3, and hence $\langle r^3\rangle$ is always normal in $G_{\alpha,\beta}$.

Below we will show that there exist other families of orientably-regular maps of type $\{6,6\}$ with simple underlying graph that not only have genera covering some of the remaining gaps in the genus spectrum, but also have a dual with simple underlying graph as well. These will be obtained as covers of a particular map of type $\{6,6\}$ and genus 2, namely the map R2.5 in [15], which has rotation group $C_2\times C_6$.

One way is via a semi-direct product construction, similar to the one used before. Let n be any odd positive integer with the property that some unit t in $\mathbb{Z}_n$ has multiplicative order 6, and form the semi-direct product $G=NH$ of a cyclic group $N=\langle\, w \mid w^n=1\,\rangle$ of order n by the group $H=C_2\times C_6=\langle\, u,v \mid u^2=v^6=[u,v]=1\,\rangle$, with conjugation of N by H given by $w^u=w^{-1}$ and $w^v=w^t$. In this group, define $x=u$ and $y=wv$. Then $xyxy^{-1}=uwvuv^{-1}w^{-1}=w^{-2}uvuv^{-1}=w^{-2}$, which generates N, and it follows that x and y generate G. Clearly x has order 2, while the orders of $y=wv$ and $xy=uvw=w^{-1}uv$ are multiples of 6.

In fact, $y^6=(wv)^6=w^{1+t+t^2+t^3+t^4+t^5}$ and $(xy)^6=(uwv)^6=w^{-(1-t+t^2-t^3+t^4-t^5)}$, so y and xy have order 6 precisely when $1+t+t^2+t^3+t^4+t^5$ and $1-t+t^2-t^3+t^4-t^5$ are both congruent to 0 mod n. If these two conditions hold, then subtracting one from the other gives $2(1+t^2+t^4)\equiv 0 \bmod n$, and hence $1+t^2+t^4\equiv 0 \bmod n$. Conversely, if $1+t^2+t^4\equiv 0 \bmod n$, then $1+t+t^2+t^3+t^4+t^5=(1+t)(1+t^2+t^4)\equiv 0 \bmod n$

and $1 - t + t^2 - t^3 + t^4 - t^5 = (1-t)(1+t^2+t^4) \equiv 0 \bmod n$, so this group G can be constructed precisely when $1 + t^2 + t^4 \equiv 0 \bmod n$.

In that case, G is the rotation group of an orientably-regular map of type $\{6,6\}$, characteristic $\chi = 12n/6 - 12n/2 + 12n/6 = -2n$ and genus $n+1$. Moreover, it is easy to verify that $xy^{-2}xy^2 = w^{2(t+t^2)}$ and $xy^{-3}xy^3 = w^{2(t+t^2+t^3)}$, and it follows that the underlying graph of this map is simple if and only if $t+t^2$ and $t+t^2+t^3$ are non-zero mod n. Similarly, $xy^{-2}xy^2 = w^{2(t-t^2)}$ and $xy^{-3}xy^3 = w^{2(t-t^2+t^3)}$, so the underlying graph of the dual map is simple if and only if $t-t^2$ and $t-t^2+t^3$ are non-zero mod n. Also if the map has simple underlying graph then it must be chiral, since any automorphism of G that inverts each of x and y must conjugate $w^{-2} = xyxy^{-1}$ to $xy^{-1}xy = w^{-2}$, and therefore centralises w, but on the other hand, it must also conjugate $w^{2(t+t^2)} = xy^{-2}xy^2$ to $xy^2xy^{-2} = w^{-2(1+t^5)}$, so that $t+t^2 \equiv -(1+t^5) = -(t^2+t)t^5 \bmod n$, which is impossible.

Examples include some of the Sherk maps (of genus 8, 14, 20, 32, 38, 44, 50, 62, 68, 74, 80, 92 and 98 for example), but also others for which both the map and its dual have simple underlying graph, such as the maps C22.2, C40.2, C58.2, C92.1 and C94.2 from [13], arising when $(n,t) = (21,10)$, $(39,4)$, $(57,46)$, $(91,30)$ and $(93,37)$.

Some other classes can be constructed as follows:

Let Ψ be the group with presentation $\Psi = \langle\, x, y \mid x^2 = y^6 = (xy)^6 = 1\,\rangle$.

Then the derived group Ψ' of Ψ (which is generated by the conjugates of the element $[x,y]$) has index 12 in Ψ, with quotient Ψ/Ψ' isomorphic to $C_2 \times C_6$, which is the rotation group of the regular map R2.5. In fact, by Reidemeister-Schreier theory (or by using the `Rewrite` command in Magma), we find that the subgroup Ψ' is generated by the four elements

$$w_1 = xy^{-1}xy, \quad w_2 = xyxy^{-1}, \quad w_3 = xy^{-2}xy^2, \quad w_4 = xy^2xy^{-2},$$

subject to a single defining relation $w_2^{-1}w_4w_3^{-1}w_1w_2w_4^{-1}w_3w_2^{-1} = 1$. Note that the fifth commutator of the form $xy^{-i}xy^i$, namely $w_5 = xy^{-3}xy^3 = xy^3xy^{-3}$, is easily expressible as a product $w_3w_1^{-1}w_2^{-1}w_4$. In particular, the above generators and single defining relation for Ψ' show that the abelianisation $\Psi'/(\Psi')'$ of Ψ' is free abelian of rank 4.

Now let us move to the quotient $\Psi/(\Psi')'$ of Ψ, which we will call G, and denote its derived subgroup $\Psi'/(\Psi')'$ by K. Then G is an extension of the free abelian subgroup $K \cong \mathbb{Z}^4$ by $G/K \cong C_2 \times C_6$. Also (for notational convenience) let us keep the same symbols x and y as the generators of G, and the same symbols w_1 to w_4 as the generators of K.

