Department of Electrical & Computer Engineering
The University of Auckland
New Zealand

Robust Control of Networked Control Systems with Random Communication Delays

Seunghwan Chae

September 2013

Supervisors: *Professor Sing Kiong Nguang*
Dr. Akshya Swain

A thesis submitted in fulfillment of the requirements of Doctor of Philosophy in Engineering

Abstract

Networked control systems (NCSs) refer to a class of systems where the components such as plants, sensors, actuators and controllers are connected via a communication network. This configuration provides several advantages over the traditional point-to-point structure such as modularity, ease of maintenance and low construction cost. NCSs enable several practical applications such as unmanned aerial vehicle and remote control of the plant, which are difficult to achieve with the traditional point-to-point structure. The presence of network inevitably, however, introduces delays and data loss as signals travel through the network, meaning that the controllers in NCSs are to stabilize the system while overcoming the adverse effects caused by the presence of the network.

As continuous signals are transmitted via a network, they are converted to clusters of data called packets and these packets of data are transmitted through the network. Hence, it is natural to consider the plant and the controller in discrete-time domain where new information about the systems components are available at each sampling instance. In this thesis, discrete-time representation of the plant is considered and the controller/filter design methodologies are developed based on Lyapunov-Krasovskii functional.

The focus of the research is to develop controller and filter design methodologies for NCSs which takes the aforementioned network constraints into account. Hence the network needs to be modelled and taken into consideration when designing the controller/filter. In particular, a finite state Markov chain is used in this research to model the network-induced delays and data loss where each mode in the Markov chain corresponds to the delays in the network. Difficulties of obtaining a completely known transition probability matrix, which describes the transitions between the modes of the Markov chain, in real world is acknowledged in this research and some of the elements in the transition probability matrix are allowed to be unknown. In this thesis, a robust $\mathcal{H}_\infty$ state feedback controller design for linear and nonlinear NCSs are first developed where the transition probability matrix of the Markov chain is assumed to be completely known. Based on

theses methodologies, robust $\mathcal{H}_\infty$ state feedback controller, robust $\mathcal{H}_\infty$ filter and robust $\mathcal{H}_\infty$ dynamic output feedback controller design methodologies for NCSs are presented where the transition probability matrix is allowed to be partially known. It is shown that the case with either completely known or unknown transition probability matrix can be considered as a special case of the presented approaches.

Study of nonlinear systems is important as every real system contains nonlinearities. Takagi-Sugeno (T-S) fuzzy model has been shown to be effective in modelling nonlinear systems which describes a global nonlinear system with a series of local linear models blended using membership functions. In this thesis, robust fuzzy $\mathcal{H}_\infty$ state feedback, robust fuzzy $\mathcal{H}_\infty$ filter and robust fuzzy $\mathcal{H}_\infty$ dynamic output feedback controller design are considered where the nonlinear NCSs are described by T-S fuzzy model, with main focus on partially known transition probability matrix in the Markov chain. Special attention is given to premise variables of the plant to correctly model NCSs where there exists a network between the plant and the controller. It has been addressed that many existing literature on nonlinear NCSs modelled by T-S fuzzy model fail to acknowledge this issue, making the existing approaches impractical. Furthermore, a methodology to incorporate membership functions into the controller/filter designs via sum-of-squares approach is presented to ensure that the controller is specific for the membership functions of the system. This has not been considered in existing studies of nonlinear NCSs, making the existing results conservative as the controllers are valid for any shape of membership functions. Iterative algorithms to convert nonconvex problems into optimization problems are also presented so that existing mathematical tools can be used to obtain a controller/filter.

Finally, the effectiveness of the proposed design methodologies are demonstrated using numerical examples in this thesis. The simulation results show that the proposed design methodologies achieve the prescribed performance requirements. Comparisons with existing methodologies without considering membership functions are made for robust fuzzy $\mathcal{H}_\infty$ state feedback controller and robust fuzzy $\mathcal{H}_\infty$ dynamic output feedback controller to illustrate that incorporating membership functions results a larger stabilization region, demonstrating the advantage of the presented methodologies.

Acknowledgements

The journey I have taken in the last four years has certainly been overwhelming. As I look to realise my childhood dream, I would like to think these people, without whom I would not be where I am now.

First and foremost, I would like to express my utmost gratitude to my supervisor Professor Sing Kiong Nguang, for his support, patience and wisdom he has continuously shown throughout this research. I am truly blessed to consider him my mentor. The support he has given me has been an encouragement throughout this research. I also thank him for nurturing me to become a better researcher. Without his support, this work would not exist.

I would also like to thank my co-supervisor, Dr. Akshya Swain and the rest of the advisory committee members for their time and insights. I also thank my colleagues, Faiz Rasool, Shakir Mohd Md Saat, Mattias Krug, Claudio Camasca Ramirez and Jian Zhang for their valuable support and being a great friend for me.

I thank all the support staff members at the Department of Electrical and Computing Engineering for all the help they provided. I would like to specially thank Leonid Ostrovsky and Hanlie Van Zyl for going the extra mile to help me out.

Last but certainly not the least, I show my deepest gratitude to my family, Hun Joo Chae, Byung Joo Choi and Sol Chae, for their love, support and faith in me. I could not have made this journey without them. They taught me how to be positive and face adversity. Thank you and I love you.

THE UNIVERSITY OF AUCKLAND
SCHOOL OF GRADUATE STUDIES

Co-Authorship Form

Graduate Centre
ClockTower – East Wing
22 Princes Street, Auckland
Phone: +64 9 373 7599 ext 81321
Fax: +64 9 373 7610
Email: postgraduate@auckland.ac.nz
www.postgrad.auckland.ac.nz

This form is to accompany the submission of any PhD that contains research reported in published or unpublished co-authored work. **Please include one copy of this form for each co-authored work**. Completed forms should be included in all copies of your thesis submitted for examination and library deposit (including digital deposit), following your thesis Acknowledgements.

Please indicate the chapter/section/pages of this thesis that are extracted from a co-authored work and give the title and publication details or details of submission of the co-authored work.

Chapter 2/p24-p43/ - Robust H∞ State Feedback Control of Discrete-Time Networked Control Systems With Completely Known Transition Probability Matrix

Nature of contribution by PhD candidate	Literature survey, analysis, implementation and writing
Extent of contribution by PhD candidate (%)	80

CO-AUTHORS

Name	Nature of Contribution
Sing Kiong Nguang	Research idea, critical review, gammatical review and comments
Akshya Swain	Critical review and comments
Faiz Rasool	Critical review and comments

Certification by Co-Authors

The undersigned hereby certify that:

- the above statement correctly reflects the nature and extent of the PhD candidate's contribution to this work, and the nature of the contribution of each of the co-authors; and
- in cases where the PhD candidate was the lead author of the work that the candidate wrote the text.

Name	Signature	Date
Sing Kiong Nguang	[signature]	17/06/2013
Akshya Swain	[signature]	17/06/2013
Faiz Rasool	[signature]	13/06/2013
		Click here
		Click here
		Click here

THE UNIVERSITY OF AUCKLAND
SCHOOL OF GRADUATE STUDIES

Co-Authorship Form

Graduate Centre
ClockTower – East Wing
22 Princes Street, Auckland
Phone: +64 9 373 7599 ext 81321
Fax: +64 9 373 7610
Email: postgraduate@auckland.ac.nz
www.postgrad.auckland.ac.nz

This form is to accompany the submission of any PhD that contains research reported in published or unpublished co-authored work. **Please include one copy of this form for each co-authored work**. Completed forms should be included in all copies of your thesis submitted for examination and library deposit (including digital deposit), following your thesis Acknowledgements.

Please indicate the chapter/section/pages of this thesis that are extracted from a co-authored work and give the title and publication details or details of submission of the co-authored work.

Chapter 3/p44-p63/ - Robust H∞ State Feedback Control of Discrete-Time Networked Control Systems With Partially Known Transition Probability Matrix

Nature of contribution by PhD candidate	Literature survey, analysis, implementation and writing
Extent of contribution by PhD candidate (%)	85

CO-AUTHORS

Name	Nature of Contribution
Sing Kiong Nguang	Research idea, critical review, gammatical review and comments

Certification by Co-Authors

The undersigned hereby certify that:

- the above statement correctly reflects the nature and extent of the PhD candidate's contribution to this work, and the nature of the contribution of each of the co-authors; and
- in cases where the PhD candidate was the lead author of the work that the candidate wrote the text.

