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Novel $\chi^{(3)}$ Phenomena in Optical Fibers and Fiber Cavities

Yiqing Xu

A thesis submitted in fulfilment of the requirements for the degree of Doctor of Philosophy in Physics

The University of Auckland
July 2013
This thesis presents a series of theoretical and experimental studies of $\chi^{(3)}$ nonlinear phenomena in optical fibers and optical fiber cavities.

In the study of four-wave mixing, we derive a simple theory which is capable of accurately describing the mean gain and statistics of a fiber optical parametric amplifier pumped by a temporally incoherent light source, and experimentally verify it.

Another study of cascaded four-wave mixing shows that a cascade of $n$ $\chi^{(3)}$ processes can mimic an equivalent $\chi^{(2n+1)}$ process. Direct amplification of high order sidebands is experimentally observed, and we use this observation to explain the nonlinear mechanism behind the dispersive wave emission by a single soliton, and the new frequency components generated in a soliton-linear wave interaction.

In the study of fiber cavities, a fiber optical parametric oscillator with a multi-watt level output power and wide tunability over 400 nm around telecommunication band is demonstrated. By introducing an intracavity filtering, a high conversion efficiency fiber parametric oscillator with a maximum internal conversion efficiency of 93% is also demonstrated.

Finally, the universal physical phenomenon of symmetry breaking is studied in a passive nonlinear fiber ring cavity which is synchronously driven by a pulsed pump. We provide strong experimental evidence for the existence of symmetry breaking in this fiber cavity by observing the spectral power asymmetry, and the temporal shift of the intracavity pulse.
I would like to gratefully thank my supervisor Stuart Murdoch for the guidance and encouragement throughout my postgraduate study, and I know my words are always inadequate for this. I would like to acknowledge John Harvey and Southern Photonics for the financial support of my PhD study.

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Cheers,
Yiqing (Ray) Xu

Auckland, July 2013
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<tr>
<td>ASE</td>
<td>Amplified Spontaneous Emission</td>
</tr>
<tr>
<td>BS</td>
<td>Bragg Scattering</td>
</tr>
<tr>
<td>CS</td>
<td>Cavity Soliton</td>
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<tr>
<td>CW</td>
<td>Continuous Wave</td>
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<tr>
<td>DSF</td>
<td>Dispersion Shifted Fiber</td>
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<tr>
<td>ECL</td>
<td>External Cavity Laser</td>
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<td>EDFA</td>
<td>Erbium Doped Fiber Amplifier</td>
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<td>FOPO</td>
<td>Fiber Optical Parametric Oscillator</td>
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<td>FWM</td>
<td>Four Wave Mixing</td>
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<tr>
<td>GVD</td>
<td>Group Velocity Dispersion</td>
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<td>MI</td>
<td>Modulation Instability</td>
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<td>GNLSE</td>
<td>Generalized Nonlinear Schrödinger Equation</td>
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<td>PCF</td>
<td>Photonic Crystal Fiber</td>
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<tr>
<td>QPM</td>
<td>Quasi Phase Matching</td>
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<td>SPM</td>
<td>Self Phase Modulation</td>
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<td>SRS</td>
<td>Stimulated Raman Scattering</td>
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<td>WDM</td>
<td>Wavelength Division Multiplexer</td>
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<tr>
<td>XPM</td>
<td>Cross Phase Modulation</td>
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<tr>
<td>ZDW</td>
<td>Zero Dispersion Wavelength</td>
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Optical fibers have revolutionized telecommunications and brought humanity into the modern age of optical communications. Optical fiber systems with their unmatched bandwidth and low loss performance have demonstrated their tremendous value in long haul and high volume telecommunication. Moreover, optical fibers have provided an ideal test-bed for the study of nonlinear phenomena. As a modern physics subject, nonlinear optics is the study of the behaviour of light in media with a higher order dielectric polarizability. In this dissertation, the nonlinear media that our studies focus on is the optical fiber. The low loss, small mode field diameter and longitudinal uniformity of modern optical fibers make them an ideal medium for the study of nonlinear optics. Since the linear superposition principle of the electromagnetic field does not hold any longer, one of the most significant features of nonlinear optics is that it allows new frequency components of electromagnetic field to be generated through the nonlinear interaction of optical fields.
1.1 Background

Although the first series of nonlinear optical effects—the Kerr and Pockels effects—were demonstrated as early as the late 19th century [1, 2], they drew little attention as the observation of these phenomena required the application of a strong external electric field. Thus, nonlinear optics remained relatively unexplored for more than half a century. In the year 1960—a milestone year for the optics community, with the advent of laser-access to high intensity spatially coherent light was finally available [3]. Only one year after the demonstration of the first laser, with the aid of its high optical intensity, the first self-induced nonlinear optical effect—second harmonic generation—was experimentally demonstrated by focusing a laser beam into a quartz crystal [4]. In the 1960s, following the observation of second harmonic generation, many other nonlinear optical phenomena were subsequently investigated in $\chi^{(2)}$ bulk materials [5–9]. In the same period, optical fibers were also undergoing rapid development. The first silica glass fiber with a core-cladding structure can be dated back to the 1950s, when it was originally designed for imaging systems [10, 11]. In the late 1960s, the loss of optical fibers was significantly reduced, allowing the possibility of optical fiber communication [12, 13]. In the year 1970, these two isolated fields—nonlinear optics and fiber optics—were finally merged with the first observation of stimulated Raman emission in a liquid CS$_2$ filled fiber waveguide [14]. The others well known nonlinear phenomena such as stimulated Brillouin scattering, stimulated Raman scattering, the Kerr effect, self-phase modulation, and four-wave mixing were all demonstrated quickly one by one in silica fibers [15–19].

Due to the isotropic molecular structure of silica glass, the second order susceptibility $\chi^{(2)}$ of the polarization vanishes, leaving only the third order $\chi^{(3)}$ susceptibility to contribute to the nonlinearity of optical fibers. As such, the nonlinearity of optical fibers is significantly weaker than that of other common $\chi^{(2)}$ nonlinear materials [20, 21]. In spite of the low nonlinearity, one intriguing feature of optical fibers is that they are able to sustain and tightly confine an optical field over a long distance without suffering any significant loss. This is in contrast to the observation of nonlinear effects in traditional
bulk $\chi^{(2)}$ materials which suffer from the issues of beam focusing, spatial overlap, the tensor orientation of nonlinear materials [8, 22]. Single mode optical fibers also have the great advantage of reducing the nonlinear formalism interaction of optical fields to a one-dimensional scalar treatment [23]. This simplification, provided by optical fibers, has further consolidated them as an ideal platform for the investigation of nonlinear effects.

Since the predominant nonlinearity in optical fibers arises from the third order $\chi^{(3)}$ susceptibility, four-wave mixing is one of major objectives of our studies. Modulation instability is one of the most common $\chi^{(3)}$ four-wave mixing effects when a high intensity monochromatic pump beam propagates in the anomalous dispersion regime of a single mode fiber [24, 25]. It is a parametric process which results in the generation of two new frequency photons by annihilating two pump photons through the coupling of $\chi^{(3)}$ susceptibility [26, 27]. It requires conservation of energy and momentum, with the output photons symmetrically detuned from the frequency of the input photons. Quite distinct from the four-wave mixing processes observed in optical fibers in the 1970s [19], the phase matched condition of modulation instability is no longer dependent on the intermodal phase matching of the four waves. As a general physical phenomenon, modulation instability has not only been studied in the context of nonlinear optics [28, 29], but also in other fields such as fluid dynamics and plasma physics [30, 31].

Another area of nonlinear fiber optics which is closely related to modulation instability is the optical soliton. It is also a $\chi^{(3)}$ effect whereby an optical pulse remains unchanged or periodically breathes during propagation in the anomalous dispersion regime of an optical fiber. The optical soliton maintains a perfect balance between the dispersion and nonlinearity. Historically, the soliton was first observed as a solitary water wave propagating undistorted in a canal [32]. In 1973, the existence of solitons in optical fibers was first predicted [33], and later experimentally demonstrated in the early 1980s [34, 35]. Beside the potential application in telecommunications [36], solitons have found various applications in generating widely tunable frequency sources and broadband spectroscopy [37–41].
Chapter 1. Introduction

1.2 Recent Developments

After three decades of development, the theory of nonlinear fiber optics has been well established [42–47], and many of the phenomena have been experimentally verified in a variety of fiber systems. It is now at the stage of putting nonlinear fiber optics into practical uses. In the 1990s, the photonic crystal fiber (PCF) which is made out of pure silica with an air-filled honeycomb-like structure, was invented [48–50]. The flexible design of the PCF’s waveguide structure not only facilitates the control of the fiber dispersion, it also greatly improves the fiber nonlinearity as the light field can be more tightly confined within a smaller effective mode area. The improvement of nonlinearity in PCFs has further allowed supercontinuum generation whereby a pump laser is converted into an ultra broad spectral continuum [51–53]. Practical uses of supercontinuum generated in PCFs have been found in optical coherence tomography to achieve high resolution scanned tissue images [54]. In metrology, the ultrabroad frequency combs from supercontinuum generation can serve as an optical clockwork for the measurement of absolute optical frequencies [55]. Another application of PCFs is known as coherent anti-Stokes Raman scattering microscopy and imaging, in which molecules are identified using the pump and the Stokes sideband generated via modulation instability in PCFs [56–58].

In telecommunications, rare earth doped fiber amplifiers have exhibited their versatility in long haul transmission systems [59]. With the annual increase of the global internet data volume, soon the fiber optical network will reach the current capacity limit [60]. To overcome this limit, one simple approach is to increase the total channel bandwidth. However, due to the finite absorption and emission bandwidth of the atomic transitions of the rare earth elements, the amplification and regeneration of optical data is currently limited to the wavelength region between 1.5 and 1.6 µm. As such, there is an increasing demand for the ability to amplify optical data channels in new frequency bands. Fiber optical parametric amplifiers have been long proposed as a candidate for optical amplification, as they are capable of providing large gain over
a very broad bandwidth with low noise [61–63]. In the late 1990s, the nonlinearity of step-index fibers in the telecommunication band was significantly improved by doping the fiber core with high concentrations of germanium and fluorine [64]. These highly nonlinear fibers further facilitated the development of fiber parametric amplifiers in the telecommunication band, resulting in lower pump power requirements and increased gain bandwidth [65–67]. Currently, the broadest parameter gain amplifier that has been demonstrated has a parametric bandwidth of 155 nm in telecommunication band [68].

In the area of solitons, a new class of optical solitons called temporal cavity soliton has been recently reported in a fiber based system [69]. Unlike the conventional solitons that propagate in single pass fibers, this type of soliton circulates continuously inside a fiber ring cavity driven by an external continuous wave pump. While the fiber dispersion is balanced by the fiber nonlinearity as with the conventional solitons, the cavity loss is also balanced by an external CW pump. Cavity solitons are able to reshape and maintain their structure inside the fiber cavity due to their robust attractive nature, and have been suggested as a future platform for optical storages and optical processing.

1.3 Objectives of Thesis

The objectives of this thesis span a broad overview of nonlinear fiber optics, including the topics of parametric amplification using a temporally incoherent pump, cascaded four-wave mixing in optical fibers, fiber optical parametric oscillators and temporal symmetry breaking in a nonlinear optical fiber cavity. All these processes strongly rely on the interplay between the $\chi^{(3)}$ nonlinearity and the dispersion properties of optical fibers.

In the study of four-wave mixing in optical fibers, we aim to introduce a new model to evaluate the mean gain and gain statistics of parametric amplification pumped by a temporally incoherent pump. We also want to theoretically and experimentally study high order phase matching in four-wave mixing cascades in optical fibers. From this, we attempt to provide an explanation for the nonlinear mechanism to describe the energy
transfer that occurs during dispersive wave emission of a single soliton propagating in a
medium in the presence of higher order dispersion [70]. We then further extend the study
of cascaded four-wave mixing to cascaded Bragg scattering, and use it to explain the
generation of new frequency components in a soliton-linear wave interaction [71,72].
Considering nonlinear optics in fiber cavities, we aim to improve the output power
level and conversion efficiency of fiber optical parametric oscillators. We will also
experimentally study a synchronously pumped passive fiber ring cavity, and observe a
dissipative structure that arises as a result of temporal symmetry breaking instability.

From a more practical point of view, these topics all involve frequency conversion
either using single pass or fiber cavity configurations. A deeper understanding of these
fundamental processes is therefore of primary importance to improve the conversion
efficiency between optical field at different frequencies.

1.4 Outline of Thesis

Beside the introduction chapter, this thesis also contains one theoretical chapter, and
four experimental chapters which will demonstrate a variety of nonlinear effects that
are developed based on the fundamental concepts introduced in the theoretical chapter.
There is also a theoretical section in each of the experimental chapters to explain and
discuss some new concepts that are relevant to the corresponding experiments.

Chapter 2 provides an overview of some of the fundamental concepts of nonlinear
fiber optics that will be used in the experimental chapters that follow. Some basic linear
and nonlinear properties of optical fibers, as well as the origin of these properties, will
be revisited. We also introduce the propagation equations that are used to describe the
evolution of optical fields in optical fibers.

Chapter 3 presents a theoretical and experimental study of a fiber optical parametric
amplifier which is pumped by a temporally incoherent light source. We introduce a
simple theory in order to accurately account for the mean parametric gain and the gain
statistics of an amplified signal of the incoherently pumped parametric amplifier. Nu-
merical simulations will be performed to examine and discuss the validity of this simple theory. Experiments are conducted to verify this simple theory in telecommunication band.

Chapter 4 covers a series of studies of cascaded four-wave mixing in optical fibers. We theoretically and experimentally show that higher order cascaded four-wave mixing sidebands can experience a quasi-phase matched-like amplification when the accumulated phase matched condition of the cascade is satisfied. From this, we establish an important connection between the cascaded four-wave mixing and the dispersive wave emission from a soliton. This study leads us to investigate a reverse process of dispersive wave emission when pumping in the normal dispersion regime. We further study another cascaded parametric process called cascaded Bragg scattering, and the corresponding phase matched condition of this process will be derived and experimentally verified. We also interpret the generation of new frequency components in soliton-linear wave interactions in terms of cascaded Bragg scattering.

Chapter 5 presents a practical study on the improvement of a frequency conversion device called fiber optical parametric oscillator. We will discuss some basic properties of this device such as the gain characteristics, cavity design, threshold conditions and oscillation stability. With these, we show how to operate this device to generate a multi-watt level sideband output. We also show how to achieve very high conversion efficiency from a pump to sidebands by means of intracavity filtering.

Chapter 6 investigates a ubiquitous phenomenon in physics called symmetry breaking in a synchronously pumped passive fiber ring cavity. Some basic properties of the passive fiber ring cavity will be introduced. Numerical simulations will be performed to reveal the underlying process of temporal symmetry breaking and its dynamics. We then provide strong experimental evidence for the existence of symmetry breaking instability in both the spectral and temporal domains.

Chapter 7 presents a summary of the above chapters, and an outlook for future work and potential applications.
This chapter is intended to introduce some of the fundamental concepts and phenomena particular to optical fibers. An overview of the origin of the physical properties such as eigenmodes, chromatic dispersion, nonlinearity and Raman scattering in optical fibers is presented. Some fundamental propagation equations and their derivation are introduced with the aim of describing the behavior of propagating optical fields. Finally, we focus on how these basic properties lead to some further nonlinear effects, such as self-phase modulation, cross-phase modulation, four-wave mixing, the combined effect of parametric and Raman gain.
2.1 Fiber Modes

Optical fibers are a type of circular dielectric waveguide, commonly made of silica glass. Typically, for a step-index fiber, it has a higher refractive index $n_1$ core surrounding by a low refractive index $n_2$ cladding. Fig. 2.1 shows the refractive index profile and cross section of a step-index fiber with a core radius of $a$. The guiding mechanism of the electromagnetic field in this type of optical fiber is known as total internal reflection.

![Figure 2.1: Schematic of refractive index profile and cross section of a step index fiber.](image)

The eigenmodes are the solutions of electromagnetic field which propagates in a step-index single mode fiber, and can be found by solving the wave equation with correct boundary conditions [73]. In circular waveguides, the eigenmodes are generally in either an HE or EH mode, depending on whether the electric field component or magnetic field component has a larger contribution in the propagation direction. The propagation constant (or wave number) $\beta$ of HE and EH modes of a step-index optical fiber can be obtained by evaluating the following eigenvalue equations

EH mode:

\[
\frac{J_{l+1}(ha)}{ha J_l(ha)} = \frac{n_1^2 + n_2^2}{2n_1^2} \frac{K'(qa)}{qaK_l(qa)} + \left( \frac{l}{(ha)^2} - R \right),
\]  

(2.1)
HE mode:

\[
\frac{J_{l-1}(ha)}{J_l(ha)} = -\frac{n_1^2 + n_2^2}{2n_1^2} \frac{K'(qa)}{qaK_l(qa)} + \left( \frac{l}{(ha)^2} - R \right),
\]

(2.2)

where

\[
R = \left[ \left( \frac{n_1^2 - n_2^2}{2n_1^2} \right)^2 \left( \frac{K'_l(qa)}{qaK_l(qa)} \right)^2 + \left( \frac{l\beta}{n_1k_0} \right)^2 \left( \frac{1}{q^2a^2 + \frac{1}{h^2a^2}} \right)^2 \right]^{1/2}.
\]

(2.3)

\(J_l\) is a Bessel function of the first kind, and \(K_l\) is a modified Bessel function of the second kind. \(l\) is the order of the Bessel functions. \(h\) and \(q\) are respectively given by

\[
h^2 = n_1^2k_0^2 - \beta^2,
\]

(2.4)

\[
q^2 = \beta^2 - n_2^2k_0^2,
\]

(2.5)

where \(k_0 = 2\pi/\lambda\) is the wavenumber of the electromagnetic field in free space, and \(\lambda\) is the free space wavelength.

### 2.2 Chromatic Dispersion

In optics, chromatic dispersion is the phenomenon whereby the refractive index experienced by a propagating electromagnetic field in a medium is frequency-dependent. It plays a critical role in nonlinear optics. We will see later in this chapter how it significantly influences the interaction between optical fields of different frequencies. Since we are considering the electromagnetic field propagating solely in the fundamental spatial mode of an optical fiber, the dispersion only consists of two components: material dispersion and waveguide dispersion. The *material dispersion* originates from the frequency dependence of the linear susceptibility of the material, and it can be well described by the Sellmeier equation [74]; while, the *waveguide dispersion* can be understood by noting that the mode field diameter of the electromagnetic field in an optical fiber is frequency-
dependent, so the effective refractive index of the mode will change with frequency even in the absence of material dispersion [75]. To quantitatively account for the chromatic dispersion in optical fibers, it is customary to express the mode propagation constant $\beta(\omega)$ (or wave number) of the fundamental mode as a function of angular frequency $\omega$. By expanding $\beta(\omega)$ into a Taylor series with respect to an arbitrary frequency $\omega_0$, we obtain the expression for the propagation constant in terms of dispersion parameters as follows

$$\beta(\omega) = n(\omega)\frac{\omega}{c}$$

$$= \beta_0 + (\omega - \omega_0)\beta_1 + \frac{1}{2}(\omega - \omega_0)^2\beta_2 + \frac{1}{6}(\omega - \omega_0)^3\beta_3 + ...$$

(2.6)

(2.7)

where

$$\beta_k = \left(\frac{d^k\beta}{d\omega^k}\right)_{\omega=\omega_0}, \ (k = 0, 1, 2...).$$

(2.8)

The first order derivative $\beta_1$ is related to the Group Velocity $v_g = 1/\beta_1$ which describes the speed of an optical pulse propagating in an optical fiber. The second order derivative $\beta_2$ is known as the Group Velocity Dispersion (GVD) parameter, and it is the parameter that describes how fast a pulse broadens when it propagates in an optical fiber. In the normal dispersion region where $\beta_2 > 0$, optical pulses with longer center wavelengths travel faster than those with shorter center wavelengths; while the opposite occurs in the anomalous dispersion region where $\beta_2 < 0$. Optical pulses experience a minimum group velocity dispersion at the Zero Dispersion Wavelength (ZDW) where $\beta_2 = 0$. In fiber optics, the dispersion length is defined as

$$L_D = (|\beta_2|d\Omega^2)^{-1},$$

(2.9)

where $d\Omega$ is the 3 dB spectral width of an optical pulse. It is commonly used to quantify the distance at which dispersion starts to have a significant effect on the optical pulse.
Formally, $L_D$ is the distance at which a chirp free Gaussian pulse broadens temporally by a factor of $\sqrt{2}$.

### 2.3 Fiber Nonlinearity

The induced polarization of atoms in dielectric media is not always proportional to the strength of the electric field. In the presence of a strong electric field $E$, the response of the bound electrons to the applied field is no longer linear. The total induced electric polarization needs to be expressed in the form of a Taylor series expansion as \cite{26,76}

$$P = \varepsilon_0 \left( \chi^{(1)} \cdot E + \chi^{(2)} \cdot EE + \chi^{(3)} \cdot EEE + \cdots \right),$$

where $\varepsilon_0$ is the vacuum permittivity and $\chi^{(j)}(j = 1, 2, \ldots)$ is the $j^{th}$ order susceptibility.

The second order nonlinear term is responsible for nonlinear phenomena in nonlinear crystals, such as second harmonic generation, sum frequency generation, and difference frequency generation. However, in optical fibers, since the molecular structure of fused silica is amorphous, the even terms of $\chi^{(j)}$ vanish as the molecular arrangement of the medium is centrosymmetric, and the lowest order nonlinear effect originates from the third order susceptibility $\chi^{(3)}$. The third order nonlinear term is responsible for nonlinear phenomena, such as Third Harmonic Generation (THG), Optical Kerr Effect, Four-Wave Mixing (FWM), Raman scattering and Brillouin Scattering. As the electric field propagates in the fundamental mode in a single mode fiber, we can neglect all spatial nonlinear effects (e.g. self-focusing) in this thesis, reducing the nonlinear formalism of optical field into a one-dimensional scalar treatment. Since the wavelength range $0.5 \sim 2 \mu m$ that of interest for the study of nonlinear phenomena in optical fibers is far from the resonances of silica, for our initial analysis we assume that $\chi^{(3)}$ is frequency-independent and the nonlinear response to the electric field is instantaneous. Later, we will present the treatment of the inclusion of non-instantaneous response (the Raman effect) on $\chi^{(3)}$ in Section 2.3.4.
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We consider an induced polarization consisting only of the first and third order susceptibilities as such

\[ P(t) = P_L(t) + P_{NL}(t), \]  

(2.11)

where the induced linear and nonlinear polarizations are

\[ P_L(t) = \varepsilon_0 \int_{-\infty}^{t} \chi^{(1)}(t-t') \cdot E(t') dt', \]  

(2.12)

\[ P_{NL}(t) = \varepsilon_0 \int \int \int \chi^{(3)}(t-t_1, t-t_2, t-t_3) \cdot E(t_1) E(t_2) E(t_3) dt_1 dt_2 dt_3. \]  

(2.13)

By substituting a linearly polarized electric field consisting of \( N \) monochromatic waves oscillating at frequencies \( \omega_j \) with corresponding propagation constants \( \beta_j \) (\( j = 1 \) to \( N \))

\[ E = \frac{1}{2} \sum_{j=1}^{N} E_j \exp[i(\beta_j z - \omega_j t)] + c.c., \]  

(2.14)

into the nonlinear polarization term \( P_{NL} = \chi^{(3)} : EEE \) of Eq. (2.10), we have

\[ P_{NL} = \frac{1}{2} \sum_{j=1}^{N} P_{NLj} \exp[i(\beta_j z - \omega_j t)] + c.c., \]  

(2.15)

where the individual induced nonlinear polarization \( P_{NLj} \) of the \( j \)-th electric field consists of the products of all triple combinations of the applied electric fields. It can be explicitly written as

\[ P_{NLj} = \frac{1}{8} \varepsilon_0 \chi^{(3)} \sum_{k,l,m=1}^{N} E_k E_l E_m \exp(i\theta_{jklm}) \]

\[ + \frac{3}{8} \varepsilon_0 \chi^{(3)} \sum_{k,l,m=1}^{N} E_k E_l E_m^* \exp(i\theta_{jklm}) + c.c., \]  

(2.16)
where $\theta_{+jklm}$ and $\theta_{-jklm}$ respectively are

\begin{align}
\theta_{+jklm} &= (\beta_k + \beta_l + \beta_m - \beta_j)z - (\omega_k + \omega_l + \omega_m - \omega_j)t, \\
\theta_{-jklm} &= (\beta_k + \beta_l - \beta_m - \beta_j)z - (\omega_k + \omega_l - \omega_m - \omega_j)t.
\end{align}

The first terms on the right hand side of both $\theta_{+jklm}$ and $\theta_{-jklm}$ describe the phase mismatch (or momentum conservation), and the second terms on the right hand side describe the frequency mismatch (or conservation of energy). The terms in the nonlinear polarization $P_{NLj}$ in Eq. (2.16) containing $\theta_{+jklm}$ are responsible for sum frequency generation in which three photons transfer their energy to a single photon. Generally, the phase matched condition for sum frequency generation is difficult to satisfy in single mode optical fibers, and we will neglect this term for the rest of our analysis. The terms containing $\theta_{-jklm}$ are responsible for most of the third order nonlinear phenomena such as the Kerr effect, self-phase modulation, cross-phase modulation, and four-wave mixing. In the case of four-wave mixing process, the nonlinear polarization $P_{NLj}$ becomes a new source term that contributes to the emission of the electric field component at frequency $\omega_j$. We will discuss these phenomena in detail in the following subsections.

### 2.3.1 Kerr Effect

For a single monochromatic input field, the nonlinear polarization given by Eq. (2.16) reduces to

\[ P_{NL} = \frac{3}{4} \varepsilon_0 \chi^{(3)}|E|^2 E. \]

The refractive index is then related to the overall dielectric constant by

\[ n(\omega, |E|^2) = \sqrt{\varepsilon_L(\omega) + \varepsilon_{NL}} = \sqrt{1 + \chi^{(1)}(\omega) + \frac{3}{4} \chi^{(3)}|E|^2}, \]
where the linear dielectric constant $\epsilon_L$ originates from the linear part of the polarization $\mathbf{P}_L$, and is responsible for the chromatic dispersion. As the nonlinear dielectric constant $\epsilon_{NL}$ is relatively small compared to the linear dielectric constant, the refractive index can be approximately written as

$$n(\omega, |E|^2) = n_L(\omega) + n_{NL}|E|^2,$$  \hspace{1cm} (2.21)

where

$$n_L(\omega) = \sqrt{1 + \chi^{(1)}(\omega)},$$  \hspace{1cm} (2.22)

$$n_{NL}(\omega) = \frac{3}{8n_L(\omega)} \Re(\chi^{(3)}).$$  \hspace{1cm} (2.23)

$n_L$ is the linear index. $n_{NL}$ is the nonlinear index coefficient, and is proportional to the intensity of the input field. This phenomenon is called the optical Kerr Effect first discovered by Scottish physicist John Kerr in 1875 [1]. Typically, the nonlinear index coefficient of fused silica is $n_{NL} \approx 2.2 \sim 3.4 \times 10^{-20}$ m$^2$/W [17,21]. Due to the confinement of optical field inside an optical fiber, the intensity of the optical field depends on the power of the field as well as the fiber effective mode area. It is conventional to introduce the nonlinear coefficient

$$\gamma(\omega) = \frac{n_{NL}(\omega)\omega}{cA_{eff}},$$  \hspace{1cm} (2.24)

to describe the strength of the interaction between the optical fields in an optical fiber, where $A_{eff}$ is the effective mode area of the field at frequency $\omega$, $c$ is the speed of light [21]. Formally, the effective mode area $A_{eff}$ is defined as

$$A_{eff} = \left(\frac{\int \int_{-\infty}^{\infty} |F(x, y)|^2 dx dy}{\int \int_{-\infty}^{\infty} |F(x, y)|^4 dx dy}\right)^{1/2},$$  \hspace{1cm} (2.25)
where \( F(x, y) \) is the mode profile of the transverse field [73]. For a monochromatic field, the amount of phase shift accumulated when it propagates in an optical fiber, is not only related to the mode propagation constant \( \beta \), but also to the nonlinear wave number which is expressed as \( k_{NL} = \gamma P \), where \( P \) is the optical power of the field. Similar to the dispersion length \( L_D \), the nonlinear length

\[
L_{NL} = (\gamma P)^{-1}
\]

is also commonly used to depict the distance at which the nonlinear effect starts to play a significant role. Formally, \( L_{NL} \) is the distance at which the nonlinear phase shift \( \phi_{NL} = k_{NL}L_{NL} \) accumulated by a CW beam is \( \phi_{NL} = 1 \).