Then the action of the generators x and y by conjugation on the generators w_i of K may be given as follows:

$$\begin{aligned} {w_1}^x &= y^{-1}xyx = w_1^{-1}, & {w_2}^x &= yxy^{-1}x = w_2^{-1}, \\ {w_3}^x &= y^{-2}xy^2x = w_3^{-1}, & {w_4}^x &= y^2xy^{-2}x = w_4^{-1}, \end{aligned}$$

and

$$\begin{aligned} {w_1}^y &= y^{-1}xy^{-1}xy^2 = (y^{-1}xyx)(xy^{-2}xy^2) = w_1^{-1}w_3, \\ {w_2}^y &= y^{-1}xyx = w_1^{-1}, \\ {w_3}^y &= y^{-1}xy^{-2}xy^3 = (y^{-1}xyx)(xy^{-3}xy^3) = w_1^{-1}w_3w_1^{-1}w_2^{-1}w_4, \\ {w_4}^y &= y^{-1}xy^2xy^{-1} = (y^{-1}xyx)(xyxy^{-1}) = w_1^{-1}w_2. \end{aligned}$$

Accordingly, the generators x and y induce linear transformations of the free abelian group $K \cong \mathbb{Z}^4$ as follows:

$$x \mapsto \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad y \mapsto \begin{pmatrix} -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ -2 & -1 & 1 & 1 \\ -1 & 1 & 0 & 0 \end{pmatrix}.$$

These matrices generate a group isomorphic to $G/K \cong C_2 \times C_6$,

They can be reduced mod m for any positive integer m, giving the corresponding action of x and y on the group $K/K^{(m)}$, where $K^{(m)}$ is the (characteristic) subgroup of K generated by the mth powers of all the w_i. Such actions can be used to consider subgroups of finite index in K that are invariant under the action of x and y (or in other words, subgroups of K that are normal in G with finite quotient), just as we did in Chapters 3 to 7 for regular covers of symmetric cubic graphs.

In particular, when m is odd, we can change the basis of K from $\{w_1, w_2, w_3, w_4\}$ to $\{z_1, z_2, z_3, z_4\}$ where

$$z_1 = w_1 w_4^{-1}, \quad z_2 = w_2 w_3^{-1}, \quad z_3 = w_1 w_2^{\frac{m+1}{2}} w_3^{\frac{m-1}{2}}, \quad z_4 = w_1^{\frac{m+1}{2}} w_2 w_4^{\frac{m-1}{2}},$$

and get new matrices representing x and y, as follows:

$$x \mapsto \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad y \mapsto \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

This shows that the group $K/K^{(m)}$ can be expressed as the direct sum of two G-invariant subgroups of rank 2, say U and V, generated by the images of $\{z_1, z_2\}$ and $\{z_3, z_4\}$ respectively. Factoring out U or V gives a quotient of G of order $12m^2$, which is the rotation group of an orientable-regular map of type $\{6, 6\}$, characteristic $12m^2/6 - 12m^2/2 + 12m^2/6 = -2m^2$ and genus $m^2 + 1$.

Now the automorphism of G that inverts x and y clearly interchanges $w_1 = xy^{-1}xy$ with $xyxy^{-1} = w_2$, and similarly interchanges w_3 with w_4, and hence interchanges z_1 with z_2, and z_3 with z_4, and it follows that each of the two resulting maps is reflexible.

One of these maps has simple underlying graph, while the other does not. In the quotient obtained by factoring U, the element $z_1 = w_1 w_4^{-1} = xy^{-1}xy(xy^2xy^{-2})^{-1} = xy^{-1}xy^3xy^{-2}x$ becomes trivial, and forces $1 = xy^3xy^{-2}xxy^{-1} = xy^3xy^{-3}$, so that y^3 is centralised by x, and hence the map obtained by factoring out U has multiple edges. On the other hand, this does not happen in the quotient obtained by factoring V. For example, when $m = 3$, these maps are R10.15 (for U) and its dual (for V).

In fact it is not difficult to see that U and V are interchanged by an automorphism that interchanges xy with y^{-1}, corresponding to map duality.

But furthermore, for some values of m, the G-invariant subgroups U and V are reducible, and we can factor out a larger (or smaller) subgroup L of $K/K^{(m)}$, and get the rotation group of an orientable-regular map of type $\{6, 6\}$ with simple underlying graph.

Similarly for m even, we can find other kinds of non-trivial proper G-invariant subgroups of $K/K^{(m)}$, and factor out those. Examples obtainable in this way for which both the primal and dual maps have simple underlying graph include R10.13, C17.3, R17.20, C22.2, R28.9, R37.23, C40.2, C49.4, R49.36, R49.37, C50.3 and C50.4 (see [13]), with the rotation groups of these examples being isomorphic to extensions by $G/K \cong C_2 \times C_6$ of $(C_3)^2$, $(C_4)^2$, $(C_2)^4$, C_{21}, $(C_3)^3$, $(C_6)^2$, C_{39}, $C_4 \times C_{12}$, $C_4 \times C_{12}$, $(C_2)^3 \times C_6$, $C_7 \times C_7$ and $C_7 \times C_7$, respectively.

8.3 Main theorem

The families of orientable regular maps with simple underlying graphs presented in Section 8.1 give us the following:

Theorem 8.3.1 *For every positive integer $g \equiv 0, 1, 3, 4$ or 5 mod 6, there exists at least one orientably-regular map of genus g with simple underlying graph.*

Note that the family D (presented in sub-section 8.1.4) did not include a map of genus 4, but the map R4.2 of type $\{4, 5\}$ in [15] has simple underlying graph.

The above theorem covers 5/6 of all genera — indeed all except those congruent to 2 mod 6. Various families of maps of type $\{6, 6\}$ cover some of the remaining genera, as described in Section 8.2. It is also clear from the computational data obtained by Conder (see [13]) that there are numerous other families and examples.

In fact Conder has recently extended the determination of all orientably-regular maps up to genus 301, and there are no gaps at all in this range; in other words, for every positive integer $g \leq 301$, there exists at least one orientably-regular map of genus g with simple underlying graph.

For g in the range $2 \leq g \leq 301$ that are congruent to 2 mod 6, there is an orientably-regular map of type $\{6, 6\}$ with simple underlying graph except when $g = 2$, 86, 116, 146, 188, 206, 236, 254, 266 or 296, and in all those cases, there are maps of other types with simple underlying graph — such as the duals of R2.1 (of type $\{8, 3\}$) and R86.4 (of type $\{20, 6\}$).

This gives us some confidence to make the following conjecture, although the question of how to prove it remains wide open:

Conjecture 8.3.2 *For every non-negative integer g, there exists at least one orientably-regular map of genus g with simple underlying graph.*

Chapter 9

Concluding Remarks

The unique symmetric $(\mathbb{Z}_5)^3$-cover of the Petersen graph found in Section 6.2 is the largest known connected 3-valent graph of diameter 10. This graph was first discovered by Conder in his determination of all symmetric cubic graphs of order up to 2048, and was one of the motivations for the work in this thesis. In particular, it shows that the construction of regular covers can be used to produce large graphs of given degree and diameter. (See [60] for further details.) By our new approach introduced in Section 2.6, it is possible to find regular covers of symmetric graphs of small order with covering groups of small rank. In this way, we can produce other symmetric graphs with given degree and diameter, including some of the graphs in [61].

