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Contents

Part 1. Randomness and computable analysis

1. Computability of Ergodic Convergence (Towsner)

Henry Towsner was at the Feb Oberwolfach meeting and discussed with Nies and Bienvenu.

Let μ be a computable measure on Ω, let T be a computable, measure-preserving transformation. Henry described an example of an L^1 computable function where the limit in the sense of Birkhoff is not L^1 computable. This was in response to a question of Nies and Bienvenu.

Definition 1.1. For any function f, we write $A_N f$ for the function

$$\frac{1}{N}\sum_{i<N} f(T^i x).$$

Recall the following.

- If $f \in L^1(\mu)$ then by Birkhoff ergodic theorem,
$$\overline{f}(x) := \lim_N A_N f(x)$$
exists a.e. and $\overline{f}$ is in $L^1(\mu)$.
- If in fact $f \in L^2(\mu)$ then the $A_N f$ converge in the L^2 norm by the main ergodic theorem.

Definition 1.2. We say the mean rate of convergence of f is computable if the $A_N f$ converge computably, namely, there is a computable function p such that for every n and every $m \geq p(n)$,

$$||A_{p(n)}f - A_m f||_{L^2} < 1/n.$$

We say the pointwise rate of convergence of f is computable if there is a computable function p such that for every n and every $m \geq p(n)$,

$$\mu(\{x \mid \exists i \in [p(n), m] |A_{p(n)}f(x) - A_i(x)| > 1/n\}) < 1/n.$$

Theorem 1.3. *Let μ be a computable measure on Ω, let T be a computable, measure-preserving transformation, and let f be L^1-computable with respect to μ such that $||f||_{L^2}$ exists (i.e., $f \in L^2(\mu)$). Then the following are equivalent:*

(1) $||\overline{f}||_{L^2}$ *is computable,*
(2) *The mean rate of convergence of f is computable,*
(3) *The pointwise rate of convergence of f is computable,*
(4) $\overline{f}$ *is L^1-computable.*

Proof. For the first three, the implications (1)⇒(2) and (1)⇒(3) are shown in [2] while the implications (3)⇒(2) and (2)⇒(1) are trivial. For the equivalence with (4), it is obvious that (3) implies (4). For the converse, not that $||\overline{f}||_{L^2} \leq ||f||_{L^2}$, and in particular $||\overline{f}||_{L^2}$ is bounded. Therefore (4) implies (2). □

I don't know of any proof that (4) implies (3) in the absence of a bound on the L^2 norm. This is almost certainly still true, but a bit more work is required, and there are some techicalities in the proof which might create problems. (Specifically, it's not clear to me that every L^1 computable function is computably approximable by L^2 functions.)

An example where all four properties fail is given in [2]. Divide Ω into countable many components. On the i-th component, T is supposed to represent a "rotation" of 2^{-j} if the i-th Turing machine halts in exactly j steps, and the identity if the the i-th Turing machine never halts. (By a rotation, I mean adding 2^{-j} to each element in the component, wrapping around if we overflow out of that component.) More precisely, consider some sequence $1^i 0\sigma = \tau \in \Omega$. (That is, the sequence starts with an initial segment of exactly i 1's.) If the i-th Turing machine halts at step j,

$$T\tau = \left(2^{i+1}\sigma + 2^{-j} \mod 1\right) 2^{-(i+1)}.$$

If the i-th Turing machine never halts, $T\tau = \tau$. Consider the set $A = \bigcup_i [1^i 01]$. On any element $\tau \in [1^i 00]$, $\overline{\chi_A}(\tau)$ is 0 if the i-th Turing machine never halts and 1/2 if the i-th Turing machine does halt. In particular, given a computable f such that $||f - \overline{\chi_A}||_{L^1} < 2^{-(i+4)}$, we may computably find subintervals of $[1^i 00]$ of collective measure $2^{-(i+3)}$ such that f is defined on these subintervals, and the i-th Turing machine halts iff the average of f over these subintervals is $> 1/4$.

2. Lebesgue Density and Lebesgue differentiation

The following is from the preprint [6]; a shortened version has been submitted, but does not contain the results below.

For measurable sets $P, A \subseteq \mathbb{R}$ with A non-null, $\lambda(P|A) = \lambda(P \cap A)/\lambda(A)$ is the conditional measure (probability) of P given A. Recall that the lower density of a measurable set $P \subseteq \mathbb{R}$ at a point $z \in \mathbb{R}$ is

$$\rho(P|z) = \liminf_{h \to 0} \{\lambda(P|I) \,:\, I \text{ is an open interval, } z \in I \;\&\; |I| < h\}.$$

Intuitively, $\rho(P|z)$ gauges the fraction of space filled by P around z if we "zoom in" arbitrarily close to z.

Lebesgue's density theorem [40, page 407] says that for any measurable set P, for almost all $z \in P$ we have $\rho(P|z) = 1$. An effective version of this theorem is given by identifying a collection of effectively presented sets P and the collection of random point z for which $\rho(P|z) = 1$ for all sets P in the collection containing z as an element. Since the theorem is immediate for open sets, the simplest nontrivial effective version is obtained by choosing P to range over the collection of effectively closed subsets of $\mathbb{R}$. We call a real number $z \in \mathbb{R}$ a *density-one point* if for every effectively closed set P containing z we have $\rho(P|z) = 1$.

Definition 2.1. A non-decreasing, lower semicontinuous function $f\colon [0,1] \to \mathbb{R}$ is *interval-c.e.* if $f(0) = 0$, and $f(y) - f(x)$ is a left-c.e. real, uniformly in rationals $x < y$.

We investigate density-one points regardless of randomness. Our investigations yield more information about interval-c.e. functions.

Recall that a measurable function $g\colon [0,1] \to \mathbb{R}$ is *integrable* if $\int_{[0,1]} |g| \, \mathrm{d}\lambda$ is finite.

A real $z \in [0,1]$ is called a *Lebesgue point* of an integrable function g if

$$(*) \qquad \lim_{|Q| \to 0} \frac{1}{|Q|} \int_Q g \, \mathrm{d}\lambda = g(z),$$

where Q ranges over open intervals containing z. A real z is called a *weak Lebesgue point* if the limit in $(*)$ exists (but may be different from $g(z)$).

The Lebesgue differentiation theorem states that for any integrable function g, almost every point $z \in [0,1]$ is a Lebesgue point of g.

If g is the characteristic function $1_{\mathcal{C}}$ of a measurable set $\mathcal{C}$, then the limit $(*)$ above equals the density $\rho(\mathcal{C}|z)$. Thus, one can view the density theorem as a special case of the differentiation theorem.

As with the density theorem, an effective version of the Lebesgue differentiation theorem is obtained by specifying a collection $\mathfrak{F}$ of effectively presented integrable functions. For example, Pathak, Rojas and Simpson [55] and independently Freer et al. [25] studied the Lebesgue differentiation theorem for L^1-computable functions; Freer et al. also considered the collection of L^p-computable functions for a computable real $p \geq 1$. They showed that Schnorr randomness of z is equivalent to being a weak Lebesgue point for each such function; the implication left to right is due to Pathak et al. Here we consider bounded lower semicomputable functions; we will see that every Oberwolfach random point is a Lebesgue point of any bounded lower semicomputable function, and a weak Lebesgue point of any integrable lower semicomputable function. Indeed, we observe that the effective versions of the Lebesgue density and differentiation theorems that we consider in this paper are equivalent for *any* real number z.

Proposition 2.2. *The following are equivalent for a real $z \in [0,1]$.*

(i) *z is a density-one point.*
(ii) *z is a Lebesgue point of every bounded upper semi-computable function $g\colon [0,1] \to \mathbb{R}$.*
(iii) *z is a Lebesgue point of every bounded lower semi-computable function $g\colon [0,1] \to \mathbb{R}$.*

Since $[0,1]$ is compact, every lower semi-continuous function on $[0,1]$ is bounded from below, and every upper semi-continuous function on $[0,1]$ is bounded from above. Hence in (ii) we could merely require that the upper semi-computable function g be bounded from below, and in (iii), that the function be bounded from above.

Proof. (ii) $\Rightarrow$ (i) is immediate. Indeed, if $\mathcal{C}$ is effectively closed then the density of $\mathcal{C}$ at z is precisely the limit in $(*)$ for $g = 1_{\mathcal{C}}$, the characteristic function of $\mathcal{C}$. The function $1_{\mathcal{C}}$ is upper semi-computable and is integrable.

(iii) $\Rightarrow$ (ii). If g is upper semi-computable and integrable then $-g$ is lower semi-computable and integrable.

(i) $\Rightarrow$ (iii). Let z be a density-one point. We show, in three steps, that z is a Lebesgue point of every integrable lower semi-computable function.

First, let $g = 1_{\mathcal{C}}$ for an effectively closed set $\mathcal{C}$. If $z \in \mathcal{C}$ then the equality $(*)$ holds at z because z is a density-one point. If $z \notin \mathcal{C}$ then z is a Lebesgue point of g because the complement of $\mathcal{C}$ is open.

Second, the property of being a Lebesgue point is preserved under taking linear combinations of functions. We conclude that z is a Lebesgue point for all linear combinations of characteristic functions of effectively closed sets.

Finally, let g be any bounded lower semi-computable function. By scaling and shifting, we may assume that g is bounded between 0 and 1.

We approximate g by a step-function. For $x \in [0,1]$, let $f(x)$ be the greatest integer multiple of 2^{-k} which is bounded by $g(x)$. For all $i \leq 2^k$, let

$$\mathcal{C}_i = g^{-1}(-\infty, i \cdot 2^{-k}].$$

Each set $\mathcal{C}_i$ is effectively closed, and $f = 2^{-k} \sum_{i=1}^{2^k} 1_{\mathcal{C}_i}$. Then $||f - g||_\infty \leq 2^{-k}$, which implies that for any interval Q,

$$\left| \frac{1}{|Q|} \int_Q g \, \mathrm{d}\lambda - \frac{1}{|Q|} \int_Q f \, \mathrm{d}\lambda \right| \leq 2^{-k}.$$

Because f is a linear combination of characteristic functions of effectively closed sets, we know that for sufficiently short intervals Q containing z we have

$$\left| f(z) - \frac{1}{|Q|} \int_Q f \, \mathrm{d}\lambda \right| < 2^{-k}.$$

Because $|f(z) - g(z)| \leq 2^{-k}$ we conclude that for sufficiently short intervals Q containing z, we have

$$\left| g(z) - \frac{1}{|Q|} \int_Q g \, \mathrm{d}\lambda \right| < 3 \cdot 2^{-k}.$$

□

Question 2.3. *If z is a density-one point, is z a Lebesgue point of every* integrable *lower semicomputable function?*

Recall that a non-decreasing, lower semicontinuous function $f\colon [0,1] \to \mathbb{R}$ with $f(0) = 0$ corresponds to a measure μ_f on $[0,1)$, determined by $\mu([x,y)) = f(y) - f(x)$. The measure μ_f is absolutely continuous with respect to Lebesgue measure if and only if the function f is an absolutely continuous function. In this case, the Radon-Nikodym theorem says that $\mu_f(A) = \int_A g \, \mathrm{d}\lambda$ for some non-negative integrable function g. A real z is a Lebesgue point of g if and only if $f'(z)$ exists and equals $g(z)$, and a weak Lebesgue point if and only if $f'(z)$ exists.

If g is lower semicomputable then f is interval-c.e.

Corollary 2.4. *Every Oberwolfach random real is a weak Lebesgue point of every integrable lower semicomputable function.*

Proof. If g is lower semi-computable, then it is bounded from below, and so by adding a constant we may assume it is positive. Then apply Theorem **??**. □

A weaker version of Question 2.3 is:

Question 2.5. *Is every Oberwolfach random real a Lebesgue point of every integrable lower semicomputable function?*

Finally, we see that the relation between non-negative, integrable lower semicomputable functions and absolutely continuous interval-c.e. functions is not a correspondence. The next result shows that there is an interval-c.e. function f which is not the distribution function $\int_0^x g \, \mathrm{d}\lambda$ for any lower semicomputable function g, indeed not for any lower semicontinuous function g.

Proposition 2.6. *There is nondecreasing computable (hence, interval-c.e.) Lipschitz function f that is not of the form $f(x) = \int_0^x g \, \mathrm{d}\lambda$ for any lower semicontinuous function g.*

Proof. Let M be a computable martingale that succeeds on any $Z \in 2^{\mathbb{N}}$ failing the law of large numbers. By Theorem 4.2 of [25] (and its proof) there is a computable Lipschitz function f such that $f'(z)$ fails to exist whenever M succeeds on a binary expansion Z of z. Adding a linear term, we may assume that f is nondecreasing. Now suppose $f(x) = \int_0^x g \, \mathrm{d}\lambda$ for a lower semicontinuous function g. If C is a Lipschitz constant for f, then

$\{x\colon\ g(x) > C\}$ is a null set. Since this set is also open, it is empty. Hence g is bounded.

If Z is 1-generic relative to a representation of g then Z is a density-one point relative to this representation of g by relativizing the observation in [7] mentioned earlier on. Hence by Proposition 2.2 in relativized form, z is a Lebesgue point of g. Then $f'(z)$ exists.

On the other hand, each 1-generic Z fails the law of large numbers. So M succeeds on Z, and $f'(z)$ does not exist. Contradiction. □

3. Randomness notions and their corresponding lowness classes

3.1. **Summary of randomness notions.** The following diagram gives an overview of the randomness notions we have discussed. They are stronger than, but close to, ML-randomness. The diagram is a modification of a similar diagram in [7]. We consider properties of a given ML-random real.

$$\begin{tikzcd} \textit{Oberwolfach random} \arrow[r] & \text{all interval-c.e. functions are differentiable at the real} \arrow[r] & \text{density-one point} \arrow[r] & \text{positive density point} \arrow[d, leftrightarrow] \\ \text{not LR-hard} \arrow[u] \arrow[rrr] & & & \text{Turing incomplete} \end{tikzcd}$$

The rightmost vertical double arrow refers to a result of Bienvenu et al. [7]. Random pseudo-jump inversion implies that the implication not LR-hard $\to$ Turing incomplete is proper. In fact, by [15], the implication density-one point $\to$ positive density point is proper.

One way to separate these notions when viewed as operators on oracles would be to separate the corresponding lowness classes. Recall that an oracle A is low for a randomness notion $\mathcal{C}$ if $\mathcal{C}^A = \mathcal{C}$. More generally, A is low for a pair of randomness notions $\mathcal{C} \subseteq \mathcal{D}$ if $\mathcal{C} \subseteq \mathcal{D}^A$. Combining results in [19, 50] shows that for the pair $\mathcal{C}$ = weak 2-randomness and $\mathcal{D}$ = ML-randomness, the double lowness class coincides with K-triviality. Thus, the lowness class for any of the notions above is contained in the K-trivials.

Using the recent result of Day and Miller [14], Franklin and Ng [23] have shown that lowness for difference randomness coincides with K-triviality. We now obtain such a coincidence for two further notions in the diagram above: density-one points, and being not LR-hard.

Proposition 3.1. *Let A be K-trivial.*

(1) *A is low for the notion "density-one $\cap$ ML-random".*
(2) *A is low for the notion "non-LR-hard $\cap$ ML-random".*

Proof. Each time we need to show that if Z is not A-random in the given sense, then Z is not random in that sense. Since A is low for ML-randomness and our notions imply ML-randomness, we may assume that Z is ML-random.

(1). Suppose $Z \in \mathcal{P}$ for some $\Pi^0_1(A)$ class $\mathcal{P}$ with $\rho(Z \mid P) < 1$.
Since A is K-trivial and Z random, by the argument of Day and Miller [14] there is a Π^0_1 class $\mathcal{Q}$ with $\mathcal{P} \supseteq \mathcal{Q} \ni Z$. Then $\rho(Z \mid \mathcal{Q}) < 1$.

(2). Let MLR denote the class of ML-randoms. We modify an argument of Hirschfeldt [51, 8.5.15]. Suppose that Z is LR-hard relative to A, namely, $Z \oplus A \geq_{LR} A' \equiv_T \emptyset'$. We show $Z \geq_{LR} \emptyset'$. Suppose that $Y \in \mathsf{MLR}^Z$. Then $Y \oplus Z \in \mathsf{MLR}$ by van Lambalgen's theorem, so $Y \oplus Z \in \mathsf{MLR}^A$. This implies $Y \in \mathsf{MLR}^{Z \oplus A}$ by van Lambalgen's theorem relative to A. By our hypothesis on Z, this implies that Y is ML-random relative to $\emptyset'$. Thus, $Z \geq_{LR} \emptyset'$. $\square$

Part 2. Randomness, Kolmogorov complexity, and computability

4. Cupping Δ^0_2 DNR sets (Joseph S. Miller, June 2012)

Nies, Stephan and Terwijn [53] proved that if $Z \leq_T \emptyset'$ is Martin-Löf random and $C \in 2^\omega$ is a c.e. set, then either $Z \oplus C \geq_T \emptyset'$ or Z is Martin-Löf random relative to C. We prove an analogous result with "is Martin-Löf random" replaced by "has DNC degree".

Theorem 4.1. *Assume that $X \in 2^\omega$ has DNC degree and $C \in 2^\omega$ is a c.e. set. Either $X \oplus C \geq_T \emptyset'$ or X has DNC degree relative to C.*

Proof. Let $g\colon \omega \to \omega$ be an X-computable DNC function. By the recursion theorem, we may assume that we control an infinite sequence of positions $\{k_{n,m}\}_{n,m\in\omega}$ of the diagonal function $e \mapsto \phi_e(e)$. If n enters $\emptyset'$ at stage s, then define

$$\phi_{k_{n,m}}(k_{n,m}) = \phi^{C_s}_{k_{n,m},s}(k_{n,m}),$$

if the latter converges. If there is an n such that $m \mapsto g(k_{n,m})$ is a DNC function relative to C, then we are done. If not, define $f \leq_T X \oplus C$ such that $f(n)$ is the least s such that $(\exists m \leq s)\ g(k_{n,m}) = \phi^{C_s}_{k_{n,m},s}(k_{n,m})$ for a C-correct computation. By assumption, f is total. If n enters $\emptyset'$, then it must happen at a stage $s < f(n)$. Otherwise, we would contradict the fact that g is DNC. Therefore, $\emptyset' \leq_T f \leq_T X \oplus C$. $\square$

This result is quite similar to, and was motivated by, a beautiful theorem of Day and Reimann [16, Corollary 8.2.1]. They proved that if $X \in 2^\omega$ has PA degree and $C \in 2^\omega$ is a c.e. set, then either $X \oplus C \geq_T \emptyset'$ or $X \geq_T C$. The conclusion can fairly easily be strengthened to highlight the similarity with Theorem 4.1 (Day, personal communication, October 2011).

Theorem 4.2 (Day and Reimann)**.** *Assume that $X \in 2^\omega$ has PA degree and $C \in 2^\omega$ is a c.e. set. Either $X \oplus C \geq_T \emptyset'$ or X has PA degree relative to C.*

Proof. Apply the result of Day and Reimann to X and C. If $X \oplus C \geq_T \emptyset'$, we are done. Otherwise, $X \geq_T C$ and $X \not\geq_T \emptyset'$. Take $Y \in 2^\omega$ such that Y has PA degree and X has PA degree relative to Y, which is possible by Simpson [59, Theorem 6.5]. Note that $X \geq_T Y$, so $Y \oplus C \leq_T X \not\geq_T \emptyset'$. Applying the result of Day and Reimann to Y and C gives $Y \geq_T C$. Therefore, X has PA degree relative to C. $\square$

The proof of Theorem 4.1 could be used, with only superficial modification, to prove this result. Kučera (2011) also gave a direct proof of Day and Reimann's result.

Note that the DNC version of the original Day and Reimann result is false. In other words, we cannot replace "X has DNC degree relative to C" with "$X \geq_T C$" in Theorem 4.1. To see this, let $C \in 2^\omega$ be a low c.e. set that is not K-trivial. By the low basis theorem relative to C, there is a Martin-Löf random $Z \in 2^\omega$ such that $Z \oplus C$ is low. So Z has DNC degree and $Z \oplus C \not\geq_T \emptyset'$. But $Z \geq_T C$ would imply that C is K-trivial by Hirschfeldt, Nies and Stephan [31].

