CONTROLLER SYNTHESIS FOR POLYNOMIAL DISCRETE-TIME SYSTEMS

by

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Abstract

The polynomial discrete-time systems are the type of systems where the dynamics of the systems are described in polynomial forms. This system is classified as an important class of nonlinear systems due to the fact that many nonlinear systems can be modelled as, transformed into, or approximated by polynomial systems.

The focus of this thesis is to address the problem of controller design for polynomial discrete-time systems. The main reason for focusing on this area is because the controller design for such polynomial discrete-time systems is categorised as a difficult problem. This is due to the fact that the relation between the Lyapunov matrix and the controller matrix is not jointly convex when the parameter-dependent or state-dependent Lyapunov function is under consideration. Therefore the problem cannot possibly be solved via semidefinite programming (SDP). In light of the aforementioned problem, we establish novel methodologies of designing controllers for stabilising the systems both with and without H_∞ performance and for the systems with and without uncertainty. Two types of uncertainty are considered in this research work; 1. Polytopic uncertainty, and 2. Norm-bounded uncertainty. A novel methodology for designing a filter for the polynomial discrete-time systems is also developed. We show that through our proposed methodologies, a less conservative design procedure can be rendered for the controller synthesis and filter design.

In particular, a so-called integrator method is proposed in this research work where an integrator is incorporated into the controller and filter structures. In doing so, the original systems can be transformed into augmented systems. Furthermore, the state-dependent Lyapunov function is selected in a way that its matrix is dependent only upon the original system state. Through this selection, a convex solution to the controller design and the filter design can be obtained efficiently. However, the price we pay for incorporating the integrator into the controller and filter structures is a large computational cost, which prevents us from using this method in general. To reduce the computational requirements for our design methodologies a number of simpler classes of polynomial systems are considered.

Based on this integrator approach, we first consider the state feedback control problem. In this case, the nonlinear state feedback control is tackled first and followed by the robust control problem in which the uncertain terms are described as polytopic forms. The robust control problem with norm-bounded uncertainty is studied next. Then, we discuss the nonlinear H_∞ state feedback control problem and robust nonlinear H_∞ state-feedback control problem with polytopic and norm-bounded uncertainty. The design

ensures that the ratio of the regulated output energy and the disturbance energy is less than a prescribed performance level. The filter design is tackled next and followed by the output feedback control problem. In the output feedback control, the problem of system uncertainties and disturbances are addressed.

The existence of such controllers and a filter are given in terms of the solvability of polynomial matrix inequalities (PMIs). The problem is then formulated as sum of squares (SOS) constraints, therefore it can be solved by any SOS solvers. In this research work, SOSTOOLS is used as a SOS solver.

Finally, to demonstrate the effectiveness and advantages of the proposed design methodologies in this thesis, numerical examples are given in each designed control system. The simulation results show that the proposed design methodologies can stabilise the systems and achieve the prescribed performance requirements.

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Abbreviations

A-D	Analogue to Digital
BMI	Bilinear Matrix Inequality
D-A	Digital to Analogue
HJI	Hamilton Jacobi Inequality
LMI	Linear Matrix Inequality
LPV	Linear parameter varying
NP	Nondeterministic Polynomial
PD	Positive Definite
PMI	Polynomial Matrix Inequality
PSD	Positive Semidefinite
SDP	Semidefinite Programming
SOS	Sum of Squares
SOSP	Sum of Squares Program
TS	Takagi Sugeno
ZOH	Zero Order Hold

Notations

The notation used in this thesis is quite standard. R^n and $R^{n\times m}$ denote respectively the set of $n \times 1$ vectors, and the set of all $n \times m$ matrices. The superscript T denotes the transpose and $*$ is used to represent the transposed symmetric entries in the matrix inequalities. In addition I represents the identity matrix and $L_2[0, \infty]$ is the space of square summable vector sequence over $[0, \infty]$. The$\|.\|_{[0,\infty]}$ denotes the $L_2[0, \infty]$ norm over $[0, \infty]$ defined as $\|f(x)\|^2_{[0,\infty]} = \sum_0^\infty \|f(x)\|^2$. The positive definiteness of the matrix $Q(x(k))$ is denoted by $Q(x(k)) > 0$ and $Q(x(k)) < 0$ denotes the negative definiteness of the $Q(x(k))$. Sometimes, for the notation simplicity we write $P(x_+)$ to denote $P(x(k + 1))$.

To my parents, parents-in-law, family, lovely wife and kids…

Chapter 1

Introduction

The purpose of this chapter is generally to emphasise the theory of nonlinear discrete-time systems, and the theory of polynomial discrete-time systems in particular. We begin this chapter by describing the concept of nonlinear systems and nonlinear discrete-time systems. Then, available methods for stabilizing nonlinear discrete-time is provided. Furthermore, the fundamental concept of polynomial systems is given and followed by the overview of the existing literature dealing with the controller synthesis for polynomial systems. As the sum of squares method is used for solving the controller and filter design problems, hence the description of sum of squares decomposition method is also highlighted in this chapter. Next, the motivation of delivering this research work is presented and followed by the contribution of this research work. This chapter is concluded with the outline of the thesis, highlighting the summary of each chapter.

1.1 Nonlinear Systems

Nonlinear systems play a vital role in the control systems engineering point of view. This is due to the fact that in practice all plants are nonlinear in nature. This is the main reason for considering the nonlinear systems in our work. In mathematics, a nonlinear system is one that does not satisfy the superposition principle, or one whose output is not directly proportional to its input. The best example to explain nonlinearity is obviously a saturation. This condition exists because it is impossible to deliver an infinite amount of energy to any real-world system.

In general, the state equations and output equations for the nonlinear systems may be written as follows:

$$\begin{aligned}\dot{x}(t) =& f[x(t), u(t)]\\ y(t) =& g[x(t), u(t)]\end{aligned} \tag{1.1}$$

The Lorenz chaotic system is one of the example of nonlinear systems which is described as below:

$$\begin{aligned}\dot{x}_1(t) =& -10x_1(t) + 10x_2(t) + u(t)\\ \dot{x}_2(t) =& 28x_1(t) - x_2(t) + x_1(t)x_3(t)\\ \dot{x}_3(t) =& x_1(t)x_2(t) - \frac{8}{3}x_3(t)\end{aligned} \tag{1.2}$$

Notice that the terms $x_1(t)x_3(t)$ and $x_1(t)x_2(t)$ exist in the equation (1.2), hence the system (1.2) is nonlinear in nature . In the sequel, the nonlinear discrete-time systems is introduced because it will be considered in this research work.

1.2 Nonlinear Discrete-Time Systems

Nowadays we can see that almost all controllers are implemented using computers. These kinds of controller are known as digital controllers. Basically, the use of digital controllers has rapidly increased since the first idea of using digital computers as one of the components in control systems emerged somewhere in 1950. The detailed history of this development can be found in [1]. The main reason for this development is due to the advances in hardware, hence it provides the control engineer with more powerful, reliable, faster and above all cheaper computers that could be implemented as process controllers. The another significant factor that drives the increase in development of digital controllers is the advantage of working with digital signals rather than continuous-time signals [2]. The aforementioned factors generally motivate us to deliver the research in the framework of discrete-time systems rather than continuous-time systems.

Generally, a closed loop system of computer controlled systems can be illustrated as Figure 1.1. From Figure 1.1 the output of the process $y(t)$ is a continuous-time signal. The measurements of the output signal are fed into an analog-to-digital (A-D) converter, where the continuous-time signal is converted into a digital signal - a sequence of measurements at sampling times t_k. At this point, if a digital measurement device is used, the A-D converter is no longer needed. This is true because the measurements are now taken at sampling times only. The computer interprets the converted output signals, $y(t_k)$ as a sequence of numbers, and this sequence is then used by the control algorithm to compute a sequence of digital control signals, $u(t_k)$. Notice that the process input is in continuous-time, hence a digital-to-analog (D-A) is used to transform the signals into a continuous-time signal. It is important to highlight here that between the sampling instants the system is in open loop mode. The system is synchronised by a real time clock in the computer. Consequently, the inter-sample behaviour is very often an issue and should not be disregarded. However, in many applications it is sufficient to describe the dynamic behaviour of the system at the sampling instants. At this stage, the interested signals are only at discrete-time, and this system is classified as a discrete-time system [1, 3]. We can now simply justify that if the dynamic of the process is in linear forms, then such a system is called a linear discrete-time system. Meanwhile if the behaviour of the process is nonlinear, then it is known as a nonlinear discrete-time system.

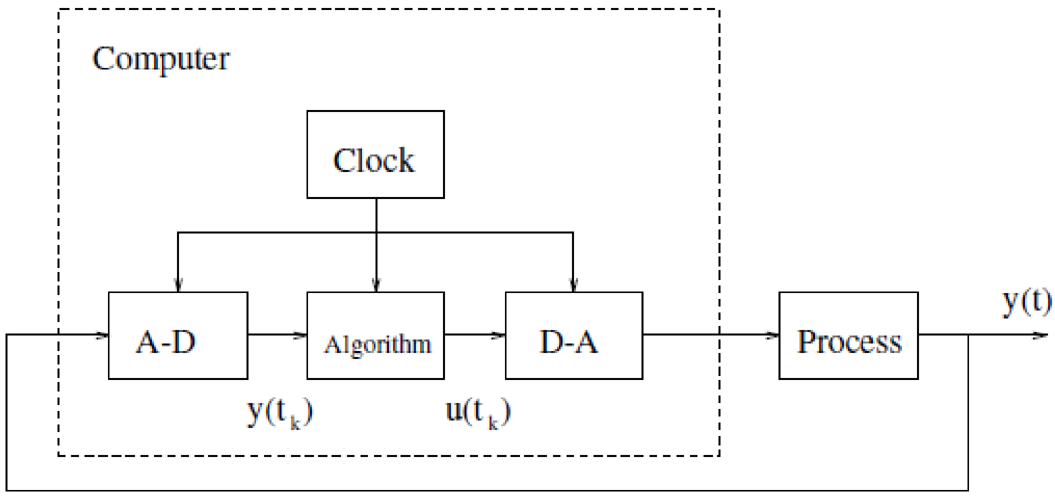

FIGURE 1.1: Schematic diagram of a computer-controlled system.

1.2.1 Discretization

With the fact that systems in this world are naturally in continuous-time, discretization shall be performed so that an approximated discrete-time system can be obtained. Listed below are the available methods in the discretization framework:

- **Euler's Forward differentiation method and Euler's Backward differentiation method**: The methods are based on the approximations of the time derivatives of the differential equation. The forward method is commonly used in developing simple simulators, whereas the backward method is normally used in discretizing simple signal filters and industrial controllers. The forward differentiation method is somewhat less accurate than the backward differentiation method, but it is simpler to use. Particularly, with nonlinear models the backward differentiation may give problems since it results in an impact equation for the output variable. In contrast, the forward differentiation method always gives an explicit equation to the solution.

- **Zero Order Hold (ZOH) method:** Using this method, it is assumed that the system has a zero order hold element on the input element of the systems. This is the case when the physical system is controlled by a computer via digital-to-analogue (D-A) converter. ZOH means that the physical input signal to the system is held fixed between the discrete points. Unfortunately, this method is relatively complicated to apply, and in practice the computer tool i.e MATLAB or LabVIEW can perform the job.

- **Tustin's method:** The discretization method is based on an integral approximation where the integral is interpreted as the area between the integrand and the time axis, and this area is approximated with trapezoids. It should be noted here that in Euler's method this area is approximated by a rectangle.

- **Tustin's method with frequency prewarping, or Bilinear transformation:** This method follows Tustin's method but with a modification so that the frequency response of an original continuous-time system and the resulting discrete-time systems have exactly the same frequency response at one or more specified frequencies.

It should be mentioned here that discretization methods are not the main focus of this thesis. The discretization is only applied in the simulation examples as to convert the continuous-time systems into discrete-time systems. To perform such a discretization, in this research work, the Euler's method is used due to its simplicity.

1.2.2 Brief Overview on The Literature of Nonlinear Discrete-time Systems

Due to the tremendous increase in digital control applications, a theory for discrete-time systems must be one of the important theories to be investigated especially for control design purposes. It is obvious that the desired performance may not be achieved if the controller design is based on the linearised model, and in many cases it is not possible to control nonlinear systems from the linearised model. Besides, the linear control theory cannot be applied in cases where: a large dynamic range of process variables is possible, multiple operating points are required, the process is operating close to its limits, small actuators cause saturation, and etc [3]. The feedback linearisation approach [4] is also cannot be extended to handle a system with parametric uncertainties. This is the major drawback of the feedback linearisation approach. With this knowledge, a significant amount of works can be found in the literature which attempts to provide a more general and a less conservative result than the linearised approach. One of the popular approaches is obviously backstepping control technique [5]. This approach is actually a combination of two popular theories, Lyapunov stability theory and the geometric method. By exploring the recursive design procedure, the time-varying uncertainties and parameter uncertainties can be incorporated in the problem formulation. However, it is difficult to find a general class of Lyapunov functions that could ensure the stability of such systems. This is the main disadvantage of backstepping control techniques, and obviously this drawback is common to all approaches that uses a constructive procedure in developing Lyapunov candidates [6].

Besides the existence of feedback linearisation and backstepping control techniques for stabilising nonlinear systems, there is one more popular method available that is widely used in control systems engineering, this method is called gain-scheduling [7]. The primary advantage of gain-scheduling for nonlinear control design is that it is usually possible to meet performance objectives over a wide range of operating conditions while still taking advantage of the wealth of tools and designers experience from linear controller synthesis. From this gain-sceduling approach, a more systematic control design technique is developed in the framework of linear parameter varying (LPV) systems with guaranteed stability and performance properties [8–10]. However it is important to highlight here that the stability and performance properties of the LPV systems only hold locally and it is well known that the application of LPV control techniques always requires one to convert the nonlinear systems into their quasi-LPV forms. These are usually the main sources of conservatism of this gain-scheduling method. Another popular approach in this area is based on the Fuzzy Takagi-Sugeno approach. It is well known that Takagi-Sugeno Fuzzy models can be used to approximate nonlinear systems [11–13]. However, in the TS fuzzy model, the premises variables are assumed to be bounded. In general, the premise variables are related to the state variables which implies that the state variables have to be bounded. This is one of major drawbacks of the TS fuzzy model approach.

Based on the above statements it is clear that there is plenty of room available to conduct study on stabilising the nonlinear discrete-time systems, and a better methodology should be proposed in order to reduce the conservatisms of the above-mentioned approaches. This motivates us to deliver the research in the framework of discrete-time systems, so that a less conservative approach can be proposed for stabilising nonlinear discrete-time systems. Before ending this section, it is important to note that the general nonlinear discrete-time systems are too complex, hence in this research work we limit the scope, where only the polynomial discrete-time systems will be considered. The reason for selecting the polynomial system will be given in the following text.

1.3 Polynomial Systems

It is well known that a wide class of nonlinear systems can be exactly represented by polynomial systems: i. e Lorenz chaotic systems. Moreover, the polynomial system has an ability to approach any analytical of nonlinear systems. These advantages explain why the polynomial system constitutes an important class of nonlinear systems and has attracted considerable attention from control researchers to involve themselves in this area, especially on the stability analysis and controller synthesis of polynomial systems [14].

The polynomial systems are the systems where the dynamic of the system is given in terms of polynomial functions or polynomial matrices. The general polynomial systems can be described as follows:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= g(x(t)) \end{aligned} \tag{1.3}$$

where $f(x(t), u(t))$ and $g(x(t))$ are in polynomials forms, and $x(t)$, $u(t)$ and $y(t)$ are respectively the states, input and the measured output.

Meanwhile, in discrete-time, the (1.3) can be written as follows:

$$\begin{aligned} x(k+1) &= f(x(k), u(k)) \\ y(k) &= g(x(k)) \end{aligned} \tag{1.4}$$

where $f(x(k), u(k))$ and $g(x(k))$ are in polynomials forms, and $x(k)$, $u(k)$ and $y(k)$ are respectively the states, input and the measured output of the system at sampling time, k. More precisely, the class of polynomial systems that is under consideration in this research work is described in terms of a state-dependent linear-like form as follows:

$$\begin{aligned} \dot{x}(t) =& A(x(t))x(t) + B(x(t))u(t) \\ y(t) =& C(x(t))x(t) \end{aligned} \tag{1.5}$$

where, $x(t) \in R^n$ is the state vectors, $u(t) \in R^m$ is the input and $y(t)$ is the measured output. $A(x(t))$, $B(x(t))$ and $C(x(t))$ are polynomial matrices of appropriate dimensions. Notice that (1.5) looks similar to the general nonlinear system, except the matrices $A(x(t))$, $B(x(t))$ and $C(x(t))$ must be of the polynomial forms.

In our work, the polynomial discrete-time system is described as follows:

$$\begin{aligned} x(k+1) =& A(x(k))x(k) + B(x(k))u(k) \\ y(k) =& C(x(k))x(k) \end{aligned} \tag{1.6}$$

where, $x(k) \in R^n$ is the state vectors, $u(k) \in R^m$ is the input and $y(k)$ is the measured output. $A(x(k))$, $B(x(k))$ and $C(x(k))$ are polynomial matrices of appropriate dimensions.

Given below are the examples of the polynomial systems:

Example 1: The Lorenz Chaotic Systems

$$\begin{aligned} \dot{x}_1(t) =& -10x_1(t) + 10x_2(t) + u(t) \\ \dot{x}_2(t) =& 28x_1(t) - x_2(t) + x_1(t)x_3(t) \\ \dot{x}_3(t) =& x_1(t)x_2(t) - \frac{8}{3}x_3(t) \end{aligned} \tag{1.7}$$

Example 2: A Tunnel Diode Circuit

$$\begin{aligned} C\dot{x}_1(t) =& -0.002x_1(t) - 0.01x_1^3(t) + x_2(t) \\ L\dot{x}_2(t) =& -x_1 - Rx_2(t) + u(t) \end{aligned} \tag{1.8}$$

where C is a capacitor value, L is inductance and R is a resistor.

One can see here that the systems given in (1.7)-(1.8) are actually nonlinear systems. These two examples illustrate the validity of the statement that we claimed earlier that many nonlinear systems can be represented by polynomial forms. It is also important to stress here that in this research we are not focusing on the method of discretizing the nonlinear systems to yield their discrete-time version. But, our main focus is to perform the controller synthesis for polynomial discrete-time systems.

1.3.1 Recent Work on Polynomial Systems

Our focus in this research work is on the controller synthesis of polynomial discrete-time systems. However, in this section, the results of polynomial continuous-time systems are also discussed. This is due to the fact that some results of the discrete-time systems are based or extended from the continuous-time systems. In this regard, we present the following recent development on the controller synthesis for polynomial systems. It is worth mentioning here that we limit the results to the one that considers a Sum of Squares (SOS) decomposition method and Linear Matrix Inequalities (LMIs) only. This is because the SOS decomposition method will be considered in this research work and it is actually complementary to the LMIs approach. A detailed description regarding the SOS decomposition method will be provided later in this chapter.

The controller synthesis or stabilisation problem is one of the important areas in the research of polynomial systems. Therefore, considerable attention has been devoted to this framework; for instances, see [16–21, 24–27]. In this present work, numerous techniques have been proposed to address the controller design problem for the polynomial systems. The brief of the proposed techniques is described below:

1. **Dissipation inequalities and SOS:** Dissipativity theory is known to be one of the most successful methods of analysing and synthesising the nonlinear control systems [15]. Mathematically speaking, this method is known as dissipation inequalities and has major advantages on the analysis and design of nonlinear systems. This might be due to the fact that the investigation of a possibly large number of differential equations, given by the control system description, is reduced to a small number of algebraic inequalities. Hence, the complexity of the analysis and design task is usually essentially reduced. In [16], the dissipation inequalities together with the SOS programming have been utilised to stabilise such polynomial systems. In particular, the authors represent their systems to be of descriptor systems or differential-algebraic systems where the functions are described by polynomial functions. They have managed to obtain the affine dissipation inequalities by the proposed method, hence the inequalities can be solved computationally via SOS programming. However, the process of achieving the affine dissipation inequalities varies for different types of problem. This means that the proposed method might not work for other problems.

2. **Kronecker products and LMIs:** The stabilisation of polynomial systems using Kronecker products method can be found in [17–19]. In the present papers, the polynomial systems have been simplified using the Kronecker product and power of vectors and matrices. Moreover, a new stability criterion for polynomial systems has been developed. A sufficient condition for the existence of the proposed controller is given in terms of the linear matrix inequalities (LMIs). The proposed controller can be applied to high order polynomial systems. This is the main advantage of this method. The strength of this approach comes from the solid theoretical results on the Kronecker products and the power of vectors and matrices.

3. **Semi-tensor products:** The semi-tensor product of matrices is a generalisation of the conventional matrix product in the case where the column number of the first factor matrix is not the same as the row number of the second factor matrix.

A brief survey for the related materials can be found in [20]. The advantage of this method is the general polynomial systems can be considered without any homogenous assumption. In [20], a method to stabilise the polynomial systems has been developed. The present paper first proposes a sufficient condition for a polynomial to be positive definite. Then, the formula for the time derivation of a candidate polynomial Lyapunov function with respect to a polynomial systems is provided. Through the sufficient condition, the candidate of the Lyapunov function can be checked for the positive definiteness, and its derivative to be negative definite. The sufficient condition is given by a set of linear algebraic inequalities. However, using this method, to choose a suitable candidate for the Lyapunov function is hard because there is no unique way to choose that Lyapunov function. The incorrect selection of the Lyapunov candidate leads the linear algebraic inequalities to have no solution. This is the main challenge of applying this method in the framework of controller synthesis for polynomial systems.

4. **Theory of moments and SOS:** Interesting work on the polynomial stabilisation that utilises a theory of moments can be found in [21]. It has been known for a long time that the theory of moments is strongly related to-and in fact, in duality with the theory of nonnegative polynomials and Hilbert's 17th problem on the representation of nonnegative polynomials [22]. In the light of this duality relationship, the author in [21] study the problem of polynomial systems stabilisation. They managed to show that the global solution to the problem can be obtained in a less conservative way than the available approaches, and the solution can be solved easily by SDP [23]. However, in order to achieve a convex solution to the controller synthesis problem, the Hermite stability criterion is used rather than the Lyapunov stability theorem. In doing so, the controller matrix can be decoupled from the Lyapunov matrix and the solvability conditions of the proposed controller are developed through a hierarchy of convex LMI relaxations. As stated by the authors, this methodology suffers from the large number of constraints in a PMI which consequently leads to the need for reliable numerical software to handle the problem.

5. **Fuzzy method and SOS:** T-S fuzzy model is well known to be good at approximating such nonlinear systems. Using this approach, [24] presented a SOS approach for modelling and control of nonlinear dynamical systems using polynomial fuzzy systems. A polynomial Lyapunov functions has been proposed in this work rather than the quadratic Lyapunov function. Hence the result is more general and less conservative than available LMI based approaches of T-S fuzzy modelling and control. Furthermore, a sufficient condition of the existence of the controller is given by polynomial matrix inequalities and formulated as SOS constraints. On

the other hand, [25] proposed an improved sum-of-squares (SOS)-based stability analysis result for the polynomial fuzzy-model based control system, formed by a polynomial fuzzy model and a polynomial fuzzy controller connected in a closed loop. Two cases, namely perfect and imperfect premise matching, are considered. Under the perfect premise matching, the polynomial fuzzy model and polynomial fuzzy controller share the same premise membership functions. While different sets of membership functions are employed, it falls into the case of imperfect premise matching. Based on the Lyapunov stability theory, improved SOS-based stability conditions are derived to determine the system stability and facilitate the controller synthesis. approach. The application of polynomial fuzzy T-S approach to the two-link robot arm can be found in [26]. Meanwhile, for static output control, the result can be found in [27]. However, in the TS fuzzy model, the premises variables are assumed to be bounded. In general, the premise variables are related to the state variables which implies that the state variables have to be bounded. This is one of the major drawbacks of the TS fuzzy model approach.

6. **Lyapunov method and SOS:** This is the common method that is widely applied in the literature for stabilising polynomial systems. This method is used in this research work and therefore the complete literature of this framework is provided in the following text.

1.3.1.1 On Literature Of Controller Synthesis For Polynomial Systems: The Lyapunov Method and SOS Decomposition Approach

It is well known that Lyapunov's stability theory [28] is one of the most fundamental pillars in control theory. Although this method was introduced more than hundred years ago, it remains popular among control researchers. This success is owed to its simplicity, generality, and usefulness. The Lyapunov's stability is a method that was developed for analysis purposes. However, it has become of equal importance for control designs over the last decades [6]. The Lyapunov's stability theory can be generalised as follows: Let us consider the problem of solving the stability for an equilibrium of a dynamical systems, $\dot{x} = f(x)$ using the Lyapunov function method. It is clear that to find a stability using the Lyapunov method, we need to find a positive definite function of Lyapunuv function, $V(x)$ defined in some region of the state space containing the equilibrium point whose derivative $\dot{V} = \frac{dv}{dx} f(x)$ is negative semidefinite along the system trajectories. Take the linear case for instance, $\dot{x} = Ax$, these conditions amount to finding a positive definite matrix P, such that $A^T P + PA$ is negative definite [29]; then the associated Lyapunov function is given by $V(x) = x^T Px$. Meanwhile for discrete-time systems, consider the following systems, $x(k+1) = f(x(k))$, we need to search the positive definite function of Lyapunov function, $V(x)$ defined in some region of the state space containing the equilibrium point whose difference of the Lyapunov function, $\Delta V = x^T(k+1)Px(k+1) - x^T(k)Px(k)$ is negative semidefinite along the system trajectories. The associated Lyapunov function is given by $V(x) = x^T Px$.

Since the SOS decomposition technique introduced about 10 years ago [30], the system analysis for polynomial systems can be performed more efficiently because it helps to answer many difficult questions on system analysis that were hard to answer before. The popularity of this method grew quickly among the community of control researchers because the algorithmic analysis of nonlinear systems can be delivered using a most popular method, which is the Lyapunov method (as discussed earlier). Generally, the most interesting and important point that was never seen until recently is that both the amount of proving the certificates of the Lyapunov function $V(x)$ and $-\dot{V}(x)$ can be reduced to the SOS [30]. Notice that for small systems, the construction of the Lyapunov function can be done manually. The difficulty of this construction is solely dependent upon the analytical skills of the researcher. However, when the vector field of the systems, $f(x)$ and the Lyapunov function candidate $V(x)$ are in polynomial forms, then the Lyapunov conditions are essentially is polynomial non-negativity conditions which can be NP-hard to test [45]. This is probably due to the lack of algorithmic constructions of Lyapunuv functions. However, if these nonnegativity conditions are replaced by the SOS conditions, then not only testing the Lyapunov function conditions - but also constructing the Lyapunov function can be done effectively using SDP [30]. This is the main advantage of using the SOS decomposition approach because the solution is indeed tractable. We will further describe the detail of the SOS decomposition method later.

The recent results in the framework of state feedback control synthesis for polynomial systems which utilises SOS decomposition method can be referred in [31–34]. In particular, [31, 32] propose the polynomial systems to be represented as a state-dependent linear-like form, and the state-dependent Lyapunov functions is proposed to be in terms of polynomial vector fields. The introduction of the state-dependent Lyapunov function or parameter-dependent Lyapunov function arises due to the fact that a quadratic Lyapunov function is always inadequate to stabilise the polynomial systems. Furthermore, the sufficient conditions to the problem is formulated as state-dependent LMIs and solved using the SOS-SDP based programming method. It is well known that to optimise the control problem for polynomial systems is hard because the solution in there is always not jointly convex. In this present papers, such a nonconvexity is avoided by assuming the Lyapunov matrix, $P(x)$ to be dependent upon the states $x(t)$ whose dynamics are not directly affected by the control input, i.e states whose corresponding rows in input matrix, $B(x)$ are zero. This, however, leads the result to be conservative.

The more recent and less conservative results can be found in [33]. In this paper, the effect of the nonlinear terms that exist in the problem formulation is described as an index, so that the control problem can be transformed into a tractable solution and can be possibly solved via SDP [23]. The optimisation approach is proposed to find a zero optimum of this index and solved using SOS programming effectively. However, to render a convex solution, the authors follow the same assumption as made in [32]. An improved version of the aforementioned approach can be found in [34] where an additional matrix variable is introduced to decouple the Lyapunov matrix from the system matrices. Therefore, the controller design can be performed in a more relaxed way and the proposed methodology can be extended to the robust control problem of polynomial systems. However, to obtain a convex solution, the non convex term is bounded by an upper bound. Therefore, the stability can only be guaranteed within the bound region.

Sometimes it is difficult to synthesise a controller that works globally. Besides, in a restricted region, local controllers often provide a better solution than global controllers. Some developments of this field can be found in [35, 36]. In [35], a rational Lyapunov functions of states was used to synthesise the polynomial systems. The variation of states is bounded, and the domain of attraction was embedded in the specified region by the nonlinear vector. With this, the state feedback controller is established and formulated as a set of polynomial matrix inequalities and solved using any SOS programming. The coupling between system matrices and the Lyapunov matrix causes the results to be quite conservative in general. Hence, [36] relaxes this issue by introducing a slack variable matrix. In doing so, the Lyapunov matrix is decoupled from the system matrices. Now, the parameterisation of the resulting controllers is independent of the Lyapunov-matrix variables. This allows them to extend their result to construct robust controller for uncertain polynomial systems using state-dependent Lyapunov functions.

With the knowledge that the full state variables are not always accessible in practical nonlinear systems and the dynamic output feedbacks result in high-order controllers which may not be practical in industry, the static output feedback design attracts much attention among practitioners. Some developments of this area that utilised the SOS decomposition approach can be found in [37–40]. The systems discussed in [37] are represented in a state-dependent linear-like form. More precisely, the authors assumed that the control input matrix has some zero rows and the Lyapunov function only depends on states whose corresponding rows in control matrix are zeros, that is, the state dynamics are not directly affected by the control input. This assumption leads to the conservatism of the controller design. The latest results of this area can be found in [38–40], where an iterative algorithm based on SOS has been proposed in order to convert the nonconvex problem into a convex problem of polynomial systems synthesis, so that it can be solved using SDP efficiently. The authors in [38–40] have managed to show that their approach is less conservative than the available approaches and provides more general results in this field. But the main disadvantage of this approach is the selection of the initial polynomial function, $\epsilon(x)$, which is hard to choose because it is unknown.

The above-mentioned results are dedicated to solving the polynomial continuous-time systems. In regard to the polynomial discrete-time systems, there are only few results available which utilises the SOS decomposition method in their approach. The first result is proposed in [41], where the authors employ a state-dependent polynomial Lyapunov function as their Lyapunov candidates. Then, some transformations are required to represent the system with the introduction of new matrices (in polynomial). Furthermore YALMIP and PENOPT for PENBMI [42, 43] have been utilised to solve the problem. However, the main drawback of this approach is that the selected new matrices are not unique and hence difficult to choose. The most recent result was addressed by [44], where the nonconvex term is bounded by an upper bound value, then optimisation is carried out to find a zero optimum for the nonlinear term. Here, the problem was formulated as SOS and could be solved by using any SOS solvers. By bounding the nonconvex terms, the controller that resulted from this method can only guarantee the closed-loop stability within bounds. This is similar to the method proposed in [33, 34]. Therefore, they share the same weaknesses as encountered in [33, 34].

1.3.2 Sum of Squares(SOS) Decomposition

A brief overview of a SOS decomposition method is given in this section. A more detail description of the SOS decomposition method can be referred in [30].

Generally, proving a nonnegativity of multivariable functions is considered as one of the important aspects in control systems engineering. This problem is similar to the problem of proving the nonnegativity of a Lyapunov function. If the nonlinear system is concerned, then it is hard to prove the nonnegativity of such systems. Basically, the problem is to prove that

$$F(x_1, ..., x_n) \geq 0 \quad x_1, \ldots, x_n \in \Re \tag{1.9}$$

A great amount of research has been devoted to proving (1.9). However, up to now there is no unique solution to the problem in (1.9).

Thus, some limit should be applied to the possible functions $F(.)$, while at the same time making the problem general enough to guarantee the applicability of the results.

It has been shown in [30], that considering the case of polynomial functions is a good compromise for this issue.

Definition 1.1. [30]: A form is a polynomial where all the monomials have the same degree $d := \sum_i \alpha_i$. In this case, the polynomial is homogenous of degree d, since it satisfies $f(\lambda x_1, \ldots, \lambda x_n) = \lambda^d f(x_1, \ldots, x_n)$.

It should be highlighted here that the general problem of testing global positivity of a polynomial function is NP-hard problem (when the degree is at least four) [45]. Therefore, a problem with a large number of variables will have unacceptable behaviour for any method that is guaranteed to obtain the right answer in every possible instance. This is actually the main drawback of theoretically powerful methodologies such as the quantifier elimination approach [46, 47].

The question now is: are there any conditions to guarantee the global positivity of a tested polynomial time functions? This question underlines the existence of SOS decomposition approach [30] as one condition to guarantee the global positivity of polynomial functions.

It is obvious that a necessary condition to satisfy a polynomial $F(x)$ in (1.9) is that the degree of the polynomial in the homogeneous case must be even. Hence, a simple sufficient for a real-valued function $F(x)$ to be positive everywhere is given by the existence of a SOS decomposition:

$$F(x) = \sum_i {f_i}^2(x) \tag{1.10}$$

It can been seen that if $F(x)$ can be written as (1.10), the nonnegativity of $F(x)$ can be guaranteed. It is stated in [30] that for the problem to make sense, some restriction on the class of functions f_i has to be imposed again. Otherwise, we need to always define f_1 to be the square root of F, but, this results the condition both useless and trivial.

It has been shown in [30] that $F(x)$ is a SOS polynomial if and only if there exists a positive definite matrix Q such that

$$F(x) = z^T Q z \tag{1.11}$$

where $z(x)$ is the vector of all monomials of degree less than or equal to the half degree of $F(x)$. This is the idea given in [51] and it can be shown to be conservative in general. The main reason is that since the variables z_i are not independent, the representation (1.11) might not be unique, and Q may be positive definite or positive semi-definite for some representations but not for others. Similar issues appear in the analysis of quasi-LPV systems; refer [52]. However, using identically satisfied constraints that relate the z_i variables among themselves, it is easily shown that there is a linear subspace of matrices Q that satisfy (1.11). If the intersection of this subspace with the positive semidefinite matrix cone is nonempty for the original function, F is guaranteed to be SOS and therefore psd. So, if F can indeed be written as the SOS of polynomials, then expanding in monomials will provide the representation (1.11). The following example explains this concept.

Example 1 [30] : Consider the quartic form in two variables described below, and define $z_1; = {x_1}^2 . z_2 := {x_2}^2, z_3 := x_1 x_2$:

$$\begin{aligned} F(x_1, x_2) &= 2{x_1}^4 + 2{x_1}^3 x_2 - {x_1}^2 {x_2}^2 + 5{x_2}^4 \\ &= \begin{bmatrix} {x_1}^2 \\ {x_2}^2 \\ x_1 x_2 \end{bmatrix}^T \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} {x_1}^2 \\ {x_2}^2 \\ x_1 x_2 \end{bmatrix} \\ &= \begin{bmatrix} {x_1}^2 \\ {x_2}^2 \\ x_1 x_2 \end{bmatrix}^T \begin{bmatrix} 2 & -\lambda & 1 \\ -\lambda & 5 & 0 \\ 1 & 0 & -1+2\lambda \end{bmatrix} \begin{bmatrix} {x_1}^2 \\ {x_2}^2 \\ x_1 x_2 \end{bmatrix} \end{aligned} \tag{1.12}$$

Take for instance $\lambda = 3$. In this case,

$$Q = L^T L, L = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix} \tag{1.13}$$

Therefore we have the sum of squares decomposition:

$$F(x_1, x_2) = \frac{1}{2}((2{x_1}^2 - 3{x_2}^2 + x_1 x_2)^2 + ({x_2}^2 + 3x_1 x_2)^2) \tag{1.14}$$

Parrilo [30] also observed that the existence of (1.11) can be cast as a semidefinite programming [23]. This is the most important property that distinguishes its from other approaches. This feature is proved to be critical in the application to many control related problem. How does it works? Basically by expanding the $z^T Q z$ and equating the coefficient of the resulting monomials to the ones in $F(x)$, we obtain a set of affine relations in the elements of Q. We know that, for $F(x)$ being a SOS is equivalent to $Q \geq 0$, the problem of finding Q then which proves $F(x)$ is an SOS can certainly be cast as a semidefinite program.

Thus, although checking the nonnegativity of $F(x)$ is NP-hard when the degree in $F(x)$ is 4 as stated before, but, checking whether $F(x)$ can be written as SOS is definitely tractable - it can be formulated as a semidefinite program, which has worst-case polynomial time complexity as mentioned in the previous paragraph. Authors in [30], produced significance results in suggesting that the relaxation is not too conservative in general. It must be noted here that as the degree of $F(x)$ is increased or its variables number is increased, then the computational complexity for testing whether $F(x)$ is a SOS is significantly increased. Nonetheless, the complexity overload is still a polynomial function of these parameters.

In general, the conversion from SOS decomposition to the semidefinite programming can be manually done for small size instance or tailored for specific problem classes. However, this such conversion is cumbersome in general. Thus the software is absolutely necessary to aid in converting them. Specifically, the relaxation uses Gram Matrix methods to efficiently transform the NP-hard problem, into linear matrix inequalities (LMIs) [29]. These can in turn be solved in polynomial time with semidefinite programming (SDP) [23, 29]. To date, there exist several freely available toolboxes to formulate these problems in Matlab, for example SOSTOOLS [53], YALMIP [54], CVX [55], and GLoptiPoly [56]. Whereas SOSTOOLS is specifically designed to address polynomial nonnegativity problems, the latter toolboxes have further functionality, such as modules to solve the dual of the SOS problem, the moment problem.