### 2.3.2 Self-Phase and Cross-Phase Modulation

For an optical pulse with a temporal intensity profile of \( P(0, T) = |E(0, T)|^2 \), the nonlinear phase shift \( \phi_{NL} \) experienced by the pulse becomes time-dependent. This phenomenon is known as **Self-Phase Modulation** (SPM) [18]. The amount of nonlinear phase shift accumulated across the pulse after propagating in an optical fiber of length \( L \) can be simply expressed as

\[
\phi_{NL}(z = L, T) = \gamma P(z = 0, T)L,
\]

where \( P(z = 0, T) \) is the input temporal intensity profile of the optical pulse. Furthermore, as the nonlinear phase shift temporally varies, it induces an instantaneous frequency change across the pulse

\[
\delta\omega(T) = -\frac{\partial \phi_{NL}}{\partial T} = -\gamma \frac{\partial}{\partial T} P(z = 0, T)L.
\]

Similar to self-phase modulation, another optical effect called **Cross-Phase Modulation** (XPM) refers to the nonlinear phase shift experienced by one optical field induced by
another optical field \[77\]. If the total optical field consists of two pulses with carrier frequencies \(\omega_1\) and \(\omega_2\), and their spectral widths are sufficiently narrow such that \(\Delta \omega_j \ll \omega_j (j = 1, 2)\), then by substituting the electric field

\[
E = \frac{1}{2} \left[ E_1 \exp(-i\omega_1 t) + E_2 \exp(-i\omega_2 t) \right] + c.c., \tag{2.29}
\]

(for simplicity, we assume the two propagation constants are identical) into the nonlinear polarization of Eq. (2.16), we obtain

\[
P_{NLj} = \frac{3}{8} \epsilon_0 \chi^{(3)} \left\{ E_1|E_1|^2 \exp(-i\omega_1 t) + E_2|E_2|^2 \exp(-i\omega_2 t) + 2E_1|E_2|^2 \exp(-i\omega_1 t) + 2E_2|E_1|^2 \exp(-i\omega_2 t) + E_1E_2^* \exp[-i(2\omega_1 - \omega_2) t] + E_2E_1^* \exp[-i(2\omega_2 - \omega_1) t] \right\} + c.c. \tag{2.30}
\]

We can see the first two terms are responsible for the SPM of the two pulses. The middle two terms are responsible for cross-phase modulation whereby one of the propagating pulses experiences a phase modulation from the other. The nonlinear phase shift induced by XPM is twice as strong as SPM. Thus, the nonlinear phase shift experienced by each pulse after propagating a distance \(L\) becomes

\[
\phi_{NL1} = \gamma \left[ P_1(z = 0, T) + 2P_2(z = 0, T) \right] L, \tag{2.31}
\]

\[
\phi_{NL2} = \gamma \left[ P_2(z = 0, T) + 2P_1(z = 0, T) \right] L. \tag{2.32}
\]

In Chapter 4, we will be frequently dealing with XPM induced by the bichromatic pump pairs. The last two terms in Eq. (2.30) generate new frequencies at \(2\omega_1 - \omega_2\) and \(2\omega_2 - \omega_1\) as a result of four-wave mixing. We will discuss this phenomenon in the next subsection.
2.3.3 Four-Wave Mixing

According to Eq. (2.16), a strong coupling between four monochromatic waves can occur if the phase matched condition is satisfied such that $\theta_{ijklm} = 0$ or

$$\omega_k + \omega_l = \omega_m + \omega_j,$$  \hspace{1cm} (2.33)
$$\beta_k + \beta_l = \beta_m + \beta_j.$$ \hspace{1cm} (2.34)

These two equations can be simply seen as energy and momentum conservation. This elementary four wave interaction driven by the $\chi^{(3)}$ nonlinear polarization, is called Four-Wave Mixing (FWM). Depending on the direction of the energy flow between the four waves, these processes can be either classified as nonlinear Bragg Scattering (BS), Phase Conjugation (PC) or Modulation Instability (MI). All these processes are optical parametric processes with no energy exchanged between optical fields and medium [76]. Among these processes, we concentrate especially on modulation instability in this thesis. In fiber optics, it is a phenomenon whereby a strong continuous wave (CW) or quasi-CW pump wave breaks up into a fast modulational envelope in temporal domain. In the spectral domain, it manifests itself as the generation of two sidebands [25, 35]. The up and down frequency shifted sidebands are called the anti-Stokes (at frequency $\omega_a$) and Stokes (at frequency $\omega_s$) sidebands, respectively, with the frequency relation as

$$\Omega = \omega_a - \omega_p = \omega_p - \omega_s.$$ \hspace{1cm} (2.35)

The requirement for momentum conversion sets the linear phase mismatch condition of the process as

$$\Delta k_L = \beta_a + \beta_s - 2\beta_p = 0.$$ \hspace{1cm} (2.36)

From a quantum mechanical viewpoint, as shown in Fig. 2.2, two incident pump photons which are degenerate at frequency $\omega_k = \omega_l = \omega_p$, are annihilated while two new photons
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Figure 2.2: Schematic of four-wave mixing with two pump input photons at frequency $\omega_p$, and Stokes and anti-Stokes photons at frequencies $\omega_s$ and $\omega_a$, respectively.

are simultaneously generated at frequencies $\omega_p - \Omega$ and $\omega_p + \Omega$ so as to conserve energy and momentum. Recall in the previous subsections, the Kerr effect induces an extra nonlinear wave number $k_{NL} = \gamma P$. Here, the extra nonlinear wave numbers are caused by the SPM of the pump and the XPM on the sidebands induced by the pump. This set the nonlinear phase mismatch of modulation instability as

$$\Delta k_{NL} = k_{a,NL} + k_{s,NL} - 2k_{p,NL},$$

$$= 2\gamma P + 2\gamma P - 2\gamma P = 2\gamma P.$$ \hspace{1cm} (2.37)

The initial amplitudes of the two sidebands are too small to induce significant SPM or XPM. By taking the nonlinear phase mismatch into account, conservation of momentum now yields

$$\Delta k_L + \Delta k_{NL} = \beta_a + \beta_s - 2\beta_p + 2\gamma P = 0.$$ \hspace{1cm} (2.38)

2.3.4 Stimulated Raman Scattering

The Raman effect is an inelastic scattering of incident light by excitations of matters (e.g. molecular vibrations). This phenomenon was first discovered by Sir C. V. Raman in
1928 [78]. As illustrated in Fig. 2.3, when an incident photon at frequency $\omega_p$ interacts with a molecule, it can either be scattered by a phonon excitation at frequency $\Omega$ and accompanied by an emission of a frequency down shifted Stokes photon at frequency $\omega_s = \omega_p - \Omega$ [Fig. 2.3 (a), Stokes scattering], or scattered by a removal of energy from an excited state and accompanied by an emission of a frequency up shifted photon at frequency $\omega_a = \omega_p + \Omega$ [Fig. 2.3 (b), anti-Stokes scattering]. Fig. 2.4 shows the measured complex Raman susceptibility $\tilde{\chi}^{(3)}(\Omega)$ of fused silica as a function of frequency shift with respect to the pump frequency. The imaginary parts of $\tilde{\chi}^{(3)}(\Omega)$ corresponds to the Raman gain with a peak gain between 13 and 15 THz. Using the Raman effect in optical fibers, a strong pump can be continuously converted into a Stokes wave via a stimulated process known as stimulated Raman scattering (SRS). This stimulated process has been widely used for signal amplification in Raman amplifiers [79, 80]. The full nonlinear response of the third order susceptibility induced by the Kerr and the Raman effect is in the form of [81]

$$\chi^{(3)}(t-t_1, t-t_2, t-t_3) = \chi^{(3)} h(t-t_1) \delta(t_1-t_2) \delta(t-t_3),$$  \hspace{1cm} (2.39)
where \( h(t) \) is the nonlinear response function written as

\[
h(t) = (1 - f_R)\delta(t) + f_R h_R(t). \tag{2.40}
\]

The first term on the right hand side is responsible for the instantaneous (\( \sim 5 \) fs) Kerr response of the bound electrons, and the second term is responsible for the delayed Raman response of the molecular vibration of fused silica. The temporal variation \( h_R(t) \) is the Raman response function, and its Fourier transform is the complex Raman susceptibility \( \tilde{\chi}^{(3)}_R(\Omega) \) as shown in Fig. 2.4. The Raman gain coefficient is given by

\[
g_R(\Omega) = \gamma f_R \Re[\tilde{\chi}^{(3)}_R(\Omega)], \tag{2.41}
\]

where \( \Omega = \omega_p - \omega_s \). \( f_R \) is the fractional contribution of the Raman susceptibility to the

![Figure 2.4: Measured real and imaginary part of complex Raman susceptibility of fused silica [82].](image-url)
total nonlinearity, and experimentally measured to be \( f_R \approx 0.18 \) in silica fibers \([82, 83]\). By substituting Eq. (2.39) into Eq. (2.13), the nonlinear polarization becomes

\[
P_{NL}(t) = \varepsilon_0 \chi^{(3)} E(t) \int_{-\infty}^{t} h(t-t') E'(t') E(t') dt'.
\] (2.42)

The individual induced nonlinear polarizations \( P_{NL,j} \) of Eq. (2.16) now become

\[
P_{NL,j} = \frac{3}{4} \varepsilon_0 \chi^{(3)} E \left[ (1 - f_R)|E|^2 + f_R \int_{-\infty}^{t} h_R(t-t')|E|^2 dt' \right].
\] (2.43)

Later in this chapter, the impact of Raman scattering on four-wave mixing will be described in detail.

### 2.4 Propagation Equations

#### 2.4.1 Nonlinear Schrödinger Equation

This subsection outlines the derivation of the nonlinear Schrödinger equation which is frequently used to describe the evolution of the optical field that propagates in an optical fiber. Fundamentally, the propagation of the electromagnetic field is well known to be governed by Maxwell’s equations. In mks units, these equations can be written as \([84]\)

\[
\nabla \cdot \mathbf{D} = \rho_f, \quad \text{(2.44)}
\]

\[
\nabla \cdot \mathbf{B} = 0, \quad \text{(2.45)}
\]

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t'}, \quad \text{(2.46)}
\]

\[
\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t'}, \quad \text{(2.47)}
\]
where \( D \) and \( B \) are the electric and magnetic flux densities induced by the incident electromagnetic field via

\[
\begin{align*}
D &= \varepsilon_0 E + P, \\
B &= \mu_0 H + M,
\end{align*}
\]

(2.48) (2.49)

where \( P \) is the induced electric polarization, and \( \mu_0 \) is the vacuum permeability. As fused silica is a nonmagnetic, charge-free, isotropic medium, the free charge density \( \rho_f \), the current density \( J_f \) and the magnetic polarization \( M \) are all equal to zero. By substituting Eq. (2.47), (2.48), (2.49), and (2.11) into the curl of Eq. (2.46), we obtain the propagation equation of the electric field

\[
\nabla^2 E = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} E + \mu_0 \frac{\partial^2}{\partial t^2} (P_L + P_{NL}),
\]

(2.50)

which can also be expressed in the form of a Helmholtz equation in the Fourier domain. By substituting Eq. (2.43) into Eq. (2.50) and taking the Fourier transform, we obtain

\[
\nabla^2 \tilde{E} = - \left[ n^2(\omega) + \frac{3}{4} \tilde{\chi}^{(3)} \left( 1 - f_R \right) + \tilde{\chi}^{(3)}_R (\Delta \omega) \right] \frac{\omega^2}{c^2} \tilde{E}.
\]

(2.51)

The general solution of Eq. (2.51) can be written in the form of

\[
\tilde{E}(x, y, z, \omega - \omega_0) = \tilde{A}(z, \omega - \omega_0) F(x, y) \exp[i\beta(\omega_0)z] + c.c.,
\]

(2.52)

where \( \tilde{A}(z, \omega - \omega_0) \) is the Fourier transform of \( A(z, t) \) which is the envelope of the slowly varying electric field with a carrier frequency of \( \omega_0 \) along the propagation direction, \( F(x, y) \) is the fiber mode profile of the transverse field. Neglecting the nonlinear effect on the mode profile, and making the slowly varying envelope approximation \( \partial^2 \tilde{A}/\partial z^2 \approx 0 \), an equation describing \( \tilde{A}(z, \omega) \) can be obtained by substituting Eq. (2.52) into Eq. (2.51).
After separating the variable $F(x, y)$, it becomes

$$2i\beta(\omega_0) \frac{\partial \tilde{A}}{\partial z} - \beta(\omega_0)^2 \tilde{A} = -\left[ \beta(\omega)^2 + \frac{3}{4} \chi^{(3)}|\omega|^2 \frac{1}{c^2} A_{\text{eff}} \right] \left( 1 - f_R + \chi^{(3)}(\omega) \right)|\tilde{A}|^2 \tilde{A}, \quad (2.53)$$

where $A_{\text{eff}}$ is the effective mode area of the transverse field profile $F(x, y)$ as defined by Eq. (2.25). We approximate $\beta(\omega)^2 - \beta(\omega_0)^2 \approx 2\beta(\omega_0)[\beta(\omega) - \beta(\omega_0)]$, then rearrange and replace the variables of the second term on the right hand side with Eq. (2.23) and Eq. (2.24), and finally take the inverse Fourier transform back into temporal domain. This gives us the Generalized Nonlinear Schrödinger Equation (GNLSE) which fully describes pulse propagation along optical fibers in the presence of higher order dispersion, the Kerr effect, and Raman scattering

$$\frac{\partial A}{\partial z} = i \sum_{n \geq 1} \frac{\beta_n}{n!} \frac{\partial^n A}{\partial t^n} + i\gamma \left[ (1 - f_R)|A|^2 + f_R \int_{-\infty}^{t} \chi^{(3)}(t - t')|A|^2 dt' \right] A. \quad (2.54)$$

$\beta_n$ is the $n^{th}$ order dispersion coefficient given by Eq. (2.8), $\gamma$ is the nonlinear coefficient, and $f_R$ is the fractional contribution of the Raman susceptibility to the total nonlinearity. The physical meaning of each term in the GNLSE is to account for the dispersion (the first term on the right), and nonlinear phase shift as well as Raman gain (the second term on the right) experienced by the slowly varying envelope propagating in an optical fiber. As the pulse width studied in this dissertation is relatively broad (> 1 ps), we neglect the higher order nonlinear effect, i.e., self-steepening effect which results from the intensity dependence of the group velocity [85]. Numerical simulations of optical pulse propagation are commonly implemented by solving the GNLSE using the well known split-step Fourier method [21, 86]. For a detailed explanation of the implementation of the split-step Fourier method, interested readers can refer to the Appendix B of Ref. [21].
2.4.2 Coupled-Mode Equations

Although the GNLSE Eq. (2.24) fully describes the nonlinear propagation of optical pulses in optical fibers, it is often convenient and efficient to model the interactions between CW optical fields at discrete frequencies using Coupled-Mode Equations [45]. From Eq. (2.16), the nonlinear polarization of the individual fields in Eq. (2.14) can be written as

$$P_{NLj} = \frac{3}{4} \varepsilon_0 \chi^{(3)} \sum_{k,l,m=1}^N \eta_{lm} E_k E_l E_m^* \exp(i\Delta k_{jklm} z),$$  \hspace{1cm} (2.55)

where $\sum_{k,l,m=1}^N$ denotes the summation over all the possible permutations such that $\omega_j = \omega_k + \omega_l - \omega_m$. $\Delta k_{jklm} = \beta_k + \beta_l - \beta_j - \beta_m$ represents the linear phase mismatch of the different four-wave mixing terms of all the fields considered in the system, and $\eta_{lm} = (1 - f_R) + f_R \tilde{\chi}^{(3)}_R (\omega_l - \omega_m)$. By substituting the nonlinear polarization into the propagation equation Eq. (2.50), the evolution of the electric field of each frequency component $E_j$ then becomes

$$\frac{dE_j}{dz} = i\frac{\mu_0 \omega_j^2}{2\beta_j} P_{NLj}.$$  \hspace{1cm} (2.56)

Assuming all the fields are propagating with a same transverse mode profile $F(x, y)$, the evolution of the electric fields can be further simplified into

$$\frac{dA_j}{dz} = i\gamma \sum_{k,l,m=1}^N \left[ (1 - f_R) + f_R \tilde{\chi}^{(3)}_R (\omega_l - \omega_m) \right] A_k A_l A_m^* \exp(i\Delta k_{jklm} z).$$  \hspace{1cm} (2.57)

where again the summation $\sum_{k,l,m=1}^N$ is over all the terms that satisfy the frequency relation $\omega_j = \omega_k + \omega_l - \omega_m$. This can be understood as the amplitude and phase evolution of the individual mode at frequency $\omega_j$ depends on the sum of all the permutations whose frequencies satisfy the above relation. We will use these coupled-mode equations extensively to describe the evolution of CW and quasi-CW optical fields in this thesis.
2.5 Modulation Instability and the Raman Effect

Modulation instability is a parametric four-wave mixing process whereby a strong CW pump is broken up into a fast modulational envelope in temporal domain, by beating with the two new generated sidebands [24, 25, 35]. The symmetric detunings of the two spectral sidebands mean that energy is always conserved. Momentum conservation of photons or phase matching is achieved by the dispersion and nonlinearity of single mode fibers. This phenomenon has not only been observed in fiber optics, but also in plasma physics and fluid dynamics [30, 31, 87]. We first show the derivation of the small signal solution of the two sidebands by subjecting a strong undepleted CW pump to small perturbations and linearizing the differential coupled-mode equations of the two sidebands. We include the effects of both the Kerr and Raman scattering as in a general fiber both effects are always present. Later, we include the effects of pump depletion in the evolution of the pump and sidebands.

2.5.1 Undepleted Pump Linear Analysis

The starting point of our analysis is to define the the interaction of pump, and Stokes and anti-Stokes waves at frequencies \( \omega_p, \omega_p - \Omega, \) and \( \omega_p + \Omega, \) with complex amplitudes \( A_p, A_s, \) and \( A_a, \) respectively. The total electric field can be expressed as

\[
A(z, t) = \left[ A_p(z) + A_s(z) \exp(i \Omega t) + A_a(z) \exp(-i \Omega t) \right] \exp(-i \omega_p t). \tag{2.58}
\]

To illustrate the parametric gain experienced by a small signal under the influence of a strong pump field, we restrict our analysis to the undepleted pump regime where \( |A_a|, |A_s| \ll |A_p|, \) and neglect the generation of higher order sidebands. In this analysis,
the coupled-mode equations from Eq. (2.57) can reduce to

\[
\frac{dA_p}{dz} = iγA_pA_p^*A_p', \quad (2.59)
\]

\[
\frac{dA_a}{dz} = iγ\left\{\left[2 - f_R + f_R\overline{\chi}^{(3)}(Ω)\right]A_pA_p^*A_a + \left[1 - f_R + f_R\overline{\chi}^{(3)}(Ω)\right]A_pA_p^*A_a^*\exp(-iΔk_Lz)\right\}, \quad (2.60)
\]

\[
\frac{dA_s}{dz} = iγ\left\{\left[2 - f_R + f_R\overline{\chi}^{(3)}(Ω)\right]A_pA_p^*A_s + \left[1 - f_R + f_R\overline{\chi}^{(3)}(Ω)\right]A_pA_p^*A_s^*\exp(-iΔk_Lz)\right\}, \quad (2.61)
\]

where the linear mismatch \(Δk_L\) is defined as Eq. (2.36). The first equation shows that the pump field only experiences its own self-phase modulation. The first terms on the right hand side of the last two equations are the cross-phase modulation terms experienced by the two sidebands, and the second terms are the four-wave mixing terms. It is convenient to write all fields in the form of \(A_{p,a,s}(z) = \sqrt{P_{p,a,s}(z)} \exp[iφ_{p,a,s}(z)]\), where \(\sqrt{P_{p,a,s}(z)}\) and \(φ_{p,a,s}(z)\) are the amplitudes and the phases of the fields, respectively. The analytical solution for the pump can be easily shown to be

\[
A_p(z) = \sqrt{P_p(0)} \exp[iγP_p(0)z]. \quad (2.62)
\]

We set \(P_p = P_p(z) = P_p(0)\) for an undepleted pump. By substituting the solution of the pump Eq. (2.62) back into Eq. (2.60) and (2.61), the differential equations of the anti-Stokes and Stokes wave can be rewritten as

\[
\frac{dA_s}{dz} = iγP_p\left\{(1+q')A_s + q'A_s^*\exp[i(2γP_p - Δk_L)z]\right\}, \quad (2.63)
\]

\[
\frac{dA_s}{dz} = iγP_p\left\{(1+q)A_s + qA_s^*\exp[i(2γP_p - Δk_L)z]\right\}, \quad (2.64)
\]

where \(q = 1 - f_R + f_R\overline{\chi}^{(3)}(Ω)\) is the Fourier transform of \(h(t)\) in Section 2.3.4 that describes the relative contributions of the Kerr and Raman effect. By performing a change
of variables with \( A_{a,s}(z) = B_{a,s}(z) \exp[i(2\gamma P_p - \Delta k_L)z] \) and substituting into Eq. (2.63) and (2.64), we obtain

\[
\frac{d}{dz} \begin{pmatrix} B^*_a \\ B_s \end{pmatrix} = i \begin{pmatrix} -\gamma q P_p - \Delta k_L/2 & -\gamma q P_p \\ \gamma q P_p & \gamma q P_p + \Delta k_L/2 \end{pmatrix} \begin{pmatrix} B^*_a \\ B_s \end{pmatrix},
\] (2.65)

The general solution of the vector differential equation above is in the form of

\[
\begin{pmatrix} B^*_a(z) \\ B_s(z) \end{pmatrix} = C_+ \exp(\lambda_+ z) \overline{\nu}_+ + C_- \exp(\lambda_- z) \overline{\nu}_-, \tag{2.66}
\]

where \( \lambda_\pm \) and \( \overline{\nu}_\pm \) are the eigenvalues and eigenvectors, respectively. \( C_\pm \) can be obtained from the initial conditions. By solving the characteristic equation of Eq. (2.65), we find the eigenvalues as

\[
\lambda_\pm = \pm \gamma P_p \sqrt{\kappa(2q - \kappa)}, \tag{2.67}
\]

where \( \kappa = -\Delta k_L/2\gamma P_p \) is the normalized linear phase mismatch. The eigenvectors can then be obtained by substituting the eigenvalues into Eq. (2.65) as

\[
\overline{\nu}_\pm = \begin{pmatrix} 1 \\ (\mp i R - q + \kappa)/q \end{pmatrix}, \tag{2.68}
\]

where \( R = \lambda_\pm/\gamma P_p = \sqrt{\kappa(2q - \kappa)} \). For \( B_{a,s} \) to undergo an exponential gain, it requires the real part of the eigenvalues \( \lambda_\pm \) to be positive. By substituting Eq. (2.67) and Eq. (2.68) into Eq. (2.66) and applying initial conditions at \( z = 0 \), we obtain

\[
C_\pm = \frac{1}{2R} \left[ \pm iq B^*_a(0) + (R \pm iq \mp i\kappa)B_s(0) \right]. \tag{2.69}
\]
Chapter 2. Fundamental Concepts

By substituting Eq. (2.69) into Eq. (2.66) again, the exact solutions of the two sidebands in the undepleted pump regime are given by [88]

\[
B_s(z) = \frac{1}{2R} \left[ \left[ iqB_s^*(0) + (R + iq - i\kappa)B_s(0) \right] \exp(\gamma P_p Rz) \right.
\]
\[
+ \left[ -iqB_s^*(0) + (R - iq + i\kappa)B_s(0) \right] \exp(-\gamma P_p Rz) \right], \tag{2.70}
\]

\[
B^*_s(z) = \frac{1}{2R} \left[ \left[ (R - iq + i\kappa)B^*_s(0) - iqB_s(0) \right] \exp(\gamma P_p Rz) \right.
\]
\[
+ \left[ (R + iq - i\kappa)B^*_s(0) + iqB_s(0) \right] \exp(-\gamma P_p Rz) \right]. \tag{2.71}
\]

Fiber optical parametric amplifiers utilize the parametric gain experienced by a signal injected at a sideband wavelength to amplify the signal. From Eq. (2.70) and Eq. (2.71), we derive the small-signal Raman parametric intensity gain of an injected signal as

\[
G(z) = \left| \cosh(\gamma P_p Rz) \pm \frac{i(\kappa - q)}{R} \sinh(\gamma P_p Rz) \right|^2. \tag{2.72}
\]

The hyperbolic sine term is positive if the sideband is injected on the anti-Stokes side of the pump, and is negative if the sideband is injected on the Stokes side. As such, this results in a gain asymmetry between a signal injected at the Stokes or the anti-Stokes frequency. For a large parametric gain \((\gamma P_p \Re(R)z \gg 1)\), Eq. (2.72) can reduce to

\[
G(z) = \frac{1}{4} \left| 1 \pm \frac{i(\kappa - q)}{R} \right|^2 \exp(2\gamma P_p \Re(R)z). \tag{2.73}
\]

This gives the combined intensity Raman parametric gain coefficient [89]

\[
g = 2\gamma P_p \Re(R). \tag{2.74}
\]

We plot in Fig. 2.5 (a) the normalized parametric gain coefficient \(g/2\gamma P_p\) as a function of normalized linear phase mismatch \(\kappa\) in the absence of the Raman effect \((q = 1)\). The region where sidebands experience parametric gain is \(0 < \kappa < 2\), with the maximum gain
Figure 2.5: (a) Normalized parametric gain coefficient $g/2\gamma P_p$ as a function of normalized linear phase mismatch $\kappa$ in the absence of the Raman effect. (b) Normalized Raman parametric gain coefficient $g/2\gamma P_p$ as a function of normalized linear phase mismatch $\kappa$ and the sidebands frequency detuning $\Omega/2\pi$.

occurs at $\kappa = 1$. To illustrate the consequence of the Raman effect on parametric gain, we also plot in Fig. 2.5 (b) the normalized combined Raman parametric gain $g/2\gamma P_p$ as a function of normalized linear phase mismatch $\kappa$ and the sidebands frequency detuning.
When Raman scattering is included, the imaginary part of $q$ is nonzero because $\tilde{\chi}^{(3)}_{R}(-\Omega)$ is complex. Thus, the real part of $\lambda_\pm$ does not necessarily vanish outside the region $0 < \kappa < 2$. This results in the gain region extended into $\kappa < 0$ and $\kappa > 2$ which can be clearly seen in Fig. 2.5 (b).