We now consider another theorem of Nies, Stephan and Terwijn [53]. They proved that if $Z \leq_T \emptyset'$ is Martin-Löf random relative to A, then A is GL_1 (i.e., $A' \leq A \oplus \emptyset'$). Any Z that is Martin-Löf random relative to A has DNC degree relative A, so the following theorem generalizes their result.

Theorem 4.3. *Assume that $X \in 2^\omega$ is Δ^0_2 and has DNC degree relative to $A \in 2^\omega$. Then A is GL_1.*

Proof. Let $g\colon \omega \to \omega$ be an X-computable DNC function relative to A. Because g is Δ^0_2, there is a computable $h\colon \omega^2 \to \omega$ such that $(\forall n)\ g(n) = \lim_{s\to\infty} h(n,s)$. By the relativized recursion theorem, we may assume that we A-computably control an infinite *computable* sequence of positions $\{k_n\}_{n\in\omega}$ of the diagonal function relative to A, i.e., $e \mapsto \phi^A_e(e)$. If n enters A' at stage s, then let $\phi^A_{k_n}(k_n) = h(k_n, s)$. Define $f \leq_T \emptyset'$ such that $f(n)$ is the least s such that $(\forall t \geq s)\ g(k_n) = h(k_n, t)$. If n enters A', then it must happen at a stage $s < f(n)$. Otherwise, we would contradict the fact that g is DNC relative to A. Therefore, $A' \leq_T A \oplus f \leq_T A \oplus \emptyset'$. □

Kučera and Slaman (1989) built an incomplete c.e. set that cups every Δ^0_2 DNC degree to $\emptyset'$. Bienvenu, Greenberg, Kučera, Nies and Turetsky (2012) showed that, in fact, every superhigh c.e. set has this property. Their proof uses Kolmogorov complexity; Kučera gave an alternate and purely computability-theoretic proof. We improve their result further by showing that any non-low c.e. set cups every Δ^0_2 DNC degree to $\emptyset'$.

Corollary 4.4. *If $C \in 2^\omega$ is a non-low c.e. set and $X \in 2^\omega$ has Δ^0_2 DNC degree, then $X \oplus C \equiv_T \emptyset'$.*

Proof. By Theorem 4.1, either $X \oplus C \geq_T \emptyset'$ or X has DNC degree relative to C. In the latter case, Theorem 4.3 implies that C is GL_1, hence low. This is not true, so $X \oplus C \geq_T \emptyset'$. Clearly, $X \oplus C \leq_T \emptyset'$. □

Note that if $C \in 2^\omega$ is a low set, then the low basis theorem relativized to C gives us an X that has DNC (even PA) degree such that $X \oplus C$ is low. Therefore, the corollary is tight: no low (c.e.) set cups every Δ^0_2 DNC degree to $\emptyset'$.

5. Cupping Δ^0_2 DNR sets (Bienvenu, Kučera, et al., Feb. 2012)

The following was obtained during the Research in Pairs stay at MFO, of Bienvenu, Greenberg, Kučera, Nies and Turetsky. It also provides a short proof of a 1989 result by Kučera and Slaman who built a c.e. incomplete set that cups all Δ^0_2 DNR sets above $\emptyset'$. The idea to use Kolmogorov complexity K is due to Bienvenu. There also is a new proof not using K but much shorter than the original construction; this is due to Kučera. See Subsection 5.2.

5.1. **A proof using Kolmogorov complexity.**

Lemma 5.1. *Let B be the function $B(n) = \min\{t \in \mathbb{N} \mid \forall s > t,\ K(s) \geq n\}$. Any function dominating B computes $\emptyset'$.*

Theorem 5.2. *Let A be a c.e. set such that $K^A(\sigma) \leq^+ f(K^{\emptyset'}(\sigma))$ for some Δ^0_2 function f. Then A joins every Δ^0_2 DNR set above $\emptyset'$.*

Taking $f(x) = x + O(1)$ in the theorem, this shows that any LR-hard c.e. degree joins every Δ^0_2 DNR set above $\emptyset'$, and in particular there exists an incomplete such c.e. set.

Proof. A first remark: the bigger the function f, the stronger the result, so we can assume that f is increasing and $f(n) > 4n$ for all n. Let A be such a set and D be a set of DNR degree. We use a result of Kjos-Hanssen, Merkle and Stephan: D having DNR degree is equivalent to D computing a sequence (σ_n) of strings such that $K(\sigma_n) \geq n$. D being $\emptyset'$-computable, using $\emptyset'$, we can compute the sequence $\sigma_{2f(n)}$, thus $K^{\emptyset'}(\sigma_{2f(n)}) \leq^+ K^{\emptyset'}(n) \leq^+ 2\log n$. By the assumption on A, $K^A(\sigma_{f(2n)}) \leq^+ f(2\log n) \leq^+ f(n)$. On the other hand $K(\sigma_{2f(n)}) \geq 2f(n)$. Informally this means that A contains a lot of information about the $\sigma_{2f(n)}$ (it makes the Kolmogorov complexity of $\sigma_{2f(n)}$ drop from at least $2f(n)$ to at most $f(n)$).

We now show how to use $A \oplus D$ to compute $\emptyset'$, using Lemma 5.1. Given n, use $A \oplus D$ to do the following. First, using D, compute the sequence σ_i, and using A look for an index i such that $K^A(\sigma_i) \leq i - 2n$. Such an i exists as $K^A(\sigma_{2f(n)}) \leq^+ f(n) \leq^+ 2f(n) - 2n$. Finding such an i means finding a program p of length at most $i - 2n$ for the A-universal machine $\mathbb{U}^A$. Let u be the use of A in the computation $\mathbb{U}(p) = \sigma_i$ and let t_n be the settling time of $A \upharpoonright u$. We claim that for any sufficiently large n, $t_n \geq B(n)$, which by Lemma 5.1 will prove the result. Let thus s be any integer bigger that t_n. First, notice that

$$K(\sigma_i) \leq^+ |p| + K(s) \tag{1}$$

Indeed, if one knows p and s, one can compute $U^{A_s}(p) = \mathbb{U}^A(p) = \sigma_i$ (the first equality comes from the definition of s). Since $K(\sigma) \geq i$ and $|p| \leq n - i$, it follows that

$$K(s) \geq^+ K(\sigma_i) - |p| \geq^+ i - (i - 2n) \geq^+ 2n \tag{2}$$

And thus for n large enough $K(s) \geq n$.

□

For the definition of JT-reducibility $\leq_{JT}$ see [51, 8.4.13]. We say that A is JT-hard if $\emptyset' \leq_{JT} A$.

Lemma 5.3. *The following are equivalent for any set A.*
(i) A is JT-hard
(ii) There exists a computable order h such that $K^A \leq^+ h(K^{\emptyset'})$

Proof. First suppose that A is JT-hard. Consider the universal oracle machine $\mathbb{U}$ for prefix complexity and consider the $\emptyset'$-partial computable function $S : \sigma \mapsto \mathbb{U}^{\emptyset'}(\sigma)$. By definition of JT-hardness, there exists a computable order g and a family (T_σ) of uniformly A-c.e. finite sets such that $\mathbb{U}^{\emptyset'}(\sigma) \in T_\sigma$ and $|T_\sigma| \leq g(|\sigma|)$. Let now x be any string and set $K^{\emptyset'}(x) = n$. This by definition means that $x = \mathbb{U}^{\emptyset'}(\sigma)$ for some σ of length n. Relative to A, x can be described by σ, and its index in the A-enumeration of T_σ. Thus $K^A(x) \leq^+ 2|\sigma| + 2|T_\sigma| \leq^+ 2n + 2g(n) \leq^+ h(K^{\emptyset'}(x))$, where $h(n) = 2n + 2g(n)$, which is a computable order, as wanted.

Conversely, suppose $K^A \leq^+ h(K^{\emptyset'})$ for some computable order h. Let f be a given $\emptyset'$-partial computable function. By definition of K, we have

$K^{\emptyset'}(f(n)) \leq^+ K^{\emptyset'}(n) \leq^+ 2\log n$ and thus by assumption $K^A(f(n)) \leq h(2\log n + c)$ for some constant c. To get an A-trace for $f(n)$, it thus suffices to A-enumerate all x's such that $K^A(x) \leq h(2\log n + c)$, and we know that there are at most $g(n) = 2^{h(2\log n+c)}$. Thus all $\emptyset'$-partial computable functions have an A-traced with size bounded by $g + O(1)$, which precisely means that A is JT-hard.

□

Theorem 5.4. *If A is a superhigh c.e. set, then for any Δ^0_2 DNR set X, one has $A \oplus X \geq_T \emptyset'$.*

Proof. It is known [51, Thm. 8.4.16 and Cor. 8.4.27] that for c.e. sets, superhighness is equivalent to JT-hardness. Thus we can apply Lemma 5.3 to get a computable order h such that $K^A \leq^+ h(K^{\emptyset'})$. Now, by relativizing Theorem 5.2 to A, we immediately obtain that $A \oplus X \geq_T \emptyset'$. □

In the following we analyze the hypothesis of Theorem 5.2 and show it is equivalent to a property we call fairly highness.

Proposition 5.5. *The following are equivalent for a Δ^0_2 set A.*

(i) A is high

(ii) For each pc functional Γ, we can trace $\Gamma^{\emptyset'}$ by an A-c.e. trace with a finite bound.

(i)→(ii): $\emptyset'$ is low relative to A. Hence it has an enumeration relative to A such that $\Gamma^{\emptyset'}(x)$ becomes undefined only finitely often.

(ii) →(i): Let $\Gamma^Z(x)$ be the number of steps it takes $J^Z(x)$ to converge (may be undefined). There is an A-c.e. trace T_x for $\Gamma^{\emptyset'}$. Let $M(x) = \max T_x$, then $M \leq_{\mathrm{T}} A'$. Hence $\emptyset'' \leq_{\mathrm{T}} A'$.

The hypothesis of Theorem 5.2 is equivalent to the following condition on a set A.

Definition 5.6. A is *medium high* if there exists a $\emptyset'$-computable order h such that A c.e.–traces with bound h every $\emptyset'$-partial computable function.

For a c.e. set S, being medium low (see Section 6) is equivalent to that J^S can be traced with a bound computable in S. Hence, by pseudo jump inversion (letting $S \equiv_T \emptyset'$), if A is Δ^0_2, then by Theorem 6.2, A is medium high iff $\emptyset'$ is medium low relative to A. In particular there is a c.e. set A that is medium high but not superhigh. Thus, for Δ^0_2 sets, this property lies properly between superhighness and highness.

5.2. **An alternative recursion-theoretic proof.** *Notation*: For an expression E which is approximable during stages s, we denote by $E[s]$ its value by the end of stage s. Let $m_x(n)$ be the modulus function of $A = W_x$, computable from A Further, let $A'[s]$ be the set of those i for which $\Phi_{i,s}(A\restriction_s)(i)$ is convergent. Thus, $A'(y)[s] = 1$ iff $y \in A'[s]$. Note that $A'[s]$ is an approximation to A' relative to A. We analogously denote by $(\sigma)'[s]$ (for $\sigma \in 2^{<\omega}$) the set of those i for which $\Phi_{i,s}(\sigma)(i)$ is convergent, and, consequently, $(\sigma)'(y)[s] = 1$ iff $y \in (\sigma)'[s]$.

Theorem 5.7. *If A is a nonlow$_1$ c.e. set then A joins to $\emptyset'$ all Δ^0_2 DNC functions.*

Proof. The main idea is to use a permitting argument at $\emptyset'$-level. When $e \in \emptyset'$ we can eventually verify that at some step s and simultaneously indicate that up to step s all relevant approximations to f at some arguments (see later) are still not stable and equal to their final values, while when $e \notin \emptyset'$, since $A' >_T \emptyset'$, A' eventually has to permit a situation when approximations to f at some argument are already stable and equal to a final value. We substantially use DNC-ness of a given Δ^0_2 function f to do that.
Let f be a DNC Δ^0_2 function, $A = W_x$ a nonlow$_1$ c.e. set and F a computable function such that $\lim F(y, s) = f(y)$.
For $\sigma \preceq A[s]$ for some s, $t(\sigma)$ denotes the least j for which $A[j] \upharpoonright_{|\sigma|}= \sigma$. Note that if $\sigma \prec A$ then $t(\sigma) = m_x(|\sigma|)$.
We use Recursion Theorem to get for any e (uniformly in e)

- indices of partial computable functions $a(e, i)$ such that $J(a(e, i))\downarrow$ if and only if $e \in \emptyset'$, and if $e \in \emptyset'$ properties described below hold.

Case $e \in \emptyset'$: let $e \in \emptyset'[\mathtt{at}\, s_0]$, $\eta = A \upharpoonright_{s_0} [s_0]$. For $0 \le j \le s_0$ let $\tau_j = A \upharpoonright_j [s_0]$. Make first $J(a(e, i))\downarrow$ for all $a(e, i)$ and, second, for those $i < s_0$ for which $(\tau_j)'(i)[t(\tau_j)] = 1$ for some $i < j \le s_0$, take the least such j, say j_0, and make $J(a(e, i)) = F(a(e, i), t(\tau_{j_0}))$ (an output value of others $J(a(e, i))$ is not relevant).
Claim. $\emptyset' \le_T A \oplus f$. Given e, using $A \oplus f$, search for the least s such that
- either $e \in \emptyset'[s]$, so that $e \in \emptyset'$, or
- for some $i < s$, $\sigma_0 \preceq A \upharpoonright_s$, $i < |\sigma_0|$ we have: σ_0 is the shortest $\sigma \preceq A \upharpoonright_s$ for which $(\sigma)'(i)[t(\sigma)] = 1$, $i < |\sigma|$ and $F(a(e, i), t(\sigma_0)) = f(a(e, i))$, in which case $e \notin \emptyset'$.

Verification: Let $e \in \emptyset'[\mathtt{at}\ s_0]$. We have to show that we cannot in steps $s < s_0$ mistakenly decide $e \notin \emptyset'$. Take all $i < s_0$ for which there is $\sigma_j = A \upharpoonright_j$ such that $i < j$ and $(\sigma_j)'(i)[t(\sigma_j)] = 1$ and for those i's let $k(i)$ denote the least such j. We could mistakenly decide $e \notin \emptyset'$ only when $t(\sigma_{k(i)}) < s_0$. But for such $\sigma_{k(i)}$ necessarily $\sigma_{k(i)} \prec \eta = A \upharpoonright_{s_0} [s_0]$ (i.e. $\sigma_{k(i)} = \tau_{k(i)}$) and we have prevented to do a mistake by making $J(a(e, i)) = F(a(e, i), t(\sigma_{k(i)}))$ thereby forcing $f(a(e, i)) \neq J(a(e, i)) = F(a(e, i), t(\sigma_{k(i)}))$.
If $e \notin \emptyset'$, then there is a step s at which we can make a decision $e \notin \emptyset'$. To see it let $H(x)$ be the $\emptyset'$-computable modulus function of the $\lim F(x, s) = f(x)$. Then there is an i such that $i \in A'[\mathtt{at}\ t]$ and $t > H(a(e, i))$, since otherwise the function $H(a(e, .))$ would dominate the A-modulus of A', a contradiction with nonlowness of A. This together with the fact that $F(a(e, i), k) = f(a(e, i))$ for all $k \ge H(a(e, i))$ finishes the proof.

□

6. Medium lowness

Kučera and Nies worked in Prague, May. Later their work was improved by Faizramonov.

Recall that a set A *superlow* if there is a computable function g such that $A'(x) = \lim_s g(x, s)$ with the number of changes for x bounded by a computable function h.

We now partially relativize to A the superlowness of A itself.

Definition 6.1. We call a set A *medium low* if there is a computable function g such that $A'(x) = \lim_s g(x,s)$ with the number of changes for x bounded by a function $h \leq_{\mathrm{T}} A$.

Clearly, we have the implications

$$\text{superlow} \Rightarrow \text{medium low} \Rightarrow \text{low}.$$

These are proper implications.

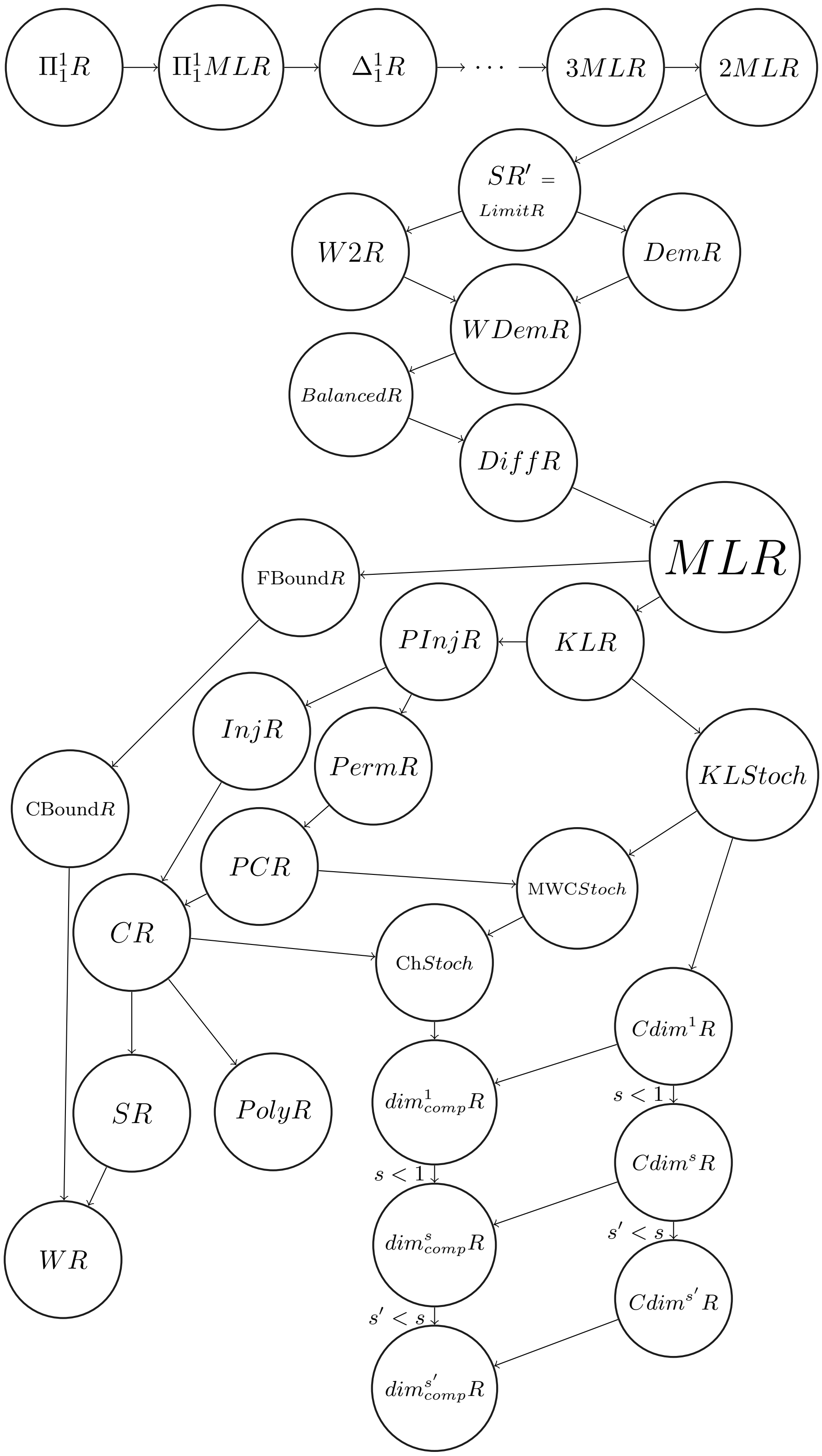
Theorem 6.2. *(i) There is a c.e. medium low set A that is not superlow. (Faizramonov)*
(ii) There is a c.e. low set that is not medium low.