In fact, using this approach to find all the symmetric abelian regular covers of higher valency graphs is more difficult, because of the larger range of classes of 'universal' actions. But it is still possible to deal with particular examples, such as the complete graph K_5, which is a 2-arc-transitive, 4-valent graph of order 5. The full automorphism group of K_5 is isomorphic to the symmetric group S_5, which acts 2-arc-transitively, and the only arc-transitive subgroups are $\mathrm{AGL}(1,5)$ and A_5, which are 1-arc-regular and 2-arc-regular respectively. The vertex-stabilizer subgroup in $\mathrm{AGL}(1,5)$ is isomorphic to C_4. We can use the free product $G = C_2 \star C_4 = \langle x, y \mid x^2 = y^4 = 1\rangle$ as universal group, and find that there exists a normal subgroup N of G of index 20 such that $G/N \cong \mathrm{AGL}(1,5)$, and studying the structure of N can lead to the construction of regular covers of K_5.

We would like to conclude this thesis by mentioning a number of open problems and potential future projects arising from this topic:

1) Find other families of regular maps with simple underlying graphs.

2) Classify the arc-transitive abelian regular covers of the complete graph K_5 and the octahedron graph (which have 6 and 7 co-tree edges, respectively).

3) Can our approach be generalized to classify the vertex-transitive abelian regular covers of vertex-transitive cubic graph?

4) Construct higher valency symmetric graphs that are suitable candidates for entries in the degree-diameter table (for general graphs, symmetric graphs, vertex-transitive graphs, or Cayley graphs).

Appendix

In Chapter 7, we classified all of the symmetric abelian regular covers of the Heawood graph, by considering all possible lifts of the subgroup $C_7 \rtimes_3 C_6$ of the automorphism group, and we listed all the possibilities for an abelian covering group of finite prime power order in the summary Table 7.3.

In this Appendix, we give specific generators for the G_1-invariant subgroups of $K/K^{(m)}$ described in Table 7.3, when $m = 7^{\ell}$ for some ℓ.

Generating sets of G_1-invariant subgroups

Rank 1

- Two subgroups with invariants $[7^t]$ for each $t > 0$

.. one is of type $5_{d,e}$, with layer sequence $({}^{v}T_0, {}^{w-v}T_1, K_w)$, and is generated by the images of $\{x_1\}$;

.. one is of type $6_{d,e}$, with layer sequence $({}^{v}T_0, {}^{w-v}T_2, K_w)$, and is generated by the images of $\{x_2\}$;

Rank 2

- Two subgroups with invariants $[7^s, 7^t]$ for each $t > s \geq 1$

.. one is of type $7_{c,d,e}$, with layer sequence $({}^{u}T_0, {}^{v-u}T_1, {}^{w-v}T_3, K_w)$, and generated by the images of $\{x_1,\ x_2{}^{7^{t-s}}\}$;

.. one is of type $8_{c,d,e}$, with layer sequence $({}^{u}T_0, {}^{v-u}T_2, {}^{w-v}T_3, K_w)$, and generated by the images of $\{x_2,\ {x_1}^{7^{t-s}}\}$;

- One subgroup with invariant $[7, 7^t]$ for each $t > 1$

.. one is of type $9_{d,e}$, with layer sequence $({}^vT_0, {}^{w-v-1}T_1, T_4, K_w)$, and generated by the images of $\{x_1,\ {x_3}^{7^{t-1}}\}$;

- One subgroup with invariant $[7^t, 7^t]$ for each $t \geq 1$

.. one is of type $10_{d,e}$, with layer sequence $({}^vT_0, {}^{w-v}T_3, K_w)$, and generated by the images of $\{x_1,\ x_2\}$;

- One subgroup with invariant [7, 7]

.. one is of type 11_e, with layer sequence $({}^{w-1}T_0, T_4, K_w)$;

Rank 3

- Two subgroups with invariants $[7, 7^s, 7^t]$ for each pair (s, t) with $t > s > 1$

.. one is of type $12_{c,d,e}$, with layer sequence $({}^uT_0, {}^{v-u}T_1, {}^{w-v-1}T_3, T_5, K_w)$, and generated by the images of $\{x_1,\ {x_2}^{7^{t-s}},\ {x_3}^{7^{t-1}}\}$;

.. one is of type $13_{c,d,e}$, with layer sequence $({}^uT_0, {}^{v-u}T_2, {}^{w-v-1}T_3, T_5, K_w)$, and generated by the images of $\{x_2,\ {x_1}^{7^{t-s}},\ {x_3}^{7^{t-1}}\}$;

- Three subgroups with invariants $[7, 7, 7^t]$ for each $t > 1$

.. one is of type $14_{d,e}$, with layer sequence $({}^vT_0, {}^{w-v-1}T_1, T_5, K_w)$, and generated by the images of $\{x_1,\ {x_2}^{7^{t-1}},\ {x_3}^{7^{t-1}}\}$;

.. one is of type $15_{d,e}$, with layer sequence $({}^vT_0, {}^{w-v-1}T_2, T_5, K_w)$, and generated by the images of $\{x_2,\ {x_1}^{7^{t-1}},\ {x_3}^{7^{t-1}}\}$;

.. one is of type $16_{d,e}$, with layer sequence $({}^vT_0, {}^{w-v-1}T_1, T_6, K_w)$, and generated by the images of $\{x_1,\ {x_3}^{7^{t-1}},\ {x_4}^{7^{t-1}}\}$;

- One subgroup with invariant $[7, 7^t, 7^t]$ for each $t > 1$

.. one is of type $17_{d,e}$, with layer sequence $({}^vT_0, {}^{w-v-1}T_3, T_5, K_w)$, and generated by the images of $\{x_1,\ x_2,\ {x_3}^{7^{t-1}}\}$;

- Two subgroups with invariants [7, 7, 7]

.. each one is of type 18_e or 19_e, with layer sequences $({}^{w-1}T_0, T_5, K_w)$ and $({}^{w-1}T_0, T_6, K_w)$, respectively;

Rank 4

- Two subgroups with invariants $[7, 7, 7^s, 7^t]$ for each pair (s, t) with $t > s > 1$

