Equilibrium Quality Choices with Generalized Smooth Cost Function

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Abstract

We show that the effect of credible quality commitment on quality choice with Bertrand and Cournot competition in the product market for quadratic cost of quality function (Aoki (2000)) holds for more general cost functions. Specifically, we compare the quality choices with sequential and simultaneous quality choices when cost of quality q is kq^n where k is a positive constant and n is any integer greater than 2. The first mover will always choose to produce higher quality, even when cost of quality increases very rapidly (n is large). All previously identifies qualitative comparisons between Bertrand and Cournot competition also extend to the generalized cost function.

JEL Classification: L1, D4, C7

Key Words: sequential vs. simultaneous choices, Bertrand and Cournot competition, vertical quality differentiation

1 Introduction

The purpose of this paper is to generalize the results of Aoki (2000) to a more general smooth cost function. Specifically, we compare sequential and simultaneous quality choices by duopolists when their products are differentiated by vertical quality when cost of quality q is,

$$C(q) = kq^n,$$

where k is a positive constant and n is a positive integer. Previous analysis in Aoki(1995,2000) was restricted to $n = 2$, quadratic cost of quality.

It was shown in Aoki (2000) that independent of type of competition in the sales stage (Bertrand or Cournot), first mover always chooses a quality higher than the second mover. For any pair of qualities, revenue (not profit) is greater for producer of the higher quality. One expects that if cost increases sufficiently with quality (large n in our cost function), profit can be greater for the producer of lower quality. We show that this will never occur at equilibrium qualities. Because the cost function is smooth, marginal cost also increases with n, and therefore higher equilibrium quality decreases with greater n. The reduction in revenue will be smaller than increase in cost and thus it remains more profitable to be the higher quality producer even for very large n.

We also show that relationship between simultaneous and sequential quality choices with Bertrand and Cournot competition claimed in Aoki (2000) all still hold with the generalalized cost function. This is because these properties are determined by the sign of best-response correspondences' slopes. The sign of the slopes only depend on revenue function and thus are independent of cost function specification.

The paper proceeds as follows: In the next section we present the underlying demand system (model of vertical quality differentiation) and the whole framework. In section 3 we will characterize the equilibrium under the two timing scenarios when firms engage in Bertrand competition at the sales stage. In section 4 we will do the same for Cournot competition.

2 The Model

We employ the standard model of vertical quality differentiation with heterogeneous consumers (Gabszewicz and Thisse (1980)). There are two firms which produce vertically differentiated products. In the first stage, firms choose qualities. Products are produced and sold in the second stage. We consider two possible strategies: prices (Bertrand) or quantities (Cournot). We characterize the subgame perfect Nash equilibrium in pure strategies for each game.

There are two identical firms, 1 and 2. In the first stage each firm chooses quality level, q_i, of its product. Cost of quality q_i is,

$$C(q_i) = kq_i^n,$$

where $k > 0$, and $n \geq 2$ is an integer for $i = 1, 2$. We consider two cases:

i) **simultaneous choice**: q_i's are chosen simultaneously

ii) **sequential choice**: q_1 is chosen first and revealed, then q_2 is chosen.

In both cases, quality choices are made common knowledge at the end of the first stage. Thus for a given pair of quality choices (determined in the first stage), the second stage game becomes identical under the two timing scenarios.

In the second stage, firms either set prices or quantities simultaneously. There is a continuum of consumers indexed by t which is uniformly distributed on [0,1]. A consumer with index $t \in [0, 1]$ consuming one unit of product with quality q_i at price p_i has surplus of $v(q_i, p_i; t) = q_i t - p_i$. A consumer will buy one unit of a product when surplus is positive and greater than the surplus from consuming the other product.

Specifically, a type t consumer will purchase the lower quality product, say q_j, if and only if $0 \leq v(q_j, p_j; t) > v(q_i, p_i; t)$ and will purchase the higher quality product, q_i, if and only if $0 \leq v(q_i, p_i; t) \geq v(q_j, p_j; t)$. Quantity of high quality product sold is $x_i(p_i, p_j) = 1 - t_i(p_i, p_j)$ where $v(q_i, p_i; t_i(p_i, p_j)) = v(q_j, p_j; t_i(p_i, p_j))$. Quantity of low quality product sold is $x_j = t_i(p_i, p_j) - t_j(p_j, p_i)$ where $0 = v(q_j, p_j; t_j(p_i, p_j))$.

For any pair of qualities, q_j and q_i, quantities (x_i, x_j) or prices (p_i, p_j) may be the choice variables. With Bertrand competition, firm i chooses p_i to maximize revenue $p_i x_i$. With Cournot competition, it will choose x_i. Since the focus of the paper is the timing of quality choices, we omit derivation of equilibrium of the second stage, which is very straightforward (Gabszewicz and Thisse (1980), Shaked and Sutton (1982)) contain detailed analysis of the first stage. Aoki (2001) shows uniqueness by presenting a complete characterization of the best-response correspndences for Bertrand and Cournot competitions.

3 Quality Choices under Bertrand Competition

Subgame perfect Nash equilibrium requires that all actions constituting the equilibrium strategies are optimal at beginning of each subgame. For the purpose of our analysis, this implies that we only need to examine the second stage revenue generated by the Nash equilibrium actions (prices or quantities) of the second stage subgame. There is a subgame corresponding to each possible pair of qualities, (q_1, q_2).

Because the firms are otherwise identical, what is relevant for characterization of revenue and profit the two qualities,independent of firm identities. So we often denote qualities as (q_H, q_L) with $q_H \geq q_L$. There are two subgames with this quality pair: $q_1 = q_H$ and $q_2 = q_H$. Then we define the equilibrium revenue function of the second stage subgame with quality pair (q_H, q_L) by $R^t(q_H, q_L), t = H, L$. $R^t(q_H, q_L)$ is the revenue of firm with quality q_t. We summarize the properties of the revenue function below.[1]

Lemma 1 *When there is Bertrand competition at the sales stage, the equilibrium revenue function is continuous* $\forall (q_H, q_L)$, $q_H \geq q_L$ *and twice continuously differentiable* $\forall q_H \neq q_L$.

$$\begin{cases} R^H(q_H, q_L) = \frac{4(q_H)^2(q_H - q_L)}{(4q_H - q_L)^2}, \\ R^L(q_H, q_L) = \frac{q_H q_L (q_H - q_L)}{(4q_H - q_L)^2}. \end{cases} \tag{1}$$

$$\begin{cases} R^L_L \gtreqless 0 \Leftrightarrow q_L \lesseqgtr \frac{4}{7} q_H, R^L_H > 0, R^L_{LL} < 0, R^L_{LH} = R^L_{HL} > 0, \\ R^H_H > 0, R^H_L < 0, R^H_{HH} < 0, R^H_{HL} = R^H_{LH} > 0. \end{cases} . \tag{2}$$

In the subgame (q_i, q_j), firm i's revenue is $R^H(q_i, q_j)$ for $q_i \geq q_j$ and $R^L(q_j, q_i)$ for $q_i \leq q_j$. A typical revenue function for firm 1 is depicted in Figure 1. [2] When firm 2's quality is q_2, firm 1's revenue is depicted by the thick revenue line; the thin revenue line depicts firm 1's revenue when firm 2's quality is slightly larger, say q_2'. Basically, with Bertrand competition, both firms prefer qualities to be further apart. There is marginal gain when qualities become further apart ($R^L_H > 0$ and $R^H_L < 0$). Similarly, (the absolute value of) marginal loss ($R^L_L < 0$ or $R^H_L < 0$) becomes smaller as qualities become further apart ($R^L_{HL} > 0$, $R^H_{LH} > 0$).