The following was introduced in [20, Def. 27]

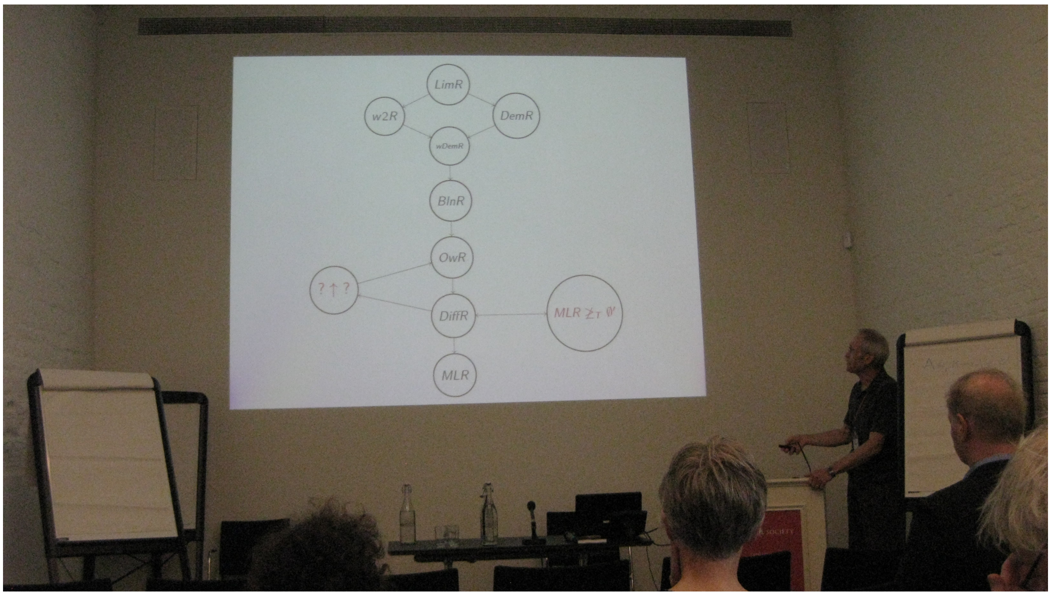
Definition 6.3. A set S is *ω-c.e.-jump dominated* if there is an ω-c.e. function $g(x)$ such that $J^S(x) \leq g(x)$ for every x such that $J^S(x)$ is defined.

[20] showed that superlowness implies ω-c.e.-jump dominated. The converse implication holds for r.e. sets but not in general, for instance because each Demuth random set is ω-c.e.-jump dominated by the proof of [51, Thm. 3.6.26]. We can also define a partial relativization of being ω-c.e.-jump dominated, where g is only ω-c.e. by S. The same relationships hold for superlow vs. ω-c.e.-jump dominated.

7. Randomness Zoo (Antoine Taveneaux)



A full version is available at http://calculabilite.fr/randomnesszoo.pdf.



Tonda Kučera at the Incomputable conference in Chicheley Hall, June 2012.

Definition 7.1 (**MLR**: Martin-Löf Randomness (Definition 3.2.1 in [43])). (1)

A Martin-Löf test, a ML-test for short, is a uniformly c.e. sequence $(G_m)_{m\in\mathbb{N}}$ of open sets such that $\forall n\ \mu(G_m) \leq 2^{-m}$.

(2) A set R fails the test if $R \in \cap_{m\in\mathbb{N}} G_m$ otherwise R passes the test.

(3) R is Martin-Löf Random if R passes each ML-test.

7.1. **Weaker than MLR.**

Definition 7.2 (Scan rule function). For a partial function $f : 2^{<\omega} \to \{\text{scan}, \text{select}\} \times \mathbb{N}$ we denote $n : 2^{<\omega} \to \mathbb{N}$ and $\delta : 2^{<\omega} \to \{\text{scan}, \text{select}\}$.

And we say that f is a scan rule if for all $\sigma, \rho \in 2^{<\omega}$ such that $\sigma \prec \rho$ we have $n(\sigma) \neq n(\rho)$.

The sequence of string observed by f on $Z \in cs$ is defined by(if f is well defined on this points):

$$V_f^A(0) = A(n(\varepsilon))$$

$$V_f^A(k+1) = V_f^A(k).A(n(V_f^A(k)))$$

and bits selected by f are:

$$T_f^A(0) = \varepsilon$$

$$T_f^A(n+1) = \begin{cases} T_f^A(n) & \text{if } \delta(V_f^A(n)) = \text{scan} \\ T_f^A(n).V_f^A(n+1) & \text{if } \delta(V_f^A(n)) = \text{select} \end{cases}$$

and we say that f is well defined on Z if $V_f^A(k)$ is well define for all k and $(T_f^A(n))$ converge to an infinite string and we denote this infinite string by T_f^A.

Definition 7.3 (Martingale)**.** A martingale is a function $\mathcal{M} : 2^{<\omega} \to \mathbb{R}^+ \cup 0$ such that for all $\sigma \in 2^{<\omega}$

$$\mathcal{M}(\sigma) = \frac{\mathcal{M}(\sigma 0) + \mathcal{M}(\sigma 1)}{2}$$

We say that $\mathcal{M}$ succeed on $Z \in 2^\omega$ if

$$\limsup_{n\to\infty} \mathcal{M}(Z \restriction n) = \infty$$

Definition 7.4 (**KLR**: KolmogorovLoveland randomness (Definition in [37] and [41]))**.** $R \in 2^\omega$ is KL-random if for any partial computable scan rule function f and any partial computable martingale $\mathcal{M}$ such that f is well defined on R the martingale martingale $\mathcal{M}$ does not succeed on T_f^A.

Definition 7.5 (**KLStoch**: KolmogorovLoveland stochasticity (Definition in [41]))**.** Let $\#0 : 2^{<\omega} \to \mathbb{N}$ the function giving the number of "0" in a string.

A sequence R is KolmogorovLoveland stochastic if for all partial computable scan rule f such that f is well defined on Z we have:

$$\lim_{n\to\infty} \frac{\#0\left(T_f^A \restriction n\right)}{n} = \frac{1}{2}$$

Definition 7.6 (**MWCStoch**: Mises-Wald-Church stochasticity (Definition in [63] and [12]))**.** A sequence R is Mises-Wald-Church stochastic if for all partial computable monotonic scan rule function f is well defined on Z we have:

$$\lim_{n\to\infty} \frac{\#0\left(T_f^A \restriction n\right)}{n} = \frac{1}{2}$$

Definition 7.7 (**ChStoch**: Church stochasticity (Definition in [63] and [12]))**.** A sequence R is Church stochastic if for all total computable monotonic scan rule function f we have:

$$\lim_{n\to\infty} \frac{\#0\left(R_{f(0)}R_{f(1)}\dots R_{f(n)}\right)}{n} = \frac{1}{2}$$

Definition 7.8 (**PInjR**: partial injective randomness (Definition in [48]))**.** A sequence R is partial injective random if for any total computable injective function $g : \mathbb{N} \to \mathbb{N}$ and any partial computable martingale $\mathcal{M}$ this martingale is defined and does not succeed on the sequence $R_{f(1)}R_{f(2)}\dots R_{f(n)}R_{f(n+1)}\dots$.

Definition 7.9 (**InjR**: injective randomness (Definition in [48, 8]))**.** A sequence R is injective random if for any total computable injective function $g : \mathbb{N} \to \mathbb{N}$ and any total computable martingale $\mathcal{M}$ this martingale does not succeed on the sequence $R_{f(1)}R_{f(2)}\dots R_{f(n)}R_{f(n+1)}\dots$.

Definition 7.10 (**PermR**: partial permutation randomness (Definition in [8]))**.** A sequence R is partial permutation random if for any total computable bijective function $g : \mathbb{N} \to \mathbb{N}$ and any partial computable martingale $\mathcal{M}$ this martingale is defined and does not succeed on the sequence $R_{f(1)}R_{f(2)}\dots R_{f(n)}R_{f(n+1)}\dots$.

Definition 7.11 (**PCR**: partial computable randomness (Definition in [1])). A sequence R is partial computable random if for all partial computable martingale $\mathcal{M}$ if $\mathcal{M}(R \upharpoonright n)$ is define for all n and $\mathcal{M}$ does not succeed on R.

Definition 7.12 (**CR**: computable randomness (Definition in [57])). A sequence R is computable random if for all total computable martingale $\mathcal{M}$ this martingale succeed on R.

Definition 7.13 (**SR**: Schnorr randomness(Definition in [57])). A Schnorr test is a uniformly c.e. sequence $(G_m)_{m\in\mathbb{N}}$ of open sets such that $\forall n\ \mu(G_m) = 2^{-m}$.

R is Schnorr random if for any Schnorr test $(G_m)_{m\in\mathbb{N}}$ $R \notin \cap_{m\in\mathbb{N}} G_m$

Definition 7.14 (**FBoundR**: finitely bounded randomness (Definition in [10])). R is finitely bounded random if R passes any Martin-Löf test (U_n) such that for every n, $\#U_n < \infty$ (with $\#U_n$ the number of sting enumerated in U_n).

Definition 7.15 (**CBoundR**: computably bounded randomness (Definition in [10])). A Martin-Löf test (U_n) is computably bounded if there is some total computable function f such that $\#U_n \leq f(n)$ for every n.

R is computably bounded random if R passes every computably bounded Martin-Löf test.

Definition 7.16 (**WR**: weakly randomness (Definition in [38])). R is weakly random if $R \in U$ for every Σ^0_1 set $U \subseteq 2^\omega$ of measure 1.

Definition 7.17 (**PolyR**: polynomial randomness (Definition in [65])). A sequence R is polynomially random if any martingale computable in polynomial time $\mathcal{M}$ do not succeed on R.

Definition 7.18 ($\mathbf{dim}^s_{\mathbf{comp}}\mathbf{R}$: computable s-randomness (Definition in [42] and [45])). A computable s-test is a uniformly computable sequence $(G_m)_{m\in\mathbb{N}}$ of computable open sets such that for all n

$$\sum_{x\in G_m} 2^{-s|x|} \leq 2^{-n}.$$

R is computably s-random if for all $s' < s$ and computable s-tests (G_m) we have:

$$R \notin \bigcap_{m\in\mathbb{N}} G_m$$

Definition 7.19 ($\mathbf{Cdim}^s\mathbf{R}$: constructive s-randomness (Definition in [42] and [45])). A constructive s-test is a uniformly computable sequence $(G_m)_{m\in\mathbb{N}}$ of computable enumerable open sets such that for all n

$$\sum_{x\in G_m} 2^{-s|x|} \leq 2^{-n}.$$

R is computably s-random if for all $s' < s$ and computable s-tests (G_m) we have:

$$R \notin \bigcap_{m\in\mathbb{N}} G_m$$

7.2. **Stronger than MLR.**

Definition 7.20 (**DiffR**: difference randomness (Definition in [23]))**.** A difference test is given by a sequence $(V_m)_{m\in\mathbb{N}}$ of uniformly c.e. sets and a Π^0_1 set P such that $\mu(P\bigcap V_m)\leq 2^{-m}$ for every m.

A sequence R is difference random if for any difference test $(V_m)_{m\in\mathbb{N}}, P$ we have

$$R\notin P\bigcap(\cap_{m\in\mathbb{N}}V_m)\,.$$

Definition 7.21 (**BalancedR**: balanced randomness (Definition in [20]))**.** A balanced test is a sequence $(V_m)_{m\in\mathbb{N}}$ of c.e. sets such that $V_i=W_{f(i)}$ for some 2^n-c.e. function f and $\mu(V_m)\leq 2^{-m}$for every m.

A sequence R is balanced random if R passes any balanced test.

Definition 7.22 (**WDemR**: weak Demuth randomness (Definition in [17]))**.** Demuth test is a sequence $(V_m)_{m\in\mathbb{N}}$ of c.e. sets such that $V_i=W_{f(i)}$ for some ω-c.e. function f and $\mu(V_m)\leq 2^{-m}$for every m.

A sequence R is weak Demuth random if for any Demuth test (V_m) we have $R\notin\cap_{m\in\mathbb{N}}V_m$.

Definition 7.23 (**DemR**: Demuth randomness (Definition in [17]))**.** A Demuth test is a sequence $(V_m)_{m\in\mathbb{N}}$ of c.e. sets such that $V_i=W_{f(i)}$ for some ω-c.e. function f and $\mu(V_m)\leq 2^{-m}$for every i.

A sequence R is Demuth random if for any Demuth test (V_m) we have $R\notin V_m$ for almost all m.

Definition 7.24 (**W2R**: weak 2-randomness (Definition in [35]))**.** A sequence R is weak 2-random if $R\notin U$ for every Π^0_2 set $U\subset 2^\omega$ of measure 0.

Definition 7.25 (**LimitR**: limit randomness (Definition in [39]))**.** A limit test is a sequence $(V_m)_{m\in\mathbb{N}}$ of c.e. sets such that $V_i=W_{f(i)}$ for some Δ^0_2-computable function f and $\mu(V_m)\leq 2^{-m}$for every m.

A sequence R is limit random if for any limit test (V_m) we have $R\notin V_m$ for almost all m.

Definition 7.26 (Δ^1_1**R**: Δ^1_1 randomness (Definition in [44]))**.** R is Δ^1_1-random if R avoids each null Δ^1_1-class.

Definition 7.27 (Π^1_1**MLR**: Π^1_1-Martin-Löf Randomness (Definition in [32]))**.** A Π^1_1-Martin-Löf test is a sequence $(G_m)_{m\in\mathbb{N}}$ of open sets such that $\forall n\ \mu(G_m)\leq 2^{-m}$ and the relation $\{\langle m,\sigma\rangle|[\sigma]\subseteq G_m\}$ is Π^1_1

R is Π^1_1-Martin-Löf Random if R passes each Π^1_1-ML-test.

Definition 7.28 (Π^1_1**R**: Π^1_1-Randomness (Definition in [32]))**.** R is Π^1_1-random if R avoids each null Π^1_1-class.

8. Complexity of recursive splittings of random sets

Ng, Nies and Stephan investigated random sets which preserve as much information as possible when subjected to a recursive splitting. In each high Turing degree we build a Schnorr random set such that one can preserve the Turing degree in both halves of the splitting by any infinite and co-infinite recursive set. We build a Martin-Löf random set so that one half can still preserve most of the information because it is Turing above Ω. Due to the Theorem of van Lambalgen, some information is necessarily lost.

8.1. **Schnorr random sets.**

Theorem 8.1. *Every high Turing degree contains a Schnorr random set Z such that $Z \leq_T R \cap Z$ for every infinite r.e. set R. Thus, if R is recursive then $Z \equiv_T R \cap Z$.*

Proof. Let g be a function in the given Turing degree which dominates all recursive functions. Now one can define a further dominating function f such that $f(n)$ is the sum of all $\varphi_k(n)$ where $k \leq n$ and $\varphi_k(n)$ is computed within $g(n)$ computation steps. Furthermore, one can inductively define for each n and each $k = 0, 1, \ldots, n$ that $E_{n,k}$ is the set of the first $2n+5$ elements of

$$A = W_{k,g(f(n))} - \{0, 1, \ldots, f(n)\} - \bigcup_{(n',k'):(n',k') <_{lex} (n,k) \wedge k' \leq n'} E_{n',k'};$$

whenever they exist; if they do not exist then $E_{n,k} = \emptyset$. In the case that $E_{n,k}$ has $2n+5$ elements, the corresponding entries are used as follows: Bit 0 is used to code one bit of $\{(x, y) : g(x) = y\}$ in order to encode the graph of g; bit $k+1$ is used to code whether $\varphi_k(n+1)$ contributes to $f(n+1)$ or not (where $k = 0, 1, \ldots, n, n+1$), bit $k+n+3$ is used to code whether $E_{n+1,k}$ is empty (where $k = 0, 1, \ldots, n, n+1$). So in total $2n+5$ coding bits are used. This determines how Z is coded on these entries. By [51, Lemma 7.5.1] there is a fixed g-recursive martingale L with rational values which dominates all recursive martingales up to a multiplicative constant. The other entries of Z are chosen so that L does not increase.

The resulting set is Schnorr random, as L has after up to $g(n)$ bets at most the value $2^{(3n+2)^2}$; for being covered by this martingale in the Schnorr sense, one could require that some recursive martingale obtains this value infinitely often already after $h(n)$ many steps for some recursive function h. See Franklin and Stephan [24, Proposition 2.2] for additional information.

Let W_e be an infnite set. As g dominates all recursive functions, one can find for almost all n more than $(2n+5)^3$ elements in W_e above $g(n)$ in time $g(f(n))$. Therefore, $E_{n,e}$ is non-empty for almost all n. So when starting with a sufficiently large n, using Z and the enumeration of W_e one can find all the entries for $E_{m,e}$ with $m \geq n$, and therefore compute the function g relative to $Z \cap W_e$. This permits to compute Z. Thus $Z \leq_T Z \cap W_e$. □

8.2. **Martin-Löf random sets.**

Theorem 8.2. *There is a Martin-Löf random set Z such that $\Omega \leq_T Z \cap R$ or $\Omega \leq_T Z - R$ for every recursive set R.*

Proof. The set Z is constructed in several steps:

- A construction of an r-maximal set S with complement $E_0 \cup E_1 \cup E_2 \cup \ldots$ where the parts $E_0, E_1, E_2, \ldots$ are finite sets with $\max E_n < \min E_{n+1}$ which each maximise their e-state; this ensures that S is r-maximal;
- Letting $Z(a_k) = \Omega(k)$ for the k-th bit a_k in the complement of S;
- Taking a set P which is PA-complete and low for Ω;

- Defining the bits in S according to a set which is random relative to $P \oplus \Omega$.

Once Z is constructed, it is shown that the resulting set Z is Martin-Löf random and that for every recursive splitting, one half of it is Turing above Ω.

First the construction of the r-maximal set S is given. One defines the n-th e-state of a set D as the sum $3^n a_0 + 3^{n-1} a_1 + \ldots 3a_{n-1} + a_n$ where each a_k is 2 if φ_k is defined on D and takes always values from $\{1, 2, \ldots\}$ and a_k is 1 if φ_k is defined on D and takes each time the value 0 and $a_k = 0$ otherwise. Furthermore, one can also consider an approximation to the e-state at some given time. Now one will make sets E_n, approximated as $E_{n,t}$ such that $E_{n,t+1} \neq E_{n,t}$ such that their e-state has the parameter n as well and that $E_{n,t+1} \neq E_{n,t}$ only when some E_m with $m \leq n$ increases its e-state. Furthermore, when the e-state of E_n is p then E_n has $n^2 \cdot 2^{3^{n+1}-p}$ elements and $\max(E_{n,t}) < \min(E_{n+1,t})$ for all n, t. Initially each $E_{n,0}$ is an interval of length $2^{3^n} \cdot n^2$ and its e-state is 0. All elements below t which are not on an interval $E_{n,t}$ at stage t are enumerated into S.

Second, let $E = E_0 \cup E_1 \cup E_2 \cup \ldots$ and let a_k be the k-th element of E (in ascending order). Now let $Z(a_k) = \Omega(k)$. Note that whenever $a_k \in E_n$ then $a_{k+1} \in E_n \cup E_{n+1}$.

Third, Miller [18, 47] showed that there is a set P which is low for Ω and PA-complete.

Fourth, let V be a set which is Martin-Löf random relative to $P \oplus \Omega$ and for $x \in S$, let $Z(x) = V(x)$. This completes the construction of Z.

For the further proof, note that one can compute the list of members of E_{n+1} if one knows the list of members of $E_0 \cup E_1 \cup \ldots \cup E_n$ as well as the final e-state of E_{n+1}: this is just done by simulating the construction until the values of $E_{m,t}$ have stabilised as E_m for $m \leq n$ and the e-state of $E_{n+1,t}$ has stabilised at the corresponding value. More generally, the e-states of $E_0, E_1, \ldots, E_{n+1}$ permit to compute the sets $E_0, E_1, \ldots, E_{n+1}$ explicitly. Now, the number of bits contained in $E_0 \cup E_1 \cup \ldots \cup E_n$ is at least $n^3/8$ while the e-states of $E_0, E_1, \ldots, E_n, E_{n+1}$ together can be described with $\log(3) \cdot \frac{n(n+1)}{2}$ bits. Therefore, the positions of $E_0, E_1, \ldots, E_n, E_{n+1}$ are reached at some stage t before the left-r.e. approximation of Ω reaches on $E_0 \cup E_1 \cup \ldots \cup E_n$ its final values. This fact an used to compute Ω from Z in an iterative manner: Knowing the bits of Ω coded on $E_0 \cup E_1 \cup \ldots \cup E_n$ permits to compute the position of E_{n+1} which then again permits to look up the bits of Ω coded on E_{n+1} from Z. So $\Omega \leq_T Z$ by only taking into consideration the positions in E. As S is r-maximal, it holds for any recursive set R that almost all elements of E are either inside R or outside R; depending on which case holds, one can compute Ω from either $Z \cap R$ or $Z - R$.

