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Sound Approximate Reasoning about Saturated Conditional Probabilistic Independence under Controlled Uncertainty

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CDMTCS-428
October 2012

Centre for Discrete Mathematics and
Theoretical Computer Science

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October 10, 2012

Abstract

Knowledge about complex events is usually incomplete in practice. Zeros can be utilized to capture such events within probability models. In this article, Geiger and Pearl's conditional probabilistic independence statements are investigated in the presence of zeros. Random variables can be specified to be zero-free, i.e., to disallow zeros in their domains. Zero-free random variables provide an effective mechanism to control the degree of uncertainty caused by permitting zeros. A finite axiomatization for the implication problem of saturated conditional independence statements is established under controlled uncertainty, relative to discrete probability measures. The completeness proof utilizes special probability models where two events have probability one half. The special probability models enable us to establish an equivalence between the implication problem and that of a propositional fragment in Cadoli and Schaerf's $\mathcal{S}$-3 logic. Here, the propositional variables in $\mathcal{S}$ correspond to the random variables specified to be zero-free. The duality leads to an almost linear time algorithm to decide implication. It is shown that this duality cannot be extended to cover general conditional independence statements. All results subsume classical reasoning about saturated conditional independence statements as the idealized special case where every random variable is zero-free. In the presence of controlled uncertainty, zero-free random variables allow us to soundly approximate classical reasoning about saturated conditional independence statements.

Keywords: Algorithm, Approximation, Axiomatization, Complexity, Conditional Independence, Discrete probability measure, Implication, $\mathcal{S}$-3 logic, Propositional logic, Zero

1 Introduction

The concept of conditional independence is important for capturing structural aspects of probability distributions, for dealing with knowledge and uncertainty in Artificial Intelligence, and for learning and reasoning in intelligent systems [1]. Application areas include natural language processing, speech processing, computer vision, robotics, computational biology, and error-control coding [2, 1, 3]. A conditional independence (CI) statement $I(Y, Z \mid X)$ represents the independence of two sets of random variables relative to a third: given three mutually disjoint subsets X, Y, and Z of a set S of random variables, if we have knowledge about the state of X, then knowledge about the state of Y does not provide additional evidence for the state of Z and vice versa. An important problem is the implication problem, which is to decide for an arbitrary set S of random variables, and an arbitrary set $\Sigma \cup \{\varphi\}$ of CI statements over S, whether every probability model that satisfies every CI statement in Σ also satisfies φ. The significance of this problem is due to its relevance for building Bayesian networks [1]. The implication problem for CI statements is not axiomatizable by a finite set of Horn rules [4]. However, it is possible to express CI statements using polynomial likelihood formulae, and reasoning about polynomial inequalities is axiomatizable [2, 5]. Recently, the implication problem of stable CI statements [6, 7] has been shown to be axiomatizable by a finite set of Horn rules, and to be *coNP*-complete to decide [8, 9]. Here, stability means that the validity of $I(Y, Z \mid X)$ over S implies the validity of every $I(Y, Z \mid X')$ where $X \subseteq X' \subseteq S - YZ$. An important efficient subclass of stable CI statements are saturated conditional independence (SCI) statements. These are CI statements $I(Y, Z \mid X)$ over S that satisfy $XYZ = S$, i.e., where the union of X, Y and Z is S. Geiger and Pearl have established an axiomatization for the implication problem of SCI statements by a finite set of Horn rules [10].

Example 1 *Let {c(onference), p(aper), s(peaker), a(ctivity), l(ocation), o(rganizer)} denote a set of random variables that model information about conferences, their organizers, papers presented by speakers at the conference, and social activities taking place at some locations during the conference. Let* Σ *consist of* $I(psal, o \mid c)$ *and* $I(ps, al \mid co)$*, and let* φ *be* $I(ps, alo \mid c)$*. Then* Σ *implies* φ*. Indeed,* φ *can be inferred from* Σ *using Geiger and Pearl's axiomatization: a single application of the so-called weak contraction rule to the two SCI statements in* Σ *results in* φ*.* ∎

It is known that the implication of SCI statements is equivalent to Boolean implication for a fragment $\mathfrak{F}$ of Boolean propositional logic [11, 12, 13]. We illustrate this equivalence by translating Example 1 into the fragment $\mathfrak{F}$.

Example 2 *Let* $\{c', p', s', a', l', o'\}$ *denote a set of propositional variables. Let* Σ' *consist of the formulae* $\neg c' \vee (p' \wedge s' \wedge a' \wedge l') \vee o'$ *and* $\neg c' \vee \neg o' \vee (p' \wedge s') \vee (a' \wedge l')$*, and let* φ' *be the formula* $\neg c' \vee (p' \wedge s') \vee (a' \wedge l' \wedge o')$*. It is not difficult to see that* φ' *is logically implied by* Σ' *in Boolean propositional logic.* ∎

The study of conditional independence statements has largely been a study of probability models that are free from zeros. In practice, zeros occur frequently, and in form of many different types. For examples, structural zeros appear when some events do not exist, and sampling zeros occur when events exist but are currently unknown. The primary goal of this article is to analyze the implication problem of saturated conditional independence statements in the presence of zeros, relative to discrete probability measures. For this purpose, the simplest interpretation of zeros is utilized. That is, any occurrence of a zero as a marked "value" of some random variable, denoted by ζ, is interpreted as *no information.* While there is a potential loss in representing knowledge with this interpretation, it is possible to model zeros whose type itself is unknown. For example, it is difficult to decide whether a zero occurrence under the random variable *maiden name* is a structural or sampling zero, and even more so when the gender of the person is unknown.

It is a fundamental question to ask when a CI statement $I(Y, Z \mid X)$ is satisfied by a probability model that can feature occurrences of zeros. In consistency with the no information interpretation of zeros, the independence of Y-events and Z-events should be conditional on total X-events only, i.e., X-events in which no zeros feature. Indeed, if some X-event does feature a zero on some random variable, then one should not ask of Y- and Z-events to be independent. Under this definition the implication problem of SCI statements requires a careful re-examination, as illustrated by the following example.

Example 3 *In Example 1, Σ does not imply φ when zeros are permitted to occur. A probability model that has the following two events*

conference	*paper*	*speaker*	*organizer*	*activity*	*location*
ICM	*23 problems*	*Hilbert*	ζ	*petanque*	*tuileries*
ICM	*4 problems*	*Landau*	ζ	*rowing*	*river cam*

and assigns both of them a probability of one half satisfies both SCI statements in Σ, but violates φ. In fact, the occurrence of the zero marker ζ in the o-events means that $I(ps, al \mid co)$ is satisfied. ∎

In practice it would be convenient to control the degree of uncertainty introduced by permitting occurrences of zeros. For this purpose, we introduce zero-free random variables which do not contain the zero marker ζ in their domains. There are different reasons to declare random variables zero-free: for some applications some random variables are such that zeros are not expected to occur at all, and for some applications it is convenient to exclude events that do feature zeros on some random variables. The goal of this article is to study the implication problem of SCI statements under controlled uncertainty, i.e., when the set of zero-free random variables can be specified. The absence of zero markers from probability models is the idealized special case where all random variables are declared zero-free. It will be shown that zero-free random variables provide an effective mechanism to approximate classical reasoning about SCI statements, i.e., reasoning in the absence of zeros. More precisely, the implication of SCI statements under the set S_ζ of zero-free random variables is equivalent to Cadoli and Schaerf's $\mathcal{S}$-3 implication of propositional formulae in $\mathfrak{F}$. Here, $\mathcal{S}$ consists of those propositional variables that correspond to the zero-free random variables, i.e., those in S_ζ. $\mathcal{S}$-3 truth assignments assign opposite truth values to propositional variables in $\mathcal{S}$, and there are three possibilities to assign truth values to propositional variables outside of $\mathcal{S}$ and their negations: either opposite truth values are assigned, or both can be assigned *true* [14]. The following example illustrates this equivalence, and complements the previous examples.

Example 4 *For Example 2 and $\mathcal{S} = \{c', p', s', a', l'\}$, Σ' does not $\mathcal{S}$-3 imply φ'. In fact, the $\mathcal{S}$-3 truth assignment where true is assigned to a literal if and only if the literal is in $\{c', \neg p', \neg s', o', \neg o', \neg a', \neg l'\}$ is a model for Σ', but not for φ'. This $\mathcal{S}$-3 truth assignment corresponds to the two-event probability model from Example 1: assign true to a propositional variable and false to its negation whenever the events match on the random variable and are different from zero, assign false to a propositional variable and true to its negation whenever the events do not match on the random variable, and assign true to a propositional variable and true to its negation whenever the events on the random variable are both zero.* ∎

Organization. We outline the main contributions of this article in Section 2. In Section 3 we introduce, in the presence of zeros, notions such as probability models and (saturated) conditional independence statements, as well as zero-free random variables. Geiger and Pearl's finite axiomatization for the implication of saturated conditional independence statements in the absence of zeros is repeated, and it is shown that one of the inference rules, the weak contraction rule, is not sound in the presence of zeros. Replacing the weak contraction rule by a new sound rule allows us in Section 4 to establish an axiomatization, by a finite set of Horn rules, for the implication of saturated conditional independence statements under controlled uncertainty, relative to discrete probability measures. The completeness proof utilizes special probability models, consisting of only two events each of probability one half. It follows that special probability models suffice to decide the implication problem. In Section 5 the special probability models are utilized to establish an equivalence between the implication of saturated conditional independence statements under controlled uncertainty and $\mathcal{S}$-3 implication of formulae in the propositional fragment $\mathfrak{F}$. It is also shown that this equivalence cannot be extended to cover general conditional independence statements. It is established in Section 6 that the implication problem of saturated conditional independence statements under controlled uncertainty is also equivalent to the Boolean implication of formulae in $\mathfrak{F}$, but this equivalence requires the elimination of some formulae from the input instance. The latter equivalence is then utilized to establish an algorithm to decide the implication problem in time almost linear in the size of the input. We conclude in Section 7 where we also discuss options for future work.

2 Contribution

This section is dedicated to an outline of the contributions made in this article.

As a first contribution, the article introduces conditional independence statements in the presence of zeros. For this purpose we interpret occurrences of zeros uniformly as having *no information* available about some property of some event. The disadvantage of this interpretation is a loss in the representation of knowledge whenever more information is available about the property of the event, for example, that a zero is a structural zero. There are, however, several advantages associated with this interpretation. It is a very simple interpretation that is easy to understand. It does not require different treatments of different types of zeros. In particular, if it is unknown which type of zero occurs, we can still represent this knowledge using the no information interpretation. In practice, it will be unlikely to know the correct type of all occurrences of zeros. For a conditional independence statement $I(Y, Z \mid X)$ to be satisfied by a probability model in the presence of zeros, we still require that given some event $\mathbf{x}$ of X, learning the event $\mathbf{y}$ of Y is independent from the event $\mathbf{z}$ of Z. However, we restrict this requirement to events $\mathbf{x}$ of X that are total, that is, where no occurrences of zeros feature on any property in X. This requirement is consistent with the no information interpretation of zeros and the notion of conditional independence: if we have no information about some property of $\mathbf{x}$, then we must not require $\mathbf{y}$ to be independent of $\mathbf{z}$.

As a second contribution, the article introduces *zero-free* random variables. Intuitively, zero-free random variables do not permit any occurrences of zeros. Therefore, by declaring a random variable to be zero-free one discards events that are uncertain on this random variable. In practice, there will be a natural tradeoff between the desire to eliminate as much uncertainty as possible and the desire to capture as many events as possible. There can be different reasons to declare random variables zero-free. It may be unlikely that zeros will ever occur for some random variable, or it may be desirable to exclude any events in which zeros do feature on some random variable. In any case, zero-free random variables provide a simple means to control the degree of uncertainty in probability models. The article shows in technical detail how zero-free random variables can be used to soundly approximate classical reasoning about saturated conditional independence statements. Here, we mean by classical reasoning the idealized special case where no zeros ever occur in a probabilistic model, i.e., where all random variables are zero-free.