.. one is of type $20_{c,d,e}$, with layer sequence $({}^{u}T_0, {}^{v-u}T_1, {}^{w-v-1}T_3, T_7, K_w)$, and generated by the images of $\{x_1,\ {x_2}^{7^{t-s}},\ {x_3}^{7^{t-1}},\ {x_4}^{7^{t-1}}\}$;

.. one is of type $21_{c,d,e}$, with layer sequence $({}^{u}T_0, {}^{v-u}T_2, {}^{w-v-1}T_3, T_7, K_w)$, and generated by the images of $\{x_2,\ {x_1}^{7^{t-s}},\ {x_3}^{7^{t-1}},\ {x_4}^{7^{t-1}}\}$;

- Three subgroups with invariants $[7, 7, 7, 7^t]$ for each $t > 1$

.. one is of type $22_{d,e}$, with layer sequence $({}^{v}T_0, {}^{w-v-1}T_1, T_7, K_w)$, and generated by the images of $\{x_1,\ {x_2}^{7^{t-1}},\ {x_3}^{7^{t-1}},\ {x_4}^{7^{t-1}}\}$;

.. one is of type $23_{d,e}$, with layer sequence $({}^{v}T_0, {}^{w-v-1}T_2, T_7, K_w)$, and generated by the images of $\{x_2,\ {x_1}^{7^{t-1}},\ {x_3}^{7^{t-1}},\ {x_4}^{7^{t-1}}\}$;

.. one is of type $24_{d,e}$, with layer sequence $({}^{v}T_0, {}^{w-v-1}T_1, T_8, K_w)$, and generated by the images of $\{x_1,\ {x_3}^{7^{t-1}},\ {x_4}^{7^{t-1}},\ {x_5}^{7^{t-1}}\}$;

- One subgroup with invariant $[7, 7, 7^t, 7^t]$ for each $t > 1$

.. one is of type $25_{d,e}$, with layer sequence $({}^{v}T_0, {}^{w-v-1}T_3, T_7, K_w)$, and generated by the images of $\{x_1,\ x_2,\ {x_3}^{7^{t-1}},\ {x_4}^{7^{t-1}}\}$;

- Two of type $[7, 7, 7, 7]$

.. each one is of type 26_e or 27_e, with layer sequences $({}^{w-1}T_0, T_7, K_w)$ and $({}^{w-1}T_0, T_8, K_w)$, respectively;

Rank 5

- Fourteen subgroups with invariants $[7, 7, 7, 7^s, 7^t]$ for each pair (s, t) with $t > s > 1$

.. seven are of type $28_{c,d,e}{}^*$, with layer sequence $({}^{u}T_0, {}^{v-u}T_1, {}^{w-v-1}T_3, T_9, K_w)$, and generated by the images of $\{x_1,\ {x_2}^{7^{t-s}}{x_6}^{\alpha 7^{t-1}},\ {x_3}^{7^{t-1}},\ {x_4}^{7^{t-1}},\ {x_5}^{7^{t-1}}\}$ for $0 \le \alpha \le 6$;

.. seven are of type $29_{c,d,e}{}^*$, with layer sequence $({}^{u}T_0, {}^{v-u}T_2, {}^{w-v-1}T_3, T_9, K_w)$, and generated by the images of $\{x_2 {x_6}^{\alpha 7^{t-1}},\ {x_1}^{7^{t-s}},\ {x_3}^{7^{t-1}},\ {x_4}^{7^{t-1}},\ {x_5}^{7^{t-1}}\}$ for $0 \leq \alpha \leq 6$;

• Fifteen subgroups with invariants $[7, 7, 7, 7, 7^t]$ for each $t > 1$

.. one is of type $30_{d,e}$, with layer sequence $({}^vT_0, {}^{w-v-1}T_1, T_9, K_w)$, and generated by the images of $\{x_1,\ {x_2}^{7^{t-1}},\ {x_3}^{7^{t-1}},\ {x_4}^{7^{t-1}},\ {x_5}^{7^{t-1}}\}$;

.. seven are of types $S_{d,e}$ for $31 \leq S \leq 37$, with layer sequences $({}^vT_0, {}^{w-v-1}T_1, T_\alpha, K_w)$, and generated by the images of $\{x_1,\ {x_3}^{7^{t-1}},\ {x_4}^{7^{t-1}},\ {x_5}^{7^{t-1}},\ (x_2^{\alpha-10}x_6)^{7^{t-1}}\}$ for $10 \leq \alpha \leq 16$;

.. seven are of type $38_{d,e}{}^*$, with layer sequence $({}^vT_0, {}^{w-v-1}T_2, T_9, K_w)$, and generated by the images of $\{x_2 {x_6}^{\alpha 7^{t-1}},\ {x_1}^{7^{t-1}},\ {x_3}^{7^{t-1}},\ {x_4}^{7^{t-1}},\ {x_5}^{7^{t-1}}\}$ for $0 \leq \alpha \leq 6$;

• Seven subgroups with invariants $[7, 7, 7, 7^t, 7^t]$ for each $t > 1$

.. seven are of type $39_{d,e}{}^*$, with layer sequence $({}^vT_0, {}^{w-v-1}T_3, T_9, K_w)$, and generated by the images of $\{x_1,\ x_2 {x_6}^{\alpha 7^{t-1}},\ {x_3}^{7^{t-1}},\ {x_4}^{7^{t-1}},\ {x_5}^{7^{t-1}}\}$ for $0 \leq \alpha \leq 6$;

• Eight subgroups with invariants $[7, 7, 7, 7, 7]$

.. each one is of type S_e for $40 \leq S \leq 47$, with layer sequences $({}^{w-1}T_0, T_\alpha, K_w)$ for $9 \leq \alpha \leq 16$ respectively;

Rank 6

• Two subgroups with invariants $[7, 7, 7, 7, 7^s, 7^t]$ for each pair (s, t) with $t > s > 1$

.. one is of type $48_{c,d,e}$, with layer sequence $({}^uT_0, {}^{v-u}T_1, {}^{w-v-1}T_3, T_{17}, K_w)$, and generated by the images of $\{x_1,\ {x_2}^{7^{t-s}},\ {x_3}^{7^{t-1}},\ {x_4}^{7^{t-1}},\ {x_5}^{7^{t-1}},\ {x_6}^{7^{t-1}}\}$;

.. one is of type $49_{c,d,e}$, with layer sequence $({}^uT_0, {}^{v-u}T_2, {}^{w-v-1}T_3, T_{17}, K_w)$, and generated by the images of $\{x_2,\ {x_1}^{7^{t-s}},\ {x_3}^{7^{t-1}},\ {x_4}^{7^{t-1}},\ {x_5}^{7^{t-1}},\ {x_6}^{7^{t-1}}\}$;