In this work we use SOSTOOLS to perform this conversion for our problem formulation. Hence we will describe the working principles of this software in the following section.

Basically the polynomial case is a well-analysed problem, first studied by David Hilbert more than century ago [48]. He raised a very popular and important questions in his famous list of twenty-three unsolved problem which was presented at the International Congress of Mathematicians in Paris, 1900, dealing with the representation of a definite form as a SOS. Hilbert also noted that not every positive semidefinite (psd) polynomial (or form) is sos. However, [49] has proved that the numerical examples seem to indicate that the gap between the SOS and nonnegativity polynomial is small. A complete characterisation has been outlined by Hilbert in explaining when these two classes are equivalent. There are three cases which the equality holds: 1. The case with two variables ($n = 2$), 2. The familiar case of quadratic form ($i.e, m = 2$), 3. A surprising case, where $P_{3,4} = \sum_{3,4}$, refer [50] for detailed explanations.

Before ending this section, the following lemma is presented and it is useful for our main results later.

Lemma 1.2. *[32] Let $F(x)$ be an $N \times N$ symmetric polynomial matrix of degree $2d$ in $x \in R^n$. Furthermore, let $Z(x)$ be a column vector whose entries are all monomials in x with a degree no greater than d, and consider the following conditions:*

1. $F(x) \geq 0$ *for all* $x \in R^n$

2. *$v^T F(x) v$ is a SOS, where $v \in R^N$*

3. *There exists a positive semidefinite matrix Q such that* $v^T F(x) v = (v \otimes Z(x))^T Q (v \otimes Z(x))$, *with $\otimes$ denoting the Kronecker product*

It is clear that $F(x)$ being a SOS implies $F(x) \geq 0$, but the converse is generally not true. Furthermore, statement (2) and statement (3) are equivalent.

1.3.2.1 SOSTOOLS

SOSTOOLS is a free, third-party MATLAB toolbox specially designed to handle and solve the SOS programs. The techniques behind it are based on the SOS decomposition for multivariate polynomials, which can be efficiently computed using semidefinite programming frameworks. The availability of SOSTOOLS gives a great advantage to the researchers that involved in SOS polynomial frameworks. Moreover, the SOSTOOLS gives a new direction for solving many hard problems such as global, constrained, and boolean optimisation due to the fact that these technique provide a convex relaxations approach.

The working principles of SOSTOOLS is shown in figure 1. Basically, the SOSTOOLS will automatically convert the SOS program (SOSP) into semidefinite programs (SDPs). Then, it calls the SDP solver, and converts the SDP solution back to the solution of the SOS programs. In this way the details of the reformulation are abstracted from the user, who can work at the polynomial object level. The user interface of SOSTOOLS has been designed to be simple, easy to use, and transparent while keeping a large degree of flexibility. The current version of SOSTOOLS uses either SeDuMi or SDPT3, both of which are free MATLAB add-ons, as the SDP solver. A detailed description about how SOSTOOLS works can be found in the SOSTOOLS user's guide [57].

1.4 Research Motivation

This section provides the reasons that prompted us to conduct research in the framework of controller synthesis for polynomial discrete-time systems. The key motivations for this thesis come from several sources. The most general motivation comes from the fact that polynomial systems appear in a wide range of applications. This is due to the fact that many nonlinear systems can be modelled as, transformed into, or approximated by polynomial systems. The polynomial systems do not only exist in process control and systems biology, but also appear in many other fields of application, for instance in mechatronic systems, and laser physics; see [58, 59]. A few well known polynomial

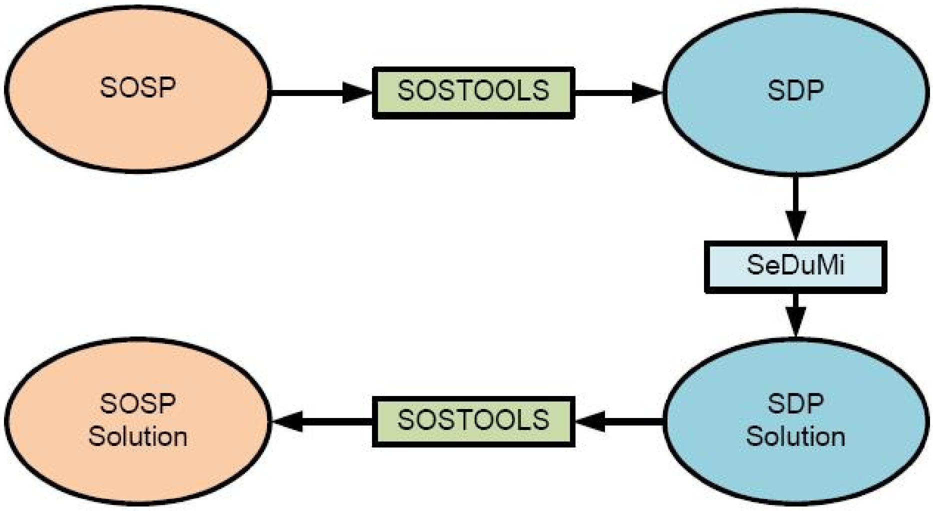

FIGURE 1.2: Diagram depicting how SOS programs (SOSPs) are solved using SOSTOOLS.

system are captured in Table 1.1 (borrowed from [16]). From this table, one can see that polynomial systems can show a very rich variety of dynamic behaviour. On the other hand, the table also depicts that polynomial systems maybe in general very difficult to study. Therefore, this class of system is considered in this thesis.

TABLE 1.1: Example of Polynomial Systems

System	Description
$\dot{x} = x^2$	Finite escape behaviour
Lorenz system	Chaos
Brockett integrator	Discontinuous time
Van der Pol system	limit cycles
Artstein Circle	Nonsmooth control
MY conjecture	Global stability

The second motivation arises from the fact that the state-dependent or parameter-dependent Lyapunov function is widely used in the framework of stability analysis and controller design for nonlinear systems. It has been shown recently that the state-dependent Lyapunov function provides a great advantage when dealing with the controller synthesis for polynomial systems [31–34]. This leads to our belief that the utilisation of the state-dependent Lyapunov function method should also be effective in designing the controller for polynomial discrete-time systems.

However, with the utilisation of state-dependent Lyapunov functions, the controller design for polynomial discrete-time systems becomes very difficult. This is due to the fact that the relation between the Lyapunov function and the controller matrix is no longer jointly convex. This problem will be highlighted in detail in Chapter 2. In continuous-time system, the aforementioned problem can be avoided by assuming that the Lyapunov matrix is only dependent upon the control input whose corresponding rows are zero [33]. Unfortunately, for discrete-time systems, although the same assumption is made, the problem still exists. A possible way to resolve this problem is given in [44], but the results suffer from some conservatism and such conservatism has been discussed earlier. In this thesis, we attempt to relax the problem by incorporating an integrator into the controller structures. In particular, we called this method as integrator method.

Due to the problem that is discussed above, only a few results are available in the area of controller synthesis in the context of polynomial discrete-time systems [41, 44]. As our discussion in the literature earlier shows, the results from both papers are suffering from their own conservatism. This consequently motivates us to carry out work on polynomial discrete-time systems stabilisation. Hence a more general and less conservative result can be provided than the available approaches.

Furthermore, it is also necessary for us to consider the robust controller design for polynomial discrete-time systems because to date, to the author's knowledge, no results have been presented in this framework that consider the SOS programming technique. For this context, the polytopic uncertainty and norm-bounded uncertainty will be considered because both of them are commonly appear in the real world. Besides, the norm-bounded uncertainty is not fully studied in the area of polynomial systems. This is another motivation that leads us to consider the norm-bounded uncertainty in our research work.

The final and somewhat peripheral motivation is that many control design problems are normally formulated in terms of inequalities rather than simple equalities. Moreover, a lot of problems in control engineering can be formulated as Polynomial Matrix Inequalities (PMIs) feasibility problems. Using the SOS decomposition method, such PMIs can be further formulated as SOS constraints [30]. The SOS inequalities framework provides a tractable method to solve the problem through an analytical solution. Furthermore, the advantage of formulating the problem in terms of the SOS inequalities is the availability of toolboxes which are compatible with MATLAB that are capable for solving the feasibility and optimisation problems by interior point methods. All of these toolboxes are actually SOS-SDP based and the general concept regarding them has already been explained earlier.

1.5 Contribution of the Thesis

The focus of this thesis is to establish novel methodologies for robust stabilisation, control with disturbance attenuation and filter design for a class of polynomial discrete-time systems. The polytopic uncertainties and norm-bounded uncertainties are considered in this research work and the proposed controller should able to handle the appearance of such uncertainties.

The main contribution arises from the incorporation of an integrator into the controller structures. In doing so, a convex solution to the polynomial discrete-time systems stabilisation with the utilisation of a state-dependent Lyapunov function function can be obtained in a less conservative way than the available approaches. In the light of this integrator method, the problem of robust control and robust H_∞ control for polynomial discrete-time systems are tackled. The integrator method is also applied to the filter design problem.

In this thesis, we first highlight the problem of the controller design for polynomial discrete-time systems when the state-dependent Lyapunov function is under consideration. Motivated by this problem, we propose a novel method in which an integrator is proposed to be incorporated into the controller structures. Then, we show that the original systems with the proposed controller can be described in augmented forms. In addition, by choosing the Lyapunov matrix to be only dependent upon the original system's states, a convex solution to the robust control problem and robust H_∞ control problem for polynomial discrete-time systems can be rendered in a less conservative way than available approaches.

In light of the integrator method, we propose a novel methodology for designing a robust nonlinear controller in which the polytopic and norm-bounded uncertainty are under consideration. It should be noted here that, to date, no result is available in this framework that utilises SOS programming for polynomial discrete-time systems. Furthermore, the interconnection between the nonlinear H_∞ control problem and the robust nonlinear H_∞ control problem is provided through a so-called $'scaled'$ systems. This allows us to efficiently solve the robust H_∞ control problem with the existence of norm-bounded uncertainties. Next, we show that by exploiting the integrator method, a filter design methodology can also be established for polynomial discrete-time systems. Furthermore, by applying the integrator method, the output feedback controller is developed for polynomial discrete-time systems with and without H_∞ performance and also with and without uncertainties.

Finally, to demonstrate the effectiveness and advantages of the proposed design methodologies of this thesis, some numerical examples are given. The simulation results also show that the proposed design methodologies can achieve the stability requirement or the prescribed performance index.

1.6 Thesis Outline

The contents of the thesis are as follows:

Chapter 2 describes a nonlinear feedback controller design for polynomial discrete-time systems. In this chapter, the problems of designing a controller for polynomial discrete-time systems are highlighted first. Then, a novel method for solving the problem is proposed. Furthermore, we show that the results can be directly extended to the robust control problem with polytopic uncertainty. The existence of the proposed controller is given in terms of the solvability of polynomial matrix inequalities (PMIs), which are formulated as SOS constraints and can be solved by the recently developed SOS solvers. The effectiveness of the proposed method is confirmed through a simulation example.

Chapter 3 demonstrates a robust nonlinear feedback controller design with the existence of norm-bounded uncertainties. We show that the uncertainties are tackled by applying an upper bound technique. The effectiveness of the proposed method is validated through a demonstrative example.

Chapter 4 presents a nonlinear H_∞ state feedback control for polynomial discrete-time systems. The H_∞ performance is needed to be fulfilled in this chapter while at the same time the system's stability is guaranteed. A less conservative result than the available approaches is obtained through the utilisation of an integrator approach. The result is then directly extended to the robust nonlinear H_∞ control problem with polytopic uncertainties. The sufficient conditions for the existence of the proposed controller is given in terms of the solvability of SOS inequalities and solved using SOSTOOLS. A tunnel diode circuit is used to demonstrate the validity of the proposed approach.

In **Chapter 5**, we study the robust nonlinear H_∞ state feedback control for polynomial discrete-time systems with the appearance of norm-bounded uncertainties in the system's state and input. The *'scaled'* system is introduced in this chapter, and the interconnection between the nonlinear H_∞ control problem (described in Chapter 3) and the robust nonlinear H_∞ control problem is established. We show that the robust nonlinear H_∞ control problem is solvable only if the *'scaled'* system is solvable. The effectiveness of the proposed methodology is demonstrated through a tunnel diode circuit.

Chapter 6 deals with the problem of filtering design for a class of polynomial discrete-time systems. By utilising the integrator method, a possible solution to the filter design problem is presented. Solutions to the filter design have been derived in terms of PMIs, which are formulated as SOS constraints. A numerical example is given along with the theoretical presentation.

In the above chapters, we assume that all the states are available for feedback which is not true in many practical cases. Therefore, in **Chapter 7** we investigate the nonlinear H_∞ output feedback control for polynomial discrete-time systems. The problems of designing the output feedback controller are given in this chapter. Then, a novel method is proposed in order to overcome those problems. The results are then directly extended to the robust H_∞ control problem with polytopic uncertainty. The sufficient conditions for the existence of such a controller is given in terms of the solvability of SOS inequalities and can be solved by the recently developed SOS solver.

In **Chapter 8**, motivated by the results illustrated in Chapter 5, we develop a methodology for robust controller design with H_∞ performance with the existence of norm-bounded uncertainties. Again, in this chapter we show that the robust nonlinear H_∞ control problem is solvable only if the $'scaled'$ system is solvable.

Concluding remarks are given and suggestions for future research work are discussed in **Chapter 8**. Finally, some mathematical background knowledge that is used throughout this research is provide in the **Appendix**.

Chapter 2

Nonlinear Control for Polynomial Discrete-Time Systems

2.1 Introduction

The controller design for polynomial discrete-time systems is a hard problem. This is due to the fact that the relation between the Lyapunov function and the controller matrix is always not jointly convex. In continuous-time systems, a convex solution can be achieved by restricting the Lyapunov function to be the only function of states whose corresponding rows in the control matrix are zeroes and whose inverse is of a certain form [32–34]. Unfortunately, this leads to the results being conservative. In discrete-time systems, the nonconvex problem remains persistent although the same restriction is applied. The attempt to design a state feedback controller for polynomial discrete-time systems can be found in [44]. The proposed methodology suffers from several sources of conservatism and such conservatisms have been discussed in the preceding chapter.

Motivated by the results in [44] and the problem that has been mentioned above, this chapter attempts to convexify the state feedback control problem for polynomial discrete-time systems in a less conservative way and consequently leads to a less conservative controller design procedure for polynomial discrete-time systems. To be precise, in our work a less conservative design procedure is achieved by incorporating an integrator into the controller structure. In doing so, an original system can be transformed into an augmented system, and the Lyapunov function can be selected to be only dependent upon the original system states. This consequently causes the solution of controller synthesis for polynomial discrete-time systems to be convex and therefore can possibly be solved via SDP. It is important to note here that the resulting controller is given in terms of a rational matrix function of the augmented system. The sufficient condition for the existence of our proposed controller is given in terms of the solvability condition of PMIs, which is formulated as SOS constraints. The problem, then, can be solved by the recently developed SOS solvers.

The rest of this chapter is organised as follows: Section 2.2 provides the main results in which the problem of designing a state feedback controller is highlighted first, then a novel method is proposed to overcome the problem. The results are then directly extended to the robust control problem with polytopic uncertainty. The validity of our proposed approach is illustrated using a simulation example in Section 2.3. Conclusions are given in Section 2.4.

2.2 Main Result

In this section, we present the robust nonlinear feedback controller design for polynomial discrete-time systems with an integrator. The significance of incorporating the integrator into the controller structure can be seen in this section. We begin this section by synthesising the controller without the existence of uncertainties, and the result is subsequently extended to the robust controller design with the existence of polytopic uncertainties.

2.2.1 Nonlinear feedback control design

Consider the following dynamic model of a polynomial discrete-time system:

$$x(k+1) = A(x(k))x(k) + B(x(k))u(k) \tag{2.1}$$

where $x(k) \in R^n$ is a state vector and $u(k)$ is an input. $A(x(k))$, and $B(x(k))$ are polynomial matrices of appropriate dimensions.

For system (2.1), the state feedback controller is proposed as follows:

$$u = K(x(k))x(k). \tag{2.2}$$

For this purpose, we use the standard assumption for the state feedback control where all states vector $x(k)$ are available for feedback. The following theorem is established for the system (2.1) with the controller (2.2).

Theorem 2.1. *The system* (2.1) *is asymptotically stable if*

1. *There exist a positive definite symmetric polynomial matrix, $P(x(k))$ and polynomial matrix, $K(x(k))$ such that*

$$\begin{aligned}&-(A(x(k))+B(x(k))K(x(k)))^T P^{-1}(x_+)(A(x(k))+B(x(k))K(x(k)))\\&+P^{-1}(x(k))>0,\end{aligned}\tag{2.3}$$

or

2. *There exist a positive definite symmetric polynomial matrix, $P(x(k))$, polynomial matrices, $K(x(k))$ and $G(x(k))$ such that*

$$\begin{bmatrix} G^T(x(k))+G(x(k))-P(x(k)) & * \\ A(x(k))G(x(k))+B(x(k))K(x(k))G(x(k)) & P(x_+)\end{bmatrix}>0.\tag{2.4}$$

Proof: Select a Lyapunov function as follows:

$$V(x(k))=x^T(k)P^{-1}(x(k))x(k)\tag{2.5}$$

Then the difference of (2.5) along (2.1) with (2.2) is given by

$$\begin{aligned}\Delta V(x(k))&=V(x(k+1))-V(x(k))<0\\&=x^T(k+1)P^{-1}(x_+)x(k+1)-x^T(k)P^{-1}(x(k))x(k)\\&=\big(A(x(k))x(k)+B(x(k))K(x(k))x(k)\big)^T P^{-1}(x_+)\big(A(x(k))x(k)\\&+B(x(k))K(x(k))x(k)\big)-x^T(k)P^{-1}(x(k))x(k)\\&=x^T(k)\big[(A^T(x(k))+K^T(x(k))B^T(x(k)))P^{-1}(x_+)(A(x(k))\\&+B(x(k))K(x(k)))-P^{-1}(x(k))\big]x(k)\end{aligned}\tag{2.6}$$

Now we have to show that (2.4) $\Leftrightarrow$ (2.3): (Necessity) Choose $G(x(k))=G^T(x(k))=P(x(k))$. (Sufficiency) Suppose (2.4) holds, thus $G^T(x(k))+G(x(k))>P(x(k))>0$. This implies that $G(x(k))$ is nonsingular. Since $P(x(k))$ is positive definite, hence the inequality

$$(P(x(k))-G(x(k)))^T P^{-1}(x(k))\,(P(x(k))-G(x(k)))>0\tag{2.7}$$

holds. Therefore establishing

$$G^T(x(k))P^{-1}(x(k))G(x(k))>G(x(k))+G^T(x(k))-P(x(k)),\tag{2.8}$$

and therefore we have

$$\begin{bmatrix} G^T(x(k))P^{-1}(x(k))G(x(k)) & * \\ A(x(k))G(x(k)) + B(x(k))K(x(k))G(x(k)) & P(x_+) \end{bmatrix} > 0. \tag{2.9}$$

Next, multiply (2.9) on the right by $diag[G^{-1}(x(k)), I]$ and on the left by $diag[G^{-1}(x(k)), I]^T$, we get

$$\begin{bmatrix} P^{-1}(x(k)) & * \\ A(x(k)) + B(x(k))K(x(k)) & P(x_+) \end{bmatrix} > 0. \tag{2.10}$$

Applying the Schur complement to (2.10), we arrive at (2.3). Knowing that (2.3) holds, we have $\Delta V(x(k)) < 0, \forall x \neq 0$, which implies that the system (2.1) with (2.2) is globally asymptotically stable. This completes the proof. $\Delta\Delta\Delta$

Remark 2.2. The advantages of formulating the problem of the form (2.4) are twofold:

1. The Lyapunov function is decoupled from the system matrices. Therefore, the selection of the polynomial feedback control law can be chosen to be a polynomial of arbitrary degree, which improves the solvability of the nonlinear matrix inequalities by the SOS solver. This also allows the method to be extended to the robust control problem.

2. The number of $P(x(k))$ can be reduced significantly in the problem formulation.

The introduction of this new polynomial matrix method is first proposed in [60] for linear cases, and has been adopted by [44] for nonlinear cases. It is also important to note here that this new polynomial matrix, $G(x(k))$ is not constrained to be symmetrical.

It is worth mentioning that the conditions given in Theorem 2.1 are in terms of state dependent polynomial matrix inequalities (PMIs). Thus, solving this inequality is computationally hard because one needs to solve an infinite set of state-dependent PMIs. To relax these conditions, we utilise the SOS decomposition approach as described in [30] and have the following proposition:

Proposition 2.3. *The system (2.1) is asymptotically stable if there exist a symmetric polynomial matrix, $P(x(k))$, polynomial matrices $L(x(k))$ and $G(x(k))$, and constants $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that the following conditions hold for all $x \neq 0$*

$$v^T[P(x(k)) - \epsilon_1 I]v \qquad \text{is a SOS} \tag{2.11}$$

$$v_1^T\left[M(x(k)) - \epsilon_2 I\right]v_1^T \qquad \text{is a SOS} \tag{2.12}$$

where,

$$M(x(k)) = \begin{bmatrix} G^T(x(k)) + G(x(k) - P(x(k)) & * \\ A(x(k))G(x(k)) + B(x(k))L(x(k)) & P(x_+) \end{bmatrix} \tag{2.13}$$

Meanwhile, v and v_1 are free vectors with appropriate dimensions and $L(x(k)) = K(x(k))G(x(k))$. Moreover, the nonlinear feedback controller is given by $K(x(k)) = L(x(k))G^{-1}(x(k))$.

Proof: The proof for this proposition can be obtained easily by following the technique given in the proof section of Theorem 2.1. Then by Proposition 1.2 (given in Chapter 1), if the inequalities described in (2.11)-(2.12) are feasible, it implies that inequality (2.4) is true. The proof ends. $\Delta\Delta\Delta$

Remark 2.4. Proposition 2.3 provides a sufficient condition for the existence of a state feedback controller and is given in terms of solutions to a set of parameterised polynomial matrix inequalities (PMIs). Notice that $P(x_+)$ appears in the PMIs, therefore the inequalities are not convex because $P(x_+) = P(x(k+1)) = P(A(x(k))x(k) + B(x(k))K(x(k))x(k))$. Therefore it is very difficult to directly solve Proposition 2.3 because the PMIs need to be checked for all combination of $P(x(k))$ and $K(x(k))$, which results in solving an infinite number of polynomial matrix inequalities. A possible way to resolve this problem has been proposed in [44] in which a predefined upper bound is used to limit the effect of the nonconvex term. However, this predefined upper bound is hard to determine beforehand, and the closed loop stability can only be guaranteed within a bound region. Motivated by this fact, we propose a novel approach in which the aforementioned problem can be removed by incorporating the integrator into the controller structure. This consequently provides a less conservative result on the similar underlying issue. The details of our method are given in the following text.

2.2.2 The integrator approach

In this section, we show that by incorporating the integrator into the controller structure, the controller synthesis for polynomial discrete-time systems can be convexified efficiently and therefore a less conservative design procedure can be achieved.

The nonlinear feedback controller with the integrator is proposed as follows:

$$\left.\begin{aligned} x_c(k+1) =& x_c(k) + A_c(x, x_c) \\ u(k) =& x_c(k) \end{aligned}\right\} \tag{2.14}$$

where x_c is an additional state or controller state. The $A_c(x, x_c)$ is an input function of the integrator and $u(k)$ is an input to the system. Here, the objective is to stabilise the system (2.1) with the controller (2.14).

The system (2.1) with the controller (2.14) can be described as follows:

$$\hat{x}(k+1) = \hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}(\hat{x}(k))A_c(x, x_c) \tag{2.15}$$

where,

$$\hat{A}(\hat{x}(k)) = \begin{bmatrix} A(x(k)) & B(x(k)) \\ 0 & 1 \end{bmatrix}; \quad \hat{B}(\hat{x}(k)) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad \hat{x}(k) = \begin{bmatrix} x(k) \\ x_c(k) \end{bmatrix}; \tag{2.16}$$

Next, we assume $A_c(x, x_c)$ to be of the form $A_c(x, x_c) = \hat{A}_c(\hat{x}(k))\hat{x}(k)$. Therefore (2.15) can be re-written as follows:

$$\hat{x}(k+1) = \hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}(\hat{x}(k))\hat{A}_c(\hat{x}(k))\hat{x}(k) \tag{2.17}$$

where $\hat{A}(\hat{x}(k))$, and $\hat{B}(\hat{x}(k))$ are as described in (2.16). Meanwhile $\hat{A}_c(\hat{x}(k))$ is a $1\times(n+1)$ polynomial matrix, where n is the original state number.

Remark 2.5. The above idea of introducing an additional dynamic is not new, see [61, 62]. However, the authors in [61, 62] used this method to overcome the problem of designing a robust controller for linear systems with norm-bounded uncertainties. In contrast, we propose this method to convexify the controller synthesis problem for polynomial discrete-time systems. Generally, the method corresponds to dynamic state feedback rather than the static state feedback. Notice that using this method, a simple form for the $\hat{B}(\hat{x}(k))$ can be obtained and it is always in the form of $[0, 1]^T$.

The sufficient conditions for the existence of our proposed controller are given in the following theorem:

Theorem 2.6. *The system (2.1) is stabilizable via the nonlinear feedback controller of the form (2.14) if there exist a symmetric polynomial matrix, $\hat{P}(x(k))$, polynomial matrices $\hat{L}(\hat{x}(k))$ and $\hat{G}(\hat{x}(k))$ such that the following conditions are satisfied for all $x \neq 0$*

$$\hat{P}(x(k)) > 0 \tag{2.18}$$

$$M_1(\hat{x}(k)) > 0 \tag{2.19}$$

where,

$$M_1(\hat{x}(k)) = \begin{bmatrix} \hat{G}^T(\hat{x}(k)) + \hat{G}(\hat{x}(k)) - \hat{P}(x(k)) & * \\ \hat{A}(\hat{x}(k))\hat{G}(\hat{x}(k)) + \hat{B}(\hat{x}(k))\hat{L}(\hat{x}(k)) & \hat{P}(x_+) \end{bmatrix} \tag{2.20}$$

Moreover, the nonlinear feedback controller is given by

$$\begin{aligned} x_c(k+1) =& x_c(k) + A_c(x, x_c) \\ u(k) =& x_c(k) \end{aligned}$$

where, $A_c(x, x_c) = \hat{A}_c(\hat{x}(k))\hat{x}(k)$ *with* $\hat{A}_c(\hat{x}(k)) = \hat{L}(\hat{x}(k))\hat{G}^{-1}(\hat{x}(k))$*.*

Proof: The Lyapunov function is chosen to be as follows:

$$\hat{V}(\hat{x}(k)) = \hat{x}^T(k)\hat{P}^{-1}(x(k))\hat{x}(k) \tag{2.21}$$

Based on the Lyapunov stability theory [63], the closed-loop system (2.1) with (2.14) is stable if there exists a Lyapunov function (2.21) > 0 such that

$$\begin{aligned} &\Delta\hat{V}(\hat{x}(k)) = \hat{x}(k+1)^T\hat{P}^{-1}(x_+)\hat{x}(k+1) - \hat{x}^T(k)\hat{P}^{-1}(x(k))\hat{x}(k) < 0 \\ &= \big(\hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}(\hat{x}(k))\hat{A}_c(\hat{x}(k))\hat{x}(k)\big)^T\hat{P}^{-1}(x_+)\big(\hat{A}(\hat{x}(k))\hat{x}(k) \\ &+ \hat{B}(\hat{x}(k))\hat{A}_c(\hat{x}(k))\hat{x}(k)\big) - \hat{x}^T(k)\hat{P}^{-1}(x(k))\hat{x}(k) \\ &= \hat{x}^T(k)\big[\big(\hat{A}^T(\hat{x}(k)) + \hat{A}_c^T(\hat{x}(k))\hat{B}^T(\hat{x}(k))\big)\hat{P}^{-1}(x_+)\big(\hat{A}(\hat{x}(k)) + \hat{B}(\hat{x}(k))\hat{A}_c(\hat{x}(k))\big) \\ &- \hat{P}^{-1}(x(k))\big]\hat{x}(k). \end{aligned} \tag{2.22}$$

Now, suppose (2.19) is feasible, then by the procedures shown in Theorem 2.1 and applying the Schur complement to it, we can easily arrive at

$$\big[\big(\hat{A}^T(\hat{x}(k))+\hat{A}_c^T(\hat{x}(k))\hat{B}^T(\hat{x}(k))\big)\hat{P}^{-1}(x_+)\big(\hat{A}(\hat{x}(k))+\hat{B}(\hat{x}(k))\hat{A}_c(\hat{x}(k))\big)-\hat{P}^{-1}(x(k))\big] < 0. \tag{2.23}$$

Knowing that (2.23) holds, then $\Delta\hat{V}(\hat{x}(k)) < 0$, which implies that the system (2.1) with the controller (2.14) is globally asymptotically stable. $\Delta\Delta\Delta$

Again, to solve Theorem 2.6 is, however, computationally hard because it requires solving an infinite set of state-dependant PMIs. To relax this problem we utilise the SOS decomposition based SDP method and have the following corollary.

Corollary 2.7. *The system (2.1) is stabilizable via the nonlinear feedback controller of the form (2.14) if there exist a symmetric polynomial matrix,* $\hat{P}(x(k))$*, polynomial matrices* $\hat{L}(\hat{x}(k))$ *and* $\hat{G}(\hat{x}(k))$*, and constants* $\epsilon_1 > 0$*, and* $\epsilon_2 > 0$ *such that the following conditions are satisfied for all* $x \neq 0$

$$v_3^T[\hat{P}(x(k)) - \epsilon_1 I]v_3 \qquad \text{is a SOS} \tag{2.24}$$

$$v_4^T\left[M_1(\hat{x}(k)) - \epsilon_2 I\right]v_4 \qquad \text{is a SOS} \tag{2.25}$$

where,

$$M_1(\hat{x}(k)) = \begin{bmatrix} \hat{G}^T(\hat{x}(k)) + \hat{G}(\hat{x}(k)) - \hat{P}(x(k)) & * \\ \hat{A}(\hat{x}(k))\hat{G}(\hat{x}(k)) + \hat{B}(\hat{x}(k))\hat{L}(\hat{x}(k)) & \hat{P}(x_+) \end{bmatrix} \tag{2.26}$$

and, v_3 and v_4 are free vectors with appropriate dimensions. Moreover, the nonlinear feedback controller is given by

$$\begin{aligned} x_c(k+1) =& x_c(k) + A_c(x, x_c) \\ u(k) =& x_c(k) \end{aligned}$$

where, $A_c(x, x_c) = \hat{A}_c(\hat{x}(k))\hat{x}(k)$ with $\hat{A}_c(\hat{x}(k)) = \hat{L}(\hat{x}(k))\hat{G}^{-1}(\hat{x}(k))$.

Proof: The proof for this section follows directly from the combination of proof given in Theorem 2.6 and Proposition 1.2. The proof ends. ∇∇∇

Remark 2.8. The advantages of formulating the problem of the form of Corollary 2.7 are twofold:

1. The solution given by Corollary 2.7 is convex, hence allows the problem to be solved computationally via SDP. This is true because the term in $\hat{P}(x_+)$ is now jointly convex. To prove this, refer to the Lyapunov function (2.21) where the state-dependent Lyapunov matrix, $\hat{P}(x(k))$ is only dependent upon the original system states. Therefore, we have $\hat{P}(x_+) = P(A(x(k))x(k) + B(x(k))x_c(k)$, where $x_c(k)$ is an additional or controller state. This consequently makes the terms in $\hat{P}(x(k))$ are jointly convex. One can see that using this integrator method, the nonconvex term does not required to be bounded by the upper bound value. Therefore, our result hold globally. In contrast, to achive a convex solution to the controller synthesis of polynomial discrete-time systems, [44] the nonconvex term must be bounded by a predefined upper bound value, hence their results are local results. Based on the mentioned reason, our method provides a less conservative design procedure for controller synthesis of polynomial discrete-time systems than [44].

2. In [44], in order to render a convex solution to the controller design problem, the Lyapunov function must be selected as such its matrix only dependent upon the input matrix whose corresponding rows are zeros. In contrast, using our proposed method, no assumption is required in order to achieve a convex solution to the controller design problem of polynomial discrete-time system This is because the original system input matrix, $B(x(k))$ is governed in the system matrices of augmented system (2.15) and the Lyapunov function is now dependent upon the $\hat{B}(x(k))$ which is always $[0, 1]^T$.

2.2.3 Robust nonlinear feedback control design

The results presented in the previous section assume that the system's parameters are known exactly. In this section, we investigate how the above method can be extended to systems in which the parameters are not exactly known. Here the uncertain terms are described as polytopic forms.

Consider the following system

$$x(k+1) = A(x(k), \theta)x(k) + B(x(k), \theta)u(k) \tag{2.27}$$

where the matrices $\cdot(x(k), \theta)$ are defined as follows

$$\begin{aligned} A(x(k), \theta) &= \sum_{i=1}^{q} A_i(x(k))\theta_i, \\ B(x(k), \theta) &= \sum_{i=1}^{q} B_i(x(k))\theta_i. \end{aligned} \tag{2.28}$$

$\theta = \left[\theta_1, \ldots, \theta_q\right]^T \in \mathbb{R}^q$ is the vector of constant uncertainty and satisfies

$$\theta \in \Theta \triangleq \left\{ \theta \in \mathbb{R}^q : \theta_i \geq 0, i = 1, \ldots, q, \sum_{i=1}^{q} \theta_i = 1 \right\}. \tag{2.29}$$

With the controller (2.14), we have the following system:

$$\hat{x}(k+1) = \hat{A}(\hat{x}(k), \theta)\hat{x}(k) + \hat{B}(\hat{x}(k), \theta)u(k) \tag{2.30}$$

where,

$$\hat{A}(\hat{x}(k), \theta) = \sum_{i=1}^{q} \hat{A}_i(\hat{x}(k))\theta_i; \quad \hat{B}(\hat{x}(k), \theta) = \sum_{i=1}^{q} \hat{B}_i(\hat{x}(k))\theta_i. \tag{2.31}$$

with $\hat{A}(\hat{x}(k))$ and $\hat{B}(\hat{x}(k))$ are as described in (2.16). We further define the following parameter-dependent Lyapunov function

$$\tilde{V}(x(k)) = \hat{x}^T(k) \left(\sum_{i=1}^{q} \hat{P}_i(x(k))\theta_i \right)^{-1} \hat{x}(k). \tag{2.32}$$

With the results given in the previous section, we can directly propose the main result for robust nonlinear feedback control with polytopic uncertainties and they are given as follows:

Proposition 2.9. *Given constants $\epsilon_1 > 0$ and $\epsilon_2 > 0$, the system (2.27) with the nonlinear feedback controller (2.14) is stable for $x \neq 0$ and $i = 1, ..., q$ if there exist* *common polynomial matrices $\hat{G}(\hat{x}(k))$ and $\hat{L}(\hat{x}(k))$ and a symmetric polynomial matrix, $\hat{P}_i(x(k))$ such that the following conditions hold for all $x \neq 0$*

$$v_5^T[\hat{P}_i(x(k)) - \epsilon_1 I]v_5 \qquad \textit{is a SOS} \tag{2.33}$$

$$v_6^T\left[M_2(\hat{x}(k)) - \epsilon_2 I\right]v_6 \qquad \textit{is a SOS} \tag{2.34}$$

where,

$$M_2(\hat{x}(k)) = \begin{bmatrix} \hat{G}^T(\hat{x}(k)) + \hat{G}(\hat{x}(k)) - \hat{P}_i(x(k)) & * \\ \hat{A}_i(\hat{x}(k))\hat{G}(\hat{x}(k)) + \hat{B}_i(\hat{x}(k))\hat{L}(\hat{x}(k)) & \hat{P}_i(x_+) \end{bmatrix} \tag{2.35}$$

Proof: This proposition follows directly as a convex combination of several systems of the form (2.27) for a common (2.14). $\Delta\Delta\Delta$

2.3 Numerical Example

In this section two design examples are provided in order to demonstrate the validity of our proposed approach.

2.3.1 Nonlinear feedback control design

Consider the following system,

$$x(k+1) = \begin{bmatrix} -1.4x_1^2(k) + x_2(k) \\ 0.3x_1(k) + 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u. \tag{2.36}$$

Remark 2.10. For the system described in (2.36), if we choose the control input, $u(k)$ to be of the form $u(k) = u(k) + 1$, then the system dynamic of (2.36) is equivalent to the Hennon map system. It is well known that for the given parameters (described in (2.36)), the Hennon map behaves chaotically. Refer Figure 2.1 for the open loop response of the system (2.36).