As a third contribution, the article investigates the feasibility of the implication problem associated with classes of conditional independence statements under controlled uncertainty, relative to discrete probability measures. The implication problem for the general class of conditional independence statements does not enjoy a finite axiomatization by Horn clauses, already in the idealized special case where all random variables are zero-free [4]. Moreover, the implication problem of stable conditional independence statements [6, 7] is *coNP*-complete to decide [8]. For these reasons, the focus is directed towards the large efficient subclass of saturated conditional independence (SCI) statements. These are statements $I(Y, Z \mid X)$ where the set union XYZ of X, Y and Z covers the underlying set S of random variables. Instances of this implication problem are denoted by $\Sigma \models_{S_\zeta} \varphi$ where $\Sigma \cup \{\varphi\}$ denotes a finite set of SCI statements and S_ζ is the set of zero-free random variables from S. Usually, we omit the subscript S and write $\Sigma \models \varphi$ whenever $S_\zeta = S$. Our first main technical result establishes an axiomatization $\mathfrak{Z}$, by a finite set of Horn rules, for the implication of SCI statements in the presence of controlled uncertainty. The fact that φ can be inferred from Σ by $\mathfrak{Z}$ is denoted by $\Sigma \vdash_{\mathfrak{Z}} \varphi$. Geiger and Pearl's well-known axiomatization $\mathfrak{G}$ of SCI statements represents the idealized special case of our axiomatization $\mathfrak{Z}$ where all random variables are zero-free. In the presence of controlled uncertainty, three of the four inference rules in Geiger and Pearl's axiomatization are still sound. The weak contraction rule, however, is only sound when some random variables are known to be zero-free. A simple restriction of the weak contraction rule to zero-free random variables does not suffice to gain completeness of the inference rules under controlled uncertainty. We establish a more powerful form of restriction on the weak contraction rule that does result in the axiomatization $\mathfrak{Z}$. Some special cases of the inference rule are discussed. From this discussion new insight is derived that enables us to establish a completeness argument that is very different from the one used in Geiger and Pearl's special case. In fact, for our completeness argument we construct special probability models in which two events are assigned probability one half. As a corollary, the implication problem for SCI statements under controlled uncertainty is equivalent to that over special probability models. We use $\Sigma \models_{2,S_\zeta} \varphi$ to denote an instance of the implication problem over special probability models.

As a fourth contribution, the article establishes a characterization of the implication problem in purely logical terms. SCI statements φ are known to correspond to formulae φ' in a propositional fragment $\mathfrak{F}$. Assuming that all random variables are zero-free, there is an equivalence between the implication of SCI statements and the Boolean implication of formulae in $\mathfrak{F}$ [11, 12, 13]. In this article, we establish an equivalence between the implication of SCI statements under controlled uncertainty and Cadoli and Schaerf's $\mathcal{S}$-3 implication of formulae in $\mathfrak{F}$. Here, $\mathcal{S}$ is the set of propositional variables that correspond to random variables declared zero-free. Instances of the implication problem in $\mathcal{S}$-3 logic are denoted by $\Sigma' \models^3_{\mathcal{S}} \varphi'$, and in Boolean logic by $\Sigma' \models_{BL} \varphi'$. In particular, in the special case where all random variables are zero-free, $\mathcal{S}$-3 implication coincides with Boolean implication. The result shows formally that classical reasoning about SCI statements can be soundly approximated by the use of zero-free random variables.

As a fifth contribution, the article establishes a characterization of the implication problem in algorithmic terms. For an SCI statement $\varphi = I(Y, Z \mid X)$ we write φ_c for X, $\varphi_{i,1}$ for Y and $\varphi_{i,2}$ for Z. Our axiomatization $\mathfrak{Z}$ is exploited to show that for any set S of random variables, and any set S_ζ of zero-free random variables from S, and any set $\Sigma \cup \{\varphi\}$ of SCI statements over S, $\Sigma \models_{S_\zeta} \varphi$ if and only if $\Sigma[\varphi_c S_\zeta] \models_S \varphi$, where $\Sigma[\varphi_c S_\zeta]$ consists of all those SCI statements $\sigma \in \Sigma$ where $\sigma_c \subseteq \varphi_c \cup S_\zeta$. This allows us to establish an

$$\mathcal{O}(|\Sigma| + \min\{k_{\Sigma[\varphi_c S_\zeta]}, \log \bar{p}_{\Sigma[\varphi_c S_\zeta], S_\zeta}\} \times |\Sigma[\varphi_c S_\zeta]|)$$

time algorithm for deciding whether $\Sigma \models_{S_\zeta} \varphi$. Herein, $|\Sigma|$ denotes the total number of random variables occurring in Σ, k_Σ denotes the cardinality of Σ, and $\bar{p}_{\Sigma,S_\zeta}$ denotes the minimum of the two numbers of sets in $IDepB_{\Sigma,S_\zeta}(\varphi_c)$ that have non-empty intersection with $\varphi_{i,1}$, and with $\varphi_{i,2}$, respectively. The independence basis $IDepB_{\Sigma,S_\zeta}(X)$ of a set X of random variables from S, with respect to Σ and S_ζ, is the set of all minimal, non-empty sets W of random variables from S that are independent of $S - XW$, given X. Hence, $\Sigma \models_{S_\zeta} I(Y, Z \mid X)$ if and only if Y is the union of elements of the independence basis of X with respect to Σ and S_ζ. The upper bound illustrates the impact of the set S_ζ of zero-free random variables on the time-complexity of deciding the implication problem. The equivalences established in this article for the implication problem of saturated conditional independence statements under controlled uncertainty are summarized in Table 1.

As a six contribution, the article establishes that the equivalence between SCI implication under controlled uncertainty and the $\mathcal{S}$-3 implication for $\mathfrak{F}$-formulae cannot be

1. $\Sigma \vdash_{\mathfrak{Z}} \varphi$ .... φ can be inferred from Σ by $\mathfrak{Z}$
2. $\Sigma \models_{2,S_\zeta} \varphi$ .... Σ S_ζ-implies φ over special probability models
3. $\Sigma' \models^3_{\mathcal{S}} \varphi'$ .... Σ' $\mathcal{S}$-3 implies φ'
4. $\Sigma[\varphi_c S_\zeta] \models_S \varphi$ .... $\Sigma[\varphi_c S_\zeta]$ implies φ
5. $\Sigma[\varphi_c S_\zeta] \vdash_{\mathfrak{G}} \varphi$ .... φ can be inferred from $\Sigma[\varphi_c S_\zeta]$ by $\mathfrak{G}$
6. $\Sigma[\varphi_c S_\zeta] \models_{2,S} \varphi$ .... $\Sigma[\varphi_c S_\zeta]$ implies φ over special probability models
7. $(\Sigma[\varphi_c S_\zeta])' \models_{BL} \varphi'$ .... $(\Sigma[\varphi_c S_\zeta])'$ classically implies φ'

Table 1: Characterizations of $\Sigma \models_{S_\zeta} \varphi$ with set S_ζ of zero-free random variables from set S of random variables

extended to cover arbitrary conditional independence statements. This non-extendibility result is established already for the idealized special case where every random variable is zero-free. That is, the equivalence between SCI implication and Boolean implication of $\mathfrak{F}$-formulae cannot be extended to cover general conditional independence statements. This non-extendibility result fits well into previous results from the literature. In [4], Studený showed that the implication of CI statements is different from the implication of *embedded multivalued dependencies* [15], demonstrating the non-extendibility between the implication of SCI statements and the implication of multivalued dependencies [11, 13]. In [16], Sagiv et al. showed that the equivalence between the implication of multivalued dependencies and the Boolean implication of $\mathfrak{F}$-formulae cannot be extended to an equivalence between the implication of embedded multivalued dependencies and the Boolean implication of any propositional fragment [16]. These results left open the possibility that the equivalence between the implication of SCI statements and the Boolean implication of $\mathfrak{F}$-formulae can be extended to cover general CI statements. Our result shows that this is impossible.

Therefore, the article establishes equivalences between the implication of SCI statements and zero-free random variables and the $\mathcal{S}$-3 implication of $\mathfrak{F}$-formulae, and, by the results in [17], the implication of multivalued dependencies and `NOT NULL` attributes. These equivalences are very special since any duality between two of these frameworks fails when extended to more expressive frameworks such as general CI statements and embedded multivalued dependencies. These achievements are illustrated in Figure 1. We also note that the implication problem of embedded multivalued dependencies is undecidable [18, 19] and not axiomatizable by a finite set of Horn rules [15].

3 Preliminaries

We use the framework of Geiger and Pearl [10]. We denote by S a finite set of distinct symbols $\{a_1, \ldots, a_n\}$, called *random variables.* A *domain mapping* is a mapping that

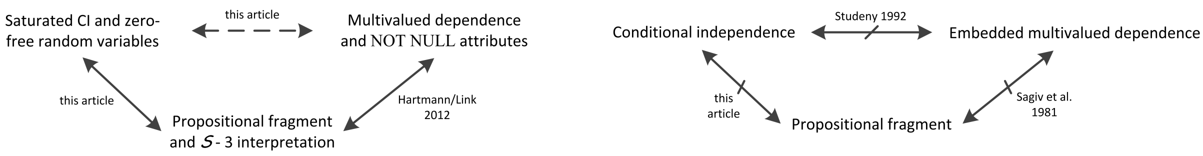

Figure 1: Summary of equivalences between implication problems and their failures for more expressive frameworks

associates a set, $dom(a)$, with each random variable a. The set $dom(a)$ is called the *domain* of a and each of its elements is called an *event* for a. For $X \subseteq S$ we say that $\mathbf{x}$ is an event of X, if $\mathbf{x} \in \prod_{a\in X} dom(a)$. For an event $\mathbf{x} = (\mathbf{a}_1, \ldots, \mathbf{a}_k)$ of X with $\mathbf{a}_i \in dom(a_i)$, we write $\mathbf{x}(a_i)$ for the event $\mathbf{a}_i$ over a_i. For some $Y \subseteq X$ we write $\mathbf{x}(Y)$ for the projection of $\mathbf{x}$ onto Y, that is, $\mathbf{x}(Y) = \prod_{a\in Y}\{\mathbf{x}(a)\}$.

3.1 Zeros and zero-free random variables

In theory one can assume that events always exist and are even known. In practice, these assumptions fail frequently. Indeed, it can happen in most samples that some events do not exist, or that some existing events are currently unknown. In the first case, one speaks commonly of structural zeros, and in the latter case of sampling zeros. In practice, it is often difficult to tell whether a given zero is a structural zero or a sampling zero. The goal of this article is to investigate the properties of saturated conditional probabilistic independence in the presence of zeros.

Unless we say otherwise we assume that each domain $dom(a)$ contains the element ζ, which we call the *zero marker*. Although we include ζ in the domain of random variables, we prefer to think of ζ as a marker. More precisely, an occurrence $\mathbf{x}(a) = \zeta$ of the zero marker ζ in some event $\mathbf{x}$ denotes that no information is currently available about the event of random variable a in $\mathbf{x}$. We say that the event $\mathbf{e}$ over S is X-*total*, if $\mathbf{e}(a) \neq \zeta$ for all $a \in X$. The interpretation of the zero marker ζ as no information means that an event does either not exist (structural zero), or an event exists but is currently unknown (sampling zero). The disadvantage of using this interpretation is a loss in knowledge when representing known structural zeros or known sample zeros in form of the zero marker. However, if we do not know whether a zero is a structural or a sampling zero, then we can still denote it in form of the zero marker. In fact, the no information interpretation is the most primitive interpretation of occurrences of zero. It is a goal of this article to investigate the properties of saturated conditional probabilistic independence under the no information interpretation of zeros.

It is an advantage to gain control over the occurrences of zeros. For this purpose we introduce *zero-free random variables*. If a random variable a of S is declared to be *zero-free*, then $\zeta \notin dom(a)$. For a given S we define S_ζ to be the set of random variables of S that are zero-free. It is a goal of this article to investigate the properties of saturated conditional probabilistic independence under controlled uncertainty, i.e., in the presence of zero-free random variables. Indeed, it will turn out that zero-free random variables provide an effective means to not just control the degree of uncertainty, but also to soundly approximate classical reasoning about saturated conditional probabilistic independence.

3.2 Saturated conditional independence and controlled uncertainty

A *probability model* over $(S = \{a_1, \ldots, a_n\}, S_\zeta)$ is a pair (dom, P) where dom is a domain mapping that maps each a_i to a finite domain $dom(a_i)$, and $P : dom(a_1) \times \cdots \times dom(a_n) \rightarrow [0, 1]$ is a probability distribution having the Cartesian product of these domains as its sample space. Note that $\zeta \notin dom(a_i)$ whenever $a_i \in S_\zeta$.