• Nine subgroups with invariants $[7, 7, 7, 7, 7, 7^t]$ for each $t > 1$

.. one is of type $50_{d,e}$, with layer sequence $({}^vT_0, {}^{w-v-1}T_1, T_{17}, K_w)$, and generated by the images of $\{x_1,\ {x_2}^{7^{t-1}},\ {x_3}^{7^{t-1}},\ {x_4}^{7^{t-1}},\ {x_5}^{7^{t-1}},\ {x_6}^{7^{t-1}}\}$;

.. one is of type $51_{d,e}$, with layer sequence $({}^vT_0, {}^{w-v-1}T_2, T_{17}, K_w)$, and generated by the images of $\{x_2,\ {x_1}^{7^{t-1}},\ {x_3}^{7^{t-1}},\ {x_4}^{7^{t-1}},\ {x_5}^{7^{t-1}},\ {x_6}^{7^{t-1}}\}$;

.. seven are of type $52_{d,e}{}^*$, with layer sequence $({}^vT_0, {}^{w-v-1}T_1, T_{18}, K_w)$, and generated by the images of $\{x_1 {x_8}^{\alpha 7^{t-1}}\} \cup V^{(7^{t-1})}$ for $0 \leq \alpha \leq 6$;

• Seven subgroups with invariants $[7^{t-1}, 7^{t-1}, 7^{t-1}, 7^{t-1}, 7^{t-1}, 7^t]$ for each $t > 2$

.. seven are of type $53_{d,e}{}^*$, with layer sequence $({}^vT_0, T_1, {}^{w-v-1}T_{18}, K_w)$, and generated by the images of $\{x_1 {x_8}^{\alpha 7^{t-1}}\} \cup V^{(7)}$ for $0 \leq \alpha \leq 6$;

• One subgroup with invariant $[7, 7, 7, 7, 7^t, 7^t]$ for each $t > 1$

.. one is of type $54_{d,e}$, with layer sequence $({}^vT_0, {}^{w-v-1}T_3, T_{17}, K_w)$, and generated by the images of $\{x_1,\ x_2,\ {x_3}^{7^{t-1}},\ {x_4}^{7^{t-1}},\ {x_5}^{7^{t-1}},\ {x_6}^{7^{t-1}}\}$;

• One subgroup with invariant $[7^{t-1}, 7^{t-1}, 7^{t-1}, 7^{t-1}, 7^t, 7^t]$ for each $t > 1$

.. one is of type $55_{d,e}$, with layer sequence $({}^vT_0, T_4, {}^{w-v-1}T_{18}, K_w)$, and generated by the images of $\{z_2 {w_7}^{-14},\ x_3\} \cup V^{(7)}$;

• One subgroup with invariant $[7^{t-1}, 7^{t-1}, 7^{t-1}, 7^t, 7^t, 7^t]$ for each $t > 1$

.. one is of type $56_{d,e}$, with layer sequence $({}^vT_0, T_6, {}^{w-v-1}T_{18}, K_w)$, and generated by the images of $\{z_2 {w_7}^{-14},\ x_3,\ x_4\} \cup V^{(7)}$;

• One subgroup with invariant $[7^{t-1}, 7^{t-1}, 7^t, 7^t, 7^t, 7^t]$ for each $t > 1$

.. one is of type $57_{d,e}$, with layer sequence $({}^vT_0, T_8, {}^{w-v-1}T_{18}, K_w)$, and generated by the images of $\{z_2 {w_7}^{-14},\ x_3,\ x_4,\ x_5\} \cup V^{(7)}$;

• Seven subgroups with invariants $[7^{t-1}, 7^t, 7^t, 7^t, 7^t, 7^t]$ for each $t > 1$

.. seven are of type $58_{d,e}{}^*$, with layer sequence $({}^vT_0, T_{14}, {}^{w-v-1}T_{18}, K_w)$, and generated by the images of $\{z_2 {w_7}^{-14},\ x_3,\ x_4,\ x_5,\ (v_5 v_6^3)(x_6 {v_6}^{-1})^{\alpha 7^{t-1}}\} \cup V^{(7)}$ for $0 \leq \alpha \leq 6$;

• One subgroup with invariant $[7, 7, 7, 7, 7, 7]$

.. one is of type 59_e, with layer sequence $({}^{w-1}T_0, T_{17}, K_w)$;

• One subgroup with invariant $[7^t, 7^t, 7^t, 7^t, 7^t, 7^t]$ for each $t \geq 1$

.. one is of type $60_{d,e}$, with layer sequence $({}^{w-1}T_0, {}^{w-v}T_{18}, K_w)$, and generated by the images of $V^{(1)}$;

Rank 7

• Fourteen subgroups with invariants $[7, 7, 7, 7, 7, 7^s, 7^t]$ for each $t > s \geq 2$

.. seven are of type $61_{c,d,e}{}^*$, with layer sequence $({}^uT_0, {}^{v-u}T_1, {}^{w-v-1}T_3, T_{19}, K_w)$, and generated by the images of $\{x_1 {x_8}^{\alpha 7^{t-1}},\ {x_2}^{7^{t-s}},\ (x_6 x_7^{-1})^{7^{t-1}}\} \cup V^{(7^{t-1})}$ for $0 \le \alpha \le 6$;

.. seven are of type $62_{c,d,e}{}^{*}$, with layer sequence $({}^{u}T_0, {}^{v-u}T_2, {}^{w-v-1}T_3, T_{19}, K_w)$, and generated by the images of $\{x_2,\ {x_1}^{7^{t-s}} {x_8}^{\alpha 7^{t-1}},\ (x_6 x_7^{-1})^{7^{t-1}}\} \cup V^{(7^{t-1})}$ for $0 \le \alpha \le 6$;

• One subgroup with invariant $[7^{s-1}, 7^{s-1}, 7^{s-1}, 7^{s-1}, 7^{s-1}, 7^{s}, 7^{t}]$ for each pair (s,t) with $t > s \ge 2$

.. one is of type $63_{c,d,e}$, with layer sequence $({}^{u}T_0, {}^{v-u}T_1, T_4, {}^{w-v-1}T_{20}, K_w)$, and generated by the images of $\{z_2 {w_7}^{-14},\ {x_3}^{7^{t-s}},\ {x_8}^{7^{t-s+1}}\} \cup V^{(7^{t-s+1})}$;