Name	Signature	Date
Sing Kiong Nguang		17/06/2013
		Click here
		Click here
		Click here
		Click here
		Click here

THE UNIVERSITY OF AUCKLAND
SCHOOL OF GRADUATE STUDIES

Co-Authorship Form

Graduate Centre
ClockTower – East Wing
22 Princes Street, Auckland
Phone: +64 9 373 7599 ext 81321
Fax: +64 9 373 7610
Email: postgraduate@auckland.ac.nz
www.postgrad.auckland.ac.nz

This form is to accompany the submission of any PhD that contains research reported in published or unpublished co-authored work. **Please include one copy of this form for each co-authored work**. Completed forms should be included in all copies of your thesis submitted for examination and library deposit (including digital deposit), following your thesis Acknowledgements.

Please indicate the chapter/section/pages of this thesis that are extracted from a co-authored work and give the title and publication details or details of submission of the co-authored work.

Chapter 4/p64-p80/ - Robust H∞ Filtering for Discrete-time Networked Control Systems With Partially Known Transition Probability Matrix

Nature of contribution by PhD candidate	Literature survey, analysis, implementation and writing
Extent of contribution by PhD candidate (%)	85

CO-AUTHORS

Name	Nature of Contribution
Sing Kiong Nguang	Research idea, critical review, gammatical review and comments

Certification by Co-Authors

The undersigned hereby certify that:

- the above statement correctly reflects the nature and extent of the PhD candidate's contribution to this work, and the nature of the contribution of each of the co-authors; and
- in cases where the PhD candidate was the lead author of the work that the candidate wrote the text.

Name	Signature	Date
Sing Kiong Nguang	[signature]	17/06/2013
		Click here
		Click here
		Click here
		Click here
		Click here

THE UNIVERSITY OF AUCKLAND
SCHOOL OF GRADUATE STUDIES

Co-Authorship Form

Graduate Centre
ClockTower – East Wing
22 Princes Street, Auckland
Phone: +64 9 373 7599 ext 81321
Fax: +64 9 373 7610
Email: postgraduate@auckland.ac.nz
www.postgrad.auckland.ac.nz

This form is to accompany the submission of any PhD that contains research reported in published or unpublished co-authored work. **Please include one copy of this form for each co-authored work**. Completed forms should be included in all copies of your thesis submitted for examination and library deposit (including digital deposit), following your thesis Acknowledgements.

Please indicate the chapter/section/pages of this thesis that are extracted from a co-authored work and give the title and publication details or details of submission of the co-authored work.

Chapter 5/p81-p93/ - Robust H∞ Dynamic Output Feedback Control of Discrete-time Networked Control Systems With Partially Known Transition Probability Matrix

Nature of contribution by PhD candidate	Literature survey, analysis, implementation and writing
Extent of contribution by PhD candidate (%)	80

CO-AUTHORS

Name	Nature of Contribution
Sing Kiong Nguang	Research idea, critical review, gammatical review and comments
Dan Huang	Critical review and comments

Certification by Co-Authors

The undersigned hereby certify that:

- the above statement correctly reflects the nature and extent of the PhD candidate's contribution to this work, and the nature of the contribution of each of the co-authors; and
- in cases where the PhD candidate was the lead author of the work that the candidate wrote the text.

Name	Signature	Date
Sing Kiong Nguang	[signature]	17/06/2013
Dan Huang	黄丹	13/06/2013
		Click here
		Click here
		Click here
		Click here

Last updated: 25 March 2013

THE UNIVERSITY OF AUCKLAND
SCHOOL OF GRADUATE STUDIES

Co-Authorship Form

Graduate Centre
ClockTower – East Wing
22 Princes Street, Auckland
Phone: +64 9 373 7599 ext 81321
Fax: +64 9 373 7610
Email: postgraduate@auckland.ac.nz
www.postgrad.auckland.ac.nz

This form is to accompany the submission of any PhD that contains research reported in published or unpublished co-authored work. **Please include one copy of this form for each co-authored work**. Completed forms should be included in all copies of your thesis submitted for examination and library deposit (including digital deposit), following your thesis Acknowledgements.

Please indicate the chapter/section/pages of this thesis that are extracted from a co-authored work and give the title and publication details or details of submission of the co-authored work.

Chapter 7/p104-p131/ - Robust Fuzzy H∞ State Feedback Control of Nonlinear Networked Control Systems With Completely Known Transition Probability Matrix

Nature of contribution by PhD candidate	Literature survey, analysis, implementation and writing
Extent of contribution by PhD candidate (%)	80

CO-AUTHORS

Name	Nature of Contribution
Sing Kiong Nguang	Research idea, critical review, gammatical review and comments
Wenqin Wang	Critical review and comments

Certification by Co-Authors

The undersigned hereby certify that:

- ❖ the above statement correctly reflects the nature and extent of the PhD candidate's contribution to this work, and the nature of the contribution of each of the co-authors; and
- ❖ in cases where the PhD candidate was the lead author of the work that the candidate wrote the text.

Name	Signature	Date
Sing Kiong Nguang	[signature]	17/06/2013
Wenqin Wang	wenqin wang	13/06/2013
		Click here
		Click here
		Click here
		Click here

Last updated: 25 March 2013

THE UNIVERSITY OF AUCKLAND
SCHOOL OF GRADUATE STUDIES

Co-Authorship Form

Graduate Centre
ClockTower – East Wing
22 Princes Street, Auckland
Phone: +64 9 373 7599 ext 81321
Fax: +64 9 373 7610
Email: postgraduate@auckland.ac.nz
www.postgrad.auckland.ac.nz

This form is to accompany the submission of any PhD that contains research reported in published or unpublished co-authored work. **Please include one copy of this form for each co-authored work**. Completed forms should be included in all copies of your thesis submitted for examination and library deposit (including digital deposit), following your thesis Acknowledgements.

Please indicate the chapter/section/pages of this thesis that are extracted from a co-authored work and give the title and publication details or details of submission of the co-authored work.

Chapter 8/p132-p147/ - Robust Fuzzy H∞ Filtering of Nonlinear Networked Control Systems With Partially Known Transition Probability Matrix

Nature of contribution by PhD candidate	Literature survey, analysis, implementation and writing
Extent of contribution by PhD candidate (%)	85

CO-AUTHORS

Name	Nature of Contribution
Sing Kiong Nguang	Research idea, critical review, gammatical review and comments

Certification by Co-Authors

The undersigned hereby certify that:

- the above statement correctly reflects the nature and extent of the PhD candidate's contribution to this work, and the nature of the contribution of each of the co-authors; and
- in cases where the PhD candidate was the lead author of the work that the candidate wrote the text.

Name	Signature	Date
Sing Kiong Nguang	[signature]	17/06/2013
		Click here
		Click here
		Click here
		Click here
		Click here

THE UNIVERSITY OF AUCKLAND
SCHOOL OF GRADUATE STUDIES

Co-Authorship Form

Graduate Centre
ClockTower – East Wing
22 Princes Street, Auckland
Phone: +64 9 373 7599 ext 81321
Fax: +64 9 373 7610
Email: postgraduate@auckland.ac.nz
www.postgrad.auckland.ac.nz

This form is to accompany the submission of any PhD that contains research reported in published or unpublished co-authored work. **Please include one copy of this form for each co-authored work**. Completed forms should be included in all copies of your thesis submitted for examination and library deposit (including digital deposit), following your thesis Acknowledgements.

Please indicate the chapter/section/pages of this thesis that are extracted from a co-authored work and give the title and publication details or details of submission of the co-authored work.

Chapter 9/p148-p165/ - Robust Fuzzy H∞ Dynamic Output Feedback Control of Nonlinear Networked Control Systems With Partially Known Transition Probability Matrix

Nature of contribution by PhD candidate	Literature survey, analysis, implementation and writing
Extent of contribution by PhD candidate (%)	85

CO-AUTHORS

Name	Nature of Contribution
Sing Kiong Nguang	Research idea, critical review, gammatical review and comments

Certification by Co-Authors

The undersigned hereby certify that:

- ❖ the above statement correctly reflects the nature and extent of the PhD candidate’s contribution to this work, and the nature of the contribution of each of the co-authors; and
- ❖ in cases where the PhD candidate was the lead author of the work that the candidate wrote the text.