Definition 1 *The expression $I(Y, Z \mid X)$ where X, Y, and Z are disjoint subsets of S is called a* conditional independence *(CI)* statement *over (S, S_ζ). If $XYZ = S$, we call $I(Y, Z \mid X)$ a* saturated *CI (SCI) statement. Let (dom, P) be a probability model over (S, S_ζ). A CI statement $I(Y, Z \mid X)$ is said to* hold for (dom, P) *if for every total event $\mathbf{x}$ of X, for every event $\mathbf{y}$ of Y, and for every event $\mathbf{z}$ of Z,*

$$P(\mathbf{y}, \mathbf{z}, \mathbf{x}) \cdot P(\mathbf{x}) = P(\mathbf{y}, \mathbf{x}) \cdot P(\mathbf{z}, \mathbf{x}). \quad (1)$$

Equivalently, (dom, P) *is said to* satisfy $I(Y, Z \mid X)$. $\blacksquare$

Remark 1 *The satisfaction of CI statements $I(Y, Z \mid X)$ only requires equation (1) to hold for* total *events $\mathbf{x}$ of X. The reason is that the independence for an event of Y and an event of Z is conditional on the event $\mathbf{x}$ of X. However, in case there is* no information *about some event of $\mathbf{x}$, then there should not be any requirement on the independence for an event of Y and an event of Z.* $\blacksquare$

Remark 2 *If every random variable is declared to be zero-free, i.e. when $S_\zeta = S$, then Definition 1 reduces to the standard definition of CI statements [10, 1].* $\blacksquare$

Remark 3 *Studený [4] showed that, already in the special case where $S_\zeta = S$, the implication problem of CI statements cannot be axiomatized by a finite set of Horn rules of the form*

$$I(Y_1, Z_1 \mid X_1) \wedge \cdots \wedge I(Y_k, Z_k \mid X_k) \rightarrow I(Y, Z \mid X)\ .$$

However, it is possible to express CI statements using polynomial likelihood formulae, and reasoning about polynomial inequalities is axiomatizable [2, 5]. Recently, the implication problem of stable CI statements [6, 7] has been shown to be axiomatizable by a finite set of Horn rules, and to be coNP-complete to decide [8, 9]. Here, stability means that the validity of $I(Y, Z \mid X)$ over (S, S_ζ) implies the validity of every $I(Y, Z \mid X')$ where $X \subseteq X' \subseteq S - YZ$. For the special case where $S_\zeta = S$ it is known that saturated CI statements form an expressive subclass of stable CI statements with good computational properties [10]. The goal of this article is to investigate the computational properties of SCI statements under controlled uncertainty.

Example 5 *Let $\{c(onference), p(aper), s(peaker), a(ctivity), l(ocation), o(rganizer)\}$ denote the set S of random variables from Example 1, let $S_\zeta = \{c, p, s, e, l\}$, let Σ consist of $I(psal, o \mid c)$ and $I(ps, al \mid co)$, and φ be $I(ps, alo \mid c)$. We may define the following probability model (dom, P) over (S, S_ζ): $dom(c) = \{ICM, AAAI\}$, $dom(p) = \{23\ problems, 4\ problems\}$, $dom(s) = \{Hilbert, Landau\}$, $dom(o) = \{\zeta, Klein, Cantor\}$, $dom(a) = \{petanque, rowing\}$, $dom(l) = \{tuileries, river\ cam\}$, and define P by assigning the probability one half to each of the following two events over S:*

conference	*paper*	*speaker*	*organizer*	*activity*	*location*
ICM	*23 problems*	*Hilbert*	ζ	*petanque*	*tuileries*
ICM	*4 problems*	*Landau*	ζ	*rowing*	*river cam*

.

It follows that (dom, P) satisfies $I(psal, o \mid c)$ and $I(ps, al \mid co)$, but violates $I(ps, alo \mid c)$.

For the remainder of the article we will be interested in saturated CI statements. Let $\Sigma \cup \{\varphi\}$ be a set of SCI statements over (S, S_ζ). We say that Σ *S_ζ-implies* φ, denoted by $\Sigma \models_{S_\zeta} \varphi$, if every probability model over (S, S_ζ) that satisfies every SCI statement in Σ also satisfies the SCI statement φ. The *implication problem* for SCI statements in the presence of controlled uncertainty is defined as the following problem.

PROBLEM:	Implication problem
INPUT:	(S, S_ζ), Set $\Sigma \cup \{\varphi\}$ of SCI statements over (S, S_ζ)
OUTPUT:	Yes, if $\Sigma \models_{S_\zeta} \varphi$; No, otherwise

Example 6 *For S, S_ζ, $\Sigma \cup \{\varphi\}$ from Example 5, the probability model (dom, P) over (S, S_ζ) is a witness that Σ does not S_ζ-imply φ.*

For Σ we let $\Sigma^*_{S_\zeta} = \{\varphi \mid \Sigma \models_{S_\zeta} \varphi\}$ be the *semantic closure* of Σ, i.e., the set of all SCI statements S_ζ-implied by Σ. In order to determine the S_ζ-implied SCI statements we use a syntactic approach by applying inference rules. These inference rules have the form

$$\frac{\text{premise}}{\text{conclusion}}\text{condition}$$

and inference rules without any premises and any condition are called axioms. The premise consists of a finite set of SCI statements, and the conclusion is a singleton SCI statement. The condition of the rule is simple in the sense that it stipulates a simple syntactic restriction on the application of the rule. Instead of using this graphical representation, we could also state our rules in the form of Horn rules, as in Remark 3. For example, the restricted weak contraction rule ($\mathcal{R}$) from Table 3 can be stated as:

$$\forall S \forall S_\zeta \subseteq S \forall U, V, W, X, Z \subseteq S \forall Y \subseteq S_\zeta$$
$$((I(ZW, YUV \mid X) \wedge I(UZ, VW \mid XY)) \rightarrow I(Z, UVWY \mid X)).$$

An inference rule is called *sound*, if every probability model over (S, S_ζ) that satisfies every SCI statement in the premise of the rule also satisfies the SCI statement in the conclusion of the rule, given that the condition is satisfied. We let $\Sigma \vdash_{\mathfrak{R}} \varphi$ denote the

$$\frac{}{I(S-X,\emptyset \mid X)}$$ (saturated trivial independence, $\mathcal{T}$)	$$\frac{I(Y,Z \mid X)}{I(Z,Y \mid X)}$$ (symmetry, $\mathcal{S}$)
$$\frac{I(ZW,Y \mid X) \qquad I(Z,W \mid XY)}{I(Z,YW \mid X)}$$ (weak contraction, $\mathcal{C}$)	$$\frac{I(Y,ZW \mid X)}{I(Y,Z \mid XW)}$$ (weak union, $\mathcal{W}$)

Table 2: Axiomatization $\mathfrak{G} = \{\mathcal{T}, \mathcal{S}, \mathcal{C}, \mathcal{W}\}$ of SCI statements over S

inference of φ from Σ by the set $\mathfrak{R}$ of inference rules. That is, there is some sequence $\gamma = [\sigma_1, \ldots, \sigma_n]$ of SCI statements such that $\sigma_n = \varphi$ and every σ_i is an element of Σ or results from an application of an inference rule in $\mathfrak{R}$ to some elements in $\{\sigma_1, \ldots, \sigma_{i-1}\}$. For Σ, let $\Sigma^+_{\mathfrak{R}} = \{\varphi \mid \Sigma \vdash_{\mathfrak{R}} \varphi\}$ be its *syntactic closure* under inferences by $\mathfrak{R}$. A set $\mathfrak{R}$ of inference rules is said to be *sound* (*complete*) for the implication of SCI statements under controlled uncertainty, if for every S, every $S_\zeta \subseteq S$ and for every set Σ of SCI statements over (S, S_ζ) we have $\Sigma^+_{\mathfrak{R}} \subseteq \Sigma^*_{S_\zeta}$ ($\Sigma^*_{S_\zeta} \subseteq \Sigma^+_{\mathfrak{R}}$). The (finite) set $\mathfrak{R}$ is said to be a (finite) *axiomatization* for the implication of SCI statements under controlled uncertainty if $\mathfrak{R}$ is both sound and complete.

In the idealized special case where all random variables are zero-free, Geiger and Pearl have established a finite axiomatization for the implication of SCI statements [10]. That is, if for any given S, S_ζ is assumed to be S, then the set $\mathfrak{G}$ of inference rules from Table 2 forms an axiomatization for the implication of SCI statements in form of a finite set of Horn rules.

The following lemma shows that $\mathfrak{G}$ does not form a finite axiomatization for the implication of SCI statements under controlled uncertainty.

Lemma 1 *The weak contraction rule ($\mathcal{C}$) is not sound for the implication of SCI statements under controlled uncertainty.*

Proof It suffices to find some probability model (dom, P) over (S, S_ζ) that satisfies $I(ZW,Y \mid X)$ and $I(Z,W \mid XY)$, but violates $I(Z,YW \mid X)$. Such a probability model has been defined in Example 5 where $S = \{s,a,p,e,l,o\}$, $S_\zeta = \{s,a,p,e,l\}$, $Z = \{a,p\}$, $W = \{e,l\}$, $Y = \{o\}$, $X = \{s\}$.

4 A finite axiomatization of SCI statements in the presence of controlled uncertainty

The goal of this section is to prove that the set $\mathfrak{Z} = \{\mathcal{T}, \mathcal{S}, \mathcal{R}, \mathcal{W}\}$ of inference rules from Table 3 is a finite axiomatization for the implication of SCI statements under controlled uncertainty. We will first show the soundness of the inference rules, in particular the restricted weak contraction rule. We will then give several remarks that provide insight

$\dfrac{}{I(S-X,\emptyset \mid X)}$ (saturated trivial independence, $\mathcal{T}$)	$\dfrac{I(Y,Z \mid X)}{I(Z,Y \mid X)}$ (symmetry, $\mathcal{S}$)
$\dfrac{I(ZW,YUV \mid X) \qquad I(UZ,VW \mid XY)}{I(Z,UVWY \mid X)} Y \subseteq S_\zeta$ (restricted weak contraction, $\mathcal{R}$)	$\dfrac{I(Y,ZW \mid X)}{I(Y,Z \mid XW)}$ (weak union, $\mathcal{W}$)

Table 3: Axiomatization $\mathfrak{Z} = \{\mathcal{T}, \mathcal{S}, \mathcal{R}, \mathcal{W}\}$ with set S_ζ of zero-free random variables over S

into the expressivity of this rule. Subsequently, we will define the notion of an independence basis for a given set of random variables under a given set of SCI statements and zero-free random variables. This notion will allow us to establish the completeness of $\mathfrak{Z}$. The completeness argument constructs a special probability model in which two events of probability one half are defined. The argument shows, in particular, that special probability models suffice to decide any instance of the implication problem for SCI statements under controlled uncertainty. This result will be fundamental for establishing the characterization of the implication problem in terms of $\mathcal{S}$-3 logic in Section 5.

4.1 Soundness

Lemma 2 *The inference rules ($\mathcal{T}$), ($\mathcal{S}$), ($\mathcal{R}$) and ($\mathcal{W}$) are sound for the implication of SCI statements under controlled uncertainty.*

Proof The soundness of ($\mathcal{T}$), ($\mathcal{S}$), and ($\mathcal{W}$) can be shown just like the case where all random variables are zero-free [10, 1]. It remains to establish the soundness of the restricted contraction rule ($\mathcal{R}$). Let (dom, P) be a probability model over (S, S_ζ) that satisfies the SCI statements $I(ZW, YUV \mid X)$ and $I(UZ, VW \mid XY)$, and let $Y \subseteq S_\zeta$. We need to show that (dom, P) also satisfies the SCI statement $I(Z, UVWY \mid X)$. Let $\mathbf{x}$ be a total event of X, and let $\mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ be events of Y, Z, U, V, and W, respectively. Since $Y \subseteq S_\zeta$ holds, we conclude that $\mathbf{y}$ is a total event of Y. Since (dom, P) satisfies the SCI statement $I(ZW, YUV \mid X)$ the following holds

$$P(\mathbf{x}) \cdot P(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v},\mathbf{w}) = P(\mathbf{x},\mathbf{z},\mathbf{w}) \cdot P(\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v}).$$

The marginalization on $UXYZ$ yields:

$$P(\mathbf{x}) \cdot P(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u}) = P(\mathbf{x},\mathbf{z}) \cdot P(\mathbf{x},\mathbf{y},\mathbf{u}). \tag{2}$$

Since (dom, P) satisfies the SCI statement $I(UZ, VW \mid XY)$ the following holds

$$\begin{aligned} P(\mathbf{x},\mathbf{y}) \cdot P(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v},\mathbf{w}) &= P(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u}) \cdot P(\mathbf{x},\mathbf{y},\mathbf{v},\mathbf{w}) \\ &= P(\mathbf{x},\mathbf{y}) \cdot P(\mathbf{z},\mathbf{u} \mid \mathbf{x},\mathbf{y}) \cdot P(\mathbf{x},\mathbf{y},\mathbf{v},\mathbf{w}). \end{aligned}$$