Firm i's first stage payoff[3] is

$$\Pi^i(q_i, q_j) = \begin{cases} \Pi^H(q_i, q_j) = R^H(q_i, q_j) - C(q_i) & q_i \geq q_j \\ \Pi^L(q_j, q_i) = R^L(q_j, q_i) - C(q_i) & q_i \leq q_j \end{cases}$$

[1] Superscript on functions denote firm identity and subscripts denote partial derivatives.

[2] Derivation of all figures in Aoki (1995) can be applied for this general cost function.

[3] For defining firm i's profit, we list the argment as (q_i, q_j), when defining profit of firm with quality q_t, we list the argment (q_H, q_L).

There are always two local maxima to the firm's payoff maximization problem: one below and one above the rival's quality level, denoted by $q^L(q_j) \leq q_j$ and $q^H(q_j) \geq q_j$, respectively. (See Figure 1.) These are constrained best-response correspondences. $q^L(q_j)$ solves,

$$\max_{q_L} \Pi^L(q_j, q_L) \text{ subject to } q_L \leq q_j,$$

and $q^H(q_j)$ solves

$$\max_{q_H} \Pi^H(q_H, q_j) \text{ subject to } q_H \geq q_j.$$

Since $R_L^L(q_H, 0) = \frac{4}{27}$ and $C'(0) = 0$, for any q_j, $q^L(q_j)$ is an interior solution that satisfies the first-order condition, [4]

$$R_L^L(q_j, q_L) - C'(q_L) = \frac{q_j^2(4q_j - 7q_L)}{(4q_j - q_L)^3} - nk(q_L)^{n-1} = 0. \quad (3)$$

However since $R_H^H(q_j, q_j) = \frac{4}{9}$, the best-response correspondence satisfies the first order condition,

$$R_H^H(q_H, q_j) - C'(q_H) = \frac{4q_H(4q_H^2 - 3q_H q_j + 2q_j^2)}{(4q_H - q_j)^3} - nk(q_H)^{n-1} = 0, \text{ for } q_j \leq (\frac{4}{9nk})^{\frac{1}{n-1}}. \quad (4)$$

It will be a corner solution,

$$q^H(q_j) = q_j \text{ for } q_j \geq (\frac{4}{9nk})^{\frac{1}{n-1}}.$$

When q_j is very large, both revenue and marginal revenue from choosing a higher quality is very low. In this case, the profit maximising quality higher or equal to that of rival is to choose the equal quality. Obviously this cannot be the global best-response since profit is negative.

It follows from the signs of R_j^i's in (2) and the envelope theorem that $\Pi^i(q^H(q_j), q_j) = \Pi^H(q^H(q_j), q_j)$ is decreasing in q_j while $\Pi^i(q^L(q_j), q_j) =$

[4] Note local concavity of $\Pi^i(q_i, q_j)$ in $\forall q_i \neq q_j$, $C'(0) = 0$, and $\lim_{q_i \to \infty} \Pi^i(q_i, q_j) = -\infty$.

$\Pi^L(q_j, q^L(q_j))$ is increasing in q_j. Thus, there is a unique $\hat{q}$ such that $\Pi^i(q^L(\hat{q}), \hat{q}) = \Pi^i(q^H(\hat{q}), \hat{q})$ and $\beta^i(q_j) = q^H(q_j)$ for $q_j \leq \hat{q}$ and $\beta^i(q_j) = q^L(q_j)$ for $q_j \geq \hat{q}$. Total differentiation of the first-order condition of maximization and (2) yields the following:

Lemma 2 *When there is Bertrand competition in the second stage, the first stage best-response correspondence is,*

$$\beta^i(q_j) = \begin{cases} q^H(q_j) > q_j \text{ and increasing for } q_j \leq \hat{q}, \\ q^L(q_j) < q_j \text{ and increasing for } q_j \geq \hat{q}. \end{cases}$$

The best-response correspondences are depicted in Figure 2. (The iso-profit curves are those of firm 1 and the arrows indicate direction of increasing profit). From the previous observation, $\hat{q} \leq (\frac{4}{9nk})^{\frac{1}{n-1}}$.

The intercept of $q^H(\cdot)$, i.e., $q^H(0)$, is the firm's quality choice when rival has the minimum quality possible, 0. This is a bench mark quality level and we denote this by q^m. From first order condition of maximizing $\Pi^i(q, 0) = \Pi^H(q, 0)$, we have

$$q^m = (\frac{1}{4nk})^{\frac{1}{n-1}}.$$

Quality is smaller when cost is larger (k larger) and cost increases more quickly (n larger).

We have the following existence and characterisation of equilibrium when firms choose qualities simultaneously.

Proposition 1 *For any n, there are two pure strategy Nash equilibria (E^1_{SIM} and E^2_{SIM}) to the simultaneous quality choice game when there is Bertrand competition at the sales stage.*

(i) The equilibrium qualities of the two equilibria are the same: $E^1_{SIM} = (q^H_{SIM}, q^L_{SIM})$ and $E^2_{SIM} = (q^L_{SIM}, q^H_{SIM})$ with $q^H_{SIM} > q^L_{SIM}$.

(ii) Both equilibrium qualities are decreasing in n.

(iii) It is more profitable to be producing the higher quality. That is, profit for firm i is higher at equilibrium E^i_{SIM}.

Proof of (i): We need to show that,

Lemma 3 *The two best-response correspondences, $\beta^1(\cdot)$ and $\beta^2(\cdot)$, intersect.*

Proof of Lemma: Since $q_H(\cdot)$ is increasing and $q_H(0) = q^m$, $q^H(\cdot)$ and $q^L(\cdot)$ must intersect if

$$\beta^i(q^m) = q^L(q^m), \tag{5}$$

i.e., $\hat{q} < q^m$. Then from symmetry, the best-response correspondences must intersect (there will be two symmetric intersections). Equation (5) follows from,

$$\Pi^i(q^L(q^m), q^m) = \Pi^L(q^m, q^L(q^m)) > \Pi^i(q^H(q^m), q^m) = \Pi^H(q^H(q^m), q^m).$$

Since marginal profit is positive at $q_i = 0$ and $\Pi^i(0, q^m) = \Pi^L(q^m,) = 0$, and therefore $\Pi^i(q', q^m) > 0$ for some q' and it suffices to show that $\Pi^i(q^H(q^m), q^m) < 0$. We can write $q^H(q^m) = \alpha q^m$ for some $\alpha > 1$. Substituting this and $q^m = (\frac{1}{4nk})^{\frac{1}{n-1}}$ into the profit function,

$$\begin{aligned}
\Pi^i(q^H(q^m), q^m) &= \frac{4\alpha^2(\alpha-1)}{(4\alpha-1)^2} q^m - k(\alpha q^m)^n \\
&= \left(\frac{4\alpha^2(\alpha-1)}{(4\alpha-1)^2} - k\alpha^n (q^m)^{n-1}\right) q^m \\
&= \left(\frac{4\alpha^2(\alpha-1)}{(4\alpha-1)^2} - \frac{\alpha^n}{4n}\right) q^m
\end{aligned}$$

This will be negative if $f_1(\alpha) < f_2(\alpha)$ where

$$f_1(a) = (a-1)n, \quad f_2(a) = \frac{a^{n-1}(4a-1)^2}{16}.$$

It is straightforward to show that $f_1(1) < f_2(1)$ and $0 < f_1'(a) < f_2'(a)$ for $a \geq 1$. Then it must be $f_1(\alpha) < f_2(\alpha)$ for all $\alpha \geq 1$. $\Box$
Proof of (ii): Given concavity of the revenue functions, the best-response correspondences will move down as n increases.$\Box$
Proof of (iii): We first prove the following lemma.