• Seven subgroups with invariants $[7^{s-1}, 7^{s-1}, 7^{s-1}, 7^{s-1}, 7^{s-1}, 7^{s}, 7^{t}]$ for each pair (s,t) with $t > s > 2$

.. seven are of type $64_{c,d,e}{}^{*}$, with layer sequence $({}^{u}T_0, {}^{v-u}T_2, T_3, {}^{w-v-1}T_{19}, K_w)$, and generated by the images of $\{x_2,\ {x_1}^{7^{t-s}} {x_8}^{\alpha 7^{t-s+1}},\ (x_6 x_7^{-1})^{t-s+1}\} \cup V^{(7^{t-s+1})}$ for $0 \le \alpha \le 6$;

• Two subgroups with invariants $[7^{s-1}, 7^{s-1}, 7^{s-1}, 7^{s-1}, 7^{s}, 7^{s}, 7^{t}]$ for each pair (s,t) with $t > s > 1$

.. one is of type $65_{c,d,e}$, with layer sequence $({}^{u}T_0, {}^{v-u}T_1, T_6, {}^{w-v-1}T_{20}, K_w)$, and generated by the images of $\{x_1,\ {x_3}^{7^{t-s}},\ {x_4}^{7^{t-s}},\ x_8^{7^{t-s+1}}\} \cup V^{(7^{t-s+1})}$;

.. one is of type $66_{c,d,e}$, with layer sequence $({}^{u}T_0, {}^{v-u}T_2, T_5, {}^{w-v-1}T_{19}, K_w)$, and generated by the images of $\{x_2,\ {x_1}^{7^{t-s}},\ {x_3}^{7^{t-s}},\ (x_6 x_7^{-1})^{7^{t-s+1}}\} \cup V^{(7^{t-s+1})}$;

• Two subgroups with invariants $[7^{s-1}, 7^{s-1}, 7^{s-1}, 7^{s}, 7^{s}, 7^{s}, 7^{t}]$ for each pair (s,t) with $t > s > 1$

.. one is of type $67_{c,d,e}$, with layer sequence $({}^{u}T_0, {}^{v-u}T_1, T_8, {}^{w-v-1}T_{20}, K_w)$, and generated by the images of $\{x_1,\ {x_3}^{7^{t-s}},\ {x_4}^{7^{t-s}},\ {x_5}^{7^{t-s}},\ x_8^{7^{t-s+1}}\} \cup V^{(7^{t-s+1})}$;

.. one is of type $68_{c,d,e}$, with layer sequence $({}^{u}T_0, {}^{v-u}T_2, T_7, {}^{w-v-1}T_{19}, K_w)$, and generated by the images of $\{x_2,\ (x_1 {x_8}^{14})^{7^{t-s}},\ {x_3}^{7^{t-s}},\ {x_4}^{7^{t-s}},\ (x_6 x_7^{-1})^{7^{t-s+1}}\} \cup V^{(7^{t-s+1})}$;

• Fourteen subgroups with invariants $[7^{s-1}, 7^{s-1}, 7^{s}, 7^{s}, 7^{s}, 7^{s}, 7^{t}]$ for each pair (s,t) with $t > s > 1$

.. seven are of type $69_{c,d,e}{}^{*}$, with layer sequence $({}^{u}T_0, {}^{v-u}T_1, T_{14}, {}^{w-v-1}T_{20}, K_w)$, and generated by the images of $\{x_1,\ {x_3}^{7^{t-s}},\ {x_4}^{7^{t-s}},\ {x_5}^{7^{t-s}},\ (v_5 {v_6}^{3})^{7^{t-s}} (x_6 {v_6}^{-1})^{\alpha 7^{t-1}},\ {x_8}^{7^{t-s+1}}\} \cup$

$V^{(7^{t-s+1})}$ for $0 \leq \alpha \leq 6$;

.. seven are of type $70_{c,d,e}{}^*$, with layer sequence $({}^{u}T_0, {}^{v-u}T_2, T_9, {}^{w-v-1}T_{19}, K_w)$, and generated by the images of $\{x_2 x_6{}^{\alpha 7^{t-s}}, (z_2 w_7{}^{-14})^{7^{t-s}}, x_3{}^{7^{t-s}}, x_4{}^{7^{t-s}}, x_5{}^{7^{t-s}}, (x_6 x_7{}^{-1})^{7^{t-s+1}}\} \cup V^{(7^{t-s+1})}$ for $0 \leq \alpha \leq 6$;

• Fourteen subgroups with invariants $[7^s, 7^{t-1}, 7^{t-1}, 7^{t-1}, 7^{t-1}, 7^{t-1}, 7^t]$ for each pair (s,t) with $t \geq s+2 > 2$

.. seven are of type $71_{c,d,e}{}^*$, with layer sequence $({}^{u}T_0, T_1, {}^{v-u-1}T_{18}, {}^{w-v}T_{19}, K_w)$, and generated by the images of $\{z_2 w_7{}^{\alpha 7^{t-1}-14},\ (x_6 x_7^{-1})^{7^{t-s}}\} \cup V^{(7)}$ for $0 \leq \alpha \leq 6$;

.. seven are of type $72_{c,d,e}{}^*$, with layer sequence $({}^{u}T_0, T_1, {}^{v-u-1}T_{18}, {}^{w-v}T_{20}, K_w)$, and generated by the images of $\{x_1 x_8{}^{14+\alpha 7^{t-s-1}},\ x_8{}^{7^{t-s}}\} \cup V^{(7)}$ for $0 \leq \alpha \leq 6$;

• Seven subgroups with invariants $[7^{s-1}, 7^s, 7^s, 7^s, 7^s, 7^s, 7^t]$ for each pair (s,t) with $t > s+1 > 2$

.. seven are of type $73_{c,d,e}{}^*$, with layer sequence $({}^{u}T_0, {}^{v-u}T_1, T_{18}, {}^{w-v-1}T_{20}, K_w)$, and generated by the images of $\{x_1 x_8{}^{\alpha 7^{t-s}},\ x_8{}^{7^{t-s+1}}\} \cup V^{(7^{t-s})}$ for $0 \leq \alpha \leq 6$;

• One subgroup with invariant $[7^{s-1}, 7^s, 7^s, 7^s, 7^s, 7^s, 7^t]$ for each pair (s,t) with $t > s > 1$