These equations imply the following equality:

$$P(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v},\mathbf{w}) = P(\mathbf{z},\mathbf{u} \mid \mathbf{x},\mathbf{y}) \cdot P(\mathbf{x},\mathbf{y},\mathbf{v},\mathbf{w}). \tag{3}$$

Moreover, the marginalization of

$$P(\mathbf{x},\mathbf{y}) \cdot P(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v},\mathbf{w}) = P(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u}) \cdot P(\mathbf{x},\mathbf{y},\mathbf{v},\mathbf{w})$$

on $UVWXY$ yields:

$$\begin{aligned} P(\mathbf{x},\mathbf{y}) \cdot P(\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v},\mathbf{w}) &= P(\mathbf{x},\mathbf{y},\mathbf{u}) \cdot P(\mathbf{x},\mathbf{y},\mathbf{v},\mathbf{w}) \\ &= P(\mathbf{x},\mathbf{y}) \cdot P(\mathbf{u} \mid \mathbf{x},\mathbf{y}) \cdot P(\mathbf{x},\mathbf{y},\mathbf{v},\mathbf{w}). \end{aligned}$$

These equations imply the following equality:

$$P(\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v},\mathbf{w}) = P(\mathbf{u} \mid \mathbf{x},\mathbf{y}) \cdot P(\mathbf{x},\mathbf{y},\mathbf{v},\mathbf{w}). \tag{4}$$

Using the three equalities above, we can make the following derivation:

$$\begin{aligned} P(\mathbf{x}) \cdot P(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u},\mathbf{v},\mathbf{w}) &\overset{(3)}{=} P(\mathbf{x}) \cdot P(\mathbf{u},\mathbf{z} \mid \mathbf{x},\mathbf{y}) \cdot P(\mathbf{x},\mathbf{y},\mathbf{v},\mathbf{w}) \\ &= P(\mathbf{x}) \cdot P(\mathbf{u},\mathbf{z} \mid \mathbf{x},\mathbf{y}) \cdot P(\mathbf{x},\mathbf{y}) \cdot P(\mathbf{v},\mathbf{w} \mid \mathbf{x},\mathbf{y}) \\ &= P(\mathbf{x}) \cdot P(\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{z}) \cdot P(\mathbf{v},\mathbf{w} \mid \mathbf{x},\mathbf{y}) \\ &\overset{(2)}{=} P(\mathbf{x},\mathbf{z}) \cdot P(\mathbf{x},\mathbf{y},\mathbf{u}) \cdot P(\mathbf{v},\mathbf{w} \mid \mathbf{x},\mathbf{y}) \\ &= P(\mathbf{x},\mathbf{z}) \cdot P(\mathbf{u} \mid \mathbf{x},\mathbf{y}) \cdot P(\mathbf{x},\mathbf{y}) \cdot P(\mathbf{v},\mathbf{w} \mid \mathbf{x},\mathbf{y}) \\ &= P(\mathbf{x},\mathbf{z}) \cdot P(\mathbf{u} \mid \mathbf{x},\mathbf{y}) \cdot P(\mathbf{x},\mathbf{y},\mathbf{v},\mathbf{w}) \\ &\overset{(4)}{=} P(\mathbf{x},\mathbf{z}) \cdot P(\mathbf{x},\mathbf{y},\mathbf{v},\mathbf{w},\mathbf{u}). \end{aligned}$$

We have shown that (dom, P) also satisfies the SCI statement $I(Z, UVWY \mid X)$.

Example 7 *Consider again Example 5 where* $S = \{c,p,s,a,l,o\}$*,* Σ *consists of* $I(psal, o \mid c)$ *and* $I(ps, al \mid co)$*, and* φ *is* $I(ps, alo \mid c)$*. However, instead of* $S_\zeta = \{c,p,s,a,l\}$ *let* $S_\zeta = \{o\}$*. Then* $\Sigma \models_{S_\zeta} \varphi$*. This can be shown by a simple inference using the restricted weak contraction rule* $(\mathcal{R})$*:*

$$\frac{I(psal, o \mid c) \qquad I(ps, al \mid co)}{I(ps, alo \mid c)} \{o\} \subseteq S_\zeta \quad .$$

Since $\mathfrak{Z}$ *is sound,* $\Sigma \models_{S_\zeta} \varphi$*.*

The restricted weak contraction rule $(\mathcal{R})$ is the only new inference rule necessary to establish an axiomatization of saturated conditional independence under controlled uncertainty. Indeed, under controlled uncertainty, the restricted weak contraction rule $(\mathcal{R})$ takes on the role that the weak contraction rule $(\mathcal{C})$ played in the classical case where all random variables are zero-free. There are several special cases which provide further insight into the expressivity of the restricted weak contraction rule $(\mathcal{R})$.

Remark 4 *The first special case we consider for* $(\mathcal{R})$ *is when* $Y = \emptyset$*. Then the restricted weak contraction rule* $(\mathcal{R})$ *becomes*

$$\frac{I(ZW, UV \mid X) \qquad I(UZ, VW \mid X)}{I(Z, UVW \mid X)}$$

and we call this rule the Boolean rule $(\mathcal{B})$*. Indeed, for* $X \subseteq S$ *we can define* $IDep_{\Sigma,S_\zeta}(X) = \{M \mid I(M, N \mid X) \in \Sigma_3^+\}$*. The Boolean rule shows that for* $M, N \in IDep_{\Sigma,S_\zeta}(X)$ *we also have* $M \cap N, M \cup N, M - N \in IDep_{\Sigma,S_\zeta}(X)$*.*

Remark 5 *The second special case we consider for* $(\mathcal{R})$ *is when* $UV = \emptyset$*. Then the restricted weak contraction rule* $(\mathcal{R})$ *becomes*

$$\frac{I(ZW, Y \mid X) \qquad I(Z, W \mid XY)}{I(Z, WY \mid X)} \, Y \subseteq S_\zeta$$

and we call this rule the weak zero contraction rule $(\mathcal{Z})$*. The rule* $(\mathcal{Z})$ *results from Geiger and Pearl's weak contraction rule* $(\mathcal{C})$ *by adding the condition* $Y \subseteq S_\zeta$ *which is necessary to guarantee soundness of the rule in the presence of zeros. Note that the inference in Example 7 applies the weak zero contraction rule* $(\mathcal{Z})$*.*

Remark 6 *One may hypothesize that the system* $\{\mathcal{T}, \mathcal{S}, \mathcal{Z}, \mathcal{W}\}$ *forms an axiomatization for the implication of saturated conditional independence in the presence of zeros. However, the system is incomplete. Indeed,* $(\mathcal{R})$ *is sound, but cannot be derived from* $\{\mathcal{T}, \mathcal{S}, \mathcal{Z}, \mathcal{W}\}$ *since* UV *in* $(\mathcal{R})$ *is not required to form a subset of* S_ζ*. The weak zero contraction rule* $(\mathcal{Z})$*, however, cannot capture such an inference in general.*

Remark 7 *Finally, consider the special case when* $S_\zeta = S$*. Then the restricted weak contraction rule* $(\mathcal{R})$ *becomes*

$$\frac{I(ZW, YUV \mid X) \qquad I(UZ, VW \mid XY)}{I(Z, UVWY \mid X)}.$$

In this special case, it already follows that $(\mathcal{R})$ *can be derived from the weak union* $(\mathcal{W})$*, symmetry* $(\mathcal{S})$*, and weak contraction rule* $(\mathcal{C})$*. Indeed, just substitute in* $(\mathcal{C})$ *the set* YUV *for the set* Y*. This is not a surprise, since* $\mathfrak{G}$ *is complete for the implication of SCI statements in the special case where* $S_\zeta = S$*.*

4.2 The independence basis

For some S, some $S_\zeta \subseteq S$, and some set Σ of SCI statements over (S, S_ζ), and some $X \subseteq S$ let $IDep_{\Sigma,S_\zeta}(X) := \{Y \subseteq S - X \mid \Sigma \vdash_{\mathfrak{Z}} I(Y, Z \mid X)\}$ denote the set of all $Y \subseteq S - X$ such that $I(Y, Z \mid X)$ can be inferred from Σ by $\mathfrak{Z}$. The soundness of the Boolean rule from Remark 4 implies that

$$(IDep_{\Sigma,S_\zeta}(X), \subseteq, \cup, \cap, (\cdot)^{\mathcal{C}}, \emptyset, S - X)$$

forms a finite Boolean algebra where $(\cdot)^{\mathcal{C}}$ maps a set W to its complement $S - XW$. Recall that an element $a \in P$ of a poset $(P, \sqsubseteq, 0)$ with least element 0 is called an *atom* of $(P, \sqsubseteq, 0)$ precisely when $a \neq 0$ and every element $b \in P$ with $b \sqsubseteq a$ satisfies $b = 0$ or $b = a$ [20]. Further, $(P, \sqsubseteq, 0)$ is said to be *atomic* if for every element $b \in P - \{0\}$ there is an atom $a \in P$ with $a \sqsubseteq b$. In particular, every finite Boolean algebra is atomic [20]. Let $IDepB_{\Sigma,S_\zeta}(X)$ denote the set of all atoms of $(IDep_{\Sigma,S_\zeta}(X), \subseteq, \emptyset)$. We call $IDepB_{\Sigma,S_\zeta}(X)$ the *independence basis* of X with respect to Σ and S_ζ. The importance of this notion for the implication problem of SCI statements under controlled uncertainty is manifested in the following result.

Theorem 1 *Let Σ be a set of SCI statements over (S, S_ζ). Then $\Sigma \vdash_{\mathfrak{Z}} I(Y, Z \mid X)$ if and only if $Y = \bigcup \mathcal{Y}$ for some $\mathcal{Y} \subseteq IDepB_{\Sigma,S_\zeta}(X)$.*

Proof Let $Y \in IDep_{\Sigma,S_\zeta}(X)$. Since every element b of a Boolean algebra is the union over those atoms a with $a \subseteq b$ [20] it follows that $Y = \bigcup \mathcal{Y}$ for $\mathcal{Y} = \{W \in IDepB_{\Sigma,S_\zeta}(X) \mid W \subseteq Y\}$.

Vice versa, let $Y = \bigcup \mathcal{Y}$ for some $\mathcal{Y} \subseteq IDepB_{\Sigma,S_\zeta}(X)$. Since $I(W, W' \mid X) \in \Sigma^+_{\mathfrak{Z}}$ holds for every $W \in \mathcal{Y}$ successive applications of the Boolean and symmetry rules result in $I(Y, Z \mid X) \in \Sigma^+_{\mathfrak{Z}}$.

Example 8 *Recall Example 7 where $S = \{c, p, s, a, l, o\}$, $S_\zeta = \{o\}$, and Σ consists of $I(psal, o \mid c)$ and $I(ps, al \mid co)$, and φ is $I(ps, alo \mid c)$. It follows that $IDepB_{\Sigma,S_\zeta}(c) = \{\{p, s\}, \{a, l\}, \{o\}\}$. According to Theorem 1, φ is S_ζ-implied by Σ.*

Example 9 *Let now be $S = \{c, p, s, a, l, o\}$, $S_\zeta = \{c, p, s, a, l\}$, and Σ be as in the previous example, i.e., consist of $I(psal, o \mid c)$ and $I(ps, al \mid co)$, and φ is $I(ps, alo \mid c)$. It follows that $IDepB_{\Sigma,S_\zeta}(c) = \{\{p, s, a, l\}, \{o\}\}$. According to Theorem 1, φ is not S_ζ-implied by Σ.*

4.3 Completeness

Theorem 2 *The set $\mathfrak{Z}$ is complete for the implication of SCI statements under controlled uncertainty.*

Proof Let $\Sigma \cup \{I(Y, Z \mid X)\}$ be a set of SCI statements over (S, S_ζ), and suppose that $I(Y, Z \mid X)$ cannot be inferred from Σ by $\mathfrak{Z}$. We will show that $I(Y, Z \mid X)$ is not S_ζ-implied by Σ. For this purpose, we will construct a probability model over (S, S_ζ) that satisfies all SCI statements of Σ, but violates $I(Y, Z \mid X)$.