The last part of the proof is to show that Z is Martin-Löf random. Assume that this is not the case. Then there is a P-recursive martingale M succeeding on Z, as P is PA-complete.

If one relaxes M to being partial-recursive, then one can in addition permit that M - while processing the input from Z, computes the set E_{n+1} whenever it has processed all members of $E_0 \cup E_1 \cup \ldots \cup E_n$ and found the corresponding values of Ω and used the time the left-approximation takes on them to get the position of E_{n+1}. Therefore the martingale knows whenever it is betting on an entry of Z belonging to S or belonging to E; that is, there is a partial-recursive function γ such that $\gamma(Z(0)Z(1)\dots Z(n)) = S(n+1)$ and γ might be wrong or undefined if the input is not a prefix of Z. The martingale evaluates γ prior to betting. Now let

$$\begin{aligned} q_n &= \frac{M(Z(0)Z(1)\dots Z(n)Z(n+1))}{M(Z(0)Z(1)\dots Z(n))}; \\ s_n &= \prod_{m\in\{0,1,\dots,n\}\cap S} q_m; \\ e_n &= \prod_{m\in\{0,1,\dots,n\}\cap E} q_m. \end{aligned}$$

As the limit superior of $s_n \cdot e_n$ is ∞, it follows that (a) the limit superior of the s_n is ∞ or (b) the limit superior of the e_n is ∞. Now it is shown that in both cases (a) and (b) a contradiction can be derived; for this contradiction, note that by construction Ω is Martin-Löf random relative to P, V is Martin-Löf random relative to $\Omega \oplus P$ and then by the Theorem of van Lambalgen Ω is Martin-Löf random relative to $V \oplus P$.

Case (a): the limit superior of the s_n is ∞. In this case, one could make another modification of M such that the resulting martingale $\tilde{M}^{P\oplus\Omega}$ abstains from betting at n when $n \in E$ (but uses Ω to retrieve the value) and bets on $V(n)$ when $n \in S$ (where $V(n) = Z(n)$). This permits to show that V is not Martin-Löf random relative to $P \oplus \Omega$, a contradiction.

Case (b): the limit superior of the e_n is ∞. In this case, one could make another modification of M such that the resulting partial $P \oplus V$-recursive martingale $\tilde{M}^{P\oplus V}$ bets on $\Omega(n+1)$ the value $q_{a_{n+1}} = e_{a_{n+1}}/e_{a_n}$ where this share is computed from $Z(0)Z(1)\dots Z(a_{n+1}-1)$ where in turn this string can be retrieved using γ and the oracle $P \oplus V$ and $\Omega(0)\Omega(1)\dots\Omega(n)$. Hence $\tilde{M}^{P\oplus V}$ takes on $\Omega(0)\Omega(1)\dots\Omega(n)$ the value $e_n = q_{a_0}q_{a_1}\dots q_{a_n}$ and therefore Ω is not Martin-Löf random relative to $P \oplus V$, a contradiction.

This case-distinction then shows that Z, other than assumed, is indeed Martin-Löf random. □

Remark 8.3. *This proof can be generalised such that for given set Y there is a Martin-Löf random Z such that either $Y \leq_T Z \cap R$ or $Y \leq_T Z - R$ for every recursive set R.*

The next small result shows that one can split every complete Turing degree into Turing incomplete ML-random degrees. Ng, Nies and Stephan would like to thank Yu Liang for a simplification of the proof of Theorem 8.4.

Theorem 8.4. *Every Turing degree above that of Ω contains a Martin-Löf random set of the form $X \oplus Y$ such that X is low and Y is Turing incomplete.*

Proof. Let $Z \geq_T \Omega$ be in the given Turing degree and let X be a low Martin-Löf random set, for example a half of Ω. Relativising the Theorem of Kučera and Gács to X gives that there is a set Y which is Martin-Löf random relative to X and which satisfies $Z \equiv_T X \oplus Y$. Now X is Martin-Löf random relative to Y by the Theorem of van Lambalgen and therefore $X \not\leq_T Y$; hence Y is Turing incomplete. □

9. Indifference for weak 1-genericity

Frank Stephan and Jason Teutsch answered in the affirmative the open question 8.4 in Adam Day's paper (the printed url is `http://sigmaone.files.wordpress.com/2011/11/day_genericity.pdf`).

Theorem 9.1. *No 2-generic set X can compute a set which is indifferent for X with respect to weak 1-genericity.*

Proof. First, assume by way of contradiction that X is 2-generic and I^X is an algorithm which produces an infinite set for X which is indifferent for X with respect to weak 1-genericity.

One can extend this definition to I^σ being the elements which can be enumerated into I relative to σ without querying the oracle outside the domain of σ. Note that I^X is the union over all I^σ with $\sigma \preceq I$.

Furthermore, one can now define an extension function ψ such that $\psi(\sigma)$ is the first extension τ of σ found for which there is an $x \in I^\tau$ with $|\sigma| < x < |\tau|$. This is a partial-recursive extension function. Note that every prefix σ of X satisfies that $\psi(\sigma)$ is defined. Hence, by 2-genericity there is a prefix $\eta \preceq X$ such that every extension σ of η is in the domain of ψ. Now one defines for any string of the form $\eta\vartheta$ a function $f(\eta\vartheta)$ which is defined inductively with $\vartheta_0, \vartheta_1, \ldots, \vartheta_n$ being all strings of length $|\vartheta|$ and $\tau_0, \tau_1, \ldots, \tau_n$ being chosen such that the following equations hold:

$$\begin{aligned} \eta\vartheta_0\tau_0 &= \psi(\eta\vartheta_0); \\ \eta\vartheta_{m+1}\tau_0\tau_1\ldots\tau_m\tau_{m+1} &= \psi(\eta\vartheta_{m+1}\tau_0\tau_1\ldots\tau_m) \text{ for all } m < n; \\ f(\eta\vartheta) &= \eta\vartheta\tau_0\tau_1\ldots\tau_n. \end{aligned}$$

Note that in this definition, the extending part of $f(\eta\vartheta)$ only depends on the length of ϑ and not of the actual bits; furthermore, $I^{f(\eta\vartheta)}$ contains an element x such that $|\eta\vartheta| < x < |f(\eta\vartheta)|$.

Now one partitions the natural numbers in intervals $\{x : x < |\eta|\}$ and $J_0, J_1, \ldots$ and there is on each interval J_n a string κ_n of length $|J_n|$ so that $f(\eta\vartheta) = \eta\vartheta\kappa_n$ whenever ϑ has the length $\min(J_n) - \min(J_0)$.

Now let for any σ extending η the function $g(\sigma)$ be $f(\sigma 0^k)$ for the first k such that there is an n with $|\sigma 0^k| = \min(J_n)$; if σ is a prefix of η then let $g(\sigma) = f(\eta)$; if σ is incomparable to η then let $g(\sigma) = \sigma 0$. Note that g enforces that every weakly 1-generic set with prefix η equals to κ_n on J_n for some n.

Let σ_n be the prefix of X of length $\min(J_n)$. Let H be the set of all n such that X restricted to J_n equals κ_n; note that for each $n \in H$ there is an element $a_n \in I^X \cap J_n$. Now let Y be the symmetric difference of X and $\{a_n : n \in H\}$. Note that the set Y does not coincide with κ_n on J_n for any n. As η is a prefix of Y, Y is not weakly 1-generic.

However, the symmetric difference of X and Y is a subset of I^X. Hence I^X cannot be indifferent for X with respect to weak 1-genericity. This contradiction completes the proof of the theorem. □

10. Turing degrees of computably enumerable ω-c.a.-tracing sets

D. Diamondstone and A. Nies started discussions in December 2011. This work now includes Joe Zheng. For background on tracing see [51, Sections 8.2,8.4].

10.1. Introduction.

The following (somewhat weak) highness property was introduced by Greenberg and Nies [29]; it coincides with the class $\mathcal{G}$ in [51, Proof of 8.5.17].

Definition 10.1. A set A is *ω-c.a.-tracing* if each function $f \leq_{\mathrm{wtt}} \emptyset'$ has a A-c.e. trace $(T_x^A)_{x\in\mathbb{N}}$ such that $|T_x^A| \leq 2^x$ for each x.

A stronger condition is that every Δ^0_2 function f must be traced:

Definition 10.2. A set A is *Δ^0_2-tracing* if each Δ^0_2 function f has a A-c.e. trace $(T_x^A)_{x\in\mathbb{N}}$ such that $|T_x^A| \leq 2^x$ for each x.

One also says that $\emptyset'$ is c.e. traceable *by* A.

10.1.1. *The Tewijn-Zambella argument.* By an argument of Terwijn and Zambella (see [51, Thm. 8.2.3]), in both cases the bound 2^x can be replaced by any order function without changing the tracing property. The two condition on a class $\mathcal{C}$ of total functions being traced which makes this argument work is the following.

(∗) *If $f \in \mathcal{C}$ and r is a computable function, then the function $x \to f\restriction_{r(x)}$ (tuples suitably encoded by numbers) is also in $\mathcal{C}$.*

10.1.2. *Double highness properties.* These two classes relate nicely to highness for pairs of randomness notions by results in [20, 4]. Recall that for randomness notions $\mathcal{C} \supset \mathcal{D}$, we let $\mathrm{High}(\mathcal{C}, \mathcal{D})$ be the class of oracles A such $\mathcal{C}^A \subseteq \mathcal{D}$ (A is strong enough to push $\mathcal{C}$ inside $\mathcal{D}$). The following is obtained by combining [21, 4]:

Theorem 10.3. *Let A be an oracle.*

(a) $A \in \mathrm{High}(\mathsf{MLR}, \mathsf{Demuth}) \Leftrightarrow A$ *is ω-c.a. tracing.*
(b) $A \in \mathrm{High}(\mathsf{MLR}, \mathsf{SR}[\emptyset']) \Leftrightarrow A$ *is Δ^0_2 tracing.*

10.1.3. *Extension to randomness notions in between the two extremes.* It is known that $\mathsf{SR}[\emptyset']$) is the same as limit random (the set has to pass all Demuth-like tests where the number of version changes is merely finite; see [39] for the formal definition). Let α be a computable limit ordinal and recall the definition of an α-c.a. function; $\alpha-\mathsf{Demuth}$ is Demuth randomness extended to tests with versions of test components α-c.a. That is, the tests have the form $[W_{f(m)}]^{\prec}_{m\in\mathbb{N}}$ where f is α-c.a. This is studied, for instance, in the last sections of [**?**], and [52]. The table suggests an extension to notions in between the two extremes, both on the randomness and the tracing side:

Conjecture 10.4. *Let α be a computable limit ordinal possibly with some additional closure properties such as closure under $+$. Then*

$$A \in \mathrm{High}(\mathsf{MLR}, \alpha - \mathsf{Demuth}) \textit{ iff } A \textit{ is } \alpha\textit{-c.a. tracing.}$$

In full this is only known if the class of α-c.a. function satisfies the condition $(*)$ in 10.1.1; for instance, this is the case when $\alpha = \omega^n$ for some $n \in \omega$. See Subsection 10.4 for more detail.

We say A is (weak) Demuth cuppable if there is a (weak) Demuth random set Z such that $A \oplus Z \geq_T \emptyset'$. Diamondstone and Nies also noticed that for each c.e. set A,

$$K\text{-trivial} \Rightarrow \text{weak Demuth noncuppable} \Rightarrow \text{superlow},$$

For the first implication, K-trivial even implies ML-noncuppable by a recent result of Day/Miller [15]. The second one is shown in Figueira et al. [21]. They use that for c.e. sets A, superlow $=$ ω-c.e. jump dominated in the sense of [21]; the negation of this second property implies High(MLR, weak Demuth). Thus if a c.e. set A is not superlow, Ω^A is weakly Demuth random and cups A above $\emptyset'$.

Demuth traceability was introduced in [5]. For instance, all superlow c.e. sets are Demuth traceable. They also observed that for *any* set,

$$\text{Demuth traceable} \Rightarrow \text{Demuth noncuppable} \Rightarrow \text{not } \omega\text{-c.a. tracing}.$$

For the first implication see [5]. The second implication follows from Theorem 10.3 above by taking Ω^A.

Question 10.5. *Characterize weak Demuth noncuppability, and Demuth noncuppability, in recursion theoretic terms.*

10.2. **Can an ω-c.a. tracing set be close to computable?** Being ω-c.a. tracing was investigated in [20]. They showed:

Fact 10.6. *No superlow set is ω-c.e.-tracing.*

On the other hand, lowness is possible. By [20, Journal version Cor. 2.5] there is an ω-c.e.-tracing low ML-random set. Here we build a c.e. such set.

Proposition 10.7. *Some low c.e. set A is ω-c.a. tracing.*

Proof. (Idea) We obtain $(f_e)_{e\in\mathbb{N}}$ a list of all ω-c.a. functions as follows. Let $\langle e\rangle$ be the e-th wtt reduction procedure (namely, e_0 indicates a Turing functional and e_1 is a computable bound on the use). At stage s we have an approximation $f_e(x)[s] = \langle e\rangle^{\emptyset'}(x)[s]$, with value 0 if this is undefined. The function f_e is given by $f_e(x) = \lim_s \langle e\rangle^{\emptyset'}(x)[s]$.

We build a c.e. oracle trace $(T^Z_x)_{x\in\mathbb{N}}$ with a fixed computable bound $h(x)$. We meet

$$P_{e,x}\colon f_e(x) \in T^A_x \ (e \leq x).$$

We also meet the usual lowness requirements L_i.

Fix an effective priority ordering of the requirements.

Strategy for $P_{e,x}$ at stage s: when there is a new value $y = f_e(x)[s]$, change A to remove the previous value (if any), unless its A-use is restrained by a stronger priority L-type requirement. Put $y \in T^A_x$ with large use on A.

Strategy for L_i at stage s: if $J^A(i)$ converges newly (by convention with use $\leq s$), restrain all weaker P requirements from changing $A\restriction_s$.

Claim 10.8. *There is a computable bound for $|T^A_x|$.*

□

We give another formal explication of the idea that a c.e. set A can be high in one sense and low in another. This answers a question asked in [21, Rmk. 31] because for c.e. sets, array rec. = c.e. traceable. For detail see Zheng's master thesis.

Theorem 10.9. *Some ω-c.a. tracing c.e. set A is c.e. traceable.*

Note that the class of c.e. traceable sets contains all the superlow c.e. sets, but not all low c.e. sets. Thus the two results are independent.

Also note that A cannot be Δ^0_2 tracing: by Barmpalias [3], such a set is not weakly arrary recursive (for each Δ^0_2 function g there is $h \le_{\mathrm{T}} A$ such that $\exists^\infty x h(x) > g(x)$), hence not c.e. traceable.

Proof. Use a $\emptyset''$ tree construction. To make A c.e. traceable we meet requirements

$$S_i\colon\ \Phi_i^A \text{ total} \Rightarrow \text{build c.e. trace } (V_x)_{x\in\mathbb{N}} \text{ for } \Phi_i^A,$$

where $|V_x|$ has fixed computable bound. Guess at Φ_i^A total on the tree via exp stages; A becomes low_2.

To make A ω-c.a. tracing, meet requirements $P_{e,x}$ as above. Use same strategies as above but on the tree. A sty $\beta\colon P_{e,x}$ below the infinitary outcome of a strategy $\alpha\colon S_i$ can only start once $\Phi_i^A \upharpoonright_y$ has converged for large enough y, so that its A changes don't make $|V_y|$ to large. For this we need the advance bound on the number of times $f_e(x)$ can change. □

10.3. **Can a non-ω-c.a. tracing set be close to $\emptyset'$?** For any Δ^0_2 set A, superhighness is equivalent to JT-hardness by [51, 8.4.27], which of course implies ω-c.a. tracing. Thus, every superhigh Δ^0_2 set is ω-c.a. tracing. The following is pretty sharp then.

Theorem 10.10. *Some high c.e. set A is not ω-c.a. tracing.*

Proof. Define an ω-c.a. function g. We meet the requirements

$$N_e\colon \exists x\, g(x) \notin T^A_{e,x},$$

where $(T^Z_{e,x})_{e,x\in\omega}$ is uniform listing of all oracle c.e. traces with bound x.

We also meet the usual highness requirements

$$P_r\colon \emptyset''(r) = \lim_n A^{[r]}(n),$$

by the usual coding into the r-th column of A.

Use methods from the tree construction of a high minimal pair as in [60]. Guess on the tree whether $\emptyset''(r) = 0$. This yields a notion of α-correct A-computations, where α is a string.

Strategy for $\alpha\colon N_e$. Pick large x. Define $g(x)$ large. Whenever $g(x) \in T^A_{e,x}$ via an α-correct computation, then increase $g(x)$ and initialize weaker requirements.

Verification. Need to check that g is indeed ω-c.a. As long as $\alpha\colon N_e$ is not initialized, all relevant computations $y \in T^A_{e,x}$ are A-correct, so it will increase $g(x)$ at most x times. □

For an alternative, full proof, again see Zheng's master thesis. There, the α correct computations are replaced by a more informative tree of strategies.

10.4. **Traceability in absence of the condition of Terwijn and Zambella.** We generalize Definitions 10.1 and 10.2.

Definition 10.11. Let $\mathcal{C}$ be a class of total functions defined on $\mathbb{N}$. Let $h\colon \mathbb{N} \to \mathbb{N}$. We say that a set A is *$\mathcal{C}$-tracing with bound h* if each function $f \in \mathcal{C}$ has a A-c.e. trace $(T^A_x)_{x\in\mathbb{N}}$ such that $|T^A_x| \le h(x)$ for each x.

Note that by the Terwijn Zambella argument, the order function h is immaterial if Condition $(*)$ of Subsection 10.1.1 holds for $\mathcal{C}$.

Definition 10.12. Let $\mathcal{C}$ be a class of total functions defined on $\mathbb{N}$.

- A *$\mathcal{C}$-Demuth test* has the form $[W_{f(m)}]^{\prec}_{m\in\mathbb{N}}$ where f is in $\mathcal{C}$.
- We say that Z is *$\mathcal{C}$-Demuth random* if for each such test we have $Z \notin [W_{f(m)}]^{\prec}$ for almost all m.

We give a fine analysis of Theorem 10.3 in this more general setting. We show the double highness notion $A \in \mathrm{High}(\mathsf{MLR}, \mathcal{C} - \mathsf{Demuth})$ implies A is $\mathcal{C}$-tracing at bound 2^m. However, we need tracing at slightly better bound, such as $2^m m^{-2}$, to reobtain the double highness notion. Thus there is a gap in the absence of condition $(*)$.

The first part is a straightforward modification of the proof of [4, Thm. 3.6].

Lemma 10.13. $A \in \mathrm{High}(\mathsf{MLR}, \mathcal{C} - \mathsf{Demuth}) \Rightarrow$
A is $\mathcal{C}$-tracing with bound 2^n.

The second part is a modification of the proof of the corresponding result [20, Prop. 32]; also see [4, Thm. 3.6].

Lemma 10.14. *A is $\mathcal{C}$ tracing with a bound g such that $\sum g(n)2^{-n} < \infty$* $\Rightarrow$
$A \in \mathrm{High}(\mathsf{MLR}, \mathcal{C} - \mathsf{Demuth})$.

Proof. Fix a $\mathcal{C}$ Demuth test $[W_{f(m)}]^{\prec}_{m\in\mathbb{N}}$. Let $(T_m)_{m\in\mathbb{N}}$ be a A-c.e. trace for f with bound g. Define an A Solovay test $(\mathcal{S}^A_m)$ as follows: for each $k \in T_m$, enumerate the open set $[W_{f(k)}]^{\prec}$ into $\mathcal{S}^A_m$ as long as its measure is $\le 2^{-k}$. Clearly $\sum_m \lambda \mathcal{S}^A_m \le \sum_m g(m) < \infty$. Thus no set that is in infinitely many $\mathcal{S}^A_m$ can be ML-random relative to A. ☐

We now look at subclasses of ω-c.a., and also classes $\mathcal{C}$ containing ω-c.a.