### 2.5.2 Effect of Pump Depletion

In the analysis of the previous subsection the pump power was assumed to be undepleted. Thus, the analytical expression Eq. (2.72) for a small signal gain can accurately predict the gain level in the undepleted pump regime, as well as the gain asymmetry between the Stokes and anti-Stokes sidebands [89]. In order to account for the effect of pump depletion and the power recurrence of two sidebands in the evolution of modulation instability [45, 90], the coupled-mode equations Eq. (2.57) needs to be solved numerically with three frequencies propagating in a single mode fiber.

![Figure 2.6](image)

**Figure 2.6:** Normalized power conversion of the pump (dash-dot curve) and the Stokes (and anti-Stokes) sidebands (solid curve) as a function of normalized pump power without Raman scattering at the phase mismatch of $\kappa = 1$ (a) and $\kappa = 0.25$ (b).

Fig. 2.6 shows the the evolution of the pump and the two sidebands in a single pass scheme without Raman scattering, as a function of normalized fiber length $\xi = \gamma P_p L$
where $\gamma$ is the fiber nonlinear coefficient, $P_p$ is the input pump power, and $L$ is the fiber length. The power conversion of the two sidebands along the propagation are identical as both sidebands experience identical amounts of parametric gain. The initial power ratio of the pump to the sidebands is set to $10^{11}$. Fig. 2.6 (a) shows the power evolution of the pump and the two sidebands at the phase mismatch of $\kappa = 1$ (perfectly phase matched). The maximum conversion efficiency for the pump to each sideband at this mismatch is only 28%. After the maximum conversion, the power of the sidebands starts converting back into the pump, and a periodic exchange of energy between the pump and the sidebands is observed. On the contrary, 100% depletion of the pump can been seen in Fig. 2.6 (b) when the phase mismatch is $\kappa = 0.25$ with 50% of the pump power converted into each sideband. This improvement of the sidebands conversion can be understood as a process with an initial smaller mismatch than phase matched linear mismatch ($\kappa < 1$) can actually become phase matched as the pump power depletes.

![Figure 2.7: Normalized power conversion of the pump (dash-dot curve), Stokes (dashed curve) and anti-Stokes (solid curve) sidebands as a function of normalized length $\xi$ at the phase mismatches of $\kappa = 1$ (a) and $\kappa = 0.25$ (b) in the presence of Raman scattering. The detuning of the sidebands is $\Omega = 20$ THz.](image)

With the inclusion of Raman scattering the power evolution shown in Fig. 2.7 becomes more complicated. As the combined Raman-parametric gain strongly depends
on the (positive and negative) detuning of the sidebands, it results in an asymmetric evolution between the Stokes and the anti-Stokes sidebands \[83, 91\]. To illustrate how the Raman effect influences the power evolution of the parametric process, we plot in Fig. 2.7 the power evolution, with Raman scattering included, at a sideband frequency detuning of \(\Omega = 20\) THz. In the presence of the Raman effect, one can see that the power conversion of the two sidebands become strongly asymmetric. After the first maximum conversion point, the pump power starts periodically depleting while the Stokes wave grows continuously with a periodic modulation. The period of the pump and sidebands recurrences in Fig. 2.7 are significantly shorter than that without the Raman effect in Fig. 2.6. We note that both with and without Raman scattering, very strong conversion for the pump to the sidebands can occur when the normalized linear mismatch \(\kappa < 1\) (\(\kappa = 0.25\) for Fig. 2.6) and 2.7. We will see how this maximum conversion mismatch facilitates to achieve high conversion efficiency in fiber parametric oscillators in Chapter 5.

### 2.6 Summary

- Optical fibers and the determination of their eigenmodes have been introduced.

- We have given a brief definition of fiber dispersion parameters.

- The origin of fiber nonlinearity and its consequent nonlinear effects (such as the Kerr effect, SPM, XPM, Raman scattering, and FWM) have been presented.

- The generalized nonlinear Schrödinger equation Eq. (2.54) and the coupled-mode equations Eq. (2.57) have been introduced to describe the optical field evolution in optical fibers.

- We have explained the gain characteristics of modulation instability and how this gain is influenced by the effects of pump depletion as well as the Raman effect.
Historically, nonlinear phenomena in optics have mostly been studied using lasers as coherent pump sources. The high spatial coherence of lasers allows them to be focused to small spot sizes and hence achieved a high optical intensity. This aids the observations of nonlinear effects over a short interaction length. As early as the 1960s, soon after the first laser was invented [3], lasers were used for the study of optical second harmonic generation and parametric amplification in nonlinear crystals [4, 92, 93]. In the 1970s, the first series of nonlinear fiber optics experiments were also demonstrated.
In this chapter, we present a simple theory which is capable of predicting the mean gain of an incoherently pumped parametric amplifier. We experimentally verify this simple model, and show that the measured gain spectrum of the incoherently pumped fiber parametric amplifier can be accurately predicted. The parametric gain slope of the incoherently pumped amplifier can be up to seven times higher than that of a coherently pumped parametric amplifier with an equivalent pump power. By further extending our simple theory, we are also able to accurately predict the gain statistics of an amplified coherent seed signal, and provide a more complete explanation of the high power events that arise from incoherently pumped parametric amplification. Lastly, we experimentally measure the gain statistics of a coherent signal after the incoherently pumped parametric amplifier, and verify that they follow our simple theory.
3.1 Theory

3.1.1 Temporally Incoherent Light

We consider linearly polarized thermal light emitted from a large number of uncorrelated atoms. At an arbitrary observation position, the total electric field can be regarded as a sum of the electric fields emitted from individual atoms

$$A(t) = \sum_{j=1}^{\infty} A_j \exp[i(2\pi v_j t + \phi_j)], \quad (3.1)$$

where $A_j$, $v_j$ and $\phi_j$ are the amplitude, the frequency and the phase of the electric field emitted by the $j$-th atom, respectively. One of the most distinct characteristics of incoherent light is that it exhibits totally random amplitude and phase fluctuations as the electric field emitted by each atom is totally uncorrelated. The time scale of the intensity fluctuations is known as coherence time $t_{coh}$ and is related to the 3 dB spectral width $d\Omega$ of the incoherent light source by

$$t_{coh} \approx \frac{2\pi}{d\Omega}. \quad (3.2)$$

As the intensity and phase of the incoherent light fluctuate randomly, the power fluctuations can only be described statistically over a large ensemble. The instantaneous power of the incoherent light can be written as $P(t) = |A(t)|^2$ with the mean power

$$\langle P \rangle = \langle |A(t)|^2 \rangle = \overline{P(t)}. \quad (3.3)$$

The distribution of the amplitude of the electric field $A(t)$ in the temporal domain obeys a Rayleigh probability density function [99]

$$p_A(A) = \frac{A}{\sigma^2} \exp\left(-\frac{A^2}{2\sigma^2}\right), \quad (3.4)$$
where \( \sigma^2 = \langle |A(t)|^2 \rangle / 2 \). By making the transformation of \( P = A^2 \), or \( A = \sqrt{P} \), we can rewrite Eq. (3.4) as

\[
p_P(P) = p_A(A = \sqrt{P})|dA|/dP,
\]

\[
= \frac{\sqrt{P}}{\sigma^2} \exp \left( -\frac{A^2}{2\sigma^2} \right) \frac{1}{2\sqrt{P}},
\]

\[
|dA|/dP,
\]

\[
(3.5)
\]

\[
p_P(P) = \frac{1}{P} \exp \left( -\frac{P}{\langle P \rangle} \right).
\]

\[
(3.6)
\]

Eq. (3.7) shows that the statistics of the intensity fluctuations of an incoherent light source follows an exponential distribution. The only parameter of the exponential distribution is the mean power \( \langle P \rangle \) of the incoherent light source. Note that this probability density function is independent of the spectral width \( d\Omega \) of the light source. The spectral width

\[
d\Omega \text{ (or coherent time } t_{coh} \text{) only affects the time scale of the intensity fluctuations. To illustrate the temporal intensity fluctuations of an incoherent light source, we plot in Fig. 3.1 the temporal intensity of a 10 GHz bandwidth incoherent light source which has a Gaussian spectral profile and a mean power of 1 W.}
\]

Figure 3.1: Temporal intensity of an incoherent light source with a bandwidth of \( d\Omega = 10 \text{ GHz} \) and average power \( \langle P \rangle = 1 \text{ W} \), and its equivalent mean power CW light.
3.1.2 Gain Spectrum of an Incoherently Pumped Parametric Amplifier

We limit our study of incoherently pumped parametric amplifiers to the amplification of a coherent seed signal only. We also assume that the amplifiers approximately satisfy the following three conditions:

- The length of the parametric gain fiber is much shorter than the dispersive walk-off length between the pump and the seed signal $L \ll L_W = t_{coh}(|\beta_2|\Omega)^{-1}$, where $\beta_2$ is the second order dispersion parameter at the pump frequency, and $\Omega$ is the frequency detuning of the seed signal from the pump frequency.

- The length of the parametric gain fiber is much shorter than the dispersion length of the pump $L \ll L_D = (|\beta_2|d\Omega^2)^{-1}$.

- The effect of the finite linewidth of the incoherent pump on the phase matched parametric process can be neglected.

The first condition can be understood as the condition in which the interaction length between the pump and the coherent seed signal is short compared to the temporal walk-off length between the seed and the temporal fluctuations of the pump. Under this condition, we can assume that the incoherent pump field experienced by the coherent seed signal is temporally stationary. The second condition assumes that the incoherent pump intensity fluctuation pattern does not temporally evolve during the propagation, such that the coherent seed signal experiences an undistorted fluctuation pattern. The third condition is required to avoid the parametric gain suppression as a result of the phase mismatch that occurs for a spectrally broad pump [100]. Also, the spectral broadening of the incoherent pump during the propagation is inevitable due to SPM [101, 102]. Thus, a full calculation of this effect is only possible via numerically integration of the GNLSE Eq. (2.54), which we will present in the experimental section. Provided these three conditions are satisfied, we can approximate the mean parametric gain of the coherent seed as

$$\langle G \rangle = \int_0^\infty p_P(P)G(P)dP,$$  \hspace{1cm} (3.8)
where $G(P)$ is the signal gain evaluated from a parametric amplifier pumped by a CW coherent pump (power $P$). $G(P)$ is calculated numerically from the coupled-mode equations Eq. (2.57), and includes the effects of both the Kerr nonlinearity and Raman scattering. This gain calculation needs to include the effect of gain saturation, or the integration of Eq. (3.8) will not converge. From here it is possible to evaluate Eq. (3.8) to calculate the mean parametric gain of a coherent seed signal in a parametric amplifier pumped by a temporally incoherent pump. Any temporal evolution of the pump, walk-off between the signal and the pump fluctuations, or dephasing of the parametric process due to the pump’s finite spectral width, will result in a reduction in the observed gain. In practice, this means that as the linewidth of the incoherent pump increased, the mean parametric gain will decrease. Thus, the maximum gain predicted by Eq. (3.8) is attainable only when the three conditions listed above are satisfied.

To illustrate how significantly the mean parametric gain of a fiber parametric amplifier increases when it is pumped with a temporally incoherent pump, we calculate the parametric gain spectrum by numerically solving Eq. (3.8) with an average pump power of 4 W and a coherent seed signal of 0.1 $\mu$W injected on the anti-Stokes side of the pump (positive detuning), and plot it in Fig. 3.2. The parametric gain medium of the amplifier is a 100 m long dispersion shifted fiber with dispersion parameters of $\beta_2 = -0.10$ ps$^2$/km, $\beta_3 = 0.15$ ps$^3$/km, and $\beta_4 = -7.1 \times 10^{-4}$ ps$^4$/km at the pump wavelength, and a fiber nonlinear coefficient of $\gamma = 2.53$ W$^{-1}$km$^{-1}$. In comparison, the parametric gain spectrum of a coherently pumped amplifier is also calculated with the same seed level and average pump power, and plotted in the same figure. Clearly, it can be seen that the maximum mean gain has been substantially increased from 5 dB (for a coherently pumped amplifier) to 40 dB (for an incoherently pumped amplifier), and the gain region has also been doubled from 0 $\sim$ 4 THz (coherently pumped) to 0 $\sim$ 8 THz (incoherently pumped). It can be readily understood that this dramatic increase of the parametric gain is due to the presence of the high intensity fluctuations in the incoherent pump. For the same reason, the shift of the peak gain is also the result of high intensity fluctuations of the pump that preferentially phase match larger frequency shift seed signals.
Figure 3.2: Mean parametric gain of a coherent seed signal as a function of (positive) seed detuning for incoherently and coherently pumped amplifiers at an average pump power of 4 W. Solid curve (blue) is the theoretical gain obtained using Eq. (3.8), and dashed curve (green) is calculated from the coupled-mode equations Eq. (2.57).

To further illustrate how the gain of the incoherently pumped parametric amplifier depends on the average pump power, we calculate the mean parametric gain as a function of the average pump power using Eq. (3.8) and plot it in Fig. 3.3. The detuning of the coherent seed signal is set to the peak of the gain spectrum evaluated at each average pump power, and the seed level is again 0.1 µW. Also plotted is the coherent parametric gain (dashed curve) pumped with an equivalent average pump power. At low pump power, the incoherently pumped amplifier has a parametric gain slope of 15.4 dB/W in the region indicated by blue dots. This is seven times higher than the coherently pumped gain slope which is only 2.2 dB/W indicated by green dots. In the region $\langle P \rangle > 7.5$ W, the incoherently pumped parametric gain starts saturating owing
to the effect of gain saturation of parametric process discussed in Section 2.5. Thus, to achieve a high gain slope, we need to operate our incoherently pumped parametric amplifier in the low pump power region where the gain saturation is insignificant.

![Figure 3.3: Mean parametric gain of a coherent signal as a function of average pump power for incoherently and coherently pumped amplifiers. Solid curve (blue) is the theoretical gain obtained using Eq. (3.8), and dashed curve (green) is calculated from the coupled-mode equations Eq. (2.57).](image)

**3.1.3 Gain Statistics of an Incoherently Pumped Parametric Amplifier**

As the incoherent pump of the parametric amplifier possesses random intensity fluctuations, the coherent seed signal after the amplification inevitably inherits the same characteristics from the pump, and loses its fidelity during the amplification process. In order to quantitatively describe the statistics of the amplified seed signal, here we derive an expression for the probability distribution of the parametric gain of the coherent seed signal...
signal. Under the same assumptions presented in the previous subsection, and with the statistics of the incoherent pump \( p_p(P) \) given by Eq. (3.7), the probability distribution for the parametric gain can be determined by making a simple variable transformation as

\[
p_G(g) = \int_0^{\infty} \delta[g - G(P)] p_p(P) dP,
\]

(3.9)

where

\[
\delta[g - G(P)] = \sum_j \frac{\delta[P_j - G^{-1}(g)]}{|dG/dP|_{P_j}}.
\]

(3.10)

Here \( G(P) \) is again the signal gain of a coherently pumped parametric amplifier calculated from the CW coupled-mode equations Eq. (2.57), and \( P_j \) is the \( j \)-th zero-crossing of the function \( g - G(P) \). The integration of Eq. (3.9) can be further simplified to

\[
p_G(g) = \sum_j p_p(P_j) \left| \frac{dG}{dP} \right|_{P_j}^{-1}.
\]

(3.11)

Using this equation, the probability distribution of parametric gain \( g \) experienced by a coherent seed signal can be directly calculated. To illustrate the statistics of the amplified signal, we calculate the gain distributions \( p_G(g) \) for the same amplifier in the previous subsection using Eq. (3.11) for several positive (anti-Stokes) detunings of 1, 2, 4, and 6 THz with respect to the pump frequency with an average pump power of \( \langle P \rangle = 4 \) W and a seed level of 100 \( \mu \)W, and plot them in Fig. 3.4. These curves clearly show how the gain statistics depend on the frequency detuning of the coherent seed signal. As the frequency detuning of the seed signal increases, the distribution develops an increasingly longer tail structure with correspondingly higher probability of high gain events. The sharp features seen in the probability distributions at 4 and 6 THz detunings are due to the complex shape of the \( G(P) \) curve at these detunings. The physical reason for such gain behaviour is that the phase matched condition is power-dependent. A seed with a
Figure 3.4: Theoretical incoherently pumped parametric gain distribution at signal detunings of 1, 2, 4 and 6 THz calculated from Eq. (3.9) with an average power $\langle P \rangle = 4$ W and a seed level of 100 $\mu$W. The inset shows the corresponding mean parametric gain at these detunings.

smaller frequency detuning is preferential phase matched by the low power events, and this results in the increase of the probability in the low gain region; while a seed with a larger detuning frequency is phase matched by high power events, and this results in the corresponding increase of the probability in high gain region and results in a long tailed structure. In Table. 3.1, we tabulate the probability of gain fluctuations in excess of 30 and 40 dB as a function of seed frequency detuning. It shows that at 1 THz detuning, there is a negligible chance ($\sim 10^{-11}$) of the coherent seed experiencing a gain fluctuation in excess of 40 dB, whereas at 4 THz detuning this probability has increased to 0.4%. The gain distribution at 6 THz detuning illustrates the amplification of the coherent seed by the high power pump fluctuations. 98% of the time the power fluctuations of the incoherent pump are not sufficiently strong to phase match the parametric process at this
detuning, and the signal experiences a net loss as a result of stimulated Raman scattering; nonetheless 0.04% of the time, the pump fluctuations are sufficiently high-powered to result in a net gain in excess of 40 dB, 1000 times higher than the mean amplified signal power and 10,000 times higher than the medium amplified signal power. We note that while the curves presented here consider only the signal injected on the high frequency side (anti-Stokes) of the pump, an analysis of the gain distributions of signals injected on the low frequency side (Stokes) shows qualitatively similar behaviour.

<table>
<thead>
<tr>
<th>Seed detuning [THz]</th>
<th>$p(g &gt; 30 , \text{dB})$</th>
<th>$p(g &gt; 40 , \text{dB})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1 \times 10^{-6}$</td>
<td>$1 \times 10^{-11}$</td>
</tr>
<tr>
<td>2</td>
<td>$9 \times 10^{-4}$</td>
<td>$1 \times 10^{-5}$</td>
</tr>
<tr>
<td>4</td>
<td>$1.5 \times 10^{-2}$</td>
<td>$4 \times 10^{-3}$</td>
</tr>
<tr>
<td>6</td>
<td>$8 \times 10^{-4}$</td>
<td>$4 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

Table 3.1: Probability of gain fluctuations in excess of 30 dB and 40 dB as a function of seed detuning.

### 3.2 Experiment

#### 3.2.1 Temporally Incoherent Light Source

To experimentally generate a temporally incoherent source, we use the setup indicated by the green dashed box in Fig 3.5. The amplified spontaneous emission (ASE) light emitted from a 500 mW C-band Erbium doped fiber amplifier (EDFA) is collimated into free space and filtered by a set of diffraction gratings with a groove density of 900 line/mm. As the diffraction grating is polarization sensitive, the filtered ASE light becomes horizontally polarized, with an approximately Gaussian spectral shape. To experimentally confirm that the instantaneous power statistics of our incoherent light follows an exponential distribution as predicted by Eq. (3.7), the 3 dB spectral width of the filtered ASE light is adjusted to 0.1 nm with a center wavelength at 1555nm. This corresponds to 12 GHz bandwidth, which is detectable by a 50 GHz photodiode/oscilloscope system. In order to increase the power of the incoherent light to meet the requirement for our
parametric amplifier, the 12 GHz incoherent light is amplitude-modulated into 10 ns square pulses of 33% duty cycle from a pattern generator (PG) and pre-amplified by a lower power EDFA. A 5 W high power EDFA is then used to amplified our final incoherent pump to a watt-level. After each EDFA the pump is filtered by a 1 nm band pass filter (BPF) to remove the unwanted ASE signal. The instantaneous power of the final amplified incoherent pump is measured with the 50 GHz oscilloscope. In Fig. 3.6, a histogram of the measured instantaneous power is plotted on top of the theoretical probability distribution of $P/\langle P \rangle$. This measured histogram consists of 20,000 sampled points. Clearly, the measured statistics of the instantaneous power shows no significant divergence from the theoretical statistics of a polarized thermal source given by Eq. (3.7). This demonstrates that to a good approximation our filtered ASE light source is temporally incoherent. Using this pump source, we are able to measure the mean gain and the gain statistics of an incoherently pumped parametric amplifier.
Figure 3.6: Experimentally measured histogram of the instantaneous power fluctuations of the 12 GHz incoherent pump as a function of $P/\langle P \rangle$ (circles), superimposed with the theoretical probability distribution of (solid line). The relative frequency of each data bin is defined as the number of events in that bin divided by the total number of events.

### 3.2.2 Experimental Setup for Gain Spectrum Measurement

With the temporally incoherent pump source described in the previous subsection, we build our incoherently pumped parametric amplifier using telecommunication components as shown in Fig. 3.5. The parametric gain medium is a 100 m long DSF with the same parameters used in the theoretical section. The incoherent pump is launched into the DSF from the 95% input port of a 95/5 fused fiber coupler. A coherent seed signal from an external cavity laser (ECL) is injected into the DSF from the 5% input port of the fiber coupler. Polarization controllers (PC) are used to ensure that the pump and the seed propagate co-linearly in the DSF. Output spectra are measured by an optical spectrum analyzer (OSA). The mean parametric gain is measured from the seed level.
at the output of the amplifier with and without the incoherent pump. By tuning the frequency of the ECL, we can obtain the parametric gain spectrum of the incoherently pumped amplifier. The mean gain as a function of the average pump power can also be measured by adjusting the output power of the EDFA with the seed signal set to the peak of the gain spectrum for each pump power.

### 3.2.3 Results of Experimental Gain Spectrum

Fig. 3.7 shows the experimentally measured parametric gain spectra of the incoherently pumped amplifier at an average pump power of 4 W. The power level of the seed signal is 0.1 µW injected on the high frequency (anti-Stokes) side of the pump. We first measure the gain spectrum of an amplifier pumped by a narrow linewidth 12 GHz (0.1 nm) pump in which the three conditions given in Section 3.1.2 are reasonably well satisfied. The measured gain spectrum of the 12 GHz linewidth amplifier (solid blue circles) follows closely the curve of our simple model (dashed curve) calculated using Eq. (3.11). The measured gain spectrum of the incoherently pumped amplifier is significantly higher compared to that of the coherently pumped amplifier (theoretically calculated, dash-dot curve). The peak of the experimentally measured gain lies $\sim 7$ dB below the maximum of the simple model prediction. We attribute the difference to the finite linewidth of the pump. To demonstrate the influence of the linewidth of the incoherent pump on the gain spectrum, the gain spectrum of a 100 GHz (0.8 nm) linewidth pump amplifier (solid red diamonds) is also measured and plotted in Fig. 3.7. The reduction of the gain is found to be much stronger in the gain spectrum of the 100 GHz linewidth pump, with the peak $\sim 20$ dB below the theoretical maximum. To fully model these effects, numerical simulations are performed using the GNLSE with 400 ensembles in a temporal window of 1 ns, and plotted as open circles and open diamonds in Fig. 3.7. Good agreement between the experimental measurements and numerical simulation can be clearly seen.

The small discrepancy is attributed to the depolarization between the pump and the seed signal that occurs during propagation along the 100 m DSF.
Figure 3.7: Measured mean parametric gain spectra. Green dashed curve is the theoretical gain obtained using Eq. (3.8). The open circles (open diamonds) are the full GNLSE simulation and the solid circles (solid diamonds) are the experimentally measured gain spectra. Dash-dot curve is the gain spectrum of coherently pump parametric amplifier.

In Fig. 3.8 we plot the measured incoherently pumped mean parametric gain as a function of average pump power at a pump linewidth of 12 GHz (blue solid circles). The seed signal is again 0.1 µW injected at the peak of the gain spectrum at each average pump power. Again, the experimental results are in very good agreement with our simple theory (blue solid curve). Also plotted in Fig. 3.8 are the experimentally measured (green open circles) and theoretical (green dash curve) gain curve of the coherently pumped parametric amplifier. The superior mean gain of the incoherently pumped parametric amplifier is clearly notable when compared with the coherently pumped amplifier. The measured incoherently pumped gain slope of 14.3 dB/W in the low power region is seven times greater than the coherently pumped gain slope of 1.9 dB/W.
This clearly demonstrates that incoherently pumped parametric amplifiers can be built to operate with very high mean gain at low pump power. However, the drawback of the incoherently pumped amplifier is that the amplified coherent seed signal inevitably loses its fidelity, and the output signal will have a probability distribution which strongly depends on the frequency detuning of the seed signal. We will demonstrate this gain probability distribution in the next subsection.

Figure 3.8: Measured mean parametric gain as a function of pump power. Solid curve is the theoretical gain of the incoherently pumped amplifier calculated using the simple model. Dashed curve is the theoretical gain of the CW pumped amplifier calculated using Eq. (2.72), solid circles and solid diamonds are the experimentally measured gain of the amplifiers with the pump linewidth of 12 GHz and 100 GHz, respectively. Open circles are the experimentally measured gain of the CW pumped amplifier.
3.2.4 Experimental Setup for Gain Statistics Measurement

We measure the gain statistics of the incoherently pumped parametric amplifier with a 12 GHz linewidth pump using the same experimental setup as shown in Fig. 3.5 with the components in red dash-dot box included. The average incoherent pump pulse power is set to 4 W, and the power level of the coherent seed signal is set to 100 µW injected on the high frequency side (anti-Stokes) of the pump. The amplified coherent signal at the output of the amplifier is separated from the pump and the idler using another free space diffraction grating, and fed into the 50 GHz photodiode/oscilloscope system. As the dynamic range of the oscilloscope is only ~ 20 dB, a variable attenuator (VA) is placed before the photodiode, and measurements are repeated at a range of attenuations to increase the dynamic range. Each measurement presented consists of over 500,000 sampled points from which it is sufficient to extract the rare events with probability as low as 0.0002%. We then calculate the parametric gain of the amplified coherent seed signal by dividing the real time signal level at the output by the input coherent seed level.

3.2.5 Result of Experimental Gain Statistics

The experimentally measured parametric gain distributions at 1, 2, 4 and 6 THz detunings are plotted in Fig. 3.9. The experimental measurements of the gain distributions are in very good agreement with the theoretical predictions. As predicted, the general trend of these plots is that as the frequency detuning of the seed signal increases, increasingly high gain intensity fluctuations are possible. This effect can clearly be witnessed in the long tailed structure of 6 THz detuning distribution. The absence of the sharp features of the theoretical probability distribution in the experimentally measured distributions are due to the coarseness of the frequency binning used (~ 2 dB/bin).
3.3 Summary and Discussion

In this chapter, we have presented a study of the gain characteristics of a fiber optical parametric amplifier pumped by a temporally incoherent pump. A simple theory has been introduced to directly calculate the mean parametric gain for a coherent seed signal. We have experimentally demonstrated that this simple theory is in excellent agreement with the measured incoherently pumped parametric gain. The parametric
gain slope of the incoherently pumped amplifier is seven times higher than that of a coherently pumped amplifier with an equivalent average power in the non-saturated gain region. The remarkable increase of parametric gain is the result of high power intensity fluctuations of the incoherent pump. The discrepancy between our simple theory and the experimental results is attributed to the finite linewidth of the incoherent pump used. This has been experimentally verified by replacing the 12 GHz linewidth pump by a 100 GHz linewidth pump. The experimental gain measurement of the amplifier pumped by a broader spectral width 100 GHz pump demonstrates a decrease in mean parametric gain.