Name	Signature	Date
Sing Kiong Nguang	[signature]	17/06/2013
		Click here
		Click here
		Click here
		Click here
		Click here

Contents

List of Figures

1

Introduction

1.1 Introduction to Networked Control Systems

In the last decade, there has been a significant advance in communication systems, which subsequently changed the way people carry out their daily lives. With wider bandwidth available on mobile networks, smartphones are becoming common, allowing people to access Internet at their fingertips. The days when a mobile phone is used to make calls are long gone. Widespread of broadband Internet allows people to get in touch with another and even see each other no matter where they are around the world, as long as there is an Internet connection. Not only did these advances make direct contributions to the way people carry out their lives, it also opened a lot of new doors that had not been possible before in many disciplines of engineering.

In control systems, this advance resulted in a new area of research called networked control systems (NCSs) where the control loop is closed via a communication network. The purpose of NCSs is to control **through** the network, not control **of** the network. This setup provides several advantages over the traditional point-to-point architecture such as modularity, quick and easy maintenance, integrated diagnostics and low construction cost [1–8]. Typical examples of NCSs are remote control systems, factory automation, unmanned vehicle navigation and motor vehicles, as shown in Figure 1.1. There has been several common-bus network architectures introduced to facilitate NCSs but NCSs can also use existing architectures to transfer information. This particular setup is useful in practice as there no longer is the need that the controller is close to the plant, allowing controllers around the world to be gathered in one location. The special setup of NCSs makes the system modularized therefore when faults occur in the system, the faulty module can simply be replaced/repaired. Furthermore, since the system can be configured to use existing networks, the construction cost is low as there no longer is copper wire connecting the system components. This also makes the system mobile and combined with modularity, it makes maintenance of NCSs easy. For these reasons, NCSs have been receiving increasing attention in recent years.

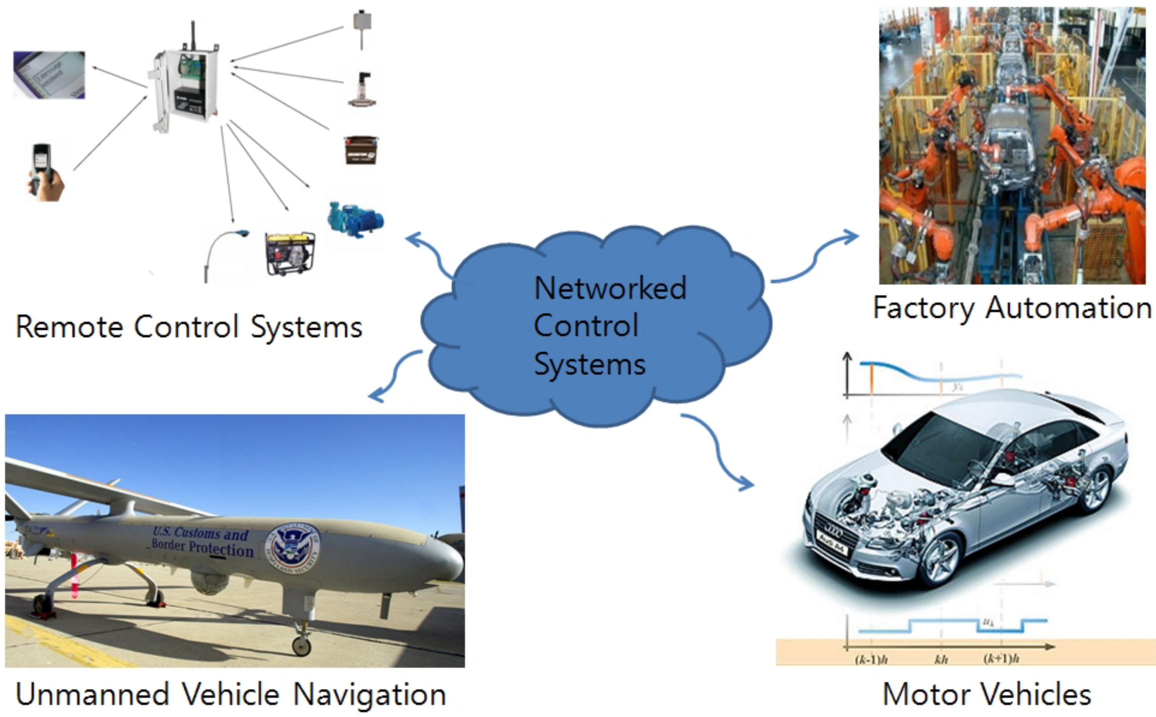

Figure 1.1: Typical examples of networked control systems

Configuration of NCSs can be broken down into two main categories shown as follows [9]:

1. **Direct structure**. This structure is the typical structure considered in the research of NCSs where the system components such as plant, sensor, controller and actuator are connected through a communication network as shown in Figure 1.2. The controller and the plant may be in different physical locations and directly connected via a network. The sensor data is transmitted via the network in packets. The

controller will compute the control signal based on the packet data received and transmits the control signal in packets via the network. In such implementation, a single hardware may be used for multiple controllers.

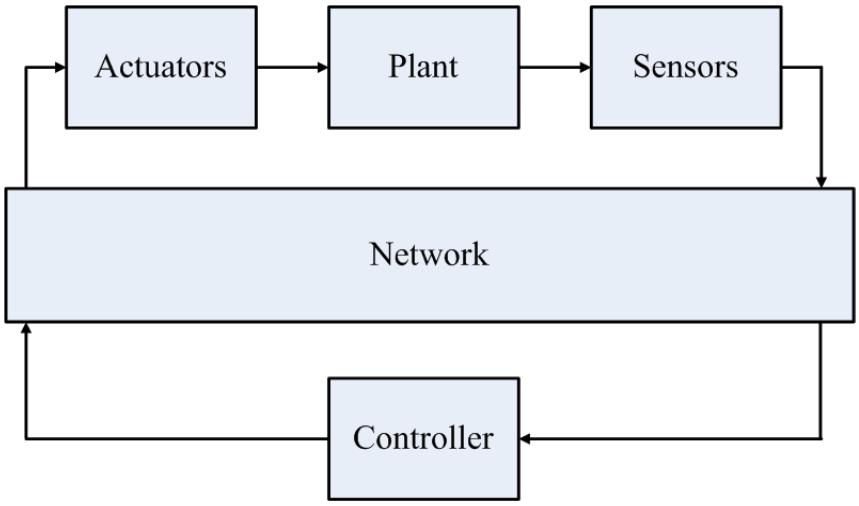

Figure 1.2: Direct structure of networked control systems

2. **Hierarchial structure**. This type of structure consists two controllers as shown in Figure 1.3. The main controller sends the reference signal to the remote system where the local controller will use this reference signal to control the plant. In this structure, the main controller may run at longer sampling period than the local system. The main controller can be implemented to handle multiple several remote systems.

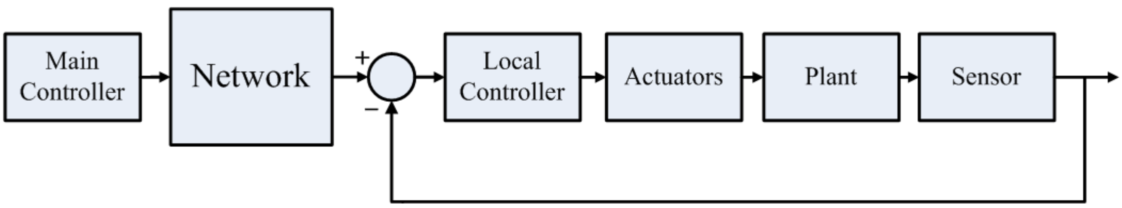

Figure 1.3: Hierarchial structure of networked control systems

Note that the control methodologies for the direct structure can still be applied to the hierarchial structure by treating the local system as a plant. For this reason, most of existing studies in NCSs have focused on the direct structure [9].

In NCSs, as already explained, the signal between the sensor and the controller or between the controller and the actuator is transmitted via a network. Traditional controller designs used in point-to-point architecture, therefore, cannot be used directly as they do not consider the presence of the network and assume instantaneous transfer of signal within the system. The presence of network makes the analysis more complicated as it introduces constraints that need to be considered. In general, the introduction of network degrades the performance of the system and worse, destabilize the system, meaning that the controller in NCSs needs to minimize the adverse effects of the network.

Figure 1.4 shows an example of NCSs in medicine called teleoperation. In this example, the surgeon is performing a surgery from a remote location and any information is transmitted via a network. The surgeon receives visual information as well as the patient's vitals via the network and performs the surgery using the robot located with the patient. Any command given by the surgeon is transmitted via the network and the robot follows the command it receives over the network. Instead of traditional surgery where the patient and the surgeon is in the same room, this special setup allows the patient to be remotely located while receiving treatment from a doctor anywhere in the world. This is a typical example of remote control that is now possible with NCSs.

