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# New Unit Root Asymptotics in the Presence of Deterministic Trends

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# NEW UNIT ROOT ASYMPTOTICS IN THE PRESENCE OF DETERMINISTIC TRENDS[1]

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**Abstract**

Recent work by the author (1998) has shown that stochastic trends can be validly represented in empirical regressions in terms of deterministic functions of time. These representations offer an alternative mechanism for modelling stochastic trends. It is shown here that the alternate representations affect the asymptotics of all commonly used unit root tests in the presence of trends. In particular, the critical values of unit root tests diverge when the number of deterministic regressors $K \to \infty$ as the sample size $n \to \infty$. In such circumstances, use of conventional critical values based on fixed $K$ will lead to rejection of the null of a unit root in favour of trend stationarity with probability one when the null is true. The results can be interpreted as saying that serious attempts to model trends by deterministic functions will always be successful and that these functions can validly represent stochastically trending data even when lagged variables are present in the regressor set, thereby undermining conventional unit root tests.

*Keywords:* Deterministic trends; Divergent critical values; Large $K$ asymptotics; Test failure; Unit root distributions.

*JEL Classification:* C22

# 1. Introduction

Since Nelson and Plosser (1982) there has been a vast amount of empirical work concerned with the issue of testing difference stationarity against trend stationarity. In constructing such tests it is now common empirical practice to work with a general maintained hypothesis embodying alternative specifications to a unit root model that include a variety of deterministic trends and trend break functions. The latter offer some interesting alternative explanations of data nonstationarity in terms of structural shifts. As is now well understood, the presence of such deterministic functions in the regression affects the asymptotic distribution of all the usual statistical tests for a unit root and does so under both null and local alternative hypotheses. This means, of course, that the critical values of the tests change with the specification of the deterministic trend functions, necessitating the use of different statistical tables according to the precise specification of the fitted model. Figure 1 shows the asymptotic distributions of the coefficient based test in a regression with polynomial trends of degrees $p = 0, 1, 2, 5$. Clearly, there is substantial sensitivity in the distribution as the trend degree changes.

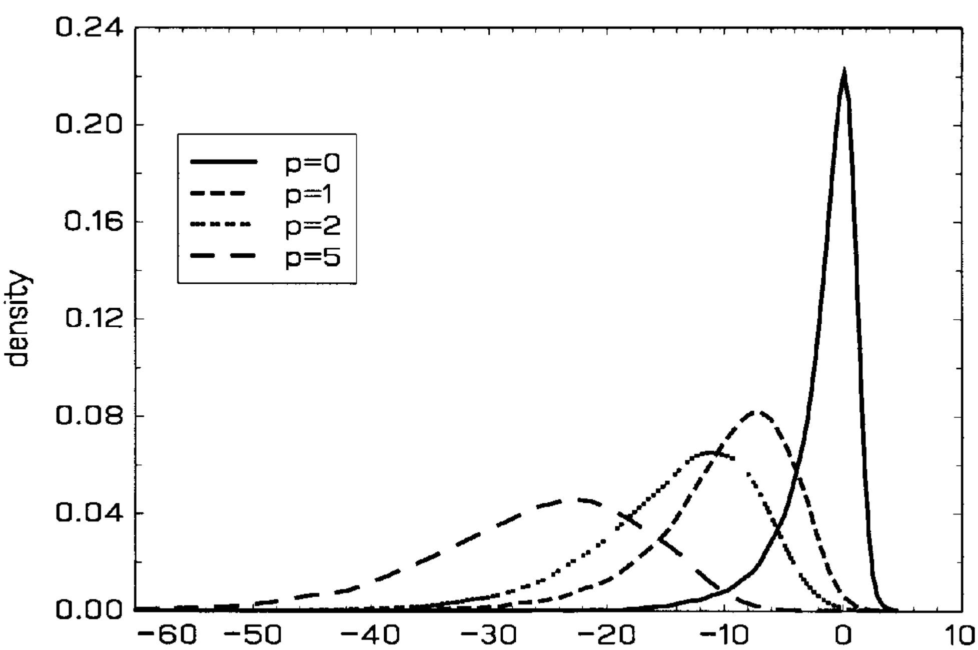


Figure 1: Densities of $\frac{\int_0^1 W_X dW}{\int_0^1 W_X^2}$ with[1] $X' = (1, r, ..., r^p)$.

In a recent paper, the author (1998) has shown that deterministic trend specifications are not necessarily alternatives to a unit root model at all. More precisely, unit root processes have limiting forms in terms of Brownian motion and continuous

[1] Here $W$ is standard Brownian motion and $W_X$ is the $L_2$ projection residual of $W$ on $X$.

stochastic processes such as Brownian motion have valid mathematical representations entirely in terms of deterministic functions. It is therefore possible to model a unit root process in the limit with an $R^2$ of unity by regression on deterministic trends. This result would appear to have certain important implications for unit root modelling and testing. In particular, it indicates that one could mistakenly 'reject' a unit root model in favour of a trend 'alternative' when in fact the alternative model is nothing other than an alternate representation of the unit root process itself.

The purpose of the present paper is to make the heuristic argument in the last paragraph precise. The paper is organized as follows. Section 2 gives the background needed for the present development. Section 3 gives some preliminary theory and a main result for unit root asymptotics when the number of deterministic regressors ($K$) tends to infinity. Section 4 shows how to derive joint limit theory for a unit root autoregression when both the sample size ($n$) and $K$ tend to infinity under the side condition that $\frac{K}{n} \to 0$. Section 5 concludes, outlines some extensions, and discusses some of the implications of the theory for applied work. Proofs are collected together in Section 6 and notational conventions are summarised in Section 7.

## 2. Background Asymptotics

The development in this paper will concentrate on a unit root time series $y_t = \Sigma_1^t u_s$, whose increments $u_t$ form a stationary time series with zero mean, finite absolute moments to order $p > 2$, and long run variance $\sigma^2 > 0$, and which satisfies the functional law

$$\frac{y_{[n\cdot]}}{\sqrt{n}} \Rightarrow B(\cdot) \equiv BM(\sigma^2), \tag{1}$$

for which primitive conditions are well known (e.g., see Phillips and Solo, 1992). It is convenient also to use the Hungarian strong approximation (e.g. Csörgõ and Horváth, 1993) to $y_t$, according to which we can construct a expanded probability space with a Brownian motion $B(\cdot)$ for which

$$\sup_{0\le k\le n} |y_k - B(k)| = o_{a.s.}(n^{1/p}),$$

or

$$\sup_{0\le k\le n} \left|\frac{y_k}{\sqrt{n}} - B\left(\frac{k}{n}\right)\right| = o_{a.s.}(1). \tag{2}$$

This gives the direct representation

$$\frac{y_{t-1}}{\sqrt{n}}=B(\frac{[nr]}{n}) + o_{a.s.}(1), \tag{3}$$

for $(t-1)/n \le r < t/n,\ t > 1$.

Phillips (1998) studied the asymptotic properties of regressions of $y_t$ on deterministic regressors of the type

$$y_t = \sum_{k=1}^{K} \widehat{b}_k \varphi_k(\frac{t}{n}) + \widehat{u}_t \tag{4}$$

or, equivalently (with $\widehat{a}_k = n^{-1/2}\widehat{b}_k$),

$$\frac{y_t}{\sqrt{n}} = \sum_{k=1}^{K} \widehat{a}_k \varphi_k(\frac{t}{n}) + \frac{\widehat{u}_t}{\sqrt{n}} \tag{5}$$

when the regressors $\varphi_k$ are the eigenfunctions of the covariance kernel, $\sigma^2 r \wedge s$, of the Brownian motion $B$. These functions have the form

$$\varphi_k(r) = \sqrt{2} \sin\left[(k - 1/2)\, \pi r\right] \tag{6}$$

and constitute a complete orthonormal system in $L_2[0,1]$. When combined with the eigenvalues

$$\lambda_k = \frac{4}{(2k-1)^2 \pi^2},$$

of the covariance kernel, these functions deliver an orthonormal representation of the Brownian motion $B$, viz.

$$B(r) = \sqrt{2} \sum_{k=1}^{\infty} \frac{\sin\left[(k - 1/2)\, \pi r\right]}{(k - 1/2)\pi} \xi_k, \tag{7}$$

where the components $\xi_k$ are independently and indentically distributed (iid) as $N(0, \sigma^2)$. This series representation of $B(r)$ is convergent almost surely and uniformly in $r \in [0,1]$. The reader is referred e.g. to Shorack and Wellner (1986) for more details on orthonormal representations of stochastic processes and to Phillips (1998) for further discussion of (7) and related representations.

Let $\widehat{a}_K = (\widehat{a}_k)$ be the coefficients and $\varphi_{Kt} = \left(\varphi_k\left(\frac{t}{n}\right)\right)$ be the $K-$vector of regressors in (5). Let $c_K \in \mathbb{R}^K$ be any vector with $c_K' c_K = 1$, $t_{c_K' \widehat{a}_K}$ be the usual least squares regression t-ratio for the linear combination of coefficients $c_K' a_K$, and let $R^2$ be the regression coefficient of determination. The following two results largely come from Phillips (1998) and give the asymptotic properties of these statistics when $K$ is fixed and when $K \to \infty$. Lemma 2.1 extends some of the early work on spurious regressions contained in Phillips (1986) and Durlauf and Phillips (1988). Lemma 2.2 deals with complete limit representations and shows that the empirical regression (5) succeeds in reproducing the entire $L_2$ orthonormal representation (i.e., (7) above) of $B(\cdot)$ when $K \to \infty$ as $n \to \infty$ provided that $\frac{K}{n} \to 0$.