10.4.1. *Subclasses of ω-c.a.* For a computable order function g, let $\mathcal{C}$ be the class of functions $f \le_{\mathrm{wtt}} \emptyset'$ such that some computable approximation for $f(x)$ has at most $g(x)$ changes. Then A is $\mathcal{C}$ tracing with bound 2^m if each function in $\mathcal{C}$ has an A-c.e. trace $(T^A_x)_{x\in\mathbb{N}}$ such that $|T^A_x| \le 2^x$ for each x. We also say that A is *g-c.a.-tracing* with bound 2^m. More generally, we could have a class $\mathcal{D}$ of computable functions instead of a single g, and we say A is $\mathcal{D}$-c.a. tracing with the obvious meaning.

Let Z be ML-random. By [21, Thm 23], Z is ω-c.a. tracing iff Z is $2^n h(n)$-c.a. tracing, where h is an arbitrary (say, slowly growing) order function.

In contrast, for c.e. sets A, there is a proper hierarchy of being g-c.a. tracing, for faster and faster growing computable functions g:

Theorem 10.15. *Let g be computable. Then there is a computable function h and a c.e. set A that is g-c.a. tracing, but not h-c.a. tracing.*

Proof. Modifying the proof of Proposition 10.7, we can build a g-c.a. tracing c.e. set A that is superlow. Hence this set is not h-c.a. tracing for an appropriate faster growing h. $\square$

By the Terwijn Zambella argument and the two lemmas above, we have:

Proposition 10.16. *Let $\mathcal{D}$ be a class of computable functions such that for each function $g \in \mathcal{D}$, the function*

$$\hat{g}(n) = \sum_{i \leq n \log^2 n} g(i)$$

is also in $\mathcal{D}$. Let $\mathcal{C}$ be the class of $\mathcal{D}$-c.a. functions. Then $A \in \mathrm{High}(\mathsf{MLR}, \mathcal{C} - \mathsf{Demuth}) \Leftrightarrow A$ is $\mathcal{C}$-tracing.

An example of such a $\mathcal{D}$ is the class of computable functions bounded by a function of type $2^n \cdot q(n)$ where q is a polynomial.

10.4.2. *Classes $\mathcal{C}$ containing ω-c.a.*

Proposition 10.17. *The class of $\mathcal{C}$ of ω^n-c.a. functions satisfies the condition $(*)$. Hence $A \in \mathrm{High}(\mathsf{MLR}, \mathcal{C} - \mathsf{Demuth}) \Leftrightarrow A$ is $\mathcal{C}$-tracing.*

Proof. One modifies [52, Lemma 5.2] where $n = 2$. This can be expanded suitably to ω^n. $\square$

Part 3. Higher randomness

11. Notions stronger than Π^1_1-ML-randomness (Chong, Nies and Yu)

Chong, Nies and Yu worked in Singapore in April. They considered analogs or randomness notions stronger than Martin-Löf's in the realm of effective descriptive set theory. For background see [51, Ch. 9]. As there, the higher analog of a concept is generally obtained by replacing "c.e." with "Π^1_1" everywhere. The notation for a higher analog is obtained by underlining the previous notation. For instance, $\underline{\Omega}$ is the higher analog of Chaitin's Ω, and $\underline{\mathsf{MLR}}$ denotes Π^1_1-Martin-Löf randomness.

The following can be seen as the higher analog of Turing reducibility. The number of stages of an oracle computation is a computable ordinal, yet the use on the oracle is finite.

Definition 11.1 (Hjorth and Nies [32]). A fin-h *reduction procedure* is a partial function $\Phi\colon 2^{<\omega} \to \omega^{<\omega}$ with Π^1_1 graph such that $\mathrm{dom}(\Phi)$ is closed under prefixes and, if $\Phi(x)\downarrow$ and $y \preceq x$, then $\Phi(y) \preceq \Phi(x)$. We write $f = \Phi^Z$ if $\forall n \exists m\, \Phi(Z\restriction_m) \succeq f\restriction_n$, and $f \le_{\text{fin-h}} Z$ if $f = \Phi^Z$ for some fin-h reduction procedure Φ.

Bienvenu, Greenberg and Monin (July) have observed that this isn't the right notion for most purposes. One shouldn't have the closure under prefixes in most results below. They call the more general version $\le_{hT}$ (higher Turing).

They may add a summary of their work here, which will appear independently.

Definition 11.2 (Bienvenu, Greenberg and Monin). A *hT reduction procedure* is a partial function $\Phi\colon 2^{<\omega} \to \omega^{<\omega}$ with Π^1_1 graph such that if $\Phi(x)\downarrow$ and $y \preceq x$, then $\Phi(y) \preceq \Phi(x)$. We write $f = \Phi^Z$ if $\forall n \exists m\, \Phi(Z\restriction_m) \succeq f\restriction_n$, and $f \le_{hT} Z$ if $f = \Phi^Z$ for some hT reduction procedure Φ.

11.1. Higher analog of weak 2-randomness.

The following was introduced in [51, Problem 9.2.17].

Definition 11.3. A *generalized* Π^1_1*-ML-test* is a sequence $(G_m)_{m\in\mathbb{N}}$ of uniformly Π^1_1 open classes such that $\bigcap_m G_m$ is a null class. Z is Π^1_1*-weakly 2-random* if Z passes each generalized Π^1_1-ML-test.

Clearly the class $\bigcap_m G_m$ is Π^1_1, so Π^1_1-randomness implies Π^1_1-weak 2 randomness.

[51, Problem 9.2.17] asked whether the new notion coincides with Π^1_1-ML-randomness.

Theorem 11.4. *If x is the leftmost path of a Σ^1_1-closed set of reals, then x is not Π^1_1-weak 2 random.*

Proof. Let $T \subseteq 2^{<\omega}$ be a Σ^1_1-tree.

For any $n \in \omega$ and $\alpha < \omega_1^{\mathrm{CK}}$, let

$$U_{n,\alpha} = \{\sigma \mid \exists z(z \text{ is the leftmost path in } T[\alpha] \wedge \sigma = z\restriction n+1)\}.$$

Define

$$U_{n,<\alpha} = \bigcup_{\beta<\alpha} U_\beta$$

and

$$U_n = \bigcup_{\alpha<\omega_1^{\mathrm{CK}}} U_{n,\alpha}.$$

The following facts are obvious.

(1) For any n and $\alpha < \omega_1^{\mathrm{CK}}$, $U_{n+1,\alpha} \subseteq U_{n,\alpha}$;
(2) For any n and $\alpha < \omega_1^{\mathrm{CK}}$, $\mu(U_{n,\alpha}) < 2^{-n}$;
(3) $x \in \bigcap_{n\in\omega} U_n$;
(4) For any n, $\alpha < \omega_1^{\mathrm{CK}}$ and real z, if $z \in U_{n,<\alpha} \setminus U_{n,\alpha}$, then $z \notin U_{n,\beta}$ for any $\beta \geq \alpha$.

Now suppose that $\mu(\bigcap_{n\in\omega} U_n) > 0$, then there must be some σ_0 so that

$$\mu(\bigcap_{n\in\omega} U_n \cap [\sigma_0]) > \frac{3}{4} \cdot 2^{-|\sigma_0|}.$$

Let $n_0 = |\sigma_0| + 2$. Then there must be some least $\alpha_0 < \omega_1^{\mathrm{CK}}$ so that

$$\mu(U_{n_0,\leq\alpha_0} \cap [\sigma_0]) > \frac{3}{4} \cdot 2^{-|\sigma_0|}.$$

By (2),

$$\mu(U_{n_0,<\alpha_0} \cap [\sigma_0]) > \frac{1}{2} \cdot 2^{-|\sigma_0|}.$$

By (1) and (4),

$$\mu(\bigcap_{n>n_0} U_{n,<\alpha_0} \cap [\sigma_0]) > \frac{1}{4} \cdot 2^{-|\sigma_0|}.$$

So there must be some $\sigma_1 \succ \sigma_0$ so that

$$\mu(\bigcap_{n>n_0} U_{n,<\alpha_0} \cap [\sigma_1]) > \frac{3}{4} \cdot 2^{-|\sigma_1|}.$$

Let $n_1 = |\sigma_1| + 2$. Then there must be some least $\alpha_1 < \alpha_0$ so that

$$\mu(U_{n_1,\leq\alpha_1} \cap [\sigma_1]) > \frac{3}{4} \cdot 2^{-|\sigma_1|}.$$

Repeat the same method, we obtained a descending sequence $\alpha_0 > \alpha_1 > \cdots$, which is a contradiction. □

In the computability setting, let Z be ML-random. Then by a result of Hirschfeldt and Miller (see [51, 5.3.16]),

Z is weakly 2-random $\Leftrightarrow$ $Z, \emptyset'$ form a Turing minimal pair.

We cannot expect this to hold in the higher setting, because by Gandy's basis theorem, there is a Π^1_1-random set $Z \leq_{\mathrm{T}} \mathcal{O}$. However, the result of Hirschfeldt and Miller actually shows that if Z is not weakly 2-random, then there is a c.e. incomputable set A below Z. This carries over and yields a characterization of the Π^1_1-weakly 2-random sets withing the ML-random sets.

Theorem 11.5. *Consider the following for a Π^1_1-ML-random set Z.*

(i) *Z is Π^1_1-weakly 2-random*

(ii) *If $A \le_{hT} Z$ for a Π^1_1 set A, then A is hyperarithmetical.*

We have that (ii)→(i).

Proof. We can adapt some of the theory of cost functions to the higher setting. Suppose $Z \in \bigcap_m G_m$ where $(G_m)_{m\in\mathbb{N}}$ is a generalized Π^1_1-ML-test. Let $c(x,\alpha) = \lambda G_{x,\alpha}$. This is a higher cost function with the limit condition. Hence there is a Π^1_1 but not Δ^1_1 set A obeying c by the higher analog of [51, 5.3.5]. By the higher analog of the Hirschfeldt-Miller method in the version [51, 5.3.15], we may conclude that $A \le_{\text{fin-h}} Z$.

(Maybe you, the reader, are puzzled why this doesn't show that the Π^1_1 random set $Z \le_{\mathrm{T}} \mathcal{O}$ obtained by Gandy's basis theorem we discussed earlier fin-h bounds a properly Π^1_1 set? The answer is that, unlike the case of Δ^0_2 sets in the computability setting, not for every set $Z \le_{\mathrm{T}} \mathcal{O}$ there is a generalized Π^1_1-ML-test (G_m) with $\bigcap G_m = \{Z\}$.) ☐

We now attempt a partial solution to [51, Problem 9.2.17], showing that as operators sending oracles to classes, the two randomness notions differ.

Theorem? 11.1. *Π^1_1 weak 2-randomness and Π^1_1 randomness differ relative to some low oracle.*

Proof. Let $\underline{\Omega}_0$ be the bits of $\underline{\Omega}$ in the even positions, and let $\underline{\Omega}_1$ be the bits of $\underline{\Omega}$ in the odd positions. Clearly $\underline{\Omega}_0$, $\underline{\Omega}_1$ are both Π^1_1-Martin-Löf random and fin-h incomparable. By the van Lambalgen theorem for Π^1_1-randomness [32], if $\underline{\Omega}_0$ is Π^1_1 random, then $\underline{\Omega}_1$ is not Π^1_1 random relative to $\underline{\Omega}_0$. It now suffice to show that $\underline{\Omega}_1$ is Π^1_1 weakly 2-random relative to $\underline{\Omega}_0$. Since it is Π^1_1 ML-random relative to $\underline{\Omega}_0$, by the theorem 11.5 relative to $\underline{\Omega}_0$ this amounts to showing that $\mathcal{O}^{\underline{\Omega}_0} \not\le_{\text{fin-h}} \underline{\Omega}$. (Can someone show this? Seems to be analogous to the fact [51] 3.4.15) that halves of Ω are not superlow.). ☐

11.2. The higher analog of difference randomness.

Definition 11.6. A Π^1_1-*difference-test* is a sequence $(G_m)_{m\in\mathbb{N}}$ of uniformly Π^1_1 open classes together with a closed Σ^1_1 class $\mathcal{C}$ such that $\lambda(\mathcal{C} \cap G_m) \le 2^{-m}$ for each m. We say that Z is Π^1_1-*difference random* if Z passes each such test in that $Z \notin \mathcal{C} \cap \bigcap_m G_m$.

Theorem 11.7. *Let Z be a Π^1_1-ML-random set. Then*

$$Z \text{ is } \Pi^1_1\text{-difference random} \Leftrightarrow \underline{\Omega} \not\le_{\text{fin-h}} Z.$$

Note: again, $\le_{\text{fin-h}}$ has to be replaced by $\le_{hT}$ to make this work (Bienvenu, Greenberg and Monin).

Proof. $\Leftarrow$: By contraposition. Suppose that $Z \in \mathcal{C} \cap \bigcap_m G_m$ for a Π^1_1-difference test $\mathcal{C}, (G_m)_{m\in\mathbb{N}}$. Define a higher Solovay test $\mathcal{S}$ as follows. When m enters $\mathcal{O}$ at stage β, we may within $L_{\omega_1^{CK}}$ compute $\gamma \ge \beta$ such that $2^{-m} \ge \lambda(\mathcal{C}_\gamma \cap G_{m,\beta})$. (More precisely, the function $h\colon \omega_1^{CK} \times \omega \to \omega_1^{CK}$ mapping β, m to γ if m enters at stage β, and to 0 otherwise, is Σ^0_1 over $L_{\omega_1^{CK}}$.) By [56, 1.8.IV], determine an open Δ^1_1 class $\mathcal{D}_m \supseteq (\mathcal{C}_\gamma \cap G_{m,\beta})$ such that $\lambda \mathcal{D}_m \le 2^{-m+1}$, and enumerate $\mathcal{D}_m$ into $\mathcal{S}$.

Let $g(m) = \lambda\alpha.[Z \in G_{m,\alpha}]$. Note that $g \leq_{\text{fin-h}} Z$. Since Z is Π^1_1 ML-random, we have $Z \notin \mathcal{D}_m$ for almost every m. Hence we have $m \in \mathcal{O} \leftrightarrow m \in \mathcal{O}_{g(m)}$ for a.e. m, which shows that $\mathcal{O} \leq_{\text{fin-h}} Z$.
$\Rightarrow$: Suppose that $\underline{\Omega} = \Gamma^Z$ for a fin-h reduction procedure Γ. Choose $c \in \mathbb{N}$ such that $2^{-n} \geq \lambda\{Y : \underline{\Omega}\restriction_{n+c} \preceq \Gamma^Y\}$. Let

$$V_{n,=\alpha} = [\{\sigma\colon\ [\Gamma^\sigma_\alpha \succ \underline{\Omega}_\alpha\restriction_{n+c} \wedge \lambda\{\tau\colon\ \Gamma^\tau_\alpha \succ \underline{\Omega}_\alpha\restriction_{n+c}\} \leq 2^{-k}]\}]^{\prec}.$$

Let $V_{n,\gamma} = \bigcup_{\alpha\leq\gamma} V_{n,=\alpha}$, and $\mathcal{D} = \bigcup_n \bigcup_\alpha \{V_{n,=\alpha}\colon\ \exists\beta > \alpha[\underline{\Omega}_\alpha < \underline{\Omega}_\beta]$. Note that $\mathcal{D}$ is Π^1_1 open, and V_n is Π^1_1 open uniformly in n. By definition we have $2^{-n} \geq \lambda V_n \setminus \mathcal{D}$. Clearly $Z \in \bigcap_n V_n \setminus \mathcal{D}$. Thus Z is not higher difference random. $\square$

11.3. **Randomness and density.**

Definition 11.8. We define the (lower) *Lebesgue density* of a set $\mathcal{C} \subseteq \mathbb{R}$ at a point x to be the quantity

$$\rho(x|\mathcal{C}) := \liminf_{\gamma,\delta\to 0^+} \frac{\lambda([x-\gamma, x+\delta] \cap \mathcal{C})}{\lambda([x-\gamma, x+\delta])}.$$

For $x \in \mathbb{R}$ and $m \in \omega$ we denote by $[x\restriction_m)$ the interval of the form $[k2^{-m}, (k+1)2^{-m})$ containing x. The *dyadic density* of a set $\mathcal{C} \subseteq \mathbb{R}$ at a point x is

$$\rho_2(x|\mathcal{C}) := \liminf_{n\to\infty} \frac{\lambda([x\restriction_n) \cap \mathcal{C})}{\lambda([x\restriction_n))}.$$

The following result it the higher analog of a result on difference randomness due to [7]. Only the usual notational changes to the proofs are necessary, which we omit.

Theorem 11.9. *Let x be a Π^1_1 Martin-Löf random real. Then $x \not\geq_{\text{fin-h}} \mathcal{O}$ iff x has positive density in some closed Σ^1_1 class containing x.*

As in [7], the result is derived from a lemma which actually shows that the Σ^1_1 class is the same on both sides.

Lemma 11.10. *Let x be a Π^1_1 Martin-Löf random real. Let $\mathcal{C}$ be a closed Σ^1_1 class containing x. The following are equivalent:*

(i) *x fails a Π^1_1 difference test of the form $((U_n)_{n\in\mathbb{N}}, \mathcal{C})$.*
(ii) *x has lower Lebesgue density zero in $\mathcal{C}$, i.e., $\rho(x|\mathcal{C}) = 0$.*

The following was first discussed in April in Paris (Bienvenu, Monin and Nies). The proof below is due to Yu. It does the weaker case of dyadic density, but could be adapted to full Lebesgue density.

Proposition 11.11. *Let $x \in 2^{\mathbb{N}}$ be Π^1_1-random. Suppose $x \in \mathcal{C}$ for a Σ^1_1 class $\mathcal{C}$. Then $\underline{\lim}_{n\to\infty} \lambda_{[x\restriction n]}(\mathcal{C}) = 1$.*

Proof. Suppose otherwise. Then for some rational $p < 1$, there are infinitely many n such that $\lambda_{[x\restriction n]}(\mathcal{C}) < p$. Define a function f that is Σ_1 over $L_{\omega_1^x}[x]$ as follows: for each k, $f(k)$ is the $<_{L[x]}$-least pair (m_k, α_k) so that $m_k > k$

and $\lambda_{[x\upharpoonright m_k]}\mathcal{C}[\alpha_k]) < p$. Then f is a total function. So there must be an ordinal $\gamma < \omega_1^x = \omega_1^{\mathrm{CK}}$ so that $\lambda_{[x\upharpoonright m_k]}\mathcal{C}[\alpha_k] < p$ for every k. This implies

$$\underline{\lim}_{n\to\infty}\lambda_{[x\upharpoonright n]}\mathcal{C}[\gamma] < p.$$

Since $\mathcal{C} \subseteq \mathcal{C}[\gamma]$, we have $x \in \mathcal{C}[\gamma]$. But $\mathcal{C}[\gamma]$ is a Δ^1_1 set, so x has density 1 in $\mathcal{C}[\gamma]$, a contradiction. □

11.4. **A higher version of Demuth's Theorem.**

Theorem 11.12. *If x is Π^1_1-random and $y \leq_h x$ is not hyperarithmetic, then there is a Π^1_1 random real $z \equiv_h y$.*

Proof. Suppose that x is Π^1_1-random and $y \leq_h x$ is not hyperaritmetic. Then there is a there is some recursive ordinal α, an nondcreasing function $f \leq_T \emptyset^{(\alpha)}$ and a recursive function Ψ so that $\lim_{n\to\infty} f(n) = \infty$ and for every n,

$$y(n) = \Psi^{x\upharpoonright f(n)\oplus\emptyset^{(\alpha)}\upharpoonright f(n)}(n)[f(n)].$$

For each $u \in 2^{<\omega}$, let

$$l(u) = \sum_{\tau\in 2^{|u|}\wedge\tau<u} \left(\sum\{2^{-|\sigma|}\colon\ \sigma \in 2^{f(|u|)} \wedge \Psi^{\sigma\oplus\emptyset^{(\alpha)}\upharpoonright f(|u|)}[f(|u|)] \upharpoonright |u| = \tau\}\right)$$

and

$$r(u) = l(u) + \sum\{2^{-|\sigma|}\colon\ \sigma \in 2^{f(|u|)} \wedge \Psi^{\sigma\oplus\emptyset^{(\alpha)}\upharpoonright f(|u|)}[f(|u|)] \upharpoonright |u| = \tau\},$$

where $\tau < u$ means that τ is in the left of u.