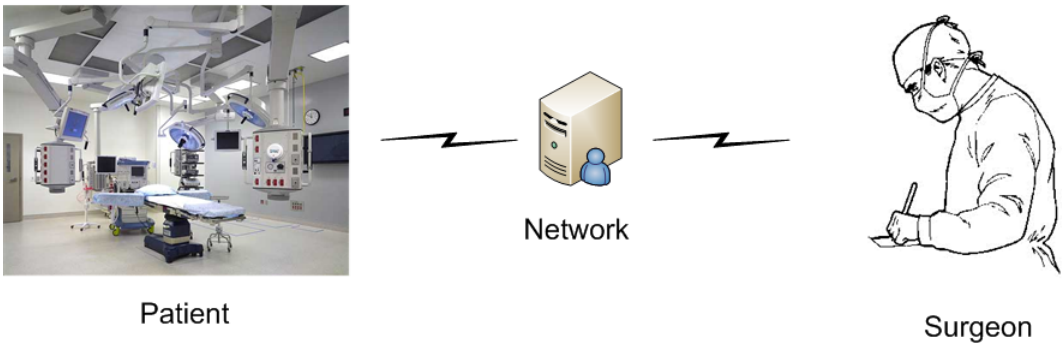

Figure 1.4: An example of networked control systems in remote surgery

In the previous example, machines and cameras (sensor) record the information about the patient (plant) and send the information through the network. The surgeon (controller) makes decisions (control signal) based on this information and it is passed to the robots (actuator) where they execute the decision the surgeon makes. As shown in this analogy, NCSs provide opportunities that have not been possible in the past. However, in order to utilize the advantages of NCSs, such as low installation cost, mobility and modularity, one needs to understand the constraints/challenges of introducing a network as a medium in control systems.

1.2 Constraints in Networked Control Systems

As mentioned in the previous section, the introduction of network degrades the performance of the overall system. This means that a specific controller design scheme for

NCSs is required [1–5, 10–12] and requires one to understand the fundamental issues arising from introducing a network. As information is transmitted via a network in NCSs, the same kind of constraints that appear in communication networks are also present in NCSs.

In the traditional point-to-point structure where the system components are connected via copper wire, infinite transmission bandwidth is assumed. This, along with the fact that the controller is located close to the plant, means that information is instantaneously received as soon as it is transmitted. However, in NCSs, information is put together in a packet and is transmitted through the network at time instances. Not only does it take a certain amount of time for these packets to travel through the network, they need to be created from the transmitter and interpreted at the receiver's end. These combined time corresponds to network-induced delays and it may be constant, time varying or random, depending on the network topologies being employed. Furthermore, due to various reasons such as buffer overload or noise, the packets may be lost and never received by the receiver. This phenomenon, called packet dropouts, results a piece of information never being available on the other side of the network. The existing theories that does not consider these constraints may not achieve stability of NCSs [1–5, 10–12]. These common issues in NCSs can be easily understood by a simple analogy of pigeon post as shown in Figure 1.5. The pigeons in this analogy represent the packets and the message they carry are the information that is being passed through the network. It is easy to see that, it will take a certain amount of time for the pigeons to deliver the message to the destination. The amount of time it takes depends on the pigeons and conditions of the sky that the pigeons fly. At the time instance t_1 in Figure 1.5, it takes $\tau(t_1)$ amount of time for the blue pigeon to deliver the message, illustrating the network-induced delays. At t_2 time instant, the red pigeon, while traveling to deliver the message, was shot down by a hunter and never arrived at the destination. This phenomenon in NCSs is the packet dropout where the information in the packet is lost. These two are the most important and common issues that appear in NCSs. In the case of green and black pigeons, because of the random network-induced delays, the green pigeon that was sent before the black pigeon arrived after the black pigeon. From the destination, it seems that the green pigeon carries a newer message and may discard the message the black pigeon is carrying. This issue, called packet reordering is caused by the random communication delays and its effect is similar to packet dropout, in that the message the black pigeon is carrying is lost.

When a signal is transmitted via a network in a form of packets, each packet inevitably experience network-induced delays and is subject to packet dropout. Figure 1.6 shows typical issues that may arise in NCSs. Both origin and destination are time-driven

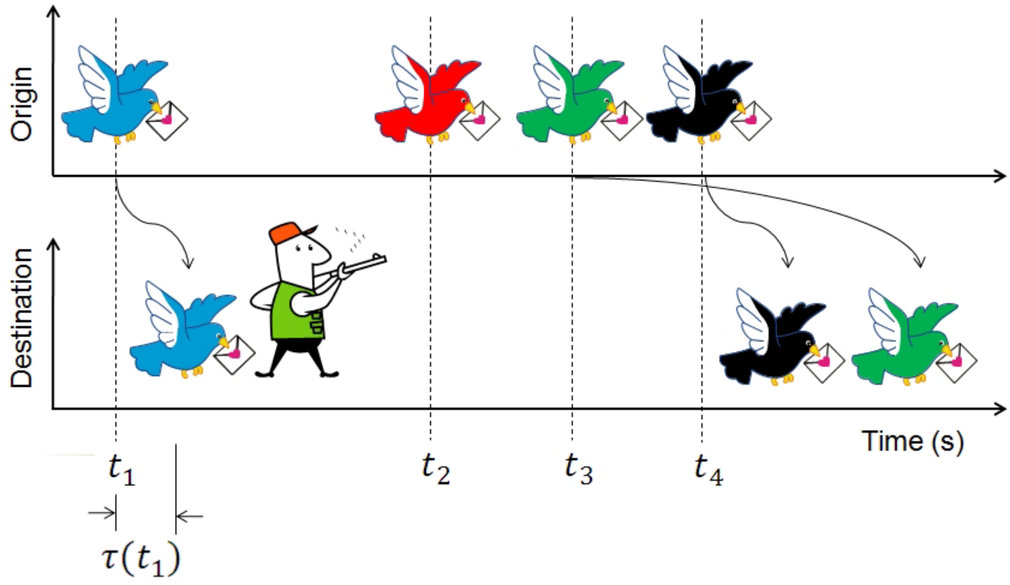

Figure 1.5: An analogy of constraints in networked control systems

for the purpose of this explanation. At time instance, t_{k-1}, the origin transmits a packet, highlighted in blue. This packet is received by the destination at time $t_{k-1} + \tau_{sc}^{k-1}$, where τ_{sc}^{k-1} is the network-induced delay at t_{k-1}. The destination uses a time-driven component therefore it will only obtain the new information from the buffer at time t_k. Unlike the traditional point-to-point structure, the transfer of information is not instantaneous; therefore by the time the destination receives the signal about the origin, the information is not up-to-date. As seen in the figure, at time t_k, when the destination receives the packet highlighted in blue, the actual information at the origin is different, highlighted in red. This means that the action taken at the destination is based on the past information: at time t_k, the signal received by the destination contains the information at time t_{k-1}. This is the most important issue in NCSs which leads to degradation of performance or instability. Packet dropout exacerbate this issue even further. For example, at time t_k, the origin sends another packet, highlighted in red. If packet dropout occurs and this packet is lost while it travels through the network, the most recent data at time t_{k+1} at the destination is still the same packet it received at time t_k, which contains the information of origin at time t_{k-1}, highlighted in blue. Note that in time-driven components, these delays will be multiple integers of the sampling period.