**2.1 Lemma** *For fixed $K$, as $n \to \infty$ we have:*

(a) $c_K' \widehat{a}_K \Rightarrow c_K' \left[\int_0^1 \varphi_K B\right] \overset{d}{=} N(0, c_K' \Lambda_K c_K)$,

(b) $n^{-2} \sum_{t=1}^{n} \widehat{u}_t^2 \Rightarrow \int_0^1 B_{\varphi_K}^2$,

(c) $n^{-1/2} t_{c_K' \widehat{a}_K} \Rightarrow c_K' \left[\int_0^1 \varphi_K B\right] / \left(\int_0^1 B_{\varphi_K}^2\right)^{1/2}$,

(d) $R^2 \Rightarrow 1 - \int_0^1 B_{\varphi_K}^2 / \int_0^1 B^2$,

*where* $B_{\varphi_K}(\cdot) = B(\cdot) - \left(\int_0^1 B\varphi_K'\right)\left(\int_0^1 \varphi_K\varphi_K'\right)^{-1} \varphi_K(\cdot)$ *is the* $L_2$ *-projection residual of* $B$ *on* $\varphi_K$, $\Lambda_K = diag(\lambda_1, ..., \lambda_K)$, *and* $\lambda_k$ *is the eigenvalue of the covariance function* $\sigma^2 r \wedge s$ *corresponding to* $\varphi_k$.

**2.2 Lemma** *As* $K \to \infty$, $c_K'\Lambda_K c_K$ *tends to a positive constant* $\sigma_c^2 = c'\Lambda c$, *where* $c = (c_k)$, $\Lambda = diag(\lambda_1, \lambda_2, ...)$ *and* $c'c = 1$. *Moreover, if* $K \to \infty$ *and* $K/n \to 0$ *as* $n \to \infty$, *we have:*

(a) $n^{-1}\sum_{t=1}^n \varphi_{Kt}\varphi_{Kt}' = I_K + O\left(\frac{K}{n}\right)$;

(b) $c_K'\widehat{a}_K \Rightarrow N(0, \sigma_c^2)$;

(c) $n^{-2}\sum_{t=1}^n \widehat{u}_t^2 \xrightarrow{p} 0$,

(d) $n^{-1/2} t_{c_K'\widehat{a}_K}$ *diverges,*

(e) $R^2 \xrightarrow{p} 1$.

**2.3 Remarks**

(a) In Lemma 2.2, the condition $\frac{K}{n} \to 0$ ensures that the matrix $n^{-1}\sum_{t=1}^n \varphi_{Kt}\varphi_{Kt}'$ is positive definite and, as $n \to \infty$, it differs from the matrix

$$\int_0^1 \varphi_K(s)\varphi_K(s)' ds = I_K,$$

where $\varphi_K(s) = (\varphi_k(s))$ by a term of $O\left(\frac{K}{n}\right) = o(1)$ as $n \to \infty$.

(b) As discussed in Phillips (1998), the divergence of the $t-$ratio $t_{c_K'\widehat{a}_K}$ confirms that the coefficients of the deterministic regressors will inevitably be deemed significant as $n \to \infty$. This divergence also applies, but at a slightly reduced rate, when robust standard errors are used in the construction of the $t-$ ratio (Phillips, 1998).

(c) Since $R^2 \xrightarrow{p} 1$, the empirical regression successfully reproduces the full orthonormal representation of the limit Brownian motion corresponding to the dependent variable $y_t$. This outcome also applies to regressions on linearly independent deterministic functions other than the orthonormal set $\{\varphi_k\}$. Thus, modelling of stochastic trends by deterministic functions will inevitably be successful in large samples of data in the sense that the alternate representations in terms of these functions will be confirmed in statistical testing.

## 3. Main Results

This section extends the analysis of regressions of the form (5) by the inclusion of a lagged dependent variable in the regression. The model conforms to the usual setting for testing the presence of a unit root against trend stationarity. Thus, we consider the typical autoregression with trend equation

$$y_t = \widehat{\rho} y_{t-1} + \sum_{k=1}^{K} \widehat{b}_k \varphi_k (\frac{t}{n}) + \widehat{u}_{t,K}, \tag{8}$$

or such an equation augmented with lagged differences in the case of augmented Dickey Fuller ($ADF$) tests. Our focus of interst will be the limit behavior of coefficient based and t-ratio based unit root tests. At a substantial level of generality regarding the increments $u_t$, semiparametric $Z$ tests (Phillips, 1987; Phillips and Perron, 1988; and Ouliaris, Park and Phillips, 1988) and $ADF$ tests (Said and Dickey, 1984; Xiao and Phillips, 1998) have the same limit distributions. In particular, the coefficient tests behave as

$$Z_\rho, ADF_\rho \Rightarrow \frac{\int_0^1 W_{\varphi_K} dW}{\int_0^1 W_{\varphi_K}^2}, \tag{9}$$

and the t-ratio tests as

$$Z_t, ADF_t \Rightarrow \frac{\int_0^1 W_{\varphi_K} dW}{\left(\int_0^1 W_{\varphi_K}^2\right)^{\frac{1}{2}}}, \tag{10}$$

where $W_{\varphi_K}(\cdot) = W(\cdot) - \left(\int_0^1 W\varphi_K'\right)\left(\int_0^1 \varphi_K \varphi_K'\right)^{-1} \varphi_K(\cdot)$ is the $L_2$ -projection residual of $W$ on $\varphi_K$ and where $W$ is standard Brownian motion.

The limit distribution (9) is shown in Fig. 2 for a selection of values of $K$. The situation is analogous to that of Fig. 1, which shows a corresponding selection for the case where $\varphi_K$ is a polynomial of degree of $K$. In both cases, the limit distributions are highly sensitive to the inclusion of additional deterministic regressors. Interestingly, the distributions shown in Figs. 1 and 2 are very similar even though the deterministic regressors are quite different.

Our purpose now is to find the limit form of these distributions as $K \to \infty$. Our analysis will first use sequential limit theory, in which we consider the limit behavior of (9) and (10) as $K \to \infty$. This is equivalent to taking limits as $n \to \infty$, followed by $K \to \infty$, which we denote as $(n, K \to \infty)_{\text{seq}}$. Subsequently, we will show that the same results apply under the more general framework of joint limits whereby $n, K \to \infty$ simultaneously, which we denote as $(n, K \to \infty)$, under the condition that $\frac{K}{n} \to 0$. A general approach to multi-index asymptotics has been developed recently in Phillips and Moon (1999) which gives some useful background theory. In particular, this reference provides some conditions under which sequential and joint limit behavior is the same. Unfortunately, the theory in Phillips and Moon (1999) cannot be applied directly here because the multi-indexed random quantities are not constituted from panels with iid cross section observations, as they are in that paper.

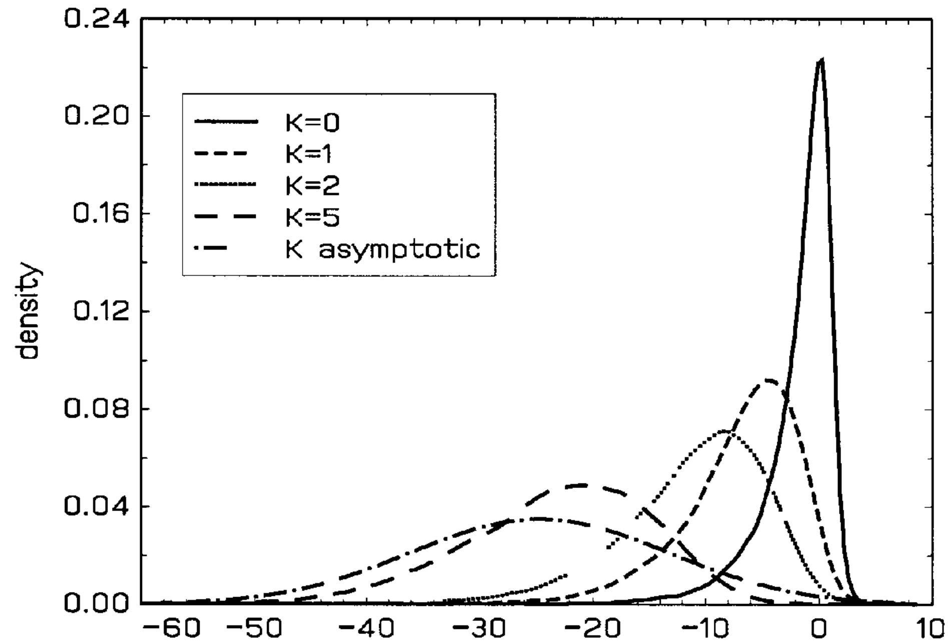


Figure 2: Unit Root Densities $\frac{\int_0^1 W_{\varphi_K} dW}{\int_0^1 W_{\varphi_K}^2}$.

Section 4 therefore provides some alternative limit theory that applies in the present case where the data is not multidimensional but involves two indexes which tend to infinity as $(n, K) \to \infty$.

The following two lemmas characterize the limit behavior, as $K \to \infty$, of the Brownian functionals that appear in the numerator and denominator of the unit root distributions (9) and (10).

**3.1 Lemma** *As* $K \to \infty$

(a) $\int_0^1 W_{\varphi_K} dW \to_p -\frac{1}{2}$;

(b) $K \int_0^1 W_{\varphi_K}^2 \to_p \frac{1}{\pi^2}$.

**3.2 Lemma** *As* $K \to \infty$

(a) $\sqrt{K}\left(\int_0^1 W_{\varphi_K} dW + \frac{1}{2}\right) \Rightarrow \frac{1}{\pi} N(0,1)$;

(b) $\sqrt{K}\left(K \int_0^1 W_{\varphi_K}^2 - \frac{1}{\pi^2}\right) \Rightarrow \frac{1}{\pi^2} N\left(0, \frac{2}{3}\right)$.

*Joint convergence of (a) and (b) also holds and the limit distributions are independent.*

The limit behavior of the unit root test statistics now follows directly and is given in the next result.