Next, with the incorporation of an integrator into the controller structure, (2.36) can be written as follows:

$$\hat{x}(k+1) = \begin{bmatrix} -1.4x_1 & 1 & 1 \\ 0.3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_c(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} A_c(x, x_c) \tag{2.37}$$

Furthermore, we choose $\epsilon_1 = \epsilon_2 = 0.01$ and using the procedure described in Corollary 2.7, and with the degree of $\hat{P}(x(k))$ and $\hat{G}(\hat{x}(k))$ set to be 4 and $\hat{L}(\hat{x}(k))$ is chosen to be

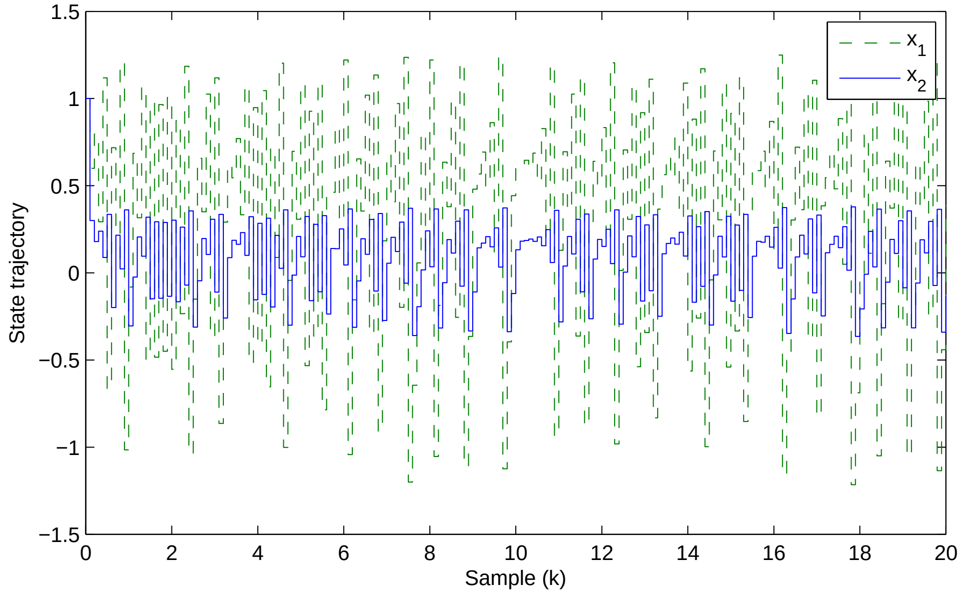

FIGURE 2.1: The trajectory for the open-loop uncertain Hennon Map with a=1.4 and b=0.3.

in the degree of 8, a feasible solution is obtained. The $P(x(k))$ in this work is defined as a symmetrical N x N polynomial matrix whose $(i, j) - th$ entry is given by

$$p_{ij}(x(k)) = p_{ij}^0 + p_{ijg}m(k)^{(1:l)} \tag{2.38}$$

where $i = 1, 2, ..., n$, $j = 1, 2, ..., n$, and $g = 1, 2, ...d$ where n is a number of states, d is the total monomial numbers and $m(k)$ is all monomials vector in $(x(k))$ from degree of 1 to degree of l, where l is a scalar even value. For example if $l = 2$, and $x(k) = [x_1(k), x_2(k)]^T$, then $p_{11} = p_{11} + p_{112}x_1 + p_{113}x_2 + p_{114}x_1^2 + p_{115}x_1x_2 + p_{116}x_2^2$. This representation is more general compared to [44] because of a higher value of l, a more relaxation in the SOS problem can be achieved. The simulation result has been plotted in Figure 2.2 for the initial value of $[x_1, x_2] = [1, 1]$. From Figure 2.2, the controller stabilises the system states to the desired operating region. The controller output response is given in Figure 2.3.

Remark 2.11. It is important to highlight here that using the approach proposed in [44], no solution could be obtained for this example. This confirms that our approach is less conservative than [44].

2.3.2 Robust nonlinear feedback control design

This section illustrates the results of robust nonlinear feedback control for uncertain polynomial discrete-time systems with polytopic uncertainties.

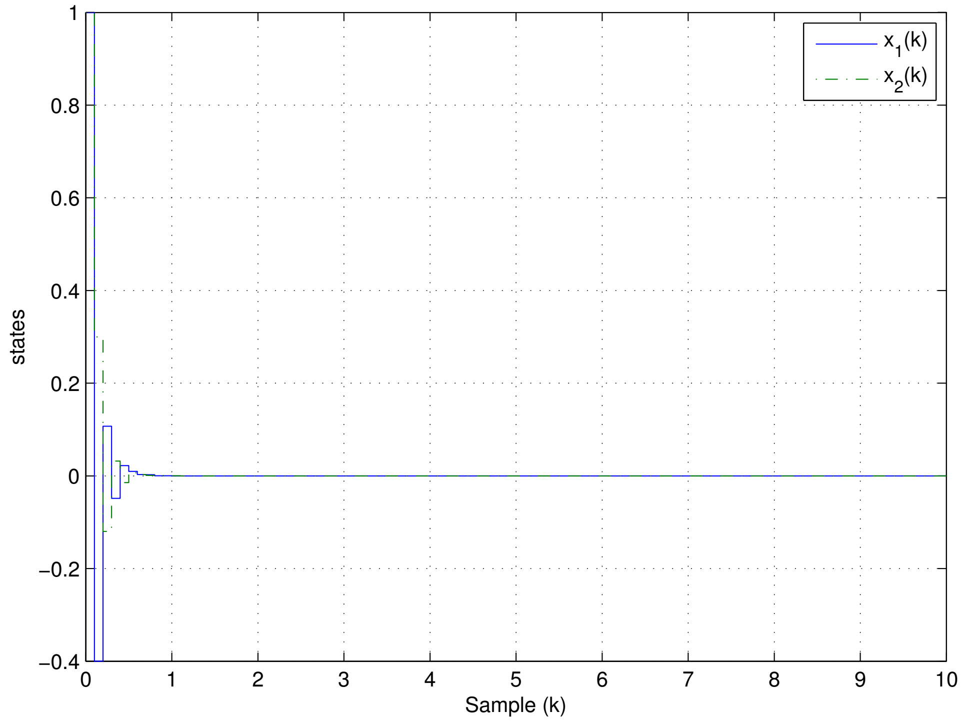

FIGURE 2.2: Response of plant states for nonlinear feedback controller.

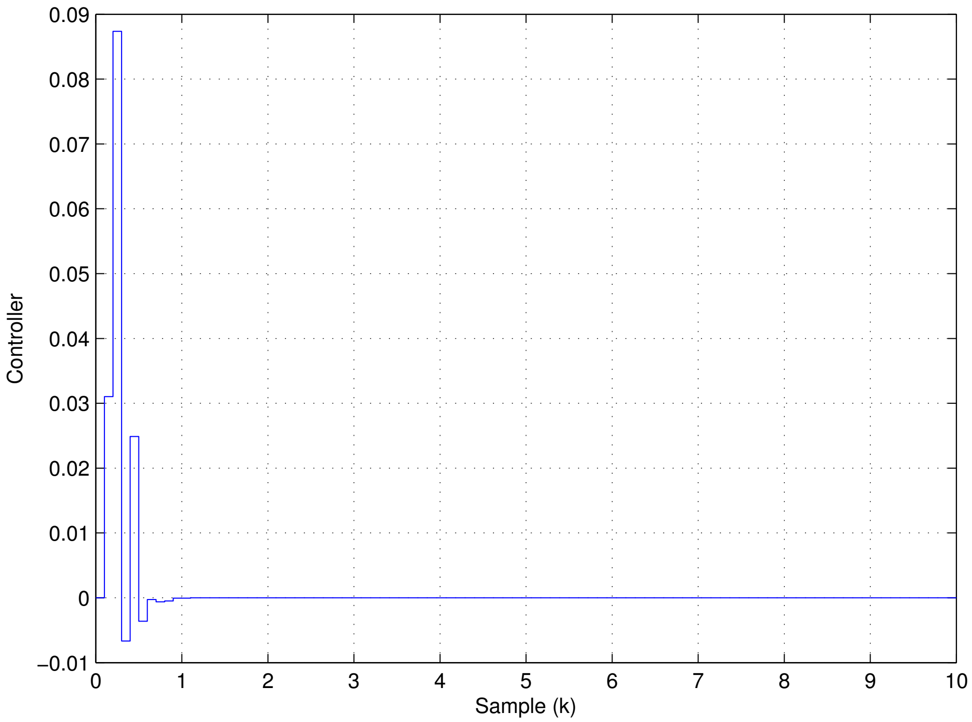

FIGURE 2.3: Controller responses for nonlinear feedback control.

The dynamics of one polynomial discrete-time system is described as follows:

$$x(k+1) = \begin{bmatrix} -ax_1^2(k) + x_2(k) \\ bx_1(k) + 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k). \tag{2.39}$$

where $a = 1.4$, $b = 0.3$. $x_1(k)$ and $x_2(k)$ are the state variables, and $u(k)$ is the control input associated with the system. Next, we assume there is a $\pm 10\%$ change from their nominal values in its parameter. Therefore, (2.39) can be written in the form of upper, $x_U(k+1)$ and lower bounds, $x_L(k+1)$ as below:

$$\begin{aligned} x_U(k+1) &= \begin{bmatrix} -1.26x_1(k) & 1 \\ 0.33 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k) \\ x_L(k+1) &= \begin{bmatrix} -1.54x_1(k) & 1 \\ 0.27 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1.1 \\ 0 \end{bmatrix} u(k) \end{aligned} \tag{2.40}$$

Then, with the integrator in the controller structure, the equation (2.40) can be rewritten as follows:

$$\begin{aligned} \hat{x}_U(k+1) &= \begin{bmatrix} -1.26x_1(k) & 1 & 1 \\ 0.33 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_c(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} A_c(x, x_c) \\ \hat{x}_L(k+1) &= \begin{bmatrix} -1.54x_1(k) & 1 & 1.1 \\ 0.27 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_c(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} A_c(x, x_c) \end{aligned} \tag{2.41}$$

Next, we select $\epsilon_1 = \epsilon_2 = 0.01$ and implementing the procedure outlined in the Proposition 2.9, we initially choose the degree of the $\hat{P}_1(x(k))$ and $\hat{P}_2(x(k))$ to be 2. The common polynomial matrices $\hat{G}(\hat{x}(k))$ and $\hat{L}(\hat{x}(k))$ are also selected to be in the degree of 2, but no feasible solution can be achieved. Then, the degree of $\hat{P}_1(x(k)), \hat{P}_2(x(k))$ and $\hat{G}(\hat{x}(k))$ are increased to 4. Meanwhile, the degree of $\hat{L}(\hat{x}(k))$ is increased to 8. With this arrangement a feasible solution is obtained. The simulation results for this example are given in Figure 2.4 between two vertices of (2.41) with the initial condition of $[1 \quad 1]$. It is obvious from Figure 2.4 that our proposed controller stabilises the system well.

Remark 2.12. In [44], the results are produced without the consideration of the uncertainties in the systems's structure. In contrast, we provide the solution for robust control problem with the existence of the polytopic uncertainties.

Remark 2.13. It is confirmed by the simulation examples that by incorporating the integrator into the controller structure, a less conservative design procedure can be rendered. However, the price we pay is the large computational cost, which prevent us from using this method for a high-order systems due to the lack of memory space in our machine. This is the main source of conservatism of this integrator method because the size of system matrices will increase. For example if the original size of one system matrix is 2×2, using this integrator method, the matrix dimension will increase to 3×3. Therefore, if we are dealing with the higher order systems, the computational complexity becomes more severe.

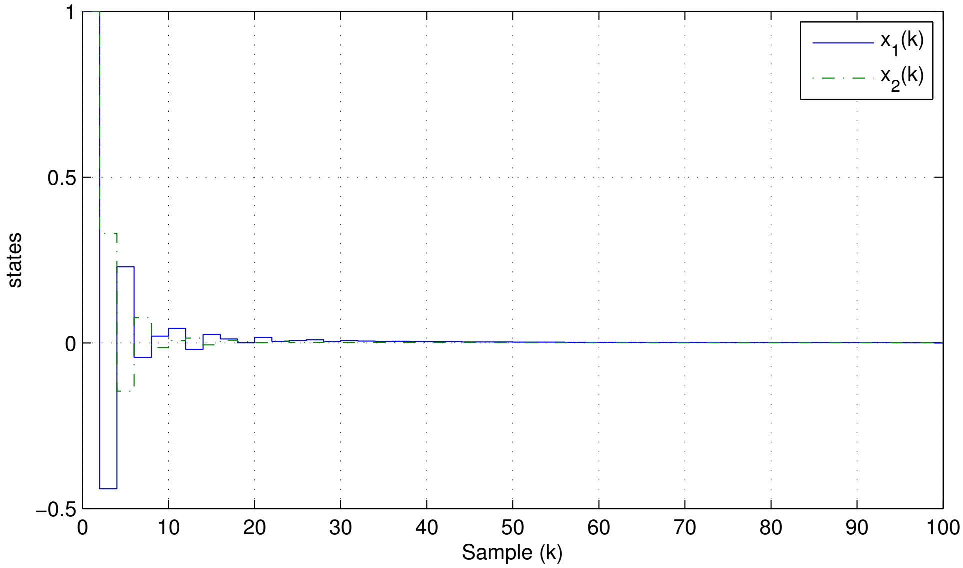

FIGURE 2.4: Response of the plant states for the uncertain system.

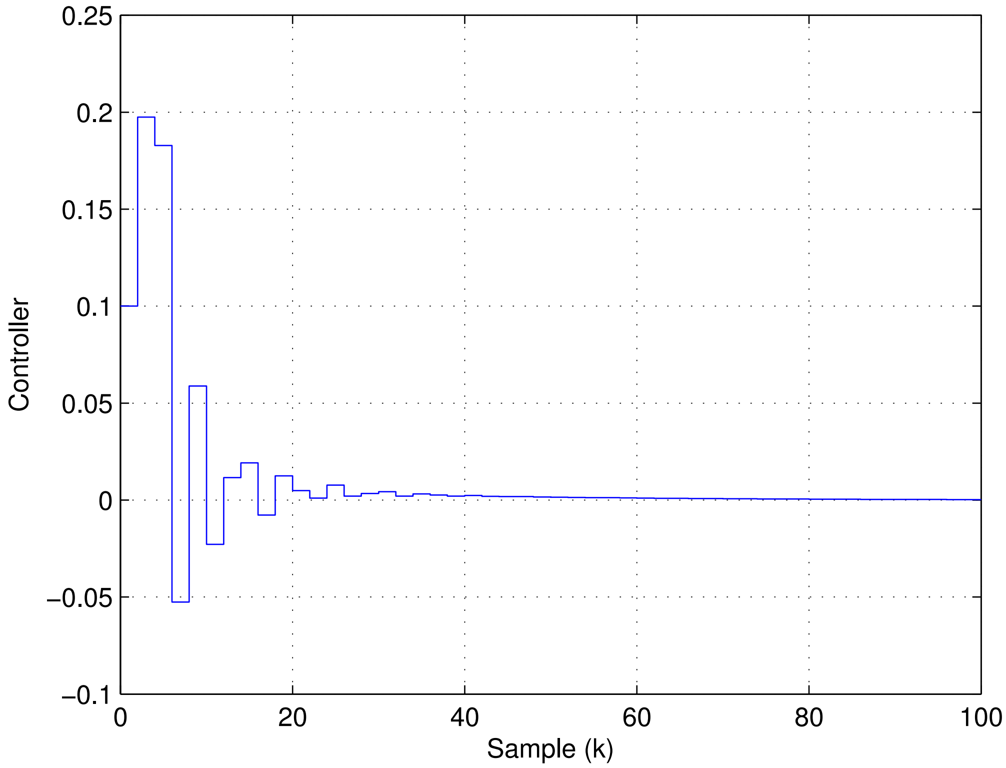

FIGURE 2.5: Controller responses for the uncertain system.

2.4 Conclusion

This chapter has examined the problem of designing a nonlinear feedback controller for polynomial discrete-time systems without and with polytopic uncertainties using a state-dependent Lyapunov functions. More precisely, a nonlinear feedback controller with integrator is proposed to stabilise such uncertain systems. It has been shown that with the incorporation of the integrator into the controller structure, a less conservative design procedure can be achieved. This is because the non-convex term, $P(x(k+1))$ does not need to be assumed to be bounded and the Lyapunov function does not have to be of a certain form in order to render a convex solution as required in [44]. However, the price we pay is the large computational cost which prevents us applying this method to the high order systems. The existence of the proposed nonlinear feedback controller is given in terms of solvability conditions of the polynomial matrix inequalities (PMIs), which are formulated as SOS constraints. The resulting controller gains are in the rational matrix function of the augmented states. Numerical examples have been provided to demonstrate the validity of this integrator approach.

Chapter 3

Robust Nonlinear Control for Polynomial Discrete-Time Systems With Norm-Bounded Uncertainty

3.1 Introduction

One of the important attributes of a good control system design is that the closed loop systems remain stable in the presence of uncertainty [64, 65]. Generally, the uncertainty could result from the system simplification or simply from parameter inaccuracies [66]. The existence of such uncertainty could degrade the performance of the system significantly especially for practical systems and may even lead to instability. Hence, it must be handled efficiently in order to design controllers which operate in a real environment.

In the previous chapter, the robust nonlinear feedback controller design was performed with the existence of the polytopic uncertainties. In contrast, in this chapter we attempt to stabilise the polynomial discrete-time system with the existence of norm-bounded uncertainties. The motivation of this chapter arises due to the fact that the norm-bounded uncertainties exist in many real systems, and most uncertain control systems can be approximated by systems with norm-bounded uncertainties. A vast literature is available in the framework of robust control in which the uncertainty is modelled as norm-bounded: see [13, 67–69]. However, none of the listed papers consider the polynomial discrete-time system in their work. Furthermore, to date, to the author's knowledge, no result has been proposed in the framework of robust control problems with norm-bounded uncertainties using SOS decomposition method.

Therefore, in this chapter, we attempt to design a robust nonlinear feedback controller for uncertain polynomial discrete-time system with norm-bounded uncertainty. The controller should ensure the uncertain system is robustly stable. Here, robustly stable means that the uncertain system is stable about the origin for all admissible uncertainties. We show that by incorporating the integrator into the controller structure, the robust control problem can be converted into a convex solution in a less conservative design procedure. In this work, motivated by the results in [70], the uncertainty is bounded by an upper bound. The existence of the proposed robust nonlinear feedback controller is given in terms of the solvability conditions of PMIs, which is formulated as SOS conditions. The SOS conditions are then solved using SOSTOOLS [53].

3.2 System Description

Consider the following dynamic model of an uncertain polynomial discrete-time system:

$$x(k+1) = A(x(k))x(k) + \Delta A(x(k))x(k) + B_u(x(k))u(k) + \Delta B_u(x(k))u(k) \tag{3.1}$$

where $x(k) \in R^n$ is the state vectors, $u(k) \in R^m$ is the input. $A(x(k))$ and $B_u(x(k))$ are polynomial matrices of appropriate dimensions. Meanwhile $\Delta A(x(k))$ and $\Delta B_u(x(k))$ represent the uncertainties in the system and satisfy the following assumption.

Assumption 3.1. The admissible parameter uncertainties considered here are assumed to be norm-bounded, and described by the following form:

$$\begin{bmatrix} \Delta A(x(k)) & \Delta B_u(x(k)) \end{bmatrix} = H(x(k))F(x(k)) \begin{bmatrix} E_1(x(k)) & E_2(x(k)) \end{bmatrix} \tag{3.2}$$

where $H(x(k))$, $E_1(x(k))$ and $E_2(x(k))$ are known polynomial matrices of appropriate dimensions, and $F(x(k))$ is an unknown state-dependent matrix function which satisfies,

$$\left\| F^T(x(k))F(x(k)) \right\| \leq I. \tag{3.3}$$

To avoid the nonconvexity in $P(x(k+1))$, the following nonlinear controller is proposed:

$$\left.\begin{aligned} x_c(k+1) =& x_c(k) + A_c(x, x_c) \\ u(k) =& x_c(k) \end{aligned}\right\} \tag{3.4}$$

where $A_c(x, x_c)$ is the input function of the integrator, x_c is the controller state and $u(k)$ is the input to the system. It has been shown in Chapter 2 that by selecting the controller of the form (3.4), a convex solution to $P(x(k+1))$ can be rendered efficiently.

Next, we assume $A_c(x, x_c) = \hat{A}_c(\hat{x}(k))\hat{x}(k)$, where $\hat{A}_c(\hat{x}(k))$ is a new polynomial matrix with dimension $1 \times (n+1)$. n is an original states number. Now, the system (3.1) with the controller (3.4) can be written as follows:

$$\hat{x}(k+1) = \hat{A}(\hat{x}(k))\hat{x}(k) + \Delta\hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}_u(\hat{x}(k))\hat{A}_c(\hat{x}(k))\hat{x}(k) \tag{3.5}$$

where

$$\begin{aligned}
\hat{A}(\hat{x}(k)) &= \begin{bmatrix} A(x(k)) & B(x(k)) \\ 0 & 1 \end{bmatrix}; \quad \hat{B}_u(\hat{x}(k)) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \\
\Delta\hat{A}(\hat{x}(k)) &= \hat{H}(\hat{x}(k))\hat{F}(\hat{x}(k))\hat{E}(\hat{x}(k)) = \begin{bmatrix} H(x(k)) \\ 0 \end{bmatrix} F(x(k)) \begin{bmatrix} E_1(x(k)) & E_2(x(k)) \end{bmatrix}; \\
\hat{x} &= \begin{bmatrix} x(k) \\ x_c(k) \end{bmatrix}.
\end{aligned} \tag{3.6}$$

The objective here is to design a nonlinear feedback controller of the form (3.4) such that the system (3.1) with (3.4) is robustly stable. Here, robustly stable means that the uncertain system (3.1) is asymptotically stable about the origin for all admissible uncertainties.

3.3 Main Results

Theorem 3.2. *The system (3.1) is stabilisable via the nonlinear feedback control of the form (3.4) if there exist a symmetric polynomial matrix, $\hat{P}(x(k))$, polynomial matrices $\hat{L}(\hat{x}(k))$ and $\hat{G}(\hat{x}(k))$, and a polynomial function $\epsilon(\hat{x}(k) > 0$ such that the following conditions hold for all $x \neq 0$*

$$\hat{P}(x(k)) > 0 \tag{3.7}$$

$$\begin{bmatrix} -\big(\hat{G}^T(\hat{x}(k)) + \hat{G}(\hat{x}(k)) \\ \quad -\hat{P}(x(k))\big) & * & * \\ \hat{E}(\hat{x}(k))\hat{G}(\hat{x}(k)) & -\epsilon(\hat{x}(k))I & * \\ \hat{A}(\hat{x}(k))\hat{G}(x(k)) \\ +\hat{B}_u(\hat{x}(k))\hat{L}(\hat{x}(k)) & 0 & -P(x_+) + \epsilon(\hat{x}(k))\hat{H}(\hat{x}(k))\hat{H}^T(\hat{x}(k)) \end{bmatrix} < 0 \tag{3.8}$$

Moreover, the nonlinear controller is given by

$$\begin{aligned} x_c(k+1) =& x_c(k) + A_c(x, x_c) \\ u(k) =& x_c(k) \end{aligned}$$

where, $A_c(x, x_c) = \hat{A}_c(\hat{x}(k))\hat{x}(k)$ *with* $\hat{A}_c(\hat{x}(k)) = \hat{L}(\hat{x}(k))\hat{G}^{-1}(\hat{x}(k))$.

Proof: Select a state dependentl Lyapunov function of the form

$$\hat{V}(\hat{x}(k)) = \hat{x}^T(k)\hat{P}^{-1}(x(k))\hat{x}(k) \tag{3.9}$$

The difference between $\hat{V}(x(k+1)$ and $\hat{V}(x(k))$ of (3.9) along (3.5) is given by

$$\begin{aligned} \Delta\hat{V}(x(k)) &= \hat{V}(x(k+1) - \hat{V}(x(k)) \\ &= \hat{x}^T(k+1)\hat{P}^{-1}(x(k+1))\hat{x}(k+1) - \hat{x}^T(k)\hat{P}^{-1}(x(k))\hat{x}(k) \\ &= \hat{x}^T(k)\big[\Omega(\hat{x}(k))\big]\hat{x}(k) \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} \Omega(\hat{x}(k)) = &\big(\hat{A}(\hat{x}(k)) + \hat{B}_u(\hat{x}(k))\hat{A}_c(\hat{x}(k)) + \hat{H}(\hat{x}(k))\hat{F}(\hat{x}(k))\hat{E}(\hat{x}(k))\big)^T\hat{P}^{-1}(x_+) \\ &\big(\hat{A}(\hat{x}(k)) + \hat{B}_u(\hat{x}(k))\hat{A}_c(\hat{x}(k)) + \hat{H}(\hat{x}(k))\hat{F}(\hat{x}(k))\hat{E}(\hat{x}(k))\big) - \hat{P}^{-1}(x(k)) \end{aligned} \tag{3.11}$$

Next, we have to show that $\Omega(\hat{x}(k)) < 0$. To show this, suppose the inequality (3.8) holds. Then, from the block $(1,1)$ of (3.8), we have $\hat{G}^T(\hat{x}(k)) + \hat{G}(\hat{x}(k)) > \hat{P}(x(k)) > 0$. This implies that $\hat{G}(\hat{x}(k))$ is nonsingular, and since $\hat{P}(x(k))$ is positive definite, hence

$$\big(\hat{P}(x(k)) - \hat{G}(\hat{x}(k))\big)^T\hat{P}^{-1}(x(k))\big(\hat{P}(x(k)) - \hat{G}(\hat{x}(k))\big) > 0 \tag{3.12}$$

holds. Therefore establishing

$$\hat{G}^T(\hat{x}(k))\hat{P}^{-1}(x_+)\hat{G}(\hat{x}(k)) \geq \hat{G}^T(\hat{x}(k)) + \hat{G}(\hat{x}(k)) - \hat{P}(x(k)). \tag{3.13}$$

This immediately gives

$$\begin{bmatrix} \begin{array}{c}-\hat{G}^T(\hat{x}(k))\hat{P}^{-1}(x(k))) \\ \times(\hat{G}(\hat{x}(k))\end{array} & * & * \\ \hat{E}(\hat{x}(k))\hat{G}(\hat{x}(k)) & -\epsilon(\hat{x}(k))I & * \\ \begin{array}{c}\hat{A}(\hat{x}(k))\hat{G}(x(k)) \\ +\hat{B}_u(\hat{x}(k))\hat{L}(\hat{x}(k))\end{array} & 0 & -P(x_+) + \epsilon(\hat{x}(k))\hat{H}(\hat{x}(k))\hat{H}^T(\hat{x}(k)) \end{bmatrix} < 0. \tag{3.14}$$

Then, multiply on the right of (3.14) by $diag[G^{-1}(\hat{x}(k)), I, I]$ and on the left by $diag[G^{-1}(\hat{x}(k)), I, I]^T$, and knowing that $\hat{L}(\hat{x}(k)) = \hat{A}_c(\hat{x}(k))\hat{G}(\hat{x}(k))$, we arrive at

$$\begin{bmatrix} -\hat{P}^{-1}(x(k))) & * & * \\ \hat{E}(\hat{x}(k)) & -\epsilon(\hat{x}(k))I & * \\ \hat{A}(\hat{x}(k)) + \hat{B}_u(\hat{x}(k))\hat{A}_c(\hat{x}(k)) & 0 & -P(x_+) + \epsilon(\hat{x}(k))\hat{H}(\hat{x}(k))\hat{H}^T(\hat{x}(k)) \end{bmatrix} < 0. \tag{3.15}$$

Similarly,

$$\begin{bmatrix} -\hat{P}^{-1}(x(k))) + \frac{1}{\epsilon(\hat{x}(k))}\hat{E}^T(\hat{x}(k))\hat{E}(\hat{x}(k)) & * \\ \hat{A}(\hat{x}(k)) + \hat{B}_u(\hat{x}(k))\hat{A}_c(\hat{x}(k)) & -P(x_+) + \epsilon(\hat{x}(k))\hat{H}(\hat{x}(k))\hat{H}^T(\hat{x}(k)) \end{bmatrix} < 0. \tag{3.16}$$

Furthermore, knowing that

$$\begin{aligned} 0 \leq x(k)^T & \begin{bmatrix} \frac{\hat{E}^T(x(k))}{\epsilon(\hat{x}(k))^{1/2}} \\ -\epsilon(\hat{x}(k))^{1/2}\hat{H}(x(k))\hat{F}(x(k)) \end{bmatrix} \\ & \times \begin{bmatrix} \frac{\hat{E}(x(k))}{\epsilon(\hat{x}(k))^{1/2}} & -\epsilon(\hat{x}(k))^{1/2}\hat{F}^T(x(k))\hat{H}^T(x(k)) \end{bmatrix} x(k), \end{aligned} \tag{3.17}$$

we have

$$\begin{aligned} & \begin{bmatrix} \frac{1}{\epsilon(\hat{x}(k))}\hat{E}^T(x(k))\hat{E}(x(k)) & 0 \\ 0 & \epsilon(\hat{x}(k))\hat{H}(x(k))\hat{H}^T(x(k)) \end{bmatrix} \\ & \geq \begin{bmatrix} 0 & \hat{E}^T(x(k))\hat{F}^T(x(k))\hat{H}^T(x(k)) \\ \hat{H}(x(k))\hat{F}(x(k))\hat{E}(x(k)) & 0 \end{bmatrix}. \end{aligned} \tag{3.18}$$

Therefore, using (3.18), the equation (3.16) can now be written as follows:

$$\begin{bmatrix} -\hat{P}^{-1}(x(k))) & * \\ \hat{A}(\hat{x}(k)) + \hat{B}_u(\hat{x}(k))\hat{A}_c(\hat{x}(k)) + \hat{H}(\hat{x}(k))\hat{F}(\hat{x}(k))\hat{E}(\hat{x}(k)) & -P(x_+) \end{bmatrix} < 0. \tag{3.19}$$

Then, utilising the Schur complement to (3.19), results in

$$\Omega(\hat{x}(k)) < 0 \tag{3.20}$$

where $\Omega(\hat{x}(k))$ is as described in (3.11). Knowing that (3.20) holds, then we have $\Delta\hat{V}(x(k)) < 0, \forall x \neq 0$, which implies that the system (3.1) with (3.4) is robustly stable. The proof ends. ∇∇∇

Unfortunately it is hard to solve Theorem 3.2 because it is given in terms of state-dependent PMIs. Solving this inequality is computationally hard because it requires solving an infinite set of PLMIs. The SOS based SDP method can provide a computational relaxation for the sufficient condition of Theorem 3.2. Therefore, the modified

SOS based conditions of Theorem 3.2 are given as follows:

Proposition 3.3. *The system (3.5) is stabilisable via the nonlinear feedback control of the form (3.4) if there exist a symmetric polynomial matrix, $\hat{P}(x(k))$, polynomial matrices $\hat{L}(\hat{x}(k))$ and $\hat{G}(\hat{x}(k))$, a polynomial function $\epsilon(\hat{x}(k)) > 0$, and constants $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that the following conditions hold for all $x \neq 0$*

$$v_1^T \left[\hat{P}(x(k)) - \epsilon_1 I\right] v_1 \quad \text{is SOS} \tag{3.21}$$

$$-v_2^T \left[M(\hat{x}(k)) + \epsilon_2 I\right] v_2 \quad \text{is SOS} \tag{3.22}$$

where,

$$M(\hat{x}(k)) = \begin{bmatrix} -(\hat{G}(\hat{x}(k)) + \hat{G}^T(\hat{x}(k)) - \hat{P}(x(k))) & * & * \\ \hat{E}(\hat{x}(k))\hat{G}(\hat{x}(k)) & -\epsilon(\hat{x}(k))I & * \\ \hat{A}(\hat{x}(k))\hat{G}(x(k)) + \hat{B}_u(\hat{x}(k))\hat{L}(\hat{x}(k)) & 0 & -P(x_+) + \epsilon(\hat{x}(k))\hat{H}(\hat{x}(k))\hat{H}^T(\hat{x}(k)) \end{bmatrix} \tag{3.23}$$

and v_1 and v_2 are free vectors in appropriate dimensions. Moreover, the nonlinear controller is given by

$$\begin{aligned} x_c(k+1) =& x_c(k) + A_c(x, x_c) \\ u(k) =& x_c(k) \end{aligned}$$

where, $A_c(x, x_c) = \hat{A}_c(\hat{x}(k))\hat{x}(k)$ with $\hat{A}_c(\hat{x}(k)) = \hat{L}(\hat{x}(k))\hat{G}^{-1}(\hat{x}(k))$.

Proof: This can be carried out via similar technique to the proof shown in Theorem 3.2 in conjunction with the Proposition 1.2 (given in Chapter 1).

Remark 3.4. It is always necessary to include the SOS constraint i.e $\epsilon_1 > 0$ to guarantee the negative definiteness of the inequalities (3.21)-(3.22) [71].

Remark 3.5. The drawback of [41] is the original systems must be transformed or decomposed into certain forms, however, those decompositions are hard to determine and there are not unique. The solvability of their stabilisation problem depends on the choice of the decomposition matrices, and the transformation matrix. Furthermore, to render a convex solution to the controller synthesis problem, the Lyapunov function must be of a special form and dependent upon the zeros row of the control matrix. If there is no zero rows in the control matrix, then they need to transform their original systems to introduce as many rows as possible in their control matrix, which often leads to nonlinear descriptor systems. As stated in their paper, there is no systematic procedure to generate them till now. In [44], a predefined upper bound on the nonconvex term needs to be determined using an optimisation approach. If the optimisation fails to give a feasible solution to that problem, then the overall algorithm will fail to provide a feasible solution too. The results obtain in [44] are local results and the Lyapunov function can only be a function of states whose corresponding rows in control matrix is zeros. In summary, our method may yield less conservative results than [41, 44] because the nonconvex term does not require to be bounded, our results are global results, and the Lyapunov function does not depend on the zeros row of the control matrix.

3.4 Numerical Example

In this section, we provide a design example to demonstrate the validity of our proposed approach.

Consider the following uncertain polynomial discrete-time system,

$$x(k+1) = A(x(k))x(k) + \Delta A(x(k))x(k) + B_u(x(k))u(k) + \Delta B_u(x(k))u(k) \tag{3.24}$$

where

$$A(x(k)) = \begin{bmatrix} 1 & -T \\ T(1+ax_1x_2) & 1-T \end{bmatrix}; \quad B_u(x(k)) = \begin{bmatrix} 0 \\ 0.05 \end{bmatrix}. \tag{3.25}$$

with $T = 0.05$ and $a = 1$. We assume the parameters a vary $\pm 30\%$ of their nominal values. Expressing (3.24) of the form (3.1), we have

$$A(x(k)) = \begin{bmatrix} 1 & -T \\ T(1+ax_1x_2) & 1-T \end{bmatrix}; \quad B_u(x(k)) = \begin{bmatrix} 0 \\ 0.05 \end{bmatrix}; \quad H(x(k)) = \begin{bmatrix} 0 \\ 0.3 \end{bmatrix};$$
$$E_1(x(k)) = \begin{bmatrix} ax_1x_2 & 0 \end{bmatrix}; \quad E_2(x(k)) = 0. \tag{3.26}$$

Furthermore, with the incorporation of an integrator into the controller structure, (3.26) can be written as an augmented system as follows:

$$\hat{x}(k+1) = \begin{bmatrix} 1 & -T & 0 \\ T(1+ax_1x_2) & 1-T & 0.05 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_c \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} A_c(x, x_c) + \Delta\hat{A}(\hat{x}(k)) \tag{3.27}$$

and the uncertainties are described as follows:

$$\Delta\hat{A}(\hat{x}(k)) = \begin{bmatrix} 0 \\ 0.3 \\ 0 \end{bmatrix} \hat{F}(\hat{x}(k)) \begin{bmatrix} ax_1x_2 & 0 & 0 \end{bmatrix}. \tag{3.28}$$

In this example, we choose $\epsilon_1 = \epsilon_2 = 0.01$. The $\epsilon(\hat{x}(k))$ is selected to be of $2\big(x_1^2(k) + x_2^2(k) + x_c^2(k)\big)$. Then, using the procedure described in Proposition 3.3, with the degree of $\hat{P}(x(k))$ and $\hat{G}(\hat{x}(k))$ are 4 and $\hat{L}(\hat{x}(k))$ is chosen to be in the degree of 8, a feasible solution is obtained. The simulation result has been plotted in Figure 3.1. The initial value for this example is $[x_1, x_2] = [-0.5, 0.5]$. The controller response is shown in Figure 3.2.

Remark 3.6. The selection of the design variable $\epsilon(\hat{x}(k))$ is a very crucial in this approach. A wrong selection of this value will drive the solution to be infeasible. In this work, the selection of the $\epsilon(\hat{x}(k))$ value is delivered using a trial and error method.

Remark 3.7. In this example we use Simulink to help us to compute $\hat{A}_c(\hat{x}(k))$ from the obtained $\hat{G}(\hat{x}(k))$ and $\hat{L}(\hat{x}(k))$. Due to the large polynomial matrices that resulted from $\hat{A}_c(\hat{x}(k))$, $\hat{G}(\hat{x}(k))$ and $\hat{L}(\hat{x}(k))$, those values are omitted here.

Remark 3.8. The computational complexity of our proposed method increases with the increment of system's order. This is because the augmented system is used in the problem formulation rather than original system matrices. This is the drawback of this approach because it requires a very large of memory spaces in order to accommodate the computational issue of high order polynomial systems.

3.5 Conclusion

The problem of designing a robust nonlinear feedback controller for polynomial discrete-time systems with norm-bounded uncertainty has been examined in this chapter. The main contribution of this chapter stems from the fact that the norm-bounded uncertainty is considered rather than polytopic uncertainties. It has been shown that by incorporating the integrator into the controller structure, a convex solution can be rendered in less conservative design procedures. Furthermore, the uncertainties are bounded by an upper bound. The sufficient conditions for the existence of the proposed controller are provided in terms of the solvability of polynomial matrix inequalities. The PMIs are then formulated as SOS constraints. Hence, the problem can be solved via any SOS solvers. A numerical example has been provided to demonstrate the validity of this integrator approach. This result can also be viewed as an extension of [70] which treats linear uncertain discrete-time systems.