.. one is of type $74_{c,d,e}$, with layer sequence $({}^{u}T_0, {}^{v-u}T_2, T_{17}, {}^{w-v-1}T_{19}, K_w)$, and generated by the images of $\{x_2,\ (x_1 x_8{}^{14})^{7^{t-s}},\ x_3{}^{7^{t-s}},\ x_4{}^{7^{t-s}},\ x_5{}^{7^{t-s}},\ x_6{}^{7^{t-s}}\} \cup V^{(7^{t-s+1})}$;

• Seven subgroups with invariants $[7, 7, 7, 7, 7, 7, 7^t]$ for each $t > 1$

.. seven are of type $75_{d,e}{}^*$, with layer sequence $({}^{v}T_0, {}^{w-v-1}T_1, T_{19}, K_w)$, and generated by the images of $\{x_1 x_8{}^{\alpha 7^{t-1}},\ (x_6 x_7^{-1})^{7^{t-1}}\} \cup V^{(7^{t-1})}$ for $0 \leq \alpha \leq 6$;

• Two subgroups with invariants $[7^s, 7^s, 7^s, 7^s, 7^s, 7^s, 7^t]$ for each pair (s,t) with $t \geq s+1 \geq 2$

.. one is of type $76_{d,e}$, with layer sequence $({}^{v}T_0, {}^{v-u}T_1, {}^{w-v}T_{20}, K_w)$, and generated by the images of $\{x_1,\ x_8{}^{7^{t-s}}\} \cup V^{(7^{t-s})}$;

.. one is of type $77_{d,e}$, with layer sequence $({}^{v}T_0, {}^{v-u}T_2, {}^{w-v}T_{19}, K_w)$, and generated by the images of $\{x_2,\ (x_6 x_7^{-1})^{7^{t-s}}\} \cup V^{(7^{t-s})}$;

• Seven subgroups with invariants $[7^{t-1}, 7^{t-1}, 7^{t-1}, 7^{t-1}, 7^{t-1}, 7^{t-1}, 7^t]$ for each $t > 2$

.. seven are of type $78_{d,e}{}^*$, with layer sequence $({}^vT_0, T_1, {}^{w-v-1}T_{19}, K_w)$, and generated by the images of $\{z_2 {w_7}^{\alpha 7^{t-1}-14},\ (x_6 x_7^{-1})^7\} \cup V^{(7)}$ for $0 \leq \alpha \leq 6$;

• Seven subgroups with invariants $[7, 7, 7, 7, 7, 7^t, 7^t]$ for each $t > 2$

.. seven are of type $79_{d,e}{}^*$, with layer sequence $({}^vT_0, {}^{w-v-1}T_3, T_{19}, K_w)$, and generated by the images of $\{x_1 {x_8}^{\alpha 7^{t-1}},\ x_2,\ (x_6 x_7^{-1})^{7^{t-1}}\} \cup V^{(7^{t-1})}$ for $0 \leq \alpha \leq 6$;

• Nine subgroups with invariants $[7^{t-1}, 7^{t-1}, 7^{t-1}, 7^{t-1}, 7^{t-1}, 7^t, 7^t]$ for each $t > 1$

.. seven are of type $80_{d,e}{}^*$, with layer sequence $({}^vT_0, T_3, {}^{w-v-1}T_{19}, K_w)$, and generated by the images of $\{z_2 {w_7}^{\alpha 7^{t-1}-14},\ x_2,\ (x_6 x_7^{-1})^7\} \cup V^{(7)}$ for $0 \leq \alpha \leq 6$;

.. one is of type $81_{d,e}$, with layer sequence $({}^vT_0, T_4, {}^{w-v-1}T_{19}, K_w)$, and generated by the images of $\{z_2 {w_7}^{-14},\ x_3,\ (x_6 x_7^{-1})^7\} \cup V^{(7)}$;

.. one is of type $82_{d,e}$, with layer sequence $({}^vT_0, T_4, {}^{w-v-1}T_{20}, K_w)$, and generated by the images of $\{x_1,\ x_3,\ {x_8}^7\} \cup V^{(7)}$;

• Two subgroups with invariants $[7^s, 7^{t-1}, 7^{t-1}, 7^{t-1}, 7^{t-1}, 7^t, 7^t]$ for each pair (s, t) with $t > s + 1 > 1$

.. one is of type $83_{c,d,e}$, with layer sequence $({}^uT_0, T_4, {}^{v-u-1}T_{18}, {}^{w-v}T_{19}, K_w)$, and generated by the images of $\{z_2 {w_7}^{-14},\ x_3,\ (x_6 x_7^{-1})^{7^{t-s}}\} \cup V^{(7)}$;

.. one is of type $84_{c,d,e}$, with layer sequence $({}^uT_0, T_4, {}^{v-u-1}T_{18}, {}^{w-v}T_{20}, K_w)$, and generated by the images of $\{z_2 {w_7}^{-14},\ x_3,\ {x_8}^{7^{t-s}}\} \cup V^{(7)}$;

• Three subgroups with invariants $[7^{t-1}, 7^{t-1}, 7^{t-1}, 7^{t-1}, 7^t, 7^t, 7^t]$ for each $t > 1$

.. one is of type $85_{d,e}$, with layer sequence $({}^vT_0, T_5, {}^{w-v-1}T_{19}, K_w)$, and generated by the images of $\{z_2 {w_7}^{-14},\ x_2,\ x_3,\ (x_6 x_7^{-1})^7\} \cup V^{(7)}$;

.. one is of type $86_{d,e}$, with layer sequence $({}^vT_0, T_6, {}^{w-v-1}T_{19}, K_w)$, and generated by the images of $\{z_2 {w_7}^{-14},\ x_3,\ x_4,\ (x_6 x_7^{-1})^7\} \cup V^{(7)}$;

.. one is of type $87_{d,e}$, with layer sequence $({}^vT_0, T_6, {}^{w-v-1}T_{20}, K_w)$, and generated by the images of $\{z_2 {w_7}^{-14},\ x_3,\ x_4,\ {x_8}^7\} \cup V^{(7)}$;

• Two subgroups with invariants $[7^s, 7^{t-1}, 7^{t-1}, 7^{t-1}, 7^t, 7^t, 7^t]$ for each pair (s, t) with

$t > s+1 > 1$

.. one is of type $88_{c,d,e}$, with layer sequence $({}^{u}T_0, T_6, {}^{v-u-1}T_{18}, {}^{w-v}T_{19}, K_w)$, and generated by the images of $\{z_2{w_7}^{-14},\ x_3,\ x_4,\ (x_6x_7^{-1})^{7^{t-s}}\} \cup V^{(7)}$;