**3.3 Theorem** *As* $K \to \infty$

(a) $\frac{\int_0^1 W_{\varphi_K} dW}{\int_0^1 W_{\varphi_K}^2} \sim -\frac{\pi^2 K}{2}, \quad \frac{\int_0^1 W_{\varphi_K} dW}{(\int_0^1 W_{\varphi_K}^2)^{\frac{1}{2}}} \sim -\frac{\pi\sqrt{K}}{2};$

(b) $\frac{1}{\sqrt{K}}\left(\frac{\int_0^1 W_{\varphi_K} dW}{\int_0^1 W_{\varphi_K}^2} + \frac{\pi^2}{2}K\right) \Rightarrow N\left(0, \pi^2 + \frac{1}{6}\pi^4\right);$

(c) $\left(\frac{\int_0^1 W_{\varphi_K} dW}{(\int_0^1 W_{\varphi_K}^2)^{\frac{1}{2}}} + \frac{\pi}{2}\sqrt{K}\right) \Rightarrow N\left(0, 1 + \frac{\pi^2}{24}\right).$

**3.4 Discussion**

(a) Both the coefficient and t-ratio forms of the unit root limit distributions diverge to $-\infty$ as $K \to \infty$. The critical values of these distributions that are used in statistical tests of unit root distributions also diverge. Thus, in test situations which are better approximated by $K \to \infty$ as $n \to \infty$, we can expect that conventional unit root tests relying on fixed $K$ asymptotics will inevitably reject the null hypothesis in favour of trend stationarity. Such situations may be relevant in cases where a serious attempt is being made to model a nonstationary series using deterministic regressors. Thus, while the conventional asymptotics that rely on unit root distributions with detrended Brownian motion functionals like $\int_0^1 W_{\varphi_K} dW / \int_0^1 W_{\varphi_K}^2$ are formally correct for the given value of $K$ in such a regression, they may be considered less relevant than the large $K$ asymptotics in view of the serious effort being put into modeling the nonstationarity in terms of deterministic functions. In such situations, the large $K$ asymptotics indicate that we can expect to reject a stochastic trend in favor of the trend stationary alternative.

(b) Apparently, the limiting forms of both the coefficient and t-ratio forms of the unit root distributions are normal as $K \to \infty$ when appropriately centred and scaled. The coefficient limit theory shown in part (b) of theorem 3.3 indicates that scaling by $\frac{1}{\sqrt{K}}$ as well as recentering is required to achieve a well defined limit distribution. Part (c) indicates that only recentering of the t-ratio limit theory is required. Thus, the t-ratio test statistic is appropriately scaled, but diverges to minus infinity as $K \to \infty$.

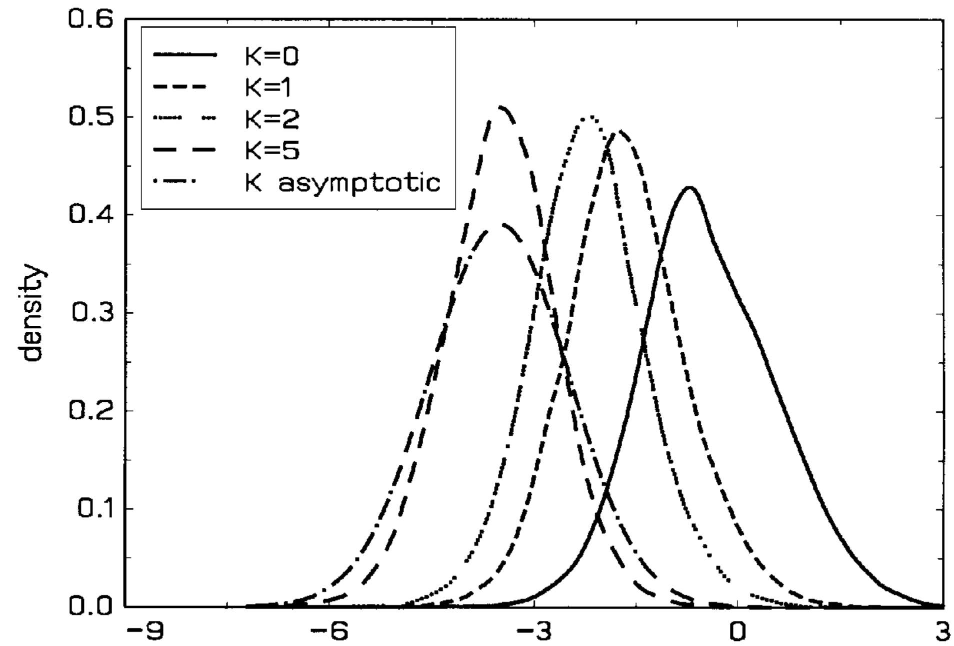


Figure 3: Unit Root $t$–ratio Densities $\frac{\int_0^1 W_{\varphi_K} dW}{(\int_0^1 W_{\varphi_K}^2)^{\frac{1}{2}}}$.

(c) Both the limit normal distributions for the coefficient and t-ratio cases are corroborated in the numerical results shown in Figs. 2 and 3. Each of these figures shows the new limiting (large $K$ asymptotic) normal approximation that applies for $K = 5$ The approximations are surprisingly good for such a small value of $K$.

## 4. Joint Limit Theory as $(n, K \to \infty)$

We now explore the conditions under which the main results above apply when $K \to \infty$ as $n \to \infty$. Our approach is to show that the sequential limit results as $(n, K \to \infty)_{\text{seq}}$ that are used in the earlier derivations hold also for joint limits as $(n, K \to \infty)$ provided that the condition $\frac{K}{n} \to 0$ holds. We will confine our attention here to extending lemma 3.1 and the divergence result given in theorem 3.3(a).

We start by noting (see Phillips and Moon, 1999) that a multi-indexed sequence $X_{K,n}$ converges in probability jointly to $X$, written $X_{K,n} \to_p X$ as $(n, K \to \infty)$, if

$$\lim_{n,K\to\infty} P\left\{||X_{K,n} - X|| > \varepsilon\right\} = 0 \;\; \forall \varepsilon > 0. \tag{11}$$

To establish (11) it is sufficient to show that as $(n, K \to \infty)$

$$E||X_{K,n} - X||^2 \to 0.$$

Using this approach we can establish the joint limits in probability for the component statistics of unit root tests arising from the regression (8). The two statistics of primary interest are: (i) the residual moment matrix

$$y_{-1}'Q_K y_{-1}, \quad Q_K = I - \Phi_K \left(\Phi_K' \Phi_K\right)^{-1} \Phi_K',$$

where

$$\Phi_K' = \left(\varphi_{K1}, ..., \varphi_{Kn}\right),$$

and $\varphi_{Kt} = \left(\varphi_k\left(\frac{t}{n}\right)\right)$ as before; and (ii) the sample covariance $y_{-1}'Q_K u$.

We will look at the leading case where $u_t$ is $iidN\left(0,\sigma^2\right)$ so that we need not have to be concerned with serial correlation corrections in the analysis that follows. The normality assumption simplifies the derivation but is not essential and could be replaced by a fourth moment condition. The results for the more general case can be expected to follow in a similar way, albeit with more complex derivations that allow for the form of the parametric or nonparametric serial correlation corrections used in the test statistics. Also, in lemma 4.1 below we use the rate condition $\frac{K^4}{n} \to 0$ in proving part (c) of the lemma. This rate places a stronger restriction on the allowable expansion path for $K$ than the requirement $\frac{K}{n} \to 0$ that is used elsewhere in the paper and is mainly the result of the line of proof adopted for part (c). It seems likely that it could be relaxed to the weaker condition if an alternative embedding argument was used in deriving the limit theory instead.

The limiting forms of the main statistics are given in the following result.

**4.1 Lemma** *As* $(n, K \to \infty)$ *with* $\frac{K}{n} \to 0$, *we have:*

(a) $E\left(\frac{K}{n^2} y_{-1}'Q_K y_{-1}\right) = \frac{\sigma^2}{\pi^2} + O\left(\frac{K}{n} + \frac{1}{K}\right)$;

(b) $E\left(\frac{1}{n} y_{-1}'Q_K u\right) = -\frac{\sigma^2}{2} + O\left(\frac{K}{n} + \frac{1}{K}\right)$;

*As* $(n, K \to \infty)$ *with* $\frac{K^4}{n} \to 0$, *we have:*

(c) $\frac{K}{n^2} y_{-1}'Q_K y_{-1} \to_p \frac{\sigma^2}{\pi^2}$;

(d) $\frac{1}{n} y_{-1}'Q_K u \to_p -\frac{\sigma^2}{2}$.

The joint limit behavior of the unit root test statistics now follows directly and we give the analogue of theorem 3.3 (a).

**4.2 Theorem** *If* $(n, K \to \infty)$ *and* $\frac{K^4}{n} \to 0$, *then*

$$Z_\rho, ADF_\rho \sim -\frac{\pi^2 K}{2}, \quad Z_t, ADF_t \sim -\frac{\pi\sqrt{K}}{2}.$$

**4.3 Discussion** This limit theory is obviously very different from that of the conventional $Z$ and $ADF$ asymptotics for fixed $K$. If the conditions of theorem 4.2 are relevant to the modeling approach, then the use of conventional critical values from (9) and (10) for fixed $K$ will lead to the false rejection of a unit root.

## 5. Conclusions

Earlier work by the author (1998) showed that serious attempts to model a stochastic trend in terms of deterministic functions will always be successful, and is indeed capable of producing an $R^2$ of unity in the limit. The present contribution shows that this outcome remains true even when a lagged dependent variable is present in the regression. In consequence, deterministic functions and lagged variables are seen to jointly compete for the explanation of a stochastic trend in a time series. In such a competition, the results confirm that the deterministic functions will always be successful, even when the correct model for the trend involves a unit root autoregression.

One way of interpreting these asymptotic results is as follows. The more serious is the attempt to model a stochastic trend by deterministic functions then the more successful it will be, leading ultimately to the rejection of alternative explanations of the trend like those provided by unit root processes. In interpreting the results in this way, it is important to recognise that careful design of a deterministic trend function, for any given realization of a time series, is certain to lead to a good deal of the low frequency variation in the series being explained. Examples of such careful deterministic trend modeling abound in recent empirical work, especially in the application of models with breaking trends. Such careful model design is, in practice, essentially equivalent to the choice of a large number, $K$, of agnostic orthonormal regressors like $\varphi_k$ in performing regressions like those in (8). As such, one may expect that regression asymptotics like those given in theorem 3.3 for multi-index asymptotics with $(n, K \to \infty)$ may well be more relevant to the practical implementation of unit root tests than conventional asymptotics for $n \to \infty$ with fixed $K$. It would be useful to perform some simulations to investigate these issues further in finite samples of data.