It is noteworthy that, in the case when the system component on the receiver is time-driven, the packet dropout can be seen as prolonged delay of more than one sampling instance. In Figure 1.6, the packet sent at time t_k has arrived the destination but the delay is longer than one sampling period. In this case, as there was no new information at time t_{k+1}, the destination uses the most recent data successfully transmitted, highlighted

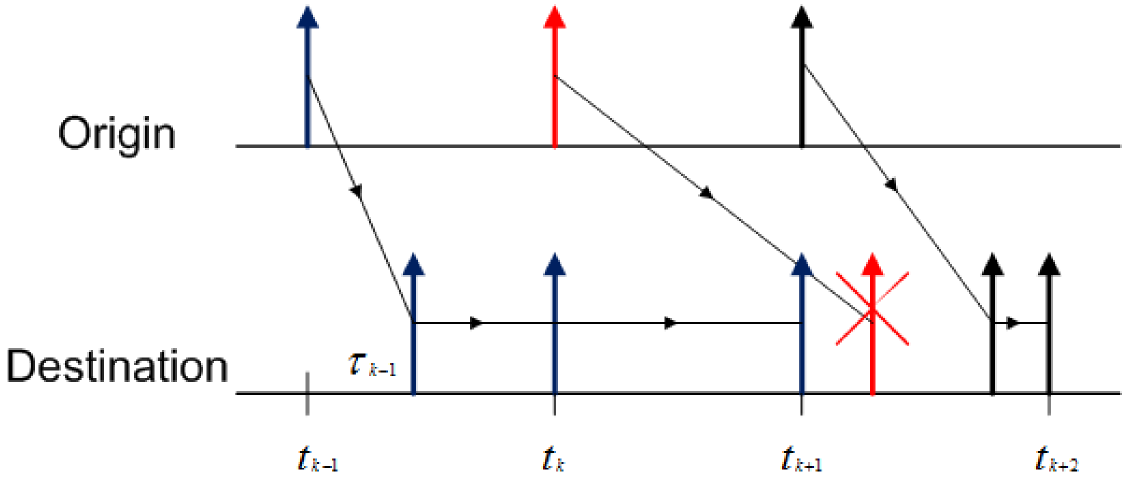

Figure 1.6: Timing of information transfer of a network

in blue. At time t_{k+2} the destination receives two packets, highlighted in red and black. Since the packet highlighted in black is the most recent, the destination will use the packet at time t_{k+1}, effectively creating packet dropout. As explained in the next chapter, this characteristic can be used to model packet dropouts in relation to network-induced delays.

The type of delays, whether it is constant, time varying or random, largely depends on the types topologies that networks use. The following paragraphs introduce two types of networks with different medium access control protocols [9].

The first type of network is called cyclic service network and this type of network protocols such as IEEE standard 802.4, SAE token bus, PROFIBUS, IEEE standard 805.5, fiber distributed data interface (FDDI) architectures and FireWire transmits the signal in cyclical order [13–16]. In these networks, the delays are deterministic and periodic. However, in practice, this periodic property may experience variations and this property can be destroyed. For example, when packet dropouts occur, as previously explained, the delay effectively increases by one sampling period, no longer satisfying periodic delay. Furthermore, discrepancies in clock generators also contribute towards the loss of periodic property in cyclic service network.

The second type of network is where many of the common networks such as Ethernet and Internet belong to, called random access network, where the delays are stochastic [17]. This type of network is more common in the real world hence many researchers have focused on this type of network. Since the delays are stochastic, probability theory is usually employed to model such network. The next section presents some of the existing studies on NCSs such as network modelling approaches and various controller designs.

1.3 Studies on Networked Control Systems

The concept of using a network to transmit signals within the system is novel but the presence of delays in the system has been considered since the 18^{th} century. These systems are called time delay systems and they differ from NCSs as time delay systems contain delays within the plant, instead of between system components [10, 18–26].

It is unclear when networked control systems were first developed but many believe that Bosch GmbH engendered the concept in 1983 when it studied feasibility of using networked devices within an automobile. It developed Control Area Network (CAN), a network medium dedicated for NCSs, which is widely used in car manufacturing in Europe [3]. The terminology, NCSs, however, did not appear until Gregory C. Walsh coined the term in 1998 when he provided a closed-loop structure where the controller and the sensor are connected via a serial communication channel.

The majority of the research in NCSs are regarding stability analysis [2, 5, 27–36] and the controller needs to minimize the adverse impact of the presence of network to achieve stability. As a consequence, the study of modelling these constraints is very important in the stability analysis of NCSs. More specifically, many existing approaches focus on modelling the network-induced delays as this is the most prominent constraint in NCSs. There are several ways researchers have attempted to incorporate the constraints in NCSs, mainly network-induced delays and packet dropouts and some of the approaches are presented in this section. Also presented are some of the controller design approaches that have been published in recent years to provide a brief overview of existing studies in NCSs.

1.3.1 Modelling of Network in NCSs

One obvious approach in NCSs is to design a controller based on the maximum delay of the network. Even though this kind of design is easier to implement, as only the maximum delay needs to be known, it is unrealistic to expect optimal performance. Furthermore, these kind of controllers are often conservative and overdesigned since the controller is based on the maximum delay with a small likelihood of occurrence. For these reasons, many researchers focus on modelling the network and design a controller based on the model of the network to ensure stability, resulting in delay-dependent controllers. The following paragraphs introduce some of the existing methodologies developed to model the network-induced delays and packet dropouts. It is noteworthy that the majority of existing papers incorporate model of network-induced delays as they are the most prominent constraint in the real world.

Bernoulli distributed white sequence has been widely used used to model the packet dropout [37–42] where discrete-time system with time-driven components are considered. When packet dropout occurs, the signal received in the last sampling instance can be used by the system. Bernoulli distributed white sequence, however, not only can it be used to model the packet dropout, it can also be used to model network-induced delays as shown in [43–45]. In a discrete-time system, the packet dropout and delay more than one sampling period are identical and the network-induced delays can be modelled by using this property.

Bernoulli distributed white sequence is essentially a binary switching sequence. In this sense, a Markov chain with two modes can essentially achieve the same outcome where one mode represents successful transmission and another a packet dropout as shown in [38, 46–48]. In these aforementioned papers, a homogeneous Markov chain is used to model the packet dropouts and since a Markov chain may consist more than 2 modes, it can be used to model successive dropouts as shown in [47, 48].

In [49], the authors focused on modelling IP network delays with generalized exponential distribution where round-trip time delays are measured and modelled. The authors overcome the difficulties of obtaining parameters of the model in real-time by treating the IP network stochastic behaviour as a parameter variation of the system transfer function. The authors show that the distribution is right skewed for IP network delays.

Since a network itself is a system, a system identification approach can be used to model the network-induced delays. In [18], the network-induced packet delays are modelled by either autoregressive moving average (ARMA) or autoregressive integrated moving average (ARIMA) model depending on whether the delay is stationary or nonstationary. The authors suggest that ARMA model is used for stationary random delays and ARMA for either nonstationary or weakly stationary delays. In this paper, a "black box" approach incorporated as the model identification relies only on the data itself instead of the prior information about the system. Autocorrelation function and partial autocorrelation function are used to determine whether the delay is stationary as the autocorrelation coefficients of stationary series decay quickly. In [50], a heuristic model based on the deviation-lag function (DLF) is developed to model the packet queueing delays. Unlike [18] where either ARMA or ARIMA model is used depending on the types of the delays, whether stationary or nonstationary, the authors show that using DLF can capture the statistical features of both types.

As explained previously, the network-induced delays of many existing networks such as CAN, Ethernet and Internet are stochastic in nature. This means that Markov chain, which also exhibit stochastic nature, is ideal in modelling the network-induced delays. For this reason, Markov chain has been widely used to model the delays [8, 11, 27, 34, 51–60] where each mode in the Markov chain corresponds to each delay in the network. In discrete-time NCSs, where the network-induced delays exists in integer multiples of the sampling period, Markov chain provides an excellent way to model the delays. It is worthwhile to note that the aforementioned papers require a completely known transition probability matrix, which describes the likelihood of transition among the modes. However, obtaining a completely known transition probability matrix in real world is either impossible or very costly. Due to this consideration, a methodology for designing a controller for Markovian jump linear system with partially known transition probability matrix in non-NCSs problem has been investigated in [61–63]. These methodologies still have room for improvement as they completely discard the unknown probabilities. Furthermore, these aforementioned papers do not consider NCSs where Markov chain is used to model the network-induced delays. Throughout this thesis, the network-induced delays are modelled by a homogenous finite state Markov chain, with the focus on partially known transition probability matrix. Refer to Chapter 2 and 3 for more information on how a Markov chain may be used to model the network in this thesis and how partially known transition probability matrix is handled to design a controller, despite knowing all transition probabilities.

A Markov chain is a random process which consists of a countable set of values and change the state at intervals. One of the key characteristics of Markov chain is the memorylessness in the state transitions, in that the next state depends only on the current state and not the past states. At each time instance, a transition to a new state occurs and its likelihood is defined by a probability called state transition probability, shown as follows:

$$p_{ij} := \mathbf{Prob}\{S_{k+1} = j | S_k = i\}$$

where S_{k+1} and S_k are the states at time instance $k+1$ and k respectively. This value, p_{ij}, describes the likelihood of state transition from i to j at time instance $k+1$.