.. one is of type $89_{c,d,e}$, with layer sequence $({}^{u}T_0, T_6, {}^{v-u-1}T_{18}, {}^{w-v}T_{20}, K_w)$, and generated by the images of $\{z_2{w_7}^{-14},\ x_3,\ x_4,\ {x_8}^{7^{t-s}}\} \cup V^{(7)}$;

- Three subgroups with invariants $[7^{t-1}, 7^{t-1}, 7^{t-1}, 7^t, 7^t, 7^t, 7^t]$ for each $t > 1$

.. one is of type $90_{d,e}$, with layer sequence $({}^{v}T_0, T_7, {}^{w-v-1}T_{19}, K_w)$, and generated by the images of $\{z_2{w_7}^{-14},\ x_2,\ x_3,\ x_4,\ (x_6x_7^{-1})^7\} \cup V^{(7)}$;

.. one is of type $91_{d,e}$, with layer sequence $({}^{v}T_0, T_8, {}^{w-v-1}T_{19}, K_w)$, and generated by the images of $\{z_2{w_7}^{-14},\ x_3,\ x_4,\ x_5,\ (x_6x_7^{-1})^7\} \cup V^{(7)}$;

.. one is of type $92_{d,e}$, with layer sequence $({}^{v}T_0, T_8, {}^{w-v-1}T_{20}, K_w)$, and generated by the images of $\{z_2{w_7}^{-14},\ x_3,\ x_4,\ x_5,\ {x_8}^7\} \cup V^{(7)}$;

- Two subgroups with invariants $[7^s, 7^{t-1}, 7^{t-1}, 7^t, 7^t, 7^t, 7^t]$ for each pair (s,t) with $t > s+1 > 1$

.. one is of type $93_{c,d,e}$, with layer sequence $({}^{u}T_0, T_8, {}^{v-u-1}T_{18}, {}^{w-v}T_{19}, K_w)$, and generated by the images of $\{z_2{w_7}^{-14},\ x_3,\ x_4,\ x_5,\ (x_6x_7^{-1})^{7^{t-s}}\} \cup V^{(7)}$;

.. one is of type $94_{c,d,e}$, with layer sequence $({}^{u}T_0, T_8, {}^{v-u-1}T_{18}, {}^{w-v}T_{20}, K_w)$, and generated by the images of $\{z_2{w_7}^{-14},\ x_3,\ x_4,\ x_5,\ {x_8}^{7^{t-s}}\} \cup V^{(7)}$;

- Fifteen subgroups with invariants $[7^{t-1}, 7^{t-1}, 7^t, 7^t, 7^t, 7^t, 7^t]$ for each $t > 1$

.. eight of them, each one is of type $S_{d,e}$ for $95 \leq S \leq 102$, with layer sequences $({}^{v}T_0, T_\alpha, {}^{w-v-1}T_{19}, K_w)$, and generated by the images of $\{z_2{w_7}^{-14},\ x_3,\ x_4,\ x_5,\ x_2^{\alpha-9}x_6\ ,\ (x_6x_7^{-1})^7\} \cup V^{(7)}$ for $9 \leq \alpha \leq 16$;

.. seven are of type ${103_{d,e}}^*$, with layer sequence $({}^{v}T_0, T_{14}, {}^{w-v-1}T_{20}, K_w)$, and generated by the images of $\{z_2{w_7}^{-14},\ x_3,\ x_4,\ x_5,\ (v_5{v_6}^3)(x_6{v_6}^{-1})^{\alpha 7^{t-1}},\ {x_8}^7\} \cup V^{(7)}$ for $0 \leq \alpha \leq 6$;

- Fourteen subgroups with invariants $[7^s, 7^{t-1}, 7^t, 7^t, 7^t, 7^t, 7^t]$ for each pair (s,t) with $t > s+1 > 1$

.. seven are of type ${104_{c,d,e}}^*$, with layer sequence $({}^{u}T_0, T_{14}, {}^{v-u-1}T_{18}, {}^{w-v}T_{19}, K_w)$, and generated by the images of $\{z_2{w_7}^{-14},\ x_3,\ x_4,\ x_5,\ (v_5{v_6}^3)(x_6{v_6}^{-1})^{\alpha 7^{t-s-1}},\ (x_6{x_7}^{-1})^{7^{t-s}}\} \cup V^{(7)}$ for $0 \le \alpha \le 6$;

.. seven are of type $105_{c,d,e}{}^*$, with layer sequence $({}^uT_0, T_{14}, {}^{v-u-1}T_{18}, {}^{w-v}T_{20}, K_w)$, and generated by the images of $\{z_2{w_7}^{-14},\ x_3,\ x_4,\ x_5,\ (v_5{v_6}^3)(x_6{v_6}^{-1})^{\alpha 7^{t-1}},\ (x_8)^{7^{t-s}}\} \cup V^{(7)}$ for $0 \le \alpha \le 6$;

• One subgroup with invariant $[7^{t-1}, 7^t, 7^t, 7^t, 7^t, 7^t, 7^t]$ for each $t > 1$

.. one is of type $106_{d,e}$, with layer sequence $({}^vT_0, T_7, {}^{w-v-1}T_{19}, K_w)$, and generated by the images of $\{z_2{w_7}^{-14},\ x_2,\ x_3,\ x_4,\ x_5,\ x_6\} \cup V^{(7)}$;

• Two subgroups with invariants $[7^s, 7^t, 7^t, 7^t, 7^t, 7^t, 7^t]$ for each pair (s,t) with $t \ge s+1 > 1$

.. one is of type $107_{c,d,e}$, with layer sequence $({}^uT_0, {}^{v-u}T_{18}, {}^{w-v}T_{19}, K_w)$, and generated by the images of $\{{x_6}^{7^{t-s}}\} \cup V^{(1)}$;

.. one is of type $108_{c,d,e}$, with layer sequence $({}^uT_0, {}^{v-u}T_{18}, {}^{w-v}T_{20}, K_w)$, and generated by the images of $\{{x_8}^{7^{t-s}}\} \cup V^{(1)}$;

• Two subgroups with invariants $[7^t, 7^t, 7^t, 7^t, 7^t, 7^t, 7^t]$ for each $t > 0$

.. one is of type $109_{d,e}$, with layer sequence $({}^vT_0, {}^{w-v}T_{19}, K_w)$, and generated by the images of $V^{(1)} \cup \{x_6\}$;

... one is of type $110_{d,e}$, with layer sequence $({}^vT_0, {}^{w-v}T_{20}, K_w)$, and generated by the images of $V^{(1)} \cup \{x_8\}$.

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