The fact that deterministic trend regressors and lagged regressors can both be used to model unit root processes raises some important modelling issues that are not discussed here. Two obvious issues are parsimony and forecasting. From both these perspectives, there may be good reasons for prefering the simplicity of lagged variable regressors to the complexity of deterministic trend/trend break representations. Criteria for choosing between such representations for trending time series have been explored recently in Phillips and Ploberger (1996) and Phillips (1996). A recent analysis of how trending data affects the capacity to reproduce the properties of the optimal predictor is given in Ploberger and Phillips (1998,1999). It is shown there that increasing the dimension of the parameter space carries a price in terms of the quantitative bound of how close we can come to the 'true' data generating process and, in consequence, how close we can reproduce the properties of the optimal predictor. It is further shown that this price goes up when we have trending data and when we use trending regressors. These considerations should play an important role in the choice between deterministic trend regressors and lagged variables in modeling trending data.

# 6. Technical Appendix and Proofs

**6.1 Proof of Lemma 2.1** See Phillips (1998), theorem 3.1.

**6.2 Proof of Lemma 2.2** Let $\Phi_{Kt} = \varphi_{Kt}\varphi_{Kt}'$ and note that $\Phi_{Kt} = \Phi_K\left(\frac{t}{n}\right)$ is a continuously differentiable matrix function with bounded derivatives of all orders in view of (6). Write, for $\frac{t-1}{n} \leq s < \frac{t}{n}$ ,

$$\begin{aligned}\Phi_{Kt} &= \Phi_K\left(\frac{t}{n}\right) = \Phi_K(s) + \Phi_K^{(1)}(s^*)\left(\frac{t}{n} - s\right) \\ &= \varphi_K(s)\,\varphi_K(s)' + \Phi_K^{(1)}(s^*)\left(\frac{t}{n} - s\right),\end{aligned}$$

with $s^*$ on the line segment between $\frac{t}{n}$ and $s$ for each component of $\Phi_K$. Then, since $K < n$, we have

$$\begin{aligned}n^{-1}\sum_{t=1}^{n}\varphi_{Kt}\varphi_{Kt}' &= \sum_{t=1}^{n}\int_{\frac{t-1}{n}}^{\frac{t}{n}}\varphi_K(s)\,\varphi_K(s)'\,ds \\ &\quad+\sum_{t=1}^{n}\int_{\frac{t-1}{n}}^{\frac{t}{n}}\Phi_K^{(1)}(s^*)\left(\frac{t}{n} - s\right)ds \\ &= \int_0^1\varphi_K(s)\,\varphi_K(s)'\,ds \\ &\quad+\sum_{t=1}^{n}\int_{\frac{t-1}{n}}^{\frac{t}{n}}\Phi_K^{(1)}(s^*)\left(\frac{t}{n} - s\right)ds \\ &= I_K + \sum_{t=1}^{n}\int_{\frac{t-1}{n}}^{\frac{t}{n}}\Phi_K^{(1)}(s^*)\left(\frac{t}{n} - s\right)ds.\end{aligned}$$

The elements of $\Phi_K^{(1)}$ are uniformly bounded above, so that

$$\sup_{1\leq i,j\leq K}\ \sup_{1\leq t\leq n}\ \sup_{s\in\left[\frac{t-1}{n},\frac{t}{n}\right]}\left|\Phi_{K,i,j}^{(1)}(s)\right| \leq M$$

for some finite $M > 0$ that is independent of $K$. Also $\left|\frac{t}{n} - s\right| \leq \frac{1}{n}$ uniformly for $s \in \left[\frac{t-1}{n}, \frac{t}{n}\right]$ . Using the matrix norm $\|A\| = \max_i \sum_{j=1}^{K}|a_{ij}|$ , we have

$$\begin{aligned}\left\|\sum_{t=1}^{n}\int_{\frac{t-1}{n}}^{\frac{t}{n}}\Phi_K^{(1)}(s^*)\left(\frac{t}{n} - s\right)ds\right\| &\leq \sum_{t=1}^{n}\int_{\frac{t-1}{n}}^{\frac{t}{n}}\left\|\Phi_K^{(1)}(s^*)\right\|\left|\frac{t}{n} - s\right|ds \\ &\leq \frac{K}{n}M\int_0^1 ds = O\left(\frac{K}{n}\right) = o(1).\end{aligned}$$

Part (a) follows directly. Parts (b), (c), (d) and (e) are proved in Phillips (1998) theorem 3.3. ■

**6.2 Proof of Lemmas 3.1 and 3.2** It is simplest to derive these two results together. To prove Part (a), we obtain an approximate representation of the stochastic integral $\int_0^1 W_{\varphi_K} dW$ which reveals its limiting form and shows that as $K \to \infty$

$$E\left(\int_0^1 W_{\varphi_K} dW\right) \to -\frac{1}{2}, \tag{12}$$

and

$$Var\left(\int_0^1 W_{\varphi_K} dW\right) \to 0, \tag{13}$$

giving the stated result. Start by writing

$$\begin{aligned}\int_0^1 W_{\varphi_K} dW &= \int_0^1 WdW - \left(\int_0^1 W\varphi_K'\right)\left(\int_0^1 \varphi_K\varphi_K'\right)^{-1}\left(\int_0^1 \varphi_K dW\right) \\ &= \int_0^1 WdW - \left(\int_0^1 W\varphi_K'\right)\left(\int_0^1 \varphi_K dW\right), \end{aligned} \tag{14}$$

whose expectation is

$$-E\left[\left(\int_0^1 W\varphi_K'\right)\left(\int_0^1 \varphi_K dW\right)\right]. \tag{15}$$

To evaluate (15), use the orthonormal representation (7) which we write in the form

$$W(r) = \varphi_K(r)' \Lambda_K^{\frac{1}{2}} \xi_K + \varphi_\perp(r) \Lambda_\perp^{\frac{1}{2}} \xi_\perp, \tag{16}$$

where $\xi_K, \xi_\perp$ are vectors of independent standard normal variates, $\Lambda_K = diag(\lambda_1, ..., \lambda_K)$, $\Lambda_\perp = diag(\lambda_{K+1}, ...)$, $\varphi_K' = (\varphi_1, ..., \varphi_K)$ and $\varphi_\perp' = (\varphi_{K+1}, ...)$. Then, by the $L_2$ orthogonality of $\varphi_K$ and $\varphi_\perp$, we have

$$\int_0^1 \varphi_K W = \Lambda_K^{\frac{1}{2}} \xi_K, \tag{17}$$

and since $\varphi_K$ is continuous we can apply integration by parts to $\int_0^1 \varphi_K dW$ giving

$$\int_0^1 \varphi_K dW = \varphi_K(1) W(1) - \int_0^1 \varphi_K^{(1)} W. \tag{18}$$

It follows that (15) is

$$\begin{aligned} & -E\left[\xi_K' \Lambda_K^{\frac{1}{2}}\left(\varphi_K(1) W(1) - \int_0^1 \varphi_K^{(1)} W\right)\right] \\ = & -E\left[\xi_K' \Lambda_K^{\frac{1}{2}} \varphi_K(1) \xi_K' \Lambda_K^{\frac{1}{2}} \varphi_K(1)\right] + E\left[\xi_K' \Lambda_K^{\frac{1}{2}}\left(\int_0^1 \varphi_K^{(1)}(r) \varphi_K(r)'\right) \Lambda_K^{\frac{1}{2}} \xi_K\right] \end{aligned}$$

$$
\begin{aligned}
&= -tr\left[E\left(\xi_K\xi_K'\right)\Lambda_K^{\frac{1}{2}}\varphi_K(1)\varphi_K(1)'\Lambda_K^{\frac{1}{2}}\right] + tr\left[E\left(\xi_K\xi_K'\right)\Lambda_K^{\frac{1}{2}}\left(\int_0^1\varphi_K^{(1)}(r)\varphi_K(r)'\right)\Lambda_K^{\frac{1}{2}}\right] \\
&= -tr\left[\Lambda_K\varphi_K(1)\varphi_K(1)'\right] + tr\left[\Lambda_K\left(\int_0^1\varphi_K^{(1)}(r)\varphi_K(r)'\right)\right] \\
&= -\varphi_K(1)'\Lambda_K\varphi_K(1) + \int_0^1\varphi_K(r)'\Lambda_K\varphi_K^{(1)}(r). \qquad (19)
\end{aligned}
$$

As $K \to \infty$, (19) tends to

$$
\begin{aligned}
&-\varphi(1)'\Lambda\varphi(1) + \int_0^1\varphi(r)'\Lambda\varphi^{(1)}(r) \\
= &-\sum_{k=1}^{\infty}\lambda_k\varphi_k(1)^2 + \sum_{k=1}^{\infty}\lambda_k\int_0^1\varphi_k(r)\varphi_k^{(1)}(r).
\end{aligned}
$$

Now

$$
\int_0^1\varphi_k(r)\varphi_k^{(1)}(r) = \left[\varphi_k(r)^2\right]_0^1 - \int_0^1\varphi_k(r)\varphi_k^{(1)}(r),
$$

so that

$$
\int_0^1\varphi_k(r)\varphi_k^{(1)}(r) = \frac{1}{2}\left[\varphi_k(1)^2 - \varphi_k(0)^2\right] = \frac{1}{2}\varphi_k(1)^2 = 1,
$$

because

$$
\varphi_k(1)^2 = \left(\sqrt{2}\sin\left(k-\frac{1}{2}\right)\pi\right)^2 = 2,
$$

for all $k$. Hence,

$$
\begin{aligned}
E\left(\int_0^1 W_{\varphi_K}dW\right) &= -2\sum_{k=1}^{\infty}\lambda_k + \sum_{k=1}^{\infty}\lambda_k = -\sum_{k=1}^{\infty}\lambda_k \\
&= -\sum_{k=1}^{\infty}\frac{1}{\left(k-\frac{1}{2}\right)^2\pi^2} \\
&= -\frac{1}{2},
\end{aligned}
$$

since

$$
\begin{aligned}
\sum_{k=1}^{\infty}\frac{1}{\left(k-\frac{1}{2}\right)^2} &= 4\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} \\
&= 4\left(\sum_{k=1}^{\infty}\frac{1}{k^2} - \sum_{k=1}^{\infty}\frac{1}{(2k)^2}\right) \\
&= 4\left(1-\frac{1}{2^2}\right)\sum_{k=1}^{\infty}\frac{1}{k^2} \\
&= \frac{\pi^2}{2}, \qquad (20)
\end{aligned}
$$

which gives the result stated above in (12) .