We have also shown that the probability distribution of the parametric gain of a coherent input signal can be evaluated by making a transformation of our simple theory. Using the transformation of our simple model, we predict that the gain distribution of a large frequency detuning seed signal exhibits a long tailed structure. This structure is due to the fact that the seed signals at the large frequency detuning are preferentially phase matched with the high power events. We have experimentally demonstrated that the measured gain statistics are in good agreement with our theoretical predictions by directly observing the amplified events. By varying the frequency detuning of the seed signal, we can precisely control the statistics of these gain fluctuations and this control may find applications in the design of future parametric devices based on incoherently pumping.

Applications that require high parametric gain, but not necessarily high signal fidelity, could benefit from the use of an incoherently pumped parametric amplifier. The amplified signal of an incoherently pumped amplifier could also serve as a useful source with an adjustable quantity of intensity noise. For example, the ability to generate a signal with a known, adjustable intensity noise distribution could be used for testing optical communication systems. A deeper understanding of the statistics of incoherently pumped amplification process, could also aid the future investigation of the nature of rogue waves in optical fibers [103].
Optical frequency comb sources have found a wide range of applications, for instance, arbitrary-wave form generation, telecommunication carriers and high precision measurements [104–107]. The generation of frequency combs in nonlinear systems are traditionally explained in terms of either discrete four-wave mixing of monochromatic waves, or the evolution of a single ultrafast pulse. Examples such as the octave-spanning frequency comb generated in microresonators are described in term of discrete four-wave mixing [108], whereas supercontinuum generation in optical fibers is described in term of the evolution of a single ultrafast pulse with a continuous spectral
envelope [109]. Another paradigm of nonlinear optics is the emission of dispersive waves (also called Cherenkov radiation) by optical solitons in optical fibers [70, 110], the contribution of which is known to be central for the extension of supercontinua towards the blue region [52, 109]. It is generally held that this process occurs because the radiation spectrum in the normal dispersion regime is in resonance with the soliton (i.e. the radiation evolves in phase with the soliton) [52, 70]. Yet, this explanation is linear in nature, and the origin of the nonlinear polarization oscillating at the dispersive wave frequency and the associated energy conservation laws are still unspecified.

In this chapter, we identify phase matched cascaded four-wave mixing as the nonlinear optical mechanism that allows dispersive waves and other symmetry breaking phenomena to occur in the presence of higher order dispersion. A unique feature of cascaded wave mixing is that sequential frequency generation can mimic the effect of a higher order nonlinear susceptibility [111, 112]. Although this feature has been known since the 1970s, it has only previously been demonstrated in bulk crystals and gas-phase systems with a small number of harmonic components generated [113–115]. We here show for the first time to our knowledge that arbitrarily high $\chi^{(2n+1)}$ nonlinear mixing processes can be directly mimicked by cascading $n \chi^{(3)}$ processes in conventional single mode optical fibers. The identified process relies on the higher order dispersion of optical fibers, and is a driving mechanism in nonlinear spectral symmetry breaking owing to the selective amplification of a single higher order sideband as verified by experiments utilizing a bichromatic excitation field. We highlight the intrinsic connection between this process and the dispersive wave emission mechanism both theoretically and experimentally, and present simple energy conservation equations to describe this process. The cascaded formalism also allows us to identify a reverse process whereby a nonlinear wave propagating in the normal dispersion regime excites a dispersive wave in the anomalous dispersion regime. We demonstrate this reverse process both numerically and experimentally.

Later in this chapter, we further extend our study of cascaded four-wave mixing to a more general cascaded process in which an initial weak probe signal can be translated to
a different frequency. This process is analog to the nonlinear Bragg scattering parametric process whereby a weak probe signal is efficiently switched to an idler that satisfies the fundamental phase matched condition. In a cascaded Bragg scattering, this weak probe signal can be even translated to a higher order idler via cascading \( n \) elementary Bragg scattering processes. We derive the analytical phase matched condition of this cascaded process and experimentally verify it. We show that this cascaded Bragg scattering process can provide us with a more fundamental explanation for the generation of the new frequency waves often observed during a soliton-linear wave interaction [39,71,72,116,117].

4.1 Cascaded Four-Wave Mixing

4.1.1 Theory

We consider the propagation of two strong CW pump beams \( E_{-p} \) and \( E_{+p} \) at two different frequencies \( \omega_p < \omega_{+p} \), with a center frequency at \( \omega_c = (\omega_{-p} + \omega_{+p})/2 \) in a single mode fiber. The frequency separation of the two pump beams is \( \Delta = \omega_{+p} - \omega_{-p} > 0 \). As shown in Fig. 4.1, through the \( \chi^{(3)} \) nonlinear interaction, the fields will drive the generation of

![Figure 4.1: Schematic of cascaded four-wave mixing frequency arrangement.](image-url)
new spectral components with frequencies $\omega_{-1} = \omega_p - \Delta$ and $\omega_{+1} = \omega_p + \Delta$ through the nonlinear mixing terms $E_{-1} \propto E_p^2 E_{+p}^*$ and $E_{+1} \propto E_p^2 E_{-p}^*$, respectively. Subsequently, the generated frequency components may interact with each other and/or with the pumps to generate new frequency components through a cascade of $\chi^{(3)}$ processes. The frequency of the $n$-th order component of the cascade is $\omega_{\pm n} = \omega_{\pm p} \pm n\Delta$, where $n$ is a positive integer.

For simplicity, we initially neglect Raman scattering in the analysis below, and restrict our consideration to the case of six interacting waves–two pumps, and two pairs of sidebands where the second pair arises from a cascaded process involving the pumps and the first pair of sidebands. Note that the relevant results are easy to generalize to arbitrary orders and will be presented at the end. We begin by assuming the pumps remain undepleted. In this case, the complex amplitudes of the pumps evolve according to $A_{\pm p}(z) = A_{\pm p}(0) \exp[i\gamma(P_{\pm p} + 2P_{\mp p})z]$. Under the assumption that the power of the two pumps is significantly higher than that of the sidebands, we can assume that the evolution of the field at $\omega_{-1} = 2\omega_p - \omega_{+p}$ and $\omega_{+1} = 2\omega_p - \omega_{-p}$ is dominated by the terms $A_{-p}^2 A_{+p}^*$ and $A_{+p}^2 A_{-p}^*$, respectively. By substituting these frequency components into the coupled-mode equations Eq. (2.57), the evolution of $A_{-1}$ and $A_{+1}$ can be written as

$$\frac{\partial A_{-1}}{\partial z} = i\gamma (2P_{-p} + 2P_{+p})A_{-1} + A_{-p}^2(z)A_{+p}^*(z) \exp(i\Delta k_{L-1} z),$$

$$\frac{\partial A_{+1}}{\partial z} = i\gamma (2P_{+p} + 2P_{-p})A_{+1} + A_{+p}^2(z)A_{-p}^*(z) \exp(i\Delta k_{L+1} z),$$

where $\Delta k_{L,\pm 1} = 2\beta(\omega_{\pm 2p}) - \beta(\omega_{\pm p}) - \beta(\omega_{\pm 1})$ is the linear phase mismatch. Writing all of the sidebands in the form of $A_j = B_j \exp[i\gamma(2P_{-p} + 2P_{+p})z]$, we simplify Eq. (4.1) and (4.2) into

$$\frac{\partial B_{-1}}{\partial z} = i\gamma A_{-p}^2(0)A_{+p}^*(0) \exp(i\kappa_{-1} z),$$

$$\frac{\partial B_{+1}}{\partial z} = i\gamma A_{+p}^2(0)A_{-p}^*(0) \exp(i\kappa_{+1} z),$$

58
where \( \kappa_{\pm 1} = 2\beta(\omega_{\pm p}) - \beta(\omega_{\mp p}) - \beta(\omega_{\pm 1}) + \gamma(P_{\mp p} - 2P_{\pm p}) \). Eq. (4.3) and (4.4) can be readily integrated with initial conditions \( B_{\pm 1}(0) = 0 \) to yield

\[
B_{-1}(z) = \gamma A_{-p}^2(0)A_{+p}^*(0) \frac{\exp(ik_{-1}z) - 1}{k_{-1}},
\]

\[
B_{+1}(z) = \gamma A_{+p}^2(0)A_{-p}^*(0) \frac{\exp(ik_{+1}z) - 1}{k_{+1}}.
\]

Clearly the amplitudes \( B_{-1} \) and \( B_{+1} \) are oscillatory functions with a period determined by the phase mismatches \( \kappa_{-1} \) and \( \kappa_{+1} \). In the next step, these generated frequency components (whether oscillatory or not) will mix to generate new frequency components at \( \omega_{-2} \) and \( \omega_{+2} \). There exists several routes to generate the new frequency components. For instance, \( \omega_{-2} = \omega_{-p} + \omega_{-1} - \omega_{+p} = \omega_{+p} + \omega_{-1} - \omega_{+1} = 2\omega_{-p} - \omega_{+1} = 2\omega_{-1} - \omega_{+p} \). Here, we expect the dominant terms to be those that involve two pumps and neglect the rest of the terms. We restrict our consideration to the processes \( \omega_{-2} = \omega_{-p} + \omega_{-1} - \omega_{+p} \) and \( \omega_{+2} = \omega_{+p} + \omega_{+1} - \omega_{-p} \). Similarly, the evolution of \( A_{-2} \) and \( A_{+2} \) can also be written as

\[
\frac{\partial B_{-2}}{\partial z} = 2i\gamma A_{-p}(0)A_{+p}^*(0)B_{-1}(z) \exp(ik_{-2}z),
\]

\[
\frac{\partial B_{+2}}{\partial z} = 2i\gamma A_{+p}(0)A_{-p}^*(0)B_{+1}(z) \exp(ik_{+2}z),
\]

where \( \kappa_{\pm 2} = \beta(\omega_{\pm p}) - \beta(\omega_{\mp p}) + \beta(\omega_{\pm 1}) - \beta(\omega_{\pm 2}) + \gamma(P_{\mp p} - P_{\pm p}) \). The extra factor of 2 comes from the degeneracy of \( E_{\pm 2} \propto E_{\pm p}E_{\pm 1}E_{\mp p}^* \) and \( E_{\pm 2} \propto E_{\pm 1}E_{\pm p}E_{\mp p}^* \). By substituting Eq. (4.5) and (4.6) into Eq. (4.7) and Eq. (4.8), the evolution equations for \( B_{-2} \) and \( B_{+2} \) read

\[
\frac{\partial B_{-2}}{\partial z} = 2i\gamma^2 A_{-p}^3(0)A_{+p}^2(0) \frac{\exp(ik_{-1}z) - 1}{k_{-1}} \exp(ik_{-2}z),
\]

\[
\frac{\partial B_{+2}}{\partial z} = 2i\gamma^2 A_{+p}^3(0)A_{-p}^2(0) \frac{\exp(ik_{+1}z) - 1}{k_{+1}} \exp(ik_{+2}z).
\]
Direct integration with initial conditions $B_{±2}(0) = 0$ yields

$$B_{-2}(z) = C_{-2} \left[ \exp\left[ i(\kappa_{-1} + \kappa_{-2})z \right] - 1 \right] \frac{1 - \exp(i\kappa_{-2}z)}{\kappa_{-1}(\kappa_{-1} + \kappa_{-2})}, \quad (4.11)$$

$$B_{+2}(z) = C_{+2} \left[ \exp\left[ i(\kappa_{+1} + \kappa_{+2})z \right] - 1 \right] \frac{1 - \exp(i\kappa_{+2}z)}{\kappa_{+1}(\kappa_{+1} + \kappa_{+2})}, \quad (4.12)$$

where the constants $C_{-2} = 2\gamma^2A_{-p}^3(0)A_{p}^2(0)$ and $C_{+2} = 2\gamma^2A_{+p}^3(0)A_{-p}^2(0)$. There are three conditions for which $|B_{-2}|$ ($|B_{+2}|$) can grow coherently: (i) $\kappa_{-1} = 0$ ($\kappa_{+1} = 0$), (ii) $\kappa_{-2} = 0$ ($\kappa_{+2} = 0$), (iii) $\kappa_{-1} + \kappa_{-2} = 0$ ($\kappa_{+1} + \kappa_{+2} = 0$). The first two conditions correspond to the phase matching of the elementary processes whereas the third one is associated with the cascaded process. The third condition can be expanded as

$$\kappa_{-1} + \kappa_{-2} = 3\beta(\omega_{-p}) - 2\beta(\omega_{+p}) - \beta(\omega_{-2}) - 3\gamma P_{-p} + 2\gamma P_{+p} = 0, \quad (4.13)$$

$$\kappa_{+1} + \kappa_{+2} = 3\beta(\omega_{+p}) - 2\beta(\omega_{-p}) - \beta(\omega_{+2}) - 3\gamma P_{+p} + 2\gamma P_{-p} = 0, \quad (4.14)$$

from which it is clear that the cascaded phase matching involves only the pump frequencies and the sideband frequency under study, but not the intermediate four-wave mixing frequencies. More generally, assuming that the subsequent sidebands are always generated through interactions that involve only one photon from the previous sidebands and two pump photons, the cascaded four-wave mixing phase matched condition for an arbitrary $n$-th order sideband of a frequency

$$\omega_{±n} = (n + 1)\omega_{±p} - n\omega_{±p} = \omega_{±p} ± n\Delta \quad (4.15)$$

can be written as

$$\sum_{j=1}^{n} \kappa_{j} = n\left[ \beta(\omega_{±p}) - \beta(\omega_{±p}) - \gamma(P_{±p} - P_{±p}) \right] - \left[ \beta(\omega_{±n}) + \gamma P_{±p} - \beta(\omega_{±p}) \right] = 0. \quad (4.16)$$
Thus, when the sum of the mismatches of the \( n \) elementary four-wave mixing processes is zero, the \( n \)-th order sideband will be phase matched and experience amplification.

To further explore the cascaded phase matched condition, we neglect the nonlinear terms of Eq. (4.16) which are typically smaller than the linear terms. By expanding the propagation constant \( \beta(\omega) \) with respect to the center frequency \( \omega_c \) of the two pumps up to third order in dispersion, from Eq. (4.16) we obtain the frequency separation of two pumps required to phase match the \( n \)-th sideband at frequency \( \omega_n \):

\[
\Delta = \left| \frac{3\beta_2}{\beta_3(n + 1/2)} \right|.
\] (4.17)

The fact that in conventional optical fibers \( \beta_3 > 0 \) suggests that phase matching an arbitrary order sideband is possible as long as the two pumps lie at the same side of the zero dispersion wavelength (ZDW). With the frequency separation \( \Delta \) given by Eq. (4.17), the frequency of the phase matched sideband \( \omega_{\pm n} \) becomes

\[
\omega_{\pm n} = \omega_c - \frac{3\beta_2}{\beta_3}.
\] (4.18)

We note that this frequency is identical to the frequency of the dispersive wave emitted by a soliton at frequency \( \omega_c \) [70, 110]. Furthermore, by expanding the second order dispersion parameter \( \beta_2 \) with respect to the ZDW as such \( \beta_2(\omega_c) = (\omega_c - \omega_{\text{ZDW}})\beta_3 \) where \( \omega_{\text{ZDW}} \) is the frequency of the ZDW, Eq. (4.18) can be reduced to

\[
\omega_{\pm n} - \omega_c = -3\delta\omega,
\] (4.19)

where \( \delta\omega = \omega_c - \omega_{\text{ZDW}} \) is the frequency detuning of the center of two pumps from the ZDW. Eq. (4.19) implies that the phase matched frequency of an arbitrary order cascaded sideband always has a frequency detuning from the center frequency of two pumps that is three times the detuning from \( \omega_c \) to the ZDW.
4.1.2 Numerical Modeling

To verify the amplification of the higher order sidebands in a four-wave mixing cascade, we perform numerical simulations based on the GNLSE given by Eq. (2.54). We consider two 20 W CW pumps propagating in an 80 m long dispersion shifted fiber (DSF) with a ZDW of 1553.8 nm, dispersion parameters $\beta_3 = 0.16 \text{ps}^3/\text{km}$, $\beta_4 = -7.0 \times 10^{-4} \text{ps}^4/\text{km}$ at the ZDW, and a nonlinear coefficient $\gamma = 2.53 \text{W}^{-1}\text{km}^{-1}$. We initially investigate the amplification of the second order ($n = \pm 2$) cascaded sidebands in the anomalous and normal dispersion regimes. For simplicity, we also neglect the Raman effect. In order to fulfill the higher order phase matched condition, we obtain the frequency separation $\Delta$ of the two pumps by solving the Eq. (4.16) with the cascaded order $n = \pm 2$ and the center frequency $\omega_c$. We first plot in Fig. 4.2 (a) and (b) the output spectra for cases

![Figure 4.2: Simulations of phase matching at the $n = 2$ and $n = -2$ order sidebands in an 80 m DSF input (red circles) and output (blue circles) for bichromatic pumping in the (a) anomalous and (b) normal dispersion regimes. Dashed line indicates the ZDW.](image)

where Eq. (4.16) is perfectly fulfilled for the second order high ($n = 2$) and low ($n = -2$) frequency sidebands, respectively. In Fig. 4.2 (a) the center frequency $\omega_c$ is chosen to lie in the anomalous dispersion regime with $\omega_c/2\pi = 188.5 \text{ THz (1591.5 nm)}$ and $\Delta/2\pi = 5.4 \text{ THz}$, whilst the simulation shown in Fig. 4.2 (b) corresponds to the case where
both pumps are in the normal dispersion regime with $\omega_c/2\pi = 196.6$ THz (1525.9 nm) and $\Delta/2\pi = 4.3$ THz. In both cases the power spectra at the output of the fiber exhibit significant spectral asymmetry with the dominant sideband of the four-wave mixing cascade precisely the second order phase matched component $\omega_2$ [Fig. 4.2 (a)] and $\omega_{-2}$ [Fig. 4.2 (b)].

![Figure 4.3](image)

Figure 4.3: Simulated power evolution along DSF distance of (a) the first and second order sidebands and (b) the two pumps in the anomalous dispersion pumping.

To get more insight into the evolution of the spectral components for the $n = 2$ order, we plot the power of the sidebands ($\omega_{+1}, \omega_{+2}$) and pumps ($\omega_{-p}, \omega_{+p}$) as a function of the fiber distance in Fig. 4.3 (a) and (b), respectively. From the evolution of the sidebands' power, we note that amplitude of the first order sideband $\omega_{+1}$ remains small throughout the entire fiber length as it is not phase matched, and displays an oscillation with
a coherence length of $L_{coh}^{(1)} = \frac{2\pi}{\kappa_{+1}} \approx 3.1 \text{ m}$, where $\kappa_{+1}$ is defined as in the previous subsection. Meanwhile, the second order sideband $\omega_{+2}$ undergoes a coherent amplification with a modulation of $L_{coh}^{(1)}$ superimposed. Note that although the underlying process is perfectly phase matched at each propagation length, the driving four-wave mixing polarization term is proportional to the amplitude of the first order sideband $\omega_{+1}$ and consequently oscillates with the same coherence length. This is the origin of the modulation observed in second order sideband. Throughout the propagation, the spectral amplitudes of all other cascaded components are too small to be presented on the scale in Fig. 4.3 (a). In addition to the amplification of the phase matched sideband, the low frequency pump is also amplified as shown in Fig. 4.3 (b). This exchange of pump powers highlights an equivalent $\chi^{(5)}$ interaction whereby the annihilation of three $\omega_{+p}$ pump photons is associated with the creation of a single sideband photon at $\omega_{+2}$ and also two $\omega_{-p}$ pump photons, as ruled by energy conservation $3\omega_{+p} = \omega_{+2} + 2\omega_{-p}$.

Figure 4.4: Schematic of $2n + 2$ photons interaction via cascaded four-wave mixing.

Up to now, we have only considered the low order cascades ($n = \pm 2$). We would like to stress that the above analysis can be extended for an arbitrary high order of cascaded four-wave mixing. The schematic in Fig. 4.4 illustrates how by cascading $n \chi^{(3)}$ elementary four-wave mixing processes, an equivalent $\chi^{(2n+1)}$ higher order process can be directly mimicked. This process requires both pumps to be in the anomalous dispersion regime (or normal dispersion regime). The photon energy of the $n$-th order cascade can be simply calculated using Eq. (4.15). To illustrate this generality, we simulate the
excitation of a higher order cascaded amplification by adjusting the pump frequency detuning as described in the theory presented in the previous subsection.

Fig. 4.5 (a) shows a particular case where a denser frequency comb is generated by two closely spaced pumps with a frequency separation of $\Delta/2\pi = 0.87$ THz in the anomalous dispersion regime (at the same center frequency $\omega_c = 188.5$ THz). The pump power is again 20 W for each pump. With these parameters, the phase matched condition is now fulfilled for the $n = 15$ order sideband according to Eq. (4.16), and the amplification of the sideband at $\omega_{s,15}$ can be clearly seen. Fig. 4.5 (c) shows the temporal evolution of the sinusoidal modulated pulse train which originates from the beating of two pumps, and subsequently evolves into a frequency comb through the cascaded process. As these pulses propagate in the anomalous dispersion regime, the individual subpulse undergoes periodic stages of compression and expansion due to soliton breathing [21]. The spectral asymmetry of the frequency comb correspondingly induces a temporal drift of the sinusoidal modulated pulse train. The frequency shift of the phase matched sideband in Fig. 4.5 (a) is found at the frequency 13.5 THz, and is in good agreement with the frequency shift predicted by Eq. (4.18), which is also the frequency shift of the dispersive wave.

To further confirm that the frequency shift of cascaded phase matched sideband is the frequency shift of the dispersive wave, we simulate and plot in Fig. 4.5 (b) and (d), respectively, the output spectrum (red) and temporal evolution of a single cycle pulse of the sinusoidal modulated pulse train with the same peak power $P = 80$ W (soliton order of $N = 1.87$). Clearly, the continuous spectrum after the propagation follows precisely the envelope of the discrete frequency comb, and the dispersive wave emitted by the soliton exactly coincides with the phase matched sideband of the discrete frequency comb. These results provide us a more natural interpretation of the emission of a dispersive wave by a soliton perturbed by higher order dispersion. The dispersive wave emitted by soliton-like pulses perturbed by higher order dispersion arises from the cascaded four-wave mixing between all the high and low spectral components of the soliton which-in pairs about the central frequency $\omega_c$-contribute to pump the phase
matched process. This is consistent with Eq. (4.18) where the frequency $\omega_{\pm n}$ of the phase matched sideband is independent of the frequency separation $\Delta$ of the pump pair, and depends only on the detuning from the center frequency $\omega_c$. We can now also explain another intrinsic feature of dispersive wave emission when pumping in the
anomalous dispersion regime known as spectral recoil, whereby the center frequency of the pump spectrum is spectrally shifted in the direction opposite to that of the dispersive wave \([70, 110, 118]\). It manifests as that a propagating soliton loses energy by emitting the dispersive waves into the normal dispersion regime at the same time the frequency center of the soliton is recoiled further into the anomalous dispersion regime. By recalling from Fig. 4.3 (b), the amplification of the phase matched sideband is associated with the simultaneous amplification (attenuation) of the pump which is spectrally further from (closer to) the phase match sideband. This effectively leads to a corresponding shift of the center frequency of the pump waves.

4.2 Observation of Cascaded Four-Wave Mixing

4.2.1 Experimental Setup

We now experimentally study cascaded four-wave mixing in a conventional silica single mode fiber using a bichromatic quasi-CW pump as well as a pulsed pump. We first investigate the bichromatic quasi-CW pumping cascade. Fig. 4.6 shows the schematic of our experimental setup. Our bichromatic pump is derived from two telecommunication band tunable external cavity lasers (ECL) which are amplitude-modulated with a 2.5% duty cycle to form 5 ns quasi-CW pulses in order to obtain sufficient peak power after amplification for the excitation of high order cascades. For the purpose of the nonlinear experiment, these quasi-CW pulses can be well considered as CW pumps. Then the power of the two modulated beams are separately amplified by two Erbium doped fiber amplifiers (EDFA). The amplified spontaneous emission (ASE) light is separately filtered out by two 0.5 nm band pass filters (BPF). The two amplified quasi-CW pulse trains are then combined with a wavelength division multiplexer (WDM). The optical path lengths before the WDM are adjusted so that the two modulated pulse trains are synchronized after the WDM. Then the bichromatic pump is launched into a 54 m long DSF with the same dispersion parameters and nonlinear coefficient as the fiber used in the modeling.
section. Two polarization controllers (PC) are used to ensure the two pumps co-linearly propagate in the DSF. The output of the DSF is split by a 99/1 fiber fused coupler, with 1% fed into an optical spectrum analyzer (OSA) and 99% into a power meter, where the spectrum and the power are recorded, respectively.

To further confirm the interpretation of dispersive wave emission in terms of cascaded four-wave mixing, we also conduct a second series of experiments using a pulsed pump. Fig. 4.7 shows the schematic setup for the pulsed pump experiment. The pulses are derived from a commercial (Alnair Lab, MLLD-100) 10 GHz repetition rate, telecommunication band tunable hybrid mode-locked laser diode. The pulse wavelength is tunable over C and L bands. The pulses are then amplified by an EDFA to increase the peak power of the pulses. A band pass filter which consists of a set of free space prisms is used to filter out the ASE of the amplified pulses after the EDFA. The pulse duration is measured using frequency resolved optical gating, and is approximately 1 ps. The
amplified pulses are directly coupled into the same DSF, and the output spectrum is analyzed again with the OSA.

Figure 4.7: Experimental setup for observation of dispersive waves. BPF: band pass filter, EDFA: Erbium doped fiber amplifier, MLLD: mode-locked laser diode, OSA: optical spectrum analyzer, PC: polarization controller.

### 4.2.2 Results

Our experimental methodology is based on maximizing the spectral power of a particular choice of sideband order \( n \) by tuning the wavelength of the ECL and adjusting the input power of the two pumps. The experimental parameters for the pump powers and wavelengths are listed in Table 4.1. \( n \) is the order of the phase matched sideband, \( P_{+p} \) is the high frequency pump power, \( P_{-p} \) is the low frequency pump power, \( \omega_c \) is the center frequency of the two pump, \( \Delta_e \) is the experimental frequency separation of the two pumps, and \( \Delta_t \) is the theoretically predicted frequency separation of the two pumps calculated using Eq. (4.16).

![Diagram showing experimental setup](image)

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<th>( P_{-p} ) (W)</th>
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<td>-4</td>
<td>11</td>
<td>40</td>
<td>194.6 (1540.6 nm)</td>
<td>1.2</td>
<td>1.5</td>
</tr>
</tbody>
</table>

Table 4.1: Experimental parameters and theoretical detunings of pumps and sidebands at different orders.
Chapter 4. Cascaded Four-Wave Mixing

phase matched amplification of the \( n = -2 \) and \( n = +2 \) order sidebands. In both cases, the second order sidebands exceed the spectral power of the first order intermediate sidebands by nearly 15 dB. We also numerically calculate the output spectra (circles) for the conditions specified in Table 4.1 using the GNLSE Eq. (2.54) with Raman scattering included. Excellent agreement is found between the experimental measurements and theoretical predictions. The small discrepancy is attributed to the depolarization of the two pumps when propagating in the DSF. Fig. 4.9 (a) and (b) show the spectra of the \( n = -4 \) and \( n = +6 \) order phase matched sidebands as well as the numerically calculated spectra. Both results clearly display the amplification of the predicted sidebands.