Let $IDepB_{\Sigma,S_\zeta}(X) = \{W_1, \ldots, W_k\}$, in particular $S = XW_1 \cdots W_k$. Since $I(Y, Z \mid X) \notin \Sigma^+_{\mathfrak{Z}}$ we conclude by Theorem 1 that Y is not the union of some elements of $IDepB_{\Sigma,S_\zeta}(X)$. Consequently, there is some $i \in \{1, \ldots, k\}$ such that $Y \cap W_i \neq \emptyset$ and $W_i - Y \neq \emptyset$ hold. Let $T := \bigcup_{j \in \{1,\ldots,i-1,i+1,\ldots,k\}} W_j \cap S_\zeta$, and $T' := \bigcup_{j \in \{1,\ldots,i-1,i+1,\ldots,k\}} W_j - S_\zeta$. In particular, S is the disjoint union of X, T, T', and W_i. For every $a \in S - S_\zeta$ we define $dom(a) = \{\mathbf{0}, \mathbf{1}, \zeta\}$; and for every $a \in S_\zeta$ we define $dom(a) = \{\mathbf{0}, \mathbf{1}\}$. We define the following two events $\mathbf{e}_1$ and $\mathbf{e}_2$ of S. We define $\mathbf{e}_1(a) = \mathbf{0}$ for all $a \in XW_iT$, $\mathbf{e}_1(a) = \zeta$

for all $a \in T'$. We further define $\mathbf{e}_2(a) = \mathbf{e}_1(a)$ for all $a \in XTT'$, and $\mathbf{e}_2(a) = \mathbf{1}$ for all $a \in W_i$. As probability measure we define $P(\mathbf{e}_1) = P(\mathbf{e}_2) = 0.5$. It follows from the construction that (dom, P) does not satisfy $I(Y, Z \mid X)$.

It remains to show that (dom, P) satisfies every SCI statement $I(V, W \mid U)$ in Σ. Suppose that for some total event $\mathbf{u}$ of U, $P(\mathbf{u}) = 0$. Then equation (1) will always be satisfied. If $P(\mathbf{u}, \mathbf{v}) = 0$ or $P(\mathbf{u}, \mathbf{w}) = 0$ for some total event $\mathbf{u}$ of U, and for some event $\mathbf{v}$ of V or for some event $\mathbf{w}$ of W, then $P(\mathbf{u}, \mathbf{v}, \mathbf{w}) = 0$. Then equation (1) is also satisfied. Suppose that for some total event $\mathbf{u}$ of U, $P(\mathbf{u}) = 0.5$. If for some event $\mathbf{v}$ of V and for some event $\mathbf{w}$ of W, $P(\mathbf{u}, \mathbf{v}) = P(\mathbf{u}, \mathbf{w}) = 0.5$, then $P(\mathbf{u}, \mathbf{v}, \mathbf{w}) = 0.5$, too. It remains to consider the case where $\mathbf{u}$ is a total event of U such that $P(\mathbf{u}) = 1$. In this case, the construction of the probability model tells us that $U \subseteq XT$. Consequently, we can apply the weak union $(\mathcal{W})$ and symmetry rules $(\mathcal{S})$ to $I(V, W \mid U) \in \Sigma$ to infer $I(V - XT, W - XT \mid XT) \in \Sigma^+_{\mathfrak{Z}}$. Theorem 1 also shows that $I(W_i, TT' \mid X) \in \Sigma^+_{\mathfrak{Z}}$. However, for $Z' = V - XTT'$ and $W' = W - XTT'$ we have $W_i = Z'W'$; and for $U' = (V - XT) \cap T'$ and $V' = (W - XT) \cap T'$ we have $T' = U'V'$. Thus, applying the restricted contraction rule $(\mathcal{R})$ to $I(W'Z', TU'V' \mid X)$ and $I(U'Z', V'W' \mid XT)$ we infer $I(Z', U'V'W'T \mid X) \in \Sigma^+_{\mathfrak{Z}}$, since $T \subseteq S_\zeta$. Hence, $I(V - XTT', TT'(W - XTT') \mid X) \in \Sigma^+_{\mathfrak{Z}}$. It follows from Theorem 1 that $V - XTT'$ is the union of elements from $IDepB_{\Sigma,S_\zeta}(X)$. Suppose first that $V - XTT' = W_i$. Then, $P(\mathbf{u}, \mathbf{v}) = 0.5$, $P(\mathbf{u}, \mathbf{w}) = 1$ and $P(\mathbf{u}, \mathbf{v}, \mathbf{w}) = 0.5$. Otherwise, $V - XTT' = \emptyset$. Then, $P(\mathbf{u}, \mathbf{v}) = 1$, $P(\mathbf{u}, \mathbf{w}) = 0.5$ and $P(\mathbf{u}, \mathbf{v}, \mathbf{w}) = 0.5$. This concludes the proof.

Remark 8 *In the idealized special case where all random variables are zero-free, Geiger and Pearl's completeness proof of* $\mathfrak{G}$ *for the implication of SCI statements constructs a probability model with* $2^{|X|+1}$ *events, where* $I(Y, Z \mid X) \notin \Sigma^+_{\mathfrak{G}}$ *[10]. For this special case, a new completeness proof of* $\mathfrak{G}$ *was given recently that constructs a probability model of two events both of probability one half [11]. The construction for the completeness of* $\mathfrak{Z}$ *for the general case, as given in the proof of Theorem 2, reduces to the recent construction from [11] for the idealized special case where all random variables are zero-free. Even in this special case, the original construction in [10] resulted in a probability model with two events only for marginal SCI statements of the form* $I(Y, Z \mid \emptyset) \notin \Sigma^+_{\mathfrak{G}}$.

We illustrate the construction in the completeness argument on our running example

Example 10 *Let* $S = \{c, p, s, a, l, o\}$, $S_\zeta = \{c, p, s, a, l\}$, *and* Σ *be as in Example 9, i.e., consist of* $I(psal, o \mid c)$ *and* $I(ps, al \mid co)$, *and* φ *is* $I(ps, alo \mid c)$. *Recall that* $IDepB_{\Sigma,S_\zeta}(c) = \{\{p, s, a, l\}, \{o\}\}$. *Applying the construction from the completeness proof, we define the following probability model:* $dom(c) = dom(p) = dom(s) = dom(a) = dom(l) = \{\mathbf{0}, \mathbf{1}\}$, *and* $dom(o) = \{\zeta, \mathbf{0}, \mathbf{1}\}$; *and* P *assigns probability one half to the following two events:*

c	p	s	a	l	o
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	ζ
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	ζ

.

Suitable substitutions of these events result in the probability model shown in Example 3.

4.4 Special probability models

We call a probability model (dom, P) over (S, S_ζ) *special*, if for every $a \in S_\zeta$, $dom(a)$ consists of two elements, for every $a \in S - S_\zeta$, $dom(a)$ consists of two elements and the zero marker ζ, and there are two events $\mathbf{e}_1, \mathbf{e}_2$ over S such that $P(\mathbf{e}_1) = 0.5 = P(\mathbf{e}_2)$. We say that Σ S_ζ*-implies* φ *in the world of special probability models*, denoted by $\Sigma \models_{2,S_\zeta} \varphi$, if every special probability model over (S, S_ζ) that satisfies every SCI statement in Σ also satisfies the SCI statement φ. The following variant of the implication problem for SCI statements emerges.

PROBLEM:	Implication problem in the world of special probability models
INPUT:	Schema (S, S_ζ), Set $\Sigma \cup \{\varphi\}$ of SCI statements over (S, S_ζ)
OUTPUT:	Yes, if $\Sigma \models_{2,S_\zeta} \varphi$; No, otherwise

The proof of Theorem 2 implies the following result.

Corollary 1 *The implication problem for SCI statements under controlled uncertainty coincides with the implication problem for SCI statements under controlled uncertainty in the world of special probability models.*

Proof Let $\Sigma \cup \{\varphi\}$ be a set of SCI statements over (S, S_ζ). We need to show that $\Sigma \models_{S_\zeta} \varphi$ if and only if $\Sigma \models_{2,S_\zeta} \varphi$. If it does not hold that $\Sigma \models_{2,S_\zeta} \varphi$, then it also does not hold that $\Sigma \models_{S_\zeta} \varphi$ since every special probability model is a probability model. Vice versa, if it does not hold that $\Sigma \models_{S_\zeta} \varphi$, then it does not hold that $\Sigma \vdash_{\mathfrak{Z}} \varphi$ since $\mathfrak{Z}$ is sound for the implication of SCI statements under controlled uncertainty. However, the proof of Theorem 2 shows how to construct a special probability model over (S, S_ζ) that satisfies every SCI statement in Σ but does not satisfy φ. Hence, it does not hold that $\Sigma \models_{2,S_\zeta} \varphi$.

Corollary 1 shows that it suffices to check special probability models in order to decide the implication problem for SCI statements under controlled uncertainty.

5 Logical Characterization

In this section we establish the equivalence between the implication of SCI statements under controlled uncertainty and the implication of formulae in a propositional fragment $\mathfrak{F}$ within Cadoli and Schaerf's well-known approximation logic $\mathcal{S}$-3 [14]. After repeating the syntax and semantics of $\mathcal{S}$-3 logic, we define a mapping of SCI statements to formulae in $\mathfrak{F}$. The core proof argument establishes an equivalence between special probability models, introduced in the previous section, and special $\mathcal{S}$-3 truth assignments. It is shown that this equivalence cannot be extended to also cover general conditional independence statements. In the following section we will also establish an equivalence of the implication problem to Boolean implication of $\mathfrak{F}$-formulae, which requires the elimination of some elements from the input instance.

5.1 Syntax and semantics of $\mathcal{S}$-3 logic

Schaerf and Cadoli [14] introduced $\mathcal{S}$-3 logics as "a semantically well-founded logical framework for sound approximate reasoning, which is justifiable from the intuitive point of view, and to provide fast algorithms for dealing with it even when using expressive languages". For a finite set L of propositional variables, let L^* denote the *propositional language* over L, generated from the unary connective $\neg$ (negation), and the binary connectives $\wedge$ (conjunction) and $\vee$ (disjunction). Elements of L^* are also called formulae of L, and usually denoted by φ', ψ' or their subscripted versions. Sets of formulae are denoted by Σ'. We omit parentheses if this does not cause ambiguity.

Let L^ℓ denote the set of all literals over L, i.e., $L^\ell = L \cup \{\neg a' \mid a' \in L\}$. Let $\mathcal{S} \subseteq L$. An $\mathcal{S}$-3 truth assignment of L is a total function $\omega : L^\ell \to \{\mathbb{F}, \mathbb{T}\}$ that maps every propositional variable $a' \in \mathcal{S}$ and its negation $\neg a'$ into opposite truth values ($\omega(a') = \mathbb{T}$ if and only if $\omega(\neg a') = \mathbb{F}$), and that does not map both a propositional variable $a' \in L - \mathcal{S}$ and its negation $\neg a'$ into *false* (we must not have $\omega(a') = \mathbb{F} = \omega(\neg a')$ for any $a' \in L - \mathcal{S}$). Accordingly, for each propositional variable $a' \in L$ and each $\mathcal{S}$-3 truth assignment ω of L there are the following possibilities:

- $\omega(a') = \mathbb{T}$ and $\omega(\neg a') = \mathbb{F}$,
- $\omega(a') = \mathbb{F}$ and $\omega(\neg a') = \mathbb{T}$,
- $\omega(a') = \mathbb{T}$ and $\omega(\neg a') = \mathbb{T}$ (only if $a' \in L - \mathcal{S}$).

$\mathcal{S}$-3 truth assignments generalize both, standard 2-valued truth assignments as well as the 3 truth assignments of Levesque [21]. That is, a 2-valued truth assignment is an $\mathcal{S}$-3 truth assignment where $\mathcal{S} = L$, while a 3 truth assignment is an $\mathcal{S}$-3 truth assignment with $\mathcal{S} = \emptyset$.

An $\mathcal{S}$-3 truth assignment $\omega : L^\ell \to \{\mathbb{F}, \mathbb{T}\}$ of L can be lifted to a total function $\Omega : L^* \to \{\mathbb{F}, \mathbb{T}\}$. This lifting has been defined as follows [14]. An arbitrary formula φ' in L^* is firstly converted (in linear time in the size of the formula) into its corresponding formula φ'_N in *Negation Normal Form* (NNF) using the following rewriting rules: $\neg(\varphi' \wedge \psi') \mapsto (\neg\varphi' \vee \neg\psi')$, $\neg(\varphi' \vee \psi') \mapsto (\neg\varphi' \wedge \neg\psi')$, and $\neg(\neg\varphi') \mapsto \varphi'$. Therefore, negation in a formula in NNF occurs only at the literal level. The rules for assigning truth values to NNF formulae are as follows:

- $\Omega(\varphi') = \omega(\varphi')$, if $\varphi' \in L^\ell$,
- $\Omega(\varphi' \vee \psi') = \mathbb{T}$ if and only if $\Omega(\varphi') = \mathbb{T}$ or $\Omega(\psi') = \mathbb{T}$,
- $\Omega(\varphi' \wedge \psi') = \mathbb{T}$ if and only if $\Omega(\varphi') = \mathbb{T}$ and $\Omega(\psi') = \mathbb{T}$.