One may view $\sum_{\sigma\in 2^{f(|u|)}\wedge\Psi^{\sigma\oplus\emptyset^{(\alpha)}\upharpoonright f(|u|)}[f(|u|)]\upharpoonright|u|=\tau} 2^{-|\sigma|}$ as a kind of "measure" for τ.

For each n, let

$$l_n = l(y \upharpoonright n), \text{ and } r_n = r(y \upharpoonright n).$$

Then $l_n \leq l_{n+1} \leq r_{n+1} \leq r_n$ for every n.

Since y is not hyperarithmetic, it is not difficult to see that $\lim_{n\to\infty} r_n = 0$. So there is a unique real

$$z = \bigcap_{n\in\omega}(l_n, r_n).$$

Obviously $z \leq_T y \oplus \emptyset^{(\alpha)}$. We leave readers to check that $y \leq_T z \oplus \emptyset^{(\alpha)}$. So $z \equiv_h y$.

Suppose that z is not Δ^1_1-random. Then there must be some recursive ordinal $\beta < \alpha$ and a $\emptyset^{(\beta)}$-ML-test $\{V_n\}_{n\in\omega}$ so that $z \in \bigcap_{n\in\omega} V_n$. Let

$$\begin{aligned}\hat{V}_n = \{u \mid\ &\exists\nu(\nu \text{ is the } k\text{-th string in } V_n\wedge\\ &\exists p \in \mathbb{Q}\exists q \in \mathbb{Q}(l(u) \leq p < q \leq r(u)\wedge[p,q] \subseteq [\nu]\wedge q-p > r(u)-2^{-n-k-2})\}.\end{aligned}$$

Since $z \in V_n$, we have that $y \in \hat{V}_n$ for every n. Note that $\{\hat{V}_n\}_{n\in\omega}$ is $\emptyset^{(\beta+1+\alpha)}$-r.e.

Let

$$U_n = \{\sigma \mid \exists\tau \in \hat{V}_n(|\sigma| = f(|\tau|) \wedge \Phi^{\sigma\oplus\emptyset^{(\alpha)}\upharpoonright f(|\tau|)}[f(|\tau|)] \upharpoonright |\tau| = |\tau|)\}.$$

Then $\{U_n\}_{n\in\omega}$ is $\emptyset^{(\beta+1+\alpha)}$-r.e and $x \in \bigcap_{n\in\omega} U_n$. Note that for every n,

$$\mu(U_n) \leq \mu(V_n) + \sum_{k\in\omega} 2^{-n-k-2+1} < 2^{-n} + 2^{-n} = 2^{-n+1}.$$

Then $\{U_{n+1}\}_{n\in\omega}$ is a $\emptyset^{(\beta+1+\alpha)}$-ML-test. So x is not a Δ^1_1-random, a contradiction. □

An immediate conclusion of the proof of Theorem 11.12 is:

Corollary 11.13. *For any Π^1_1-random real z, if $x \leq_h z$ is not hyperarithmetic, then x is Π^1_1-random relative to some measure μ.*

11.5. **Separating lowness for higher randomness notions.** In [66], Yu gave a new proof of the the following theorem.

Theorem 11.14 (Martin and Friedman)**.** *For any Σ^1_1 tree T_1 which has uncountably many infinite paths, $[T_1]$ has a member of each hyperdegree greater than or equal to the hyperjump.*

Lemma 11.15. *Given any two uncountable Σ^1_1 sets of reals A_0 and A_1, for any real $z \geq_h \mathcal{O}$, there are reals $x_0 \in A_0$ and $x_1 \in A_1$ so that $x_0 \oplus x_1 \equiv_h z$.*

Proof. Fix a real $z \geq_h \mathcal{O}$, two uncountable Σ^1_1-sets A_0 and A_1. So there are two recursive trees $T_0, T_1 \subseteq 2^{<\omega} \times \omega^{<\omega}$ so that $A_i = \{x \mid \exists f \forall n(x \upharpoonright n, f \upharpoonright n) \in T_i\}$ for each $i \leq 1$. We may assume that neither A_0 nor A_1 contains a hyperarithmetic real. We also fix a recursive tree $T_2 \subseteq \omega^{<\omega}$ so that $[T_2]$ is uncountable but does not contain a hyperarithmetic infinite path. Let f be the leftmost path in T_2. Then $f \equiv_h \mathcal{O}$.

Given a finite string $\sigma \in 2^{<\omega}$, we say that σ is *splitting* on a tree $T \subseteq 2^{<\omega} \times \omega^{<\omega}$ if for any $j \leq 1$, $T_{\sigma^\frown j} = \{(\sigma', \tau') \mid (\sigma' \succeq \sigma^\frown j \vee \sigma' \prec \sigma^\frown j) \wedge (\sigma', \tau') \in T\}$ contains an infinite path. For any $i \leq 1$ and $(\sigma, \tau) \in T_i$, define

$$T_{i,(\sigma,\tau)} = \{(\sigma', \tau') \in T_i \mid (\sigma', \tau') \succeq (\sigma, \tau) \vee (\sigma', \tau') \prec (\sigma, \tau)\}.$$

We shall construct a sequence $(\sigma_{i,0}, \tau_{i,0}) \prec (\sigma_{i,1}, \tau_{i,1}) \prec \cdots$ from T_i for each $i \leq 1$. The idea is to apply a mutually coding argument. In other words, we will use $\sigma_{0,j}$ to code $z(j)$, $f(j)$ and $\tau_{1,j-1}$, and $\sigma_{1,j}$ to code $\tau_{0,j}$ for each $j \in \omega$.

At stage 0, let $(\sigma_{i,0}, \tau_{i,0}) = (\emptyset, \emptyset)$ for $i \leq 0$. Without loss of generality, we may assume that $(\emptyset, \emptyset)$ is a splitting node in both T_0 and T_1.

At stage $s+1$:

Substage 1, let $\sigma^0_{0,s+1}$ be the left-most finite splitting string extending $\sigma_{0,s}^\frown z(s)$ on $T_{0,(\sigma_{0,s},\tau_{0,s})}$. So we code $z(s)$ here. Then we prepare to code $\tau_{1,s}$. Let $n^0_{s+1} = |\tau_{1,s}| - |\tau_{1,s-1}|$. Inductively, for any $k \in [1, n^0_{s+1}]$, let $\sigma^k_{0,s+1}$ be the left-most finite splitting string extending $(\sigma^{k-1}_{0,s+1})^\frown 1$ on $T_{0,(\sigma_{0,s},\tau_{0,s})}$ so that there are $\tau_{0,s+1}(k + |\tau_{0,s}|)$-many splitting nodes between $\sigma^{k-1}_{0,s+1}$ and $\sigma^k_{0,s+1}$. Let $\sigma^{n^0_{s+1}+1}_{0,s+1}$ be the left-most finite splitting string extending $(\sigma^{n^0_{s+1}}_{0,s+1})^\frown 1$ on $T_{0,(\sigma_{0,s},\tau_{0,s})}$ so that there are $f(s)$-many splitting nodes between$\sigma^{n^0_s+1}_{0,s+1}$ and $\sigma^{n^0_{s+1}+1}_{0,s+1}$. So we code $f(s)$ here. Inductively, for $j \leq 1$, let $\sigma^{n^0_{s+1}+1+j+1}_{0,s+1}$ be the next splitting string in $T_{0,(\sigma_{0,s},\tau_{0,s})}$ extending $(\sigma^{n^0_{s+1}+1+j}_{0,s+1})^\frown 1$. This coding tells us that the action at this stage for $(\sigma_{0,s+1}, \tau_{0,s+1})$-part is finished. Define $\sigma_{0,s+1} = \sigma_{0,s+1}^{n^0_{s+1}+3}$. Let $\tau_{0,s+1} \in \omega^{|\sigma_{0,s+1}|}$ be the leftmost finite string so that the tree $T_{0,(\sigma_{0,s+1},\tau_{0,s+1})}$ has an infinite path.

Substage 2. Let $\sigma^0_{1,s+1} = \sigma_{1,s}$ and $n^1_{s+1} = |\tau_{0,s+1}| - |\tau_{0,s}|$. Inductively, for any $k \in [1, n^1_{s+1}]$, let $\sigma^k_{1,s+1}$ be the left-most finite splitting string extending $(\sigma^{k-1}_{1,s+1})^\frown 1$ on $T_{1,(\sigma_{1,s},\tau_{1,s})}$ so that there are $\tau_{0,s+1}(k + |\tau_{0,s}|)$-many splitting nodes between $\sigma^{k-1}_{1,s+1}$ and $\sigma^k_{1,s+1}$. So $\tau_{0,s+1}$ is coded. Inductively, for $j \leq 1$, let $\sigma^{n^1_{s+1}+j+1}_{1,s+1}$ be the next splitting string in $T_{1,(\sigma_{1,s},\tau_{1,s})}$ extending $(\sigma^{n^1_{s+1}+j}_{1,s+1})^\frown 1$. This coding tells us that the action at this stage for $(\sigma_{1,s+1}, \tau_{1,s+1})$-part is finished. Define $\sigma_{1,s+1} = \sigma^{n^1_{s+1}+2}_{1,s+1}$. Let $\tau_{1,s+1} \in \omega^{|\sigma_{1,s+1}|}$ be the leftmost finite string so that the tree $T_{1,(\sigma_{1,s+1},\tau_{1,s+1})}$ has an infinite path. So we code $\tau_{0,s+1}$ into $\sigma_{1,s+1}$.

This finishes the construction at stage $s + 1$.

Let $x_i = \bigcup_{s\in\omega} \sigma_{i,s}$ for $i \leq 1$. Obviously $z \geq_h x_0 \oplus x_1$.

Now we use x_0 and x_1 to decode the coding construction. The decoding method is a finite injury which is quite similar to the new proof of Theorem 11.14. We want to construct a sequence ordinals $\{\alpha_s\}_{s\in\omega}$ Δ_1-definable in $L_{\omega_1^{x_0\oplus x_1}}[x_0 \oplus x_1]$ so that $\lim_{s\to\omega} \alpha_s = \omega_1^{\mathrm{CK}}$. Once this is done, then it is obvious to decode the construction and so $x_0 \oplus x_1 \geq_h z$.

As in the proof of Theorem 11.14, we may fix a Σ_1 enumeration of $\{T_i[\alpha]\}_{i\leq 2,\alpha<\omega_1^{\mathrm{CK}}}$ over $L_{\omega_1^{\mathrm{CK}}}$ so that for $i \leq 1$,

- $T_i[0] = T_i$; and
- $T_i[\alpha] \subseteq T_i[\beta]$ for $\omega_1^{\mathrm{CK}} > \alpha \geq \beta$; and
- $T_i[\omega_1^{\mathrm{CK}}] = \bigcap_{\alpha<\omega_1^{\mathrm{CK}}} T_i[\alpha]$; and
- $T_i[\omega_1^{\mathrm{CK}}]$ has no dead node; and
- $A_i = \{x \mid \exists f \forall n (x \upharpoonright n, f \upharpoonright n) \in T_i[\omega_1^{\mathrm{CK}}]\}$.

Since $[T_i]$ does not contain a hyperarithmetic infinite path, we have that $[T_i[\omega_1^{\mathrm{CK}}]] = [T_i]$ for $i \leq 2$.

We perform almost the same decoding construction as in the proof of Theorem 11.14. At every stage s, we have a guess for the parameters defined in the coding construction up to stage s. Also we may need to correct our guess by searching a big ordinal α_s to redefine those parameters turning out to be incorrect at stage α_s (if necessary). Since neither x_0 nor x_1 is the leftmost real in T_0 or T_1, these parameters can only be redefined at most finitely many times. So every parameter will be stable after stage α_s for large enough s. Then using the same method as in the proof of Theorem 11.14, we can find f at stage $\alpha = \lim_{s\to\omega} \alpha_s$. The only extract effort is to decode $\tau_{i,s}$. But this is just like to decode $f(s)$ without any new insight.

So $\mathcal{O} \equiv_h f \leq_h x_0 \oplus x_1$.

□

By [11], being low for Π^1_1 randomness is equivalent to being low for Δ^1_1 randomness and being not cuppable above $\mathcal{O}$ by a Π^1_1 random. So it suffices to define a low for Δ^1_1 random that is cuppable.

Lemma 11.16. *There is an uncountable Σ^1_1 set A in which every real is Δ^1_1-traceable.*

Proof. This follows directly from the proof of Theorem 4.7 in [58]. □

By [11], each Δ^1_1-traceable real is low for Δ^1_1-random. By [32], the Π^1_1-random reals form a Σ^1_1 set. Then by Lemma 11.16 and 11.15, there is a real x which is low for Δ^1_1-randomness but $x \oplus y \geq_h \mathcal{O}$ for some Π^1_1-random real y. We may conclude:

Theorem 11.17. *Lowness for Δ^1_1-randomness does not imply lowness for Π^1_1-randomness. And lowness for Δ^1_1-Kurtz-randomness does not imply lowness for Π^1_1-Kurtz-randomness.*

Remark: Lemma 11.15 can be used to answer Question 58 in [26] and Question 3 in [58]. Solutions were announced by Friedman and Harrington, but never published.

12. Some notes on Δ^1_2- and Σ^1_2-randomness

Input by Yu. (October)

12.1. **Within** ZFC**.** The ground model for Σ^1_2-theory is $L_{\delta^1_2}$ where δ^1_2 is the least ordinal which can not be Δ^1_2-definable. We have Gandy-Spector theorem for Σ^1_2-sets over $L_{\delta^1_2}$. The generalized Turing jumps with $L_{\delta^1_2}$ are Π^1_1-singletons.

Within ZFC, very limited interesting results can be obtained for the randomness notions. The following result was proved by Kechris under PD but it turns out to be a theorem under ZFC.

Theorem 12.1 (Kechris [36])**.** *Given a measurable Σ^1_2 set $A \subseteq 2^\omega$, both the sets $\{p \in \mathbb{Q} \mid \mu(A) > p\}$ and $\{p \in \mathbb{Q} \mid \mu(A) \geq p\}$ are Σ^1_2.*

Again the following result was proved by Kechris under PD which is unnecessary.

Proposition 12.2 (Kechris [36])**.** *Suppose that $A \subseteq 2^\omega$ is a Σ^1_2 set with a positive measure, then for any n, there is a Δ^1_2 perfect set $B_n \subseteq A$ so that $\mu(A) < \mu(B_n)+2^{-n}$. Moreover, if $\mu(A)$ is Δ^1_2, then $\{(n,x) \mid x \in B_n \wedge n \in \omega\}$ is Δ^1_2.*

An immediate conclusion of Proposition 12.2 is:

$$\Delta^1_2\text{-randomness} = \Delta^1_2\text{-ML-randomness} = \Pi^1_2\text{-randomness}.$$

By a similar enumeration over $L_{\delta^1_2}$, we have a universal Σ^1_2-ML-test. By a forcing argument over Δ^1_2-closed positive measure sets, we may show that the collection of Δ^1_2-random reals is not $\mathbf{\Sigma}^0_3$. So Δ^1_2-randomness is different than Σ^1_2-ML-randomness.

Fix a Π^1_2-tree T presenting a Π^1_2-closed set only containing Σ^1_2-ML-random reals. The leftmost path of T can be covered by a generalized Σ^1_2-ML-test. So Σ^1_2-ML-randomness is different with strong Σ^1_2-ML-randomness.

12.2. **Outside *ZFC*.** If $V = L$, then for every real x, the set $\{y \mid x \not\leq_{\Delta^1_2} y\}$ is countable and so null. But if we believe that L must be small, then we have regular results as following.

Theorem 12.3 (Kechris [36]). *Suppose that every Σ^1_2 set of reals is measurable. If x is a real so that $\{y \mid x \leq_{\Delta^1_2} y\}$ has positive measure, then x is Δ^1_2;*

Theorem 12.3 was proved by Kechris under PD. Stern [62] observed that it can be proved under the much weaker assumption: by Solovay [61], the assumption in the Theorem is a consequence of "$(\omega_1)^L < \omega_1$".

Definition 12.4. A real x is *L-random* if for any Martin-Löf test $\{U_n\}_{n\in\omega}$ in L, $x \notin \bigcap_{n\in\omega} U_n$.

L-randomness was essentially introduced by Solovay [61]. The following fact is obvious.

Proposition 12.5. *The followings are equivalent:*

(i) *x is L-random;*
(ii) *x does not belong to any Borel null set coded in L;*
(iii) *x is a $\mathbb{R} = (\mathbf{T}, \leq)$-generic real over L, where $\mathbb{R}$ is random forcing.*

The following theorem is an analog of Π^1_1-randomness theory.

Theorem 12.6 (Stern [62]). *For any real x, x is Δ^1_2-random and $\delta^{1,x}_2 = \delta^1_2$ if and only if x is L-random;*

If L is small, then Σ^1_2-randomness is the same as L-randomness.

Theorem 12.7 (Stern [62]). *Assume that the set of non-L-random reals is null. Then the set of non-L-random reals is the largest null Σ^1_2 set and so Σ^1_2-randomness is the same as L-randomness.*

Solovay [61] proves that the set of non-L-random reals is null if and only if every Σ^1_2-set is measurable. So if every Σ^1_2-set is measurable, then strong Σ^1_2-ML randomness is different than Σ^1_2-randomness. This sheds some light on the corresponded Π^1_1-randomness problem.

12.3. **Lowness.** Very little is known.

If $V = L$, then lowness for Δ^1_2-randomness=lowness for Σ^1_2-ML-randomness=Δ^1_2-ness. But they should not be treated as "regular results".

Note that every Sacks generic real is low for L-random. So if every $\mathbf{\Sigma}^1_2$-set is measurable, then every Sacks generic real is low for Σ^1_2-random. Actually, they are precisely those reals that are "constructibly traceable" under certain assumptions.

12.4. **Within L.** Within L, nothing is interesting for higher-up randomness notions. The main point is that there is a Δ^1_2-well ordering over reals in L. So for $n \geq 2$, to study Σ^1_n-randomness, we have to appeal to some axioms to make the universe "regular".

All the proofs can be found in the forthcoming book [13]. Draft available on Yu's web site.

Part 4. Complexity of Equivalence relations

13. Complexity of arithmetical equivalence relations

Sy Friedman, Katia Fokina, André Nies worked Vienna, Jan. 2012. This research also involves discussions at the Oberwolfach February meeting with D. Cenzer, J. Knight, J. Liu, V. Harizanov, A. Nies. More recently (April 2012) Russell Miller and Selwyn Ng joined in this line of research, along with Nies' MSc student Egor Ianovski.

For equivalence relations E, F with domain ω, we write $E \leq_1 F$ if there is a computable 1-1 function g such that $xEy \Leftrightarrow g(x)Fg(y)$.

13.1. There is a 1-complete Π^0_1 equivalence relation.

We begin with some examples of Π^0_1 equivalence relations.

- Elementary equivalence of automatic structures for the same finite signature (according to Khoussainov, known to be undecidable by some Russian result); the same with the extended language allowing $\exists^\infty$.
- Isomorphism of automatic equivalence structures/ trees of height 2. This is Π^0_1 complete in the set sense by Kuske, Liu and Lohrey (TAMS, to appear), but not known to be Π^0_1 complete for eqrels.
- If f is a binary computable function then let $E_f xy \leftrightarrow \forall i\, f(x,i) = f(y,i)$, which clearly is Π^0_1. Ianovski, Miller, Nies and Ng [34] have shown that every Π^0_1 eqrel is of this form. This contrasts with Marchenkov's result (1970s) that there is no universal negative enumeration (Reference?).