Lemma 4 *For (q_H, q_L), $q_H > q_L$, if $q_L = q^L(q_H)$ and $q_H \leq (\frac{7}{27nk})^{\frac{1}{n-1}}$, then*

$$\Pi^H(q_H, q_L) > \Pi^L(q_H, q_L).$$

Proof of Lemma: If we let $q^L = q$, then $q^H = \alpha q$ for some $\alpha > 1$. We need to show

$$\Pi^H(\alpha q, q) = R^H(\alpha q, q) - C(\alpha q) > \Pi^L(\alpha q, q) = R^L(\alpha q, q) - C(q).$$

The differences in revenues and costs are,

$$\Delta R = R^H(\alpha q, q) - R^L(\alpha q, q) = \frac{(\alpha - 1)q\alpha}{4\alpha - 1},$$
$$\Delta C = C(\alpha q) - C(q) = (\alpha^n - 1)kq^n.$$

We need to show

$$\frac{\Delta C}{\Delta R} = \frac{(4\alpha - 1)(\alpha^n - 1)k}{\alpha(\alpha - 1)} q^{n-1} < 1, \tag{6}$$

Given the assumption on $q_H = \alpha^{n-1}q^{n-1}$,

$$\frac{\Delta C}{\Delta R} < \frac{(4\alpha - 1)(\alpha^n - 1)k}{\alpha(\alpha - 1)\alpha^{n-1}} \frac{7}{27nk}$$
$$= \left(4 - \frac{1}{\alpha}\right)\left(1 + \frac{1}{\alpha^2} + \cdots + \frac{1}{\alpha^{n-1}}\right)\frac{7}{27n}.$$

From (2) and $q = q^L(\alpha q)$, we know that $\alpha > \frac{7}{4}$. For any $\alpha > \frac{7}{4}$,

$$\frac{\Delta C}{\Delta R} < 4\frac{1 - (\frac{4}{7})^{n-1}}{\frac{3}{7}}\frac{7}{27n}.$$

It is straightforward to show that right-hand side is less than 1 for any $n \geq 2$.
□

Now we apply this Lemma to $q_L = q^L_{SIM}$ and $q_H = q^H_{SIM}$. Equilibrium is on both $\beta^2(\cdot) = q^L(\cdot)$ and $\beta^1(\cdot) = q^H(\cdot)$. In particular, it satisfies the first condition of the Lemma, $q_L = q^L(q_H)$. Since $q^H(\cdot)$ is increasing, $\beta^1(\cdot) = q^L(\cdot)$ for $q \geq (\frac{7}{27nk})^{\frac{1}{n-1}}$, and $q^H((\frac{7}{27nk})^{\frac{1}{n-1}}) = (\frac{7}{27nk})^{\frac{1}{n-1}}$, and it follows that $q^H_{SIM} = q^H(q^L_{SIM}) = \alpha q \leq (\frac{7}{27nk})^{\frac{1}{n-1}}$. This is the second condition of the Lemma. Applying the Lemma, we get

$$\Pi^H(q^H_{SIM}, q^L_{SIM}) > \Pi^L(q^H_{SIM}, q^L_{SIM}).$$

□

We now characterize the equilibrium under sequential choice $E_{SEQ} = (q^1_{SEQ}, q^2_{SEQ})$. Firm 1 will choose quality first, and then firm 2 will choose its quality after observing the choice of firm 1. q^1_{SEQ} solves,

$$\max_{q_1} \Pi^1(q_1, q_2) \text{ subject to } q_2 \in \beta^2(q_1).$$

The solution will be one of the two local optima. One maximum, F^H, involves firm 1 (first mover), choosing the higher quality:

$$\max_{q_1} \Pi^H(q_1, q^L(q_1)) \text{ subject to } q_1 \geq q^L(\hat{q}),$$

while firm 1 chooses the lower quality in the other local maximum, F^L:

$$\max_{q_1} \Pi^L(q^H(q_1), q_1) \text{ subject to } q_1 \leq \hat{q}.$$

(Both points are depicted in Figure 3.) One or both local maximum can be corner solutions. Although cost is higher at F^H, first mover will always produce the higher quality:

Proposition 2 *For any n, F^H is the global optimum, i.e., $E_{SEQ} = F^H$.*

Proof: Let $F^H = (q_H^H, q_L^H)$ and $F^L = (q_L^L, q_H^L)$. We need to show that $\Pi^1(F^H) = \Pi^H(F^H) > \Pi^1(F^L) = \Pi^L(F^L)$. F^L is on $\beta^2(\cdot) = q^H(\cdot)$ but $G^L = (q_H^L, q_L^L)$ will be on $\beta^1(\cdot) = q^H(\cdot)$. Since the two firms are identical, $\Pi^1(F^L) = \Pi^2(G^L)$. We can find a point $X = (x_H, x_L)$ on $\beta^2(\cdot) = q^L(\cdot)$ such that $\Pi^2(G^L) = \Pi^2(X) = \Pi^L(X)$.

We apply Lemma 4 to $q_H = x_H, q_L = x_L$. We chose X to be on $q^L(\cdot)$ and it satisfies the first condition, $x_L = q^L(x_H)$. We note regarding G^L and X that (i) both points are on the same iso-profit curve, (ii) the iso-profit curve is tangent to an upward sloping $\beta^1(\cdot)$ at G^L, (iii) the same iso-profit curve has slope 0 at X. This implies X must lie southwest of G^L. In particular, $x_H \leq (\frac{7}{27})^{\frac{1}{n-1}}$. The Lemma implies $\Pi^H(X) > \Pi^L(X)$. But $\Pi^1(X) = \Pi^H(X)$ and by definition of F^H, $\Pi^1(F^H) \geq \Pi^1(X)$. We have shown now,

$$\Pi^1(F^H) = \Pi^H(F^H) \geq \Pi^H(X) > \Pi^L(X) = \Pi^2(G^L) = \Pi^L(G^L) = \Pi^1(F^L).$$

□

Because the best-response correspondence is upward sloping (Lemma 2), E_{SEQ} lies to the southwest of E^1. By definition, the profit for firm 1 is greater under sequential choice than under simultaneous choice. From the iso-profit curves, it is easy to see that the profit for firm 2 is lower under sequential quality choice. Summarizing,

Proposition 3 *When there is Bertrand competition at the sales stage and firms choose qualities sequentially, the first mover will choose a quality higher than that of the rival, i.e.,*

$$q_{SEQ}^1 > q_{SEQ}^2.$$

The qualities in the equilibrium under sequential choice are (pair wise) lower than the qualities of the simultaneous quality choice, i.e.,

$$q_{SIM}^H > q_{SEQ}^1 \text{ and } q_{SIM}^L > q_{SEQ}^2.$$

The firm supplying the superior product earns greater profit and the firm supplying the inferior product earns less profit under sequential choice.