An $\mathcal{S}$-3 truth assignment ω is a *model* of a set Σ' of L-formulae if and only if $\Omega(\sigma'_N) = \mathbb{T}$ holds for every $\sigma' \in \Sigma'$. We say that Σ' $\mathcal{S}$-3 *implies* an L-formula φ', denoted by $\Sigma' \models^3_{\mathcal{S}} \varphi'$, if and only if every $\mathcal{S}$-3 truth assignment that is a model of Σ' is also a model of φ'.

5.2 The propositional fragment $\mathfrak{F}$

As a first step towards the anticipated duality we define the propositional fragment that corresponds to SCI statements. Let $\phi : S \to L$ denote a bijection between a set S of random variables and the set $L = \{a' \mid a \in S\}$ of propositional variables. In particular, for $S_\zeta \subseteq S$ let $\mathcal{S} = \phi(S_\zeta)$. Thus, zero-free random variables correspond to propositional variables interpreted classically.

We extend ϕ to a mapping Φ from the set of SCI statements over (S, S_ζ) to the set $\mathfrak{F} \subseteq L^*$. For an SCI statement $I(Y, Z \mid X)$ over (S, S_ζ), let $\Phi(I(Y, Z \mid X))$ denote the formula

$$\bigvee_{a \in X} \neg a' \vee \left(\bigwedge_{b \in Y} b' \right) \vee \left(\bigwedge_{c \in Z} c' \right),$$

which we abbreviate by

$$\bigvee_{a \in X} \neg a' \vee \bigwedge_{b \in Y} b' \vee \bigwedge_{c \in Z} c' \ .$$

As usual, disjunctions over zero disjuncts are interpreted as $\mathbb{F}$ and conjunctions over zero conjuncts are interpreted as $\mathbb{T}$. We will simply denote $\Phi(\varphi) = \varphi'$ and $\Phi(\Sigma) = \{\sigma' \mid \sigma \in \Sigma\} = \Sigma'$.

Example 11 *Let $S = \{c, p, s, a, l, o\}$, $S_\zeta = \{c, p, s, a, l\}$, and Σ be as before, i.e., consist of $I(psal, o \mid c)$ and $I(ps, al \mid co)$, and φ is $I(ps, alo \mid c)$. As a corresponding instance in terms of $\mathcal{S}$-3 logics we obtain:*

- $L = \{c', p', s', a', l', o'\}$ *and* $\mathcal{S} = \{c', p', s', a', l'\}$,
- $\Sigma' = \{\neg c' \vee (p' \wedge s' \wedge a' \wedge l') \vee o', \neg c' \vee \neg o' \vee (p' \wedge s') \vee (a' \wedge l')\}$,
- $\varphi' = \neg c' \vee (p' \wedge s') \vee (a' \wedge l' \wedge o')$.

Note that these are just the formulae from Example 2.

5.3 Special truth assignments

We will now show that for any set $\Sigma \cup \{\varphi\}$ of SCI statements over (S, S_ζ) there is a probability model $\pi = (dom, P)$ over (S, S_ζ) that satisfies Σ and violates φ if and only if there is a truth assignment ω_π that is an $\mathcal{S}$-3 model of Σ' but not an $\mathcal{S}$-3 model of φ'. For arbitrary probability models π it is not obvious how to define the truth assignment ω_π. However, the key to showing the correspondence between probability models and truth assignments is Corollary 1. Corollary 1 tells us that for deciding the implication problem $\Sigma \models_{S_\zeta} \varphi$ it suffices to examine special probability models (instead of arbitrary probability models). For a special probability model $\pi = (dom, \{\mathbf{e}_1, \mathbf{e}_2\})$, however, we can define its corresponding special $\mathcal{S}$-3 truth assignment ω_π of L by defining for each $a' \in L$,

$$\omega_\pi(a') \quad = \quad \begin{cases} \mathbb{T} & \text{, if } \mathbf{e}_1(a) = \mathbf{e}_2(a) \\ \mathbb{F} & \text{, otherwise} \end{cases} \quad \text{, and}$$

$$\omega_\pi(\neg a') \quad = \quad \begin{cases} \mathbb{T} & \text{, if } \mathbf{e}_1(a) = \zeta = \mathbf{e}_2(a) \text{ or } \mathbf{e}_1(a) \neq \mathbf{e}_2(a) \\ \mathbb{F} & \text{, otherwise} \end{cases} \quad .$$

Note that the truth assignment is Boolean for each variable in $\mathcal{S}$, that is, for each variable that corresponds to a zero-free random variable.

Example 12 *Recall Example 5 where* $S = \{c, p, s, a, l, o\}$*,* $S_\zeta = \{c, p, s, a, l\}$*, and any special probability model* π *defined by*

conference	*paper*	*speaker*	*organizer*	*activity*	*location*
ICM	*23 problems*	*Hilbert*	ζ	*petanque*	*tuileries*
ICM	*4 problems*	*Landau*	ζ	*rowing*	*river cam*

shows that Σ*, consisting of* $I(psal, o \mid c)$ *and* $I(ps, al \mid co)$*, does not* S_ζ*-imply* φ*, which is* $I(ps, alo \mid c)$*. The special* $\mathcal{S}$*-3 truth assignment* ω_π *is given by*

- $\omega_\pi(c') = \mathbb{T}$ *and* $\omega_\pi(\neg c') = \mathbb{F}$*,*
- $\omega_\pi(p') = \mathbb{F}$ *and* $\omega_\pi(\neg p') = \mathbb{T}$*,*
- $\omega_\pi(s') = \mathbb{F}$ *and* $\omega_\pi(\neg s') = \mathbb{T}$*,*
- $\omega_\pi(o') = \mathbb{T}$ *and* $\omega_\pi(\neg o') = \mathbb{T}$*,*
- $\omega_\pi(a') = \mathbb{F}$ *and* $\omega_\pi(\neg a') = \mathbb{T}$*, and*
- $\omega_\pi(l') = \mathbb{F}$ *and* $\omega_\pi(\neg l') = \mathbb{T}$*.*

5.4 Equivalence between satisfaction by probability models and by truth assignments

Next we justify the definition of the special truth assignment and that of the Boolean fragment $\mathfrak{F}$ in terms of the special probability models.

Lemma 3 *Let* $\pi = (\text{dom}, \{\mathbf{e}_1, \mathbf{e}_2\})$ *be a special probability model, and let* φ *denote an SCI statement over* (S, S_ζ)*. Then* π *satisfies* φ *if and only if* ω_π *is an* $\mathcal{S}$*-3 model of* φ'*.*

Proof Let $\varphi = I(Y, Z \mid X)$ and $\varphi' = \bigvee_{a \in X} \neg a' \vee \bigwedge_{b \in Y} b' \vee \bigwedge_{c \in Z} c'$. Suppose first that π satisfies φ. We need to show that ω_π is an $\mathcal{S}$-3 model of φ'. Assume that $\omega_\pi(\neg a') = \mathbb{F}$ for all $a \in X$. According to the special truth assignment we must have $\zeta \neq \mathbf{e}_1(a) = \mathbf{e}_2(a) \neq \zeta$ for all $a \in X$. That means $P(\mathbf{e}_1(X)) = 1$. Suppose that for some $b \in Y$ we have $\omega_\pi(b') = \mathbb{F}$. Then $\mathbf{e}_1(b) \neq \mathbf{e}_2(b)$ according to the special truth assignment. Then $P(\mathbf{e}_1(XY)) = P(\mathbf{e}_1) = 0.5$. However, since $\mathbf{e}_1(X)$ is X-total and π satisfies φ, we must have $P(\mathbf{e}_1(XZ)) = 1$. Hence, for every $c \in Z$, we have $\mathbf{e}_1(c) = \mathbf{e}_2(c)$. This means that for all $c \in Z$ we have $\omega_\pi(c') = \mathbb{T}$. This shows that ω_π is an $\mathcal{S}$-3 model of φ'.

Suppose ω_π is an $\mathcal{S}$-3 model of φ'. We need to show that π satisfies φ. That is, for every total event $\mathbf{x}$ of X, and every event $\mathbf{y}$ of Y, and every event $\mathbf{z}$ of Z, we must show that $P(\mathbf{x}) \cdot P(\mathbf{x}, \mathbf{y}, \mathbf{z}) = P(\mathbf{x}, \mathbf{y}) \cdot P(\mathbf{x}, \mathbf{z})$ holds. We distinguish between three cases.

Case 1. Certainly, if $P(\mathbf{x}, \mathbf{y}) = 0$ or $P(\mathbf{x}, \mathbf{z}) = 0$, then $P(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 0$, too. For the remaining cases we can therefore assume that $P(\mathbf{x}, \mathbf{y}) > 0$ and $P(\mathbf{x}, \mathbf{z}) > 0$. In particular, $P(\mathbf{x}) > 0$.

Case 2. Suppose $P(\mathbf{x}) = 1$. It follows that $\mathbf{e}_1(X) = \mathbf{x} = \mathbf{e}_2(X)$. Since $\mathbf{x}$ is a total event of X, the special truth assignment entails that $\omega_\pi(\neg a') = \mathbb{F}$ for all $a \in X$. Since ω_π is an $\mathcal{S}$-3 model of φ' we conclude that $\omega_\pi(b') = \mathbb{T}$ for all $b \in Y$, or $\omega_\pi(c') = \mathbb{T}$ for all $c \in Z$. The special truth assignment entails that $\mathbf{e}_1(XY) = (\mathbf{x}, \mathbf{y}) = \mathbf{e}_2(XY)$ or $\mathbf{e}_1(XZ) = (\mathbf{x}, \mathbf{z}) = \mathbf{e}_2(XZ)$ holds. This, however, would mean that $P(\mathbf{x}, \mathbf{y}) = 1$ or $P(\mathbf{x}, \mathbf{z}) = 1$. Since φ is saturated, it follows that exactly one of $P(\mathbf{x}, \mathbf{y})$ and $P(\mathbf{x}, \mathbf{z})$ is 1, and the other 0.5. Consequently, $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ equals $\mathbf{e}_1$ or $\mathbf{e}_2$. Hence, $P(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 0.5$. It follows that π satisfies φ.

Case 3. Suppose that $P(\mathbf{x}) = 0.5$. Then $P(\mathbf{x}, \mathbf{y}) = 0.5 = P(\mathbf{x}, \mathbf{z})$. Then $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ equals $\mathbf{e}_1$ or $\mathbf{e}_2$, as $P(\mathbf{x})$ would have to be 1 otherwise. Hence, $P(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 0.5$.

Example 13 *Recall from Example 12 that any special probability model defined by π satisfies the SCI statements in Σ, but violates φ. It is easy to verify that the special $\mathcal{S}$-3 truth assignment ω_π from Example 12 is an $\mathcal{S}$-3 model of the formulae in Σ', but not an $\mathcal{S}$-3 model of φ'.*

5.5 The equivalence between entailment

Corollary 1 and Lemma 3 allow us to establish the anticipated equivalence between SCI implication over discrete probability models and $\mathcal{S}$-3 implication for formulae in the propositional fragment $\mathfrak{F}$.

Theorem 3 *Let $\Sigma \cup \{\varphi\}$ be a set of SCI statements over (S, S_ζ), and let $\Sigma' \cup \{\varphi'\}$ denote the corresponding set of formulae over L, where $\mathcal{S}$ denotes the propositional variables that correspond to the random variables in S_ζ. Then $\Sigma \models_{S_\zeta} \varphi$ if and only if $\Sigma' \models^3_{\mathcal{S}} \varphi'$.*

Proof Based on Corollary 1 it suffices to establish an equivalence between $\Sigma \models_{2,S_\zeta} \varphi$ and $\Sigma' \models^3_{\mathcal{S}} \varphi'$.

Suppose first that $\Sigma \models_{2,S_\zeta} \varphi$ does not hold. Then there is some special probability model π over (S, S_ζ) that satisfies every SCI statement σ in Σ but violates φ. Let ω_π denote the special truth assignment associated with π. By Lemma 3 it follows that ω_π is an $\mathcal{S}$-3 model of every formula σ' in Σ' but not an $\mathcal{S}$-3 model of φ'. Consequently, $\Sigma' \models^3_{\mathcal{S}} \varphi'$ does not hold.