To prove (13), we start by writing (14) in the following form using (17) and (18)

$$\begin{aligned}\int_0^1 W_{\varphi_K} dW &= \frac{1}{2}\left(W(1)^2-1\right)-\left(\int_0^1 W\varphi_K'\right)\left(\int_0^1 \varphi_K dW\right)\\ &= \frac{1}{2}\left(W(1)^2-1\right)-\left(\xi_K'\Lambda_K^{\frac{1}{2}}\right)\left(\varphi_K(1)W(1)-\int_0^1 \varphi_K^{(1)}W\right)\\ &= \frac{1}{2}\left(W(1)^2-1\right)-\left(\xi_K'\Lambda_K^{\frac{1}{2}}\right)\left(\varphi_K(1)\varphi_K(1)'\Lambda_K^{\frac{1}{2}}\xi_K+\varphi_K(1)\varphi_\perp(1)'\Lambda_\perp^{\frac{1}{2}}\xi_\perp\right)\\ &\quad +\xi_K'\Lambda_K^{\frac{1}{2}}\left(\int_0^1 \varphi_K^{(1)}(r)\varphi_K(r)'\right)\Lambda_K^{\frac{1}{2}}\xi_K+\xi_K'\Lambda_K^{\frac{1}{2}}\left(\int_0^1 \varphi_K^{(1)}(r)\varphi_\perp(r)'\right)\Lambda_\perp^{\frac{1}{2}}\xi_\perp.\end{aligned}$$

Observe that for any $K-$ vector $b$

$$\begin{aligned}b'\left(\int_0^1 \varphi_K^{(1)}(r)\varphi_K(r)'\right)b &= b'\frac{1}{2}\left(\int_0^1 \varphi_K^{(1)}(r)\varphi_K(r)'+\int_0^1 \varphi_K(r)\varphi_K^{(1)}(r)'\right)b\\ &= b'\frac{1}{2}\left(\left[\varphi_K(r)\varphi_K(r)'\right]_0^1-\int_0^1 \varphi_K(r)\varphi_K^{(1)}(r)'+\int_0^1 \varphi_K(r)\varphi_K^{(1)}(r)'\right)b\\ &= \frac{1}{2}b'\varphi_K(1)\varphi_K(1)'b,\end{aligned}$$

so that

$$\xi_K'\Lambda_K^{\frac{1}{2}}\left(\int_0^1 \varphi_K^{(1)}(r)\varphi_K(r)'\right)\Lambda_K^{\frac{1}{2}}\xi_K=\frac{1}{2}\xi_K'\Lambda_K^{\frac{1}{2}}\varphi_K(1)\varphi_K(1)'\Lambda_K^{\frac{1}{2}}\xi_K,$$

and thus

$$\begin{aligned}\int_0^1 W_{\varphi_K} dW &= \frac{1}{2}\left(W(1)^2-1\right)-\frac{1}{2}\xi_K'\Lambda_K^{\frac{1}{2}}\varphi_K(1)\varphi_K(1)'\Lambda_K^{\frac{1}{2}}\xi_K-\xi_K'\Lambda_K^{\frac{1}{2}}\varphi_K(1)\varphi_\perp(1)'\Lambda_\perp^{\frac{1}{2}}\xi_\perp\\ &\quad +\xi_K'\Lambda_K^{\frac{1}{2}}\left(\int_0^1 \varphi_K^{(1)}(r)\varphi_\perp(r)'\right)\Lambda_\perp^{\frac{1}{2}}\xi_\perp\\ &= \frac{1}{2}\left(W(1)^2-1\right)-\frac{1}{2}\xi_K'\Lambda_K^{\frac{1}{2}}\varphi_K(1)\varphi_K(1)'\Lambda_K^{\frac{1}{2}}\xi_K\\ &\quad -\xi_K'\Lambda_K^{\frac{1}{2}}\left(\int_0^1 \varphi_K(r)\varphi_\perp^{(1)}(r)'\right)\Lambda_\perp^{\frac{1}{2}}\xi_\perp\\ &= -\frac{1}{2}+\xi_K'\Lambda_K^{\frac{1}{2}}\varphi_K(1)\varphi_\perp(1)'\Lambda_\perp^{\frac{1}{2}}\xi_\perp+\frac{1}{2}\xi_\perp'\Lambda_\perp^{\frac{1}{2}}\varphi_\perp(1)\varphi_\perp(1)'\Lambda_\perp^{\frac{1}{2}}\xi_\perp\\ &\quad -\xi_K'\Lambda_K^{\frac{1}{2}}\left(\int_0^1 \varphi_K(r)\varphi_\perp^{(1)}(r)'\right)\Lambda_\perp^{\frac{1}{2}}\xi_\perp. \qquad (21)\end{aligned}$$

Note that

$$\varphi_\perp(1)'\Lambda_\perp^{\frac{1}{2}}\xi_\perp=\sum_{k=K+1}^{\infty}\lambda_k^{\frac{1}{2}}\varphi_k(1)\xi_k\equiv N\left(0,\varphi_\perp(1)'\Lambda_\perp\varphi_\perp(1)\right),$$

and

$$\varphi_\perp(1)'\Lambda_\perp\varphi_\perp(1) = \sum_{k=K+1}^{\infty}\lambda_k\varphi_k(1)^2 = \sum_{k=K+1}^{\infty}\frac{2}{\left(k-\frac{1}{2}\right)^2\pi^2}$$

It follows that

$$\int_0^1 W_{\varphi_K}dW = -\frac{1}{2} + \xi_K'\Lambda_K^{\frac{1}{2}}\left(\int_0^1 \varphi_K^{(1)}(r)\varphi_\perp(r)'\right)\Lambda_\perp^{\frac{1}{2}}\xi_\perp + o_p\left(\frac{1}{K}\right). \quad (22)$$

Hence,

$$Var\left(\int_0^1 W_{\varphi_K}dW\right) = tr\left[\Lambda_K\left(\int_0^1 \varphi_K^{(1)}(r)\varphi_\perp(r)'\right)\Lambda_\perp\left(\int_0^1 \varphi_\perp(r)\varphi_K^{(1)}(r)'\right)\right] + o\left(\frac{1}{K}\right)$$

Now introduce the standard Brownian motion $V$ defined in terms of its orthonormal representation as

$$V(r) = \xi_K'\Lambda_K^{\frac{1}{2}}\varphi_K(r) + \varphi_\perp(r)'\Lambda_\perp^{\frac{1}{2}}\eta_\perp,$$

where $\eta_\perp' = (\eta_{K+1}, ...)$ is a vector of iid standard normal variates, independent of $\xi_K$ and $\xi_\perp$. The process $V(r)$ is independent of $\xi_\perp$. Hence, in view of the continuity of the elements of $\varphi_\perp(r)$, we may write

$$\begin{aligned}\xi_K'\Lambda_K^{\frac{1}{2}}\left(\int_0^1 \varphi_K^{(1)}(r)\varphi_\perp(r)'\right)\Lambda_\perp^{\frac{1}{2}}\xi_\perp &= \int_0^1 dV\varphi_\perp(r)'\Lambda_\perp^{\frac{1}{2}}\xi_\perp\left[1+o_p(1)\right] \\ &\equiv N\left(0, \xi_\perp'\Lambda_\perp\xi_\perp\right)\left[1+o_p(1)\right],\end{aligned}$$

as $K \to \infty$. Next, observe that

$$\xi_\perp'\Lambda_\perp\xi_\perp = \sum_{k=K+1}^{\infty}\lambda_k\xi_k^2,$$

and

$$E\left(\xi_\perp'\Lambda_\perp\xi_\perp\right) = \sum_{k=K+1}^{\infty}\lambda_k = \sum_{k=K+1}^{\infty}\frac{1}{\left(k-\frac{1}{2}\right)^2\pi^2}.$$

As $K \to \infty$,

$$KE\left(\xi_\perp'\Lambda_\perp\xi_\perp\right) \sim \frac{K}{\pi^2}\int_{K+1}^{\infty}\frac{dx}{\left(x-\frac{1}{2}\right)^2} \sim \frac{K}{\pi^2\left(K+\frac{1}{2}\right)} \to \frac{1}{\pi^2}, \quad (23)$$

and

$$Var\left(\xi_\perp'\Lambda_\perp\xi_\perp\right) = \sum_{k=K+1}^{\infty}\lambda_k^2 = O\left(\frac{1}{K^3}\right)$$

so that

$$K\xi_\perp'\Lambda_\perp\xi_\perp \to_p \frac{1}{\pi^2}. \quad (24)$$

Hence,

$$\sqrt{K}\xi_K'\Lambda_K^{\frac{1}{2}}\left(\int_0^1 \varphi_K^{(1)}(r)\,\varphi_\perp(r)'\right)\Lambda_\perp^{\frac{1}{2}}\xi_\perp \sim N\left(0, K\xi_\perp'\Lambda_\perp\xi_\perp\right)[1+o_p(1)] \quad (25)$$
$$\Rightarrow \frac{1}{\pi}N(0,1).$$

We deduce that

$$\sqrt{K}\left[\int_0^1 W_{\varphi_K}dW + \frac{1}{2}\right] \Rightarrow \frac{1}{\pi}N(0,1),$$

and

$$E\left(\int_0^1 W_{\varphi_K}dW\right) \to -\frac{1}{2},$$

as required for Part (a) of lemmas 3.1 and 3.2.