4 Quality choices under Cournot competition

The function $\tilde{R}^t(q_H, q_L), t = H, L$ denotes equilibrium revenue from the second stage with Cournot competition. The equilibrium revenue function has the following properties.

Lemma 5 *When there is Cournot competition at the sales stage, the equilibrium revenue function is continuous* $\forall(q_H, q_L)$*,* $q_H \geq q_L$ *and twice continuously differentiable* $\forall q_H \neq q_L$*.*

$$\begin{cases} \tilde{R}^L(q_H, q_L) = \frac{q_L q_H^2}{(4q_H - q_L)^2}, \\ \tilde{R}^H(q_H, q_L) = \frac{q_H(2q_H - q_L)^2}{(4q_H - q_L)^2}. \end{cases} \tag{7}$$

$$\begin{cases} \tilde{R}^L_L > 0, \tilde{R}^L_H < 0, \tilde{R}^L_{LL} > 0, \tilde{R}^L_{LH} = \tilde{R}^L_{HL} < 0, \\ \tilde{R}^H_H > 0, \tilde{R}^H_L < 0, \tilde{R}^H_{HH} < 0, \tilde{R}^H_{HL} = \tilde{R}^H_{LH} > 0. \end{cases} \tag{8}$$

Note that the second order properties of the revenue function differs from the Bertrand case. Most notably, $R^L_{LL} > 0$ and $R^L_{HL} < 0$. A typical revenue function for firm 1 is depicted in Figure 4. In Cournot competition, a firm's profit is increasing in its own quality and decreasing in rival's quality, independent of relative quality levels. Contrary to the Bertrand case, marginal *gain* ($\tilde{R}^i_i > 0$) becomes smaller when qualities become further apart ($\tilde{R}^L_{LH} < 0$ and $\tilde{R}^H_{HL} > 0$). The payoff for firm i in the first stage is,

$$\tilde{\Pi}^i(q_i, q_j) = \begin{cases} \tilde{\Pi}^H(q_i, q_j) = \tilde{R}^H(q_i, q_j) - C(q_i) & q_i \geq q_j \\ \tilde{\Pi}^L(q_j, q_i) = \tilde{R}^L(q_j, q_i) - C(q_i)v & q_i < q_j \end{cases}.$$

As in the case with Bertrand competition, the best-response correspondence is given by one of the two local maxima or constrained best-response correspondences: $\tilde{q}^H(q_j) \geq q_j$ and $\tilde{q}^L(q_j) \leq q_j$. Since $\tilde{R}^L_L(q_H, 0) = \frac{1}{16}$ and $\tilde{R}^L_L(q_H, q_H) = \frac{5}{27}$, $\tilde{q}^L(q_j)$ is an interior solution of the profit maximization for $q_j \geq (\frac{5}{27kn})^{\frac{1}{n-1}}$ satisfying,

$$\tilde{R}^L_L(q_j, q_L) - C'(q_L) = \frac{q_L^2(4q_L + q_j)}{(4q_j - q_L)^3} - nkq_L^{n-1} = 0. \tag{9}$$

and

$$\tilde{q}^L(q_j) = q_j \quad \text{for } q_j \leq (\frac{5}{27kn})^{\frac{1}{n-1}}.$$

Since $\tilde{R}^H_H(q_L, q_L) = \frac{7}{27}$, $\tilde{q}^H(q_j)$ will an interior solution for $q_j(\frac{7}{27nk})^{\frac{1}{n-1}}$ and satisfies

$$\tilde{R}^H_H(q_H, q_j) - C'(q_H) = \frac{(2q_H - q_j)(8q_H^2 - 2q_Hq_j + q_j^2)}{(4q_H - q_j)^2} - nkq_H^{n-1} = 0,$$

and

$$\tilde{q}^H(q_j) = q_j \text{ for } q_j \geq (\frac{7}{27kn})^{\frac{1}{n-1}}.$$

The iso-profit curves are depicted in Figure 5. (Arrow indicates direction of increasing profit for firm 1 and iso-profit curves are those of firm 1.) We are able to make the following partial characterization of the best-response function.

Lemma 6 *When there is Cournot competition in the second stage, the first stage best-response correspondence satisfies,*

$$\tilde{\beta}^i(q_j) = \begin{cases} \tilde{q}^H(q_j) > q_j \text{ and increasing for } q_j \leq \frac{5}{27kn}, \\ \tilde{q}^L(q_j) < q_j \text{ and decreasing for } q_j \geq \frac{7}{27kn}. \end{cases}$$

For $(\frac{5}{27nk})^{\frac{1}{n-1}} < q_j < (\frac{7}{27nk})^{\frac{1}{n-1}}$ both local maxima are interior solutions and $\beta^i(\cdot)$ must be determined by comparing $\Pi^H(q^H(q_j), q_j)$ and $\Pi^L(q_j, q^L(q_j))$. (See the iso-profit curves in Figure 5.) Qualities are strategic complements for the higher quality firm and substitutes for the lower quality firm.

We have the following characterization of equilibrium when firms choose qualities simultaneously.

Proposition 4 *For any n, there are two pure strategy Nash equilibria ($\tilde{E}^1_{SIM}$ and $\tilde{E}^2_{SIM}$) when there is Cournot competition at the sales stage.*

(i) The qualities of the two equilibria are the same: $\tilde{E}^S_{SIM} = (\tilde{q}^H_{SIM}, \tilde{q}^L_{SIM})$ and $\tilde{E}^2_{SIM} = (\tilde{q}^L_{SIM}, \tilde{q}^H_{SIM})$ with $\tilde{q}^H_{SIM} > \tilde{q}^L_{SIM}$

(ii) The higher equilibrium quality ($\tilde{q}^H$) is decreasing in n. The lower equilibrium quality ($\tilde{q}^L$) may increase or decrease with n.

(iii) In equilibrium, it is more profitable to be producing the higher quality. That is, profit for firm i is higher at equilibrium $\tilde{E}^i_{SIM}$.

Proof: To show existence, we need to show that the two best-response correspondences intersect. This will follow from

$$\tilde{\beta}^2(q^m) = \tilde{q}^L(q^m) \tag{10}$$

$$\tilde{q}^L(q^m) < \frac{5}{27nk} \tag{11}$$

These two relationships imply $\tilde{\beta}^2(q^m) < \frac{5}{27nk}$. On the other hand, since $\tilde{\beta}^1(0) = \tilde{q}^H(0) = q^m$ and $\tilde{q}^H(\cdot)$ is increasing, $\tilde{\beta}^1(\frac{5}{27nk}) > q^m$. Then the two best-response correspondences must intersect.