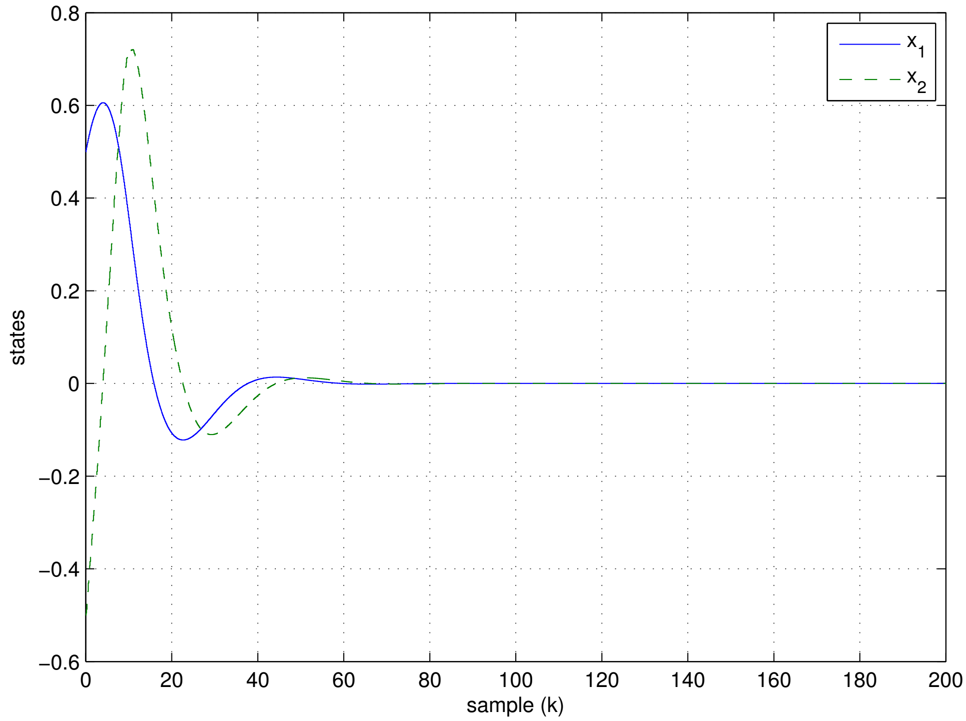

FIGURE 3.1: System states for uncertain discrete-time systems.

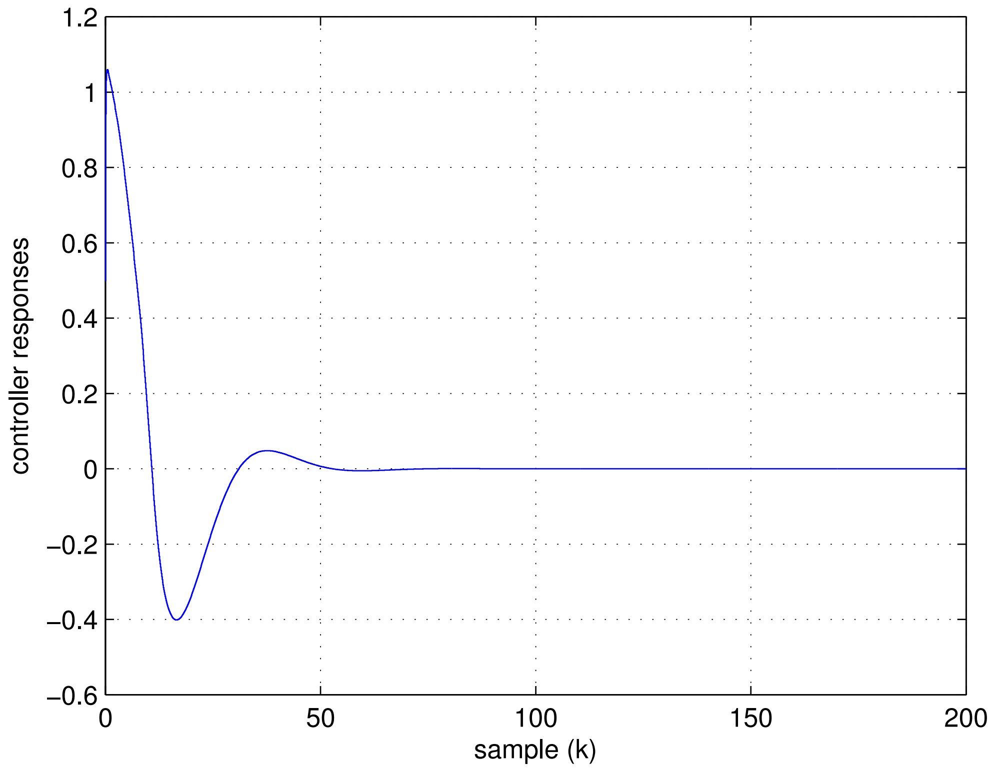

FIGURE 3.2: Controller Responses.

Chapter 4

Nonlinear H_∞ State Feedback Control for Polynomial Discrete-Time Systems

4.1 Introduction

The problem of designing a nonlinear H_∞ controller has attracted considerable attention for more than three decades; see for instance [72–75]. Generally speaking, the aim of an H_∞ control problem is to design a controller such that the resulting closed loop control system is stable and a prescribed level of attenuation from the exogenous disturbance input to the output in L_2/l_2-norm is fulfilled. There are two common approaches available to address nonlinear H_∞ control problems: One approach is based on the dissipativity theory [76] and theory of differential games [72]. The other is based on the nonlinear version of the classical bounded real lemma as developed in [15] and [77]. The underlying idea behind both approaches is the conversion of the nonlinear H_∞ control problem into the solvability form of the so-called Hamilton-Jacobi equation (HJE). Unfortunately, this representation is hard to solve and it is generally very difficult to find a global solution.

However, when the polynomial system is under consideration, then there is an approach to relax the above-mentioned problem. The approach is called sum of squares (SOS) decomposition and has been developed in [30]. Based on this SOS method, several results can be found in the framework of H_∞ control of polynomial continuous-time systems [31, 33, 34, 57]. However, unlike its continuous-time systems counterpart there are only a few results available which considers the polynomial discrete-time system with H_∞ performance objectives; see [44]. Unfortunately, the aforementioned results suffer from their own conservatism and such conservatism has been discussed in detail in Chapter 1.

In this chapter, by utilising the integrator approach as proposed in the previous chapters, we attempt to propose a less conservative design procedure than the available approaches and consequently provide a more general result in the framework of H_∞ control of polynomial discrete-time systems. The result is subsequently extended to the robust H_∞ control problem with the existence of the polytopic uncertainties. The attention here is to design a nonlinear feedback controller such that both stability and a prescribed disturbance attenuation for the closed loop polynomial discrete-time system are achieved. Furthermore, based on the SOS-based method, the existence of the proposed controller is given in terms of the solvability of polynomial matrix inequalities (PMIs), which are formulated as SOS constraints and can be solved by the recently developed SOS solvers.

4.2 System Description and Problem Formulation

4.2.1 System description

The following dynamic model of a polynomial discrete-time system is considered,

$$\left.\begin{aligned} x(k+1) =& A(x(k))x(k) + B_u(x(k))u(k) + B_w(x(k))\omega(k) \\ z(k) =& C_z(x(k))x(k) + D_{zu}(x(k))u(k) \end{aligned}\right\} \tag{4.1}$$

where $x(k) \in R^n$ is a state vector and other vectors are of monomials. $A(x(k))$, $B_u(x(k))$, $B_w(x(k))$, $C_z(x(k))$, and $D_{zu}(x(k))$ are polynomial matrices of appropriate dimensions. In addition, $z(k)$ is a vector of output signals related to the performance of the control system and $\omega(k)$ is the disturbance which belongs to $L_2[0, \infty]$.

The following nonlinear feedback controller with an integrator is proposed:

$$\left.\begin{aligned} x_c(k+1) =& x_c(k) + A_c(x, x_c) \\ u(k) =& x_c(k) \end{aligned}\right\} \tag{4.2}$$

where $A_c(x, x_c)$ is the input function of the integrator, x_c is the controller state and $u(k)$ is the input function to the system.

Remark 4.1. The main reason for selecting the controller of the form (4.2) is to ensure a convex solution to $P(x(k+1))$ can be obtained. The detailed explanation regarding this can be found in Chapter 2. Hence we omit the complete discussion here.

The system (4.1) with the controller (4.2) can now be described as follows:

$$\left.\begin{aligned} \hat{x}(k+1) &= \hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}_u(\hat{x}(k))A_c(x,x_c) + \hat{B}_\omega(\hat{x}(k))\omega(k) \\ z(k) &= \hat{C}_z(\hat{x}(k))\hat{x}(k) \end{aligned}\right\} \tag{4.3}$$

where,

$$\hat{A}(\hat{x}(k)) = \begin{bmatrix} A(x(k)) & B(x(k)) \\ 0 & 1 \end{bmatrix}; \quad \hat{B}_u(\hat{x}(k)) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad \hat{B}_\omega(\hat{x}(k)) = \begin{bmatrix} B_\omega(x(k)) \\ 0 \end{bmatrix};$$
$$\hat{C}_z(\hat{x}(k)) = \begin{bmatrix} C_z(x(k))x(k) & D_{zu}(x(k)) \end{bmatrix}; \quad \hat{x} = \begin{bmatrix} x(k) \\ x_c(k) \end{bmatrix}. \tag{4.4}$$

Next, we assume $A_c(x,x_c)$ to be of the form $A_c(x,x_c) = \hat{A}_c(\hat{x}(k))\hat{x}(k)$. Therefore, (4.3) can be re-written as follows:

$$\left.\begin{aligned} \hat{x}(k+1) &= \hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}_u(\hat{x}(k))\hat{A}_c(\hat{x}(k))\hat{x}(k) + \hat{B}_\omega(\hat{x}(k))\omega(k) \\ z(k) &= \hat{C}_z(\hat{x}(k))\hat{x}(k) \end{aligned}\right\} \tag{4.5}$$

where $\hat{A}(\hat{x}(k))$, $\hat{B}_u(\hat{x}(k))$, $\hat{B}_\omega(\hat{x}(k))$, and $\hat{C}_z(\hat{x}(k))$ are as described in (4.4).

4.2.2 Problem formulation

Given a prescribed H_∞ performance $\gamma > 0$, design a nonlinear feedback controller (4.2) such that

$$\|z(k)\|_{[0,\infty]} \leq \gamma^2 \|\omega(k)\|_{[0,\infty]} \tag{4.6}$$

and system in (4.1) with (4.2) is globally asymptotically stable.

4.3 Main Results

In this section, the result of the nonlinear H_∞ control problem is presented first. Then the result is subsequently extended to the robust nonlinear H_∞ control problem with the existence of the polytopic uncertainty.

4.3.1 Nonlinear H_∞ control problem

The sufficient conditions for the existence of our proposed nonlinear feedback controller of the form (4.2) which satisfy (4.6) are given in the following theorem:

Theorem 4.2. *Given a prescribed H_∞ performance, $\gamma > 0$, the system (4.1) is stabilisable with H_∞ performance (4.6) via the nonlinear feedback controller of the form (4.2) if there exist a symmetric polynomial matrix, $\hat{P}(x(k))$, polynomial matrices $\hat{L}(\hat{x}(k))$ and $\hat{G}(\hat{x}(k))$ such that the following conditions are satisfied for all $x \neq 0$:*

$$\hat{P}(x(k)) > 0 \tag{4.7}$$

$$M(\hat{x}(k)) > 0 \tag{4.8}$$

where,

$$M(\hat{x}(k)) = \begin{bmatrix} \hat{G}^T(\hat{x}(k)) + \hat{G}(\hat{x}(k)) - \hat{P}(x(k)) & * & * & * \\ 0 & \gamma^2 I & * & * \\ \hat{A}(\hat{x}(k))\hat{G}(x(k)) + \hat{B}_u(\hat{x}(k))\hat{L}(\hat{x}(k)) & \hat{B}_\omega(\hat{x}(k)) & \hat{P}(x_+) & * \\ \hat{C}_z(\hat{x}(k))\hat{G}(\hat{x}(k)) & 0 & 0 & I \end{bmatrix}. \tag{4.9}$$

The nonlinear feedback controller is given by

$$\begin{aligned} x_c(k+1) =& x_c(k) + A_c(x, x_c) \\ u(k) =& x_c(k) \end{aligned}$$

where, $A_c(x, x_c) = \hat{A}_c(\hat{x}(k))\hat{x}(k)$ with $\hat{A}_c(\hat{x}(k)) = \hat{L}(\hat{x}(k))\hat{G}^{-1}(\hat{x}(k))$.

Proof: A Lyapunov function of the following form is selected

$$\hat{V}(\hat{x}(k)) = \hat{x}^T(k)\hat{P}^{-1}(x(k))\hat{x}(k) \tag{4.10}$$

The difference between $\hat{V}(\hat{x}(k+1))$ and $\hat{V}(\hat{x}(k)$ along (4.5) is given by

$$\begin{aligned} \Delta\hat{V}(\hat{x}(k)) =& \hat{x}(k+1)^T\hat{P}^{-1}(x_+)\hat{x}(k+1) - \hat{x}^T(k)\hat{P}^{-1}(x(k))\hat{x}(k) \\ =& \big(\hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}_u(\hat{x}(k))\hat{A}_c(\hat{x}(k))\hat{x}(k) + \hat{B}_\omega(\hat{x}(k))\omega(k)\big)^T\hat{P}^{-1}(x_+) \\ & \times \big(\hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}_u(\hat{x}(k))\hat{A}_c(\hat{x}(k))\hat{x}(k) + \hat{B}_\omega(\hat{x}(k))\omega(k)\big) \\ & - \hat{x}^T(k)\hat{P}^{-1}(x(k))\hat{x}(k). \end{aligned} \tag{4.11}$$

Then, adding and subtracting $-z^T(k)z(k) + \gamma^2\omega^T(k)\omega(k)$ to and from (4.11), results in

$$\begin{aligned}
\Delta\hat{V}(\hat{x}(k)) &= \big(\hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}_u(\hat{x}(k))\hat{A}_c(\hat{x}(k))\hat{x}(k) + B_\omega(\hat{x}(k))\omega(k)\big)^T\hat{P}^{-1}(x_+)\\
&\times \big(\hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}_u(\hat{x}(k))\hat{A}_c(\hat{x}(k))\hat{x}(k) + B_\omega(\hat{x}(k))\omega((k))\big)\\
&- x^T(k)\hat{P}^{-1}(x(k))x(k)) - z^T(k)z(k) + \gamma^2\omega^T(k)\omega(k) + z^T(k)z(k) - \gamma^2\omega^T(k)\omega(k)\\
&= \big(\hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}_u(\hat{x}(k))\hat{A}_c(\hat{x}(k))\hat{x}(k) + \hat{B}_\omega(\hat{x}(k))\omega((k))\big)^T\hat{P}^{-1}(x_+)\\
&\times \big(\hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}_u(\hat{x}(k))\hat{A}_c(\hat{x}(k))\hat{x}(k) + \hat{B}_\omega(\hat{x}(k))\omega((k))\big) - x^T(k)\hat{P}^{-1}(x(k))x(k))\\
&+ (\hat{C}_z(\hat{x}(k))\hat{x}(k))^T(\hat{C}_z(\hat{x}(k))\hat{x}(k)) - \gamma^2\omega^T(k)\omega(k) - z^T(k)z(k) + \gamma^2\omega^T(k)\omega(k). \quad (4.12)
\end{aligned}$$

Now, (4.12) is written as

$$\Delta\hat{V}(\hat{x}(k)) = \hat{X}^T(k)\Omega(\hat{x}(k))\hat{X}(k) - z^T(k)z(k) + \gamma^2\omega^T(k)\omega(k), \quad (4.13)$$

where

$$\Omega(\hat{x}(k)) = \Theta_1(\hat{x}(k))^T\hat{P}^{-1}(x_+)\Theta_1(\hat{x}(k)) + \Theta_2(\hat{x}(k))^T\Theta_2(\hat{x}(k)) - \Lambda,$$

with

$$\begin{aligned}
\Theta_1(\hat{x}(k)) &= \begin{bmatrix}\hat{A}(\hat{x}(k)) + \hat{B}_u(\hat{x}(k))\hat{A}_c(\hat{x}(k)) & \hat{B}_\omega(\hat{x}(k))\end{bmatrix};\\
\Theta_2(\hat{x}(k)) &= \begin{bmatrix}\hat{C}_z(\hat{x}(k)) & 0\end{bmatrix}; \quad \Lambda = \begin{bmatrix}\hat{P}^{-1}(x(k)) & 0\\ 0 & \gamma^2\end{bmatrix};\\
\hat{X}(k) &= \begin{bmatrix}\hat{x}(k) & \omega(k)\end{bmatrix}^T.
\end{aligned}$$

Now, we need to show that $\hat{X}^T(k)\Omega(\hat{x}(k))\hat{X}(k) < 0$. To show this, suppose (4.8) is feasible. Then, from the block $(1,1)$ of (4.9), we have $\hat{G}^T(\hat{x}(k))+\hat{G}(\hat{x}(k)) > \hat{P}(x(k)) > 0$. This implies that $\hat{G}(\hat{x}(k))$ is nonsingular, and since $\hat{P}(x(k))$ is positive definite, hence

$$\big(\hat{P}(x(k)) - \hat{G}(\hat{x}(k))\big)^T\hat{P}^{-1}(x_+)\big(\hat{P}(x(k)) - \hat{G}(\hat{x}(k))\big) > 0 \quad (4.14)$$

holds. Therefore establishing

$$\hat{G}^T(\hat{x}(k))\hat{P}^{-1}(x(k))\hat{G}(\hat{x}(k)) \geq \hat{G}^T(\hat{x}(k)) + \hat{G}(\hat{x}(k)) - \hat{P}(x(k)). \quad (4.15)$$

This immediately gives

$$\begin{bmatrix} \hat{G}^T(\hat{x}(k))\hat{P}^{-1}(x(k))\hat{G}(\hat{x}(k)) & * & * & * \\ 0 & \gamma^2 I & * & * \\ \hat{A}(\hat{x}(k))\hat{G}(x(k)) + \hat{B}_u(\hat{x}(k))\hat{L}(\hat{x}(k)) & \hat{B}_\omega(\hat{x}(k)) & \hat{P}(x_+) & * \\ \hat{C}_z(\hat{x}(k))\hat{G}(\hat{x}(k)) & 0 & 0 & I \end{bmatrix} > 0. \tag{4.16}$$

Then, multiply (4.16) on the right by $diag[\hat{G}^{-1}(\hat{x}(k)), I, I, I]$ and on the left by $diag[\hat{G}^{-1}(\hat{x}(k)), I, I, I]^T$, and with the fact that $\hat{L}(\hat{x}(k)) = \hat{A}_c(\hat{x}(k))\hat{G}(\hat{x}(k))$, we have

$$\begin{bmatrix} \hat{P}^{-1}(x(k)) & * & * & * \\ 0 & \gamma^2 I & * & * \\ \hat{A}(\hat{x}(k)) + \hat{B}_u(\hat{x}(k))\hat{A}_c(\hat{x}(k)) & \hat{B}_\omega & \hat{P}(x_+) & * \\ \hat{C}_z(\hat{x}(k)) & 0 & 0 & I \end{bmatrix} > 0, \tag{4.17}$$

which is equivalent to

$$\begin{bmatrix} \Lambda & * & * \\ \Theta_1(\hat{x}(k)) & \hat{P}(x_+) & * \\ \Theta_2(\hat{x}(k)) & 0 & I \end{bmatrix} > 0. \tag{4.18}$$

Next, applying the Schur complement to (4.18), results in

$$\left[\Theta_1(\hat{x}(k))^T \hat{P}^{-1}(x_+)\Theta_1(\hat{x}(k)) + \Theta_2(\hat{x}(k))^T \Theta_2(\hat{x}(k)) - \Lambda\right] < 0. \tag{4.19}$$

Then, knowing that (4.19) holds, hence from (4.13), we have

$$\Delta\hat{V}(\hat{x}(k)) < -z^T(k)z(k) + \gamma^2\omega^T(k)\omega(k). \tag{4.20}$$

Then, taking a summation from 0 to ∞, yield

$$\hat{V}(\hat{x}(\infty)) - \hat{V}(\hat{x}(0)) \leq -\sum_{k=0}^{\infty} z^T(k)z(k) + \sum_{k=0}^{\infty} \gamma^2\omega^T(k)\omega(k). \tag{4.21}$$

Using the fact that $\hat{V}(x(0)) = 0$ and $\hat{V}(x(\infty)) \geq 0$, we obtain

$$\sum_{k=0}^{\infty} z^T(k)z(k) \leq \gamma^2 \sum_{k=0}^{\infty} \omega^T(k)\omega(k). \tag{4.22}$$

Hence (4.6) holds and therefore the H_∞ performance is fulfilled.

Now we prove the system (4.1) with (4.2) is asymptotically stable. To prove the stability, we set the disturbance $\omega(x(k)) = 0$. Therefore, the system (4.1) with the controller (4.2) can be described as below:

$$\hat{x}(k+1) = \hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}_u(\hat{x}(k))\hat{A}_c(\hat{x}(k))\hat{x}(k). \tag{4.23}$$

where $\hat{A}(\hat{x}(k))$, and $\hat{B}_u(\hat{x}(k))$ are as described in (4.4). From the Lyapunov function described in (4.10), the difference between $\hat{V}(\hat{x}(k+1))$ and $\hat{V}(\hat{x}(k))$ of (4.10) along (4.23) is given by

$$\begin{aligned}
&\Delta\hat{V}(\hat{x}(k)) = \hat{V}(\hat{x}(k+1)) - \hat{V}(\hat{x}(k)) \\
&= \big(\hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}(\hat{x}(k))\hat{A}_c(\hat{x}(k))\hat{x}(k)\big)^T \hat{P}^{-1}(x_+) \\
&\times \big(\hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}(\hat{x}(k))\hat{A}_c(\hat{x}(k))\hat{x}(k)\big) - \hat{x}^T(k)\hat{P}^{-1}(x(k))\hat{x}(k) \\
&= \hat{x}^T(k)\big[(\hat{A}^T(\hat{x}(k)) + \hat{A}_c^T(\hat{x}(k))\hat{B}^T(\hat{x}(k)))\hat{P}^{-1}(x_+)(\hat{A}(\hat{x}(k)) + \hat{B}(\hat{x}(k))\hat{A}_c(\hat{x}(k))) \\
&- \hat{P}^{-1}(x(k))\big]x(k).
\end{aligned} \tag{4.24}$$

From the Lyapunov stability theory [63], the system (4.23) is stable if the Lyapunov function (4.10) > 0 such that (4.24) < 0. Hence, it is obvious from (4.24) that the sufficient condition to achieve (4.24) < 0 is by having the terms in $[.] < 0$. Therefore, if

$$\big(\hat{A}^T(\hat{x}(k)) + \hat{A}_c^T(\hat{x}(k))\hat{B}^T(\hat{x}(k))\big)\hat{P}^{-1}(x_+)\big(\hat{A}(\hat{x}(k)) + \hat{B}(\hat{x}(k))\hat{A}_c(\hat{x}(k))\big) - \hat{P}^{-1}(x(k)) < 0 \tag{4.25}$$

holds, then system (4.1) with (4.2) is asymptotically stable. The asymptotic stability of the system (4.1) with (4.2) has been proven in Chapter 2 (see Theorem 2.6). Hence the complete proof is omitted here. ∇∇∇

Note that conditions (4.7) - (4.8) of Theorem 4.2 are in state-dependent polynomial matrix inequalities (PMIs). Using SOS decomposition method based on SDP [32] provides a relaxation to the problem. In addition, to ensure the positive definiteness of (4.7) and (4.8), it is often necessary to add some SOS constraints in the form of positive definite constant or polynomial i.e $\epsilon > 0$ or $\epsilon(x) > 0$. Then the inequalities of (4.7) - (4.8) can be modified into SOS and they are given as follows:

$$v_1^T[\hat{P}(x(k)) - \epsilon_1 I]v_1 \qquad \text{is a SOS} \tag{4.26}$$

$$v_2^T[M_1(\hat{x}(k)) - \epsilon_2 I]v_2 \qquad \text{is a SOS} \tag{4.27}$$

where v_1 and v_2 are vectors in appropriate dimensions, and $M_1(\hat{x}(k))$ is as defined in (4.8). Moreover, ϵ_1 and ϵ_2 are constant and are always positive i.e $\epsilon > 0$. Clearly, this will provide sufficient conditions for Equations (4.7) and (4.8) and helps the SOS conditions (4.26) and (4.27) to be feasible. Therefore, the Theorem 4.2 can be written in the form of SOS conditions and it is given by the following corollary:

Corollary 4.3. *Given a prescribed H_∞ performance, $\gamma > 0$, the system (4.1) is stabilisable with H_∞ performance (4.6) via the nonlinear feedback controller of the form (4.2) if there exist a symmetric polynomial matrix, $\hat{P}(x(k))$, polynomial matrices $\hat{L}(\hat{x}(k))$ and $\hat{G}(\hat{x}(k))$, and constants $\epsilon_1 > 0$, and $\epsilon_2 > 0$ such that the following conditions are satisfied for all $x \neq 0$:*

$$v_1^T[\hat{P}(x(k)) - \epsilon_1 I]v_1 \qquad \text{is a SOS} \tag{4.28}$$

$$v_2^T[M_1(\hat{x}(k)) - \epsilon_2 I]v_2 \qquad \text{is a SOS} \tag{4.29}$$

where $M_1(\hat{x}(k))$ is as given in (4.9), and v_1 and v_2 are free vectors in appropriate dimensions. Moreover, the nonlinear feedback controller is given by

$$\begin{aligned} x_c(k+1) =& x_c(k) + A_c(x, x_c) \\ u(k) =& x_c(k) \end{aligned}$$

where, $A_c(x, x_c) = \hat{A}_c(\hat{x}(k))\hat{x}(k)$ with $\hat{A}_c(\hat{x}(k)) = \hat{L}(\hat{x}(k))\hat{G}^{-1}(\hat{x}(k))$.

Remark 4.4. By using the controller of the form (4.2) and the Lyapunov function of the form (4.10), the nonconvexity that exist in the $P(x(k+1))$ can be removed efficiently. Therefore, Corollary 4.3 can be solved computationally via SDP. This is the major advantage of our proposed method compared to others.

4.3.2 Robust nonlinear H_∞ control problem

The results presented in the previous section assume that the system's parameters are known exactly. In this section, we investigate how the above method can be extended to systems in which the parameters are not exactly known.

Consider the following system

$$\begin{aligned} x(k+1) =& A(x(k), \theta)x(k) + B_u(x(k), \theta)u(k) + B_\omega(x(k), \theta)\omega(k) \\ z(k) =& C_z(x(k), \theta)x(k) + D_{zu}(x(k), \theta)u(k) \end{aligned} \tag{4.30}$$

where the matrices $\cdot(x(k), \theta)$ are defined as follows

$$\begin{aligned} A(x(k), \theta) &= \sum_{i=1}^{q} A_i(x(k))\theta_i; \quad B(x(k), \theta) = \sum_{i=1}^{q} B_i(x(k))\theta_i; \\ B_\omega(x(k), \theta) &= \sum_{i=1}^{q} B_{\omega i}(x(k))\theta_i; \quad C_z(x(k), \theta) = \sum_{i=1}^{q} C_{zi}(x(k))\theta_i; \\ D_{zu}(x(k), \theta) &= \sum_{i=1}^{q} D_{zui}(x(k))\theta_i. \end{aligned} \tag{4.31}$$

$\theta = \Big[\theta_1, \dots, \theta_q\Big]^T \in \mathbb{R}^q$ is the vector of constant uncertainty and satisfies

$$\theta \in \Theta \triangleq \left\{\theta \in \mathbb{R}^q : \theta_i \geq 0, i = 1, \dots, q, \sum_{i=1}^{q} \theta_i = 1\right\}. \tag{4.32}$$

With controller (4.2), we have the following system:

$$\begin{aligned} \hat{x}(k+1) =& \hat{A}(\hat{x}(k), \theta)\hat{x}(k) + \hat{B}_u(\hat{x}(k), \theta)u(k) + \hat{B}_\omega(\hat{x}(k), \theta)\omega(k) \\ z(k) =& \hat{C}_z(x(k), \theta)\hat{x}(k) \end{aligned} \tag{4.33}$$

where,

$$\begin{aligned} \hat{A}(\hat{x}(k), \theta) = \sum_{i=1}^{q} \hat{A}_i(\hat{x}(k))\theta_i; \quad \hat{B}_u(\hat{x}(k), \theta) = \sum_{i=1}^{q} \hat{B}_i(\hat{x}(k))\theta_i; \\ \hat{C}_z(x(k), \theta) = \sum_{i=1}^{q} \hat{C}_{zi}(x(k))\theta_i; \quad \hat{B}_\omega(\hat{x}(k), \theta) = \sum_{i=1}^{q} \hat{B}_i(\hat{x}(k))\theta_i. \end{aligned} \tag{4.34}$$

We further define the following parameter-dependent Lyapunov function

$$\hat{V}(\hat{x}(k)) = \hat{x}^T(k)\big(\sum_{i=1}^{q} \hat{P}_i(x(k))\theta_i\big)^{-1}\hat{x}(k). \tag{4.35}$$

where $\hat{P}(x(k))$ is as defined in Chapter 2.

With the results from the previous section, the main result for robust H_∞ synthesis can be proposed directly and given by the following proposition.

Proposition 4.5. *Given a prescribed H_∞ performance, $\gamma > 0$, constants $\epsilon_1 > 0$ and $\epsilon_2 > 0$ for $x \neq 0$ and $i = 1, ..., q$, the system (4.30) with the nonlinear feedback controller (4.2) is asymptotically stable with H_∞ performance (4.6) for $x \neq 0$ if there exist common polynomial matrices $\hat{G}(\hat{x}(k))$ and $\hat{L}(\hat{x}(k))$, and a symmetric polynomial matrix, $\hat{P}_i(x(k))$ such that the following conditions are satisfied for all $x \neq 0$:*

$$v_3^T[\hat{P}_i(x(k)) - \epsilon_1 I]v_3 \quad \textit{is a SOS} \tag{4.36}$$

$$v_4^T\left[M_2(\hat{x}(k)) - \epsilon_2 I\right]v_4 \quad \textit{is a SOS} \tag{4.37}$$

where,

$$M_2(\hat{x}(k)) = \begin{bmatrix} \hat{G}^T(\hat{x}(k)) + \hat{G}(\hat{x}(k)) - \hat{P}_i(x(k)) & * & * & * \\ 0 & \gamma^2 I & * & * \\ \hat{A}_i(\hat{x}(k))\hat{G}(x(k)) + \hat{B}_{ui}(\hat{x}(k))\hat{L}(\hat{x}(k)) & \hat{B}_{\omega i}(\hat{x}(k)) & \hat{P}_i(x_+) & * \\ \hat{C}_{zi}(\hat{x}(k))\hat{G}(\hat{x}(k)) & 0 & 0 & I \end{bmatrix} \tag{4.38}$$

Meanwhile v_3 and v_4 are free vectors in appropriate dimensions.

Proof: This proposition follows directly as a convex combination of several systems of form (4.30) for a common (4.2). $\Delta\Delta\Delta$

4.4 Numerical Examples

In this section, a tunnel diode circuit is used to validate the effectiveness of the proposed approach.

4.4.1 Nonlinear H_∞ control problem

Consider a tunnel diode circuit as shown in Figure 4.1 [78], where the characteristics of the tunnel diode are described as follows:

$$i_D(t) = 0.002v_D(t) + 0.01v_D^3(t). \tag{4.39}$$

Next, letting $x_1(t)$=$v_c(t)$ and $x_2(t)$=$i_L(t)$ be the state variables, then the circuit is

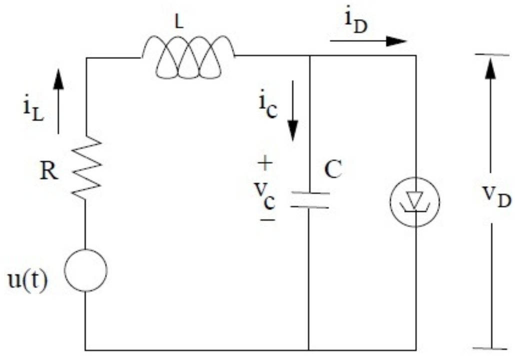

FIGURE 4.1: A Tunnel diode circuit.

governed by the following state equations:

$$\begin{aligned} C\dot{x}_1(t) &= -0.002x_1(t) - 0.01x_1^3(t) + x_2(t) \\ L\dot{x}_2(t) &= -x_1(t) - Rx_2(t) + \omega(t) + u(t) \\ z(t) &= x_2(t) + u(t) \end{aligned} \tag{4.40}$$

where $\omega(t)$ is the noise to the system, $z(t)$ is the controlled output, and we assume both $x_1(t)$=$v_c(t)$ and $x_2(t)$=$i_L(t)$ are available for feedback. Meanwhile the circuit parameter is given as follows: $C = 20mF$, $L = 1000mH$, and $R = 1\Omega$. With these parameters, the dynamic of the circuit can be written as follows:

$$\begin{aligned} \dot{x}_1(t) &= -0.1x_1(t) - 0.5x_1^3(t) + 50x_2(t) \\ \dot{x}_2(t) &= -x_1(t) - x_2(t) + \omega(t) + u(t) \\ z(t) &= x_2(t) + u(t) \end{aligned} \tag{4.41}$$

Then, the above system is sampled at $T = 0.02$ and by Euler's discretisation method, the following discrete-time nonlinear dynamic equations is obtained:

$$\begin{aligned} x_1(k+1) &= x_1(k) + T\big[-0.1x_1(k) - 0.5x_1^3(k) + 50x_2(k)\big] \\ x_2(k+1) &= x_2(k) + T\big[-x_1(t) - x_2(t) + \omega(t) + u(t)\big] \\ z(k) &= x_2(k) + u(k) \end{aligned} \tag{4.42}$$

From (4.42), the system with controller (4.2) can be written as follows:

$$\left.\begin{aligned} \hat{x}(k+1) &= \hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}_u(\hat{x}(k))\hat{A}_c(\hat{x}(k))\hat{x}(k) + \hat{B}_\omega(\hat{x}(k))\omega(k) \\ z(k) &= \hat{C}_z(\hat{x}(k))\hat{x}(k) \end{aligned}\right\} \tag{4.43}$$

where,

$$\begin{aligned} &\hat{A}(\hat{x}(k)) = \begin{bmatrix} 1+T\big(-0.1-0.5x_1^2(k)\big) & 50T & 0 \\ -T & 1-T & T \\ 0 & 0 & 1 \end{bmatrix}; \quad \hat{B}_u(\hat{x}(k)) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \\ &\hat{B}_\omega(\hat{x}(k)) = \begin{bmatrix} 0 \\ T \\ 0 \end{bmatrix}; \quad \hat{C}_z(\hat{x}(k)) = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}; \quad \hat{x}(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_c(k) \end{bmatrix}. \end{aligned} \tag{4.44}$$

Remark 4.6. For this example, we choose $\epsilon_1 = \epsilon_2 = 0.01$ and γ is selected to be 1. The open loop response is given in Figure 4.2. Then, using the procedure described in the Corollary 4.3, and with the degree of $\hat{P}(x(k))$ and $\hat{G}(\hat{x}(k))$ set to be 4 and $\hat{L}(\hat{x}(k))$ chosen to be in the degree of 8, a feasible solution is achieved. The band-limited white noise (noise power = 10) is used in the simulation. The energy ratio of the regulated output and the disturbance input noise is shown in Figure 4.3. The value shown is less than a prescribed value 1.

4.4.2 Robust nonlinear H_∞ control problem

Consider the tunnel diode circuit shown in 4.1. For this example, we assume the value of R is uncertain and given by $R = 1 \pm 30\%$. Therefore, the system can be described as

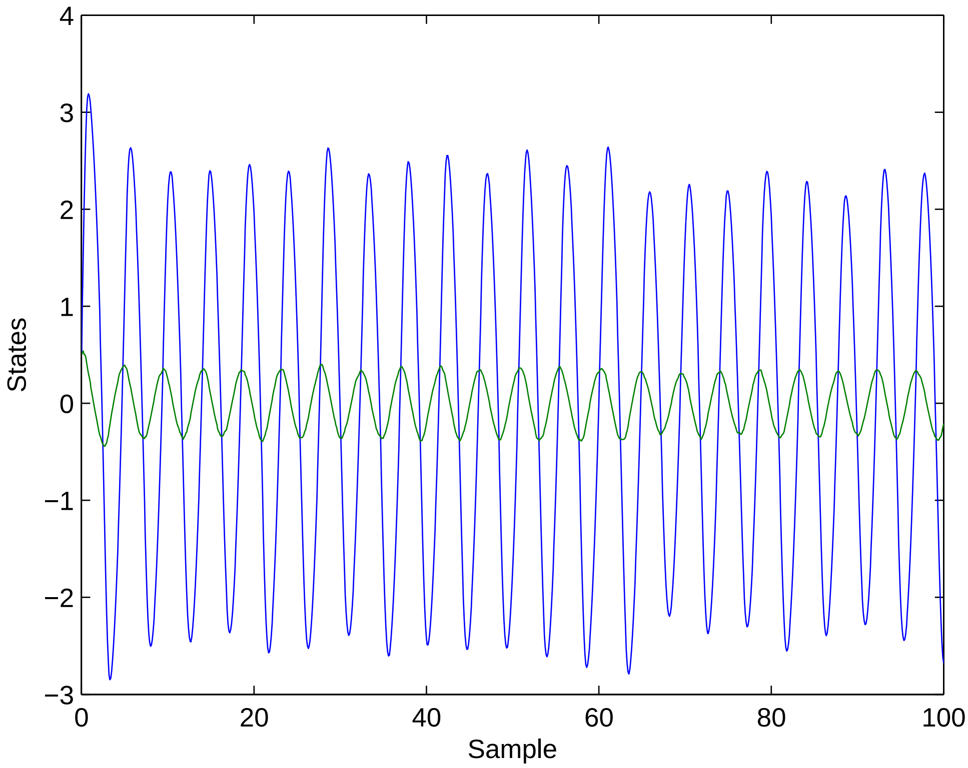

FIGURE 4.2: Open loop responses for a Tunnel diode circuit.

follows:

$$\hat{A}(\hat{x}(k)) = \begin{bmatrix} 1+T\big(-0.1-0.5x_1^2(k)\big) & 50T & 0 \\ -T & 1-RT & T \\ 0 & 0 & 1 \end{bmatrix}; \quad \hat{B}_u(\hat{x}(k)) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix};$$

$$\hat{B}_\omega(\hat{x}(k)) = \begin{bmatrix} 0 \\ T \\ 0 \end{bmatrix}; \quad \hat{C}_z(\hat{x}(k)) = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}; \tag{4.45}$$

By implementing Proposition 4.5, where γ is chosen to be 1 and the values of the positive constants ϵ_1 and ϵ_2 are fixed at 0.01, we initially choose the degree of $\hat{P}_1(x(k))$ and $\hat{P}_2(x(k))$ to be 2. The common polynomial matrices $\hat{G}(\hat{x}(k))$ and $\hat{L}(\hat{x}(k))$ are also selected to be in the degree of 2, but no feasible solution can be achieved. Then, the degree of $\hat{P}_1(x(k)), \hat{P}_2(x(k))$ and $\hat{G}(\hat{x}(k))$ are increased to 4. Meanwhile, the degree of $\hat{L}(\hat{x}(k))$ is increased to 8, and with this arrangement a feasible solution is obtained. From the simulation result shown in Figure 4.4, the ratio of the regulated output energy to the disturbance input noise energy in this example tends to be a constant value which is approximately at 1.3×10^{-4}. Thus, $\gamma = \sqrt{1.3 \times 10^{-4}} \approx 0.0114$. This implies that the L_2 gain from the disturbance to the regulated output is no greater than 0.00114.