For Part (b) of Lemma 3.1, observe that

$$\int_0^1 W_{\varphi_K}^2 = \xi_\perp'\Lambda_\perp\xi_\perp,$$

and, thus, from (24)

$$K\int_0^1 W_{\varphi_K}^2 = K\xi_\perp'\Lambda_\perp\xi_\perp \to_p \frac{1}{\pi^2}, \quad (26)$$

as required. Next, consider

$$\begin{aligned} K\int_0^1 W_{\varphi_K}^2 - \frac{1}{\pi^2} &= K\xi_\perp'\Lambda_\perp\xi_\perp - \frac{1}{\pi^2} \\ &= K\sum_{k=K+1}^{\infty}\lambda_k\left(\xi_k^2-1\right) + K\sum_{k=K+1}^{\infty}\lambda_k - \frac{1}{\pi^2} \\ &= K\sum_{k=K+1}^{\infty}\lambda_k\left(\xi_k^2-1\right) + O\left(\frac{1}{K}\right), \end{aligned}$$

since

$$\begin{aligned} K\sum_{k=K+1}^{\infty}\lambda_k - \frac{1}{\pi^2} &= \frac{K}{\pi^2}\left(\sum_{k=K+1}^{\infty}\frac{1}{\left(k-\frac{1}{2}\right)^2} - \int_{K+1}^{\infty}\frac{dx}{\left(x-\frac{1}{2}\right)^2}\right) \\ &\quad + \frac{1}{\pi^2}\left(K\int_{K+1}^{\infty}\frac{dx}{\left(x-\frac{1}{2}\right)^2} - 1\right) \\ &= \frac{K}{\pi^2}\left(\sum_{k=K+1}^{\infty}\left[\frac{1}{\left(k-\frac{1}{2}\right)^2} - \left(\frac{1}{k-\frac{1}{2}} - \frac{1}{k+\frac{1}{2}}\right)\right]\right) \\ &\quad + \frac{1}{\pi^2}\left(\frac{K}{K+\frac{1}{2}} - 1\right) \end{aligned}$$

$$= \frac{K}{\pi^2}\left(\sum_{k=K+1}^{\infty}\frac{1}{\left(k-\frac{1}{2}\right)^2\left(k+\frac{1}{2}\right)}\right)-\frac{1}{\pi^2}\left(\frac{\frac{1}{2}}{K+\frac{1}{2}}\right)$$
$$= O\left(\frac{1}{K}\right).$$

It follows that

$$\sqrt{K}\left(K\int_0^1 W_{\varphi_K}^2-\frac{1}{\pi^2}\right) = K^{\frac{3}{2}}\sum_{k=K+1}^{\infty}\lambda_k\left(\xi_k^2-1\right)+O_p\left(\frac{1}{\sqrt{K}}\right)$$
$$= \frac{1}{\pi^2}\frac{1}{\sqrt{K}}\sum_{k=K+1}^{\infty}\frac{K^2}{\left(k-\frac{1}{2}\right)^2}\left(\xi_k^2-1\right)+O_p\left(\frac{1}{\sqrt{K}}\right) \quad (27)$$

Now the variates

$$\frac{K^2}{\left(k-\frac{1}{2}\right)^2}\left(\xi_k^2-1\right)$$

are independent with uniformly bounded moments of all orders for all $k > K$ and

$$Var\left(\sum_{k=K+1}^{\infty}\frac{K^2}{\left(k-\frac{1}{2}\right)^2}\left(\xi_k^2-1\right)\right) = 2\sum_{k=K+1}^{\infty}\left(\frac{K^2}{\left(k-\frac{1}{2}\right)^2}\right)^2$$
$$= 2K^4\sum_{k=K+1}^{\infty}\left(\frac{1}{\left(k-\frac{1}{2}\right)^4}\right)$$
$$\sim 2K^4\int_{K+1}^{\infty}\frac{dx}{\left(x-\frac{1}{2}\right)^4}$$
$$= \frac{2}{3}K^4\frac{1}{\left(K+\frac{1}{2}\right)^3}$$
$$= O(K)$$

as $K\to\infty$. It follows by the Martingale central limit theorem that

$$\frac{1}{\sqrt{K}}\sum_{k=K+1}^{\infty}\frac{K^2}{\left(k-\frac{1}{2}\right)^2}\left(\xi_k^2-1\right)\Rightarrow N\left(0,\frac{2}{3}\right).$$

Hence,

$$\sqrt{K}\left(K\int_0^1 W_{\varphi_K}^2-\frac{1}{\pi^2}\right)\Rightarrow\frac{1}{\pi^2}N\left(0,\frac{2}{3}\right),$$

as required.

Finally, consider the joint distribution. From (22), (25) and (27) we have the representation

$$\begin{bmatrix}\sqrt{K}\left[\int_0^1 W_{\varphi_K}dW+\frac{1}{2}\right]\\ \sqrt{K}\left(K\int_0^1 W_{\varphi_K}^2-\frac{1}{\pi^2}\right)\end{bmatrix} = \begin{bmatrix}N\left(0,K\xi_\perp'\Lambda_\perp\xi_\perp\right)\left[1+o_p(1)\right]\\ \frac{1}{\pi^2}\frac{1}{\sqrt{K}}\sum_{k=K+1}^{\infty}\frac{K^2}{\left(k-\frac{1}{2}\right)^2}\left(\xi_k^2-1\right)+o_p(1)\end{bmatrix}, \quad (28)$$

and, since $K\xi_\perp'\Lambda_\perp\xi_\perp$ converges in probability to a constant as shown in (26), the two elements of (28) are asymptotically independent.

**6.3 Proof of Theorem 3.3** Part (a) is an immediate consequence of Lemma 3.2. In particular, we have

$$\frac{\int_0^1 W_{\varphi_K} dW}{\int_0^1 W_{\varphi_K}^2} \sim \frac{-\frac{1}{2}K}{\frac{1}{\pi^2}} + o_p(1) = O(K)$$

and

$$\frac{\int_0^1 W_{\varphi_K} dW}{\left(\int_0^1 W_{\varphi_K}^2\right)^{\frac{1}{2}}} \sim \frac{-\frac{1}{2}\sqrt{K}}{\frac{1}{\pi}} + o_p(1) = O\left(\sqrt{K}\right).$$

For Part (b), write

$$\begin{aligned}
\frac{1}{\sqrt{K}}\left(\frac{\int_0^1 W_{\varphi_K} dW}{\int_0^1 W_{\varphi_K}^2} + \frac{\pi^2}{2}K\right) &= \frac{1}{\sqrt{K}}\frac{\left(\int_0^1 W_{\varphi_K} dW + \frac{1}{2}\right) + \frac{\pi^2}{2}\left(K\int_0^1 W_{\varphi_K}^2 - \frac{1}{\pi^2}\right)}{\int_0^1 W_{\varphi_K}^2} \\
&= \frac{\sqrt{K}\left(\int_0^1 W_{\varphi_K} dW + \frac{1}{2}\right) + \frac{\pi^2}{2}\sqrt{K}\left(K\int_0^1 W_{\varphi_K}^2 - \frac{1}{\pi^2}\right)}{K\int_0^1 W_{\varphi_K}^2} \\
&\Rightarrow \frac{\frac{1}{\pi}N(0,1) + \frac{\pi^2}{2}\frac{1}{\pi^2}N\left(0,\frac{2}{3}\right)}{\frac{1}{\pi^2}} \\
&\equiv N\left(0, \pi^2 + \frac{1}{6}\pi^4\right),
\end{aligned}$$

which is the stated result. For the $t-$ ratio limit distribution

$$\begin{aligned}
\left(\frac{\int_0^1 W_{\varphi_K} dW}{\left(\int_0^1 W_{\varphi_K}^2\right)^{\frac{1}{2}}} + \frac{\pi}{2}\sqrt{K}\right) &= \frac{\left(\int_0^1 W_{\varphi_K} dW + \frac{1}{2}\right) + \frac{\pi}{2}\left(\sqrt{K}\left(\int_0^1 W_{\varphi_K}^2\right)^{\frac{1}{2}} - \frac{1}{\pi}\right)}{\left(\int_0^1 W_{\varphi_K}^2\right)^{\frac{1}{2}}} \\
&= \frac{\sqrt{K}\left(\int_0^1 W_{\varphi_K} dW + \frac{1}{2}\right) + \frac{\pi}{2}\sqrt{K}\left(\sqrt{K}\left(\int_0^1 W_{\varphi_K}^2\right)^{\frac{1}{2}} - \frac{1}{\pi}\right)}{\left(K\int_0^1 W_{\varphi_K}^2\right)^{\frac{1}{2}}} \\
&\Rightarrow \frac{\frac{1}{\pi}N(0,1) + \frac{\pi}{2}\frac{1}{2\pi}N\left(0,\frac{2}{3}\right)}{\frac{1}{\pi}} \\
&\equiv N(0,1) + \frac{\pi}{4}N\left(0,\frac{2}{3}\right) \equiv N\left(0, 1 + \frac{\pi^2}{24}\right),
\end{aligned}$$

where the third line follows by a delta method calculation involving

$$\sqrt{K}\left(\sqrt{K}\left(\int_0^1 W_{\varphi_K}^2\right)^{\frac{1}{2}} - \frac{1}{\pi}\right) \sim \frac{\pi}{2}\sqrt{K}\left(K\int_0^1 W_{\varphi_K}^2 - \frac{1}{\pi^2}\right) \Rightarrow \frac{\pi}{2}\frac{1}{\pi^2}N\left(0,\frac{2}{3}\right).$$

■

**6.4 Proof of Lemma 4.1** To prove part (a) we need to show that as $(n, K \to \infty)$ with $\frac{K}{n} \to 0$

$$E\left(\frac{K}{n^2}y_{-1}'Q_K y_{-1}\right) = \frac{\sigma^2}{\pi^2} + O\left(\frac{K}{n} + \frac{1}{K}\right).$$