Theorem 13.1. *There is a Π^0_1 equivalence relation E such that $G \leq_1 E$ for each Π^0_1 equivalence relation G.*

Proof. Let $X^{[2]}$ denote the unordered pairs of elements of X. A set $R \subseteq X^{[2]}$ is *transitive* if $\{u,v\}, \{v,w\} \in R$ and $u \neq w$ implies $\{u,w\} \in R$. We view eqrels as transitive subsets of $\omega^{[2]}$. We view the p-th r.e. set W_p as a subset of $\omega^{[2]}$.

The idea is to copy W_p as long as $\omega^{[2]} - W_p$ looks transitive. The resulting partial copy may have finite or infinite domain. If $\omega^{[2]} - W_p$ is indeed transitive then the domain is infinite.

In a sense E is a uniform disjoint sum of all these partial copies.

Uniformly in a given p define a partial computable sequence of stages by $t^p_0 = 0$ and

$$t^p_{i+1} \simeq \mu t > t^p_i \left[[0, t^p_i)^{[2]} - W_{p,t} \text{ is transitive } \right].$$

Define a computable function $g\colon\ \omega \to \omega \times \omega$ by the following construction. At each stage the domain of g is a finite initial segment of ω. At stage t, for each i with $0 < i \leq t$, if $t = t^p_{i+1}$ then add an interval of numbers L^p_i of length $\ell = t^p_i - t^p_{i-1}$ to the domain of g, and map it in an increasing fashion to the least ℓ numbers in $\{p\} \times \omega$ that are not yet in the range of g. In this way, we extend the range of g to contain $\{p\} \times [0, t^p_i)$.

Now for $x < y$ declare $\{x, y\} \in E$ if $g(x)_0 = g(y)_0 =: p$, and for the unique i such that $y \in L_i^p$, we have

$$\forall t \geq t_i^p \,\forall k \geq i\, [t = t_{k+1}^p \rightarrow \{g(x)_1, g(y)_1\} \notin W_{p,t}].$$

Clearly E is Π_1^0. To check E is transitive, suppose $\{x, y\} \in E$, $\{z, y\} \in E$, where $x < y$, and $z < y$. We may suppose that $x < z$. There are unique p, a, b, c such that $g(x) = \langle p, a\rangle, g(y) = \langle p, b\rangle$, and $g(z) = \langle p, c\rangle$. We have $y \in L_i^p$, and $z \in L_r^p$ for some $r \leq i$. Assume that there is $k \geq r$ such that for $t = t_{k+1}^p$ we have $\{a, c\} \in W_{p,t}$. Let $j = \max(i, k)$ and note that t_{j+1}^p is defined. Then $[0, t_j^p)^{[2]} - W_{p,t_{j+1}^p}$ is not transitive, because it contains $\{a, b\}, \{b, c\}$, but not $\{a, c\}$. This contradicts the definition of t_{j+1}^p. Thus $\{x, z\} \in E$.

Finally, given a Π_1^0 equivalence relation $G = \omega^{[2]} - W_p$, the sequence of stages $(t_i^p)_{i\in\mathbb{N}}$ is infinite. Hence, the (total) computable map $a \rightarrow g^{-1}(\langle p, a\rangle)$ is the required 1-reduction of G to E. $\square$

13.2. **Completeness for Σ_3^0 equivalence relations of $\equiv_1$ on the r.e. sets.** It is trivial that for each there is some Σ_n^0 complete equivalence relation, because the transitive closure of a Σ_n^0 relation is Σ_n^0. We can make it unique by requiring it to be EUH (effective universal homogeneous) in the sense of Nerode and Remmel.

In the following we look for natural examples. For Σ_3^0 we don't have to look far.

Theorem 13.2. *For each Σ_3^0 equivalence relation S, there is a computable function g such that*

$$\begin{aligned} ySz &\Rightarrow W_{g(y)} \equiv_1 W_{g(z)}, \text{ and} \\ \neg ySz &\Rightarrow W_{g(y)}, W_{g(z)} \text{ are Turing incomparable.} \end{aligned}$$

Corollary 13.3. *Many-one equivalence and 1-equivalence on indices of c.e. sets are Σ_3^0 complete for equivalence relations under computable reducibility.*

According to S. Podzorov at the Sobolev Institute Novoskibirsk, the result for m-equivalence was possibly known to the Russians by the end of the 1970s. No reference has been given yet. Sadly, Podzorov passed away in late 2012.

Note that this is significantly stronger than the mere Σ_3^0 completeness of $\equiv_m$ as a set of pairs of c.e. indices, which follows for instance because the m-complete c.e. set have a Σ_3^0 complete index set. As a further consequence, Turing equivalence on indices of c.e. sets is a Σ_3^0 hard equivalence relation for computable reducibility. However, this equivalence relation is only Σ_4^0. Ianovski et al. [34] have shown that in fact it is Σ_4^0 complete in our sense.

Proof of theorem. Since S is Σ_3^0, there is a uniformly c.e. triple sequence

$$(V_{y,z,i})_{y,z,i\in\omega,y<z}$$

of initial segments of $\mathbb{N}$ such that for each $y < z$,

$$ySz \Leftrightarrow \exists i\, V_{y,z,i} = \omega.$$

We build a uniformly c.e. sequence of sets $A_x = W_{g(x)}$ ($x \in \omega$), g computable. We meet the following coding requirements for all $y < z$ and $i \in \omega$.

$G_{y,z,i}$: $V_{y,z,i} = \omega \Rightarrow A_y \equiv_1 A_z$.

We meet diagonalization requirements for $u \neq v$,

$N_{u,v,e}$: $u = \min[u]_S \wedge v = \min[v]_S \Rightarrow A_u \neq \Phi_e(A_v)$.

where Φ_e is the e-th Turing functional, and $[x]_S$ denotes the S-equivalence class of x. Meeting these requirements suffices to establish the theorem.

The basic strategies to meet the requirements are as follows. If $V_{y,z,i} = \omega$, a strategy for $G_{y,z,i}$ "finds out" that z is S-related to the smaller y. Hence it builds a computable permutation h such that $A_y \equiv_1 A_z$ via h.

A strategy for $N_{u,v,e}$ picks a witness n, and waits for $\Phi_e(A_v; n)$ to converge. Thereafter, it ensures that this computation is stable and $A_u(n)$ does not equal its output $\Phi_e(A_v; n)$ by enumerating n into A_u if this output is 0.

The tree of strategies. To avoid conflicts between strategies that enumerate into the same set A_z, we need to provide the strategies with a guess at whether z is least in its S-equivalence class $[z]_S$. An N-type strategy will only enumerates into A_z if according to its guess, z is least in its $[z]_S$; a G-type strategy only enumerates into A_z if according to its guess, z is not least.

Fix an effective priority ordering of all requirements. We define a tree T of strategies, which is a computable subtree T of $2^{<\omega}$. We write $\alpha : R$ if strategy α is associated with the requirement R. By recursion on $|\alpha|$, we define whether $\alpha \in T$, and which is the requirement associated with α. We also define a function L mapping $\alpha \in T$ to a cofinite set $L(\alpha)$ consisting of the numbers x such that according to α's guesses, x is least in its equivalence class.

Let $L(\emptyset) = \omega$. Assign to α the highest priority requirement R not yet assigned to a proper prefix of α such that either (a) or (b) hold.

(a) R is $G_{y,z,i}$ and $z \in L(\alpha)$; in this case put both $\alpha 0$ and $\alpha 1$ on T, and define $L(\alpha 0) = L(\alpha) - \{z\}$ while $L(\alpha 1) = L(\alpha)$ (along $\alpha 0$ we know that x is no longer the least in its equivalence class)

(b) R is $N_{u,v,e}$ and $u, v \in L(\alpha)$; in this case put only $\alpha 0$ on T, and define $L(\alpha 0) = L(\alpha)$.

For strings $\alpha, \beta \in 2^{<\omega}$, we write $\alpha <_L \beta$ if there is i such that $\alpha\restriction_i = \beta\restriction_i$, $\alpha(i) = 0$ and $\beta(i) = 1$. We let $\alpha \preceq \beta$ denote that α is a prefix of β. We define a linear ordering on strings by

$$\alpha \leq \beta \text{ if } \alpha <_L \beta \text{ or } \alpha \preceq \beta.$$

Construction of a u.c.e. sequence of sets $(A_x)_{x\in\mathbb{N}}$. We declare in advance that $A_x(4m+1) = 0$ and $A_x(4m+3) = 1$ for each x, m. The construction then only determines membership of even numbers in the A_x.

We define a computable sequence $(\delta_s)_{s\in\mathbb{N}}$ of strings on T of length s. Suppose inductively that δ_t has been defined for $t < s$. Suppose $k < s$ and that $\eta = \delta_s\restriction_k$ has been defined. If $\eta : N_{u,v,e}$ let $\delta_s(k) = 0$. Otherwise $\eta : G_{y,z,i}$. Let $t < s$ be the largest stage such that $t = 0$ or $\eta \preceq \delta_t$. Let $\delta_s(k) = 0$ if $V_{y,z,i,s} \neq V_{y,z,i,t}$, and otherwise $\delta_s(k) = 1$.

The *true path* TP is the lexicographically leftmost path $f \in 2^\omega$ such that $\forall n\, \exists^\infty s \geq n\, [\delta_s \upharpoonright_n \prec f]$. To *initialize* a strategy α means to return it to its first instruction. If $\alpha : G_{y,z,i}$ we also make the partial computable function h_α built by the strategy α undefined on all inputs. At stage s, let $\mathrm{init}(\alpha, s)$ denote the largest stage $\leq s$ at which α was initialized.

An $N_{u,v,e}$ strategy α. At stages s:

(a) Appoint an unused even number $n > \mathrm{init}(\alpha, s)$ as a witness for diagonalization. Initialize all the strategies $\beta \succ \alpha$.
(b) Wait for $\Phi_e(A_v; n)[s]$ to converge with output r. If $r = 0$ then put n into A_u. Initialize all the strategies $\beta \succ \alpha$.

A $G_{y,z,i}$ strategy α. If $\alpha 0$ is on the true path then this strategy builds a computable increasing map h_α from even numbers to even numbers such that $A_y(k) = A_z(h_\alpha(k))$ for each k. Furthermore, $A_z - \mathrm{range}(h_\alpha)$ is computable. By our definitions of A_y and A_z on the odd numbers, this implies that h_α can be extended to a computable permutation showing that $A_y \equiv_1 A_z$, as required.

At stages s, if $\alpha 0 \subseteq \delta_s$, let $t < s$ be greatest such that $t = 0$ or $\alpha 0 \subseteq \delta_t$, and do the following.

(a) For each even $k < s$ such that $k \notin \mathrm{dom}(h_{\alpha,t})$ pick an unused even value $m = h_{\alpha,s}(k) > \mathrm{init}(\alpha, s)$ in such a way that h_α remains increasing.
(b) From now on, unless α is initialized, ensure that $A_z(m) = A_y(k)$. (We will verify that this is possible.)

The stage-by-stage construction is as follows. At stage $s > 0$ initialize all strategies $\alpha >_L \delta_s$. Go through substages $i \leq s$. Let $\alpha = \delta_s \upharpoonright_i$. Carry out the strategy α at stage s.

Verification. To show the requirements are met, we first check that there is no conflict between different strategies that enumerate into the same set A_z.

Claim 13.4. *Let $\alpha\colon G_{y,z,i}$. Then (b) in the strategy for α can be maintained as long as α is not initialized.*

To prove the claim, suppose a strategy $\beta \neq \alpha$ also enumerates numbers into A_z. If $\alpha 0 <_L \beta$ then β is initialized when α extends its map h_α, so the numbers enumerated by β are not in the range of h_α. If $\beta <_L \alpha 0$ then α is initialized when β is active, so again the numbers enumerated by β are not in the range of h_α. Now suppose neither hypothesis holds, so $\alpha 0 \preceq \beta$ or $\beta \prec \alpha$.

Case $\beta\colon N_{z,v,e}$. In this case $\alpha 0 \preceq \beta$ is not possible because $z \notin L(\alpha 0)$. If $\beta \prec \alpha$ then α is initialized when β appoints a new diagonalization witness.
Case $\beta\colon G_{y',z,i'}$. In this case $\alpha 0 \preceq \beta$ is not possible because $z \notin L(\alpha 0)$. If $\beta 1 \preceq \alpha$ then α is initialized each time β extends its map h_β. Finally, $\beta 0 \preceq \alpha$ is not possible because $z \notin L(\beta 0)$. This proves the claim.

Claim 13.5. *Let α be the $N_{u,v,e}$ strategy on the true path. Suppose α is not initialized after stage s. Then α only acts finitely often, and meets its requirement.*

At some stage $\geq \mathrm{init}(\alpha, s)$ the strategy α picks a permanent witness n. No strategy $\beta \prec \alpha$ can put n into A_u because $u \in L(\alpha)$. No other strategy can put n into A_u because of the initialization α carries out when it picks n. Suppose now that at a later stage t, a computation $\Phi_e(A_v; n)[t]$ converges. Since $v \in L(\alpha)$, no G-type strategy $\beta \prec \alpha$ enumerates into A_v. Thus the initialization of strategies $\gamma \succ \alpha$ carried out by α at that stage t will ensure that this computation is preserved with value different from $A_u(n)$. This proves the claim.

It is now clear by induction that each strategy α on the true path is initialized only finitely often. Thus the N-type requirements are met. Now suppose $\alpha\colon G_{y,z,i}$ and $\alpha 0$ is on the true path. Then no strategy $\beta \succeq \alpha 0$ enumerates into A_z. Thus by the initialization at stages s such that $\alpha 0 \preceq \delta_s$, the set $A_z - \mathrm{range}(h_\alpha)$ is computable. As noted earlier, this implies that h_α can be extended to a computable permutation showing that $A_y \equiv_1 A_z$. There is a computable bijection q between the set of odd numbers and the set of numbers that are odd, or even but not in the range of h_α, so that $m \in A_y \leftrightarrow q(m) \in A_z$. Now let the permutation be $q \cup h_\alpha$.

□

13.3. **Computable isomorphism of trees.** We use the terminology of Fokina et al. [22]. Thus, a tree is a structure in the language containing the predecessor function as a single unary function symbol. The root is its own predecessor. A countable tree can be represented given by a nonempty subset B of $\omega^{<\omega}$ closed under prefixes, where the predecessor function takes of the last entry of a non-empty tuple of natural numbers.

A tree has a computable presentation iff we can choose B r.e. For in that case B is the range of a partial computable 1-1 function ϕ with domain an initial segment of ω; the preimage of the predecessor function under ϕ is the required computable atomic diagram.

We let $B_e = \{\sigma\colon \exists \tau \succeq \sigma\, [\tau \in W_e]\}$, where the e-th r.e. set W_e is viewed as a subset of $\omega^{<\omega}$. Then $(B_e)_{e\in\mathbb{N}}$ is a uniform listing of all computable trees.

We say a tree has height k if every leaf has length at most k.

Corollary 13.6. *Computable isomorphism of computable trees of height 2 where every node at level 1 has out-degree at most 1 is m-complete for Σ^0_3 equivalence relations.*

Proof. Let h be a computable function such for each e, $B_{h(e)}$ is the tree

$$\emptyset \cup \{\langle x\rangle\colon\ x \in \omega\} \cup \{\langle x, 0\rangle\colon\ x \in W_e\}.$$

Clearly, $W_y \equiv_1 W_z$ iff $B_{h(y)}$ is computably isomorphic to $B_{h(z)}$. Now we apply Theorem 13.2. □

For background on computable metric spaces, see [9]. A computable metric space is *discrete* if every point is isolated. For such a space, necessarily every point is an ideal point.

Corollary 13.7. *Computable isometry of discrete computable metric spaces is m-complete for Σ^0_3 equivalence relations.*

Proof. Given a computable tree B, create a discrete computable metric M_B space as follows: if a string $\langle x\rangle$ enters B, add a point p_x. If later $\langle x, i\rangle$ enters

B for the first i, add a further point q_x. Declare $d(p_x, q_x) = 1/4$. Declare $d(p_x, p_y) = 1$ and $d(q_x, p_y) = 1$ (if q_x exists). Clearly for trees B, C as in Cor. 13.6, B is computably isomorphic to C iff M_B is computably isometric to M_C. □

Corollary 13.8. *Computable isomorphism of recursive equivalence relations where every class has at most 2 members is m-complete for Σ^0_3 equivalence relations.*

Proof. Given r.e. set A, build a computable equivalence relation R_A such that

$$A \equiv_1 B \text{ iff } R_A \equiv_{comp} R_B.$$

We may assume at most one element enters A at each stage, and only at even stages.

Let $R_A = \{\langle 2a, 2t+1\rangle : a \text{ enters } A \text{ at stage } t\}$. □

13.4. **Boolean algebras.** For a linear order L with least element, *Intalg* L denotes the subalgebra of the Boolean algebra $\mathcal{P}(L)$ generated by intervals $[a, b)$ of L where $a \in L$ and $b \in L \cup \{\infty\}$. Here ∞ is a new element greater than any element of L, and $[a, \infty)$ is short for $\{x \in L : x \geq a\}$. Note that *Intalg* L consists of all sets S of the form

$$S = \bigcup_{r=1}^{n} [a_r, b_r)$$

where $a_0 < b_0 < a_1 \ldots < b_n \leq \infty$. From a computable presentation of L as a as a linear order, we may canonically obtain a computable presentation of the Boolean algebra *Intalg* L.

Theorem 13.9. *Computable isomorphism of computable Boolean algebras is complete for Σ^0_3 equivalence relations.*

Proof. Let $(V^e)_{e\in\mathbb{N}}$ be an effective listing of the c.e. sets containing the even numbers. The relation of 1-equivalence $\equiv_1$ of c.e. sets V^e is Σ^0_3 complete by Theorem 13.2 and its proof below. We will computably reduce it to computable isomorphism of computable Boolean algebras. We define the Boolean algebra C^e to be the interval algebra of a computable linear order L^e. Informally, to define L^e, we begin with the order type ω. For each $x \in \omega$, when x enters V^k we replace x by a computable copy of $[0, 1)_{\mathbb{Q}}$. More formally,

$$L^e = \bigoplus_{x\in\omega} M^e_x,$$

where M^e_x has one element $m^k_x = 2x$, until x enters V^e; if and when that happens, we expand M^e_x to a computable copy of $[0, 1)_{\mathbb{Q}}$, using the odd numbers, while ensuring that $m^k_x = \min M^k_x$ holds in L^k. Also note that the domain of L^k is $\mathbb{N}$ because $0 \in V^k$.

Claim 13.10. $V^e \equiv_1 V^i \Leftrightarrow C^e \cong_{comp} C^i$.

$\Rightarrow$: Suppose $V^e \equiv_1 V^i$ via a computable permutation π. We define a computable isomorphism $\Phi : C^e \cong C^i$.

(a) Let $\Phi(m^e_x) = m^i_{\pi(x)}$. Once x enters V^e, we know that $\pi(x) \in V^i$. So we may always ensure that Φ restricts to a computable isomorphism of linear orders $M^e_x \cong M^i_{\pi(x)}$.
(b) Consider an element S of C^e. It is given in the form $S = \bigcup_{r=1}^n [a_r, b_r)$ where $a_0 < b_0 < a_1 \ldots < b_n$ for $a_r, b_r \in L^e \cup \{\infty\}$ as above. If $b_n < \infty$, we can compute the maximal $x \in \omega$ such that $M^e_x \cap S \neq \emptyset$. Define

$$\Phi(S) = \bigcup_{y \le x} \Phi(S \cap M^e_y).$$

Note that the set $\Phi(S \cap M^e_y)$ can be determined by (a).

If $b_n = \infty$, then let $\Phi(S)$ be the complement in L^i of $\Phi(L^e \setminus S)$.

$\Leftarrow$: Now suppose that $C^e \cong_{comp} C^i$ via some computable isomorphism Φ. We show that $V^e \le_1 V^i$ via some computable function f. Suppose we have defined $f(y)$ for $y < x$. We have $\Phi(M^e_x) = \bigcup_{r=1}^n [a_r, b_r)$ where $a_r, b_r \in L^i \cup \{\infty\}$ as above.

If $n > 1$ then M^e_x is not an atom in C^e, whence $x \in V^e$. Thus let $f(x)$ be the least even number that does not equal $f(y)$ for any $y < x$.

Now suppose $n = 1$. If $a_1 = m^i_y, b_1 = m^i_{y+1}$ then let $f(x) = y$. Otherwise, again we know M^e_x is not an atom in C^e, and define $f(x)$ as before.