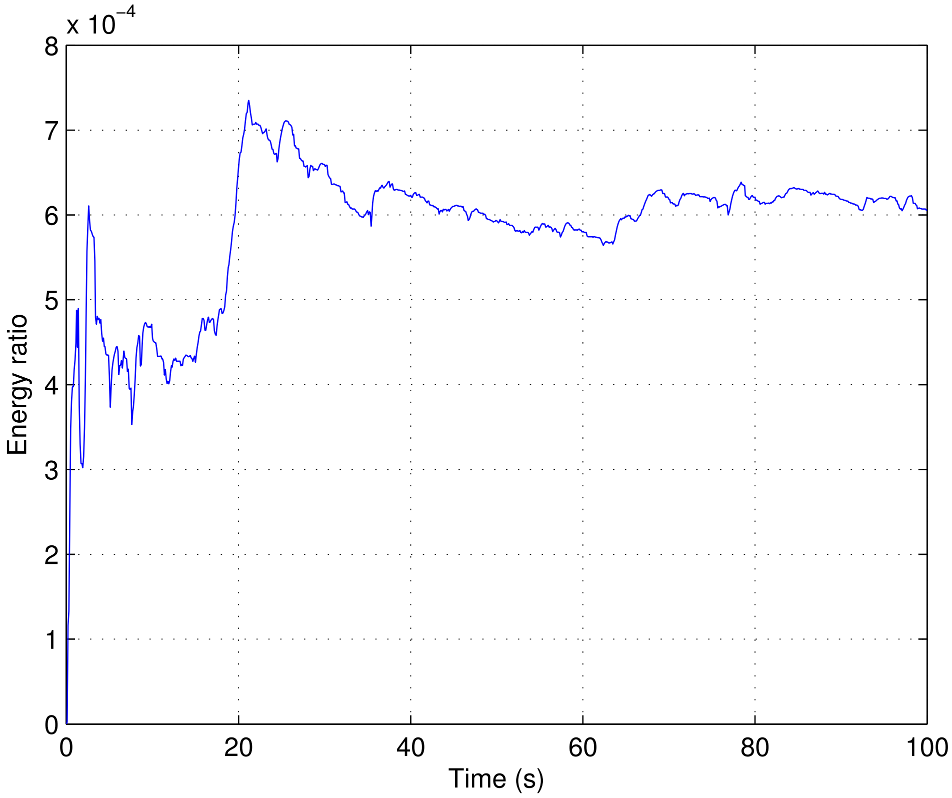

FIGURE 4.3: Ratio of the regulated output energy to the disturbance input noise energy without uncertainty.

Remark 4.7. It is worth noting here that, to date, no result has been presented in the literature which considers the robust H_∞ stabilisation for the polynomial discrete-time system.

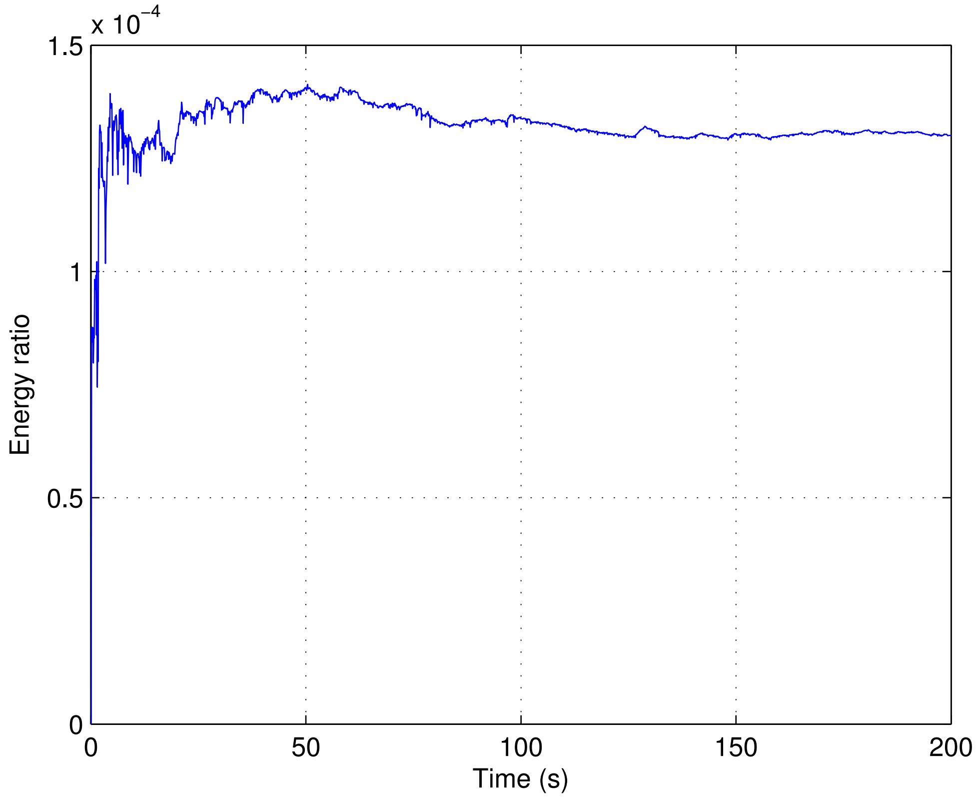

FIGURE 4.4: Energy ratio of the regulated output and the disturbance input noise with polytopic uncertainties.

4.5 Conclusion

In this chapter, we have examined the problem of designing a nonlinear H_∞ feedback controller for polynomial discrete-time systems with and without polytopic uncertainties. The main contribution of this chapter is the stability and prescribed disturbance attenuation performance of polynomial discrete-time systems with and without polytopic uncertainties are fulfilled. Furthermore, a less conservative design procedure is obtained by incorporating the integrator into the controller structure. The sufficient conditions for the existence of the proposed controller are provided in terms of the solvability of polynomial matrix inequalities, which are formulated as SOS constraints. The effectiveness of the proposed design methodology is demonstrated through a tunnel diode circuit.

Chapter 5

Robust Nonlinear H_∞ State Feedback Control for Polynomial Discrete-Time Systems With Norm-Bounded Uncertainty

5.1 Introduction

The motivation of this chapter arises from the work performed in [79]. In [79], the interconnection between robust H_∞ control problem and nonlinear H_∞ control problem is presented by introducing a $'scaled'$ nonlinear system. The solution to the problem is then given in terms of the $'scaled'$ Hamilton Jacobi inequalities (HJIs). However, it is well known that to solve the HJIs is hard because no computational tools are available for solving them. Hence, in light of this method, the $'scaled'$ polynomial discrete-time system is established in our work and using this $'scaled'$ polynomial discrete-time system, the interconnection between robust H_∞ control problem and nonlinear H_∞ control problem is developed for polynomial discrete-time systems. The sufficient conditions for the existence of the proposed controller with an integrator is given by the solvability of the PMIs which are formulated as SOS constraints. This consequently allows us to solve the problem by the recently developed SOS solvers. It is also important to note here that, to date, to the author's knowledge, no result has been presented in the framework of the robust H_∞ control problem for polynomial discrete-time systems using SOS decomposition methods in which the uncertainty is modelled as norm-bounded.

5.2 System Description and Problem Formulation

Consider the following polynomial discrete-time system with uncertainties in the state and input,

$$\left.\begin{aligned} x(k+1) =& A(x(k))x(k) + \Delta A(x(k))x(k) + B_u(x(k))u(k) + \Delta B_u(x(k))u(k) \\ &+ B_\omega(x(k))\omega(k) \\ z(k) =& C_z(x(k))x(k) + D_{zu}(x(k))u(k) \end{aligned}\right\} \tag{5.1}$$

where $x(k) \in R^n$ is the state vectors, and $u(k) \in R^m$ is the input. $A(x(k))$, $B_u(x(k))$, $C_z(x(k))$ and $D_{zu}(x(k))$ are polynomial matrices of appropriate dimensions. $z(k)$ is a vector of output signals related to the performance of the control system. $\omega(k)$ is the disturbance which belongs to $L_2[0,\infty]$. Meanwhile $\Delta A(x(k))$ and $\Delta B_u(x(k))$ represent the uncertainties in the system and satisfy the following assumption.

Assumption 5.1. The parameter uncertainties considered here are norm-bounded, and described by the following form

$$\begin{bmatrix} \Delta A(x(k)) & \Delta B_u(x(k)) \end{bmatrix} = H(x(k))F(x(k)) \begin{bmatrix} E_1(x(k)) & E_2(x(k)) \end{bmatrix} \tag{5.2}$$

where $H(x(k))$, $E_1(x(k))$ and $E_2(x(k))$ are known polynomial matrices of appropriate dimensions, and $F(x(k))$ is an unknown matrix function which satisfies,

$$\left\| F^T(x(k))F(x(k)) \right\| \leq I. \tag{5.3}$$

The nonlinear feedback controller is proposed as follows:

$$\left.\begin{aligned} x_c(k+1) =& x_c(k) + A_c(x, x_c) \\ u(k) =& x_c(k) \end{aligned}\right\} \tag{5.4}$$

where $x_c(k)$ is the state of the controller and $A_c(x, x_c)$ is the input function of the integrator. The reason for incorporating an integrator into the controller structure is to avoid the nonconvexity in $P(x(k+1))$. The effectiveness of this integrator method has already been described in Chapter 2, hence the details about the selection of the controller (5.4) are omitted here.

The robust nonlinear H_∞ control problem is defined as follows: Given any $\gamma > 0$, find a controller of the form (5.4) such that the L_2 gain from the disturbance $\omega(k)$ to the output that needs to be controlled $z(k)$ for system (5.1) with (5.4) is less than or equal to γ, i.e

$$\|z(k)\|_{[0,\infty]} \leq \gamma^2 \|\omega(k)\|_{[0,\infty]} \tag{5.5}$$

for all $w(k) \in L_2[0, \infty]$ and for all admissible uncertainties. In this situation, the system (5.1) is said to have a robust H_∞ performance (5.5).

5.3 Main Results

In this section, we show that the robust nonlinear H_∞ control problem is solvable if the nonlinear H_∞ control problem for the *'scaled'* system is solvable. We begin this section by defining the *'scaled'* system. Then, the *'scaled'* system with controller is represented as the augmented form, and followed by the methodology for solving the robust nonlinear H_∞ feedback control.

Motivated by the work performed in [79], the following *'scaled'* system is defined:

$$\left.\begin{aligned} \tilde{x}(k+1) =& A(\tilde{x}(k))\tilde{x}(k) + \begin{bmatrix} B_\omega(\tilde{x}(k)) & \frac{1}{\delta}\bar{H}(\tilde{x}(k)) \end{bmatrix} \tilde{\omega}(k) + B_u(\tilde{x}(k))u(k) \\ \tilde{z}(k) =& \begin{bmatrix} C_z(\tilde{x}(k)) \\ \delta E_1(\tilde{x}(k)) \end{bmatrix} \tilde{x}(k) + \begin{bmatrix} D_{zu}(\tilde{x}(k)) \\ \delta E_2(\tilde{x}(k)) \end{bmatrix} u(k) \end{aligned}\right\} \tag{5.6}$$

where $\tilde{x} \in R^n$ is the state, $u(k) \in R^m$ and $\tilde{\omega} \in R^{m+i}$ is the input noise. The δ is a positive constant, $\tilde{z}(k)$ is the controlled output and $\bar{H}(\tilde{x}(k)) = [H_1(\tilde{x}(k)) \quad H_1(\tilde{x}(k))]$.

Remark 5.2. Notice that the system described in (5.6) is in similar form to the system described in (4.1) (refer Chapter 3). Therefore, the methodology used to solve the system (4.1) can be applied in order to solve the *'scaled'* nonlinear H_∞ control problem of the system shown in (5.6).

The system (5.6) with controller (5.4) can be written as follows:

$$\left.\begin{aligned} \hat{x}(k+1) &= \hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}_u(\hat{x}(k))A_c(x, x_c) + \hat{B}_\omega(\hat{x}(k))\tilde{\omega}(k) \\ \tilde{z}(k) &= \hat{C}_z(\hat{x}(k))\hat{x}(k) \end{aligned}\right\} \tag{5.7}$$

where,

$$\hat{A}(\hat{x}(k)) = \begin{bmatrix} A(\tilde{x}(k)) & B_u(\tilde{x}(k)) \\ 0 & 1 \end{bmatrix}; \quad \hat{B}_u(\hat{x}(k)) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad \hat{B}_\omega(\hat{x}(k)) = \begin{bmatrix} \tilde{B}_\omega(\tilde{x}(k)) \\ 0 \end{bmatrix};$$

$$\hat{C}_z(\hat{x}(k)) = \begin{bmatrix} \tilde{C}_z(\tilde{x}(k)) & \tilde{D}_{zu}(\tilde{x}(k)) \end{bmatrix}; \quad \hat{x} = \begin{bmatrix} x(k) \\ x_c(k) \end{bmatrix}, \tag{5.8}$$

with

$$\tilde{B}_\omega(\tilde{x}(k)) = \begin{bmatrix} B_\omega(\tilde{x}(k)) & \frac{1}{\delta}\bar{H}(\tilde{x}(k)) \end{bmatrix}; \quad \tilde{C}_z(\tilde{x}(k)) = \begin{bmatrix} C_z(\tilde{x}(k)) \\ \delta E_1(\tilde{x}(k)) \end{bmatrix};$$
$$\tilde{D}_{zu}(\tilde{x}(k)) = \begin{bmatrix} D_{zu}(\tilde{x}(k)) \\ \delta E_2(\tilde{x}(k)) \end{bmatrix}. \tag{5.9}$$

Next, we assume $A_c(x, x_c)$ to be of the form $A_c(x, x_c) = \hat{A}_c(\hat{x}(k))\hat{x}(k)$. Therefore, (5.7) can be re-written as follows:

$$\left.\begin{aligned} \hat{x}(k+1) &= \hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}_u(\hat{x}(k))\hat{A}_c(\hat{x}(k))\hat{x}(k) + \hat{B}_\omega(\hat{x}(k))\tilde{\omega}(k) \\ \tilde{z}(k) &= \hat{C}_z(\hat{x}(k))\hat{x}(k) \end{aligned}\right\} \tag{5.10}$$

where $\hat{A}(\hat{x}(k))$, $\hat{B}_u(\hat{x}(k))$, $\hat{B}_\omega(\hat{x}(k))$, and $\hat{C}_z(\hat{x}(k))$ are as described in (5.8).

In view of the $'scaled'$ system (5.6), the following theorem is established.

Theorem 5.3. *Consider the system (5.1). There exists a controller of the form (5.4) such that (5.5) holds for all admissible uncertainties if there exists a positive constant, $\delta > 0$, such that (5.5) holds for system (5.6) with the same controller.*

Proof: Suppose

$$\|\tilde{z}(k)\|_{[0,\infty]} \leq \|\tilde{\omega}(k)\|_{[0,\infty]} \tag{5.11}$$

holds for (5.6) with (5.4) for all $w(k) \in L_2[0, \infty]$. Then, we need to show that

$$\|z(k)\|_{[0,\infty]} \leq \|\omega(k)\|_{[0,\infty]} \tag{5.12}$$

holds for the system (5.1) with the same controller. To show this, let choose

$$\tilde{\omega} = \begin{bmatrix} \omega(k) \\ \delta\eta(k) \end{bmatrix}, 0 \leq k \leq \infty \tag{5.13}$$

where,

$$\eta(k) = F(x(k)) \begin{bmatrix} E_1(x(k))x(k) \\ E_2(x(k))u(k) \end{bmatrix} \tag{5.14}$$

Then, it is trivial to show that (5.11) implies (5.12). The proof ends.

$\nabla\nabla\nabla$

In light of Theorem 5.3, what is left here is to solve the $'scaled'$ nonlinear H_∞ control problem given in (5.6). Therefore, the sufficient conditions for the existence of a solution to the robust H_∞ control problem is presented in the following Theorem.

Theorem 5.4. *Given a prescribed H_∞ performance, $\gamma > 0$, the system (5.1) is stabilisable with H_∞ performance (5.5) via the nonlinear feedback controller of the form (5.4) if there exist a symmetric polynomial matrix, $\hat{P}(x(k))$, polynomial matrices $\hat{L}(\hat{x}(k))$ and $\hat{G}(\hat{x}(k))$ such that the following conditions are satisfied for all $x \neq 0$:*

$$\hat{P}(x(k)) > 0 \tag{5.15}$$

$$M(\hat{x}(k)) > 0 \tag{5.16}$$

where,

$$M(\hat{x}(k)) = \begin{bmatrix} \hat{G}^T(\hat{x}(k)) + \hat{G}(\hat{x}(k)) - \hat{P}(x(k)) & * & * & * \\ 0 & \gamma^2 I & * & * \\ \hat{A}(\hat{x}(k))\hat{G}(x(k)) + \hat{B}_u(\hat{x}(k))\hat{L}(\hat{x}(k)) & \hat{B}_\omega(\hat{x}(k)) & \hat{P}(x_+) & * \\ \hat{C}_z(\hat{x}(k))\hat{G}(\hat{x}(k)) & 0 & 0 & I \end{bmatrix}. \tag{5.17}$$

The nonlinear feedback controller is given by

$$\begin{aligned} x_c(k+1) =& x_c(k) + A_c(x, x_c) \\ u(k) =& x_c(k) \end{aligned}$$

where, $A_c(x, x_c) = \hat{A}_c(\hat{x}(k))\hat{x}(k)$ with $\hat{A}_c(\hat{x}(k)) = \hat{L}(\hat{x}(k))\hat{G}^{-1}(\hat{x}(k))$.

Proof: By Theorem 5.3, the robust nonlinear H_∞ control problem is converted to the nonlinear H_∞ control problem for a *'scaled'* system. Then, by adapting Theorem 4.2, the result can be obtained easily. It is important to note here that the Lyapunov function of the following form is selected

$$\hat{V}(\hat{x}(k)) = \hat{x}^T(k)\hat{P}^{-1}(x(k))\hat{x}(k). \tag{5.18}$$

$\nabla\nabla\nabla$

Remark 5.5. The idea of choosing the Lyapunov function to be of the form (5.18) is to ensure that a convex solution of $P(x(k+1))$ can be achieved. This idea has been outlined in detail in Chapter 2, hence for the sake of simplicity, the complete explanation is omitted here.

Note that the conditions (5.15) - (5.16) of Theorem 5.4 are in state-dependent polynomial matrix inequalities (PMIs). Using the SOS decomposition method based on SDP [32] provides a relaxation of the problem. Therefore, the (5.15) - (5.16) can be modified into SOS conditions, and they are given in the following corollary:

Corollary 5.6. *Given a prescribed H_∞ performance, $\gamma > 0$, the system (5.1) is stabilisable with H_∞ performance (5.5) via the nonlinear feedback controller of the form (5.4) if there exist a symmetric polynomial matrix, $\hat{P}(x(k))$, polynomial matrices $\hat{L}(\hat{x}(k))$ and $\hat{G}(\hat{x}(k))$, and constants $\epsilon_1 > 0$, and $\epsilon_2 > 0$ such that the following conditions are satisfied for all $x \neq 0$:*

$$v_1^T[\hat{P}(x(k)) - \epsilon_1 I]v_1 \qquad \text{is a SOS} \tag{5.19}$$

$$v_2^T[M(\hat{x}(k)) - \epsilon_2 I]v_2 \qquad \text{is a SOS} \tag{5.20}$$

where $M(\hat{x}(k))$ is as given in (5.17), and v_1 and v_2 are free vectors in appropriate dimensions. Moreover, the nonlinear feedback controller is given by

$$\begin{aligned} x_c(k+1) =& x_c(k) + A_c(x, x_c) \\ u(k) =& x_c(k) \end{aligned}$$

where, $A_c(x, x_c) = \hat{A}_c(\hat{x}(k))\hat{x}(k)$ with $\hat{A}_c(\hat{x}(k)) = \hat{L}(\hat{x}(k))\hat{G}^{-1}(\hat{x}(k))$.

5.4 Numerical Example

In this section, a tunnel diode circuit is used to validate the effectiveness of the proposed approach.

Consider a tunnel diode circuit as shown in Figure 4.1 [78], where the characteristics of the tunnel diode are described as follows:

$$i_D(t) = 0.002v_D(t) + 0.01v_D^3(t). \tag{5.21}$$

Next, letting $x_1(t)$=$v_c(t)$ and $x_2(t)$=$i_L(t)$ be the state variables, then the circuit is governed by the following state equations:

$$\begin{aligned} C\dot{x}_1(t) &= -0.002x_1(t) - 0.01x_1^3(t) + x_2(t) \\ L\dot{x}_2(t) &= -x_1(t) - Rx_2(t) + \omega(t) + u(t) \\ z(t) &= x_2(t) + u(t) \end{aligned} \tag{5.22}$$

where $\omega(t)$ is the noise to the system, z(t) is the controlled output, and we assume both $x_1(t)$=$v_c(t)$ and $x_2(t)$=$i_L(t)$ are available for feedback. Meanwhile the circuit parameter is given as follows: $C = 20mF$, $L = 1000mH$, and $R = 1 \pm 30\%\Omega$. With these parameters, the dynamic of the circuit can be written as

$$\begin{aligned}
\dot{x}_1(t) &= -0.1x_1(t) - 0.5x_1^3(t) + 50x_2(t) \\
\dot{x}_2(t) &= -x_1(t) - (1+\Delta R)x_2(t) + \omega(t) + u(t) \\
z(t) &= x_2(t) + u(t)
\end{aligned} \tag{5.23}$$

Then, the above system is sampled at $T = 0.02$ and by using Euler's discretisation method, then the following discrete-time nonlinear dynamic equations is obtained:

$$\begin{aligned}
x_1(k+1) &= x_1(k) + T\big[-0.1x_1(k) - 0.5x_1^3(k) + 50x_2(k)\big] \\
x_2(k+1) &= x_2(k) + T\big[-x_1(k) - (1+\Delta R)x_2(k) + \omega(k) + u(k)\big] \\
z(k) &= x_2(k) + u(k)
\end{aligned} \tag{5.24}$$

From (5.24), the system with controller (5.4) can be written as follows:

$$\left.\begin{aligned}
\hat{x}(k+1) &= \hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}_u(\hat{x}(k))\hat{A}_c(\hat{x}(k))\hat{x}(k) + \hat{B}_\omega(\hat{x}(k))\omega(k) \\
z(k) &= \hat{C}_z(\hat{x}(k))\hat{x}(k)
\end{aligned}\right\} \tag{5.25}$$

where,

$$\begin{aligned}
&\hat{A}(\hat{x}(k)) = \begin{bmatrix} A(\tilde{x}(k)) & B_u(\tilde{x}(k)) \\ 0 & 1 \end{bmatrix}; \quad \hat{B}_u(\hat{x}(k)) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad \hat{B}_\omega(\hat{x}(k)) = \begin{bmatrix} \tilde{B}_\omega(\tilde{x}(k)) \\ 0 \end{bmatrix}; \\
&\hat{C}_z(\hat{x}(k)) = \begin{bmatrix} \tilde{C}_z(\tilde{x}(k)) & \tilde{D}_{zu}(\tilde{x}(k)) \end{bmatrix}; \quad \hat{x} = \begin{bmatrix} x(k) \\ x_c(k) \end{bmatrix},
\end{aligned} \tag{5.26}$$

and with

$$\begin{aligned}
\tilde{B}_\omega(\tilde{x}(k)) &= \begin{bmatrix} B_\omega(\tilde{x}(k)) & \frac{1}{\delta}\bar{H}(\tilde{x}(k)) \end{bmatrix}; \quad \tilde{C}_z(\tilde{x}(k)) = \begin{bmatrix} C_z(\tilde{x}(k)) \\ \delta E_1(\tilde{x}(k)) \end{bmatrix}; \\
\tilde{D}_{zu}(\tilde{x}(k)) &= \begin{bmatrix} D_{zu}(\tilde{x}(k)) \\ \delta E_2(\tilde{x}(k)) \end{bmatrix}.
\end{aligned} \tag{5.27}$$

Similarly,

$$\hat{A}(\hat{x}(k)) = \begin{bmatrix} 1+T\big[-0.1-0.5x_1^2(k)\big] & 50T & 0 \\ -T & 1-T(1+\Delta R) & T \\ 0 & 0 & 1 \end{bmatrix}; \quad \hat{B}_u(\hat{x}(k)) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix};$$

$$\hat{B}_\omega(\hat{x}(k)) = \begin{bmatrix} 0 & 0 & 0 \\ T & \frac{1}{\delta}0.3 & \frac{1}{\delta}0.3 \\ 0 & 0 & 0 \end{bmatrix}; \quad \hat{C}_z(\hat{x}(k)) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & \delta & 0 \end{bmatrix};$$

$$\hat{x}(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_c(k) \end{bmatrix}. \tag{5.28}$$

where, $\delta = 1$. From (5.28), for clarity we describe again the matrices which represent the uncertainty:

$$H(x(k)) = \begin{bmatrix} 0 \\ 0.3 \end{bmatrix}; \quad E_1(x(k)) = \begin{bmatrix} 0 & \delta \end{bmatrix}; \quad E_2(x(k)) = \begin{bmatrix} 0 \end{bmatrix}.$$

Then, applying the procedures outlined in Corollary 5.6 where the $\hat{P}(\hat{x}(k))$ and $\hat{G}(\hat{x}(k))$ are set to be degree of 4, and $\hat{L}(\hat{x}(k))$ is selected to be degree of 8. Through this set up, a feasible solution is obtained. The ratio of the regulated output energy to the noise energy is shown in Figure 5.1. It can be clearly seen from the figure that the energy ratio tends to be a constant value after $15s$ which is at approximately 4.25×10^{-4}. Hence, the γ value is equivalent to $\sqrt{4.25 \times 10^{-4}} \approx 0.0206$. This value is absolutely less than a prescribed γ value 1.

5.5 Conclusion

The problem of designing a robust nonlinear feedback controller for uncertain polynomial discrete-time systems with norm-bounded uncertainty has been examined in this chapter. In particular, the interconnection between robust nonlinear H_∞ feedback control and nonlinear H_∞ feedback control has been established through a so called $'scaled'$ system. Based on this connection, the sufficient conditions for the existence of the proposed controller for polynomial discrete-time systems are provided in terms of the solvability of SOS matrix inequalities. A numerical example has been provided to demonstrate the validity of the proposed approach. In contrast to the work delivered in [79], our method yields a solution to discrete-time systems, and the solution is given in terms of the PMIs which are formulated as SOS constraints. Therefore, the problem can be solved

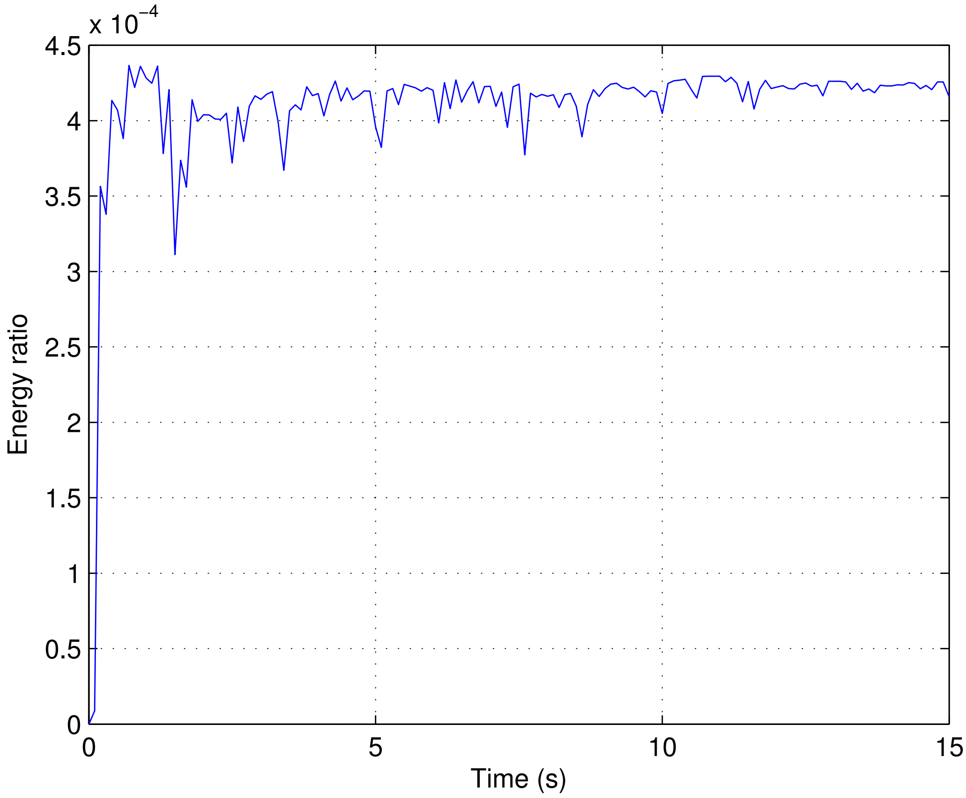

FIGURE 5.1: energy ratio: $\frac{\sum z^T z}{\sum \omega^T \omega} \leq \gamma^2$.

computationally via recently developed SOS solvers. However, in [79], the solution is expressed in terms of HJIs and it is well known to solve those HJIs is hard because no computational tools are available to help the solution.

Chapter 6

Nonlinear Filter Design for Polynomial Discrete-time Systems

6.1 Introduction

The availability of all the states for direct measurement is considered as a rare occasion in practical feedback control systems. Besides, in most cases, there is a need for a reliable estimation of unmeasurable state variables. The reliable estimation is needed especially for the synthesis of model-based controllers or for process monitoring purposes. For these purpose, a filter is usually employed in order to accurately estimate the state. The results of filter designs for nonlinear systems can be found in [80–82]. In particular, [80] proposed a solution in terms of the Hamilton Jacobi Inequalities (HJIs). It is well known that to solve the HJIs is hard because there are no computational tools available for solving them. Meanwhile, a convex solution to the filter problem has been given in [81] through the S-procedure method. In the present paper, the problem is formulated in LMI forms, and solved using LMI toolbox. However, to render a convex solution, some assumptions about nonlinear terms of the error dynamics have to be made. The assumptions might cause the results to be conservative. On the other hand, the filter design for polynomial systems have been considered in [82]. The author has shown that a convex solution can be rendered without requiring any assumptions about nonlinear terms of the error dynamics. By utilising the SOS programming approach, the convex problem can be solved efficiently. Unfortunately, the result that is proposed in this paper are only held locally.

Therefore, in this chapter, we attempt to design a filter to estimate the state of polynomial discrete-time systems. In this work, a global filter design method for polynomial discrete-time systems by using the SOS-SDP based is established without any assumptions about nonlinear terms of the error dynamics. In our work, to ensure that a convex solution to the filter design problem can be obtained, an integrator is incorporated into the filter structure. To compute the filter gains, SOS techniques have been used to reduce the problems to SDP. The effectiveness of the proposed method is confirmed through a simulation example.

6.2 System Description and Preliminaries

Consider the following polynomial discrete-time system:

$$\begin{aligned} x(k+1) &= A(x(k))x(k) \\ y &= C(x(k))x(k) \end{aligned} \tag{6.1}$$

where $x(k) \in \Re^n$ is the state, and y is the measurement. $A(x(k))$ and $C(x(k))$ are polynomial matrices with appropriate dimension.

A filter to estimate the state $x(k)$ from y is selected to be of the following form:

$$\begin{aligned} \hat{x}(k+1) =& A(\hat{x}(k))\hat{x} + L(\hat{x}(k))\big(y - \hat{y}\big) \\ \hat{y} =& C(\hat{x}(k))\hat{x}(k) \end{aligned} \tag{6.2}$$

where $\hat{x}$ is a filter state, $\hat{y}$ is a filter measurement and $L(\hat{x}(k))$ is a designed polynomial matrix with appropriate dimensions.

To study the convergence and performance of the filter (6.2), we will look at the dynamics of the estimation error defined by $e = \hat{x}(k) - x(k)$. The error dynamics is given as follows:

$$\begin{aligned} e(k+1) &= \hat{x}(k+1) - x(k+1) \\ &= A(\hat{x}) + L(\hat{x})\big(C(\hat{x})\hat{x} - C(x)x\big) - A(x)x \\ &= \big[A(\hat{x}) + L(\hat{x})C(\hat{x})\big]e + \big[A(\hat{x}) - A(x) + L(\hat{x})C(\hat{x}) - L(\hat{x})C(x)\big]x \end{aligned} \tag{6.3}$$

Now, let $\tilde{e} = [e, x]^T$, therefore, the system (6.3) can be re-written as below:

$$\tilde{e}(k+1) = \phi(x, \hat{x})\tilde{e} \tag{6.4}$$

where,

$$\phi(x, \hat{x}) = \begin{bmatrix} A(\hat{x}) + L(\hat{x})C(\hat{x}) & A(\hat{x}) - A(x) + L(\hat{x})C(\hat{x}) - L(\hat{x})C(x) \\ 0 & A(x) \end{bmatrix} \tag{6.5}$$

Theorem 6.1. *Consider the system (6.1), the error dynamics shown in (6.4) is asymptotically stable if there exist polynomial matrices $L(\hat{x})$ and $P(\tilde{e})$ such that the following conditions are satisfied:*

$$P(\tilde{e}) > 0 \tag{6.6}$$

$$\begin{bmatrix} P(\tilde{e}) & \phi^T(x,\hat{x})P^T(\tilde{e}(k+1)) \\ P(\tilde{e}(k+1))\phi(x,\hat{x}) & P(\tilde{e}(k+1)) \end{bmatrix} > 0 \tag{6.7}$$

Proof: The parameter-dependent Lyapunov function is selected as follows:

$$V(\tilde{e}) = \tilde{e}^T P(\tilde{e})\tilde{e} \tag{6.8}$$

The difference between $V(\tilde{e}(k+1))$ and $V(\tilde{e}(k))$ along (6.4) with (6.2) is given below:

$$\begin{aligned} \Delta(V(\tilde{e})) &= V(\tilde{e}(k+1)) - V(\tilde{e}) \\ &= \tilde{e}^T(k+1)P(\tilde{e}(k+1))\tilde{e}(k+1) - \tilde{e}^T P(\tilde{e})\tilde{e} \\ &= \tilde{e}^T\left[\phi^T(x,\hat{x})P(\tilde{e}(k+1))\phi(x,\hat{x}) - P(\tilde{e})\right]\tilde{e} \end{aligned} \tag{6.9}$$

Suppose (6.7) is feasible, then multiplying it to the left by $diag[I, P(\tilde{e}(k+1))]$ and to the right by $diag[I, P^T(\tilde{e}(k+1))]$ and by applying the Schur complement, we have

$$\phi^T(x,\hat{x})P(\tilde{e}(k+1))\phi(x,\hat{x}) - P(\tilde{e}) < 0 \tag{6.10}$$

Knowing that (6.10) holds, then $\Delta V(\tilde{e}) < 0$, which implies that the error dynamic (6.4) with the filter (6.2) is globally asymptotically stable. The proof ends. $\nabla\nabla\nabla$

Remark 6.2. Theorem 6.1 provides a sufficient condition for the existence of filter gains and is given in terms of solutions to a set of parameterised PMIs. However, notice that the $P(\tilde{e}(k+1))$ appears in the PMIs, therefore the inequalities are not jointly convex. One might choose to select the Lyapunov matrix to be of $P(e)$ instead of $P(\tilde{e})$. However, such a selection does not help the solution to be convex because the problem remains persistent. Hence, to directly solve Theorem 6.1 is hard because the PMIs need to be checked for all combination of $P(\tilde{e})$ and $L(\hat{x})$, which results in solving an infinite number of PMIs. In light of the aforementioned problem, in our work, it is proposed to incorporate an integrator into the filter dynamics. In doing so, a convex solution to the filter design problem for polynomial discrete-time systems can be rendered efficiently. The details of this integrator method are illustrated in the following section.