Note that $E\left(y_{-1}y_{-1}'\right) = \sigma^2 LL' = \sigma^2\Omega$, say, where $L$ is a lower triangular matrix with unity in all elements in and below the main diagonal. Then,

$$\begin{aligned}
E\left(\frac{K}{n^2}y_{-1}'Q_K y_{-1}\right) &= \frac{K}{n^2}tr\left\{\Omega\left(I - \Phi_K\left(\Phi_K'\Phi_K\right)^{-1}\Phi_K'\right)\right\} \\
&= \frac{K}{n^2}tr\Omega - \frac{K}{n^2}\sum_{k=1}^{K}\frac{1}{n}\varphi_k'\Omega\varphi_k \\
&= \sigma^2\frac{K}{n^2}\sum_{j=1}^{n} j - \frac{K}{n^2}\sum_{k=1}^{K}\frac{1}{n}\varphi_k'\Omega\varphi_k \\
&= \sigma^2\frac{K}{n^2}\frac{n(n+1)}{2} - \sigma^2 K\sum_{k=1}^{K}\frac{1}{n^2}\sum_{t,s=1}^{n}\varphi_k\left(\frac{t}{n}\right)\left(\frac{t\wedge s}{n}\right)\varphi_k\left(\frac{s}{n}\right) \\
&= \sigma^2 K\frac{1}{2} - \sigma^2 K\sum_{k=1}^{K}\int_0^1\int_0^1\varphi_k(r)(r\wedge p)\varphi_k(p)\,drdp + O\left(\frac{K}{n}\right) \\
&= \sigma^2 K\left(\frac{1}{2} - \sum_{k=1}^{K}\lambda_k\right) + O\left(\frac{K}{n}\right) \\
&= \sigma^2 K\left(\frac{1}{2} - \sum_{k=1}^{\infty}\lambda_k + \sum_{k=K+1}^{\infty}\lambda_k\right) + O\left(\frac{K}{n}\right) \\
&= \sigma^2 K\left(\frac{1}{2} - \frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{\left(k-\frac{1}{2}\right)^2} + \sum_{k=K+1}^{\infty}\lambda_k\right) + O\left(\frac{K}{n}\right) \\
&= \sigma^2 K\sum_{k=K+1}^{\infty}\lambda_k + O\left(\frac{K}{n}\right),
\end{aligned}$$

since

$$\sum_{k=1}^{\infty}\frac{1}{\left(k-\frac{1}{2}\right)^2} = \frac{\pi^2}{2},$$

from (20) above (see also Gradshteyn and Ryzhik, 1994, formula 0.234-2). Also, as in (23), we find that as $K \to \infty$

$$K\sum_{k=K+1}^{\infty}\lambda_k = \frac{1}{\pi^2} + O\left(\frac{1}{K}\right), \tag{29}$$

and the stated result follows immediately.

For part (b) we need to show that

$$E\left(\frac{1}{n}y'_{-1}Q_K u\right) \to -\frac{\sigma^2}{2}.$$

Note that

$$\begin{aligned}
E\left(\frac{1}{n}y'_{-1}Q_K u\right) &= \sigma^2\frac{1}{n}tr\left\{\left(I-\Phi_K\left(\Phi'_K\Phi_K\right)^{-1}\Phi'_K\right)(L-I)\right\} \\
&= \frac{\sigma^2}{n}tr\left\{\Phi_K\left(\Phi'_K\Phi_K\right)^{-1}\Phi'_K(I-L)\right\} \\
&= \sigma^2\frac{K}{n}-\frac{\sigma^2}{n^2}\sum_{k=1}^{K}\varphi'_k L\varphi_k \\
&= O\left(\frac{K}{n}\right)-\sigma^2\sum_{k=1}^{K}\frac{1}{n^2}\sum_{s=1}^{n}\sum_{t=s}^{n}\varphi_k\left(\frac{t}{n}\right)\varphi_k\left(\frac{s}{n}\right) \\
&= O\left(\frac{K}{n}\right)-\sigma^2\sum_{k=1}^{K}\int_0^1\varphi_k(r)\int_0^r\varphi_k(p)\,dpdr. \qquad (30)
\end{aligned}$$

Note that

$$\begin{aligned}
\int_0^1\varphi_k(r)\int_0^r\varphi_k(p)\,dpdr &= 2\int_0^1\sin\left(\left(k-\frac{1}{2}\right)\pi r\right)\int_0^r\sin\left(\left(k-\frac{1}{2}\right)\pi p\right)dpdr \\
&= 2\int_0^1\sin\left(\left(k-\frac{1}{2}\right)\pi r\right)\frac{1}{\left(k-\frac{1}{2}\right)\pi}\left[-\cos\left(\left(k-\frac{1}{2}\right)\pi p\right)\right]_0^r dr \\
&= \frac{2}{\left(k-\frac{1}{2}\right)\pi}\int_0^1\sin\left(\left(k-\frac{1}{2}\right)\pi r\right)\left[1-\cos\left(\left(k-\frac{1}{2}\right)\pi r\right)\right]dr \\
&= \frac{1}{\left(k-\frac{1}{2}\right)\pi}\left[2\int_0^1\sin\left(\left(k-\frac{1}{2}\right)\pi r\right)dr-\int_0^1\sin\left((2k-1)\pi r\right)\right]dr \\
&= \frac{1}{\left(k-\frac{1}{2}\right)\pi}\left[\frac{2}{\left(k-\frac{1}{2}\right)\pi}\left[-\cos\left(\left(k-\frac{1}{2}\right)\pi r\right)\right]_0^1\right] \\
&\quad -\frac{1}{\left(k-\frac{1}{2}\right)\pi}\left[\frac{1}{(2k-1)\pi}\left[-\cos\left((2k-1)\pi r\right)\right]_0^1\right] \\
&= \frac{1}{\left(k-\frac{1}{2}\right)^2\pi^2}\left[2-\frac{1}{2}\left[-(-1)^{2k-1}+1\right]\right] \\
&= \frac{1}{\left(k-\frac{1}{2}\right)^2\pi^2}. \qquad (31)
\end{aligned}$$

It follows from (30), (31) and (20) that

$$E\left(\frac{1}{n}y'_{-1}Q_K u\right) = -\frac{\sigma^2}{\pi^2}\sum_{k=1}^{K}\frac{1}{\left(k-\frac{1}{2}\right)^2}+O\left(\frac{K}{n}\right)$$

$$
\begin{aligned}
&= -\frac{\sigma^2}{\pi^2}\left[\frac{\pi^2}{2}+O\left(\frac{1}{K}\right)\right]+O\left(\frac{K}{n}\right) \\
&= -\frac{\sigma^2}{2}+O\left(\frac{K}{n}+\frac{1}{K}\right),
\end{aligned}
$$

giving the stated result.

For part (c) it is sufficient to show that under the stated conditions

$$
Var\left(\frac{K}{n^2}y'_{-1}Q_K y_{-1}\right) \to 0.
$$

Let $y = Lu$, where $u \equiv N(0, I_n)$. Then, $\Omega = \sigma^2 LL'$ and

$$
y'_{-1}Q_K y_{-1} = u'A_K u, \quad \text{with } A_K = L'\left[I - \Phi_K\left(\Phi'_K\Phi_K\right)^{-1}\Phi'_K\right]L.
$$

Then

$$
\begin{aligned}
Var\left(\frac{K}{n^2}y'_{-1}Q_K y_{-1}\right) &= \left(\frac{K}{n^2}\right)^2 Var\left(u'A_K u\right) \\
&= \frac{K^2}{n^4}2tr\left(A_K^2\right). \qquad (32)
\end{aligned}
$$

Evaluating $tr\left(A_K^2\right)$ we find

$$
\begin{aligned}
tr\left(A_K^2\right) &= tr\left\{\Omega\left(I-\frac{1}{n}\Phi_K\Phi'_K+O\left(\frac{K}{n^2}\right)\right)\Omega\left(I-\frac{1}{n}\Phi_K\Phi'_K+O\left(\frac{K}{n^2}\right)\right)\right\} \\
&= tr\left(\Omega^2\right)\left[1+O\left(\frac{K}{n^2}\right)\right]-\frac{2}{n}tr\left(\Phi'_K\Omega^2\Phi_K\right)+\frac{1}{n^2}tr\left(\Phi'_K\Omega\Phi_K\right)^2. \qquad (33)
\end{aligned}
$$

Clearly

$$
(tr\Omega)^2 = \sigma^4\left(\sum_{j=1}^{n} j\right)^2 = \sigma^4\left(\frac{n(n+1)}{2}\right)^2,
$$

and the $k$'th diagonal element of $\Omega^2$ is $\sigma^4\left[\sum_{j=1}^{k-1} j^2 + k^2(n-k+1)\right]$, so that

$$
\begin{aligned}
tr\left(\Omega^2\right) &= \sigma^4\sum_{k=1}^{n}\left\{\sum_{j=1}^{k-1} j^2 + k^2(n-k+1)\right\} \\
&= \sigma^4\sum_{k=1}^{n}\left\{\frac{(k-1)k(2k-1)}{6} - k^3 + k^2 + k^2 n\right\} \\
&= \sigma^4\sum_{k=1}^{n}\left\{\left(\frac{1}{3}k^3-\frac{1}{2}k^2+\frac{1}{6}k\right) - k^3 + k^2(n+1)\right\} \\
&= \sigma^4\sum_{k=1}^{n}\left\{\left(-\frac{2}{3}k^3+\frac{1}{6}k\right) + k^2\left(n+\frac{1}{2}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \left\{ -\frac{2}{3}\left(\frac{n(n+1)}{2}\right)^2 + \frac{1}{6}\frac{n(n+1)}{2} + \frac{n(n+1)(2n+1)}{6}\left(n+\frac{1}{2}\right) \right\} \sigma^4 \\
&\sim \frac{1}{6} n^4 \sigma^4. \qquad (34)
\end{aligned}
$$