Proof of (10): We need to show that

$$\tilde{\Pi}^H(\tilde{q}^H(q^m), q^m)) < \tilde{\Pi}^L(q^m, \tilde{q}^L(q^m)).$$

First we show that if α satisfies $q^H(q^m) = \alpha q^m > q^m$, then it must be that $\alpha \leq (\frac{28}{27})^{\frac{1}{n-1}}$. To show this, we first note that such α must satisfy (4), which using $(q^m)^{n-1} = \frac{1}{4nk}$ becomes,

$$\frac{(2\alpha - 1)(8\alpha^2 - 2\alpha + 1)}{(4\alpha - 1)^3} = \frac{\alpha^{n-1}}{4}.$$

Left hand side is decreasing in α and equal to $\frac{7}{27}$ when $\alpha = 1$. Right hand side, which is increasing in α is equal to $\frac{1}{4}$ when $\alpha = 1$. Both sides will be equal to α such that $\frac{1}{4} \leq \frac{\alpha^{n-1}}{4} \leq \frac{7}{27}$. The second inequality implies what we need to show.

Now we show (10), by claiming that,

$$\tilde{\Pi}^H(\alpha q^m, q^m) < \tilde{\Pi}^L(q^m, \frac{q^m}{\alpha}), \tag{12}$$

for any $\alpha \leq (\frac{28}{27})^{\frac{1}{n-1}}$. Then if we choose α as $\tilde{q}^H(q^m) = \alpha q^m$, (10) follows since $\tilde{\Pi}^L(q^m, \frac{q^m}{\alpha}) < \tilde{\Pi}^L(q^m, \tilde{q}^L(q^m))$.

In order to show (12), we define functions $f(\alpha)$ and $h(\alpha)$,

$$\begin{aligned} f(\alpha) &= \frac{\alpha(2\alpha-1)^2}{(4\alpha-1)^2} - \frac{\alpha^n}{4n}, \\ h(\alpha) &= \frac{\alpha}{(4\alpha-1)^2} - \frac{1}{\alpha^n 4n}. \end{aligned}$$

Then

$$\tilde{\Pi}^L(q^m, \frac{q^m}{\alpha}) - \tilde{\Pi}^H(\alpha q^m, q^m) = q^m(h(\alpha) - f(\alpha)).$$

Note that $h(1) - f(1) = 0$. We are interested in $h(\alpha) - f(\alpha)$ for $1 < \alpha \geq \frac{28}{27}^{\frac{1}{n-1}}$. For this, we inspect the change in the difference,

$$\begin{aligned} h'(\alpha) - f'(\alpha) &= -q^m \left\{ \tilde{P}i_L^L \frac{-1}{\alpha^2} + \tilde{P}i_H^H \right\} \\ &= q^m \left\{ -\frac{1}{\alpha^2} \left(\frac{\alpha^2(4\alpha+1)}{(4\alpha-1)^3} - \frac{1}{\alpha^{n-1}} \right) - \left(\frac{(2\alpha-1)(8\alpha^2-2\alpha+1)}{(4\alpha_1)^3} - \frac{\alpha^{n-1}}{4} \right) \right\} \end{aligned}$$

Sign of this expression is the same as sign of

$$s = (4\alpha-1)^3(1+\alpha^2(\alpha^{n-1})^2) - 4\alpha^{n-1}\alpha^2 \left((4\alpha+1) + (2\alpha-1)(8\alpha^2-2\alpha+1) \right).$$

Using $1 < \alpha \leq (\frac{28}{27})^{\frac{1}{n-1}}$, we have

$$s > (4\alpha-1)^3(1+\alpha^2) - \frac{28}{27} 4\alpha^2(8\alpha + 16\alpha^3 - 12\alpha^2).$$

This will be positive for any α such that $1 < \alpha \leq (\frac{28}{27})^{\frac{1}{n-1}}$. This implies $h(\alpha) - f(\alpha)$ is positive for the same range of α, which includes α that satisfies $\tilde{q}^H(q^m) = \alpha q^m$. This proves (12). $\Box$

Proof of (11): $\tilde{q}^L(\frac{5}{27nk} = \frac{5}{27nk}$ and $\tilde{q}^L(q_j)$ is decreasing in q_j for $q_j \geq \frac{5}{27nk}$. Since $q^m = \frac{1}{4nk} > \frac{5}{27nk}$, this implies $\tilde{q}^L(q^m) < \tilde{q}^L(\frac{5}{27nk}) = \frac{5}{27nk}$. $\Box$

Proof of (iii): Since $\tilde{R}^H_L < 0$ and $\tilde{R}^L_H > 0$, a firm's profit is monotonically increasing along the best-response correspondence ($\beta^i(\cdot)$) as rival quality decreases. (Direction of arrow in Figure 4). Thus for firm 1, $\tilde{E}^1_{SIM}$ is more profitable than $\tilde{E}^2_{SIM}$. $\Box$

We now characterize the sequential quality choice equilibrium $\tilde{E}_{SEQ} = (\tilde{q}^1_{SEQ}, \tilde{q}^2_{SEQ})$ when firm 1 chooses quality before firm 2. Again there are two local optima to the constrained maximization problem of firm 1,

$$\max_{q_1} \tilde{\Pi}^1(q_1, q_2) \text{ subject to } q_2 \in \tilde{\beta}^2(q_1).$$

One of the optima, $\tilde{F}^H$, involves the first mover firm 1 choosing the higher quality and in the other, $\tilde{F}^L = (\tilde{q}^L_1, \tilde{q}^L_2)$, firm 1 chooses the lower quality. (See Figure 4.)

The local optimum with higher quality, $\tilde{F}^H$, lies in a region of higher profit for high quality firm. Thus $\tilde{E}_{SEQ} = \tilde{F}^H$. Since firm 2's best-response function is downward sloping for $q_1 > q_2$, $\tilde{F}^H$ lies to the northwest of $\tilde{E}^1_{SIM}$. It is also clear from the direction of change along the best-response function that profit for firm 2 is lower at $\tilde{E}_{SEQ}$ than at $\tilde{E}^1_{SIM}$. So we have,

Proposition 5 *When there is Cournot competition in the sales stage and firms choose qualities sequentially, the first mover will choose a quality higher than that of the rival, i.e.,*

$$\tilde{q}^1_{SEQ} > \tilde{q}^2_{SEQ}.$$

The first mover's quality choice is also higher than the higher quality of the equilibrium under simultaneous choice. But the choice of the second mover *will be lower than the lower quality of the equilibrium under simultaneous choice, i.e.,*

$$\tilde{q}^H_{SIM} < \tilde{q}^1_{SEQ} \text{ and } \tilde{q}^L_{SIM} > \tilde{q}^2_{SEQ}.$$

The firm supplying the superior product earns greater profit under and the firm supplying the inferior product earns less profit under sequential choice.