Figure 1.7 illustrates an example of a finite state machine with 4 states in the Markov chain. For example, say the current state at time instance k is 1. At the next time instance, $k+1$, one of the following will occur; either the next state at time $k+1$ remains at 1 or it moves to any of the other states; 2, 3, or 4. The likelihood of such occurrence is given by a set of probabilities; p_{11}, p_{12}, p_{13} and p_{14} respectively. In the Markov chain that contains a finite number of states, hence the name “finite state Markov process”, these

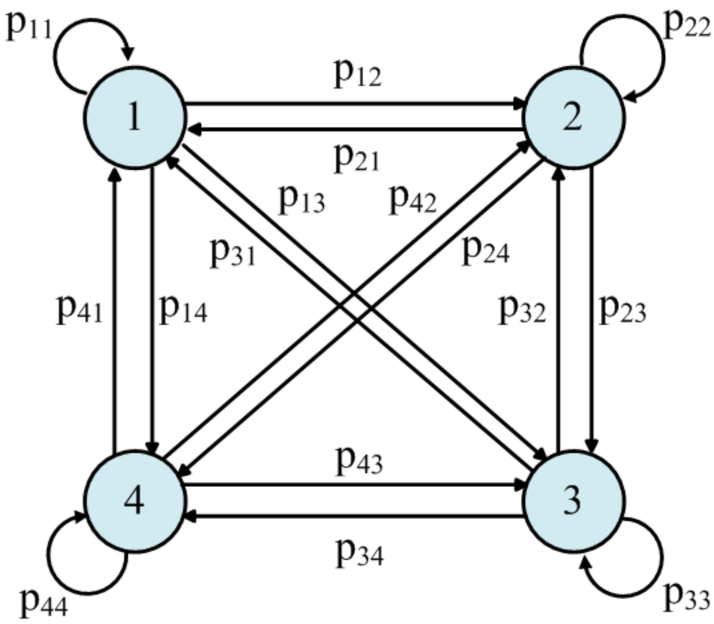

Figure 1.7: State diagram of Markov chain with 4 states

probabilities can be represented by a matrix as follows;

$$P_\tau \triangleq \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} & \cdots & p_{1s} \\ p_{21} & p_{22} & p_{23} & p_{24} & \cdots & p_{2s} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & p_{(s-1)s} \\ p_{s1} & p_{s2} & p_{s3} & p_{s4} & \cdots & p_{ss} \end{bmatrix} \quad (1.3.1)$$

where s represents the number of states in the Markov chain. Since p_{ij} are probabilities, $0 \leq p_{ij} \leq 1$ holds. Furthermore, at each time instance either transition to a new mode occurs or it remains in the current state, which means $\sum_{j=1}^{n} p_{ij} = 1$; $i = 1, 2, \ldots, n$. This fact that the summation of probabilities in each row must equal to 1 is used, as explained in Chapter 3, to deal the situation where some of the probabilities in (1.3.1) are unknown.

When a Markov chain is used to model the network-induced delays, each state, or mode, corresponds to the time-varying delay in the network. Throughout this thesis, it is assumed that the network-induced delays are time-varying but upper bounded by a constant. Furthermore, the current mode of the Markov chain is assumed to be accessible by the controller. This can easily be achieved by using a time stamp as shown in [17, 57, 64]. Based on the information the time stamp contains, the controller can determine the network- induced delay.

Another way to use Markov chain to model the network-induced delay is presented

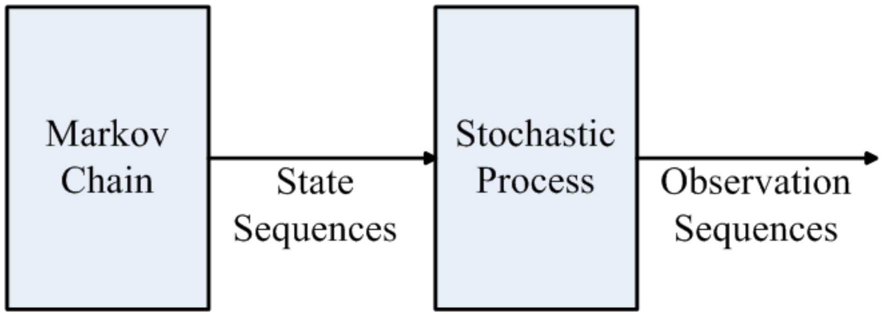

Figure 1.8: A basic structure of hidden Markov model

in [56]. This approach, named hidden Markov models, contains a Markov chain that is "hidden." It is essentially a double stochastic process with an underlying, unobservable process, and another stochastic process, from which the hidden process is to be estimated. Figure 1.8 illustrates the basic structure of a hidden Markov model.

Many of the papers on Markovian jump systems assume that the transition probability matrix is completely known [8, 11, 27, 34, 51–60]. However, obtaining a completely known transition probability matrix is either practically impossible or very expensive. This provided motivation to several researchers to consider a Markovian jump systems where the transition probability matrix is partially known [61–63]. In these papers, more general approach is presented where the Markovian jump system with completely known transition probability matrix becomes a special case of this approach. However, the methodologies presented in [61–63] still leaves room for improvement in the way the unknown transition probabilities are handled.

In [61–63], terms containing probabilities are separated and unknown probabilities are discarded. For example we have the following for any matrix P_j,

$$\sum_{j=1}^{n} p_{ij}P_j = \sum_{j\in\mathcal{S}_{\mathcal{K}}^i}^{n} p_{ij}P_j + \sum_{j\in\mathcal{S}_{\mathcal{UK}}^i}^{n} p_{ij}P_j \tag{1.3.2}$$

where

$$\mathcal{S}_{\mathcal{K}}^i \triangleq \{j : \text{if } p_{ij} \text{ is known}\}, \quad \mathcal{S}_{\mathcal{UK}}^i \triangleq \{j : \text{if } p_{ij} \text{ is unknown}\} \tag{1.3.3}$$

The above is true because the summation of all probabilities in each row of the transition probability matrix, regardless of whether they are known or not, is always one. Therefore if $\sum_{j\in\mathcal{S}_{\mathcal{K}}^i}^{n} p_{ij}P_j < 0$ and $P_j < 0, \quad \forall j \in \mathcal{S}_{\mathcal{UK}}^i$ then $\sum_{j=1}^{n} p_{ij}P_j < 0$ holds. By stating $P_j < 0, \quad \forall j \in \mathcal{S}_{\mathcal{UK}}^i$, the unknown transition probabilities are simply discarded, making the result more conservative.

Even though individual unknown probabilities cannot be obtained, the summation of unknown probabilities are bounded because the summation of all probabilities must equal to one. This means that

$$\sum_{j=1}^{n} p_{ij} = \sum_{j \in \mathcal{S}_{\mathcal{K}}^{i}}^{n} p_{ij} + \sum_{j \in \mathcal{S}_{\mathcal{UK}}^{i}}^{n} p_{ij} = \sum_{j \in \mathcal{S}_{\mathcal{K}}^{i}}^{n} p_{ij} + \Big(1 - \sum_{j \in \mathcal{S}_{\mathcal{K}}^{i}}^{n} p_{ij}\Big) = 1 \tag{1.3.4}$$

In this thesis, the above information is used to create an upper bound of the unknown probabilities so that the probability information is used in the controller design. By doing so, unlike what is shown in [61–63], the summation of parts corresponding to the known and unknown probabilities is considered, instead of separating them.

1.3.2 Controller Design for NCSs

The role of the controller in NCSs is to maintain the stability of the system as well as controlling and maintaining the system performance in the presence of the constraints imposed by the presence of the network. Several typical approaches are presented in the previous paragraphs. The following paragraphs describe some of the controller design methodologies in NCSs that have been published in recent years.

In [2, 30–33, 65–68], the authors use maximum allowable transfer time (MATI) or maximum allowable equivalent delay bound (MAEDB), which are quality of service measure for NCSs, in the stability analysis. In [33, 65], a methodology to determine MATI of an NCS to ensure that the overall system is stable is proposed. The design approach provided in these papers are to use standard control methodologies and choosing the network protocol and bandwidth to ensure that the system will remain stable when a computer network is introduced to the feedback loop. The advantage of such approach is the wide range of existing control methodologies and this approach is particularly useful when existing systems are converted to NCSs. However, it requires flexibility of choosing network protocols and bandwidth, which may not be the case in the real world. In [66, 67], the authors propose controller design based on MAEDB. In [30–32], the authors propose new scheduling protocol and show, by using MATI, that their protocol is superior.