Moreover

$$
\begin{aligned}
\frac{1}{n} tr\left(\Phi_K' \Omega^2 \Phi_K\right) &= \frac{1}{n}\sum_{k=1}^{K}\sum_{t,s=1}^{n} \varphi_k\left(\frac{t}{n}\right)\left[\Omega^2\right]_{t,s}\varphi_k\left(\frac{s}{n}\right) \\
&= n^3 \sum_{k=1}^{K}\frac{1}{n^3}\sum_{t,s=1}^{n}\varphi_k\left(\frac{t}{n}\right)\frac{1}{n}\left[\Omega^2\right]_{t,s}\varphi_k\left(\frac{s}{n}\right) \\
&= \sigma^4 n^4 \sum_{k=1}^{K}\frac{1}{n^3}\sum_{t,s,q=1}^{n}\varphi_k\left(\frac{t}{n}\right)\frac{t\wedge q}{n}\frac{q\wedge s}{n}\varphi_k\left(\frac{s}{n}\right) \\
&= \sigma^4 n^4 \sum_{k=1}^{K}\int_0^1\int_0^1\int_0^1 \varphi_k(r)(r\wedge p)(p\wedge u)\varphi_k(u)\,drdpdu + O\left(n^4\frac{K}{n}\right) \\
&= \sigma^4 n^4 \sum_{k=1}^{K}\lambda_k\int_0^1\int_0^1 \varphi_k(r)(r\wedge p)\varphi_k(p)\,drdp + O\left(n^3K\right) \\
&= \sigma^4 n^4 \sum_{k=1}^{K}\lambda_k^2\int_0^1 \varphi_k(r)^2\,dr + O\left(n^3K\right) \\
&= \sigma^4 n^4 \sum_{k=1}^{K}\lambda_k^2 + O\left(n^3K\right). \qquad (35)
\end{aligned}
$$

By a similar calculation we find

$$
\begin{aligned}
&\frac{1}{n^2} tr\left(\Phi_K'\Omega\Phi_K\right)^2 \\
&= \frac{\sigma^4}{n^2}\sum_{k,l=1}^{K}\left[\sum_{t,s=1}^{n}\varphi_k\left(\frac{t}{n}\right)(t\wedge s)\varphi_l\left(\frac{s}{n}\right)\right]\left[\sum_{t,s=1}^{n}\varphi_l\left(\frac{t}{n}\right)(t\wedge s)\varphi_k\left(\frac{s}{n}\right)\right] \\
&= \sigma^4 n^4 \sum_{k,l=1}^{K}\left[\frac{1}{n^2}\sum_{t,s=1}^{n}\varphi_k\left(\frac{t}{n}\right)\left(\frac{t\wedge s}{n}\right)\varphi_l\left(\frac{s}{n}\right)\right]\left[\frac{1}{n^2}\sum_{t,s=1}^{n}\varphi_l\left(\frac{t}{n}\right)\left(\frac{t\wedge s}{n}\right)\varphi_k\left(\frac{s}{n}\right)\right] \\
&= \sigma^4 n^4 \sum_{k,l=1}^{K}\left[\int_0^1\int_0^1 \varphi_k(r)(r\wedge p)\varphi_l(p)\,drdp\right]\left[\int_0^1\int_0^1 \varphi_l(r)(r\wedge p)\varphi_k(p)\,drdp\right] + O\left(n^4\frac{K^2}{n}\right) \\
&= \sigma^4 n^4 \sum_{k,l=1}^{K}\left[\lambda_l\int_0^1 \varphi_k(r)\varphi_l(r)\,dr\right]\left[\lambda_k\int_0^1 \varphi_l(r)\varphi_k(r)\,dr\right] + O\left(n^4\frac{K^2}{n}\right) \\
&= \sigma^4 n^4 \sum_{k=1}^{K}\lambda_k^2 + O\left(n^3K^2\right). \qquad (36)
\end{aligned}
$$

Hence, combining (33)-(36) we get

$$
\begin{aligned}
tr\left(A_K^2\right) &= \frac{1}{6}n^4\sigma^4 - n^4\sigma^4\sum_{k=1}^{K}\lambda_k^2 + O\left(n^3K^2\right) \\
&= \frac{1}{6}n^4\sigma^4 - n^4\sigma^4\frac{1}{\pi^4}\sum_{k=1}^{K}\frac{1}{\left(k-\frac{1}{2}\right)^4} + O\left(n^3K^2\right). \qquad (37)
\end{aligned}
$$

Next

$$
\begin{aligned}
\sum_{k=1}^{K}\frac{1}{\left(k-\frac{1}{2}\right)^4} &= 2^4\sum_{k=1}^{K}\frac{1}{(2k-1)^4} = 2^4\left[\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4} - O\left(\frac{1}{K^3}\right)\right] \\
&= 2^4\frac{\pi^4}{96} + O\left(\frac{1}{K^3}\right) = \frac{\pi^4}{6} + O\left(\frac{1}{K^3}\right), \qquad (38)
\end{aligned}
$$

where the formula for the infinite sum in the penultimate line is given for example in Gradshteyn and Ryzhik (1980, formula 0.234-5).

It follows from (37) and (38) that

$$
tr\left(A_K^2\right) = O\left(n^3K^2\right) + O\left(\frac{n^4}{K^3}\right).
$$

Thus, from (32)

$$
Var\left(\frac{K}{n^2}y_{-1}'Q_Ky_{-1}\right) = \frac{K^2}{n^4}2tr\left(A_K^2\right) = O\left(\frac{K^4}{n} + \frac{1}{K}\right) = o(1),
$$

and the stated result (c) follows immediately.

For part (d) we write

$$
\begin{aligned}
\frac{1}{n}y_{-1}'Q_Ku &= \frac{1}{n}y_{-1}'u - \frac{1}{n}y_{-1}'\Phi_K\left(\Phi_K'\Phi_K\right)^{-1}\Phi_K'u \\
&= \frac{1}{n}y_{-1}'u - \frac{1}{n^2}y_{-1}'\Phi_K\left[I_K + O\left(\frac{K}{n}\right)\right]\Phi_K'u \\
&= \frac{1}{n}y_{-1}'u - \frac{1}{n^{\frac{3}{2}}}y_{-1}'\Phi_K\left[I_K + O\left(\frac{K}{n}\right)\right]\frac{1}{\sqrt{n}}\Phi_K'u. \qquad (39)
\end{aligned}
$$

Now

$$
\frac{1}{n^{\frac{3}{2}}}y_{-1}'\Phi_K = \frac{1}{n}\sum_{t=1}^{n}\frac{y_{t-1}}{\sqrt{n}}\varphi_K\left(\frac{t}{n}\right)'.
$$

By virtue of Gaussianity and expanding the probability space as needed, we can embed $\frac{y_{t-1}}{\sqrt{n}}$ in a Brownian motion $B = \sigma W$, so that $\frac{y_{t-1}}{\sqrt{n}} = \sigma W\left(\frac{t-1}{n}\right)$. Then, using the representation (16), i.e.

$$
\frac{y_{t-1}}{\sqrt{n}} = \sigma W\left(\frac{t-1}{n}\right) = \sigma\left[\varphi_K\left(\frac{t-1}{n}\right)'\Lambda_K^{\frac{1}{2}}\xi_K + \varphi_\perp\left(\frac{t-1}{n}\right)\Lambda_\perp^{\frac{1}{2}}\xi_\perp\right],
$$

we have

$$\begin{aligned}
\frac{1}{n}\sum_{t=1}^{n}\varphi_K\left(\frac{t}{n}\right)\frac{y_{t-1}}{\sqrt{n}} &= \frac{1}{n}\sum_{t=1}^{n}\varphi_K\left(\frac{t}{n}\right)\left[\varphi_K\left(\frac{t-1}{n}\right)'\Lambda_K^{\frac{1}{2}}\xi_K+\varphi_\perp\left(\frac{t-1}{n}\right)'\Lambda_\perp^{\frac{1}{2}}\xi_\perp\right] \\
&= \frac{1}{n}\sum_{t=1}^{n}\varphi_K\left(\frac{t}{n}\right)\varphi_K\left(\frac{t-1}{n}\right)'\Lambda_K^{\frac{1}{2}}\xi_K+o_p(1) \\
&= \Lambda_K^{\frac{1}{2}}\xi_K+o_p(1),
\end{aligned}$$

in view of the orthogonality of $\varphi_K$ and $\varphi_\perp$. It follows that we may write (39) as

$$\begin{aligned}
\frac{1}{n}y_{-1}'Q_K u &= \frac{1}{n}y_{-1}'u-\xi_K'\Lambda_K^{\frac{1}{2}}\frac{1}{\sqrt{n}}\Phi_K'u+o_p(1) \\
&= \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left[\varphi_K\left(\frac{t-1}{n}\right)'\Lambda_K^{\frac{1}{2}}\xi_K+\varphi_\perp\left(\frac{t-1}{n}\right)'\Lambda_\perp^{\frac{1}{2}}\xi_\perp\right]u_t-\xi_K'\Lambda_K^{\frac{1}{2}}\frac{1}{\sqrt{n}}\Phi_K'u+o_p(1) \\
&= \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\xi_\perp'\Lambda_\perp^{\frac{1}{2}}\varphi_\perp\left(\frac{t-1}{n}\right)u_t+o_p(1) \\
&= \xi_\perp'\Lambda_\perp^{\frac{1}{2}}\int_0^1\varphi_\perp(r)\,dB(r)+o_p(1) \\
&= o_p(1),
\end{aligned}$$

since

$$\xi_\perp'\Lambda_\perp^{\frac{1}{2}}\left(\int_0^1\varphi_\perp(r)\varphi_\perp(r)'dr\right)\Lambda_\perp^{\frac{1}{2}}\xi_\perp=\xi_\perp'\Lambda_\perp\xi_\perp=\sum_{k=K+1}^{\infty}\lambda_k\xi_k^2=O_p\left(\frac{1}{K}\right)$$

from (24) above, thereby establishing part (d).

# 6. Notation

| | | | |
|---|---|---|---|
| $\rightarrow_{a.s.}$ | almost sure convergence | $\Rightarrow, \rightarrow_d$ | weak convergence |
| $\rightarrow_p$ | convergence in probability | $[\cdot]$ | integer part of |
| $=_d$ | distributional equivalence | $r\wedge s$ | $min(r,s)$ |
| $:=$ | definitional equality | $\equiv$ | equivalence |
| $W(r)$ | standard Brownian motion | $o_p(1)$ | tends to zero in probability |
| $BM(\sigma^2)$ | Brownian motion with variance $\sigma^2$ | $o_{a.s.}(1)$ | tends to zero almost surely |

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