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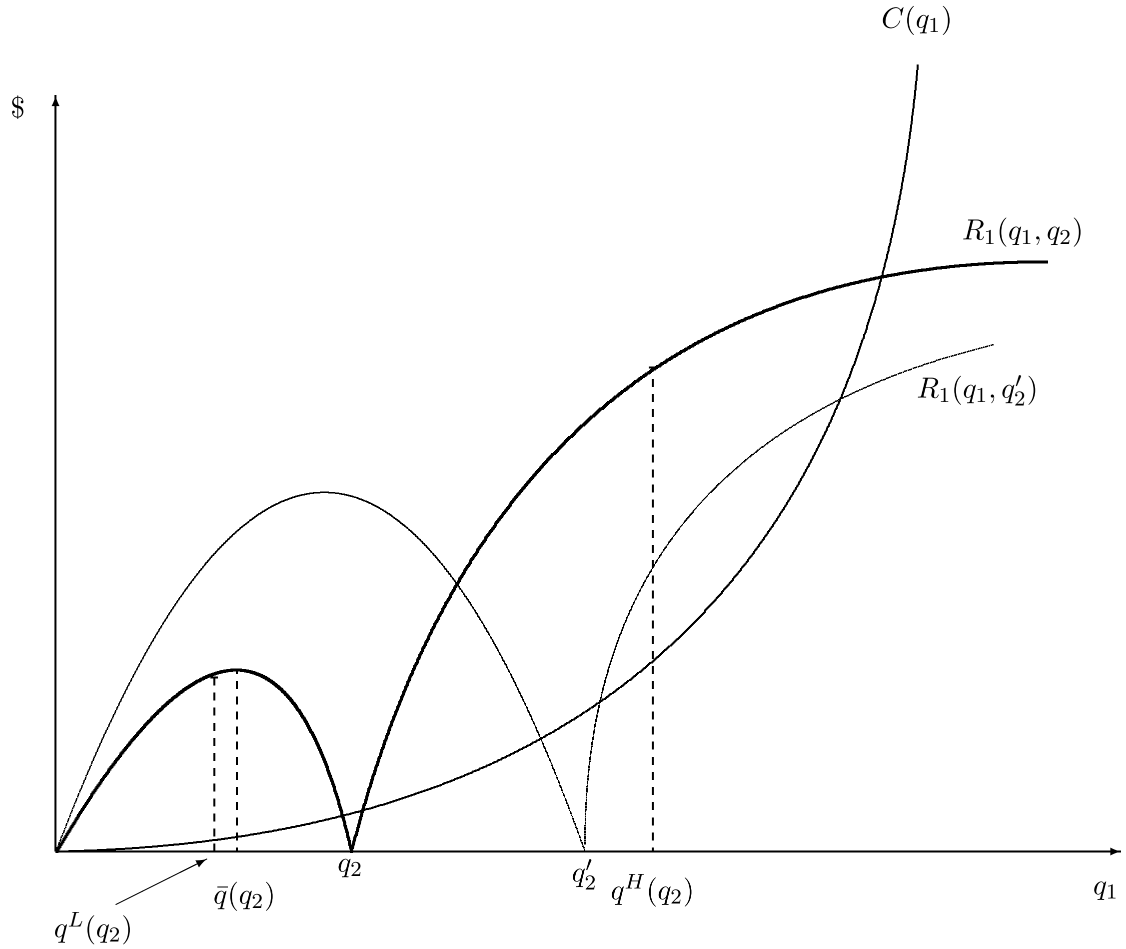

Figure 1: Frim 1's revenue and cost functions with Bertrand competition

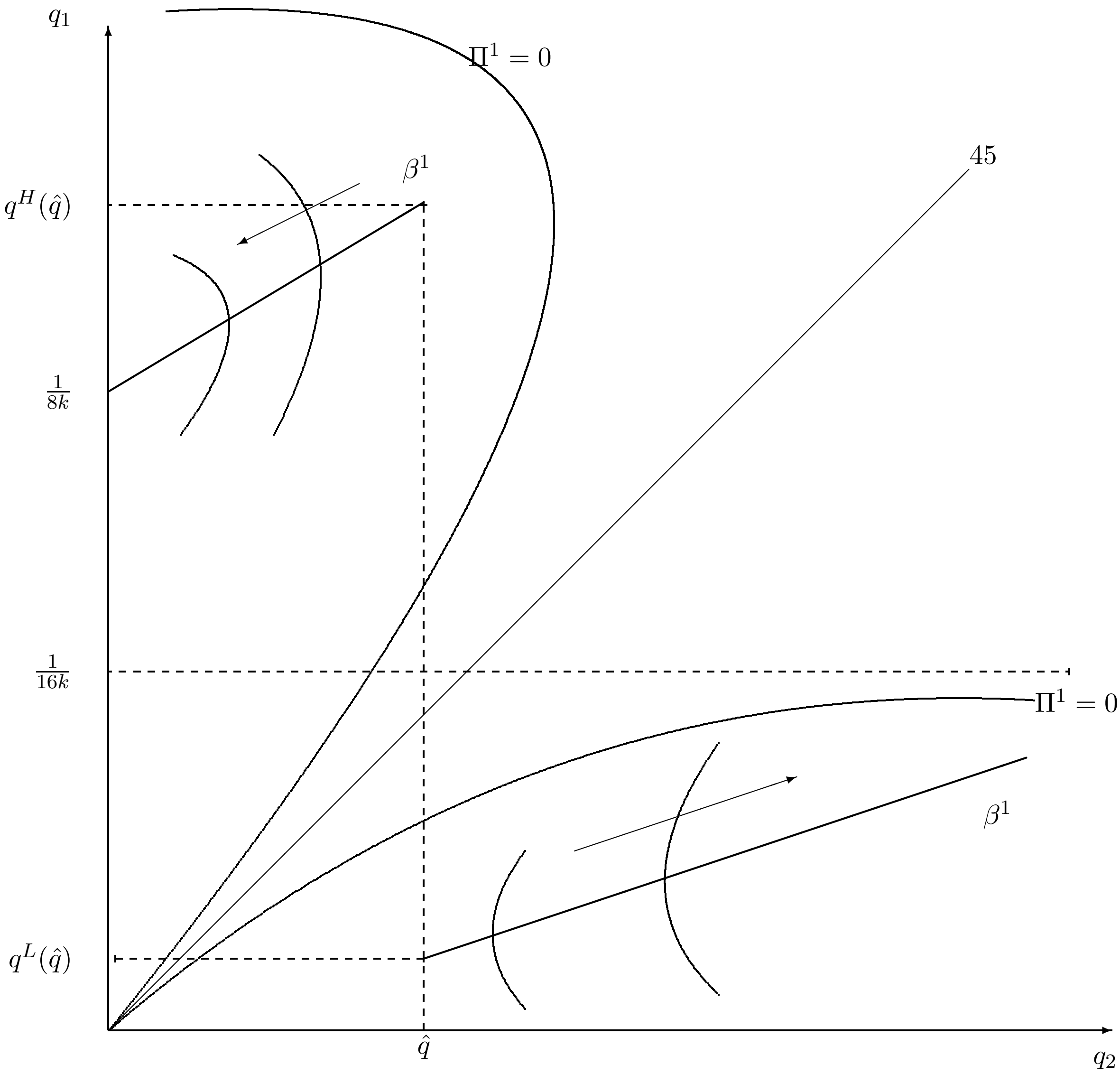

Figure 2: Firm 1's iso-profit curves and best-response correspondence with Bertrand competition