6.3 Main Results

In this section, the significance of incorporating an integrator into the filter dynamics will be illustrated.

The following nonlinear filter is proposed:

$$\begin{aligned}\hat{x}(k+1) &= A(\hat{x})\hat{x} + x_f \\ x_f(k+1) &= x_f + L(\hat{x})\big(C(\hat{x})\hat{x} - C(x)x\big)\end{aligned} \tag{6.11}$$

where, $\hat{x}$ is a filter state, x_f is an augmented filter state and $L(\hat{x})$ is a designed polynomial matrix. $A(\hat{x})$, $C(\hat{x})$ and $C(x)$ are all polynomial matrices in approproate dimensions.

Now, error is defined as follows:

$$\bar{e} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \hat{x} - x \\ x_f \end{bmatrix} \tag{6.12}$$

The error dynamics is then given by

$$\begin{aligned}\bar{x}(k+1) = \begin{bmatrix} e_1(k+1) \\ e_2(k+1) \end{bmatrix} &= \begin{bmatrix} \hat{x}(k+1) - x(k+1) \\ x_f(k+1) \end{bmatrix} \\ &= \begin{bmatrix} A(\hat{x})\hat{x} + x_f - A(x)x \\ x_f + L(\hat{x})\big(C(\hat{x})\hat{x} - C(x)x\big) \end{bmatrix} \\ &= \begin{bmatrix} A(\hat{x})e_1 + e_2 + \big(A(\hat{x}) - A(x)\big)x \\ e_2 + L(\hat{x})C(\hat{x})e_1 + \big(L(\hat{x})C(\hat{x}) - L(\hat{x})C(x)\big)x \end{bmatrix}\end{aligned} \tag{6.13}$$

Next, let define $\check{e} = [e_1, x, e_2]^T$, hence, the error dynamics described in (6.13) can be re-written as follows:

$$\check{e}(k+1) = \phi_2(x, \hat{x})\check{e} \tag{6.14}$$

where,

$$\phi_2(x, \hat{x}) = \begin{bmatrix} A(\hat{x}) & A(\hat{x}) - A(x) & 1 \\ L(\hat{x})C(\hat{x}) & L(\hat{x})C(\hat{x}) - L(\hat{x})C(x) & 1 \\ 0 & A(x) & 0 \end{bmatrix} \tag{6.15}$$

Theorem 6.3. *Consider the system (6.1), the error dynamics shown in (6.14) is asymptotically stable if there exist a symmetric polynomial matrix $P(e_1)$, polynomial matrices*

$K(\hat{x})$ and $G(\hat{x})$ such that the following conditions are satisfied:

$$P(e_1) > 0 \tag{6.16}$$

$$\begin{bmatrix} P(e_1) & \phi_2^T(x,\hat{x})G^T(\hat{x}) \\ G(\hat{x})\phi_2(x,\hat{x}) & G^T(\hat{x}) + G(\hat{x}) - P(e_1(k+1)) \end{bmatrix} > 0 \tag{6.17}$$

where,

$$G(\hat{x}) = \begin{bmatrix} G_{11}(\hat{x}) & G_{12}(\hat{x}) & G_{13}(\hat{x}) \\ G_{21}(\hat{x}) & G_{12}(\hat{x}) & G_{23}(\hat{x}) \\ G_{31}(\hat{x}) & G_{12}(\hat{x}) & G_{33}(\hat{x}) \end{bmatrix} \tag{6.18}$$

Therefore, the filter is given by

$$\begin{aligned} \hat{x}(k+1) &= A(\hat{x})\hat{x} + x_f \\ x_f(k+1) &= x_f + L(\hat{x})\big(C(\hat{x})\hat{x} - C(x)x\big) \end{aligned} \tag{6.19}$$

where,

$$L(\hat{x}) = K(\hat{x})G_{12}^{-1}(\hat{x}) \tag{6.20}$$

Proof: The Lyapunov function is selected as follows:

$$V(\check{e}) = \check{e}^T P(e_1)\check{e} \tag{6.21}$$

Then, the difference between $V(\check{e}(k+1))$ and $V(\check{e}(k))$ along (6.14) with (6.11) is given below:

$$\begin{aligned} \Delta(V(\check{e})) &= V(\check{e}(k+1)) - V(\check{e}) \\ &= \check{e}^T(k+1)P(e_1(k+1))\check{e}(k+1) - \check{e}^T P(e_1)\check{e} \\ &= \check{e}^T\big[\phi_2^T(x,\hat{x})P(e_1(k+1))\phi_2(x,\hat{x}) - P(\check{e})\big]\check{e} \end{aligned} \tag{6.22}$$

Suppose (6.17) is feasible, thus $G^T(\hat{x}) + G(\hat{x}) > P(e_1(k+1)) > 0$. This implies that $G(\hat{x})$ is nonsingular. Since $P(e_1(k+1))$ is positive definite, hence the inequality

$$\big(P(e_1(k+1)) - G(\hat{x})\big)P^{-1}(e_1(k+1))\big(P(e_1(k+1)) - G(\hat{x})\big)^T > 0 \tag{6.23}$$

holds. Therefore establishing

$$G(\hat{x}(k))P^{-1}(e_1(k+1))G^T(\hat{x}) \geq G(\hat{x}) + G^T(\hat{x}) - P(e_1(k+1)). \tag{6.24}$$

This immediately gives

$$\begin{bmatrix} P(e_1) & \phi_2^T(x,\hat{x})G^T(\hat{x}) \\ G(\hat{x})\phi_2(x,\hat{x}) & G^T(\hat{x})P^{-1}(e_1(k+1))G(\hat{x}) \end{bmatrix} > 0 \tag{6.25}$$

Next, by multiplying (6.25) on the right by $diag[I, G^{-1}(\hat{x}(k))]^T$ and on the left by $diag[I, G^{-1}(\hat{x}(k))]$, we get

$$\begin{bmatrix} P(e_1) & \phi_2^T(x,\hat{x}) \\ \phi_2(x,\hat{x}) & P^{-1}(e_1(k+1)) \end{bmatrix} > 0 \tag{6.26}$$

Then, by applying the Schur complement into (6.26), we have

$$\phi_2^T(x,\hat{x})P(e_1(k+1))\phi_2(x,\hat{x}) - P(e_1) < 0 \tag{6.27}$$

Knowing that (6.27) holds, then $\Delta V(\check{e}) < 0$, which implies that the error dynamic (6.14) with the filter (6.11) is globally asymptotically stable. The proof ends. ∇∇∇

Remark 6.4. One might wonder how the term $L(\hat{x}) = K(\hat{x})G_2^{-1}(\hat{x})$ can suddenly appear in Theorem 6.6. The fact is that a change-of-variable technique has been applied in the above proof, where $K(\hat{x}) = L(\hat{x})G_{12}(\hat{x})$. This is explicitly applied in Theorem 6.6. It is also important to note here that to allow the same value of $L(\hat{x})$ to be obtained, the polynomial matrix $G(\hat{x})$ must be enforced to be of a certain structure: see Equation (6.18). Although the $G(\hat{x})$ must be of a certain form, the results are still not too conservative because it is independent of the Lyapunov matrix.

Remark 6.5. The inequalities (6.17) of Theorem 6.6 are convex. This is true because the terms in $P(e_1(k+1))$ are jointly convex. For clarity, refer to the following expansion version of $P(e_1(k+1))$,

$$P(e_1(k+1)) = P\big[A(\hat{x})\hat{x} + x_f - A(x)x\big] \tag{6.28}$$

From (6.28), the x_f is an augmented state, hence the $P(e_1(k+1))$ provides a convex solution. Therefore, the Theorem 6.6 can possibly be solved via SDP.

Unfortunately, to solve Theorem 6.6 is hard because we need to solve an infinite set of state-dependent PMIs. To relax these conditions, we utilise a SOS decomposition approach [30] and therefore the conditions given in Theorem 6.6 can be converted into SOS conditions, which are given by the following corollary:

Corollary 6.6. *Consider the system* (6.1)*, the error dynamics shown in* (6.14) *is asymptotically stable if there exist a symmetric polynomial matrix* $P(e_1)$*, polynomial matrices* $K(\hat{x})$ *and* $G(\hat{x})$*, and positive constant* ϵ_1 *and* ϵ_2 *such that the following conditions are* *satisfied:*

$$v_1^T[P(e_1) - \epsilon_1 I]v_1 \qquad \textit{is a SOS} \tag{6.29}$$

$$v_2^T \begin{bmatrix} P(e_1) - \epsilon_2 I & \phi_2^T(x,\hat{x})G^T(\hat{x}) \\ G(\hat{x})\phi_2(x,\hat{x}) & G^T(\hat{x}) + G(\hat{x}) - P(e_1(k+1)) - \epsilon_2 I \end{bmatrix} v_2 \qquad \textit{is a SOS} \tag{6.30}$$

where, v_1 and v_2 are free vectors with appropriate dimensions, and

$$G(\hat{x}) = \begin{bmatrix} G_{11}(\hat{x}) & G_{12}(\hat{x}) & G_{13}(\hat{x}) \\ G_{21}(\hat{x}) & G_{12}(\hat{x}) & G_{23}(\hat{x}) \\ G_{31}(\hat{x}) & G_{12}(\hat{x}) & G_{33}(\hat{x}) \end{bmatrix} \tag{6.31}$$

Therefore, the filter is given by

$$\begin{aligned} \hat{x}(k+1) &= A(\hat{x})\hat{x} + x_f \\ x_f(k+1) &= x_f + L(\hat{x})\big(C(\hat{x})\hat{x} - C(x)x\big) \end{aligned} \tag{6.32}$$

where,

$$L(\hat{x}) = K(\hat{x})G_{12}^{-1}(\hat{x}) \tag{6.33}$$

Remark 6.7. It should be mentioned here that the above design procedures are dedicated to solve the full-order filter design problems. We only present the most fundamental filter design procedure without inclusion of any performance objectives or uncertainties. The above idea only provides a possible solution to the global filter design problem of polynomial discrete-time systems. Some conservatisms of the proposed method are given below:

- The slack polynomial matrix, $G(\hat{x})$ must be of a certain structure in order to achieve a feasible solution the problem. Although this conservativeness is not much severe because the slack polynomial matrix is independence form Lyapunov matrix, but it is still conservative in general.
- The computational complexity must be one of the important issues using this method. This is because the filter state that is proposed in this method has double of the order of an original state. Therefore, if we are dealing with the second order system, the the filter order must be 4th. It consequently will give burden to the computational aspects. The computational complexity becomes even worst because all matrices are in polynomial forms. Therefore, this method can only be applied to the low order of academic examples.

6.4 Numerical Example

The following polynomial system is considered:

$$x(k+1) = \begin{bmatrix} 1 & -0.01 \\ 0.01 + 0.01x_1x_2 & 1-0.01 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{6.34}$$

Then, by applying Corollary 6.6 where the $P(e_1)$ is set to be of degree of 2, polynomial matrix $G(\hat{x})$ is in degree of 4, and polynomial matrix $K(\hat{x})$ is set to be degree of 6, a feasible solution is obtained. The results of the error between the estimation state and the actual state can be seen in Figure 6.1 and 6.2. The initial condition for the actual state is $x(0) = [1 \quad 1]$ and for the estimation state is $\hat{x}(0) = [0.5 \quad 0.5]$.

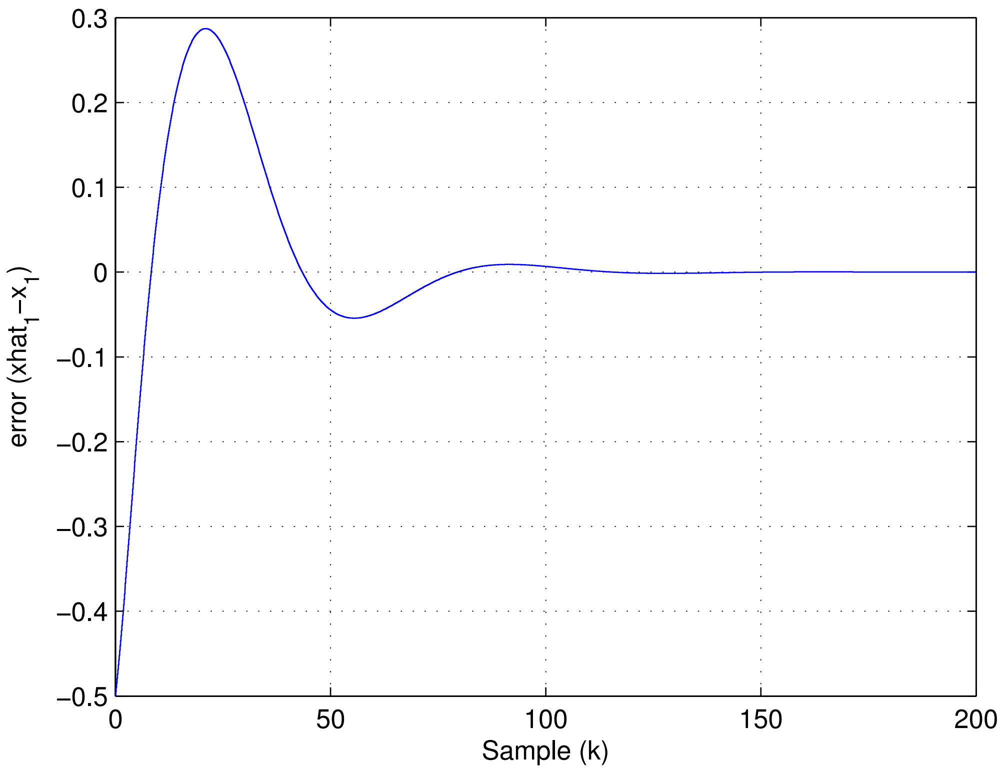

FIGURE 6.1: Trajectory of the $\hat{x}_1 - x_1$.

Remark 6.8. It should be mentioned here that the error between the actual state and the estimation state is quite large. This is due to the fact that the degree of the polynomial matrices $P(e_1)$, $G(\hat{x})$, and $K(\hat{x})$ cannot be increased further because of the limitation of the memory space of our machine. This significantly affect the feasibility of the solution because with the current set-up, the feasibility of the solution is very low which is 0.15.

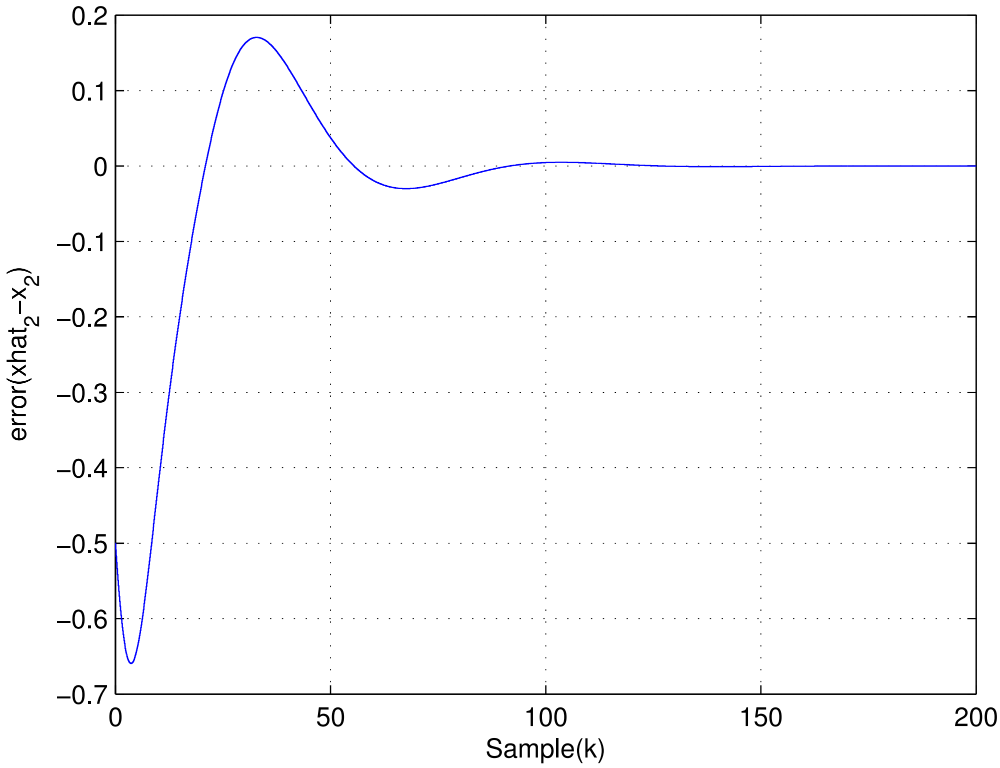

FIGURE 6.2: Trajectory of the $\hat{x}_2 - x_2$.

We believe that a better solution might be obtained by increasing the polynomial degree of the mentioned parameters, and consequently yield a better estimation.

Remark 6.9. The values of the polynomial matrices $P(e_1)$, $G(\hat{x})$, and $K(\hat{x})$ are omitted here due to the large sizes of them.

6.5 Conclusion

The filter design problem has been examined in this chapter. It has been shown that a convex solution to the problem can be obtained efficiently by incorporating an integrator into the filter structure. The existence of our proposed filter is given in terms of the solvability of the PMIs, which is formulated as SOS constraints and can be solved by any SOS solvers. In this work, SOSTOOLS has been used to solve the SOS-PMIs. The effectiveness of the proposed method is validated using a simulation example. Unlike the work performed in [82], our proposed methodology provides a solution to the global filter design. However, a current limitation of the proposed approach is the fact that these methods have been developed for the general class of polynomial discrete-time systems. Hence the computational costs are often high and the methods are usually only applicable to small academic examples. But this is also an opportunity for future research.

Chapter 7

Nonlinear H_∞ Output Feedback Control for Polynomial Discrete-time Systems

7.1 Introduction

In the chapter 2 to chapter 5, the results are under the assumption that the states are available for controller implementations which are not always true in many practical cases. Due to this reason, the static output feedback design has attracted much attention from the control practitioners. A comprehensive survey on static output feedback can be found in [83]. In [83] the authors prove that any dynamic output feedback problem can be transformed into a static output feedback problem. Hence, the static output feedback formulation provides a more general method than the full order dynamic output feedback and therefore, the static output formulation can be applied to design a full order dynamic controller. The converse, however, is not true.

Although the static output feedback control design for polynomial systems is not as widely studied as its linear counterpart, some significant attempts can still be found in [37–39]. In particular, to achieve a convex solution to the H_∞ control problem, a predefined upper bound has been introduced in [37] to limit the effect of the nonconvex term. However, this predefined upper bound is hard to determine beforehand, and the closed-loop stability can only be guaranteed within a bound region. On the other hand, in [38, 39], the existence of a nonlinear static output feedback control law is given by the solvability of polynomial matrix inequalities (PMIs), which are formulated as SOS constraints. To solve the nonconvexity that exists in the PMIs, an iterative algorithm based on the SOS decomposition has been developed. Unfortunately, it is hard to determine the first value of a slack variable matrix, $\epsilon(x)$ because it is unknown. The $\epsilon(x)$ value plays a vital role in this approach because it determines the feasibility of the problem. There is no unique way to choose this value, thus it is difficult to apply this approach.

The above-mentioned results are concerned with the polynomial continuous-time systems. To the best of the author's knowledge, when it comes to the polynomial discrete-time systems, no general result has been presented yet. However, closely related results can be found in [84, 85]. In this work, the nonlinear discrete-time systems have been approximated by Takagi-Sugeno (TS) fuzzy models, which are locally linear models connected by IF-THEN rules. To be specific, in [84, 85], a convex solution to the H_∞ control problem is achieved by selecting the Lyapunov function to be of a rational form and introducing a transformation variable that is coupled with the system output matrices. In doing so, a change-of-variable is able to be applied to the bilinear term. To note here: in [84], a quadratic Lyapunov function is used to analyse the stability of such TS fuzzy systems. It is well-known that the quadratic Lyapunov function is always inadequate to resolve the problem of nonlinear systems or linear systems with structured uncertainties [35, 60]. This is the main drawback of the quadratic Lyapunov function based approach. Motivated by this fact, [85] uses a parameter-dependent Lyapunov function and therefore this allows them to extend their approach to robust control synthesis for nonlinear systems. But, by employing the parameter-dependent approach, the consequence is that the solution to the H_∞ control problem is no longer convex. Hence, in the present paper, to render a convex solution to the problem, the Lyapunov matrix is enforced to be diagonal in structure. This selection might lead to conservative results. Furthermore, in the TS fuzzy model, the premise variables are assumed to be bounded. In general, the premise variables are related to the state variables which implies that the state variables have also to be bounded. This is one of the major drawbacks of the TS fuzzy model approach.

In this chapter, motivated by the aforementioned problems and the results in [84, 85], the problem of H_∞ control using a static output feedback controller for a class of polynomial discrete-time systems is studied. In particular, we address the problem of H_∞ control in which both stability and a prescribed H_∞ performance are required to be fulfilled. To be specific, the polynomial discrete-time system is represented in the form of state-dependent linear-like form, and a state-dependent polynomial Lyapunov function is used to represent the Lyapunov candidate. Attention is focused on the design of the nonlinear static output feedback controller with an integrator to stabilise such discrete-time systems and to ensure that the prescribed level of the H_∞ performance is fulfilled. By incorporating the integrator into the controller structure, the original system can be transformed into the augmented system and the Lyapunov matrix can be chosen to be dependent upon the original states only. Through this, the static output synthesis problem of polynomial discrete-time systems can be convexified in a less conservative way and can be solved computationally via SDP. The existence of the controller is given in terms of the solvability conditions of polynomial matrix inequalities (PMIs) which are formulated as SOS constraints and solved using SOSTOOLS [53]. It is important to note here that the resulting controller gains are in the rational matrix functions of the system output matrices and the additional augmented state. The results are then directly extended to the robust H_∞ output feedback control with polytopic uncertainty.

In comparison with the existing method of a static output feedback controller design for nonlinear discrete-time systems, there are two features of our proposed approach which deserve attention:

1. By introducing an integrator into the controller structure, a less conservative result can be obtained. This is because the nonconvex term that exists between the Lyapunov function and the controller matrix due to the utilisation of the state-dependent Lyapunov function can be convexified in a less conservative way than the available approaches. To be precise, by incorporating the integrator into the controller structure, the Lyapunov matrix does not need to be of a certain structure to render a convex solution. In contrast, to achieve this, [85] has to enforce the Lyapunov matrix to be in a diagonal form. This condition might give a conservative result. On the other hand, a predefined upper bound has been proposed in [37, 44] to limit the effect of the nonconvex term. However, this predefined upper bound is hard to determine beforehand, and the results just only hold within a bound region.
2. The Lyapunov function does not need to be of a rational form and no additional transformation matrix is needed in order to apply a change-of-variable technique to the bilinear term as required in [85].

7.2 System Description and Preliminaries

The following polynomial discrete-time system is considered,

$$\left.\begin{aligned} x(k+1) &= A(x(k))x(k) + B_u(x(k))u(k) + B_\omega(x(k))\omega(k) \\ z(k) &= C_z(x(k))x(k) + D_{zu}(x(k))u(k) \\ y(k) &= C_y(x(k))x(k) \end{aligned}\right\} \tag{7.1}$$

where $x(k) \in R^n$ is the state vectors, $u(k) \in R^m$ is the input and $y(k)$ is the measured output. $A(x(k))$, $B_u(x(k))$, $C_z(x(k))$, $D_{zu}(x(k))$ and $C_y(x(k))$ are polynomial matrices of appropriate dimensions. Meanwhile $z(k)$ is a vector of output signals related to the performance of the control system. $\omega(k)$ is the disturbance which belongs to $L_2[0,\infty]$.

For the polynomial discrete-time system described in (7.1), a static output feedback controller is proposed as

$$u(k) = K(y)y(k) \tag{7.2}$$

Before presenting the main result, the following lemma is needed,

Lemma 7.1. *[44, 60] The system (7.1) without disturbance i.e $\omega(k) = 0$ is asymptotically stable if*

1. *There exist a positive definite symmetric polynomial matrix, $P(x(k))$ and polynomial matrix, $K(x(k))$ such that*

$$\begin{bmatrix} P(x(k)) & * \\ P(x_+)A(x(k)) + P(x_+)B(x(k))K(x(k)) & P(x_+) \end{bmatrix} > 0, \tag{7.3}$$

or

2. *There exist a positive definite symmetric polynomial matrix, $P(x(k))$, polynomial matrix, $K(x(k))$ and polynomial slack matrix, $G(x(k))$ such that*

$$\begin{bmatrix} P(x(k)) & * \\ G(x(k))A(x(k)) + G(x(k))B(x(k))K(x(k)) & G(x(k)) + G^T(x(k)) - P(x_+) \end{bmatrix} > 0. \tag{7.4}$$

Proof: The detail proof can be found in [44, 60], hence omitted.

Remark 7.2. Lemma 7.1 shows that by utilising a slack variable technique i.e introducing a slack polynomial matrix, $G(x(k))$), a less conservative result can be obtained [44, 60]. This is because the Lyapunov function can be decoupled from the system matrices and therefore the controller design is independence of Lyapunov matrix. The controller design is now dependent upon the slack polynomial matrix. By employing such a slack variable technique, the parameter-dependent Lyapunov function has been used for linear uncertain systems [60] and the state-dependent Lyapunov function has been utilised for polynomial systems [44]. Although the above lemma provides a solution for the state feedback control, the method has also been applied in the framework of static output feedback control designs as shown in [84, 85]. Based on the slack variable technique and the state-dependent Lyapunov function, we attempt to derive a new method which provides a less conservative design procedure for designing a nonlinear H_∞ output feedback controller for polynomial discrete-time systems. This is delivered by incorporating an integrator into the controller structure.

7.3 Main Results

We begin this section by highlighting the problem of designing an H_∞ output feedback controller for polynomial discrete-time systems when a state-dependent Lyapunov function is under consideration. Then, a novel method is proposed to overcome that problem. Based on this novel method, a solution to the nonlinear H_∞ output feedback control problem is given. The results are subsequently extended to the robust H_∞ control with polytopic uncertainty.

7.3.1 Nonlinear H_∞ output feedback control

The following theorem provides sufficient conditions for the existence of a nonlinear static output feedback controller (7.2) for the system (7.1) without disturbance i.e $\omega(k) = 0$.

Theorem 7.3. *The system* (7.1) *without disturbance i.e* $\omega(k) = 0$ *is stabilisable asymptotically via the static output feedback control of the form* (7.2) *if there exist a symmetric polynomial matrix,* $P(x(k))$*, polynomial matrices* $K(y)$ *and* $G(x(k))$ *such that the following conditions are satisfied for all* $x \neq 0$

$$P(x(k)) > 0 \tag{7.5}$$

$$\begin{bmatrix} P(x(k)) & * \\ G(x(k))A(x(k)) & \\ +G(x(k))B_u(x(k))K(y)C_y(x(k)) & G(x(k)) + G^T(x(k)) - P(x_+) \end{bmatrix} > 0 \tag{7.6}$$

Proof: Select a Lyapunov function of the form

$$V(x(k)) = x^T(k)P(x(k))x(k) \tag{7.7}$$

The difference of the Lyapunov function (7.7) along the system (7.1) with (7.2) for $\omega(k) = 0$ is given by

$$\begin{aligned}\Delta V(x(k)) &= V(x(k+1)) - V(x(k)) < 0\\ &= \big(A(x(k))x(k) + B_u(x(k))K(y)C_y(x(k))x(k)\big)^T P(x_+)\big(A(x(k))x(k)\\ &+ B_u(x(k))K(y)C_y(x(k))x(k)\big) - x^T(k)P(x(k))x(k)\\ &= x^T(k)\big[\big(A(x(k)) + B_u(x(k))K(y)C_y(x(k))\big)^T P(x_+)\big(A(x(k))\\ &+ B_u(x(k))K(y)C_y(x(k))\big) - P(x(k))\big]x(k). \end{aligned} \tag{7.8}$$

Suppose (7.6) holds, then by Lemma 7.1, we have

$$\begin{bmatrix} P(x(k)) & * \\ P(x_+)A(x(k)) + P(x_+)B_u(x(k))K(y)C_y(x(k)) & P(x_+) \end{bmatrix} > 0. \tag{7.9}$$

Next, multiply (7.9) on the left by $diag[I, P^{-1}(x_+)]$ and on the right by $diag[I, P^{-1}(x_+)]^T$, we get

$$\begin{bmatrix} P(x(k)) & * \\ A(x(k)) + B_u(x(k))K(y)C_y(x(k)) & P^{-1}(x_+) \end{bmatrix} > 0. \tag{7.10}$$

Then, by applying the Schur complement to (7.10), we obtain

$$\begin{aligned}\big[(A(x(k)) + B_u(x(k))K(y)C_y(x(k)))^T P(x_+)\big(A(x(k)) + B_u(x(k))K(y)C_y(x(k))\big)\\ - P(x(k))\big] < 0. \end{aligned} \tag{7.11}$$

Knowing that (7.11) holds, we have $\Delta V(x(k)) < 0, \forall x \neq 0$, which implies that the system (7.1) with (7.2) is asymptotically stable. The proof completes. $\nabla\nabla\nabla$

It is worth mentioning that conditions given in Theorem 7.3 are in terms of state-dependent PMIs. Thus, solving this inequality is computationally hard because one needs to solve an infinity set of state-dependent PMIs. To relax these conditions, we utilise the SOS decomposition approach and semidefinite programming as described in [30, 57] where the conditions in Theorem 7.3 can be solved by parameterizing $P(x(k))$ and $K(y)$ in a proper polynomial form. Moreover, to render conditions given in Theorem 7.3 into tractable SOS conditions, it is often necessary to include some SOS constraints i.e $\epsilon > 0$ [57]. Therefore the static output feedback stabilisation conditions given in Theorem 7.3 can be modified into SOS conditions and they are given by the following proposition:

Proposition 7.4. *The system* (7.1) *without disturbance i.e* $\omega(k) = 0$ *is asymptotically stable via static output feedback controller* (7.2) *if there exist a symmetric polynomial* *matrix, $P(x(k))$, polynomial matrices $K(y)$ and $G(x(k))$, positive constants $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that the following conditions hold for all $x \neq 0$*

$$v_1^T[P(x(k)) - \epsilon_1 I]v_1 \qquad \textit{is a SOS} \tag{7.12}$$

$$v_2^T\left[M(x(k)) - \epsilon_2 I\right]v_2 \qquad \textit{is a SOS} \tag{7.13}$$

where,

$$M(x(k)) = \begin{bmatrix} P(x(k)) & * \\ G(x(k))A(x(k)) + G(x(k))B_u(x(k))K(y)C_y(x(k)) & G(x(k)) + G^T(x(k)) - P(x_+) \end{bmatrix}, \tag{7.14}$$

and v_1 and v_2 are free vectors in appropriate dimensions.

Proof: The proof follows directly from the proof shown in Theorem 7.3. In addition, knowing that the equations given in (7.5) and (7.6) are in symmetric form, we can apply Proposition 1.2 (statement 1 and statement 2). Therefore, if the Proposition 7.4 holds, it implies that Theorem 7.3 is true. The proof ends. $\nabla\nabla\nabla$

Remark 7.5. Unfortunately, Proposition 7.4 cannot be solved easily by SDP because

1. A change-of-variable technique cannot be applied directly to the bilinear term, $G(x(k))B_u(x(k))K(y)$ due to the existence of $B_u(x(k))$ in between of the additional slack variable matrix, $G(x(k))$ and the controller matrix, $K(y)$. One possible way to solve this problem is by imposing the Lyapunov function to be of the rational form, and introducing a transformation variable, T such that $C_yT = [I, 0]$ [84, 85]. Then, by enforcing the slack variable matrix to be of a certain form, the change-of-variable technique can be applied to the bilinear term accordingly. However, the combination of C_y and T is not unique, hence it is difficult to choose a suitable candidate for T. In light of this method, it is not hard to see that the change-of-variable technique can be applied to the $G(x(k))B_u(x(k))K(y)$ of (7.14) easily by forcing the $B_u(x(k))$ to be $[0, 1]^T$, and choosing the $G(x(k))$ to be of a certain form (as shown in [85]). In doing so, a change-of-variable can be applied to the bilinear term but the result becomes more conservative because the input matrix, $B_u(x(k))$ must always be $[0, 1]^T$.

2. The terms in $P(x_+)$ are not jointly convex. This is true because if we expand it, we have

$$\begin{aligned} P(x_+) &= P(x(k+1)) \\ &= P(A(x(k))x(k) + B_u(x(k))K(y)C(x(k))x(k)) \end{aligned} \tag{7.15}$$

   It is obvious from (7.15) that the terms in there are not jointly convex, hence it is hard to search for these values simultaneously and therefore it is hard to solve (7.15) - it is equivalent to solving some bilinear matrix inequalities (BMIs). It has been shown in [44] that one possible way to convexify this problem is by introducing a predefined upper bound to limit the effect of the nonconvex term. This predefined upper bound, however, is hard to be determined beforehand, and the closed-loop stability can only be guaranteed within a bound region. Another possible solution is by selecting the Lyapunov function to be of a quadratic form as applied in [84]. But it is well-known that the quadratic Lyapunov function is always inadequate to solving nonlinear systems.

Motivated by the above-mentioned problems and the results in [84, 85], we introduce an integrator into the static output controller structure. In doing this, the problems mentioned in Remark 7.5 can be resolved in a less conservative way than the available approaches.

Our proposed controller with an integrator is given as follows:

$$\begin{aligned} x_c(k+1) &= x_c(k) + A_c(y, x_c) \\ u(k) &= x_c(k) \end{aligned} \tag{7.16}$$

where x_c is the controller state, $u(k)$ is the input to the system and $A_c(y, x_c)$ is the input function of the integrator.

The system (7.1) with the controller (7.16) can be described as follows:

$$\left.\begin{aligned} \hat{x}(k+1) &= \hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}_u(\hat{x}(k))A_c(y, x_c) + \hat{B}_\omega(\hat{x}(k))\omega(k) \\ z(k) &= \hat{C}_z(\hat{x}(k))\hat{x}(k) \\ y(k) &= \hat{C}_y(\hat{x}(k))\hat{x}(k) \end{aligned}\right\} \tag{7.17}$$

where,

$$\hat{A}(\hat{x}(k)) = \begin{bmatrix} A(x(k)) & B(x(k)) \\ 0 & 1 \end{bmatrix}; \quad \hat{B}_u(\hat{x}(k)) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad \hat{B}_\omega(\hat{x}(k)) = \begin{bmatrix} B_\omega(x(k)) \\ 0 \end{bmatrix};$$
$$\hat{C}_z(\hat{x}(k)) = \begin{bmatrix} C_z(x(k)) & D_{zu}(x(k)) \end{bmatrix}; \quad \hat{C}_y(\hat{x}(k)) = \begin{bmatrix} C_y(x(k)) & 0 \end{bmatrix};$$
$$\hat{x} = \begin{bmatrix} x(k) \\ x_c(k) \end{bmatrix}. \tag{7.18}$$

Next, we assume $A_c(y, x_c)$ to be of the form

$$A_c(k) = \hat{K}(y, x_c)y. \tag{7.19}$$

where, $\hat{K}(y, x_c)$ is a polynomial matrix of dimensions $(n+1)\times 1$. n is an original system state number. Therefore, (7.17) can be re-written as follows:

$$\left.\begin{aligned} \hat{x}(k+1) &= \hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}_u(\hat{x}(k))\hat{K}(y, x_c)y + \hat{B}_\omega(\hat{x}(k))\omega(k) \\ z(k) &= \hat{C}_z(\hat{x}(k))\hat{x}(k) \\ y(k) &= \hat{C}_y(\hat{x}(k))\hat{x}(k) \end{aligned}\right\} \tag{7.20}$$

Here, the objective is to design a nonlinear output feedback controller of the form (7.16) such that for a given prescribed H_∞ performance $\gamma > 0$, the

$$\|z(k)\|_{[0,\infty]} \leq \gamma_2 \|\omega(k)\|_{[0,\infty]} \tag{7.21}$$

is fulfilled and the system in (7.1) with (7.16) is globally asymptotically stable.