![Figure 4.8](image_url)

Figure 4.8: Experimental results (solid curves) and theoretical calculation (circles) showing the phase matched amplification of the (a) \( n = +2 \) and (b) \( n = -2 \) order sidebands. Dashed line indicates the ZDW. Capital labels A and N indicate the anomalous and normal dispersion regimes, respectively.

To verify the proposed connection between the cascaded four-wave mixing and dispersive waves, in the second series of experiments we replace the bichromatic pump with a 10 GHz repetition rate picosecond pulse pump which is centered at the same frequencies as the center frequencies of the \( n = -4 \) and \( n = +6 \) order cascades. The input (red dash-dot) and output (green dotted) spectra are plotted in Fig. 4.10, and superimposed on top of the spectra of the discrete bichromatic pump output. Fig. 4.10
clearly shows the output pulsed pump spectra follow closely the spectral envelopes of the discrete frequency combs. Although the pump pulse duration in the anomalous dispersion regime in the experiment is 680 fs [Fig. 4.10 (a)], whereas the duration of a single cycle pulse of the bichromatic pump is 420 fs, a remarkable agreement in the position and the structure can still be seen between the dispersive wave emitted by the pulsed pump and the amplified higher order phase matched sideband of the discrete frequency comb. This further confirms our physical interpretation of dispersive wave emissions by solitons in the anomalous dispersion regime in terms of cascaded four-wave mixing.

Figure 4.9: Experimental results (solid curves) and theoretical calculation (circles) showing the phase matched amplification of the (a) $n = +6$ and (b) $n = -4$ order sidebands. Dashed line indicates the ZDW. Capital labels A and N indicate the anomalous and normal dispersion regimes, respectively.

More interestingly, the result of the pulsed pump in the normal dispersion regime also shows a similar behaviour with a dispersive-wave-like wave emitted across the ZDW in the anomalous dispersion regime indicated by the arrow in Fig. 4.10 (b). Here, the pulse center frequency was tuned to 194.6 THz corresponding to the center frequency of the bichromatic pump of the $n = -4$ order cascade, with a pulse duration of 920 fs. Again, Fig. 4.10 (b) clearly shows the asymmetric deformation of the pulsed pump spectrum,
with a new generated spectral peak coinciding with the phase matched sideband of the $n = -4$ order cascade. Unlike the extensive studies of the soliton dynamics in the anomalous dispersion regime, dispersive waves generated from a pump in the normal dispersion regime has not yet been fully investigated [119, 120]. We will further investigate the properties of dispersive wave emission when pumping in the normal dispersion regime in the next subsection.

Figure 4.10: Experimental input (dash-dot) and output (dotted) spectra of a pulsed pump in (a) anomalous and (b) normal dispersion regimes. Dashed line indicates the ZDW. Capital labels A and N indicate the anomalous and normal dispersion regimes, respectively.
4.2.3 Dispersive Waves from a Pump in Normal Dispersion Regime

Recall from Eq. (4.19), that one of the predictions of cascaded four-wave mixing is that nonlinear wave propagation in the normal dispersion regime can excite a dispersive wave in the anomalous dispersion regime. To further investigate dispersive wave emission from a pulsed pump in the normal dispersion regime, we numerically simulate the propagation of a sech pulse with 600 W peak power and 1 ps pulse duration in a 100 m standard telecommunication fiber. For simplicity, we only consider the dispersion of the fiber up to third order $\beta_3 = 0.2 \text{ ps}^3/\text{km}$. The center frequency of pump pulse is set to $\delta \omega = 6 \text{ THz}$ detuning from the ZDW. At this detuning, the corresponding second order dispersion parameter is $\beta_2 = 7.5 \text{ ps}^2/\text{km}$, placing the pump well inside the normal dispersion regime.

![Figure 4.11: Simulated dispersive wave spectral evolution when pumping in the normal dispersion regime (a) no Raman scattering included, and (b) Raman scattering included. Dashed line indicates the ZDW, and A and N indicate the anomalous and normal dispersion regimes, respectively.](image)

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dispersion regime. Fig. 4.11 (a) shows the spectral evolution of the pump pulse when it propagates in the fiber in the absence of Raman scattering. The spectrum is rapidly broadened during the first 10 m as a result of SPM. Starting at 10 m, the spectrum stretches asymmetrically toward the low frequency side, and the spectral power is translated to a wave at a detuning of $-13$ THz, inside the anomalous dispersion regime. The generation of this new frequency component behaves similarly to the emission of a dispersive wave from a soliton propagating in the anomalous dispersion regime. There is one important characteristic that allows us to claim that the new generated spectral component is a dispersive wave. This is that the power of the new spectral component does not grow from the noise, rather it requires the broadened pump spectrum to overlap with the dispersive wave frequency before power can be transferred from the pump to the dispersive wave. This is essentially the process of cascaded four-wave mixing. Solitons can propagate undistorted in the anomalous dispersion regime as a result of the balance between the Kerr nonlinearity and the fiber dispersion, whereas a pulse propagating in the normal dispersion regime is unavoidably broadened as it has the opposite sign dispersion. As such, little power ($P \sim |\beta_2|/(\gamma T^2) \approx 10$ W) is required for the formation of a soliton in the anomalous dispersion regime as well as the generation of dispersive wave. A much higher peak pump (600 W) is required for the generation of dispersive wave when pumping in the normal dispersion regime (with the same magnitude $\beta_2$ but opposite sign). This is an order of magnitude higher than that for the anomalous pumping. We note that the peak power of the pump pulse in this simulation has dropped by a factor of 4 after the first 20 m due to the temporal broadening. This means very little nonlinear evolution occurs in the remaining 80 m with the amplitude of the dispersive wave remaining almost constant. As the dispersive wave excited by a pump in the normal dispersion regime will be frequency downshifted with respect to the pump, further amplification can occur via stimulated Raman scattering. We now calculate the evolution of the dispersive wave pumped in the normal dispersion regime with Raman scattering included and plot it in Fig. 4.11 (b). The frequency shift of the dispersive wave is identical to that in the absence of Raman scattering. However, spectral power is now
Observation of Cascaded Four-Wave Mixing

continuously transferred from the pump to the dispersive wave over the last 80 m via stimulated Raman scattering. This is because the frequency shift of the dispersive wave is \(-13\) THz detuning from the pump and so it falls within the Raman gain bandwidth of silica [82].

Figure 4.12: Simulated spectrograms at a distance of (a) \(L = 17\) m and (b) \(L = 100\) m. Dashed lines indicate the ZDW.

According to Eq. (4.19), the predicted frequency shift of the dispersive wave from the pump is three times the detuning from the pump to the ZDW. Thus, the frequency shift of the dispersive wave should be \(-18\) THz in this case. However, the frequency shift of the dispersive wave that we observed in the numerical simulation is \(-13\) THz. We attribute this discrepancy between the simulation and the theoretical frequency detuning to the temporal walk-off between the initial emitted dispersive wave and the
dispersing pump pulse. To get insight into this, we plot in Fig. 4.12 (a) a spectrogram of the propagating pulse at the distance of 17 m. As expected, the pulse spectrum has started spreading temporally due to dispersion. The dispersive wave emitted into the anomalous dispersion regime even in this early stage is clearly distinguishable from the pump spectrum. Fig. 4.12 (a) shows that it has been temporally offset from the center of the pump spectrum. Thus, the initially emitted dispersive wave is temporally overlapped with only the lower frequency components of the pump spectrum which have a smaller detuning from the ZDW (indicated by the dotted line). Thus, the actual frequency shift of the dispersive wave is smaller than the predicted frequency shift given by Eq. (4.19). The output spectrogram in Fig. 4.12 (b) shows that after 100 m both of the pump and the dispersive wave have temporally dispersed with no signature of any solitonic behaviour visible.

To experimentally demonstrate the generation of a dispersive wave from a pump in the normal dispersion regime, we use a 100 m length of standard telecommunication fiber (Corning MetroCor) with an estimated nonlinear coefficient of $\gamma = 2.5 \text{ W}^{-1}\text{km}^{-1}$, and measured dispersion parameters of $\beta_2 = 6.4 \text{ ps}^2/\text{km}$, $\beta_3 = 0.12 \text{ ps}^3/\text{km}$, $\beta_4 = -9.0 \times 10^{-4} \text{ ps}^4/\text{km}$ at the pump wavelength 1568.5 nm. The input pump pulse is derived from a passively mode-locked fiber laser (Calmar FPL-02C), and amplified by a short-pulse EDFA (Keopsys PEFA-SP-C). The amplified input pump pulse is characterized using frequency-resolved optical gating (FROG) and found to be well approximated by a 850 fs chirp-free sech pulse centered at 1568.5 nm with a peak power of 575 W. The output spectrum (solid) is measured with an optical spectrum analyzer (OSA) and plotted in Fig. 4.13. The numerically simulated output spectrum (green dashed curve) using the GNLSE under the same conditions is superimposed on top of the experimental spectrum. A distinct dispersive wave is clearly visible at 1680 nm, in the anomalous dispersion regime of the fiber. Note that the spectral power of this dispersive wave is 30 dB below the pump spectral power, and is insufficient to form a soliton at that wavelength as $|\beta_2| \gg \gamma PT^2$. Clearly, this subsection demonstrates experimentally that the generation of dispersive waves is not restricted to soliton-like pumps propagating in
the anomalous dispersion regime. Dispersive waves can also be generated by a transfer of power from a pump in the normal dispersion regime due to the presence of higher order dispersion. The mechanism for this power transfer is cascaded four-wave mixing.

### 4.3 Cascaded Bragg Scattering

As introduced in Chapter 2, one of the elementary four-wave mixing processes in $\chi^{(3)}$ media is called Bragg scattering. This is a non-degenerate parametric process whereby two strong pumps instigate a sinusoidal exchange of power between a signal and an idler [121, 122]. Conservation of energy and momentum mandates that the power exchange is efficient only when the interacting waves fulfill the phase matched condition.
Bragg scattering has attracted particular interest because of its noise free characteristics that can allow quantum frequency translation \[123\]. As has been demonstrated previously, higher order $\chi^{(2n+1)}$ effects can be directly mimicked by cascading four-wave mixing in optical fibers, we now further extend this idea of cascaded four-wave mixing to a more general four-wave mixing process. Here, the amplification of a higher order idler is generated by launching a weak probe signal with the bichromatic pump at the beginning of the propagation, and the power is translated from the weak probe signal to the higher order idler via cascaded Bragg scattering.

### 4.3.1 Theory

We first derive the phase matched condition for cascaded Bragg scattering with the frequency arrangements shown in Fig. 4.14. The two pumps interact with a weak probe signal at frequency $\omega_0 = \omega_c + \Omega$, and drive the sequential generation of idlers separated by $\Delta$ via a Bragg scattering cascade. The frequency of the $n$-th order ($n = 1, 2, 3...$) idler is $\omega_n = \omega_0 \pm n\Delta$ where the sign describes the direction of the cascade: (+) refers to the scenario of Fig. 4.14 (a) in which a low-frequency signal is converted towards higher-frequencies and (−) to the opposite scenario shown in Fig. 4.14 (b) with a high-frequency
signal converted towards lower frequencies. The \( j \)-th Bragg scattering element of the cascade converts the signal \( \omega_{j-1} \) to \( \omega_j \) via the four-photon process \( \omega_j = (\omega_{zp} - \omega_{\pi p}) + \omega_{j-1} \), and is associated with an elementary phase mismatch

\[
\kappa_j = \left[ \beta(\omega_{zp}) - \beta(\omega_{\pi p}) \right] - \left[ \beta(\omega_j) - \beta(\omega_{j-1}) \right] - \gamma(P_{zp} - P_{\pi p}),
\]

(4.20)

where \( \beta(\omega) \) is the propagation constant of the fiber mode. Conventionally, efficient frequency translation in a single Bragg scattering process mandates that \( \kappa_j = 0 \). However, in the Bragg scattering cascade \( \omega_1 \rightarrow \cdots \omega_j \cdots \rightarrow \omega_n \), the \( n \)-th order sideband will also experience amplification, even if all the elementary mismatches \( \kappa_j \) are nonzero, provided that the accumulated mismatch vanishes:

\[
\sum_{j=1}^{n} \kappa_j = n \left[ \beta(\omega_{zp}) - \beta(\omega_{\pi p}) - \gamma(P_{zp} - P_{\pi p}) \right] - \left[ \beta(\omega_{zn}) - \beta(\omega_0) \right] = 0.
\]

(4.21)

Assuming the two pump powers are equal to each other, Eq. (4.21) can reduce to

\[
\sum_{j=1}^{n} \kappa_j = n \left[ \beta(\omega_{zp}) - \beta(\omega_{\pi p}) \right] - \left[ \beta(\omega_{zn}) - \beta(\omega_0) \right] = 0.
\]

(4.22)

Eq. (4.22) coincides with the direct phase matched condition of a \( \chi^{(2n+1)} \) process with \( \omega_n = \pm n\omega_{zp} \mp n\omega_{\pi p} + \omega_0 \), so that the cascade-induced frequency translation can be intuitively understood as the mimicking of a directly phase matched higher-order process.

At this point we wish to remark upon two features of Eq. (4.22): (i) the equation remains valid on the change of variable \( \beta(\omega) \rightarrow D(\omega) \), where \( D(\omega) = \beta(\omega) - \beta_0 - \beta_1 (\omega - \omega_c) \) is the reduced propagation constant expanded about \( \omega_c \) with \( \beta_k = d^k \beta / d\omega^k \big|_{\omega=\omega_c} \) and (ii) for small pump frequency detunings (\( \Delta \rightarrow 0 \)) the cascaded phase matched condition reduces to \( D(\omega_n) \approx D(\omega_0) \) for large orders \( n \). Again, by considering only the linear term of the phase mismatch and expanding \( \beta(\omega) \) as a Taylor series about \( \omega_c \) up to the fourth
order, Eq. (4.22) further reduces to

\[
\begin{align*}
n^3 \beta_4 \Delta^3 & \quad + \quad (4n^2 \beta_4 \Omega + 4n^2 \beta_3 - \beta_3) \Delta^2 \\
& \quad + \quad (6n \beta_4 \Omega^2 + 12n \beta_3 \Omega + 12n \beta_2) \Delta \\
& \quad + \quad (4 \beta_4 \Omega^3 + 12 \beta_3 \Omega^2 + 24 \beta_2 \Omega) = 0,
\end{align*}
\]

(4.23)

where \( \beta_2, \beta_3 \) and \( \beta_4 \) are the second, third and fourth order dispersion parameters evaluated at \( \omega_c \), respectively. Thanks to the cascade order number \( n \), an extra degree of freedom is provided for the detuning of the two pumps \( \Delta \) and of the probe signal \( \Omega \) to fulfill the phase matched condition. By searching for a real solution to Eq. (4.23) in terms of either \( \Delta \) or \( \Omega \) for an arbitrary order \( n \), one can determine the corresponding frequency separation of two pumps, or the detuning of the probe signal. Not surprisingly, by setting \( \Omega = \Delta/2 \) such that the probe signal is seeded by the pump \( \omega_{zp} \) and neglecting the fourth order dispersion, we obtain exactly the same root \( \Delta \) as given by Eq. (4.17) in our cascaded four-wave mixing analysis. This reveals that the cascaded four-wave mixing process studied in the previous section is essentially a special case of cascaded Bragg scattering in which one of the pumps serves as an initial probe signal.

In order to highlight the flexibility arising from the extra degree of freedom, we consider cascaded Bragg scattering in a typical telecommunication fiber with a cubic dispersion profile. In Fig. 4.15, we plot a schematic of the phase matched configurations for the fundamental \( (n = 1) \) and for the cascaded \( (\text{up to } n = 5) \) Bragg scattering sequences with the same pump frequency separation \( \Delta \). Here we show the frequency arrangements along the reduced propagation constant \( D(\omega) \), and highlight the signal-idler pairs phase matched through the cascade for various orders. For a fixed pump detuning, efficient fundamental Bragg scattering can only be achieved between a single signal-idler pair. For the cascaded process, on the other hand, a phase matched pair can be found for multiple orders \( n \), such that a signal centered at any of the phase matched frequencies shown in Fig. 4.15 will be translated to the corresponding idler via a sequence of elementary Bragg scattering processes. In fact, it is easy to see that in the limit \( \Delta \to 0 \) the signal
wave can be placed anywhere between \(0 < \Omega < \Omega_{\text{lim}}\), where \(\Omega_{\text{lim}} = \omega_{\pm n} - \omega_c = -3\beta_2/\beta_3\) is the frequency shift of the dispersive wave as given by Eq. (4.18), and a phase matched cascade is guaranteed. The ability to change the order \(n\) therefore substantially relaxes the energy-momentum conservation conditions, because the signal-idler detuning can be extended to integer multiples of \(\Delta\).

![Diagram of phase matched frequencies for various orders of Bragg scattering](image)

Figure 4.15: Phase matched frequencies for various orders of Bragg scattering. Numbers below the components identify the signal-idler pairs for a given cascade order. Intermediate idlers are not shown. The arrow lengths of the pumps, signals and idlers are arbitrary.

### 4.3.2 Numerical Modeling

To verify our analysis in the previous subsection we perform numerical simulations based on the GNLSE Eq. (2.54). For simplicity, we neglect Raman scattering in our simulations. Our simulations consider the propagation of two 20 W CW pumps centered at
Chapter 4. Cascaded Four-Wave Mixing

188.1 THz (1595 nm) with a frequency separation of \( \Delta/2\pi = 2.20 \) THz (in the anomalous dispersion dispersion regime) in a 125 m long dispersion shifted fiber (DSF) with a ZDW at 1553.5 nm, and the same higher order dispersion parameters \((\beta_3, \beta_4)\) and nonlinear coefficient as the DSF used in the previous section. At this center frequency, the dispersion parameters are \( \beta_2 = -5.4 \text{ ps}^2\text{km}^{-1}, \beta_3 = 0.18 \text{ ps}^3\text{km}^{-1} \) and \( \beta_4 = -7.0 \times 10^{-4} \text{ ps}^4\text{km}^{-1} \). Alongside the pump waves our simulation includes a weak 200 mW probe signal at 196.1 THz (1529.9 nm) (corresponding \( \Omega/2\pi = 8 \) THz). The frequency separation \( \Delta/2\pi \) and probe signal detuning \( \Omega/2\pi \) are chosen so as to exactly fulfill Eq. (4.23) for a second order cascade \((n = 2)\). In Fig. 4.16 (a) and (b), we compare the optical spectra at the fiber input and after 45 m of propagation, respectively. We can clearly see how after 45 m of propagation the input signal has been almost completely (greater than 30 dB extinction) translated to the second order idler. We also note that no significant depletion of the two pump waves is observed, as expected for the conversion of a weak signal via Bragg scattering. Apart from the injected probe signal and the idlers involved in the cascaded Bragg scattering process, there also exist small amplitude non-phase matched sidebands due to the cascaded four-wave mixing of the two pumps. As the amplitudes

![Figure 4.16: Simulated spectra of the \( n = 2 \) order cascaded Bragg scattering (a) at the DSF input and (b) after 45 m of propagation.](image-url)
of the non-phase matched cascaded four-wave mixing sidebands are small, the cascaded Bragg scattering process is not affected by the additional cascaded four-wave mixing components.

Figure 4.17: Simulated power evolution of $n = 2$ cascaded Bragg scattering along DSF distance when pumping in the anomalous dispersion regime. (a) Power of the second order idler $\omega_2$ (solid), the injected probe signal $\omega_0$ (dashed) and the first order idler $\omega_1$ (dash-dot). (b) Power evolution of two pumps.

In Fig. 4.17 (a) we plot the power evolution of the probe signal at $\omega_0$, the first order idler at $\omega_1$, and the second order idler at $\omega_2$. A quasi-sinusoidal switching can clearly be seen between the probe signal and the second order idler, again closely resembling the sinusoidal evolution associated with elementary Bragg scattering. However, here the sinusoid is modulated, owing to the fact that the nonlinear polarization driving the growth
of the second order idler is proportional to the amplitude of the non-phase matched first order idler which itself is oscillating at a coherence length \( L_{\text{coh}}^{(1)} = \frac{2\pi}{\kappa_1} \approx 11 \text{ m} \).

Fig. 4.17 (b) also shows the power evolution of the two pumps. This shows that the evolution of the two pumps is not strongly effected by the cascaded Bragg scattering. The power exchange between the two pumps once again highlights an equivalent \( \chi^{(5)} \) interaction whereby the annihilation of two \( \omega_{p} \) pump photons and an injected probe signal photon at \( \omega_0 \), is associated with the creation of a second order idler photon at \( \omega_{i_2} \) and also two \( \omega_{-p} \) pump photons by energy conservation: \( 2\omega_{p} + \omega_0 = \omega_{i_2} + 2\omega_{-p} \).

Similar to cascaded four-wave mixing, by cascading \( n \chi^{(3)} \) elementary Bragg scattering processes, a \( \chi^{(2n+1)} \) equivalent process can also be mimicked as illustrated in Fig. 4.18.

![Figure 4.18: Schematic of \( 2n + 2 \) photons interaction via cascaded Bragg scattering.](image)

We also note that simulations of higher order cascades demonstrate similar dynamics to those discussed above provided that the cascaded phase matched condition is satisfied. In this case, more energy is distributed amongst a larger number of intermediate idlers, and the total conversion efficiency is diminished. To illustrate this effect, we plot in Fig. 4.19 the simulated power evolution of a \( n = 3 \) order cascade in a 160 m DSF with the same parameters used previously. The frequency separation of the two pumps is \( \Delta/2\pi = 1.74 \text{ THz} \), and the probe signal detuning is \( \Omega/2\pi = 7.5 \text{ THz} \). Both pump power and injected probe signal power are the same that used previously. As we can see, the period of the power exchange between the probe signal and the third order idler is about 160 m, which is almost twice that of the \( n = 2 \) order cascade. The maximum conversion efficiency from the probe signal to the third order idler is only 70%.
Figure 4.19: Simulated power evolution of \( n = 3 \) cascaded Bragg scattering in a 160 m DSF distance for anomalous dispersion pumping. The third order idler \( \omega_3 \) is the solid curve, and the injected probe signal \( \omega_0 \) is the dashed curve.

Finally, it is well known that in supercontinuum generation new frequency linear waves can be generated by a collision between a soliton and a linear wave [124]. We now interpret the nonlinear mechanism behind this new frequency component generation in terms of cascaded Bragg scattering. As the spacing of the frequency comb reduces, the pump field in temporal domain approaches that of single mode-locked pulse with a continuous spectral envelope. Recall from Eq. (4.22) in the \( \Delta \to 0 \) limit, the phase matched condition reduces to \( \beta(\omega_n) = \beta(\omega_0) \). We here show this phase matched condition is the same as the phase matched condition for the new generated wave in a soliton-linear wave interaction. To demonstrate this, we numerically simulate a \( n = 6 \) order Bragg scattering cascade in a 40 m DSF with the same dispersion parameters used in the \( n = 2 \) cascade simulation. The frequency separation of the two pumps \( \Delta \) and the probe frequency detuning \( \Omega \) are chosen such that they fulfill the phase matched condition given by Eq. (4.23). As shown in Fig. 4.20, two input pumps are both 10 W with a frequency separation of \( \Delta/2\pi = 0.72 \) THz. Their center frequency \( \omega_c/2\pi \) is at 188.1 THz (in the anomalous dispersion regime). The 200 mW weak probe signal is detuned from \( \omega_c \) by \( \Omega/2\pi = 8 \) THz. The input and output spectra of the cascaded Bragg scattering
are plotted as blue circles in Fig. 4.20, and show the strong amplification of the 6-th order idler at the output. In comparison, we also simulated a collision between a pulsed pump made from a single cycle of the bichromatic pump and a probe pulse under the same conditions as the discrete frequency CW interaction. Note that unlike the discrete frequency interaction where the CW pumps and probe signal can continuously interact without walk-off, the interaction between the single cycle pulse pump and the probe pulse can only temporally overlap once in the propagation. Therefore, the fiber length and temporal offset between the single cycle pulse pump and the probe pulse are chosen such that the interaction length is maximized in the simulation. The result of this simulation is superimposed in Fig. 4.20 (red curve). Clearly, the frequency shift of the phase matched idler of the cascaded Bragg scattering coincides exactly with the frequency shift of the new generated frequency component in the soliton-linear wave interaction. The continuous spectrum (red curve) after the propagation follows closely

Figure 4.20: Simulation of a higher order Bragg scattering cascade \( (n = 6) \) in a 40 m DSF. (a) Input spectrum of a bichromatic pump and probe (blue circle) a the single cycle pulse and probe (red curve). (b) Output spectra of the discrete frequency comb (blue circle) and single cycle pulse (red curve), demonstrating the soliton-linear wave interaction. SL indicates the new frequency component generated by the soliton-linear wave interaction.
the discrete frequency combs of the cascaded Bragg scattering. In the same way that the results of the previous section allowed us to identify the nonlinear mechanism behind dispersive wave generation as cascaded four-wave mixing, these results allow us to identify the nonlinear mechanism that drives the soliton-linear wave interaction in the presence of high order dispersion as cascaded Bragg scattering.

4.3.3 Experimental Setup

To confirm our theoretical description, we perform experiments using a setup similar to the one used in the previous section. The schematic diagram is shown in Fig. 4.21. Two strong pump waves are derived by combining the CW beams from two external cavity lasers (ECL) with a 50/50 fiber fused coupler. The two pumps are then amplitude-modulated with a 10% duty cycle to yield a 9 ns quasi-CW pulse train, and amplified by a 5 W L-band Erbium doped fiber amplifier (EDFA) to obtain sufficient peak power for the excitation of Bragg scattering cascades. A weak probe signal is generated from a third ECL at C-band, also modulated to yield an identical pulse width and duty cycle as the two pump pulses. The probe signal is also amplified by a C-band EDFA, and combined and synchronized with the two strong pumps in the DSF using a fiber WDM with a cut wavelength at 1560 nm. The maximum obtainable peak power of the pumps and the signal are estimated to be 40 W and 400 mW, respectively. To demonstrate the various cascade orders, we use two different DSFs with ZDWs of 1561.5 nm (length 100 m) and 1553.5 nm (length 54 m), and the same higher order dispersion parameters \( \beta_3 = 0.16 \text{ ps}^3\text{km}^{-1} \) and \( \beta_4 = -7.0 \times 10^{-4} \text{ ps}^4\text{km}^{-1} \) at the ZDWs. The nonlinear coefficients of these two fibers are both \( \gamma = 2.5 \text{ W}^{-1}\text{km}^{-1} \). We use polarization controllers to ensure that all the waves propagate co-linearly in the DSF. We subsequently vary the wavelength of the probe signal ECL so as to optimize the amplification for a given higher order idler. This methodology yields an experimental detuning \( \Omega_e \) from the pumps center to the probe signal, which we compare with the theoretical phase matched detuning \( \Omega_t \) obtained from Eq. (4.23).

4.3.4 Results

We first perform the excitation of a $n = 2$ order cascade in the 100 m DSF whose ZDW is 1561.5 nm. The two pump wavelengths are set to 1587.9 nm and 1597.6 nm such that $\Delta/2\pi = 1.14$ THz. The corresponding dispersion parameters at the center frequency $\omega_c$ are $\beta_2 = -3.9$ ps$^2$km$^{-1}$, $\beta_3 = 0.18$ ps$^3$km$^{-1}$ and $\beta_4 = -7.0 \times 10^{-4}$ ps$^4$km$^{-1}$. The input pump power is estimated to be 6 W for both pumps, and the probe signal is estimated to be 200 mW. In Fig. 4.22 we plot the input (a) and output (b) spectra of the $n = 2$ order cascade. The circles are the numerical simulation obtained by solving the GNLSE (with Raman scattering included) with the parameters given above, and is in very good agreement with our experimental result. The experimental detuning of the $n = 2$ order
cascade is found to be optimized with $\Omega_c = 6.4$ THz (1539.7 nm), and it is also in good agreement with the theoretical prediction of $\Omega_c = 6.4$ THz calculated using Eq. (4.23). As can be seen, at the output of the DSF, the injected probe signal is efficiently switched to the second order idler with the spectral amplitude of the injected probe signal is 8 dB below the second order idler. This illustrates the efficient switching characteristics of cascaded Bragg scattering. Note that the small amplitude sidebands around the two pumps presented at the input of the DSF are the products of the non-phase matched four-wave mixing of the two pumps inside the EDFA. Their small amplitudes mean they contribute negligible effect to the cascaded process.