Suppose now that $\Sigma' \models^3_{\mathcal{S}} \varphi'$ does not hold. Then there is some truth assignment ω over L that is an $\mathcal{S}$-3 model of every formula σ' in Σ', but not an $\mathcal{S}$-3 model of the formula φ'. Define the following special probability model $\pi = (dom, \{\mathbf{e}_1, \mathbf{e}_2\})$ over (S, S_ζ). For $a \in S_\zeta$, let $dom(a) = \{\mathbf{0}, \mathbf{1}\}$; and for $a \in S - S_\zeta$, let $dom(a) = \{\mathbf{0}, \mathbf{1}, \zeta\}$. We now define $\mathbf{e}_1$ and $\mathbf{e}_2$ as follows. If $\omega(a') = \mathbb{T}$ and $\omega(\neg a') = \mathbb{F}$, then $\zeta \neq \mathbf{e}_1(a) = \mathbf{e}_2(a) \neq \zeta$. If $\omega(a') = \mathbb{T}$ and $\omega(\neg a') = \mathbb{T}$, then $\mathbf{e}_1(a) = \zeta = \mathbf{e}_2(a)$. Finally, if $\omega(a') = \mathbb{F}$ and $\omega(\neg a') = \mathbb{T}$, then $\zeta \neq \mathbf{e}_1(a) \neq \mathbf{e}_2(a) \neq \zeta$. Since ω is not an $\mathcal{S}$-3 model of φ', it follows that $\mathbf{e}_1 \neq \mathbf{e}_2$. It follows now that $\omega_\pi = \omega$. By Lemma 3 it follows that π satisfies every SCI statement σ in Σ but violates φ. Hence, $\Sigma \models_{2,S_\zeta} \varphi$ does not hold.

Example 14 *Recall from Example 12 that Σ does not $\mathcal{S}$-3 imply φ. It now simply remains to observe that Σ' does not $\mathcal{S}$-3 imply φ'.*

5.6 Non-extendibility to cover general conditional independence

It is shown in this section that the equivalence between the implication of SCI statements under controlled uncertainty and the $\mathcal{S}$-3 implication of $\mathfrak{F}$-formulae cannot be extended to cover CI statements, too. In fact, it is shown that such an extension does already not exist in the idealized special case where all random variables are zero-free. It follows that SCI statements, multivalued dependencies and the propositional fragment $\mathfrak{F}$ are all very special, as outlined in the contributions in Section 2.

The first lemma shows that the inference rule $\dfrac{I(X,Y \mid W) \quad I(X,Z \mid W)}{I(X,YZ \mid W)}$ is not sound for CI statements.

Lemma 4 *The CI statements $I(X,Y \mid W), I(X,Z \mid W)$ do not S-imply the CI statement $I(X,YZ \mid W)$.*

Proof Let $S = \{a,b,c,d\}$ and $S_\zeta = S$. Let $dom(a) = dom(b) = dom(c) = dom(d) = \{\mathbf{0},\mathbf{1}\}$. Then we define the probability measure P over S by assigning probability one fourth to each of the following four events:

a	b	c	d	P
0	**0**	**0**	**0**	1/4
0	**0**	**1**	**1**	1/4
0	**1**	**1**	**0**	1/4
0	**1**	**0**	**1**	1/4

.

Indeed, (dom, P) satisfies $I(d,c \mid a)$ and $I(d,b \mid a)$, but not the CI statement $I(d,bc \mid a)$.

Remark 9 *The probability model (dom, P) from the proof of Lemma 4 does not satisfy the CI statement $I(d,c \mid ab)$. Hence, the CI statement $I(X,Y \mid W)$ does not S-imply the CI statement $I(X,Y \mid WZ)$. Note that a set Σ of CI statements is stable when Σ is closed under applications of this rule, i.e., if $I(X,Y \mid W) \in \Sigma$, then $I(X,Y \mid WZ) \in \Sigma$, too.*

For some general CI statement $\sigma = I(Y,Z \mid X)$ over the set S of random variables, let $\sigma' = \bigvee_{a \in X} \neg a' \vee \bigwedge_{b \in Y} b' \vee \bigwedge_{c \in Z} c'$ denote the corresponding $\mathfrak{F}$-formula over the corresponding set $L = \{a' \mid a \in S\}$ of propositional variables. As for SCI statements before, $\Sigma' = \{\sigma' \mid \sigma \in \Sigma\}$ for any set Σ of CI statements. Note that XYZ does not necessarily cover all random variables in S, unlike the special case of saturated CI statements. It is shown now that in the general case, unlike the special case of saturated CI statements, the mapping $(\cdot)'$ from CI statements to $\mathfrak{F}$-formulae does not result in an equivalence between the implication problems.

Lemma 5 *There is a set S of random variables, and there is some set $\Sigma \cup \{\varphi\}$ of CI statements over S such that $\Sigma \not\models \varphi$, but $\Sigma' \models_{BL} \varphi'$.*

Proof Let $\Sigma = \{I(X,Y \mid W), I(X,Z \mid W)\}$ and $\varphi = I(X,YZ \mid W)$ over a set S of random variables. Lemma 4 has shown that Σ does not S-imply φ. However, it is easy to see that Σ' does imply φ' in Boolean propositional logic.

Lemma 5 leaves open the possibility that there might be some other mapping $\hat{(\cdot)}$ between CI statements and some propositional fragment where the corresponding implication problems are equivalent. The next lemma shows that any such mapping must necessarily be $(\cdot)'$.

Lemma 6 *For every mapping $\hat{(\cdot)}$ from CI statements σ to propositional formulae $\hat{\sigma}$, that extends the mapping $(\cdot)'$ from SCI statements to $\mathfrak{F}$-formulae, and is such that for all finite sets S of random variables, and all sets $\Sigma \cup \{\varphi\}$ of CI statements over S, $\Sigma \models \varphi$ if and only if $\hat{\Sigma} \models_{BL} \hat{\varphi}$, it holds that $\hat{(\cdot)} = (\cdot)'$. That is, for any mapping that results in an equivalence between the corresponding implication problems it necessarily holds that for all CI statements τ we have that $\hat{\tau}$ is equivalent to τ'.*

Proof We show that a truth assignment ω of L is a model of $\hat{\tau}$ if and only if ω is a model of τ'.

The soundness of the *decomposition rule* says that $\sigma = I(X, YZ \mid W)$ implies $\tau = I(X, Y \mid W)$ [10]. For $S = WXYZ$, $\sigma = I(X, YZ \mid W)$ is an SCI statement and, thus, $\hat{\sigma} = \sigma' = \bigvee_{a \in W} \neg a' \vee \bigwedge_{b \in X} b' \vee \bigwedge_{c \in YZ} c'$ implies $\hat{\tau}$. Hence, if the truth assignment ω of L is not a model of $\bigwedge_{a \in W} a'$, or ω is a model of $\bigwedge_{b \in X} b'$, then ω is a model of $\hat{\tau}$. Similarly, $\sigma = I(XZ, Y \mid W)$ implies $\tau = I(X, Y \mid W)$. Consequently, if the truth assignment ω of L is not a model of $\bigwedge_{a \in W} a'$, or ω is a model of $\bigwedge_{c \in Y} c'$, then ω is a model of $\hat{\tau}$. Therefore, if ω is not a model of $\bigwedge_{a \in W} a'$, or is a model of $\bigwedge_{b \in X} b'$ or is a model of $\bigwedge_{c \in Y} c'$, then ω is a model of $\hat{\tau}$. That is, if ω is a model of τ', then ω is a model of $\hat{\tau}$.

It remains to show that if ω is not a model of τ', then ω is not a model of $\hat{\tau}$. Let $S = WXYZ$ where W, X, Y, Z are each non-empty, $\tau_1 = I(X, Z \mid WY)$ and $\sigma = I(X, YZ \mid W)$. The soundness of the *contraction rule* means that the CI statement $\tau = I(X, Y \mid W)$ and the SCI statement $\tau_1 = I(X, Z \mid WY)$ imply the SCI statement $\sigma = I(X, YZ \mid W)$ [10]. It follows that $\hat{\tau}$ and τ_1' imply σ'. Let ω be a truth assignment of L that is not a model of τ'. That is, ω is a model of $\bigwedge_{a \in W} a'$, ω is not a model of $\bigwedge_{b \in X} b'$, and ω is not a model of $\bigwedge_{c \in Y} c'$. Consequently, ω is a model of τ_1', but not a model of σ'. Since $\hat{\tau}$ and τ_1' imply σ', ω is not a model of $\hat{\tau}$. This is what we wanted to show.

The following proposition can be shown similar to the case of SCI statements.

Proposition 1 *Let $\Sigma \cup \{\varphi\}$ be a set of CI statements over the set S of random variables. Then $\Sigma \models \varphi$ implies $\Sigma \models_2 \varphi$, and $\Sigma \models_2 \varphi$ implies $\Sigma' \models_{BL} \varphi'$.* ∎

It is now shown that it is impossible to extend the equivalence between SCI implication and the Boolean implication of $\mathfrak{F}$-formulae to a mapping between general CI statements and some propositional fragment $\hat{\mathfrak{F}}$ which results in an equivalence between CI implication and Boolean implication of $\hat{\mathfrak{F}}$-formulae.

Theorem 4 *For every mapping $\hat{(\cdot)}$ of CI statements to propositional formulae that extends the mapping $(\cdot)'$ of SCI statements to $\mathfrak{F}$-formulae there is some finite set S of random variables and there is some set $\Sigma \cup \{\varphi\}$ of CI statements over S such that $\Sigma \not\models \varphi$, but $\hat{\Sigma} \models_{BL} \hat{\varphi}$.*

Proof Assume, to the contrary of what is to be shown, that for every finite set S of random variables and for every set $\Sigma \cup \{\varphi\}$ of CI statements over S, $\Sigma \models \varphi$ if and only if $\hat{\Sigma} \models_{BL} \hat{\varphi}$. Then it follows by Lemma 6 that $\Sigma \models \varphi$ if and only if $\Sigma' \models_{BL} \varphi'$. This, however, is a contradiction to Lemma 5. Indeed, for $S = WXYZ$ with non-empty W, X, Y, Z and for $\Sigma = \{I(X, Y \mid W), I(X, Z \mid W)\}$ and $\varphi = I(X, YZ \mid W)$ over S we have $\Sigma \not\models \varphi$ and $\Sigma' \models_{BL} \varphi'$. Consequently, the assumption was wrong and there is some finite set S and there is some set $\Sigma \cup \{\varphi\}$ of CI statements over S such that exactly one of $\Sigma \models \varphi$ and $\hat{\Sigma} \models_{BL} \hat{\varphi}$ holds. Due to Lemma 1, it must be that $\Sigma \not\models \varphi$, but $\hat{\Sigma} = \Sigma' \models_{BL} \varphi' = \hat{\varphi}$.

6 Algorithmic Characterization

In this section we will establish algorithms for i) deciding the implication problem $\Sigma \models_{S_\zeta} I(Y, Z \mid X)$ for sets $\Sigma \cup \{I(Y, Z \mid X)\}$ of SCI statements in the presence of S_ζ, and ii) computing the independence basis $IDepB_{\Sigma,S_\zeta}(X)$ of a set X of random variables with respect to Σ and S_ζ. We will derive a tight worst-case upper time bound that highlights the impact of the set S_ζ of zero-free random variables. The results follow from a reduction of the implication problem to its counter-part over probabilistic models without occurrences of zero markers. The reduction itself is a consequence of our axiomatization.

6.1 Equivalence to Boolean implication

Let $\Sigma[U]$ contain only those SCI statements $I(Y, Z \mid X)$ from Σ where X is a subset of the set U of random variables.

Lemma 7 *Let $\Sigma \cup \{I(Y, Z \mid X)\}$ be a set of SCI statements, and S_ζ the set of zero-free random variables over the set S of random variables. Then the following are equivalent:*

1. $\Sigma \vdash_{\mathfrak{Z}} I(Y, Z \mid X)$,
2. $\Sigma[XS_\zeta] \vdash_{\mathfrak{Z}} I(Y, Z \mid X)$, *and*
3. $\Sigma[XS_\zeta] \vdash_{\mathfrak{G}} I(Y, Z \mid X)$.