Now, we are ready to present our main result. The sufficient conditions for the existence of our proposed controller (7.16) for the system (7.1) without disturbance i.e $\omega(k) = 0$ are given in the following corollary:

Corollary 7.6. *The system (7.1) without disturbance i.e $\omega(k) = 0$ is asymptotically stable via the nonlinear output feedback controller (7.16) if there exist a symmetric polynomial matrix, $\hat{P}(x(k))$, a polynomial function $L_{31}(y, x_c)$, a polynomial matrix $\hat{G}(\hat{x}(k))$, and positive constants $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that the following conditions hold for all $x \neq 0$*

$$v_3^T[\hat{P}(x(k)) - \epsilon_1 I]v_3 \quad \text{is a SOS} \tag{7.22}$$
$$v_4^T\left[M(\hat{x}(k)) - \epsilon_2 I\right]v_4 \quad \text{is a SOS} \tag{7.23}$$

where, v_3 and v_4 are free vectors in appropriate dimensions, and

$$M(\hat{x}(k)) = \begin{bmatrix} \hat{P}(x(k)) & * \\ \hat{G}(\hat{x}(k))\hat{A}(\hat{x}(k)) + \hat{L}(y,x_c)\hat{C}_y(\hat{x}(k)) & \hat{G}(\hat{x}(k)) + \hat{G}^T(\hat{x}(k)) - \hat{P}(x_+) \end{bmatrix}, \tag{7.24}$$

with

$$\hat{G}(\hat{x}(k)) = \begin{bmatrix} G_{11}(\hat{x}(k)) & G_{12}(\hat{x}(k)) & 0 \\ G_{21}(\hat{x}(k)) & G_{22}(\hat{x}(k)) & 0 \\ G_{31}(\hat{x}(k)) & G_{32}(\hat{x}(k)) & G_{33}(y,x_c) \end{bmatrix}; \quad \hat{L}(y,x_c) = \begin{bmatrix} 0 \\ 0 \\ L_{31}(y,x_c) \end{bmatrix}. \tag{7.25}$$

Moreover, the nonlinear output feedback controller is given by

$$\begin{aligned} x_c(k+1) &= x_c(k) + A_c(y,x_c) \\ u(k) &= x_c(k) \end{aligned} \tag{7.26}$$

where

$$A_c(y,x_c) = \hat{K}(y,x_c)\hat{C}_y(\hat{x}(k))\hat{x}(k), \quad \textit{with} \quad \hat{K}(y,x_c) = L_{31}(y,x_c)G_{33}^{-1}(y,x_c). \tag{7.27}$$

Proof: The Lyapunov function is selected to be of the form

$$\hat{V}(\hat{x}(k)) = \hat{x}^T(k)\hat{P}(x(k))\hat{x}(k), \tag{7.28}$$

and we let

$$\begin{aligned} \hat{L}(\hat{x}(k)) &= \begin{bmatrix} 0 \\ 0 \\ L_{31}(y,x_c) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ G_{33}(y,x_c)\hat{K}(y,x_c) \end{bmatrix} \\ &= \begin{bmatrix} G_{11}(\hat{x}(k)) & G_{12}(\hat{x}(k)) & 0 \\ G_{21}(\hat{x}(k)) & G_{22}(\hat{x}(k)) & 0 \\ G_{31}(\hat{x}(k)) & G_{32}(\hat{x}(k)) & G_{33}(y,x_c) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \hat{K}(y,x_c) \end{bmatrix} \\ &= \begin{bmatrix} G_{11}(\hat{x}(k)) & G_{12}(\hat{x}(k)) & 0 \\ G_{21}(\hat{x}(k)) & G_{22}(\hat{x}(k)) & 0 \\ G_{31}(\hat{x}(k)) & G_{32}(\hat{x}(k)) & G_{33}(y,x_c) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \hat{K}(y,x_c) \\ &= \hat{G}(\hat{x}(k))\hat{B}_u(\hat{x}(k))\hat{K}(y,x_c). \end{aligned} \tag{7.29}$$

Then, by applying a similar technique of the proof shown in Theorem 7.3, it is trivial to show that we have the following inequalities:

$$\hat{P}(x(k)) > 0, \quad \text{and}$$
$$M(\hat{x}(k)) > 0. \tag{7.30}$$

where, $M(\hat{x}(k))$ is as described in (7.24). Furthermore, by utilising Proposition 1.2 we can show that if the inequalities (7.22)-(7.23) are SOS, then the inequalities given in (7.30) hold. Hence, the proof is completed. ∇∇∇

Remark 7.7. Note that our proposed controller $\hat{K}(y, x_c)$ is dependant upon the augmented system output matrices. This is to ensure a feasible solution to the problem can be obtained.

Remark 7.8.

1. In order to allow us to apply a change-of-variable technique to the bilinear term $\hat{G}(\hat{x}(k))\hat{B}_u(\hat{x}(k))\hat{K}(y, x_c)$, we follow the method shown in [84, 85], but, our method provides a less conservative way because the Lyapunov function does not need to be of a rational form and no transformation matrix is required. This is due to the fact that our $\hat{B}_u(\hat{x}(k))$ is always in the vector of $[0, 0, 1]^T$. In addition, the control matrix, $B_u(x(k))$ is now governed in the system matrices, hence the Lyapunov matrix does not need to be of a special form to render a convex solution, that is, the Lyapunov function does not need to depend upon the system's states whose corresponding rows in the control matrix are zeros. Therefore, our method produces a more general result than [44].

2. The term $P(x_+)$ is now in a convex form. To explain this, refer to our proposed Lyapunov matrix (7.28), where $\hat{P}(x(k))$ is only dependent upon the original system matrices. Hence, if we expand the term $P(x_+)$, we have

$$\begin{aligned}\hat{P}(x_+) &= \hat{P}(x(k+1))\\ &= \hat{P}(A(x(k))x(k) + B(x(k))u(k))\\ &= \hat{P}(A(x(k))x(k) + B(x(k))x_c(k))\end{aligned} \tag{7.31}$$

Now it is not hard to see from (7.31) that the terms in there are jointly convex because $x_c(k)$ is an augmented state, and therefore allow us to possibly solve Corollary 7.6 via SDP.

Theorem 7.9. *Given a prescribed H_∞ performance, $\gamma > 0$, the system (7.1) is asymptotically stable via the nonlinear output feedback controller (7.16) with H_∞ performance (7.21) if there exist a symmetric polynomial matrix, $\hat{P}(x(k))$, a polynomial function $L_{31}(y, x_c)$ and a polynomial matrix $\hat{G}(\hat{x}(k))$ such that the following conditions hold for* *all* $x \neq 0$

$$\hat{P}(x(k)) > 0 \tag{7.32}$$

$$M_2(\hat{x}(k)) > 0 \tag{7.33}$$

where

$$M_2(\hat{x}(k)) = \begin{bmatrix} \hat{P}(x(k)) & * & * & * \\ 0 & \gamma^2 I & * & * \\ \hat{G}(\hat{x}(k))\hat{A}(\hat{x}(k)) + \hat{L}(y,x_c)\hat{C}_y(\hat{x}(k)) & \hat{B}_\omega(\hat{x}(k)) & \hat{G}(\hat{x}(k)) + \hat{G}^T(\hat{x}(k)) - \hat{P}(x_+) & * \\ \hat{C}_z(\hat{x}(k)) & 0 & 0 & I \end{bmatrix}, \tag{7.34}$$

with

$$\hat{G}(\hat{x}(k)) = \begin{bmatrix} G_{11}(\hat{x}(k)) & G_{12}(\hat{x}(k)) & 0 \\ G_{21}(\hat{x}(k)) & G_{22}(\hat{x}(k)) & 0 \\ G_{31}(\hat{x}(k)) & G_{32}(\hat{x}(k)) & G_{33}(y,x_c) \end{bmatrix}; \quad \hat{L}(y,x_c) = \begin{bmatrix} 0 \\ 0 \\ L_{31}(y,x_c) \end{bmatrix}. \tag{7.35}$$

Moreover, the nonlinear output feedback controller is given by

$$\begin{aligned} x_c(k+1) &= x_c(k) + A_c(y,x_c) \\ u(k) &= x_c(k) \end{aligned} \tag{7.36}$$

where

$$A_c(y,x_c) = \hat{K}(y,x_c)\hat{C}_y(\hat{x}(k))\hat{x}(k) \quad \textit{with} \quad \hat{K}(y,x_c) = L_{31}(y,x_c)G_{33}^{-1}(y,x_c). \tag{7.37}$$

Proof: Based on the Lyapunov function given in (7.28), the $V(x(k+1)) - V(x(k))$ along (7.20) is given by

$$\begin{aligned} &\Delta\hat{V}(\hat{x}(k)) = \hat{V}(\hat{x}(k+1)) - \hat{V}(\hat{x}(k)) \\ &= \hat{x}^T(k+1)\hat{P}(x_+)\hat{x}(k+1) - \hat{x}^T(k)\hat{P}(x(k))\hat{x}(k) \\ &= \big(\hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}_u(\hat{x}(k))\hat{K}(y,x_c)\hat{C}_y(\hat{x}(k))\hat{x}(k) + B_\omega(x(k))\omega(k)\big)^T \hat{P}(x_+)\big(\hat{A}(\hat{x}(k))\hat{x}(k) \\ &+ \hat{B}_u(\hat{x}(k))\hat{K}(y,x_c)\hat{C}_y(\hat{x}(k))\hat{x}(k) + B_\omega(x(k))\omega(k)\big) - \hat{x}^T(k)\hat{P}(x(k))\hat{x}(k). \end{aligned} \tag{7.38}$$

Furthermore, adding and subtracting $-z^T(k)z(k) + \gamma^2\omega^T(k)\omega(k)$ to and from (7.38), results in

$$\begin{aligned}\Delta\hat{V}(\hat{x}(k)) &= \big(\hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}_u(\hat{x}(k))\hat{K}(y,x_c)\hat{C}_y(\hat{x}(k))\hat{x}(k) + B_\omega(x(k))\omega(k)\big)^T\hat{P}(x_+)\\ &\big(\hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}_u(\hat{x}(k))\hat{K}(y,x_c)\hat{C}_y(\hat{x}(k))\hat{x}(k) + B_\omega(x(k))\omega(k)\big) - \hat{x}^T(k)\hat{P}(x(k))\hat{x}(k)\\ &- z^T(k)z(k) + \gamma^2\omega^T(k)\omega(k) + z^T(k)z(k) - \gamma^2\omega^T(k)\omega(k).\end{aligned} \tag{7.39}$$

Knowing that $z(k) = \hat{C}_z(\hat{x}(k)\hat{x}(k))$, then (7.39) becomes

$$\begin{aligned}&\Delta\hat{V}(\hat{x}(k))\\ &= \big(\hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}_u(\hat{x}(k))\hat{K}(y,x_c)\hat{C}_y(\hat{x}(k))\hat{x}(k) + B_\omega(x(k))\omega(k)\big)^T\hat{P}(x_+)\\ &\big(\hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}_u(\hat{x}(k))\hat{K}(y,x_c)\hat{C}_y(\hat{x}(k))\hat{x}(k) + B_\omega(x(k))\omega(k)\big) - \hat{x}^T(k)\hat{P}(x(k))\hat{x}(k)\\ &+ (\hat{C}_z(\hat{x}(k))\hat{x}(k))^T(\hat{C}_z(\hat{x}(k))\hat{x}(k)) - \gamma^2\omega^T(k)\omega(k) - z^T(k)z(k) + \gamma^2\omega^T(k)\omega(k).\end{aligned} \tag{7.40}$$

Now, (7.40) can be re-written as follows:

$$\Delta\hat{V}(\hat{x}(k)) = \hat{X}^T(k)\Omega(\hat{x}(k))\hat{X}(k) - z^T(k)z(k) + \gamma^2\omega^T(k)\omega(k) \tag{7.41}$$

where

$$\Omega(\hat{x}(k)) = \phi_1(\hat{x}(k))^T P(x_+)\phi_1(\hat{x}(k)) + \phi_2(\hat{x}(k))^T\phi_2(\hat{x}(k)) - \Xi$$

with

$$\begin{aligned}\phi_1(\hat{x}(k)) &= \begin{bmatrix}\hat{A}(\hat{x}(k)) + \hat{B}_u(\hat{x}(k))\hat{K}(y,x_c)\hat{C}_y(\hat{x}(k)) & \hat{B}_\omega(\hat{x}(k))\end{bmatrix};\\ \phi_2(\hat{x}(k)) &= \begin{bmatrix}\hat{C}_z(\hat{x}(k)) & 0\end{bmatrix}; \quad \hat{X}(k) = \begin{bmatrix}x(k)\\ \omega(k)\end{bmatrix}; \quad \Xi = \begin{bmatrix}\hat{P}(x(k)) & 0\\ 0 & \gamma^2\end{bmatrix}.\end{aligned}$$

Now, we need to show that $\hat{X}^T(k)\Omega(\hat{x}(k))\hat{X}(k) < 0$. To show this, suppose (7.33) is feasible. Then from the block $(3,3)$ of (7.34), we have $\hat{G}(\hat{x}(k)) + \hat{G}^T(\hat{x}(k)) > \hat{P}(x_+) > 0$. This implies that $\hat{G}(\hat{x}(k))$ is nonsingular, and since $\hat{P}(x_+)$ is positive definite, hence

$$\big(\hat{P}(x_+) - \hat{G}(\hat{x}(k))\big)\hat{P}^{-1}(x_+)\big(\hat{P}(x_+) - \hat{G}(\hat{x}(k))\big)^T > 0 \tag{7.42}$$

holds. Therefore establishing

$$\hat{G}(\hat{x}(k))\hat{P}^{-1}(x_+)\hat{G}^T(\hat{x}(k)) \geq \hat{G}(\hat{x}(k)) + \hat{G}^T(\hat{x}(k)) - \hat{P}(x_+). \tag{7.43}$$

This immediately gives

$$\begin{bmatrix} \hat{P}(x(k)) & * & * & * \\ 0 & \gamma^2 I & * & * \\ \hat{G}(\hat{x}(k))\hat{A}(\hat{x}(k)) + \hat{L}(\hat{x}(k))\hat{C}_y(x(k)) & \hat{B}_\omega(\hat{x}(k)) & \hat{G}(\hat{x}(k))\hat{P}^{-1}(x_+)\hat{G}^T(\hat{x}(k)) & * \\ \hat{C}_z(\hat{x}(k)) & 0 & 0 & I \end{bmatrix} > 0 \tag{7.44}$$

On the other hand, from (7.35) and (7.37) and with the fact that $\hat{B}_u(\hat{x}(k))$ is always $[0, 0, 1]^T$, we have

$$\begin{aligned} \hat{L}(\hat{x}(k)) &= \begin{bmatrix} 0 \\ 0 \\ L_{31}(y, x_c) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ S_{33}(y, x_c)\hat{K}(y, x_c) \end{bmatrix} \\ &= \begin{bmatrix} G_{11}(\hat{x}(k)) & G_{12}(\hat{x}(k)) & 0 \\ G_{21}(\hat{x}(k)) & g_{22}(\hat{x}(k)) & 0 \\ G_{31}(\hat{x}(k)) & G_{32}(\hat{x}(k)) & G_{33}(y, x_c) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \hat{K}(y, x_c) \end{bmatrix} \\ &= \begin{bmatrix} G_{11}(\hat{x}(k)) & G_{12}(\hat{x}(k)) & 0 \\ G_{21}(\hat{x}(k)) & G_{22}(\hat{x}(k)) & 0 \\ G_{31}(\hat{x}(k)) & G_{32}(\hat{x}(k)) & G_{33}(y, x_c) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \hat{K}(y, x_c) \\ &= \hat{G}(\hat{x}(k))\hat{B}_u(\hat{x}(k))\hat{K}_y(y, x_c). \end{aligned} \tag{7.45}$$

Next, by multiplying (7.44) on the right by $diag[I, I, G^{-1}(\hat{x}(k)), I]^T$ and on the left by $diag[I, I, G^{-1}(\hat{x}(k)), I]$, we get

$$\begin{bmatrix} \hat{P}(x(k)) & * & * & * \\ 0 & \gamma^2 I & * & * \\ \hat{A}(\hat{x}(k)) + \hat{B}_u(x(k))\hat{K}(y, x_c)\hat{C}_y(x(k)) & \hat{B}_\omega(\hat{x}(k)) & \hat{P}^{-1}(x_+) & * \\ \hat{C}_z(\hat{x}(k)) & 0 & 0 & I \end{bmatrix} > 0, \tag{7.46}$$

and similarly,

$$\begin{bmatrix} \Xi & \phi_1(\hat{x}(k))^T & \phi_2(\hat{x}(k))^T \\ \phi_1(\hat{x}(k)) & \hat{P}^{-1}(x_+) & 0 \\ \phi_2(\hat{x}(k)) & 0 & I \end{bmatrix} > 0. \tag{7.47}$$

Applying the Schur complement to (7.47), we have

$$\left(\phi_1(\hat{x}(k))^T P(x_+)\phi_1(\hat{x}(k)) + \phi_2(\hat{x}(k))^T \phi_2(\hat{x}(k)) - \Xi\right) < 0 \tag{7.48}$$

Now, knowing that (7.48) holds, then from (7.41), we have

$$\Delta \hat{V}(\hat{x}(k)) < -z^T(k)z(k) + \gamma^2 \omega^T(k)\omega(k) \tag{7.49}$$

Furthermore, taking a summation from 0 to ∞, yield

$$\hat{V}(x(\infty)) - \hat{V}(\hat{x}(0)) \leq -\sum_{k=0}^{\infty} z^T(k)z(k) + \sum_{k=0}^{\infty} \gamma^2 \omega^T(k)\omega(k), \tag{7.50}$$

and with the fact that $\hat{V}(\hat{x}(0)) = 0$ and $\hat{V}(\hat{x}(\infty)) \geq 0$, we obtain

$$\sum_{k=0}^{\infty} z^T(k)z(k) \leq \gamma^2 \sum_{k=0}^{\infty} \omega^T(k)\omega(k). \tag{7.51}$$

Thus (7.21) holds and the H_∞ performance is fulfilled.

To prove the system (7.1) with (7.16) is asymptotically stable, we set the disturbance $\omega(k) = 0$. Asymptotic stability for such polynomial discrete-time systems has already been shown in Corollary 7.6. Hence, completes the proof. $\nabla\nabla\nabla$

However to solve Theorem 7.9 is hard because we need to solve an infinity set of state-dependent PMIs. To relax these conditions, we utilise the SOS decomposition approach [30] and therefore the conditions given in Theorem 7.9 can be converted into SOS conditions and they are given in the following corollary:

Corollary 7.10. *Given a prescribed H_∞ performance, $\gamma > 0$, the system (7.1) is asymptotically stable via the nonlinear output feedback controller (7.16) with H_∞ performance (7.21) if there exist a symmetric polynomial matrix, $\hat{P}(x(k))$, polynomial function $L_{31}(y, x_c)$, polynomial matrix $\hat{G}(\hat{x}(k))$, and constants $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that the following conditions hold for all $x \neq 0$:*

$$v_5^T[\hat{P}(x(k)) - \epsilon_1 I]v_5 \quad \text{is a SOS} \tag{7.52}$$

$$v_6^T\left[M_2(\hat{x}(k)) - \epsilon_2 I\right]v_6 \quad \text{is a SOS} \tag{7.53}$$

where, v_5 and v_6 are free vectors in appropriate dimensions, and

$$M_2(\hat{x}(k)) = \begin{bmatrix} \hat{P}(x(k)) & * & * & * \\ 0 & \gamma^2 I & * & * \\ \hat{G}(\hat{x}(k))\hat{A}(\hat{x}(k)) + \hat{L}(y, x_c)\hat{C}_y(\hat{x}(k)) & \hat{B}_\omega(\hat{x}(k)) & \hat{G}(\hat{x}(k)) + \hat{G}^T(\hat{x}(k)) - \hat{P}(x_+) & * \\ \hat{C}_z(\hat{x}(k)) & 0 & 0 & I \end{bmatrix}, \tag{7.54}$$

with

$$\hat{G}(\hat{x}(k)) = \begin{bmatrix} G_{11}(\hat{x}(k)) & G_{12}(\hat{x}(k)) & 0 \\ G_{21}(\hat{x}(k)) & G_{22}(\hat{x}(k)) & 0 \\ G_{31}(\hat{x}(k)) & G_{32}(\hat{x}(k)) & G_{33}(y,x_c) \end{bmatrix}; \quad \hat{L}(y,x_c) = \begin{bmatrix} 0 \\ 0 \\ L_{31}(y,x_c) \end{bmatrix}. \tag{7.55}$$

Moreover, the nonlinear output feedback controller is given by

$$\begin{aligned} x_c(k+1) &= x_c(k) + A_c(y,x_c) \\ u(k) &= x_c(k) \end{aligned} \tag{7.56}$$

where

$$A_c(y,x_c) = \hat{K}(y,x_c)\hat{C}_y(\hat{x}(k))\hat{x}(k) \quad \textit{with} \quad \hat{K}(y,x_c) = L_{31}(y,x_c)G_{33}^{-1}(y,x_c). \tag{7.57}$$

Proof: Proof for this section follows directly from the combination of proof shown in Theorem 7.9 and Lemma 1.2. The proof ends. $\nabla\nabla\nabla$

The advantages of formulating the conditions of the nonlinear output feedback problem with prescribed H_∞ performance γ in the form of Corollary 7.10 are twofold:

1. A less conservative design procedure can be achieved (refer Remark 7.8).
2. The output feedback controller is decoupled from the Lyapunov function, hence the controller design can be performed in a more relaxed way because it is independence from the Lyapunov matrix,

7.3.2 Robust nonlinear H_∞ output feedback control

Consider the following system

$$\begin{aligned} x(k+1) =& A(x(k),\theta)x(k) + B_u(x(k),\theta)u(k) + B_\omega(x(k),\theta)\omega(k) \\ z(k) =& C_z(x(k),\theta)x(k) + D_{zu}(x(k),\theta)u(k) \\ y(k) =& C_y(x(k),\theta)x(k) \end{aligned} \tag{7.58}$$

where the matrices $\cdot(x(k),\theta)$ are defined as follows

$$
\begin{aligned}
A(x(k),\theta) &= \sum_{i=1}^{q} A_i(x(k))\theta_i; \quad B_u(x(k),\theta) = \sum_{i=1}^{q} B_i(x(k))\theta_i; \\
C_z(x(k),\theta) &= \sum_{i=1}^{q} C_{zi}(x(k))\theta_i; \quad D_{zu}(x(k),\theta) = \sum_{i=1}^{q} D_{zui}(x(k))\theta_i; \\
C_y(x(k)) &= \sum_{i=1}^{q} C_y(x(k))\theta_i; \quad B_\omega(x(k),\theta) = \sum_{i=1}^{q} B_{\omega i}(x(k))\theta_i.
\end{aligned} \tag{7.59}
$$

$\theta = \left[\theta_1, \ldots, \theta_q\right]^T \in \mathbb{R}^q$ is the vector of constant uncertainty and satisfies

$$
\theta \in \Theta \triangleq \left\{\theta \in \mathbb{R}^q : \theta_i \geq 0, i = 1, \ldots, q, \sum_{i=1}^{q} \theta_i = 1\right\}. \tag{7.60}
$$

We further define the following parameter-dependent Lyapunov function

$$
\hat{V}(\hat{x}(k)) = \hat{x}^T(k)\big(\sum_{i=1}^{q} \hat{P}_i(x(k))\theta_i\big)^{-1}\hat{x}(k). \tag{7.61}
$$

where $\hat{P}(x(k))$ is defined as in Chapter 2.

With the results from the previous section, the main result for robust H_∞ control problem can be proposed directly and given by the following corollary.

Corollary 7.11. *Given a prescribed H_∞ performance, $\gamma > 0$, and constants $\epsilon_1 > 0$ and $\epsilon_2 > 0$ for $x \neq 0$ and $i = 1, ..., q$, the system (7.58) with the nonlinear feedback controller (7.16) is asymptotically stable with H_∞ performance (7.21) for $x \neq 0$ if there exist a common polynomial matrices $\hat{G}(\hat{x}(k))$, a common polynomial function $\hat{L}_{31}(\hat{x}, y)$ and symmetric polynomial matrices, $\hat{P}_i(x(k))$ such that the following conditions are satisfied for all $x \neq 0$:*

$$
v_7^T[\hat{P}_i(x(k)) - \epsilon_1 I]v_7 \qquad \textit{is a SOS} \tag{7.62}
$$

$$
v_8^T\left[M_3(\hat{x}(k)) - \epsilon_2 I\right]v_8 \qquad \textit{is a SOS} \tag{7.63}
$$

where, v_7 and v_8 are free vectors in appropriate dimensions, and

$$M_3(\hat{x}(k)) =$$
$$\begin{bmatrix} \hat{P}_i(x(k)) & * & * & * \\ 0 & \gamma^2 I & * & * \\ \hat{G}(\hat{x}(k))\hat{A}_i(\hat{x}(k)) + \hat{L}(y,x_c)\hat{C}_{yi}(\hat{x}(k)) & \hat{B}_{\omega i}(\hat{x}(k)) & \hat{G}(\hat{x}(k)) + \hat{G}^T(\hat{x}(k)) - \hat{P}_i(x_+) & * \\ \hat{C}_{zi}(\hat{x}(k)) & 0 & 0 & I \end{bmatrix}, \tag{7.64}$$

with

$$\hat{G}(\hat{x}(k)) = \begin{bmatrix} G_{11}(\hat{x}(k)) & G_{12}(\hat{x}(k)) & 0 \\ G_{21}(\hat{x}(k)) & G_{22}(\hat{x}(k)) & 0 \\ G_{31}(\hat{x}(k)) & G_{32}(\hat{x}(k)) & G_{33}(y,x_c) \end{bmatrix}; \quad \hat{L}(y,x_c) = \begin{bmatrix} 0 \\ 0 \\ L_{31}(y,x_c) \end{bmatrix}. \tag{7.65}$$

Moreover, the nonlinear output feedback controller is given by

$$\begin{aligned} x_c(k+1) &= x_c(k) + A_c(y,x_c) \\ u(k) &= x_c(k) \end{aligned} \tag{7.66}$$

where

$$A_c(y,x_c) = \hat{K}(y,x_c)\hat{C}_y(\hat{x}(k))\hat{x}(k) \quad \textit{with} \quad \hat{K}(y,x_c) = L_{31}(y,x_c)G_{33}^{-1}(y,x_c). \tag{7.67}$$

Remark 7.12. The main drawback of this method is on the computational complexity. This drawback is common to all augmented systems method. Therefore, the method can only be applied to a small academic examples.

7.4 Numerical Example

In this section, a tunnel diode circuit is used to demonstrate the validity of our proposed approach.

7.4.1 Nonlinear H_∞ output feedback control

A Tunnel Diode Circuit: A tunnel diode circuit with input $u(t)$ is shown in Figure 7.1 [78]. The characteristics of the tunnel diode are described as follows:

$$i_D(t) = 0.002v_D(t) + 0.01v_D^3(t). \tag{7.68}$$

Next, choosing the state variables of the form $x_1(t)$=$v_c(t)$ and $x_2(t)$=$i_L(t)$, then the

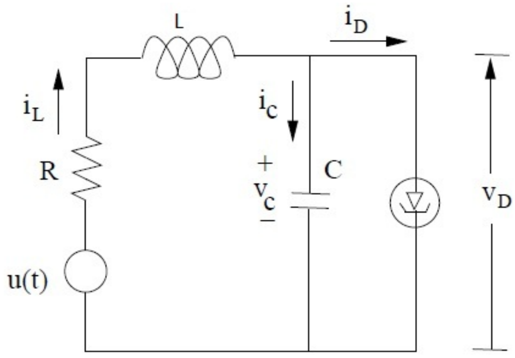

FIGURE 7.1: Tunnel diode circuit.

circuit can be represented by the following state equations:

$$\begin{aligned} C\dot{x}_1(t) &= -0.002x_1(t) - 0.01x_1^3(t) + x_2(t) \\ L\dot{x}_2(t) &= -x_1(t) - Rx_2(t) + \omega(t) + u(t) \\ y(t) &= Sx(t) \\ z(t) &= x_2(t) + u(t) \end{aligned} \tag{7.69}$$

where $\omega(t)$ is the noise to the system, $y(t)$ is the measured output, $z(t)$ is the controlled output, and $u(t)$ is the input to the circuit. In addition we assume the state $x_2(t)$=$i_L(t)$ is available for feedback. Therefore, $S = [0 \quad 1]$. The circuit parameter is given as follows: $C = 20mF$, $\epsilon = 1000mH$, and $R = 1\Omega$. With these parameters, the dynamic of the circuit can be written as follows:

$$\begin{aligned} \dot{x}_1(t) &= -0.1x_1(t) - 0.5x_1^3(t) + 50x_2(t) \\ \dot{x}_2(t) &= -x_1(t) - x_2(t) + \omega(t) + u(t) \\ y(t) &= x_2(t) \\ z(t) &= x_2(t) + u(t) \end{aligned} \tag{7.70}$$

The above system is in continuous-time, therefore to convert (7.70) into discrete-time, we sample the above system at $T = 0.02$, and by applying Euler's discretisation method, then the following discrete-time nonlinear dynamic equations is obtained:

$$\begin{aligned} x_1(k+1) &= x_1(k) + T\big[-0.1x_1(k) - 0.5x_1^3(k) + 50x_2(k)\big] \\ x_2(k+1) &= x_2(k) + T\big[-x_1(t) - x_2(t) + \omega(t) + u(t)\big] \\ y(k) &= x_2(k) \\ z(k) &= x_2(k) + u(k) \end{aligned} \tag{7.71}$$

Then, from (7.71), the system with controller (7.16) can be written as follows:

$$\left.\begin{aligned} \hat{x}(k+1) &= \hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}_u(\hat{x}(k))\hat{A}_c(\hat{x}(k))\hat{x}(k) + \hat{B}_\omega(\hat{x}(k))\omega(k) \\ z(k) &= \hat{C}_z(\hat{x}(k))\hat{x}(k) \\ y(k) &= \hat{C}_y(\hat{x}(k))\hat{x}(k) \end{aligned}\right\} \tag{7.72}$$

where,

$$\begin{aligned} \hat{A}(\hat{x}(k)) &= \begin{bmatrix} 1+T\big(-0.1-0.5x_1^2(k)\big) & 50T & 0 \\ -T & 1-T & T \\ 0 & 0 & 1 \end{bmatrix}; \quad \hat{B}_u(\hat{x}(k)) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \\ \hat{B}_\omega(\hat{x}(k)) &= \begin{bmatrix} 0 \\ T \\ 0 \end{bmatrix}; \quad \hat{C}_z(\hat{x}(k)) = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}; \quad \hat{C}_y(\hat{x}(k)) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}; \\ \hat{x}(k) &= \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_c(k) \end{bmatrix}. \end{aligned} \tag{7.73}$$

In this example, we choose $\epsilon_1 = \epsilon_2 = 0.01$ and γ is selected to be 1. Then, using the procedure described in the Corollary 7.10, and with the degree of $\hat{P}(x(k))$ and $\hat{G}(\hat{x}(k))$ set to be 4 and $\hat{L}(y, x_c)$ chosen to be in the degree of 8, a feasible solution is achieved. The SOSTOOLS returns the following values:

$$\begin{aligned} G_{33}(\hat{x}(k)) &= 14.68x_2^4 - 0.085x_2^3x_c + 0.0102x_2^2x_c^2 - 59.401x_2^2 - 0.004x_2x_c^3 + 1.828x_2x_c \\ &\quad + 0.007x_c^4 - 0.484x_c^2 + 7858.733. \\ L_{31}(\hat{x}(k)) &= -1.009x_2^4 - 0.403x_2^3x_c - 0.0026x_2^2x_c^2 - 0.00436x_2x_c^3 + 8.705x_2x_c \\ &\quad - 0.00033x_c^4 + 0.113x_c^2 - 285.65. \end{aligned} \tag{7.74}$$

The nonlinear controller is given by

$$\begin{aligned} x_c(k+1) &= x_c(k) + K(y, x_c)C(\hat{x}(k))\hat{x}(k) \\ u(k) &= x_c(k) \end{aligned} \tag{7.75}$$

where,

$$K(y, x_c) = \frac{L_{31}(y, x_c)}{G_{33}(\hat{x}(k))}. \tag{7.76}$$

It is best to note here that to calculate $K(y, x_c)$ is hard because it contains the polynomial terms; hence we use Matlab/Simulink to aid us to compute those value. It is also important to highlight here that the Lyapunov matrix $\hat{P}(x(k))$ in this example is defined to be a symmetrical N x N polynomial matrices whose $(i,j)-th$ entry is given by

$$p_{ij}(x(k)) = p_{ij}^0 + p_{ijg}m(k)^{(1:l)} \tag{7.77}$$

where $i = 1,2,...,n$, $j = 1,2,...,n$, and $g = 1,2,....d$ with n is number of states and d is total monomials numbers. Meanwhile $m(k)$ is all monomial vectors in $(x(k))$ from degree of 1 to degree of l, where l is a scalar even value. For example if $l = 2$, and $x(k) = [x_1(k), x_2(k)]^T$, then $p_{11}(x(k)) = p_{11}+p_{112}x_1+p_{113}x_2+p_{114}x_1^2+p_{115}x_1x_2+p_{116}x_2^2$. This structure is more general as compared to [44] because of a higher value of l, a more relaxation in SOS problem can be achieved. Due to the large number of values returned by SOSTOOLS for $P(x(k))$ in this example, those values are omitted.

Remark 7.13. The disturbance input signal, w(k), which was used during the simulation is the band limited white noise (noise power is 10). The simulation result for the ratio of the controlled output energy to the disturbance input noise energy obtained by using the H_∞ output feedback controller is illustrated in Figure 7.2. It can be seen from the figure that the ratio of the controlled output energy to the disturbance input noise energy is always less than a prescribed value, 1 and decreases to about 0.005. Thus, $\gamma = \sqrt{0.005} \approx 0.07$. This implies that the L_2 gain from the disturbance to the regulated output is no greater than 0.07.

Remark 7.14. To date, to the author's knowledge, no result has been presented in the framework of H_∞ output feedback control for polynomial discrete-time systems.

7.4.2 Robust nonlinear H_∞ output feedback control

Consider the tunnel diode circuit shown in 7.1. For this example, we assume the value of R is uncertain and given by $R = 1 \pm 30\%$. Therefore, the system can be described as follows:

$$\hat{A}(\hat{x}(k)) = \begin{bmatrix} 1+T\big(-0.1-0.5x_1^2(k)\big) & 50T & 0 \\ -T & 1-RT & T \\ 0 & 0 & 1 \end{bmatrix}; \quad \hat{B}_u(\hat{x}(k)) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix};$$
$$\hat{B}_\omega(\hat{x}(k)) = \begin{bmatrix} 0 \\ T \\ 0 \end{bmatrix}; \quad \hat{C}_z(\hat{x}(k)) = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}; \quad \hat{C}_y(\hat{x}(k)) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}; \tag{7.78}$$

For this example, we choose $\epsilon_1 = \epsilon_2 = 0.01$ and γ is selected to be 1. Then, using the procedure described in Corollary 7.11, and with the degree of $\hat{P}(x(k))$ and $\hat{G}(\hat{x}(k))$ set

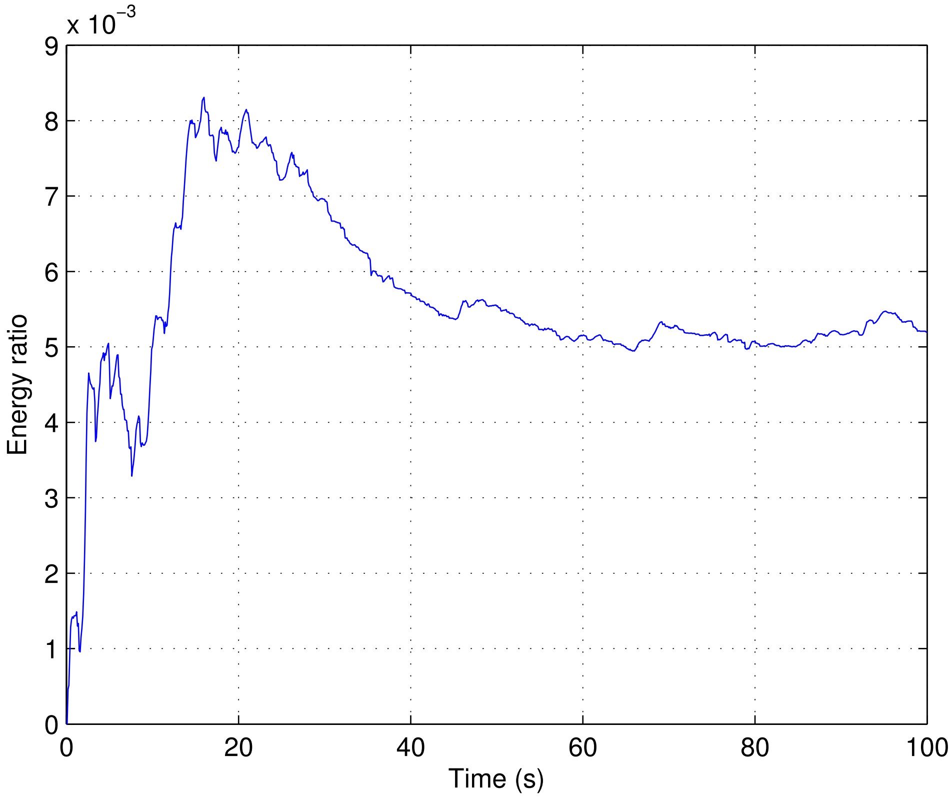

FIGURE 7.2: Ratio of the regulated output energy to the disturbance input noise energy of a tunnel diode circuit.

to be 4 and $\hat{L}(y, x_c)$ chosen to be in the degree of 6, a feasible solution is achieved. The following values are returned by SOSTOOLS:

$$\begin{aligned} G_{33}(\hat{x}(k)) &= 14.18x_2^4 - 0.076x_2^3x_c + 0.0062x_2^2x_c - 11.867x_2^2 - 0.0119x_2x_c^3 + 0.463x_2x_c \\ &\quad + 0.0022x_c^4 - 0.089x_c^2 + 328.0373. \\ L_{31}(\hat{x}(k)) &= -4.591x_2^4 + 1.169x_2^3x_c - 0.9036x_2^2x_c^2 + 0.007x_2x_c^3 + 2.0713x_2x_c \\ &\quad + 0.1765x_c^4 - 15.027x_2^2 + 0.455x_c^2 - 13.614. \end{aligned} \tag{7.79}$$

The nonlinear controller is given by

$$\begin{aligned} x_c(k+1) &= x_c(k) + K(y, x_c)C(\hat{x}(k))\hat{x}(k) \\ u(k) &= x_c(k) \end{aligned} \tag{7.80}$$

where,

$$K(y, x_c) = \frac{L_{31}(y, x_c)}{G_{33}(\hat{x}(k))}. \tag{7.81}$$

The disturbance input signal, w(k), which was used during the simulation is the band limited white noise (noise power is 2). The simulation results for the ratio of the controlled output energy to the disturbance input noise energy obtained by using the H_∞ output feedback controller is shown in Figure 7.3. It can be seen from the figure that the ratio of the controlled output energy to the disturbance input noise energy is always less than the prescribed value, 1 and decreasing to about 0.00012. Thus, $\gamma = \sqrt{0.00012} \approx 0.011$. This implies that the L_2 gain from the disturbance to the regulated output is no greater than 0.011.