![Figure 4.22: n = 2 order cascaded Bragg scattering experimental (solid line) and numerical simulation (circles) input (a) and output (b) spectra.](image)

To excite a higher order cascaded Bragg scattering, we replace the 100 m DSF with
the 54 m DSF whose ZDW is 1553.5 nm. Two pump wavelengths are now set to 1590.5 nm and 1594.5 nm such that \( \Delta/2\pi = 0.473 \) THz. The corresponding dispersion parameters at the center frequency \( \omega_c \) are \( \beta_2 = -5.1 \text{ ps}^2\text{km}^{-1} \), \( \beta_3 = 0.18 \text{ ps}^3\text{km}^{-1} \) and \( \beta_4 = -7.0 \times 10^{-4} \text{ ps}^4\text{km}^{-1} \). The input pump peak power is now increased to 10 W for both pumps. In Fig. 4.23, we plot the input and output spectra of a \( n = 6 \) order cascade, and note the large amplitude of the \( n = 6 \) order idler amidst the cascade. In Fig. 4.23 (a), the initial small amplitude comb structure around the two input pumps is again due to non-phase matched four-wave mixing of the two pumps inside the EDFA, and plays no significant role in nonlinear evolution.

![Input and Output Spectra](image-url)

**Figure 4.23:** \( n = 6 \) order cascaded Bragg scattering experimental (solid line) and numerical simulation (circles) input (a) and output (b) spectra. Solid red curve is the equivalent soliton-pulse interaction spectra.
The experimental detuning of the $n = 6$ order cascade is found to be optimized at $\Omega_e = 8.3$ THz (1525.3 nm), and is in good agreement with the theoretical prediction of $\Omega_t = 8.2$ THz. By tuning the wavelength of the injected probe signal closer to the pumps, we can excite an even higher order cascade. In Fig. 4.24 we show the input and the output spectra with the injected probe signal wavelength set to optimize the conversion of the $n = 11$ order idler. This occurs at a detuning of $\Omega_e = 7.1$ THz (1535.0 nm) from the center frequency of the pumps. Again we find that this value is in good agreement with the theoretical value $\Omega_t = 6.9$ THz obtained from Eq. (4.23). We also simulate the experiments in Fig. 4.23 and Fig. 4.24 using two CW pumps and an input CW weak
Chapter 4. Cascaded Four-Wave Mixing

probe signal under the same conditions. Excellent agreement is observed between the experimental (blue curves) and simulation results (green circles). We note that as the cascade order becomes larger, the conversion efficiency to the phase matched idler diminishes as more energy is consumed by the non-phase matched intermediate idlers.

By numerically simulating a collision between a soliton and a linear pulse with the same propagation conditions as of that the bichromatic pump and the weak probe signal used in the experiments of Fig. 4.23 (a) and Fig. 4.24 (a), we can compare the output spectra of a single soliton interaction with a linear wave to the spectra of the equivalent discrete frequency comb. The pulse duration of the input soliton is set to 1 ps sech pulse. This pulse duration is chosen to best match the pulse duration of a single cycle of the bichromatic pump $\Delta = 0.473$ THz. The pulse peak power is set to 27 W, and the center frequency is at the same as the bichromatic pump. The co-propagating probe pulse is centered at same center frequency as the experimental quasi-CW probe signal, with a pulse duration set to 2 ps, and the pulse peak power set to 600 mW. The temporal offsets between the soliton and the probe pulses at 8.3 and 7.1 THz detuning are adjusted to 1.5 and 2.6 ps, respectively, such that the interaction length between the soliton and the probe pulse is maximized during the propagation. The output spectra of the 7.1 and 8.3 THz probe detunings are plotted as solid red curves in Fig. 4.23 (b) and Fig. 4.24 (b), respectively. It can be clearly seen that the output spectra of the soliton-linear wave interactions closely follow the envelopes of the output spectra of the cascaded Bragg scattering frequency combs. We highlight that the frequency shifts of the new generated spectral components in the soliton-linear wave interaction are identical to the frequency shifts of the phase matched idlers in the cascaded Bragg scattering process. We can thus understand that cascaded Bragg scattering allows a frequency domain description of the interaction, whereby a number of pump pairs inside the spectrum of a soliton centered at $\omega_c$ translate the probe pulse to the same idler frequency via different orders of cascaded Bragg scattering.
4.4 Summary and Discussion

In this chapter, we have presented a theoretical and experimental study of cascaded four-wave mixing in optical fibers. We have derived the phase matched condition of cascaded four-wave mixing from the coupled-mode equations. The \( n \)-th order sideband experiences amplification when the accumulated phase mismatch of the individual elementary processes (up to \( n \)-th order) is zero. The frequency shift of the high order phase matched sidebands are found to be identical to the frequency shift of the dispersive wave emitted by a soliton propagating in the presence of third order dispersion. Numerical simulations have been performed to verify the amplification of the second order cascaded sidebands \( (n = \pm 2) \) in the anomalous and normal dispersion regimes. The amplitude of the phase matched second order sideband grows coherently along the fiber distance with a modulation whose period is identical to the period of the non-phase matched first order sideband. This cascaded phase matched process directly mimics a higher order \( \chi^{(2n+1)} \) effect which allows \( 2n + 2 \) photons to interact. Numerical simulations of a bichromatic pump pair with a small frequency separation have been compared to that of a single cycle bichromatic pump propagation. These simulated results provide us a more natural interpretation for the nonlinear mechanism which describes dispersive wave emission. To our knowledge, this is the first time dispersive wave emission has been described in terms of cascaded four-wave mixing.

We have also experimentally demonstrated cascaded four-wave mixing in a telecommunication fiber based system. Direct amplification of the second order sideband in both the anomalous and normal dispersion regimes have been observed. Very good agreement has been found between the discrete output spectra and simulations obtained by numerically solving the GNLSE. By adjusting the bichromatic pump frequency separation, phase matched higher order sidebands have been observed. We highlight experimentally that the frequency shift of dispersive wave generated by a pulse from a mode-locked laser exactly coincided with the frequency shift of the higher order phase matched sideband. This further confirms our interpretation of the dispersive wave in
terms of cascaded four-wave mixing.

The cascaded four-wave mixing interpretation of dispersive wave also allows us to predict and investigate dispersive waves generated in the anomalous dispersion regime when pumping in the normal dispersion regime, as the cascaded phase matched condition holds when pumping on either side of the ZDW. Our numerical simulations have shown that the emission dynamics of dispersive wave emission when pumping in the normal dispersion regime resembles the behaviour of dispersive waves generated when pumping in the anomalous dispersion regime. The frequency shift of dispersive wave when pumping in the normal dispersion regime was found to be smaller than the simple phase matched frequency prediction. We explain this fact by observing that the initial emitted dispersive wave is only temporally overlapped with the lower frequency components of the pump pulse. An experiment has also been conducted to verify the generation of a dispersive wave in the anomalous dispersion regime in a standard telecommunication fiber. We note that the generation of the dispersive wave when pumping in the normal dispersion regime requires significantly more power as the pump pulses are not solitonic and experience strong temporal dispersion.

Based on the theory of cascaded four-wave mixing, we have also extended our study to a non-degenerate four-wave mixing process called Bragg scattering. The higher order phase matched condition of a Bragg scattering cascade has been derived. This phase matched condition substantially relaxes the energy-momentum conservation conditions. Under the influence of a strong bichromatic pump, numerical simulations have shown that a weak injected probe signal can be efficiently switched into a higher order idler via the cascaded process. The cascaded Bragg scattering in optical fibers has also been experimentally demonstrated again using a telecommunication fiber based setup. The initial weak probe signal can be efficiently converted into various order idler whose frequency can be precisely predicted by the analytical phase matched condition. Finally, we also interpret the interaction between a soliton and a linear wave in terms of cascaded Bragg scattering, and show that these two approaches yield the same phase matched condition which is often discussed in supercontinuum generation [124].

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Fiber Optical Parametric Oscillators

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The optical parametric oscillator was first proposed in the early 1960s, inspired from the frequency conversion methods that had already been demonstrated in microwave devices [125–127]. The first optical parametric oscillator was experimentally demonstrated in 1965, using a \(\chi^{(2)}\) nonlinear crystal as a parametric gain medium and pumped by a visible pulsed laser [9]. With almost half a century of development, \(\chi^{(2)}\) based optical parametric oscillators have found extensive applications in both science and industry. In this chapter, we present a study of a \(\chi^{(3)}\) parametric oscillator based on the Kerr nonlinearity of silica optical fibers. Although the observation of parametric processes in optical fibers dates back to 1970s [19], it was not until the late 1980s that
the first fiber optical parametric oscillator was demonstrated in a 20 cm linear cavity pumped by a 1.0 µm Q-switch Nd:YAG pulsed laser [128]. A year later, a fiber ring cavity parametric oscillator was also reported using a 1.5 µm mode locked laser as a pump source [129].

Over the last two decades, extensive work on fiber based parametric oscillators has been reported using both CW or pulsed pumps and multiple cavity configurations. An all-fiber parametric oscillator with wide tunability (±30 THz) and high conversion efficiency has been demonstrated in telecommunication band using a quasi-CW nanosecond pulsed pump [130]. Wide tunability of a fiber parametric oscillator using a picosecond pulse pump has also been demonstrated in the telecommunication band [131]. In the visible region, a very widely tunable picosecond pumped fiber parametric oscillator with a tuning range in excess of 125 THz has been demonstrated in a butt-coupled PCF oscillator configuration [132]. In the femtosecond pumping regime, a fiber parametric oscillator with an octave-spanning tuning range has been reported using a few centimeters of PCF [133].

All optical parametric oscillators work by placing a gain medium (e.g. an optical fiber) into an optical resonator. The newly generated frequency field starts oscillating once its gain exceeds the round trip loss. The advantage of using an optical fiber as the parametric gain medium is that it provides a tight confinement for the optical field over a long interaction length. The flexibility of optical fiber design, especially the invention of photonic crystal fibers [134], offers extra degrees of freedom for the control of chromatic dispersion, and has facilitated fiber parametric oscillators that operate over increased frequency ranges [135].

In this chapter, we revisit the gain characteristics of the scalar modulation instability of the $\chi^{(3)}$ parametric process introduced in Chapter 2, and show how wide frequency tunability of this process can be achieved by pumping near the zero dispersion wavelength of a parametric gain fiber. We also discuss the choice of cavity configurations and designs that will be used to implement our fiber parametric oscillators. To evaluate the optimum operating conditions of fiber optical parametric oscillators, the intracavity
conversion dynamics of the pump and sidebands will be evaluated by numerically integrating the coupled-mode equations. We then experimentally demonstrate the ability of fiber optical parametric oscillators to operate at high average power with multiwatt level output power, and tunable over a range from 1350 nm to 1795 nm. To our knowledge, this fiber parametric oscillator is the highest power oscillator yet reported. Later, we also investigate how to achieve a high conversion efficiency fiber parametric oscillator by introducing an intracavity filtering on the feedback.

5.1 Theory

5.1.1 Tunable Parametric Gain

Unlike a laser gain medium whose gain bandwidth is determined by an atomic transition, the gain bandwidth of a parametric process is determined by the dispersion and nonlinearity of the gain fiber [76]. We recall from Section 2.5 that the parametric gain \( \lambda \) depends on the phase mismatch between the pump and the two sidebands. In the absence of Raman scattering, the parametric gain coefficient of the two sidebands is

\[
\lambda = \gamma P \sqrt{\kappa(2 - \kappa)}.
\]  

(5.1)

For the sidebands to experience parametric gain, it is necessary that the normalized linear phase mismatch \( \kappa = -\Delta k_L/2\gamma P \) falls within the range \( 0 < \kappa < 2 \). By expanding the propagation constants of linear phase mismatch \( \Delta k_L \) into a Taylor expansion with respect to the pump frequency \( \omega_p \) up to the fourth order of dispersion, we obtain

\[
\Delta k_L(\omega_p, \Omega) = \beta(\omega_p + \Omega) + \beta(\omega_p - \Omega) - 2\beta(\omega_p),
\]  

(5.2)

\[
= \beta_2(\omega_p)\Omega^2 + \frac{\beta_4}{12}\Omega^4,
\]  

(5.3)

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where \( \Omega \) is the frequency shift of the two sidebands from the pump. Note that the second order dispersion coefficient \( \beta_2(\omega_p) \) is a function of the pump frequency, and can be further expanded via another Taylor expansion around the zero dispersion wavelength (ZDW) as

\[
\beta_2(\omega_p) = \beta_3(\omega_p - \omega_{ZDW}) + \beta_4(\omega_p - \omega_{ZDW})^2 / 2, \tag{5.4}
\]

\[
\Rightarrow \beta_2(\delta \omega) = \beta_3 \delta \omega + \beta_4 \delta \omega^2 / 2, \tag{5.5}
\]

where \( \delta \omega = \omega_p - \omega_{ZDW} \) is the frequency shift of the pump with respect to the ZDW, and \( \beta_3 \) and \( \beta_4 \) are the third and fourth order dispersion coefficients evaluated at the ZDW, respectively. Since the variation of the fourth order dispersion \( \beta_4 \) is relatively small over the frequency range under study, we assume \( \beta_4 \) to be constant. The linear mismatch \( \Delta k_L(\delta \omega, \Omega) \) is now a function of \( \delta \omega \) and \( \Omega \). By substituting \( \Delta k_L(\delta \omega, \Omega) \) into Eq. (5.1), the parametric gain spectrum \( \lambda(\delta \omega, \Omega) \) becomes

\[
\lambda(\delta \omega, \Omega) = \gamma P \sqrt{\kappa(2 - \kappa)}, \tag{5.6}
\]

\[
= \left( \frac{-\Delta k^2_L(\delta \omega, \Omega)}{4} - \gamma P \Delta k_L(\delta \omega, \Omega) \right)^{\frac{1}{2}}. \tag{5.7}
\]

Maximum parametric gain is obtained at phase matching condition \( \kappa = 1 \), or

\[
\Delta k_L(\delta \omega, \Omega) + 2 \gamma P = \beta_2(\delta \omega) \Omega^2 + \frac{\beta_4}{12} \Omega^4 + 2 \gamma P = 0. \tag{5.8}
\]

By solving the equation above, we obtain the frequency shift of the phase matched sidebands as a function of the pump frequency shift with respect of the ZDW as

\[
\Omega_{pm}^2 = \frac{-6 \beta_2(\delta \omega) \pm \sqrt{36 \beta_2(\delta \omega) - 24 \beta_4 \gamma P}}{\beta_4}. \tag{5.9}
\]

The existence of real solutions requires \( \Omega_{pm}^2 > 0 \). Thanks to the chromatic dispersion of the optical fibers, a widely tunable gain region can be obtained by simply tuning
the pump wavelength over a small range in the vicinity of the ZDW. To illustrate this wide tunability of parametric gain in an optical fiber, Fig. 5.1 shows a plot of normalized parametric gain spectrum \( \lambda(\delta \omega, \Omega)/\gamma P \) as a function of the pump frequency detuning \( \delta \omega \) with respect to ZDW. The normalized parametric gain \( \lambda(\delta \omega, \Omega)/\gamma P \) is calculated using Eq. (5.7) which is directly related to the linear phase mismatch \( \Delta k_L \) between the pump and the sidebands as well as the nonlinear phase mismatch \( \Delta k_{NL} = \gamma P \). The fiber dispersion parameters used to calculate this spectrum are \( \beta_3 = 0.16 \text{ ps}^3/\text{km}, \beta_4 = -7.0 \times 10^{-4} \text{ ps}^4/\text{km} \). The nonlinear coefficient is \( \gamma = 2.5 \text{ W}^{-1}\text{km}^{-1} \), and the pump power is \( P = 100 \text{ W} \). The phase matched curve of Eq. (5.9) is superimposed as a white solid curve in Fig. 5.1. In the anomalous dispersion regime \( (\delta \omega < 0) \), very limited tuning range can be obtained as both of the dispersion terms of Eq. (5.8) are negative, and the phase matched condition

![Figure 5.1](image.png)

Figure 5.1: Normalized parametric gain spectrum as a function of pump detuning \( \delta \omega/2\pi \) in a DSF. Phase matched curve is represented by a white solid curve calculated using Eq. (5.9).
can only be fulfilled by the positive nonlinear term. As the pump goes further into the anomalous dispersion regime, the frequency shift $\Omega$ of the phase matched sidebands decreases. On the contrary, in the normal dispersion regime ($\delta \omega > 0$) the frequency shift $\Omega$ of the sideband becomes a strong function of the pump frequency shift $\delta \omega$. Widely tunable sidebands (over $\pm 20$ THz tunability) can be easily achieved by tuning the pump frequency within a small range (about 1 THz) [136, 137]. It is this simple tuning mechanism that we will exploit in the experimental oscillators presented later in this chapter.

### 5.1.2 Oscillator Cavity

Fiber optical cavities are more integrated and compact than free space resonator cavities due to the flexibility of optical fibers. Depending on the number of waves resonating in a cavity, fiber parametric oscillators can be classified into either singly, doubly or triply resonant oscillators. For a singly resonant cavity, only one of the sidebands is selected to be resonant inside the cavity. This results in a phase insensitive amplification of the resonating sideband and so no interferometric stabilization of the cavity is required. For a doubly resonant cavity, both sidebands are resonant with the cavity. For triply resonant cavities, pump and both sidebands are all resonant inside the cavity. The amplification process in both doubly and triply resonant cavities are phase sensitive and interferometric stabilizations are required [138]. In this chapter we consider only single resonant oscillator design.

Fig. 5.2 shows two generic singly resonant fiber optical parametric oscillator cavities: (a) a unidirectional (ring) cavity and (b) a Fabry-Perot (linear) cavity. For the ring cavity as shown in Fig. 5.2 (a), a pump is input into the cavity via a dichroic coupler input without significant loss. This dichroic coupler also serves as a reflector to recombine the resonating sideband with the pump at the input. An optical fiber acts as a parametric gain medium to convert the pump into two sidebands. A fraction of the output of the optical fiber is coupled out of the cavity through an output coupler while the rest
of the output is feedback into the cavity. An optical filter is commonly placed inside the cavity to select the frequency, and set the linewidth of the resonating sideband as well as to filter out the residual pump and idlers reflected from the output coupler. This ensures the oscillator is singly resonant. We will see later in this chapter how the intracavity filtering can also be used to enhance the conversion efficiency of a singly resonant fiber optical parametric oscillator. As shown in the schematic, the circulation of the resonating sideband in a ring cavity in Fig 5.2 (a) is unidirectional, and the round trip distance is identical to the cavity length. The circulation of the resonating sideband in a linear cavity is always bidirectional, as shown in Fig 5.2 (b), such that the total round trip distance is twice the cavity length. The intracavity filtering of the linear cavity usually relies on the selectivity of the output coupler. Unlike the amplification of the light field in a laser gain medium which is provided by stimulated emission of an atomic transition, parametric amplification is a phase sensitive process in which the
power coupling between the pump and the sidebands in a single mode fiber typically occurs only in the co-propagating direction. Thus, linear cavities offer no parametric gain when the resonating sideband is counter propagating to the direction of the pump. Despite this fact, they are still commonly used as they can provide a high feedback for the sidebands and significantly reduce the oscillator’s threshold power [132, 139]. Comparing these two types of cavity, we can see that the tunable intracavity filtering of the ring cavity is often easier to achieve. For this reason we choose the singly resonant ring fiber cavity as the cavity design for our fiber parametric oscillator study.

5.1.3 Threshold and Conversion Efficiency

Firstly, we define the internal conversion efficiency of an oscillator as the efficiency of the conversion from the pump to each sideband that occurs in the parametric gain fiber. For example, a 30 % internal conversion efficiency to the anti-Stokes sideband means that 30 % of the pump incident on the parametric gain fiber has been converted to the anti-Stokes sideband at the output of the fiber. We also define the external conversion efficiency of the oscillator as the ratio of the output power of each sideband to the input pump power. This definition of the external conversion efficiency takes into account both the losses and the output coupling of the oscillator.

The sideband signal at the phase matched frequency is initially at the quantum noise level [140], and experiences parametric gain when it propagates in the gain fiber. Then a fraction of the resonating sideband after one round trip is feedback to the input and serves as a new seed of the successive round trip. The oscillation of the sideband starts once the round trip gain of its power is greater than its feedback fraction $\alpha$. When this occurs the resonating sideband grows continuously until it saturates. In the CW limit, it is simple to model the pump and sidebands power of a singly resonant ring cavity using the coupled-mode equations Eq. (2.57) introduced in Chapter 2. For the sake of simplicity, we neglect the effect of Raman scattering in our initial simulations. To illustrate the saturation of the resonating phase matched sideband power, we calculate the internal
conversion efficiency of the resonating sideband along a $L = 100$ m long single mode fiber with a nonlinear coefficient $\gamma = 2.53 \text{ W}^{-1}\text{km}^{-1}$. The input pump power is $P = 12 \text{ W}$, the initial seed level is set to 1 pW for both sidebands, and the feedback fraction is $\alpha = 0.05$. After evaluating the power of the pump and the resonating sideband using Eq. (2.57), we plot their internal conversion efficiencies as a function of round trip number in Fig. 5.3 (a). For such a seed level, the power of the resonating sideband reaches a steady state after 20 round trips. We also plot the parametric gain per round trip (dot-dashed curve). The saturation of the parametric gain occurs due to the increase of the resonating sideband power and the depletion of the pump power. Thus, the oscillation reaches the equilibrium when the saturated parametric gain exactly balances the round trip loss. The evolution of the pump and the sidebands power along the fiber distance after the parametric gain is saturated, is also plotted in Fig. 5.3 (b).

![Figure 5.3](image_url)

**Figure 5.3:** (a) Internal conversion efficiencies (solid, left y-axis) and parametric gain (dash-dot, right y-axis) of the resonating sideband as a function of number of round trips. (b) Conversion efficiencies of pump (dash-dot) and sidebands (solid and dash) power along the length of the gain fiber after parametric gain saturation.

In Fig. 5.4, we plot the internal conversion efficiency of the oscillator as a function of the feedback fraction of the resonating sideband $\alpha$, and the normalized input pump power $\xi = \gamma P L$ at the phase matched condition. The output coupling of the oscillator is set to $1 - \alpha$, and we assume no other internal losses. The white dashed curve in Fig 5.4
traces the analytical threshold condition of the oscillator, and the analytical expression for the oscillator’s threshold at phase matched condition ($\kappa = 1$) is given by

$$\xi_{\text{threshold}} = \sinh^{-1} \left( \sqrt{\alpha^{-1} - 1} \right).$$  \hspace{1cm} (5.10)

This expression is obtained by calculating the pump power at which the small signal

![Figure 5.4: Internal conversion efficiency from pump to each sideband of a singly resonant ring fiber optical parametric oscillator as a function of normalized pump power $\xi$ and sideband feedback fraction. The white dashed curve shows the oscillator’s threshold condition. The white solid curve shows the oscillator’s optimum internal conversion efficiency at phase matched frequency.](image)

parametric gain in the nonlinear fiber (from Eq. 2.72) matches the round trip feedback fraction $\alpha$ of the resonating sideband. Fig. 5.4 shows that the maximum internal conversion efficiency, at the phase matched frequency, from the input pump into each output sideband approaches 28%. For higher round trip losses, the oscillator requires a corre-
spondingly higher pump power to achieve this conversion. Numerically, we find that Eq. (5.11) provides a good fit to the optimum conversion contour. This curve is plotted in Fig. 5.4 as a white solid curve:

\[
\xi_{\text{optimum}} = \sinh^{-1}(2.18 \sqrt{\alpha^{-1} - 1}).
\] (5.11)

At the pump powers beyond this optimum value, the internal conversion efficiency of the oscillator decreases. This behavior is due to the periodic exchange of the power between the three waves that occurs after the initial power transfer from the pump to the sidebands has depleted the pump. The islands of the high power conversion seen for high resonant sideband feedback fractions at \( \xi = 4 \sim 5 \), correspond to a regime in which the input power has oscillated between the pump and the sidebands more than once along the fiber length. The primary high conversion regime, marked by the solid curve of Eq. (5.11), corresponds to a single flow of power from the pump to the sidebands.

### 5.2 High Power Fiber Optical Parametric Oscillator

#### 5.2.1 Experimental Implementation

We chose to implement our high power fiber parametric oscillator in the telecommunication band. The advantages of building an oscillator in the telecommunication band are several: Firstly, the fiberized optical components are well developed and have low loss. This significantly reduces the risk of damage by the high optical power. Secondly, telecommunication fibers have a high longitudinal uniformity, which minimizes the parametric gain reduction that large frequency shift sidebands experience as a results of dispersion fluctuations \([141]\). Experimentally, our oscillator is built as an all-fiber loop with all components spliced together as shown in Fig. 5.5. The input coupler is a standard telecommunications high power wavelength division multiplexer (WDM) with a cut wavelength of 1535 nm and a nominal maximum input power of 2 W. This
Figure 5.5: Experimental setup of a fiber optical parametric oscillator. ECL: external cavity laser, AM: Mach-Zehnder amplitude modulator, PG: pattern generator, WDM: wavelength division multiplexer, DG: diffraction grating, EDFA: Erbium doped fiber amplifier, OSA: optical spectrum analyzer, PC: polarization controller.

is followed by 28 m of standard telecommunications dispersion shifted fiber (DSF) with a ZDW of 1555 nm, dispersion parameters $\beta_3 = 0.16 \text{ ps}^3/\text{km}$, $\beta_4 = -7.0 \times 10^{-4} \text{ ps}^4/\text{km}$ at the ZDW, and a nonlinear coefficient of $\gamma = 2.53 \text{ W}^{-1}\text{km}^{-1}$. The splice loss between the standard single mode fiber of the WDM and the DSF is measured to be 5%. After the DSF, a 90/10 fused fiber coupler outputs 90% of the light from the cavity. The remaining 10% is collimated into free space, and filtered by a diffraction grating with a bandwidth of 0.5 nm so that only the anti-Stokes sideband is feedback to the reflection port of the input WDM. Polarization controllers (PC) are placed before and after the WDM to ensure the input pump and the resonating sideband are co-linearly propagating along the DSF. This forms a resonant ring cavity for the anti-Stokes sideband. The pump is derived from a C-band tunable external cavity laser (ECL) that is amplitude-modulated using a Mach-Zehnder modulator to form 26 ns pulses with a duty cycle of 14% and a repetition rate synchronized to the fiber ring cavity (5.4 MHz). This modulated pulsed pump is then amplified by a high power EDFA (Keopsys, KPS 42 dBm) to an average power of 15 W. At 15 W average power into the ring cavity, we observe no sign of any damage or
degradation to either the fiber or the optical components. We estimate the pump’s peak power in the DSF to be 85 W. This set the normalized power to $\xi = 6$. Eq. (5.11) predicts an optimum feedback fraction of $-40$ dB at this power level. The optical filter allows us to set not only its center wavelength but also the attenuation of the feedback fraction. Experimentally, at each pump wavelength used, the internal conversion efficiency of the oscillator is optimized with respect to the output power of the anti-Stokes sideband by adjusting the transmission of the filter at the phase matched anti-Stokes wavelength, as well as by fine tuning the repetition rate of the pump pulses.