Proof Since $\Sigma[XS_\zeta]$ is a subset of Σ, it follows that (2) implies (1). A simple induction over the length of an inference of $I(Y, Z \mid X)$ from Σ by $\mathfrak{Z}$ shows that $I(Y, Z \mid X)$ can already be inferred from $\Sigma[XS_\zeta]$ by $\mathfrak{Z}$. Hence, (1) also implies (2). The equivalence between (2) and (3) can also be established using induction over the length of an inference. We show the interesting cases here. The first case is when, in an inference using $\mathfrak{Z}$ the restricted weak contraction rule ($\mathcal{R}$) is applied to infer $I(Z, YUVW \mid X)$ from $I(ZW, YUV \mid X)$ and $I(UZ, VW \mid XY)$. From $I(Z, YUVW \mid X) \in \Sigma[XS_\zeta]^+_{\mathfrak{Z}}$ and the inference step we conclude $I(ZW, YUV \mid X), I(UZ, VW \mid XY) \in \Sigma[XS_\zeta]^+_{\mathfrak{Z}}$. Then, by hypothesis, $I(ZW, YUV \mid X), I(UZ, VW \mid XY) \in \Sigma[XS_\zeta]^+_{\mathfrak{G}}$. Applying the symmetry rule ($\mathcal{S}$) and the weak union rule ($\mathcal{W}$) to $I(UZ, VW \mid XY) \in \Sigma[XS_\zeta]^+_{\mathfrak{G}}$ we obtain $I(Z, W \mid XYUV) \in \Sigma[XS_\zeta]^+_{\mathfrak{G}}$. Finally, we can apply the weak contraction rule ($\mathcal{C}$) to

$I(ZW, YUV \mid X), I(Z, W \mid XYUV) \in \Sigma[XS_\zeta]^+_{\mathfrak{G}}$ to infer $I(Z, YUVW \mid X) \in \Sigma[XS_\zeta]^+_{\mathfrak{G}}$. The other interesting case is when, in an inference using $\mathfrak{G}$ the weak contraction rule ($\mathcal{C}$) is applied to infer $I(Z, YW \mid X)$ from $I(ZW, Y \mid X)$ and $I(Z, W \mid XY)$. From $I(Z, YW \mid X) \in \Sigma[XS_\zeta]^+_{\mathfrak{G}}$ and the inference step we conclude $I(ZW, Y \mid X), I(Z, W \mid XY) \in \Sigma[XS_\zeta]^+_{\mathfrak{G}}$. Then, by hypothesis, $I(ZW, Y \mid X), I(Z, W \mid XY) \in \Sigma[XS_\zeta]^+_{\mathfrak{Z}}$. It is not difficult to see the following property: if $\Sigma[XS_\zeta] \vdash_{\mathfrak{Z}} I(Z, W \mid XY)$, then $\Sigma[XS_\zeta] \vdash_{\mathfrak{Z}} I(ZU, VW \mid XY')$ where $Y = Y'UV$ for some $Y' \subseteq S_\zeta$. Applying this property to $I(ZW, Y \mid X), I(Z, W \mid XY) \in \Sigma[XS_\zeta]^+_{\mathfrak{Z}}$ means that $I(ZW, Y'UV \mid X), I(ZU, VW \mid XY') \in \Sigma[XS_\zeta]^+_{\mathfrak{Z}}$ where $Y' \subseteq S_\zeta$. Hence, we can now apply the restricted weak contraction rule ($\mathcal{R}$) to $I(ZW, Y'UV \mid X), I(ZU, VW \mid XY') \in \Sigma[XS_\zeta]^+_{\mathfrak{Z}}$ where $Y' \subseteq S_\zeta$ to infer $I(Z, YW \mid X) = I(Z, Y'UVW \mid X) \in \Sigma[XS_\zeta]^+_{\mathfrak{Z}}$.

Since $\mathfrak{Z}$ is an axiomatization for the implication of SCI statements under controlled uncertainty, and $\mathfrak{G}$ is an axiomatization for the implication of SCI statements in the absence of zeros, Lemma 7 also results in an equivalence between the corresponding implication problems. Moreover, the implication of SCI statements in the absence of zeros is known to be equivalent to the Boolean implication of formulae in the propositional fragment $\mathfrak{F}$ [11]. Thus, we obtain the following corollary.

Corollary 2 *Let $\Sigma \cup \{I(Y, Z \mid X)\}$ be a set of SCI statements, and S_ζ the set of zero-free random variables over S. Then the following decision problems are equivalent:*

1. $\Sigma \models_{S_\zeta} I(Y, Z \mid X)$,
2. $\Sigma[XS_\zeta] \models I(Y, Z \mid X)$,
3. $(\Sigma[XS_\zeta])' \models_{BL} \bigvee_{a \in X} \neg a' \vee \bigwedge_{b \in Y} b' \vee \bigwedge_{c \in Z} c'$. ∎

We illustrate Corollary 2 by applying its results to our running example.

Example 15 *Recall our running example where $S = \{c, p, s, a, l, o\}$, $S_\zeta = \{c, p, s, a, l\}$, and Σ, consisting of $I(psal, o \mid c)$ and $I(ps, al \mid co)$, does not S_ζ-imply φ, which is is $I(ps, alo \mid c)$. Consequently, $\varphi_c = \{c\}$ and $\Sigma[c, p, s, a, l] = \{I(psal, o \mid c)\}$. Hence, $IDepB_{\Sigma[c,p,s,a,l],S}(c) = \{\{p, s, a, l\}, \{o\}\}$. From this we can create a special probability model π defined by*

conference	*paper*	*speaker*	*organizer*	*activity*	*location*
ICM	*23 problems*	*Hilbert*	*Klein*	*petanque*	*tuileries*
ICM	*4 problems*	*Landau*	*Klein*	*rowing*	*river cam*

.

It shows that $\Sigma[c, p, s, a, l]$ does not imply φ. Equivalently, we can define a special Boolean truth assignment ω^b_π by $\omega^b_\pi(c') = \omega^b_\pi(o') = \mathbb{T}$ and $\omega^b_\pi(p') = \omega^b_\pi(s') = \omega^b_\pi(a') = \omega^b_\pi(l') = \mathbb{F}$. This Boolean truth assignment is a witness that $(\Sigma[c, p, s, a, l])' = \{\neg c' \vee (p' \wedge s' \wedge a' \wedge l') \vee o'\}$ does not imply $\varphi' = \neg c' \vee (p' \wedge s') \vee (a' \wedge l' \wedge o')$.

6.2 Algorithm to compute the independence basis

Sagiv [22], and later Galil [23] developed almost linear time algorithms to i) compute the independence basis $IDepB_{\Sigma,S}(X)$, and ii) decide the implication problem $\Sigma \models_S \varphi$ in the idealized special case where $S_\zeta = S$. Our previous results can now be applied to utilize these algorithms for the corresponding problems in the general case where $S_\zeta \subseteq S$. The following corollary is a direct consequence of Lemma 7.

Corollary 3 *Let Σ be a set of SCI statements, S_ζ the set of zero-free random variables, and X a set of random variables over S. Then we have*

$$IDepB_{\Sigma,S_\zeta}(X) = IDepB_{\Sigma[XS_\zeta],S_\zeta}(X) = IDepB_{\Sigma[XS_\zeta],S}(X). \quad \blacksquare$$

This corollary allows us to use Galil's algorithm for computing the independence basis in a much more general framework than it was designed for.

Corollary 4 *Let Σ be a set of SCI statements, S_ζ the set of zero-free random variables, and X a set of random variables over S. Then Galil's algorithm [23] computes the independence basis $IDepB_{\Sigma,S_\zeta}(X)$ of X with respect to Σ and S_ζ in time*

$$\mathcal{O}(|\Sigma| + \min\{k_{\Sigma[XS_\zeta]}, \log p_{\Sigma[XS_\zeta],S_\zeta}\} \times |\Sigma[XS_\zeta]|).$$

Herein, $|\Sigma|$ denotes the total number of random variables occurring in Σ, k_Σ denotes the number of elements in Σ, and p_{Σ,S_ζ} the number of elements in $IDepB_{\Sigma,S_\zeta}(X)$. $\blacksquare$

6.3 Algorithm to decide the implication problem

Let Σ be a set of SCI statements, and S_ζ the set of zero-free random variables over S. If $IDepB_{\Sigma,S_\zeta}(X)$ is known, the implication problem $\Sigma \models_{S_\zeta} I(Y, Z \mid X)$ can be decided in linear time.

Corollary 5 *Using Galil's algorithm [23], the implication problem $\Sigma \models_{S_\zeta} I(Y, Z \mid X)$ of sets $\Sigma \cup \{I(Y, Z \mid X)\}$ of SCI statements and the set S_ζ of zero-free random variables over S can be decided in time*

$$\mathcal{O}(|\Sigma| + \min\{k_{\Sigma[XS_\zeta]}, \log \bar{p}_{\Sigma[XS_\zeta],S_\zeta}\} \times |\Sigma[XS_\zeta]|).$$

Herein, $|\Sigma|$ denotes the total number of random variables occurring in Σ, k_Σ denotes the cardinality of Σ, and $\bar{p}_{\Sigma,S_\zeta}$ denotes the minimum of the two numbers of sets in $IDepB_{\Sigma,S_\zeta}(X)$ that have non-empty intersection with Y, and with Z, respectively.

Proof The problem $\Sigma \models_{S_\zeta} I(Y, Z \mid X)$ is equivalent to $(\Sigma[XS_\zeta])' \models_{BL} \bigvee_{a\in X} \neg a' \vee \bigwedge_{b\in Y} b' \vee \bigwedge_{c\in Z} c'$ by Corollary 2. The last problem can be decided by Galil's algorithm in $\mathcal{O}(|\Sigma| + \min\{k_{\Sigma[XS_\zeta]}, \log \bar{p}_{\Sigma[XS_\zeta],S_\zeta}\} \times |\Sigma[XS_\zeta]|)$ time [23].

7 Conclusion and Future Work

Theoretical investigations into the logical properties of conditional independence have focused on probability models that are free from zeros. In practice, it is necessary but challenging to balance the desire of achieving as much certainty as possible and the desire to acquire as much information as possible. This article introduces effective techniques to reason efficiently about saturated conditional independence and guarantee control over the degree of uncertainty in the form of zeros. The intuitive idea is that random variables can be declared to be zero-free to approximate reasoning over zero-free probabilistic models. The article has established substantial technical evidence to support this intuitive idea, including axiomatic, algorithmic and logical characterizations of the associated implication problem. In particular, it was shown that the implication problem of saturated conditional independence under zero-free random variables is equivalent to the implication problem of a propositional fragment in Cadoli and Schaerf's $\mathcal{S}$-3 logics. Here, $\mathcal{S}$ corresponds to the set of random variables declared zero-free. Therefore, a theory has been established in which reasoning about saturated conditional independence in the absence of zeros occurs as an idealized special case. The level of approximation can be controlled effectively by declaring random variables zero-free as required by the context of the application domain. The article has also established a trinity between the implication problems of three different efficient reasoning frameworks: saturated conditional independence statements and zero-free random variables, multivalued dependencies and `NOT NULL` attributes, and the propositional fragment $\mathfrak{F}$ under $\mathcal{S}$-3 interpretations. The dualities between any two of these frameworks fail when extended to more expressive languages such as general conditional independence statements or embedded multivalued dependencies, already in the idealized special case where all random variables are zero-free, all attributes are `NOT NULL`, and all propositional variables are interpreted classically. For these more expressive languages, the associated implication problems are no longer efficient.

The results have been established with respect to the simplest interpretation of zero markers as *no information*. The technical contributions in this article demonstrate the advantages of this interpretation. Moreover, the interpretation can accommodate other interpretations of zero marker occurrences where more information is available, e.g., that some occurrence is a structural or sampling zero. The disadvantage of this interpretation is a loss in the representation of knowledge whenever more information is available about the type of some zero occurrence. This warrants further research into this subject. Sampling zeros, for example, could be given semantics by possible worlds. One may then distinguish between possible and certain conditional independence statements which are satisfied by some and all possible worlds, respectively. Research on three-valued logics is rich. Ciucci and Dubois have given a concise survey on the interpretation of the third truth value [24].

Information about conditional independence is frequently represented in the form of graphs, where nodes correspond to random variables and edges represent the independencies among the random variables. It would be interesting to investigate how the theoretical properties of (saturated) conditional independence under controlled uncertainty can help facilitate the concise representation of probability distributions over sets of random variables [25, 26, 9, 1, 3]. It would also be interesting to extend our notion of probabilistic independence to the case where the probabilities are indeterminate or imprecisely specified, using, for example, the notion of epistemic independence [27]. Various notions of logical conditional independence together with their structural and computational properties have been studied [28, 29]. Certainly, it would be interesting to approximate reasoning about these notions, too.

8 Acknowledgement

This research is supported by the Marsden Fund Council from Government funding, administered by the Royal Society of New Zealand.

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