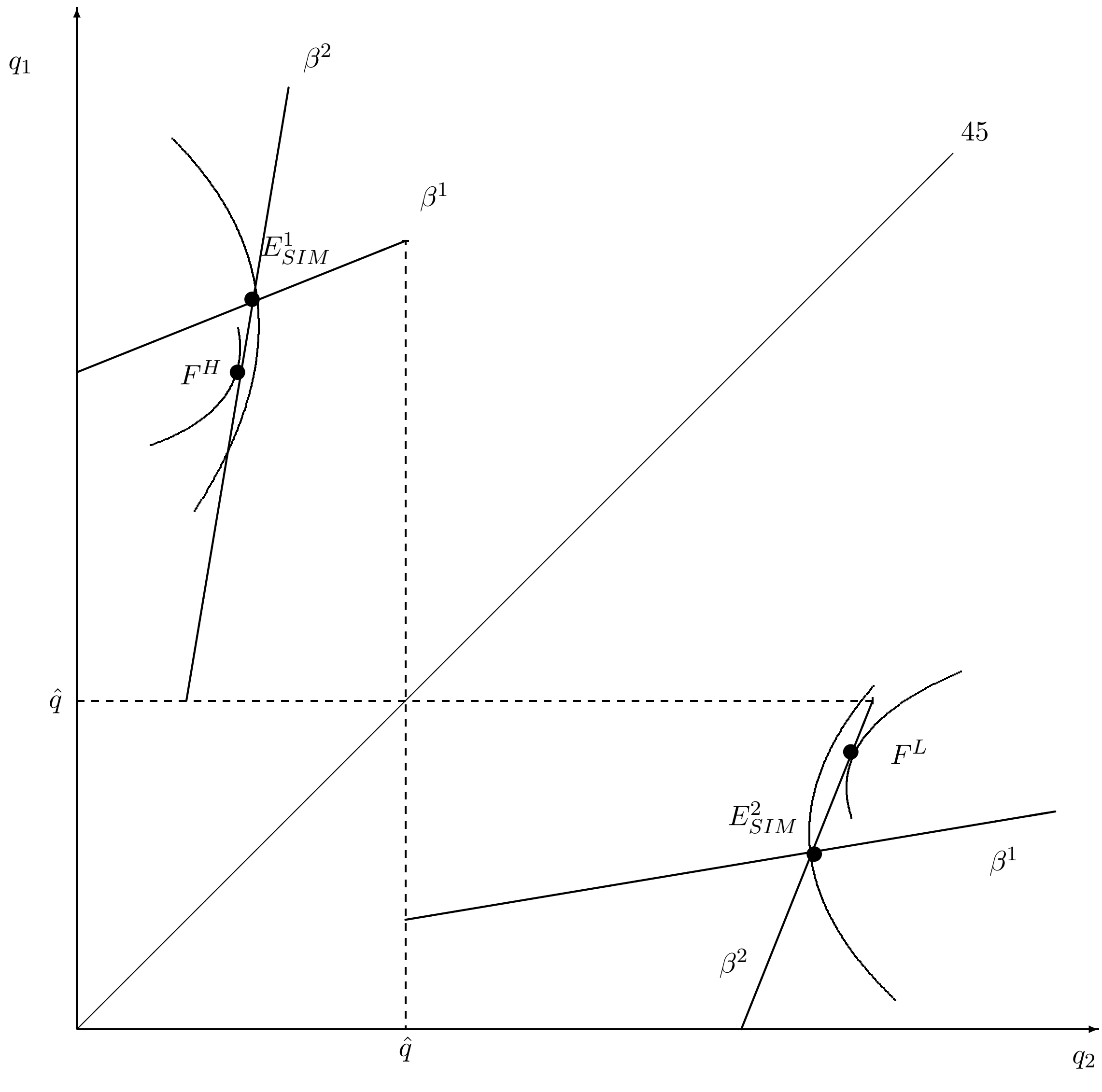

Figure 3: Best-response correspondences and equilibria with Bertrand competition

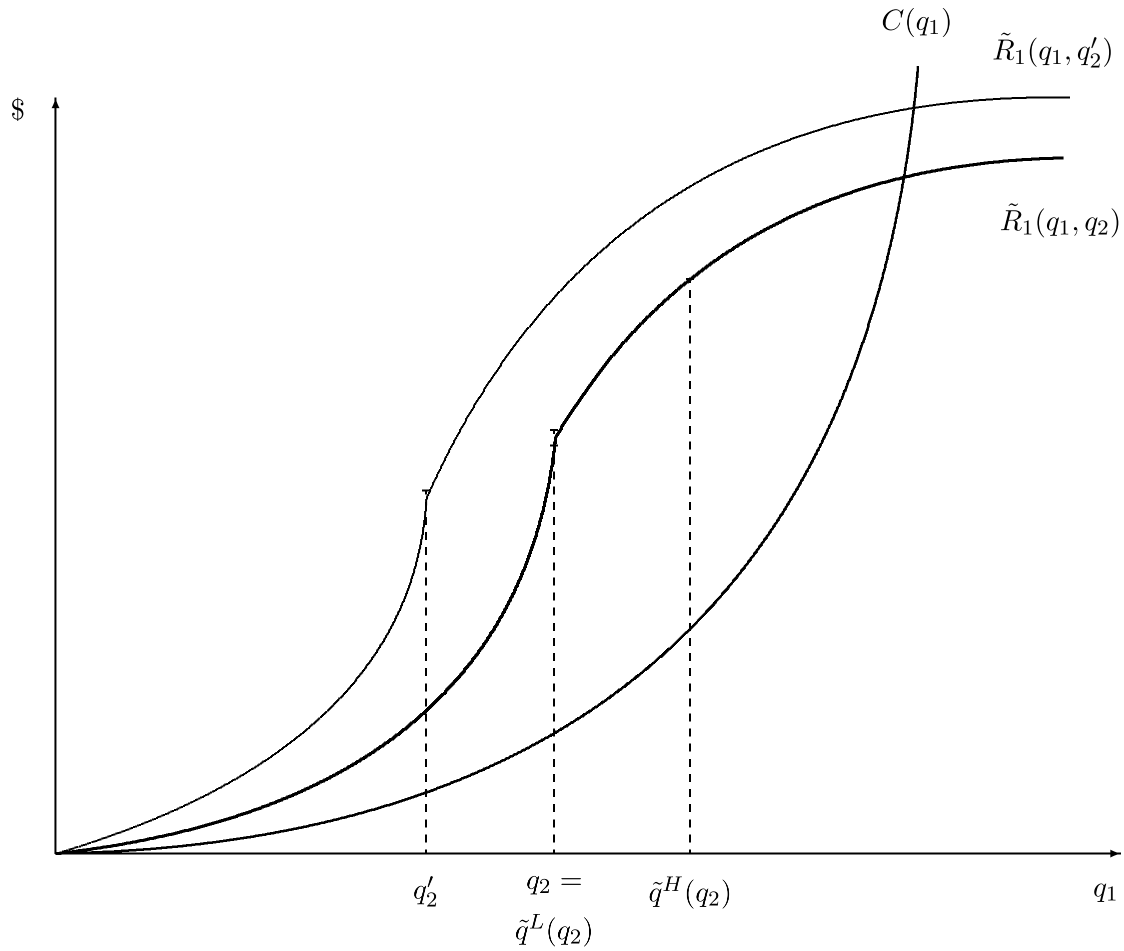

Figure 4: Frim 1's revenue and cost functions with Cournot competition