### 5.2.2 Results

![Combined spectra of the output of the fiber optical parametric oscillator at 13 different pump wavelengths between 1556 nm and 1540.7 nm. Arrows indicate the wavelength tuning direction of the pump and the sidebands.](image)

Figure 5.6: Combined spectra of the output of the fiber optical parametric oscillator at 13 different pump wavelengths between 1556 nm and 1540.7 nm. Arrows indicate the wavelength tuning direction of the pump and the sidebands.

In Fig. 5.6 we plot the spectra of the output of the oscillator at 13 different pump...
wavelengths between 1556 nm and 1540.7 nm. As the pump wavelength is tuned from the ZDW into the normal dispersion regime, the phase matched sideband frequency shift increases correspondingly as illustrated in Section 5.1.1. At each pump wavelength the intracavity filter is centered around its corresponding phase matched frequency. The optical spectrum analyzer used to record these spectra cannot operate at wavelengths above 1770 nm, and so the final two Stokes wavelengths at 1780 nm and 1795 nm are not shown. The frequency shifts of these output sidebands is well described by the phase matched curve calculated from Eq. (5.9) as plotted in Fig. 5.7.

![Graph of theoretical and experimental phase matched curve](image)

Figure 5.7: Theoretical (solid curve) and experimental (circle) phase matched curve of the output sidebands of the high power fiber parametric oscillator.

The 3 dB linewidth of the output sidebands is measured to be less than 0.5 nm at all detunings. The tuning range of this oscillator was limited by the high power EDFA, which cannot be operated below 1540 nm. Should this restriction be overcome, the tuning range of the oscillator would likely be next limited by the longitudinal dispersion.
fluctuations in the DSF, as well as the strong attenuation of the Stokes sideband above 1900 nm due to the transmission of silica fibers [130, 141].

The total average power exiting the oscillator was measured to be 11.3 W. In Fig. 5.8 we plot the measured average output power in each of the three waves as a function of the sideband frequency detuning. This plot clearly shows that both the Stokes and anti-Stokes outputs have an average power in excess of 1.9 W out to 25 THz detuning. The asymmetry of the output power between the two sidebands is mainly due to the influence of Raman scattering on the parametric process discussed in Section 2.5. Between 5 and 14 THz, the average power of the Stokes output is in excess of 3.8 W. To our knowledge, this represents the highest output power for a tunable fiber parametric oscillator yet demonstrated.
5.3  High Conversion Fiber Optical Parametric Oscillator

High conversion fiber parametric amplifiers have been well demonstrated over the last decade. As discussed in Section 2.5, it is well known that the maximum conversion of the parametric process occurs when the sideband has a smaller frequency shift than the phase matched frequency \([45, 90]\). Experimental investigations have shown that an almost complete depletion of a pump can be achieved by seeding a signal at the appropriate frequency detuning \([142, 143]\). The initial seeding is necessary for a strong depletion in a single pass fiber parametric amplifier. Without the initial seeding, the spontaneous amplified sidebands at the phase matched frequency experience the highest small signal gain and dominate over the sidebands at any other frequency within the gain bandwidth. Recall in Section 5.1.2, one advantage of the oscillator cavity is that the intracavity filtering allows selection of the sideband frequency resonant inside the cavity. Using the same technique used for the strong pump depletion in parametric amplifiers, we demonstrate the high conversion efficiency of a singly resonant fiber optical parametric oscillator by introducing an intracavity filtering.

5.3.1  Numerical Modeling

Firstly, we numerically model the dependence of the internal conversion efficiency of a fiber optical parametric oscillator on its phase mismatch and pump power. We consider a singly resonant ring fiber parametric oscillator of the same form as the singly resonant cavity used in the previous section. The ring consists of a dichroic input coupler, a parametric gain fiber, a broadband output coupler, and an intracavity filter to ensure that only one of the parametric sidebands is feedback into the reflection port of the input WDM \([144]\). For simplicity, we again initially neglect the Raman effect. This allows us to describe the evolution of the pump and the two sidebands propagating in the parametric gain fiber of the oscillator via the coupled-mode equations Eq. (2.57). In Fig. 5.9, the internal conversion efficiency of a singly resonant oscillator with a 5% feedback
High Conversion Fiber Optical Parametric Oscillator

Figure 5.9: Internal conversion efficiency from the pump to each sideband, of a singly resonant ring fiber optical parametric oscillator as a function of normalized linear mismatch $\kappa$ and normalized input pump power $\xi$ with a resonating sideband feedback fraction of 5%.

fraction, is plotted as a function of the normalized linear mismatch $\kappa = -\Delta k_L/2\gamma P$, and the normalized input pump power $\xi = \gamma PL$. We can see that the minimum threshold power for oscillation, with 5% feedback, occurs at the phase matched frequency ($\kappa = 1$) and at a normalized pump power of $\xi = 2.5$. These values agree with the standard analytic values calculated using Eq. (5.10). Fig. 5.9 also shows that at the phase matched condition $\kappa = 1$ (white dash-dot line), the maximum internal conversion efficiency to each sideband is only 28%, just as for the single-pass configuration. This occurs at the normalized pump power of $\xi \approx 3$ (white dot). This point corresponds to the maximum possible flow of power from the pump to the sidebands at the phase matched frequency. Beyond this point, the power in both of the sidebands will start to flow back into the pump. However, Fig. 5.9 also shows that if we select a linear mismatch smaller than
the phase mismatch $\kappa = 1$ higher conversion efficiencies are possible. At $\kappa \approx 0.25$ (i.e. a value $\Delta k_L$ four times smaller than that found at the phase matched frequency, marked with a white dashed line in Fig. 5.9), an internal conversion efficiency approaching 50% into each sideband is possible for $\xi = 5 \sim 6$. This corresponds to an almost complete

![Conversion efficiency](https://example.com/conversion_efficiency.png)

Figure 5.10: Internal conversion efficiency from the pump to each sideband, of a singly resonant ring fiber optical parametric oscillator as a function of normalized length with a feedback fraction of 5%. (a) is at $\kappa = 1$ and $\xi = 3$, (b) is $\kappa = 0.25$ and $\xi = 5.4$. Dash-dot, solid and dashed curves are the simulated power of the pump, feedback sideband, and non-feedback sideband, respectively.

transfer of power from the pump to the sidebands. The only drawback with operating at this linear mismatch is that the threshold power of the oscillator has increased by $\sim 20\%$ ($\xi \sim 3$) due to the reduced small-signal parametric gain available at this detuning. To visualize the evolution of the pump and sidebands inside the gain fiber of the oscillator, we plot in Fig. 5.10 the steady state evolution (gain saturated) of the three waves as they propagate along the normalized fiber length. Figs. 5.10 (a) and (b) are calculated with normalized linear mismatches and normalized powers at $\kappa = 1$, $\xi = 3$, and $\kappa = 0.25$, $\xi = 5.4$, respectively. Note that the small difference between the power of the two sidebands is because one of the sidebands is resonant inside the cavity and acts as an initial seed to stimulate the process.
Continuous tunability of the output sidebands is still possible when operating in this configuration again by simply tuning the pump wavelength. The above theory considers only a pure Kerr nonlinearity, whereas real fibers also possess a Raman contribution to the $\chi^{(3)}$ nonlinearity. In this case the coupled-mode equations used in the above simulations need to be modified to include Raman scattering (see Section 2.5). However, we find that the inclusion of Raman scattering does not significantly alter the results presented above. The feedback fraction can be adjusted according to the need of the actual setup. As we aim for a high external conversion efficiency, a small feedback fraction is used such that most of the power can be coupled out of the cavity.

5.3.2 Experimental Implementation

To experimentally investigate the optimum conversion efficiency of a fiber parametric oscillator, we use a setup which is almost identical to the one used in the previous section as shown in Fig. 5.5. The DSF is now changed to a length of 54 m, with a slightly different ZDW of 1553.8 nm, and dispersion parameters $\beta_3 = 0.16$ ps$^3$/km, $\beta_4 = -7.0 \times 10^{-4}$ ps$^4$/km at the ZDW. The pump for the oscillator is a C-band tunable external cavity laser amplitude-modulated to form 44 ns pulses with a duty cycle of 4% (repetition rate of 890 kHz synchronized with the cavity length), then amplified by a 2 W Erbium doped fiber amplifier. The maximum pump peak power is estimated to be 40 W. The pump wavelength is set to 1554.68 nm, which sets $\beta_2 = -0.1$ ps$^2$/km. The linear transmission of the input pump through the WDM, parametric gain fiber, and the output coupler to the 90% output port is measured to be 73%. The external conversion efficiency of the oscillator can thus be simply calculated by multiplying the internal conversion efficiency by this linear transmission coefficient.

5.3.3 Results

For our first experiment, the input pump peak power is set to 23 W ($\xi = 3$), slightly above the threshold power of the oscillator. The phase matched frequency of the parametric
gain can be determined from the single-pass detuning of the spontaneous sidebands in the DSF, and is measured to be 5.3 THz. The intracavity filter forces the anti-Stokes wave to oscillate at the wavelength selected, allowing us to alter the linear mismatch of the oscillator’s parametric gain. In Fig. 5.11, we plot the experimentally measured internal conversion efficiencies of the pump and the two sidebands as a function of the normalized linear mismatch. The optimum conversion occurs at $\kappa = 0.7$, where we obtain a pump depletion of 60%, with 33% converted into the Stokes sideband and 27% into the anti-Stokes sideband. At this pump level the optimum conversion efficiency is not too much higher than the conversion measured at the phase matched frequency $\kappa = 1$, where 50% of the pump has been depleted, with 28% converted into the Stokes sideband.
and 22% into the anti-Stokes sideband. The curves in Fig. 5.11 are calculated using the coupled-mode equations Eq. (2.57) in Section 2.4, with the effect of Raman scattering included. The parameter $f_R$ used in the calculation is set to $f_R = 0.18$ (see Section 2.3.4 for more detail of $f_R$). The asymmetry of the conversion efficiencies between the Stokes and the anti-Stokes waves is a result of Raman scattering. Fig. 5.12 shows the output spectra of the oscillator with the intracavity filtering set at the two frequency detunings ($\kappa = 1$ and $\kappa = 0.7$) discussed above. The spectral power of the sidebands at 3.6 THz detuning ($\kappa = 0.7$) is notably higher than at 5.3 THz detuning ($\kappa = 1$) as expected. Note that the additional higher order sidebands visible in the spectra of Fig. 5.12 are generated due to the non-phase matched four-wave mixing between the pump and the strong sidebands. Their effects on the conversion efficiency of the two sidebands should be negligible as the amplitudes of the non-phase matched sidebands are about 30 dB below the pump.

We then increase the peak power of the pump to 33 W, corresponding to normalized power of $\xi = 4.3$. This power gives a phase matched frequency shift of 5.3 THz. In Fig. 5.13 we plot the measured internal conversion efficiencies of the pump and the
sidebands as a function of the linear mismatch $\kappa$ at this input power. Fig 5.13 shows that the peak internal conversion efficiency of the oscillator has shifted to a linear mismatch of $\kappa = 0.25$. At this point the input pump is very efficiently converted into the sidebands with a power depletion of 93%. Again, due to the presence of Raman scattering, the Stokes sideband experiences a slightly higher conversion of 53% compared to the anti-Stokes conversion of 40%. This is also clearly shown in Fig. 5.14 where we plot the measured optical spectra of the output of the oscillator at the phase matched frequency [(a) 5.3 THz detuning, $\kappa = 1$] and at the optimum linear mismatch [(b) 3.2 THz detuning, $\kappa = 0.25$]. The increased internal conversion efficiency of the oscillator at the optimum linear mismatch is clearly evident. Given the linear transmission of the oscillator is
73%, our maximum external conversion efficiency is 68% with 29% converted into the anti-Stokes sideband, and 39% into the Stokes sideband. To our knowledge, this is a record high external conversion efficiency for a fiber optical parametric oscillator.

Figure 5.14: Spectra of the oscillator output taken with the anti-Stokes sideband fed back at a linear mismatch of $\kappa = 1$ (a) (5.3 THz detuning), and $\kappa = 0.25$ (b) (3.2 THz detuning) at a normalized pump power of $\xi = 4.3$.

5.4 Summary and Discussion

Fiber optical parametric oscillators based on four-wave mixing in $\chi^{(3)}$ media have been well established as efficient frequency conversion devices. In this chapter, we have demonstrated a multiwatt average output power fiber parametric oscillator built from a standard telecommunication fiber and fiber components. By singly resonating the anti-Stokes sideband in the cavity, widely tunable output from 1350 nm to 1790 nm has been demonstrated. Note that the tuning range was only limited by the high power EDFA used. The tuning range of the output could easily be modified by replacing the gain fiber with another fiber with a different zero dispersion wavelength. There is no fundamental reason why tunable fiber oscillators with significantly higher output power could not be
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built. Large mode area fibers are routinely used to build fiber lasers with output power in excess of 100 W [145]. We note that very recently a fiber parametric amplifier with 17 W average power for the parametric sidebands has been demonstrated using a large mode area fiber [146]. A large mode area fiber with correctly selected dispersion and a suitable high-power pump could in principle be used to construct a very high average output power tunable fiber oscillator.

We have also presented a study on how to achieve a high conversion efficiency in a singly resonant fiber parametric oscillator. By introducing an intracavity filtering to detune the resonant sideband from the phase matched frequency, the pump power can be efficiently converted into the power of both sidebands. A maximum internal conversion efficiency in excess of 93% has been demonstrated, with 40% of the pump converted into the anti-Stokes sideband and 53% of the pump converted into the Stokes sideband. This yields an external conversion efficiency for the oscillator of 68% (29% into the anti-Stokes sideband, and 39% into the Stokes sideband). Almost a complete depletion of pump can be achieved at the expense of a higher threshold power for the oscillator. With the recent development of highly nonlinear optical fibers, an even lower threshold power can be achieved [147]. In the presence of Raman scattering, the conversion efficiency also depends on the detuning of the sideband, and it results in an asymmetric conversion between the Stokes and anti-Stokes sidebands. Moreover, as the frequency detuning of the sideband increases, the parametric gain bandwidth becomes narrower and a more accurate tuning of the cavity filtering is required to correctly set the linear mismatch.
Temporal Symmetry Breaking

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Spontaneous symmetry breaking plays a fundamental role in physics. The phenomenon of symmetry breaking has not only been extensively studied in physical systems such as semiconductor lasers, quantum dots, photonic lattices and liquid crystals [148–151], but also in chemical reactions and developmental biology [152,153]. In Bose-Einstein condensation, similar effects which are known as macroscopic quantum self-trapping have also been observed in a double-well potential [154,155]. In nonlinear optics, a theoretical study has shown that a spatial vector multimode soliton can undergo a spatial-inversion symmetry breaking instability in a pure Kerr medium in the absence of birefringence, provided that the strength of cross-phase modulation dominates that of self-phase modulation [156]. The existence of this multimode asymmetric soliton has already been experimentally demonstrated in a CS₂ waveguide using the two components of circularly polarized light [157].
The study of symmetry breaking in optical cavities can be dated back as early as the late 1980s when the possibility was pointed out that transverse optical mode instability can lead to the spontaneous formation of stationary patterns (dissipative structures) in an initially uniform optical system [158]. One of the intriguing features of dissipative cavities is that when some of the cavity parameters exceed certain critical values, the symmetric solutions of the intracavity field become unstable with the occurrence of symmetry breaking instability [159, 160]. Temporal symmetry breaking instability in optical dissipative cavities was first theoretically studied in 1995, when it was shown that the intracavity field of a synchronously pulsed driven fiber cavity in the anomalous dispersion regime ($\beta_2 < 0$) can exhibit a temporal shift in the peak of the pulse’s position [161, 162]. Later, another independent theoretical study also rediscovered that the transverse field in a Fabry-Pérot cylindrical cavity filled with a self-focusing Kerr nonlinear medium can develop spatially asymmetric patterns [163]. However, no experimental observation has yet been reported. Recently, a very compact fiber based system in which temporal cavity solitons have been demonstrated, has attracted lots of attention owing to its simplicity [69]. This nonlinear cavity system not only has potential applications in all optical storage, wavelength conversion, and optical regeneration, but also it provides us here an ideal test-bed for our study of temporal symmetry breaking.

In this chapter, we firstly introduce a one-dimensional passive fiber ring cavity in which symmetry breaking instability will be investigated. Numerical simulations of the intracavity pulse in this fiber cavity, which is synchronously driven by a pulsed pump, are performed using a mean field model to illustrate the dynamics of the symmetry breaking process. These simulations reveal that the behaviour of the intracavity pulse depends on the pump power and cavity detuning. As a small asymmetry always exists in practice due to the presence of the odd orders of higher order dispersion, we discuss the influence of third order dispersion on the symmetry breaking process. We then experimentally demonstrate the temporal symmetry breaking in this fiber ring cavity. As a result of this instability, a temporal shift of the peak of the intracavity pulse can be observed and is accompanied with an uneven distribution of spectral power. As the
equation governing the nonlinear cavity dynamics is symmetric with respect to the frame of the propagating pulses, we show that a nearly perfect mirror image of the asymmetric stable solution can also exist. To our knowledge, this is the first experimental observation of *temporal symmetry breaking instability* yet demonstrated in nonlinear optical cavities.

### 6.1 Fiber Ring Cavity

We begin by introducing the passive fiber ring cavity where the temporal symmetry breaking instability will be investigated. A schematic of an all fiber ring cavity is shown in Fig. 6.1. This cavity is simply made of two single mode fiber couplers, and some single mode fiber between the input and output coupler. The intensity reflection and transmission coefficients of the couplers are \( \rho \) and \( \theta \) respectively (\( \rho + \theta = 1 \)). For simplicity, both fiber couplers are assumed to have identical reflection and transmission coefficients. An external coherent optical field is coupled into the cavity through the input coupler. It then propagates along the single mode fiber where it experiences chromatic dispersion and Kerr nonlinearity. Over each round trip, the optical field inside the cavity suffers losses mainly due to the input and output fiber couplers, while the new
incoming optical field is added into the intracavity field coherently and resonantly. The fiber ring cavity behaves similarly to a Fabry-Perot cavity. As the output and intracavity fields are phase sensitive, interferometric stabilization is usually required \([138]\). If the incoming optical field is coherently superimposed on the intracavity field at the input fiber coupler such that they constructively interfere, then intracavity power can be continuously built up. As for a Fabry-Perot cavity, the free spectral range (FSR) of the fiber ring cavity is defined as the frequency separation of two adjacent resonances, and is written as \(\text{FSR} = c/(nL)\), where \(c, n\) and \(L\) are the speed of light, fiber refractive index and cavity length, respectively. The finesse \(\tilde{\mathcal{g}}\) of the ring cavity is defined as the ratio of the free spectral range to the full width half maximum of the resonance peak. It can be approximated as \(\tilde{\mathcal{g}} \approx \pi/\alpha\), where \(\alpha = (\theta_1 + \theta_2)/2\) represents the total round trip loss of the intracavity field. The nonlinear resonance of the intracavity power \(P = |E|^2\) driven by an input power \(P_{in} = |E_{in}|^2\) can be described by the Airy function of optical resonators \([138]\)

\[
\frac{P}{P_{in}} = \frac{1}{\alpha \left( 1 + F \sin^2 \left[ (\phi_0 + \gamma PL)/2 \right] \right)},
\]

where \(F = \rho/(1 - \rho)^2\), \(\rho\) is the coupler reflection coefficient. \(\phi_0\) is the linear round trip phase shift of the intracavity field, and \(\gamma\) is the fiber’s nonlinear coefficient. Fig. 6.2 shows

![Resonances of the nonlinear cavity](image)

Figure 6.2: Resonances of the nonlinear cavity with \(\theta = 0.35\) and \(\gamma P_{in} L = 0.01, 0.2, 0.5, 1\).
the nonlinear resonances of the cavity as a function of linear phase shift. The tilting of
the nonlinear resonances results in the hysteresis or bistability of the fiber cavity, which
is commonly seen in many nonlinear systems with a positive feedback \[164, 165\]. To
explore the behaviour of the intracavity field further, instead of describing the evolution
of the optical field for every consecutive round trip, we here introduce the modified
Lugiato and Lefever model to determine the average evolution of the optical field in the
fiber cavity \[158, 166\]

\[
\frac{\partial E(t, \tau)}{\partial t} = \left[ -1 + i(|E(t, \tau)|^2 - \Delta) - i\eta \frac{\partial^2}{\partial \tau^2} \right] E(t, \tau) + S(\tau).
\] (6.2)

This is a well known mean field model that replaces the discrete round trip evaluation of
the intracavity field by a continuous slow time evolution in the limit of high finesse \(\gamma\). The dispersion and nonlinear coefficients in Eq. (6.2) are normalized to unity. \(\eta = \beta_2/|\beta_2|\),
is the sign of the second order dispersion coefficient. In this work, in order to observe
symmetry breaking instability, our fiber ring cavity has to be synchronously pumped
in the anomalous dispersion regime \((\eta < 0)\) with the repetition rate of the pump pulses
matched to the cavity round trip time. The normalized time scale of the evolution
can be transformed by \(t \rightarrow \alpha t/t_R\), where \(t_R\) is the cavity round trip time. The temporal
profile of the intracavity field is described in the fast time \(\tau\) frame, which is normal-
ized as \(\tau \rightarrow \tau \sqrt{2\alpha/(|\beta_2|L)}\). Eq. (6.2) is essentially a modified driven damped nonlinear
Schrödinger equation with the normalized input driving field and the normalized intra-
cavity field written as

\[
S(\tau) \rightarrow E_{in}(\tau) \sqrt{\gamma L \theta_1/\alpha^3},
\] (6.3)
\[
E(\tau) \rightarrow E(\tau) \sqrt{\gamma L/\alpha}.
\] (6.4)

\(\Delta\) is the normalized cavity detuning and defined as

\[
\Delta = \delta_0/\alpha,
\] (6.5)
where $\delta_0 = 2k\pi - \phi_0 \ll 1$ is the linear cavity round trip phase mismatch of the intracavity field. We initially restrict our analysis to a perfectly symmetric system in which Eq. (6.2) only accounts for the second order dispersion $\beta_2$, and the pump pulse profile is temporally symmetric [i.e. $S(\tau) = S(-\tau)$]. The influence of the inclusion of higher order dispersion will be further analyzed in Section 6.2.1. We can derive the steady state solution ($\partial E / \partial t = 0$) from Eq. (6.2) when the cavity is pumped by a homogeneous field (i.e. CW pumping). Eq. (6.2) can be simplified to

$$\frac{\partial E}{\partial t} = S - \left[1 + i(\Delta - |E|^2)\right]E = 0.$$  \hspace{1cm} (6.6)

We rewrite Eq. (6.6) with the input pump field $S$ as a subject

$$S = \left[1 + i(\Delta - |E|^2)\right]E,$$  \hspace{1cm} (6.7)

and define the dimensionless intracavity field power $Y = |E|^2$, and the input field intensity $X = |S|^2$. The relation between the $X$ and $Y$ is found to be a simple cubic equation

$$X = Y^3 - 2\Delta Y^2 + (\Delta^2 + 1)Y.$$  \hspace{1cm} (6.8)

![Figure 6.3: Bistable response of the intracavity power at cavity detuning $\Delta = 1, 2, 3, 4$. Dotted curves indicate the unstable state regions.](image)
The characteristics S-shape curve can be plotted from the cubic equation Eq. (6.8). Shown in Fig. 6.3 is the normalized intracavity power dependence on the input power at several cavity detunings. Bistability exists when the cavity detuning $\Delta$ is greater than a critical value $\Delta_c = \sqrt{3}$ [167].

### 6.2 Numerical Modeling

To illustrate the formation of symmetry breaking instability in the fiber ring cavity, we employ both the split step method and the Newton-Raphson method to obtain the steady state solutions of the driven damped nonlinear Schrödinger equation Eq. (6.2) [21, 168]. Although symmetry breaking instability can occur for a range of combinations of parameters, for practical purposes we choose to study the system with a normalized pulse width parameter $T_0$ about the order of magnitude of $T_0 = 1$ which is also roughly the modulation instability period. This makes symmetry breaking more easily detectable in practice. Our pump pulse has a perfectly symmetric chirp-free Gaussian profile in the form of $S(\tau) = \sqrt{X} \exp\left[-(\tau/T_0)^2\right]$, $X = 6$, and $T_0 = 2.3$ as plotted in dash-dot curve in Fig. 6.4 (a). The cavity parameters are $\Delta = 0.92$, $\alpha = 0.16$, with only second order dispersion included. The corresponding cavity finesse $\tilde{\gamma} = 20$ is sufficiently high to approximate the intracavity field by the mean field model. The above parameters are chosen such that they correspond to our experimental parameters in the later section. Under these conditions, the mean field driven damped nonlinear Schrödinger equation Eq. (6.2) is perfectly symmetric under a time reversal transformation in fast time $\tau$, yet it admits asymmetric solutions. Fig. 6.4 (a) shows the temporal intensity (solid curve) and phase (dashed curve) of the steady state asymmetric solution of Eq. (6.2) calculated with the parameters given above. As can be seen, the temporal intensity profile of the intracavity pulse is strongly asymmetric with its peak position shifted from the center of the pump pulse. Fig. 6.4 (b) shows the corresponding spectral intensity and phase of the steady state solution. The spectral power is also unevenly distributed with respect to the pump pulse spectrum (dash-dot curve). We emphasize that this symmetry breaking
Chapter 6. Temporal Symmetry Breaking

occurs without any odd order dispersions present, and only occurs due to the fact that the pulse is propagating in a driven-damped nonlinear cavity.

Figure 6.4: $\Delta = 0.92$, $X = 6$, $\alpha = 0.16$. (a) Temporal intensity profiles of the asymmetric intracavity pulse (solid curve), and its corresponding temporal phase (dashed curve, right y-axis) superimposed on the pump pulse (dash-dot curve). (b) Spectral intensity profiles of the asymmetric intracavity pulse (solid curve) and its corresponding spectral phase (dashed curve). (c) and (d) are the temporal and spectral evolution of the intracavity pulse along the normalized slow time, respectively.

To get more insight to the dynamics of the symmetry breaking process, we calculate the temporal and spectral evolution of the normalized fast time $\tau$ intracavity field as a function of the normalized slow time $t$ and plot this in Fig. 6.4 (c) and (d). The initial
intracavity field of the propagation is set to the same as the pump pulse. The evolution shows that both temporal and spectral profiles are perfectly symmetric initially with the temporal and spectral peaks located at the center of the pump pulse. Since the intracavity pulse is unstable under any perturbation, the symmetry breaking occurs as soon as the intracavity pulse starts propagating in the cavity although the shifting of the peak becomes clear between the normalized slow time \( t = 100 \sim 150 \), when the temporal pulse peak shifts from \( \tau = 0 \) to \( \tau = 0.4 \) in the fast time. The spectral power is also redistributed simultaneously with most of the power located on the high frequency side. Once the intracavity pulse reaches the steady state, it maintains its asymmetric shape.