Similar to [33, 65], where introducing a network to existing feedback loop is investigated, [49] also investigates using an existing controller to convert a non-NCS to an NCS. This particular concept is practical when replacing an existing controller is costly, inconvenient and/or time consuming. By using what the authors call gain scheduler middleware, the output of the existing controller is modified based on a gain scheduling algorithm with respect to the current network conditions. This allows the existing controllers to be used when network is inserted in the feedback loop.

In [16], the authors developed the sampling time scheduling methodology for an NCS to select a sampling period to ensure the stability of the system. This methodology selects a sampling time that is longer than the worst-case delay bound for a discrete-time NCS so that the delay does not affect the system performance. In this paper, a time-driven sensor and controller and an event-driven actuator is used. This methodology is originally applied to cyclic service networks whereby all connections of every NCS on the network are known in advance. It has been modified in [69] to CAN, which is a random access network. Furthermore, the methodology presented in [16] is expanded to multi-dimensional cases in [70, 71].

It has been shown in [72, 73] that nonlinear and perturbation theory can be used to formulate delay effects as perturbation of a continuous-time system under the assumption that there is no observation noise. The networks are restricted to be priority-based networks and priority scheduling algorithms are presented in [74]. This approach can be applied to both linear and nonlinear systems, but it requires a very small sampling period to ensure that the system can be approximated as a continuous-time system. This small sampling period usually results in congestion of network and high computation burden on components, both of which are not desirable in real world.

A problem of robust stability of NCSs with delays satisfying Bernoulli random binary distribution is investigated in [22]. In this paper, a new type of system model with stochastic parameter matrices is proposed. Exponential mean square stability of the original system based on Lyapunov functional is established. However, this work only investigates stability analysis, not the controller design methodology.

In [75], a method to formulate an NCS as a discrete-time switched system is proposed. Augmented state space representation of the system according to the delays are introduced in this paper. This means that the parameters in the system change based on the network conditions and a dynamic controller is used to control an NCS. Similarly, [76] presents stability and disturbance attenuation issues for NCSs under uncertain access delay and packet dropout using switched systems framework. As shown in [75], [76] also formulates an NCS as a discrete-time switched system so that existing methods for discrete-time switched systems such as [77–79] may still be used. However, the controller is required to run at a higher frequency than the sampling frequency which may lead to practical issues.

Queuing methodologies, which utilize deterministic [80, 81] or probabilistic information [82] for the control algorithm formulation are developed for NCSs. An observer and a predictor are used in [80, 81] to estimate the plant states and compensate for the delay based on the past output measurements stored in a first-in-first-out queue. However, this approach requires high model accuracy as the observer and the predictor are highly dependent on it. This means that the dynamic model of the plant needs to be precise. In [82], the presented methodology uses the amount of information in the queue to improve the state prediction. However, this approach is a scheme to predict state variables, not a control algorithm.

Optimal control methodology has been proven to be effective in traditional control systems theory. The purpose of such method is to minimize the cost function and this approach has yielded successful results in NCSs [17, 57, 64, 83]. The effects of the delays are treated as a linear quadratic Gaussian problem in [17]. The authors assume that the total delay is less than one sampling period and the information on the past time delays are available. The authors propose using a time stamp to store this information. A similar approach is used in [57, 64] but this approach allows network-induced delays longer than one sampling period. As it was the case with [17], [57, 64] also use time-driven sensor and even-driven controller and actuator. However, the approaches shown in these papers are not practical as the controller memory needs to be large to store the past information of delay, which is required to use the presented approach. In [83], linear quadratic regulator output feedback gain scheduling controller is designed for mobile NCSs. In this paper, the controller gain is dependent on the network condition, more specifically the delay at each time instance. The state space representation of a discrete-time system is augmented to include all the delay terms. By using this approach, the overall closed-loop system becomes a switched system. The major drawback of this approach is that the augmented system includes all the delay terms, that is, a system with long delays will increase the complexity of the problem significantly.

As explained the previous section, Markov chain is ideal in modelling the network-induced delays. By using this approach, the overall system becomes Markovian jump systems whereby each possible delay in the network is represented as a state, or mode, in the Markov chain [8, 54, 56]. A Markovian jump system can be seen as a collection of subsystems with a constant delay where each subsystem consists a controller designed for the constant delay. At each time instance, depending on the current delay of the network, one of the controllers corresponding to the subsystem will be "switched on." Hence, the controller is mode dependent, resulting in different the controller gain depending on the current delay in the network at a given time instance. In [8] a state feedback controller design for NCSs is developed where the stability analysis is based on Lyapunov function.

The existence of the controller is given in terms of solvability of bilinear matrix inequalities. In [54], interior-point algorithm is used to design stabilizing dynamic controllers. Refer to Chapter 2 and 3 for more information about how a Markov chain is used to model the network in NCSs and how a partially known transition probability matrix is considered in this thesis.

Study of nonlinear systems is important as every system contain inherent nonlinearities. Takagi-Sugeno (T-S) fuzzy model has been proven to be very effective in modelling nonlinear systems. Its success in traditional control theory has been translated to nonlinear NCSs in recent years [36, 41, 58, 66, 67, 84–91]. The attraction of T-S fuzzy model is that the global nonlinear system is modelled by local linear models. Fuzzy blending of local linear models with membership functions allow the global nonlinear plant to be modelled by a T-S fuzzy model. In [84] a parity-equation approach and a fuzzy observer based approach for fault detection of an NCS were presented. In contrast to other approaches in NCSs, this approach does not require the knowledge of exact values of network-induced delay as it addresses situations involving all possible delays. State feedback controller design for nonlinear NCSs is presented in [36, 58, 67, 85–88]. In [85], the authors take the probabilistic interval distribution of the communication delay into the controller design. They also provide a general framework where network-induced delays and packet dropout are under a unified framework. Two Markov processes are used to model the sensor-to-controller and controller-to-actuator delays in [58]. Fuzzy $\mathcal{H}_\infty$ tracking controller is presented in [67] where the controller design is based on the maximum allowable equivalent delay bound. State quantization in NCSs is considered in [86] where a robust $\mathcal{H}_\infty$ fuzzy state feedback controller is designed. In [87], a new type of state feedback controllers, named switched parallel distribution compensation controllers, are presented and it has been shown that this method provides better or at least the same results as the existing design methodologies. A sufficient condition for the existence of a fuzzy controller with network-induced delay and packet dropout are presented in [36]. Fuzzy rules are used to model the network-induced delays, which are then used to design a fuzzy state feedback controller in [88].

Compared to fuzzy state feedback controller design approach, study on output feedback controller design for nonlinear NCSs is even more scarce [41, 66, 89–91]. In [66, 91], the authors propose a static output feedback controller design. An observer based output feedback control of NCSs with multiple packet dropouts is presented in [41]. In [89] a fuzzy dynamic output feedback control of continuous-time NCS is presented where the sufficient conditions are derived by Lyapunov-Krasovskii functional. An $\mathcal{H}_\infty$ output tracking control for nonlinear discrete-time NCSs with packet dropout modelled as a Bernoulli sequence is presented in [90].

It has been shown in [92–96] that the membership functions play a vital role in the stability analysis of fuzzy model based control. However, many of the existing papers discard the membership functions in the design analysis. This is because the membership functions are nonlinear, hence popular linear matrix inequalities (LMIs) approach cannot be adopted. The traditional approach discards the membership functions in the controller design in order to formulate the conditions in LMIs. The drawback of such approach is that the controller designed is valid for any shape of membership functions, which cause conservatism. Several attempts have been made to overcome this drawback. In [92, 93], boundary information of membership function are incorporated into the controller design. However in this approach, only the lower and upper bound of the membership functions are included. In [94], the membership functions are approximated by using staircase piecewise functions. This means that in each sub-region membership functions are approximated by a scalar, allowing controller design to be formulated in terms of LMI conditions. Similar to [94], [95] use piecewise linear functions to approximate the membership functions. However, since piecewise functions are used in [94, 95], the number of sub-regions needs to be quite large in order to approximate the membership functions with accuracy. In [96], polynomial functions are used to approximate the products of membership functions and the bounds of the approximation error are incorporated into the design. This approach provides an alternative to [94, 95] by using continuous polynomial functions to approximate membership functions using sum-of-squares decomposition. Unlike [94, 95], where the membership functions are essentially discretized, polynomial function approximation provides continuous membership functions. Refer to Chapter 7 for more information about how membership functions are incorporated into the fuzzy controller design in this thesis.