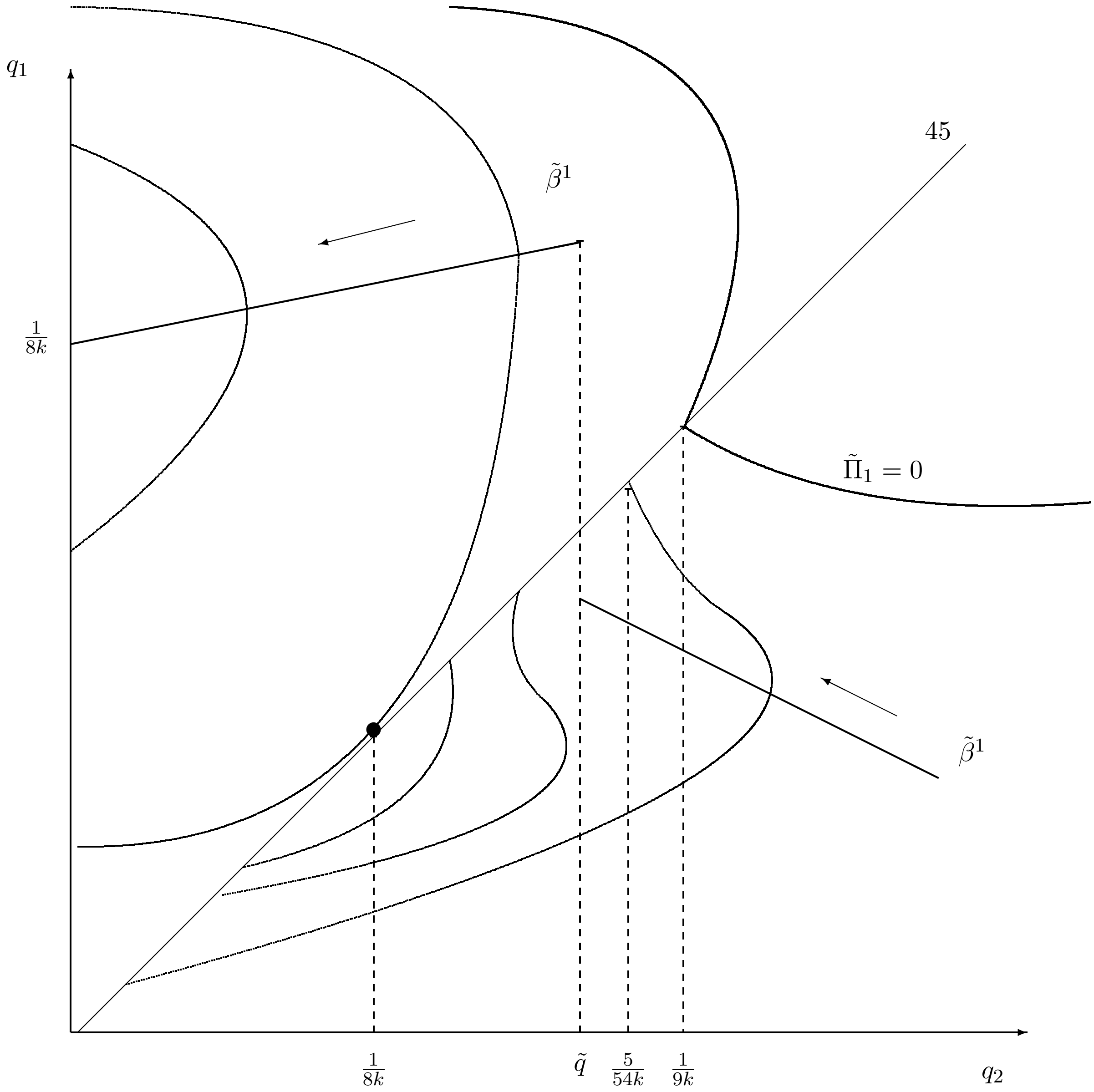

Figure 5: Firm 1's iso-profit curves and best-response correspondences with Cournot competition

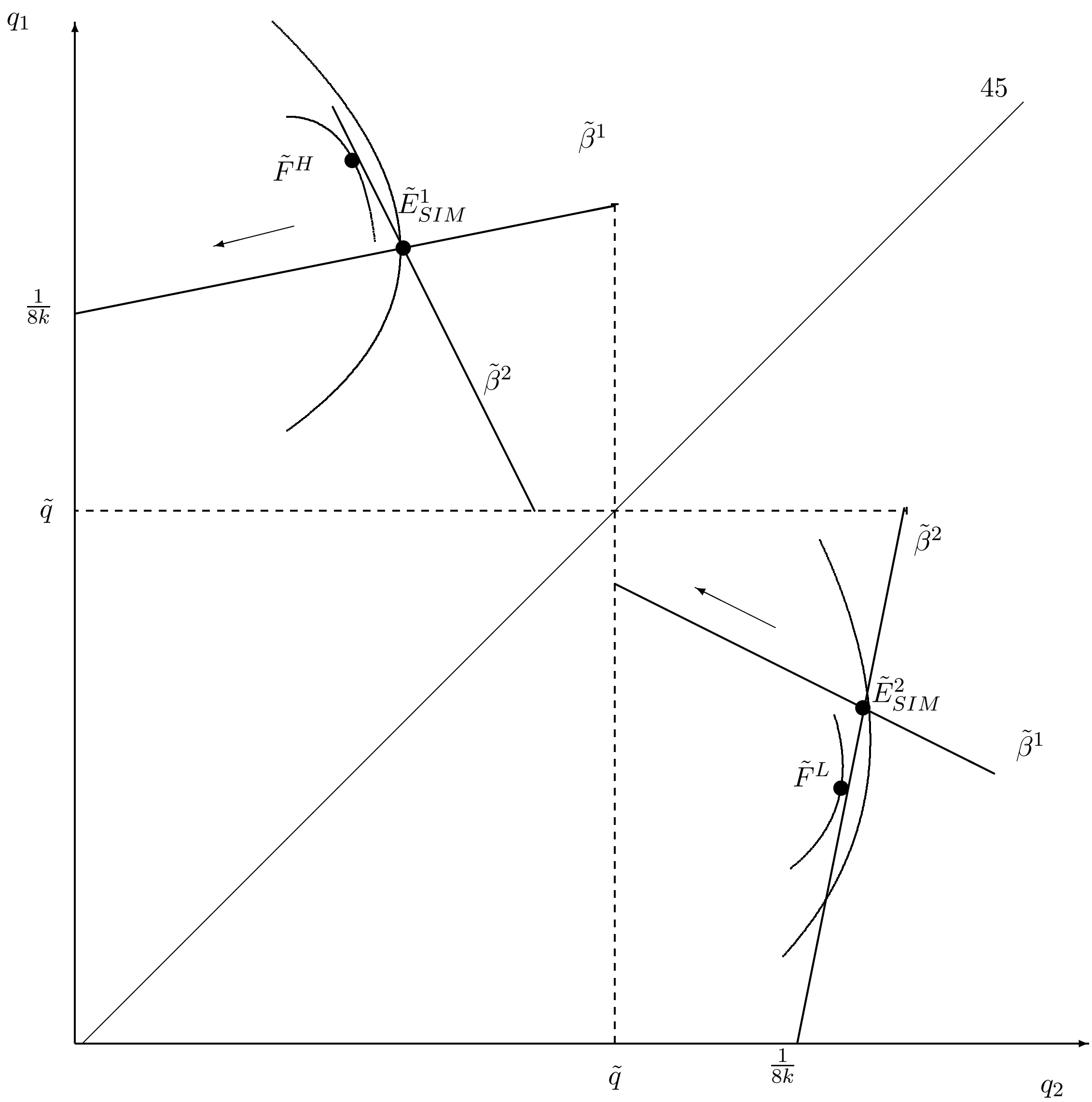

Figure 5: Best-response correspondences and equilibria with Cournot competition