Figure 6.5: \( \Delta = 0.92 \). Theoretical temporal center power \( Y(\tau = 0) \) as a function of normalized input peak power \( X \). Solid curves are the steady stable states, and dash-dot curves are the unstable states. The blue dot indicates the state corresponding to the profile in Fig. 6.4. Black, light green and blue curves represent the homogeneous, symmetric, and asymmetric states, respectively.
We further study the behaviour of the asymmetry of the intracavity pulse by varying the pump power $X$ and tracking the corresponding power of the intracavity pulse at $\tau = 0$. This point also corresponds to the temporal center of the pump pulse. We calculate the right hand side solutions (i.e. peak temporally delays) of Eq. (6.2) with a multi-dimensional Newton-Raphson solver written by Stéphane Coen [168]. The stability of the solutions can be analyzed by the solver, and we can further apply the continuation method of the Newton-Raphson solver to trace out all asymmetric states. We plot in Fig. 6.5 the temporal center power $Y(\tau = 0, X)$ of the intracavity pulse at a detuning of $\Delta = 0.92$, as a function of normalized input peak power $X$. This figure illustrates how the asymmetric temporal profile of the intracavity pulse depends on the input pump power. As the intracavity pulse breaks its symmetry, the peak of the intracavity pulse temporally delays with respect to the center of the pump pulse, simultaneously $Y(\tau = 0, X)$ drops below that of the symmetric state. By comparing this behaviour with the characteristic S-shape curve of CW pumping (black), it can be noticed that $Y(\tau = 0, X)$ of the asymmetric state neither behaves like the CW pumped steady states nor like the pulse pumped symmetric states (green). The asymmetric steady states detours from

![Figure 6.6: Theoretical temporal center power $Y(\tau = 0)$ as a function of normalized input peak power $X$ at cavity detunings $\Delta = 1.8$ (a) and $\Delta = 3.2$ (b). Solid curves are the steady states, and dash-dot curves are the unstable states.](image)

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the unstable symmetric states between the region $X = 4.6 \sim 10.5$ (dash-dot). Inside this region, any unstable states will be spontaneously attracted toward the asymmetric steady states as revealed by the evolution of intracavity pulse in Fig. 6.4. To further explore the behaviour of the intracavity pulse dynamics, we calculate $Y(\tau = 0, X)$ at another two cavity detunings $\Delta = 1.8$ and $\Delta = 3.2$ in which the intracavity pulse exhibits bistability, and plot them in Fig. 6.6 (a) and (b), respectively. Compared to $Y(\tau = 0, X)$ at $\Delta = 0.92$, most of the upper branch states are no longer stable, and they bifurcate to other branches of asymmetric steady states (indicated by blue solid curve) from the upper symmetric unstable states. As cavity detuning $\Delta$ increases, the asymmetric branch is stretched further from the symmetric upper branch, indicating a larger shift in the temporal position of the peak of the intracavity pulses.

Figure 6.7: $\Delta = 3.2$ and $X = 6$. Asymmetric steady state solution (blue curve), superimposed with a cavity soliton (red dotted curve) generated by CW pumping (power level indicated by the dash-dot line). The low state solution (green solid) is generated by a pulsed pump (black dashed curve).
To get some insight to the origin of symmetry breaking instability, we attempt to interpret the formation of symmetry breaking instability in terms of temporal cavity soliton. We first determine the asymmetric steady state solution (blue solid curve) in Fig. 6.7 with the cavity parameters of $\Delta = 3.2$ (bistable) and $X = 6$, and superimpose this onto the pump pulse profile (black dashed curve). We then work out the cavity soliton profile (red dotted curve) for the same cavity detuning, and a CW pump power of $X = 3.57$ which corresponds to the pulse pump level where the peak of the asymmetric solution is located (as indicated by a circle). We also determine the lower steady state solution (green solid curve) for a pulsed pump with the same setting ($\Delta = 3.2$ and $X = 6$). Fig. 6.7 clearly shows that the asymmetric solution is almost perfectly matched to the central portion of the cavity soliton, while the low state solution also precisely encloses the remnant of the asymmetric solution. The formation of the symmetry breaking instability can be interpreted in this way: As the upper state solution is unstable during the propagation, the intracavity field has to redistribute and self-reorganize to a more stable form, owing to the robust attractive nature of the dissipative structure—in this case, a cavity-soliton-like structure [69].

### 6.2.1 Influence of Third Order Dispersion

Up to now, our analysis of the symmetry breaking has solely considered a perfectly symmetric (in fast time $\tau$) system. In practice, optical fibers always possess high order dispersion, and this inevitably introduces some asymmetry into the system due to the odd terms of high order dispersion $\beta_{2n+1}$, ($n \geq 1$). We now include the high order dispersion into our analysis, and see how it influences the behaviour of the symmetry breaking instability. The mean field nonlinear Schrödinger equation with the high order dispersion included becomes

$$
\frac{\partial E(t, \tau)}{\partial t} = \left[ -1 + i(|E(t, \tau)|^2 - \Delta) - i\eta \frac{\partial^2}{\partial \tau^2} + \sum_{n=3}^{\infty} i^{n+1} d_n \frac{\partial^n}{\partial \tau^n} \right] E(t, \tau) + S(\tau). \tag{6.9}
$$
The higher order dispersion coefficients $d_n (n > 3)$ are normalized as

$$d_n = \frac{\beta_n L}{n! \alpha \left( |\beta_2| L \right)^{2}}. \quad (6.10)$$

We here only consider the higher order odd dispersion term contributed from the third order dispersion $\beta_3$ as the effects of even higher orders should be negligible. We repeat the same treatment of searching for all the temporal center powers $Y(\tau = 0, X)$ of the intracavity pulse using the Newton-Raphson solver with $\beta_3$ included, and plot it in Fig. 6.8 using the same parameters as used in Fig. 6.5, and an extra third order dispersion parameter $d_3 = 0.002$ (corresponding to our experimental value). On top of the

![Figure 6.8](image.png)

Figure 6.8: $\Delta = 0.92$. Theoretical temporal center power $Y(\tau = 0, X)$ with the third order dispersion $d_3 = 0.002$ included. Dashed curves are the unstable states. Green and black curve are the symmetric and asymmetric states without the third order dispersion, blue and red are the temporal delayed and temporally advanced asymmetric states with the third order dispersion included.
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\( Y(\tau = 0, X) \) curve without the third order dispersion in Fig. 6.5, we also plot in Fig. 6.8 \( Y(\tau = 0, X) \) of the temporal advanced and delayed intracavity pulses for comparison. Clearly, the degeneracy of the temporal advanced and delayed asymmetric states has been removed. The lifting of degeneracy occurs at \( X = 4 \) as the pump power increases, and the intracavity pulse is attracted toward the temporal delayed asymmetry due to the third order dispersion. Also we can see that a branch of temporal advanced asymmetric states is neither connected to the branch of temporal delayed asymmetric states nor to the branch of symmetric states, forming an isolated loop between the branches of symmetric and temporal delayed asymmetric states.

![Figure 6.9](image)

Figure 6.9: Theoretical temporal profile of \( \Delta = 0.92 \) at \( X = 6 \) with (solid curves) and without (dash-dot curves) the third order dispersion. (a) symmetric unstable state (b) asymmetry steady states.
To observe the influence of the third order dispersion on the temporal profile, we plot in Fig. 6.9 the symmetric (a) and asymmetric (b) states in both cases with and without the third order dispersion for comparison. Note that there is a small offset between the temporal profiles with and without inclusion of the third order dispersion. In the presence of the third order dispersion, the symmetric state sees the intracavity pulse is slightly advanced (and is therefore, slightly asymmetric), whereas the asymmetric state pulses are both slightly delayed. Nonetheless, the behaviour of the symmetry breaking instability in the presence of high order dispersion is essentially unchanged, and the asymmetry due to the third order dispersion is relatively small compared to that due to the symmetry breaking instability in the nonlinear cavity.

6.3 Experimental Observation

6.3.1 Experimental Setup

We perform our temporal symmetry breaking instability experiment using standard telecommunication components. The experimental setup is shown in Fig. 6.10. The pump pulses are derived from a passively mode locked fiber laser (Calmar FPL-02C) with pulse width of 500 fs, repetition rate of 20 MHz, and center wavelength at 1550 nm. The pulses are then broadened to 2.3 ps by spectral filtering with a band pass filter (BPF). 1% of the power of the pump pulses is tapped off with a 99/1 coupler and launched counterclockwisely into the fiber ring cavity to serve as a cavity detuning control beam. The remaining 99% is amplified by an Erbium doped fiber amplifier (EDFA) to increase the peak power. 5% of the power of the amplified pump pulses is split and recombined at the cavity output with a 60/40 fiber coupler, to serve as reference pulses to monitor the position of the temporal peak of the output pulses. The remaining 95% of the power is coupled clockwisely into the ring cavity through an 80/20 fiber coupler. The fiber ring cavity is made out of 10 m of single mode fiber (Corning SMF 28) with a nonlinear coefficient $\gamma \approx 1.8 \text{ W}^{-1}\text{km}^{-1}$, a second order dispersion coefficient $\beta_2 \approx -20 \text{ ps}^2/\text{km}$ and
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Figure 6.10: Experimental setup for the observation of the symmetry breaking instability. FPL: fiber passive mode-locked laser, BPF: bandpass filter, EDFA: Erbium-doped fiber amplifier, PC: polarisation controller, MS: mechanical stretcher, PFS: piezoelectric fiber stretcher, PID: feedback controller, OSA: optical spectrum analyzer, FROG: frequency-resolved optical gating.

A third order dispersion coefficient $\beta_3 \approx 0.1 \text{ ps}^3/\text{km}$ at 1550 nm. Note that the negative second order dispersion coefficient ($\eta = -1$) is necessary for the occurrence of temporal symmetry breaking. The input pump pulses are analyzed using frequency resolved optical gating (FROG) before entering the fiber cavity and confirmed to have a nearly chirp-free Gaussian profile with a full width half maximum of 2.3 ps. A variable attenuator is placed just before the input of the fiber ring cavity to adjust the pump pulse power, and the maximum peak power of the pump pulses entering the cavity is estimated to be about 20 W, which corresponds to a normalized power of $X \approx 16$. Other parameters are: the normalized pulse width $T_0 = 2.3$, and the third order dispersion parameter $d = 0.002$. The intracavity pulses and the control beam are monitored using a 99/1 coupler at the cavity output. The finesse of the cavity is measured to be $\tilde{\chi} = 20$ which justifies the use of the mean field Eq. (6.2). The cavity length is precisely adjusted with a mechanical stretcher (MS) to synchronize the cavity length to the repetition rate.
of the pump pulses to within 25 fs (5 µm). The ring cavity is phase sensitive, therefore the round trip phase shift of the cavity must thus be stabilized so as to control the cavity phase detuning $\Delta$. This is done with a servo feedback system consisting of a photodiode (New Focus front-end optical receivers Model-2001), PID controller (SRS SIM960-100 kHz analog) and a piezoelectric fiber stretcher (PFS). Polarization controllers (PC) are placed before the input of the cavity as well as inside the cavity to allow full control of the polarizations of the pump pulses and the control beam, such that the locking of the control beam to a set cavity detuning $\Delta$ also sets correspondingly the cavity detuning of the pump [138]. The spectral and temporal profiles of the intracavity pulses are then analyzed with an optical spectrum analyzer (OSA) and FROG, respectively.

### 6.3.2 Results

We first study the temporal symmetry breaking instability by measuring the asymmetric power distribution of the intracavity pulses’ spectra. The parameters of our experiments are the same as those used in the numerical modeling section. Initially, the symmetry breaking instability is investigated at a small cavity detuning of $\Delta = 0.92$, which has no bistability present. Fig. 6.11 (a) shows the temporal center power $Y(\tau = 0, X)$ of the asymmetry calculated with the third order dispersion included. The dash-dot curve indicates the unstable states, and the stable asymmetric states occur near the region of $X = 4 \sim 11$ where the solid curve detours from the dash-dot curve. Following the curve shown in Fig. 6.11 (a), the asymmetric solutions can be readily excited by simply ramping up the pump pulse power to reach the dash-dot curve region. The measured spectrum of the intracavity pulse, plotted in Fig. 6.11 (b) as a function of pump power $X$, directly reflects the asymmetric behaviour of the spectral profile. As can be seen, the spectral power starts shifting toward the high frequency side as the pump power increases. As the spectral profile of the intracavity pulse becomes asymmetric, so does the temporal profile. The experimentally measured spectral profile of the asymmetric state at $X = 6.4$ is plotted in Fig. 6.11 (c) (red dots), and it is in excellent agreement
Figure 6.11: $\Delta = 0.92$ (a) Theoretical calculation of the temporal center power of the intracavity pulses. Dash-dot part of curve is the unstable solution. (b) Measured intracavity spectra as a function of pump power. (c) Theoretical and experimental spectral profile at $X = 6.4$. Solid curve, dotted curve and dash-dot curves are the theoretical, experimentally measured and pump spectra, respectively (d) Asymmetric ratio of the output spectra as a function of pump power. Solid and dash-dot curve are the numerical simulation with and without the third order dispersion, circles are the experimental measurements.

with the theoretical prediction (blue solid). The input spectrum is also plotted (green dash-dot), and we can observe that the asymmetric state has a main peak shifted to the high frequency side, and also a broader wing present on the low frequency side.
Nearly 66% of the spectral power is distributed on the high frequency side of the pump center frequency. To quantitatively compare the measured asymmetry of the intracavity pulse to our theoretical prediction, we define the spectral asymmetric ratio as the ratio of the integrated spectral power on the high frequency side of the center frequency of the pump to that on the low frequency side. The asymmetric ratios are evaluated by integrating the spectral power in Fig. 6.11 (b) and the result is plotted in Fig. 6.11 (d). The asymmetric ratio is initially equal to 1 at low pump power (corresponding to a perfectly symmetric state), and rapidly rises from 1 to 1.9 when the pump power increases from $X = 4$, to $X = 6.4$ where the maximum asymmetry has been found. The experimental data shows good agreement with the theoretical calculation. Because of the third order dispersion, we can see in Fig. 6.11 (d) that the onset of symmetry breaking is more progressive than without the third order dispersion (dash-dot curve). One piece of strong evidence for symmetry breaking in the Fig. 6.11 (d) is that this ratio instead of growing monotonically with the pump power, starts diminishing after the maximum asymmetric state with perfect symmetry restored at $X \geq 11$. This behaviour confirms that the observed asymmetry is not simply due to the third order dispersion, but is really tied to the temporal symmetry breaking instability of the nonlinear cavity dynamics.

We next increase the cavity detuning to $\Delta = 3.2$, and perform a similar set of measurements. At this detuning, Fig. 6.12 (a) shows the theoretical bistable characteristics of the Kerr cavity and the region where the stable solutions (solid curve) exist. The symmetry of the intracavity pulses break down in the upper branch for $X > 4$ where the temporal center power $Y(\tau = 0, X)$ departs from the symmetric states branch, and follows a new branch of solutions corresponding to the symmetry breaking. To excite the asymmetric state, we follow the direction indicated by the dotted arrows in Fig. 6.12 (a). The measured spectra and their corresponding asymmetric ratios are plotted in Fig. 6.12 (b) and (d), respectively. As the pump power ramps up, the lower branch states remains perfectly symmetric with the asymmetric ratio equal to 1. When the pump power is greater than $X = 8.4$, the lower branch states no longer exist, and the intracavity spectral power automatically switches to the asymmetric branch as shown in upper plot.
Figure 6.12: \( \Delta = 3.2 \) (a) Theoretical calculation of the temporal center power of the intracavity pulses. Dash-dot part of curve is the unstable state. (b) Measured intracavity spectra as a function of pump power. (c) Theoretical and experimental spectral profile at \( X = 10 \). Solid curve, dotted curve and dash-dot curve are the theoretical, experimentally measured and pump spectra, respectively. (d) Asymmetric ratio of the output spectra as a function of pump power. Solid and dash-dot curves are the numerical simulation with and without the third order dispersion, circles are the experimental measurements. Dashed arrows indicate the switching directions.

of Fig. 6.12 (b). The asymmetric ratio of the upper branch is measured by lowering the pump power toward the bifurcation point. When the pump power is below the
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bifurcation point, the intracavity field is switched back into the symmetric state, which is clearly visible in the lower plot of Fig. 6.12 (b) at X = 4. The bifurcation behaviour of symmetry breaking can also be seen in the asymmetric ratio of the spectral power plotted in Fig. 6.12 (d) (circles are the experimental data) with bifurcation occurring at X = 4. The asymmetric ratio increases almost linearly with the pump power until X > 10 where no stable solution can be observed, and the intracavity pulse breathes periodically. The influence of the third order dispersion on the asymmetric ratio is not as profound as in the case of \( \Delta = 0.92 \), with a small deviation from that without the third order dispersion (plotted in Fig. 6.12 (d) as a dash-dot curve). At X = 10, the asymmetric spectrum of the intracavity field shown in Fig. 6.12 (c) has a maximum of 73% of the spectral power distributed on the high frequency side, and very good agreement can be found between the experimental measurement (red dotted curve) and the theoretical prediction (solid blue curve).

We have also experimentally measured the temporal profiles of the asymmetric steady states indicated by the blue dots in Fig. 6.11 (a) (X = 6.4, \( \Delta = 0.92 \)) and 6.12 (a) (X = 10, \( \Delta = 3.2 \)). Because of the presence of imperfections in our system (mainly due to third order dispersion and the desynchronization of the pump pulses), the peak of the intracavity pulses always shift spontaneously toward the high frequency side. By introducing a small amount of desynchronization of 25 fs to the cavity length, one can observe the nearly mirror image asymmetric steady states. The FROG recovery traces in Fig. 6.13 (a) and (b) show the temporal profiles of the intracavity pulses (solid curve) at the maximum asymmetric ratio with respect to the perfect symmetric states (dash-dot curve) for cavity detuning of \( \Delta = 0.92 \) and \( \Delta = 3.2 \), respectively. The delayed subpulses on the right hand side are the reference pump pulses which were split before entering the cavity and recombined at the output, providing a timing reference for the FROG measurement. Clearly, the recovered pulses from the FROG exhibit strongly asymmetric temporal profiles compared to the temporal profiles of the symmetric states, with significant temporal shifts from the center peak of the perfectly symmetric pulses. The mirror reflection profiles of the asymmetric states do not possess perfect reflection.
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We attribute this non-perfect reflection symmetry of the asymmetric pulses to the change in cavity desynchronization needed to excite each state.

![Figure 6.13](image)

Figure 6.13: Mirror reflection profiles of the FROG traces of the intracavity pulses at $\Delta = 0.92$ (a) and $\Delta = 3.2$ (b). The dash-dot curves are the perfect symmetric pulses at the low pump power states.

6.4 Summary and Discussion

In summary, symmetry breaking is a ubiquitous phenomenon that is commonly observed in physics. Here we have studied temporal symmetry breaking instability in a Kerr type one-dimensional nonlinear fiber cavity which is synchronously driven by a pulsed pump. This extremely simple nonlinear cavity provides us with a robust envi-
Summary and Discussion

ronment to investigate the dynamics of the symmetry breaking instability. By solving the driven damped nonlinear Schrödinger equation using the split-step and Newton-Raphson methods, it has shown that the intracavity pulse of this nonlinear cavity can exhibit symmetry breaking. The intracavity pulse evolves into an asymmetric steady state even though the governing equation is perfectly symmetric (in the absence of high order dispersion). By fixing the cavity detuning, we are able to trace out all the steady states for a range of pump powers using the Newton continuation method. Although the simulation results show that the degeneracy of the asymmetry is lifted due to the presence of third order dispersion, the strong asymmetry of the temporal and spectral profiles is still mainly due to the symmetry breaking instability of the nonlinear cavity dynamics.

Thanks to the simplicity of this Kerr type fiber cavity, we have experimentally observed the temporal symmetry breaking instability. The experimental evidence for the existence of symmetry breaking is demonstrated by measuring the asymmetric ratio of the spectral power of the intracavity pulse. We further confirm this symmetry breaking by demonstrating the recovery of the symmetry of the asymmetric ratio at the detuning of $\Delta = 0.92$. We have also shown that the symmetric and asymmetric states can coexist at the detuning of $\Delta = 3.2$ in which the cavity exhibits bistability. We have also directly measured the intracavity temporal profile of the asymmetric states using a FROG. A significant temporal shift of the peak of the intracavity pulses has been observed with respect to the intracavity pulses of the symmetric lower states. To our knowledge, our result is the first demonstration of the temporal symmetry breaking instability in nonlinear optical cavities. Given the importance of symmetries in physical theories, it reinforces the status of passive Kerr cavities as a paradigm of nonlinear systems subject to instabilities.
In this thesis, a broad class of $\chi^{(3)}$ nonlinear phenomena have been studied in optical fibers based systems. Thanks to the simplicity of fiber waveguide structure, the one-dimensional scalar treatment of nonlinear interaction facilitates the comparison between our experimental observations and theories. We have concentrated our studies on four distinct topics, including incoherently pumped fiber parametric amplifiers, cascaded four-wave mixing, fiber optical parametric oscillators and temporal symmetry breaking in a fiber ring cavity. In the following section, we briefly summarize the main results of each of the four experimental chapters. Finally, we outline the future research direction and some potential applications of these works.

- In Chapter 3, a fiber parametric amplifier pumped by a temporally incoherent pump has been studied in the telecommunication band. A simple theory has been derived to evaluate the maximum attainable mean parametric gain when the amplifier is seeded with a coherent signal. The experimental gain measurements are in excellent agreement with our simple model. Due to the presence of intensity fluctuations in an incoherent pump, the parametric gain slope of the incoherently pumped amplifier was measured to be seven times higher than that of a coherently pumped amplifier with an equivalent average pump power. The gain statistics of the parametric amplifier have also been experimentally measured, and show good agreement with the theoretical predictions derived from our simple model. The
L-shape gain probability distribution is a consequence of the phase matching of large frequency shift sidebands by rare high power events.

- In Chapter 4, the amplification of higher order sidebands in cascaded four-wave mixing in nonlinear fiber optics has been investigated. This amplification is the result of the accumulated phase matching of each of the elementary processes. Direct amplification of the second order sideband in the anomalous and normal dispersion regimes has been observed. The measured frequency detunings of the phase matched sidebands are in good agreements with those calculated using the analytical higher order phase matched equation. A pulse-pumped experiment has been performed to show that the frequency shift of the dispersive wave emitted by a pulsed pump exactly coincides with the frequency shift of a higher order phase matched sideband from a bichromatic pump. We believe this is the first time that the origin of dispersive wave emission has been interpreted in terms of cascaded four-wave mixing. This interpretation has led us to numerically and experimentally study the generation of dispersive waves in the anomalous dispersion regime when pumping in the normal dispersion regime. The frequency shift of the dispersive wave when pumping in the normal dispersion regime has been found to be smaller than that predicted by the higher order phase matched condition. We attributed this discrepancy to the preferential temporal overlap between the initially emitted dispersive wave and the frequency components of the pump pulse closer to the ZDW. In the context of cascaded four-wave mixing, we have also considered a cascade of nonlinear Bragg scattering whereby an initial weak probe signal is switched to a new frequency channel. We derived the phase matched condition for this process. The experimental frequency shifts of higher order idlers are in very good agreement with the frequency shifts predicted by the theoretical phase matched condition. By comparing the discrete experimental comb output spectra of the Bragg scattering cascade to the continuous output spectra of a simulated collision between a single soliton and a weak linear wave,
we have also been able to identify that the underlying nonlinear mechanism of the soliton-linear wave interaction in the presence of high order dispersion is the cascaded Bragg scattering.

- In Chapter 5, a fiber optical parametric oscillator with a mulitwatt-level average output power has been built from a standard telecommunications fiber and fiber components. Theoretically, there is no upper limit on the output power of parametric oscillators as parametric processes require no power exchange with the nonlinear medium. The only practical limit is the damage threshold of the fiber components by high optical power. By singly resonating the anti-Stokes sideband inside a fiber ring cavity, widely tunable sidebands with narrow linewidth have been generated between 1350 nm and 1790 nm. Between 1450 nm and 1670 nm, the average power of the sidebands output is in excess of 3 W. By controlling the phase mismatch of the parametric process, a high conversion efficiency fiber parametric oscillator has also been demonstrated. The control of the phase mismatch is achieved by introducing an intracavity filter to select the frequency of the oscillating sideband. A maximum internal conversion efficiency in excess of 93% has been demonstrated.

- In Chapter 6, temporal symmetry breaking instability has been numerically and experimentally studied in a fiber ring cavity which was driven synchronously by a pulsed pump. Numerical results have been performed to show how the behaviour of the intracavity field dynamics depends on the pump power by plotting the trajectory of the intracavity pulse power at the temporal center of pump pulsed. We have interpreted the formation of asymmetric states in terms of the attractive nature of cavity solitons, in which an unstable symmetric state evolves into an asymmetric steady state through self-organization. The experimentally measured spectral asymmetric ratio of the intracavity pulse has provided us with strong evidence for the existence of temporal symmetry breaking instability. Optical bistability has also been observed in the spectral asymmetric ratio at a large cavity detuning.
parameter. The temporal shifts of the peak of the intracavity pulses with respect to the pulses of the symmetric steady states have been further experimentally confirmed by FROG measurements.

The above studies have hitherto given us insight into some of the fundamental phenomena in nonlinear fiber optics. It is not yet the end of our exploration, and we outline some future work and applications in the following section.

- Incoherent nonlinear optics has been recently revisited by the optics community. Particularly, rogue waves in optical fibers have drawn wide attention due to their probabilistic nature which has also been observed in hydrodynamics and ocean waves \[31, 103, 169\]. Although the random gain fluctuations of an incoherently pumped parametric amplifier are detrimental to telecommunication data as the amplified signal loses its fidelity, the advantage of high mean gain and low average pump power could potentially find used in other applications, e.g. a source with an adjustable intensity noise with a controllable probability distribution function for testing optical communication systems. Recently, theoretical studies have shown that a coherent parametric sideband can be generated from a temporally incoherent pump in Kerr type media via a phase locking mechanism and convection. Simultaneously the other sideband with same group velocity as of the pump will absorb the temporal intensity fluctuations from the pump \[170, 171\]. Experimental demonstration of this effect would be of great interest to scientific community, as it would demonstrate that temporally coherent light can also be generated from incoherent light using a parametric process instead of using laser systems.

- Recent studies of frequency combs in microresonators have shown their high potential in spectroscopy and telecommunications \[105, 107, 172\]. Cascaded four-wave mixing plays a crucial role in the ultrabroad frequency comb generation. The nonlinear mechanism of dispersive wave generation has for the first time been interpreted in terms of cascaded four-wave mixing in this thesis. This description
is expected to provide insights into a plethora of optical systems, including Kerr combs generated in ring resonators [173]. As a nonlinear phenomenon, the cascaded mechanism could also be manifested in a wide range of other systems such as plasma physics, Bose-Einstein condensates, and hydrodynamics [174, 175].

• The development of fiber optical parametric oscillators is now relatively mature, with the possibility of commercialization in the near future. Although the tunability of fiber parametric oscillators is currently limited primarily by the longitudinal dispersion fluctuations of the gain fiber, this limitation can be overcome by increasing the fiber nonlinearity and shortening the fiber length. One simple approach would be to replace the standard commercial DSF that used in our experiment with a highly nonlinear DSF of a few meters’ order. To further simplify the pulsed pump system for the fiber parametric oscillator, the pump pulse could be generated by using a gain switched laser system [176].

• As a universal phenomenon, the symmetry breaking instability in fiber based nonlinear resonators provides a simplified environment for the studies of symmetry breaking in other macroscopic systems. The self-organized behaviour in nonlinear optics can be potentially used as an analogue system to study problems in chemical reactions and developmental biology [153].