By symmetry, we also have $V^i \le_1 V^e$, and hence $V^i \equiv_1 V^e$ by Myhill's theorem. $\square$

Now the reader might be ready to conclude that for every reasonably rich class of structures the computable isomorphism problem is Σ^0_3 complete for eqrels. But this is not so. For instance, consider the class of computable permutations of order 2 (this class itself is Π^0_2). Then the computable isomorphism relation on this class is Π^0_2. (And this is the same as the classical isomorphism relation.) This is so because we only need to figure out whether for two given permutations, both have the same number of 1 cycles, and the same number of 2 cycles.

13.5. **Almost inclusion of r.e. sets is a Σ^0_3-complete preordering.** For $X, Y \subseteq \omega$, we write $X \subseteq^* Y$ if $X \setminus Y$ is finite. We write $X =^* Y$ if $X \subseteq^* Y \subseteq^* X$. Let W_e denote the e-th r.e. set.

Theorem 13.11. *$\{\langle e, i\rangle \colon W_e \subseteq^* W_i\}$ is m-complete for Σ^0_3 preorderings.*

Proof. All sets in this proof will be r.e. Fix a non-recursive set A. By $X \sqsubseteq A$ we denote that X is a split of A, i.e., $A \setminus X$ is r.e. Let X, Y range over splits of A.

Since A is non-recursive, there is a small major subset $D \subset_{sm} A$ (see [60, pg. 194]). Then, for each $X \sqsubseteq A$, we have

$$X \subseteq^* D \leftrightarrow X \text{ is recursive}$$

(see [54, Lemma 4.1.2]). Consider the Boolean algebra

$$\mathcal{B}_D(A) = \{(X \cup D)^* \colon X \sqsubseteq A\},$$

which has a canonical Σ^0_3 presentation in the sense of [49, Section 2]. The Friedberg splitting theorem implies that every nonrecursive set can be split into two nonrecursive sets obtained uniformlhy in an r.e. index for the given set. Iterating this, we obtain a uniformly r.e. sequence of splittings $X_n \sqsubset A$ (given by r.e. indices for both the set and its complement in A) such that the sequence $(p_n)_{n\in\mathbb{N}}$ freely generates a subalgebra $\mathcal{F}$ of $\mathcal{B}_D(A)$, where $p_n = (X_n \cup D)^*$.

Now suppose that $\preceq$ is an arbitrary Σ^0_3 preordering. Let $\mathcal{I}_0$ be the ideal of $\mathcal{F}$ generated by $\{p_n - p_k \colon n \preceq k\}$. We claim that

$$n \preceq k \leftrightarrow p_n - p_k \in \mathcal{I}_0.$$

The implication "$\rightarrow$" is clear by definition. For the implication "$\leftarrow$, let $\mathcal{B}_{\preceq}$ be the Boolean algebra generated by the subsets of ω of the form $\hat{i} = \{r \colon r \preceq i\}$. The map $p_i \mapsto \hat{i}$ extends to a Boolean algebra homomorphism $g \colon \mathcal{F} \to \mathcal{B}_{\preceq}$ that sends $\mathcal{I}_0$ to 0. If $n \not\preceq k$ then $\hat{n} \not\subseteq \hat{k}$, and hence $p_n - p_k \notin \mathcal{I}_0$. This proves the claim.

Now let $\mathcal{I}$ be the ideal of $\mathcal{B}_D(A)$ generated by $\mathcal{I}_0$. Clearly $\mathcal{I}_0 = \mathcal{I} \cap \mathcal{F}$. Since $p_n = (X_n \cup D)^*$, the claim now implies that

$$n \preceq k \leftrightarrow ((X_n - X_k) \cup D)^* \in \mathcal{I}.$$

Note that $\mathcal{I}$ is a Σ^0_3 ideal. By the basic Σ^0_3 case of the ideal definability lemma in [30] (a simpler proof of this case was given in [49, Lemma 3.2]) there is $B \in [D, A]$ such that

$$(Y \cup D)^* \in \mathcal{I} \leftrightarrow Y \subseteq^* B$$

for each $Y \sqsubseteq A$. Thus

$$n \preceq k \leftrightarrow ((X_n \setminus X_k) \cup D)^* \in \mathcal{I} \leftrightarrow X_n \subseteq^* X_k \cup B.$$

Since the sequence of splittings $(X_n)_{n\in\mathbb{N}}$ is uniform, this yields the desired m-reduction. $\square$

Each equivalence relation is a preorder. Thus, as an immediate consequence, we obtain:

Corollary 13.12. $\{\langle e, i\rangle \colon W_e =^* W_i\}$ *is m-complete for Σ^0_3 equivalence relations.*

For a natural complete Σ^0_3 preordering, one can consider embeddability of subgroups of $(\mathbb{Q}, +)$.

Corollary 13.13. *Computable embeddability among computable subgroups of $(\mathbb{Q}, +)$ is m-complete for Σ^0_3 preorderings.*

Proof. We will represent a computable group by a 4-tuple of computable functions, $(e, \oplus, \ominus, I)$, where $e(x) = 1$ for the identity element and 0 elsewhere, $\oplus$ is the binary group operation, $\ominus$ a unary function taking an element to its inverse and I is the "interpretation" function which maps natural numbers to $\mathbb{Q}$. Thereby, $\oplus(x, y) = z$ iff $I(x)+I(y) = I(z)$. We do not require that I be one to one.

Let p_n be the n-th prime. For convenience, treat p_0 as 1. Code the x-th r.e. set W_x into G_x: the group generated by 1 and all $1/p_n$, $n \in W_x$. For a computable presentation, let x_r denote the last element to enter $W_{x,r}$. Use the odd numbers to encode all finite sequences of integers, non-zero even numbers of the form $2r$ to encode $1/p_{x_r}$ and 0 to encode 0. This immediately defines the behaviour of I on even numbers, so to account for the odd map the sequence $\langle m_1, m_2, \dots, m_k\rangle$ to $m_1/p_{x_1} + m_2/p_{x_2} \cdots + m_k/p_{x_k}$. Because I so defined is totally computable, it implies computable $e, \oplus, \ominus$ via I^{-1}. For instance, to compute $\oplus(x, y)$ first compute $I(x)$ and $I(y)$, and then find the least element of $I^{-1}(I(x) + I(y))$. In this vein for the rest of this proof we will abuse notation slightly and interpret $I^{-1}(x)$ as the least element of $I^{-1}(x)$.

Now suppose W_x is almost contained in W_y. We wish to show that G_x is embeddable in G_y. Let P be the product of the finitely many primes p_n with $n \in W_x \setminus W_y$, I the interpretation function of G_x and J the interpretation function of G_y.

The desired embedding of G_x in G_y is given by $a \mapsto J^{-1}(PI(b))$. Observe that since the group operations modulo I or J respectively correspond to addition of the rational numbers, it follows that this mapping is one to one as $PI(a) = PI(b)$ if and only if $I(a) = I(b)$, and it preserves the group operation as $PI(a) + PI(b) = PI(\oplus(a, b))$.

On the other hand, suppose there are infinitely many elements in W_x that are not in W_y. Note that we can define a notion of divisibility in a group in the usual way: $a|b$ iff $I(b) = cI(a)$. Any embedding of G_x into G_y must clearly preserve divisibility modulo the interpretation. That is, where f is the embedding if $I(b) = cI(a)$ then $J(f(b)) = cJ(f(a))$. Observe that since $1 = p_i/p_i$ for any i, for every $i \in W_x$, $2i|I^{-1}(1)$. We will show that this implies that no embedding is possible.

Let $i \in W_x \setminus W_y$. Let $f(I^{-1}(1)) = J^{-1}(a/b)$. As $a/b = p_i I(f(1/p_i))$, and p_i cannot appear in the denominator of any element in G_y, p_i must appear in a. However, as there are infinitely many such i, all of them coprime, no finite nominator can satisfy this requirement. $\square$

13.6. **Computable isomorphism versus classical non-isomorphism.** Note that for any two infinite and co-infinite r.e. sets W_y and W_z, the tree $B_{h(y)}$ in Cor. 13.6 is classically, but not computably, isomorphic to $B_{h(z)}$. We now strengthen the result for trees: non-equivalence even turns into classical non-isomorphism.

Theorem? 13.1. *For each Σ^0_3 equivalence relation S, there is a computable function g such that*

$$xSy \Rightarrow B_{g(x)} \cong_c B_{g(y)},$$

and

$$\neg xSy \Rightarrow B_{g(x)} \ncong B_{g(y)}$$

Proof. (Sketch) As before we meet coding requirements for $y < z$,

$G_{y,z,i}$: $V_{y,z,i} = \omega \Rightarrow T_y \cong_c T_z$.

If $\neg uSv$ we ensure that T_u is not isomorphic to T_v. To do so, if $u = \min[u]_S$ then T_u will have exactly u infinite paths. They are denoted

$$f^u_0, \dots, f^u_{u-1}.$$

We meet the requirements

$N_{u,e}$: $u = \min[u]_S \Rightarrow f^u_0\restriction_{e+1}, \dots, f^u_{u-1}\restriction_{e+1}$ are defined.

As before there are strategies $\alpha\colon N_{u,e}$. At each stage s we have approximations $f^u_{i,\alpha,s}$ of f^u_i of length $e+1$, where $f^u_{i,\alpha,s} \prec f^u_{i,\beta,s}$ for strategies $\alpha \prec \beta$. We let $f^u_{i,\emptyset,s} = \langle 2i \rangle$ for each s, which makes the paths distinct.

Fix an effective priority ordering of all requirements with $N_{u,e} < N_{u,e+1}$. We define a tree T of strategies and L as before, with the only difference that the former requirements $N_{u,v,e}$ now become $N_{u,e}$.

To *initialize* a strategy α means to return it to its first instruction. If $\alpha : G_{y,z,i}$ we also make the partial computable function h_α built by the strategy α undefined on all inputs. If $\alpha : N_{u,e}$ we make all current $f^u_{i,\alpha}$ undefined. As before, at stage s, let $\mathrm{init}(\alpha, s)$ denote the largest stage $\leq s$ at which α was initialized.

Strategy $\alpha\colon N_{u,e}$ at stage s.

If $e > 0$ let $\gamma \prec \alpha$ be the $N_{u,e-1}$ strategy, otherwise $\gamma = \emptyset$. If $f^u_{i,\alpha,s-1}$ is undefined, pick a large even number n so that for each $i < u$, $f^u_{i,\alpha,s} := f^u_{i,\gamma,s}\hat{\ } n > \mathrm{init}(\alpha, s)$. Initialize all the G-type strategies $\beta \succ \alpha$.

A $G_{y,z,i}$ *strategy* α *at stages* s. If $\alpha 0$ is on the true path then this strategy builds a computable 1-1 map h_α from all even strings in T_y to even strings in T_z preserving the length and the prefix relation.

If $\alpha 0 \subseteq \delta_s$, let $t < s$ be greatest such that $t = 0$ or $\alpha 0 \subseteq \delta_t$, and do the following.

Declare each $\eta \in T_{z,s-1}$, $\mathrm{init}(\alpha, s) < \eta$ unextendable (leaf). For each even string $\eta < s$ let k be largest such that $\eta\restriction_k \in \mathrm{dom}(h_{\alpha,t})$. If $k < |\eta|$, pick a fresh even extension $\sigma \succeq h_{\alpha,t}(\eta\restriction_k)$, where $\sigma > \mathrm{init}(\alpha, s)$ and $|\sigma| = |\eta|$, and let $h_\alpha(\eta\restriction_j) = \sigma\restriction_j$ for each j with $k < j \leq |\eta|$.

The stage-by-stage construction follows the same scheme as before. In particular, at stage $s > 0$ initialize all strategies $\alpha >_L \delta_s$. $\square$

Part 5. Others

14. Bernstein v.s Vitali

Input by Yu.

This is a result for fun concerning the question in:

http://mathoverflow.net/questions/71575/vitali-sets-vs-bernstein-sets

I need to modify the definition of Vitali set to apply recursion theory.

We call a set $V \subset 2^\omega$ to be *Vitali* if for any Turing degree $\mathbf{x}$, there is a unique real $x \in \mathbf{x} \cap V$.

We call a set $V \subset 2^\omega$ to be *Bernstein* if neither V nor $2^\omega \setminus V$ contains a perfect subset.

Both Vitali and Bernstein sets are used construct nonmeasurable sets as in classical analysis books.

Now an interesting question is which way is stronger? More precisely, over ZF, does the existence either one implies the existence of another one?

Theorem 14.1. *There is a model $\mathcal{M}$ of ZF in which there is a Bernstein set but no Vitali set.*

In [64], a model $\mathcal{M}$ of ZF was constructed so that there is a cofinal chain of Turing degrees of order type ω_1 but there is no well ordering of reals. We claim that the $\mathcal{M}$ is exactly what we want.

Fix a cofinal chain $\{\mathbf{x}_\alpha\}_{\alpha<\omega_1}$ of Turing degrees order type of ω_1.

Lemma 14.2. *There is no a Vitali set in $\mathcal{M}$.*

Proof. Assume otherwise. Then we may pick a unique real from each Turing degree $\mathbf{x}_\alpha$. Since $\{\mathbf{x}_\alpha\}_{\alpha<\omega_1}$ is cofinal, we have a well ordering of reals in $\mathcal{M}$ which is a contradiction. □

We construct a Bernstein set in $\mathcal{M}$. We define a set $B = \bigcup_{\alpha<\omega_1} B_\alpha$.

We shall let B_α only contain reals which Turing below $\mathbf{x}_\alpha$ for limit ordinal α.

Let $B_0 = \emptyset$.

At limit stage α, just let $B_\alpha = \bigcup_{\gamma<\alpha} B_\gamma$.

At stage $\alpha+1$ but α is not limit, just let $B_{\alpha+1} = B_\alpha$.

At stage $\alpha+1$ but α is limit. Let $\mathcal{A}_\alpha$ be the collection of perfect trees Turing below $\mathbf{x}_\alpha$. Let

$$B_{\alpha+1} = B_\alpha \cup \{z \mid \exists T \in \mathcal{A}_\alpha (z \in T \wedge z \not\leq_T \mathbf{x}_\alpha \wedge z \leq_T \mathbf{x}_{\alpha+1})\}.$$

Note that for each $T \in \mathcal{A}_\alpha$, there is a real $x \in [T]$ so that $x \oplus T \equiv_T \mathbf{x}_{\alpha+1}$. So $B_{\alpha+1} \cap [T]$ is not empty for each $T \in \mathcal{A}_\alpha$.

This finishes the construction.

Since $\{\mathbf{x}_\alpha\}_{\alpha<\omega_1}$ is cofinal, by the construction, $B \cap [T]$ is not empty for any perfect tree T.

Moreover, for any recursive tree T, let α be the least limit ordinal so that $T \leq_T \mathbf{x}_\alpha$. There must be some $z \in [T]$ so that $z \oplus T \equiv_T \mathbf{x}_{\alpha+2}$. Then for such a real z, $z \notin B$.

So B is a Bernstein set.

The following question seems unknown.

Question 14.3. *Is there a model $\mathcal{M}$ of ZF in which there is a Vitali set but no Bernstein set?*

15. Random linear orders and equivalence relations

Fouché and Nies discussed at the Wollic 2012 meeting in Buenos Aires. They studied how to define randomness for infinite objects other than sets of natural numbers. They didn't want to use any general background such as computable probability spaces (e.g., [27], [33]), but rather restrict attention to a simple kind of relational structures with domain $\mathbb{N}$, and define randomness for them in a direct way.

S_∞ denotes the group of permutations of $\mathbb{N}$. Fix a Borel class $\mathcal{C} \subseteq \mathcal{P}(\mathbb{N}^k)$ closed under permutations $p \in S_\infty$, such as the linear orders, or the equivalence relations. We say that a measure μ on $\mathcal{C}$ is *invariant* if for each measurable $\mathcal{D} \subseteq \mathcal{C}$,

$$\forall p \in S_\infty \, [\mu(p(\mathcal{D})) = \mu(\mathcal{D})].$$

15.1. **Linear orders on $\mathbb{N}$.** Let LO denote the class of reflexive linear orders on $\mathbb{N}$. Using methods of topological dynamics, Glasner and Weiss [28] showed that there is a unique invariant probability measure on LO. The uniqueness is the important part, as it shows that this measure is canonical.

Fouché [] showed that this "Glasner-Weiss" measure μ_{GW} is computable. For any distinct $a_0, \ldots, a_{n-1} \in \mathbb{N}$ let

$$[a_0, \ldots, a_{n-1}] = \{L \in \mathsf{LO} \colon \; a_0 <_L \ldots <_L a_{n-1}\}.$$

Clearly, for any invariant measure μ on LO, we must have

$$\mu[a_0, \ldots, a_{n-1}] = 1/n!.$$

Another approach to obtain this measure is as follows. There is a natural correspondence between LO and the class of functions $[T]$, where T is the tree

$$\{f \in \mathbb{N}^\mathbb{N} \colon \; \forall i \, f(i) \le i\}.$$

Namely, $f(i) = k$ means: If $k = 0$ put i as a new least, and if $k = i$ put i as a new greatest element. Otherwise we put i between the $k-1$–th and the k–th element of the linear order we already have on $\{0, \ldots, i-1\}$.

Now let μ be the product measure $\Pi_i \mu_i$ where μ_i is the uniform probability measure on $\{0, \ldots, i\}$.

We can now export the known randomness notions to $[T]$ and thereby to LO. For instance, every Kurtz-random linear order is of type $\mathbb{Q}$. The following analog of the Levin-Schnorr Theorem yields a characterization of ML-random linear orders via the prefix-free initial segment complexity.

Proposition 15.1. *Let $f \in [T]$. Then f is ML-random* $\Longleftrightarrow$

$$\exists b \forall n \, K(f \restriction_n) \ge \log_2(n!) - b.$$

Proof. Adapt the usual proof of Levin-Schnorr (for instance [51, 3.2.9]). Thus, let

$$\mathcal{U}_b = \{f \in [T] \colon \exists n \, K(f \restriction_n) < \log_2(n!) - b\}$$

and show that $\langle \mathcal{U}_b \rangle_{b \in \mathbb{N}}$ is a universal Martin-Löf test. ∎

Melnikov et al. [46] defined K-triviality for functions in Baire space. This now yields a notion of K-trivial linear orders. Lower c.e. functions are the generalizations of c.e. sets (what does this mean for linear orders?). We now have everything together and could study whether in the setting of LO, anything new happens in comparison with the interactions of lowness and randomness on the subsets of $\mathbb{N}$.

Note that correspondence between $[T]$ and LO is useful to get the definitions right, but it does not cohere with the permutation invariance of LO. This suggests that new things will happen not necessarily for $[T]$, but for LO.

15.2. **Equivalence relations on $\mathbb{N}$.** Here there is an invariant probability measures, though it is not unique. To define it, pick a probability computable measure γ on $\mathbb{N}$, such as the one given by

$$\gamma(\{i\}) = \frac{6}{\pi^2} i^{-2}.$$

Let μ^* be its power $\gamma^{\mathbb{N}}$ which is a computable measure on Baire space $\mathbb{N}^{\mathbb{N}}$. Mapping a function f to its kernel yields an onto map Φ from $\mathbb{N}^{\mathbb{N}}$ to the set EQ of equivalence relations on $\mathbb{N}$. Let μ on EQ be the image measure of μ^*.

Since Φ respects permutations, clearly μ is invariant. Another γ would yield a different invariant measure.

We note that if $\gamma(\{i\}) \neq 0$ for each i, then a.e. $[\mu]$ equivalence relation E is of type (∞, ∞), that is, has infinitely many classes that are all infinite. This is so because for any fixed n, k, the class of functions f with $f^{-1}(n) \subseteq \{0, \ldots, k-1\}$ has measure 0.

15.3. **Other examples.** Bakh Khoussainov has a draft (8 pages, late 2012) on random algebras in a finite signature. It's not clear whether there is an invariant measure leading to his definition.

One could also consider random Polish spaces, where the structure is given by a countable dense set with distance relations $R_{<q}xy$ denoting that $d(x, y) < q$ (where $q \in \mathbb{Q}^+$).

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