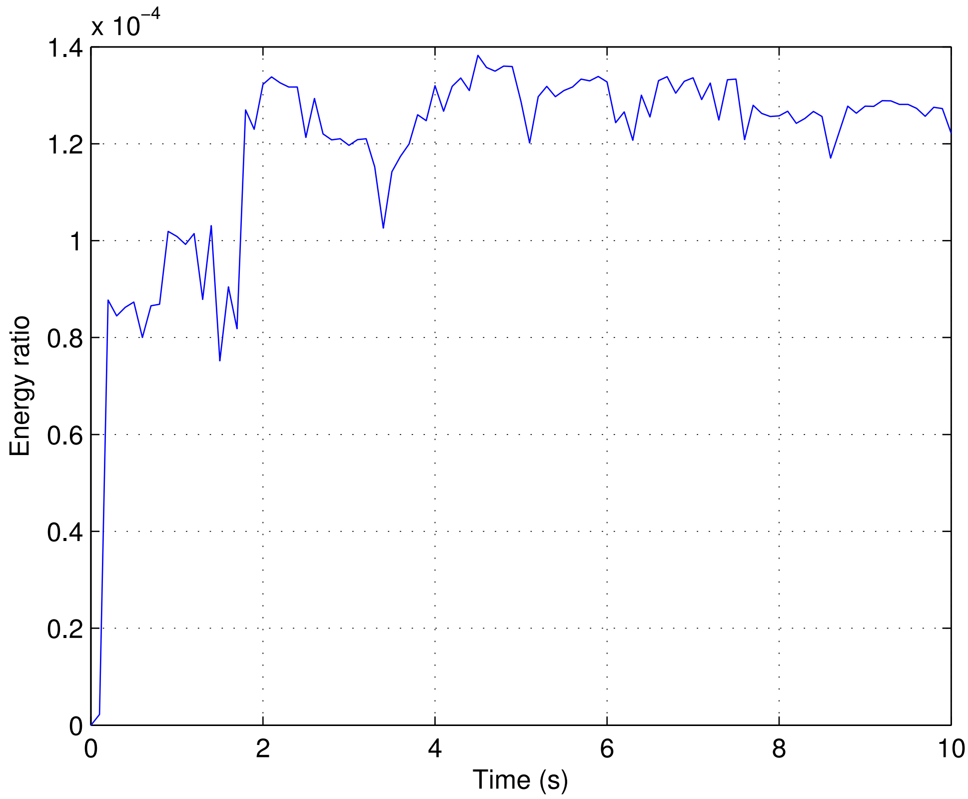

FIGURE 7.3: Ratio of the regulated output energy to the disturbance input noise energy of a tunnel diode circuit with polytopic uncertainty.

7.5 Conclusion

The problem of designing a nonlinear H_∞ output feedback controller for polynomial discrete-time systems has been examined in this chapter. It has been shown that by incorporating an integrator into the controller structures, a less conservative design methodology can be achieved. However, the price we pay is the computational burden, which prevent us from applying this method to the high order systems. Then, by applying the sum of squares approach, sufficient conditions for the existence of the proposed controller are provided in terms of the solvability of PMIs, which are formulated as SOS constraints and have been solved using SOSTOOLS. The results are then extended to the robust H_∞ control problem with polytopic uncertainty. The effectiveness of the proposed design methodology is demonstrated through a tunnel diode circuit.

Chapter 8

Robust Nonlinear H_∞ Output Feedback Control for Polynomial Discrete-time Systems with Norm-Bounded Uncertainty

8.1 Introduction

In this chapter, we attempt to design a robust nonlinear H_∞ output feedback controller for polynomial discrete-time systems. The norm-bounded uncertainty is considered in this chapter. In light of the results shown in Chapter 5, the interconnection between the robust nonlinear H_∞ output feedback control problem and the nonlinear H_∞ output feedback control problem is established through a so-called '*scaled*' system. The sufficient conditions for the existence of the proposed controller with an integrator is given by the solvability of the PMIs which are formulated as SOS constraints. The SOS conditions are then solved using SOSTOOLS. A tunnel diode circuit is then used to validate the effectiveness of the proposed methodology.

8.2 System Description and Problem Formulation

Consider the following uncertain polynomial discrete-time system:

$$\left.\begin{aligned} x(k+1) =& A(x(k))x(k) + \Delta A(x(k))x(k) + B_u(x(k))u(k) + \Delta B_u(x(k))u(k) \\ &+ B_\omega(x(k))\omega(k) \\ z(k) =& C_z(x(k))x(k) + D_{zu}(x(k))u(k) \\ y(k) =& C_y(x(k))x(k) \end{aligned}\right\} \tag{8.1}$$

where $x(k) \in R^n$ is the state vectors, and $u(k) \in R^m$ is the input and $y(k)$ is the measured output. $A(x(k))$, $B_u(x(k))$, $C_z(x(k))$, $D_{zu}(x(k))$ and $C_y(x(k))$ are polynomial matrices of appropriate dimensions. $z(k)$ is a vector of output signals related to the performance of the control system. $\omega(k)$ is the disturbance which belongs to $L_2[0,\infty]$. Meanwhile $\Delta A(x(k))$ and $\Delta B_u(x(k))$ represent the uncertainties in the system and satisfy the following assumption.

Assumption 8.1. The parameter uncertainties considered here are described by norm-bounded, and given as follows:

$$\begin{bmatrix}\Delta A(x(k)) & \Delta B_u(x(k))\end{bmatrix} = H(x(k))F(x(k))\begin{bmatrix}E_1(x(k)) & E_2(x(k))\end{bmatrix} \tag{8.2}$$

where $H(x(k))$, $E_1(x(k))$ and $E_2(x(k))$ are known polynomial matrices of appropriate dimensions, and $F(x(k))$ is an unknown matrix function which satisfies,

$$\left\|F^T(x(k))F(x(k))\right\| \leq I. \tag{8.3}$$

To ensure a convex solution of $P(x(k+1))$ can be rendered (a detail discussion about this issue can be found in Chapter 7), the following nonlinear feedback controller is proposed:

$$\left.\begin{aligned} x_c(k+1) =& x_c(k) + A_c(y, x_c) \\ u(k) =& x_c(k) \end{aligned}\right\} \tag{8.4}$$

where $A_c(y, x_c)$ is the input function of the integrator.

Problem formulation: Given any $\gamma > 0$, find a controller of the form (8.4) such that the L_2 gain from the disturbance $\omega(k)$ to the output that needs to be controlled $z(k)$ for system (8.1) with (8.4) is less than or equal to γ, i.e

$$\|z(k)\|_{[0,\infty]} \leq \gamma^2 \|\omega(k)\|_{[0,\infty]} \tag{8.5}$$

for all $w(k) \in L_2[0,\infty]$ and for all admissible uncertainties. In this situation, the system (8.1) is said to have a robust H_∞ performance (8.5).

8.3 Main results

In this section, we show that the robust nonlinear H_∞ output feedback control problem is solvable if the nonlinear H_∞ output feedback control problem for the '*scaled*' system is solvable. We begin this section by defining the '*scaled*' system. Then, the '*scaled*' system with controller is represented by the augmented form, and followed by the methodology for solving the robust nonlinear H_∞ output feedback control.

Motivated by the work performed in [79] and the results in Chapter 5, the following '*scaled*' system is defined:

$$\left.\begin{aligned} \tilde{x}(k+1) =& A(\tilde{x}(k))\tilde{x}(k) + \begin{bmatrix} B_\omega(\tilde{x}(k)) & \frac{1}{\delta}\bar{H}(\tilde{x}(k)) \end{bmatrix} \tilde{\omega}(k) + B_u(\tilde{x}(k))u(k) \\ \tilde{z}(k) =& \begin{bmatrix} C_z(\tilde{x}(k)) \\ \delta E_1(\tilde{x}(k)) \end{bmatrix} \tilde{x}(k) + \begin{bmatrix} D_{zu}(\tilde{x}(k)) \\ \delta E_2(\tilde{x}(k)) \end{bmatrix} u(k) \\ \tilde{y}(k) =& C_y(x(k))x(k) \end{aligned}\right\} \tag{8.6}$$

where $\tilde{x} \in R^n$ is the state, $u(k) \in R^m$ and $\tilde{\omega} \in R^{m+i}$ is the input noise. The δ is a positive constant, $\tilde{z}(k)$ is the controlled output and $\bar{H}(\tilde{x}(k)) = [H_1(\tilde{x}(k)) \quad H_1(\tilde{x}(k))]$.

Remark 8.2. The system described in (8.6) is in similar form to the system described in (7.1) (refer Chapter 7). Therefore, we can apply the methodology used for solving the system (7.1) for the '*scaled*' systems (8.6).

The system (8.6) with controller (8.4) can be written as follows:

$$\left.\begin{aligned} \hat{x}(k+1) &= \hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}_u(\hat{x}(k))A_c(y,x_c) + \hat{B}_\omega(\hat{x}(k))\tilde{\omega}(k) \\ \tilde{z}(k) &= \hat{C}_z(\hat{x}(k))\hat{x}(k) \\ \tilde{y}(k) &= \hat{C}_y(\hat{x}(k))\hat{x}(k) \end{aligned}\right\} \tag{8.7}$$

where,

$$\begin{aligned} \hat{A}(\hat{x}(k)) &= \begin{bmatrix} A(\tilde{x}(k)) & B_u(\tilde{x}(k)) \\ 0 & 1 \end{bmatrix}; \quad \hat{B}_u(\hat{x}(k)) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad \hat{B}_\omega(\hat{x}(k)) = \begin{bmatrix} \tilde{B}_\omega(\tilde{x}(k)) \\ 0 \end{bmatrix}; \\ \hat{C}_z(\hat{x}(k)) &= \begin{bmatrix} \tilde{C}_z(\tilde{x}(k)) & \tilde{D}_{zu}(\tilde{x}(k)) \end{bmatrix}; \quad \hat{C}_y(\hat{x}(k)) = \begin{bmatrix} C_y(x(k)) & 0 \end{bmatrix}; \\ \hat{x} &= \begin{bmatrix} x(k) \\ x_c(k) \end{bmatrix}, \end{aligned} \tag{8.8}$$

with

$$\tilde{B}_\omega(\tilde{x}(k)) = \begin{bmatrix} B_\omega(\tilde{x}(k)) & \frac{1}{\delta}\bar{H}(\tilde{x}(k)) \end{bmatrix}; \quad \tilde{C}_z(\tilde{x}(k)) = \begin{bmatrix} C_z(\tilde{x}(k)) \\ \delta E_1(\tilde{x}(k)) \end{bmatrix};$$
$$\tilde{D}_{zu}(\tilde{x}(k)) = \begin{bmatrix} D_{zu}(\tilde{x}(k)) \\ \delta E_2(\tilde{x}(k)) \end{bmatrix}. \tag{8.9}$$

Next, we assume $A_c(y, x_c)$ to be of the form $A_c(y, x_c) = \hat{A}_c(y, x_c)y$. Therefore, (8.7) can be re-written as follows:

$$\left.\begin{aligned} \hat{x}(k+1) &= \hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}_u(\hat{x}(k))\hat{A}_c(y, x_c)y + \hat{B}_\omega(\hat{x}(k))\tilde{\omega}(k) \\ \tilde{z}(k) &= \hat{C}_z(\hat{x}(k))\hat{x}(k) \\ \tilde{y}(k) &= \hat{C}_y(\hat{x}(k))\hat{x}(k) \end{aligned}\right\} \tag{8.10}$$

where $\hat{A}(\hat{x}(k))$, $\hat{B}_u(\hat{x}(k))$, $\hat{B}_\omega(\hat{x}(k))$, $\hat{C}_z(\hat{x}(k))$ and $\hat{C}_y(\hat{x}(k))$ are as described in (8.8).

In view of the '*scaled*' system (8.6), the following theorem is established.

Theorem 8.3. *Consider the system* (8.1)*. There exists a controller of the form* (8.4) *such that* (8.5) *holds for all admissible uncertainties if there exists a positive constant,* $\delta > 0$*, such that* (8.5) *holds for system* (8.6) *with the same controller.*

Proof: The proof can be shown using similar techniques as proposed in Theorem 5.3 of Chapter 5, hence omitted here. ∇∇∇

In the light of Theorem 8.3, what is left here is to solve the '*scaled*' nonlinear H_∞ control problem given in (8.6). Therefore, the sufficient conditions for the existence of a solution to the robust H_∞ output feedback control problem is presented in the following theorem.

Theorem 8.4. *Given a prescribed H_∞ performance, $\gamma > 0$, the system* (8.1) *is stabilisable with H_∞ performance* (8.5) *via the nonlinear output feedback controller of the form* (8.4) *if there exist a symmetric polynomial matrix, $\hat{P}(x(k))$, a polynomial function $L_{31}(y, x_c)$ and a polynomial matrix $\hat{G}(\hat{x}(k))$ such that the following conditions hold for all $x \neq 0$*

$$\hat{P}(x(k)) > 0 \tag{8.11}$$
$$M_2(\hat{x}(k)) > 0 \tag{8.12}$$

where

$$M_2(\hat{x}(k)) = \begin{bmatrix} \hat{P}(x(k)) & * & * & * \\ 0 & \gamma^2 I & * & * \\ \hat{G}(\hat{x}(k))\hat{A}(\hat{x}(k)) + \hat{L}(y,x_c)\hat{C}_y(\hat{x}(k)) & \hat{B}_\omega(\hat{x}(k)) & \hat{G}(\hat{x}(k)) + \hat{G}^T(\hat{x}(k)) - \hat{P}(x_+) & * \\ \hat{C}_z(\hat{x}(k)) & 0 & 0 & I \end{bmatrix}, \tag{8.13}$$

with

$$\hat{G}(\hat{x}(k)) = \begin{bmatrix} G_{11}(\hat{x}(k)) & G_{12}(\hat{x}(k)) & 0 \\ G_{21}(\hat{x}(k)) & G_{22}(\hat{x}(k)) & 0 \\ G_{31}(\hat{x}(k)) & G_{32}(\hat{x}(k)) & G_{33}(y,x_c) \end{bmatrix}; \quad \hat{L}(y,x_c) = \begin{bmatrix} 0 \\ 0 \\ L_{31}(y,x_c) \end{bmatrix}. \tag{8.14}$$

Moreover, the nonlinear output feedback controller is given by

$$\begin{aligned} x_c(k+1) &= x_c(k) + A_c(y,x_c) \\ u(k) &= x_c(k) \end{aligned} \tag{8.15}$$

where

$$A_c(y,x_c) = \hat{K}(y,x_c)\hat{C}_y(\hat{x}(k))\hat{x}(k) \quad \textit{with} \quad \hat{K}(y,x_c) = L_{31}(y,x_c)G_{33}^{-1}(y,x_c). \tag{8.16}$$

Proof: By Theorem 8.3, the robust nonlinear H_∞ output feedback control problem is converted to the nonlinear H_∞ output feedback control problem for a '*scaled*' system. Then, by adapting Theorem 7.9, the result can be obtained easily. It is important to note here that the Lyapunov function of the following form is selected

$$\hat{V}(\hat{x}(k)) = \hat{x}^T(k)\hat{P}^{-1}(x(k))\hat{x}(k). \tag{8.17}$$

∇∇∇

Remark 8.5. The idea of choosing the Lyapunov function to be of the form (8.17) is to ensure a convex solution to the terms in $P(x(k+1))$ can be achieved. This idea has been completely covered in Chapter 7, hence for the sake of simplicity, the complete explanation is omitted here.

Note that the conditions (8.11) - (8.12) of Theorem 8.4 are in state-dependent polynomial matrix inequalities (PMIs). Using the SOS decomposition method based on SDP [32] provides a relaxation for the problem. Therefore, the (8.11) - (8.12) can be modified into SOS conditions, and they are given in the following corollary:

Corollary 8.6. *Given a prescribed H_∞ performance, $\gamma > 0$, the system (8.1) is asymptotically stable via the nonlinear output feedback controller (8.4) with H_∞ performance (8.5) if there exist a symmetric polynomial matrix, $\hat{P}(x(k))$, polynomial function $L_{31}(y, x_c)$, polynomial matrix $\hat{G}(\hat{x}(k))$, and constants $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that the following conditions hold for all $x \neq 0$:*

$$v_5^T[\hat{P}(x(k)) - \epsilon_1 I]v_5 \quad \textit{is a SOS} \tag{8.18}$$

$$v_6^T\,[M_2(\hat{x}(k)) - \epsilon_2 I]\,v_6 \quad \textit{is a SOS} \tag{8.19}$$

where, v_5 and v_6 are vectors in appropriate dimensions, and

$$M_2(\hat{x}(k)) = \begin{bmatrix} \hat{P}(x(k)) & * & * & * \\ 0 & \gamma^2 I & * & * \\ \hat{G}(\hat{x}(k))\hat{A}(\hat{x}(k)) + \hat{L}(y, x_c)\hat{C}_y(\hat{x}(k)) & \hat{B}_\omega(\hat{x}(k)) & \hat{G}(\hat{x}(k)) + \hat{G}^T(\hat{x}(k)) - \hat{P}(x_+) & * \\ \hat{C}_z(\hat{x}(k)) & 0 & 0 & I \end{bmatrix}, \tag{8.20}$$

with

$$\hat{G}(\hat{x}(k)) = \begin{bmatrix} G_{11}(\hat{x}(k)) & G_{12}(\hat{x}(k)) & 0 \\ G_{21}(\hat{x}(k)) & G_{22}(\hat{x}(k)) & 0 \\ G_{31}(\hat{x}(k)) & G_{32}(\hat{x}(k)) & G_{33}(y, x_c) \end{bmatrix}; \quad \hat{L}(y, x_c) = \begin{bmatrix} 0 \\ 0 \\ L_{31}(y, x_c) \end{bmatrix}. \tag{8.21}$$

Moreover, the nonlinear output feedback controller is given by

$$\begin{aligned} x_c(k+1) &= x_c(k) + A_c(y, x_c) \\ u(k) &= x_c(k) \end{aligned} \tag{8.22}$$

where

$$A_c(y, x_c) = \hat{K}(y, x_c)\hat{C}_y(\hat{x}(k))\hat{x}(k) \quad \textit{with} \quad \hat{K}(y, x_c) = L_{31}(y, x_c)G_{33}^{-1}(y, x_c). \tag{8.23}$$

Proof: Proof for this section follows directly from the combination of proof shown in Theorem 8.4 and Lemma 1.2. The proof ends. ∇∇∇

8.4 Numerical Example

In this section, a tunnel diode circuit is used to demonstrate the validity of our proposed approach.

A tunnel diode circuit with input $u(t)$ is shown in Figure 7.1 [78]. The characteristics of the tunnel diode are described as follows:

$$i_D(t) = 0.002v_D(t) + 0.01v_D^3(t). \tag{8.24}$$

Next, choosing the state variables of the form $x_1(t)$=$v_c(t)$ and $x_2(t)$=$i_L(t)$, then the circuit can be represented by the following state equations:

$$\begin{aligned} C\dot{x}_1(t) &= -0.002x_1(t) - 0.01x_1^3(t) + x_2(t) \\ L\dot{x}_2(t) &= -x_1(t) - Rx_2(t) + \omega(t) + u(t) \\ y(t) &= Sx(t) \\ z(t) &= x_2(t) + u(t) \end{aligned} \tag{8.25}$$

where $\omega(t)$ is the noise to the system, $y(t)$ is the measured output, $z(t)$ is the controlled output, and $u(t)$ is the input to the circuit. In addition we assume the state $x_2(t)$=$i_L(t)$ is available for feedback. Therefore, $S = [0 \quad 1]$. The circuit parameter is given as follows: $C = 20mF$, $\epsilon = 1000mH$, and $R = 1 \pm 30\%\Omega$. With these parameters, the dynamic of the circuit can be written as follows:

$$\begin{aligned} \dot{x}_1(t) &= -0.1x_1(t) - 0.5x_1^3(t) + 50x_2(t) \\ \dot{x}_2(t) &= -x_1(t) - Rx_2(t) + \omega(t) + u(t) \\ y(t) &= x_2(t) \\ z(t) &= x_2(t) + u(t) \end{aligned} \tag{8.26}$$

The above system is in continuos-time, therefore to convert (8.26) into discrete-time, we sample the above system at $T = 0.02$, and by Euler's discretization method, the following discrete-time nonlinear dynamic equations is obtained:

$$\begin{aligned} x_1(k+1) &= x_1(k) + T\big[-0.1x_1(k) - 0.5x_1^3(k) + 50x_2(k)\big] \\ x_2(k+1) &= x_2(k) + T\big[-x_1(t) - Rx_2(t) + \omega(t) + u(t)\big] \\ y(k) &= x_2(k) \\ z(k) &= x_2(k) + u(k) \end{aligned} \tag{8.27}$$

From (8.27), the system with the controller (8.4) can be written as follows:

$$\left.\begin{aligned} \hat{x}(k+1) &= \hat{A}(\hat{x}(k))\hat{x}(k) + \hat{B}_u(\hat{x}(k))\hat{A}_c(\hat{x}(k))\hat{x}(k) + \hat{B}_\omega(\hat{x}(k))\omega(k) \\ z(k) &= \hat{C}_z(\hat{x}(k))\hat{x}(k) \\ y(k) &= \hat{C}_y(\hat{x}(k))\hat{x}(k) \end{aligned}\right\} \tag{8.28}$$

where,

$$\begin{aligned} \hat{A}(\hat{x}(k)) &= \begin{bmatrix} A(\tilde{x}(k)) & B_u(\tilde{x}(k)) \\ 0 & 1 \end{bmatrix}; \quad \hat{B}_u(\hat{x}(k)) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad \hat{B}_\omega(\hat{x}(k)) = \begin{bmatrix} \tilde{B}_\omega(\tilde{x}(k)) \\ 0 \end{bmatrix}; \\ \hat{C}_z(\hat{x}(k)) &= \begin{bmatrix} \tilde{C}_z(\tilde{x}(k)) & \tilde{D}_{zu}(\tilde{x}(k)) \end{bmatrix}; \quad \hat{C}_y(\hat{x}(k)) = \begin{bmatrix} C_y(x(k)) & 0 \end{bmatrix}; \\ \hat{x} &= \begin{bmatrix} x(k) \\ x_c(k) \end{bmatrix}, \end{aligned} \tag{8.29}$$

and with

$$\begin{aligned} \tilde{B}_\omega(\tilde{x}(k)) &= \begin{bmatrix} B_\omega(\tilde{x}(k)) & \frac{1}{\delta}\bar{H}(\tilde{x}(k)) \end{bmatrix}; \quad \tilde{C}_z(\tilde{x}(k)) = \begin{bmatrix} C_z(\tilde{x}(k)) \\ \delta E_1(\tilde{x}(k)) \end{bmatrix}; \\ \tilde{D}_{zu}(\tilde{x}(k)) &= \begin{bmatrix} D_{zu}(\tilde{x}(k)) \\ \delta E_2(\tilde{x}(k)) \end{bmatrix}. \end{aligned} \tag{8.30}$$

Similarly,

$$\begin{aligned} \hat{A}(\hat{x}(k)) &= \begin{bmatrix} 1 + T\big[-0.1 - 0.5x_1^2(k)\big] & 50T & 0 \\ -T & 1 - RT & T \\ 0 & 0 & 1 \end{bmatrix}; \quad \hat{B}_u(\hat{x}(k)) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \\ \hat{B}_\omega(\hat{x}(k)) &= \begin{bmatrix} 0 & 0 & 0 \\ T & \frac{1}{\delta}0.3 & \frac{1}{\delta}0.3 \\ 0 & 0 & 0 \end{bmatrix}; \quad \hat{C}_z(\hat{x}(k)) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & \delta & 0 \end{bmatrix}; \\ \hat{C}_y(\hat{x}(k)) &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}; \quad \hat{x}(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_c(k) \end{bmatrix}. \end{aligned} \tag{8.31}$$

where, $\delta = 1$.

In this example, we choose $\epsilon_1 = \epsilon_2 = 0.01$ and γ is selected to be 1. Then, using the procedure described in the Corollary 8.6, and with the degree of $\hat{P}(x(k))$ and $\hat{G}(\hat{x}(k))$ set to be 4 and $\hat{L}(y, x_c)$ chosen to be in the degree of 8, a feasible solution is achieved.

The following values are returned by SOSTOOLS:

$$
\begin{aligned}
G_{33}(\hat{x}(k)) &= 25.5662x_2^4 - 0.0351x_2^3x_c + 0.1624x_2^2x_c^2 - 38.6407x_2^2 - 0.0044x_2x_c^3 \\
&\quad + 0.0213x_2x_c + 0.01174x_c^4 - 0.5731x_c^2 + 6162.2353. \\
L_{31}(\hat{x}(k)) &= -0.0002x_2^6 + 0.0002x_2^4x_c^2 - 3.0236x_2^4 - 0.0003x_2^2x_c^4 \\
&\quad - 0.0128x_2^2x_c^2 + 11.7213x_2^2 - 0.0112x_2x_c^3 + 12.4267x_2x_c \\
&\quad - 0.0007x_c^4 + 0.1586x_c^2 - 265.2317.
\end{aligned} \tag{8.32}
$$

The nonlinear controller is given by

$$
\begin{aligned}
x_c(k+1) &= x_c(k) + K(y, x_c)C(\hat{x}(k))\hat{x}(k) \\
u(k) &= x_c(k)
\end{aligned} \tag{8.33}
$$

where,

$$
K(y, x_c) = \frac{L_{31}(y, x_c)}{G_{33}(\hat{x}(k))}. \tag{8.34}
$$

It is best to note here that to calculate $K(y, x_c)$ is hard because it contains the polynomial terms, hence we use Matlab/Simulink to aid us to compute those values. It is also important to highlight here that the Lyapunov matrix $\hat{P}(x(k))$ in this example is defined to be as a symmetrical N x N polynomial matrices whose $(i, j) - th$ entry is described in (7.77).

Remark 8.7. The disturbance input signal, $w(k)$, which was used during the simulation is the band limited white noise (noise power is 1). The simulation results for the ratio of the controlled output energy to the disturbance input noise energy is illustrated in Figure 8.1. It can be seen from the figure that the ratio of the controlled output energy to the disturbance input noise energy is always less than a prescribed value, 1 and decreasing to about 0.007. Thus, $\gamma = \sqrt{0.007} \approx 0.083$. This implies that the L_2 gain from the disturbance to the regulated output is no greater than 0.08.

8.5 Conclusion

The problem of designing a robust nonlinear H_∞ output feedback controller for polynomial discrete-time systems has been examined in this chapter. In particular, the norm-bounded uncertainty has been considered in this chapter. The interconnection between the robust nonlinear H_∞ output feedback control problem and nonlinear H_∞ output feedback control problem has been established through a so called $'scaled'$ system. The integrator is incorporated into the controller structures, hence a convex solution in $P(x(k+1))$ can be obtained efficiently. Sufficient conditions for the existence of the

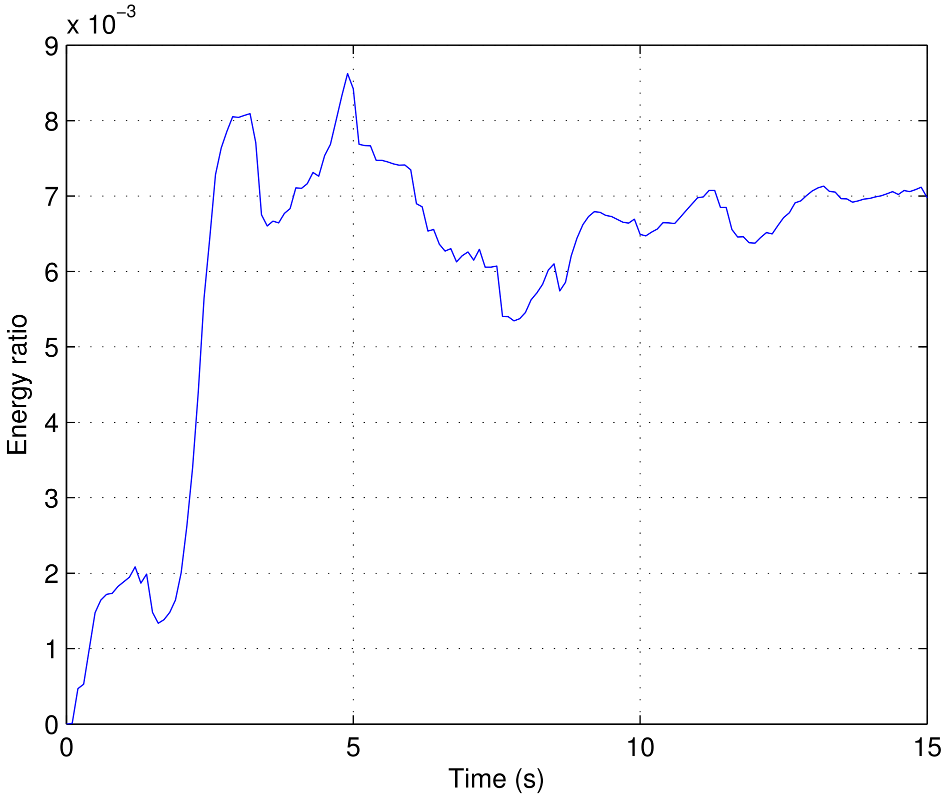

FIGURE 8.1: Ratio of the regulated output energy to the disturbance input noise energy of a tunnel diode circuit.

proposed controller are provided in terms of the solvability of PMIs, which are formulated as SOS constraints and have been solved using SOSTOOLS. The effectiveness of the proposed design methodology is demonstrated through a tunnel diode circuit.

Chapter 9

Conclusion

9.1 Summary of Thesis

This thesis proposes novel methodologies for controller synthesis and filter design for polynomial discrete-time systems. For the controller synthesis, state feedback controllers and output feedback controllers are designed with and without H_∞ performance. Polytopic uncertainties and norm-bounded uncertainties have also been considered in this thesis. To ensure a convex solution to the control design problem for polynomial discrete-time systems can be rendered efficiently, an integrator is incorporated into controller structures. It has been shown that by incorporating the integrator into the controller structures, original systems can be transformed into augmented systems and the Lyapunov function can be selected as such its matrix is only dependent upon the original state. In doing so, a nonconvex controller design problem of polynomial discrete-time systems can be converted into a convex design problem in a less conservative way than available approaches. This consequently allows the problem to be solved via SDP. The integrator method is also applied to filter design for polynomial discrete-time systems. The effectiveness and advantages of the proposed design methodologies are verified by numerical examples in every chapter. The simulation results show that the proposed design methodologies can fulfil the prescribed performance requirement.

Some conservatisms aspect of the proposed method are also explained accordingly in most of the chapters. Generally, the main problem of this integrator method is computational complexity. This problem is common to all methods that are based on the augmented systems. However, in this research work, the problem becomes more severe because all the involved matrices are defined in polynomial forms. This consequently creates a large sizes of SDP which requires a lot of memory spaces in order to solve the problem. This is the reason of selecting small sizes of systems as our numerical examples so that the feasibility problem of the controller synthesis and filter design can be performed efficiently.

To clarify the approach used in this research work, seven technical chapters have been provided. In chapter 2, the problem of designing a controller for polynomial discrete-time systems is raised. Then, a novel method, called an integrator method is proposed to solve the problem in a less conservative way than the available approaches. Based on this integrator method, a nonlinear feedback control is tackled and it has been shown that the results can be extended to the robust control problem with polytopic uncertainty. Furthermore, in chapter 3, the robust control problem with norm-bounded uncertainty for polynomial discrete-time systems is presented. The nonlinear H_∞ state feedback control problem for polynomial discrete-time systems is discussed in Chapter 4 and Chapter 5. In particular, the nonlinear H_∞ state feedback control problem is provided in Chapter 4, and the results are subsequently extended to the robust nonlinear H_∞ state feedback control problem with the existence of polytopic uncertainties. Meanwhile, Chapter 5 presents the results of the robust H_∞ control problem with the existence of norm-bounded uncertainties.

The filtering problem is presented in Chapter 6. The problem of filtering design for polynomial discrete-time systems is provided and in light of the integrator method, a possible solution to the filter design problem is given. A nonlinear output feedback controller is developed in Chapter 7 in which the stability and the H_∞ performance objectives must be fulfil. The results are then subsequently extended to the robust H_∞ control problem with polytopic uncertainties. Lastly, a robust H_∞ output feedback controller design is constructed in Chapter 8 so that both robust stability and a prescribed level of disturbance attenuation performance for the closed-loop systems are achieved. Here, the uncertain terms are described as norm-bounded.

Here is a summary of the contributions of this thesis:

- The controller synthesis for polynomial discrete-time systems are considered. The controller designs are performed with and without the performance objective i. e H_∞ control.
- A less conservative design procedure of controller synthesis is obtained by incorporating an integrator into controller structures and.
- A possible method for solving the filtering problem for polynomial discrete-time systems is also given. This is delivered with the help of an integrator approach.

- The polytopic and norm-bounded uncertainties have been considered in this research. The methodologies for solving the uncertain polynomial discrete-time systems with polytopic and norm-bounded uncertainties have been provided.

  As a result, this thesis provides a less conservative design methodology for the controller synthesis and filter design of polynomial discrete-time systems and represents a valuable and meaningful contribution to the development of an SOS-SDP based solution in the framework of polynomial discrete-time systems.

9.2 Future Research Work

In general, control of polynomial discrete-time systems still remains an open area and lots of research work needs to be conducted. Further research work, to name a few, could be directed to the following areas:

1. Time-delays system is one of the important problem in the framework of control systems engineering. In fact, to the author's knowledge, no result has been presented yet in the framework of controller synthesis for polynomial discrete-time systems with time-delays. Therefore, it is interesting to consider the design of a controller for polynomial discrete-time systems with time-delays. It is also desirable to see whether the incorporation of an integrator into the controller structure could help in solving the problem or not. The success of controller synthesis for such discrete time-delays systems could provide a potential methodology of the controller synthesis for the networked control systems in which the system or plant is represented by polynomial discrete-time systems.

2. It is noticeable from our research work that the incorporation of an integrator into the controller structure leads to the computational burden. This is due to the fact that a large number of sparse is created when solving the problem. This problem is quite common in the field of SOS programming, especially when using SOSTOOLS. Hence, research on reducing the number of spare is the most interesting one to be done in the future. The reduction of this sparse will reduce the size of SDP, hence, could provide a better future for this integrator method because it can also be applied to higher order systems than the systems used in this research work.

3. The filtering problem in this research work is not completely studied. The performance objective, such as H_∞ should be considered in the future. The robust filtering problem with polytopic and norm-bounded uncertainty is also an interesting area to be explored. As only the full-order filter design is considered in this research work, the reduced-order filter design problem must be the one that

is very important to be explored in the future. It is indeed interesting to see how the integrator method can be applied in order to develop a methodology for this reduced-order filter design methodology.

4. In this research, the observer design has not been performed. Therefore, for future research, it is highly recommended to study this observer design for polynomial discrete-time systems. The other interesting consideration for future research work is involving the fault-tolerant problem for polynomial discrete-time systems.

Appendix A

Mathematical

In this section, we will introduce some mathematical background knowledge that will be applied throughout this research.

A.1 Linear Matrix Inequality (LMI)

Due to the fact that the SOS inequality is actually complementary from the LMI, therefore it is necessary to understand the concept of LMI. In this section, a brief overview regarding the LMI theory is presented. A detail description about LMI can be referred to [29].

Since the early 1990s, with the development of interior-point method methods for solving LMI problems, the LMI method [29] has gained increased interest and emerged as a useful tools for solving a number of control problems; i.e. synthesis of gain-scheduled (parameter-varying) controllers, mixed-norm and multi objective control design, hybrid dynamical systems, and fuzzy control. Three important factors that make LMI techniques appealing:

- A variety of design specifications and constraints can be expressed as LMIs.
- Once formulated in terms of LMIs, a problem can be solved exactly by efficient convex optimisation algorithms (the "LMI solvers").
- While most problems with multiple constraints or objectives lack analytical solutions in terms of matrix equations, they often remain tractable in the LMI framework. This makes LMI-based design a valuable alternative to classical "analytical" methods

For system and control perspective, the importance of LMI optimisation stems from the fact that a wide variety of system and control problems can be recast as LMI problems. Therefore recasting a control problem as an LMI problem is equivalent to finding a "solution" to the original problem.

A LMI has the form [29]

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i > 0, \tag{A.1}$$

where $x \in R^m$ is the variable to be determined and symmetric matrices $F_i = F_i^T \in R^{n\times m}$, $i = 0, ..., m$, are given. The inequality symbol in (A.1) means that $F(x)$ is positive definite, i.e $u^T F(x)u > 0$ for all nonzero $u \in R^n$.

Even though this canonical expression (A.1) is generic, LMI rarely arise in this form in control applications. Instead, structured representation of LMIs is often used. For instance, the expression $A^T P + PA < 0$ in the Lyapunov inequality is explicitly described as a function of the matrix variable P, and A is the given matrix. In addition to saving notation, the structured representation may lead to more efficient computation.

A.2 The Schur Complement

The Schur complement [29] is standard in the LMI context. The basic idea is as follows: The LMI

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} > 0 \tag{A.2}$$

where $Q(x) = Q^T(x)$, $R(x) = R^T(x)$, and $S(x)$ depend affinely on x, is equivalent to

$$R(x) > 0, \qquad Q(x) - S(x)R^{-1}(x)S^T(x) > 0 \tag{A.3}$$

In other words, the set of nonlinear inequalities (A.3) can be represented as the LMI (A.2).

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LIST OF AUTHOR'S PUBLICATIONS

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