In NCSs, a network is present between the plant and the controller. Signals traveling between the plant and the controller inevitably experience network-induced delays. In [36, 67, 85, 87, 88], the premise variables of the plant, which are the state variables of the plant, are assumed to be measurable and the same premise variables are used by the fuzzy controller. However in NCSs the premise will inevitably experience network-induced delays, that is, if the premise variables of the controller are selected to be the same as the plant, then the fuzzy controller has be based on the delayed premise of the plant. Therefore, the fuzzy controller based on the current premise of the plant as in [36, 67, 85, 87, 88] may be impractical in NCSs.

Study of NCSs is by no means complete as existing literatures focus on a small part of the issues in NCSs. Many of the existing papers make too many impractical assumptions such as that the network-induced delays are less than the sampling period or that network traffic cannot be overloaded. Furthermore, the majority of the studies in NCSs are focused on continuous-time systems and study on nonlinear NCSs is even more scarce in both continuous-time and discrete-time domain. Even well-established approaches such as Markov process to model the network or T-S fuzzy models for nonlinear NCSs fails to address some practical considerations. Therefore, the study of NCSs is still worthwhile with many areas that still need to be addressed.

1.4 Research Motivation

Study of NCSs is an area with a huge potential as it has a wide range of applications such as unmanned vehicle, remote control systems and teleoperation. NCSs has been receiving significant research interest with recent advance in communication systems. The fact that the controller can now be remotely located is very attractive in many applications as a single hardware can now be used to control several plants around the world using NCSs. Since NCSs can use existing networks to transmit the signal, it provides low construction cost due to lack of wires. The modularity and mobility of NCSs enable easy maintenance, which is yet another attractive feature of NCSs. Even though there are numerous literature on this topic, the study is not yet comprehensive. Some papers make too many unrealistic assumptions and some neglect important practical considerations.

Since the signal is passed through a network, it is formed into packets. Hence, the continuous signal needs to be sampled to be made into packets where these packets are then transmitted via the network at time instances. This makes study of NCSs in discrete-time domain very natural. However, the majority of studies in NCSs focus on continuous-time domain with time-driven sensors and even-driven controllers and actuators. Motivated by the advantages of discrete-time domain in NCSs and the lack of extensive studies compared to the continuous-time counterpart, this research focuses on controller design for discrete-time NCSs.

As most networks, such as CAN, Ethernet and Internet, exhibit random network-induced delays, Markov chain makes an ideal way of modelling these random network-induced delays. There have been several papers published on NCSs where the network is modelled by Markov chain [8, 11, 27, 34, 51–60]. These approaches rely on the fact that the transition probability matrix, which describes the statistical distribution of the delay, is completely known. However, obtaining a completely known transition probability matrix in real world is not an easy task and it is either costly or practically impossible. In light of this, methodologies for handling a partially known transition probability matrix for non-NCSs are presented in [61–63]. However, these methodologies are very crude as the unknown transition probabilities are simply discarded. Furthermore, the terms that contain probabilities are separated into two parts and considered separately. This research attempts to establish a methodology for designing a controller for NCSs with completely known transition probability matrix first then extend to NCSs with partially known transition probability matrix. It is shown that the problem with completely known transition probability matrix can be seen as a special case of the partially known transition probability matrix problem.

Filtering is an important research issue, particularly in signal processing applications, as it provides means to estimate the state information when the plant is disturbed. Traditional approach of Kalman filtering approach may not provide satisfactory results when there are uncertainties in the system model. $\mathcal{H}_\infty$ filtering approach has been one of the most popular approaches since it does not require exact knowledge of the statics of the external noise [97–100]. This research presents a robust $\mathcal{H}_\infty$ filter design for NCSs where the network-induced delays are modelled by Markov chain whose transition probability matrix is allowed to be partially known.

Study of robust stability is a popular area of research in control systems. When modelling a real world plant, parametric uncertainties arise due to aging of devices or identification errors. Parametric uncertainties play a vital role in stability and performance and NCSs is no exception. Because of this importance, the plant model considered in this research contain parametric uncertainties.

Every system in real world exhibits nonlinearities to a certain degree. A common practice is to linearize the nonlinear system about the operating region and apply linear control theory to obtain a controller, resulting a simpler analysis. However, when the operating region is wide, it is difficult to obtain an accurate linear model. Ever since its proposition in [101], T-S fuzzy model has been proven to be very effective in modelling nonlinear systems. Fuzzy system theory allows qualitative, linguistic information about the system and formulate the nonlinear system as a collection of local linear subsystems with fuzzy blending with membership functions. It has been shown that T-S fuzzy model can still be successfully used in nonlinear NCSs [36, 41, 58, 66, 67, 84–91]. However, the amount of work that has been carried out on nonlinear NCSs is minuscule compared to that of linear NCSs and even more scarce is the study of nonlinear discrete-time NCSs. The existing literature on nonlinear NCSs modelled by T-S fuzzy model shows promising results, making the research worthwhile.

As shown in [92–96], membership functions, which creates the fuzzy blending of linear subsystems, play a vital role in stability analysis of fuzzy model based control.

However, the majority of the existing literatures discard the membership functions in the controller design since membership functions are nonlinear, therefore LMI approach cannot be used. Several researchers investigated and presented their findings of incorporating membership functions in [92–96] but none of them are for NCSs. On another note, the signals transmitted via a network in NCSs inevitably experience network-induced delays. Many of the existing literature on nonlinear NCSs described by T-S fuzzy model neglects this fact and assume that the premise variables of the controller are not experiencing delays, leading to unrealistic problem formulation of NCSs. Motivated by lack of existing studies on nonlinear NCSs described by T-S fuzzy model where membership functions are incorporated into the controller design, this research investigates and addresses this issue. This research also acknowledges the fact that the premise variables will experience network-induced delays as they are transmitted via the network. In this research, the premise variables of the controller is the time delayed version of the plant's premise variables, assuming that the plant's premise variables are measurable, providing practical consideration of NCSs. By presenting a fuzzy dynamic output feedback controller design, it is ensured that the nonlinear NCSs can still be stabilized when the premise variables are not measurable.

1.5 Objectives of the Thesis

The main focus of this thesis is to establish the foundation of study of NCSs with random network-induced delays and parametric uncertainties in both linear and nonlinear realm. Markov chain is used to model the network and the transition probability matrix is allowed to be partially known. The discrete-time model of the plant is considered as it is more natural in NCSs. Throughout this thesis, novel methodologies for $\mathcal{H}_\infty$ controller/filter design are presented for both linear and nonlinear NCSs. Based on Lyapunov-Krasovskii functionals, the sufficient conditions for the existence of the controller/filter are presented. A T-S fuzzy model is used to describe the nonlinear NCSs and a less conservative fuzzy controller/filter design is proposed by incorporating membership functions into the controller design. The premise variables of the plant is allowed to be different to the premise variables of the controller in this thesis. In summary, the objectives of the thesis are as follows:

- Investigate using a Markov chain with partially known transition probability matrix in the controller/filter design of NCSs

- Develop a novel methodology for a robust $\mathcal{H}_\infty$ state feedback controller design for discrete-time linear NCSs where the network is modelled by a finite state homogeneous Markov chain with completely known transition probability matrix
- Extend the previous methodology to state feedback controller design for discrete-time linear NCSs with partially known transition probability matrix
- Develop a robust $\mathcal{H}_\infty$ filter and a robust $\mathcal{H}_\infty$ dynamic output feedback controller design methodology for linear NCSs with partially known transition probability matrix
- Establish a T-S fuzzy model for discrete-time nonlinear NCSs and investigate incorporating membership functions in the controller design
- Develop a new method of designing a robust fuzzy $\mathcal{H}_\infty$ state feedback controller for nonlinear NCSs where the plant is modelled by a T-S fuzzy model and membership functions are incorporated into the fuzzy controller design
- Develop a robust fuzzy $\mathcal{H}_\infty$ filter and a robust fuzzy $\mathcal{H}_\infty$ dynamic output feedback controller design methodologies for nonlinear NCSs described by T-S fuzzy model with partially known transition probability matrix and membership functions incorporated into the controller/filter design

In order to demonstrate the effectiveness of the proposed methodologies, numerical examples are provided. Simulation results show that the proposed methodologies achieve stability and prescribed performance index. In nonlinear NCSs, comparisons to methodologies without incorporating membership functions are made to illustrate that incorporating membership functions yield larger stabilization region.

1.6 Outline of the Thesis

Following the introduction, Chapter 2 presents a robust $\mathcal{H}_\infty$ state feedback controller design methodology for linear NCSs. The network-induced delays are modelled by a finite state Markov chain whose transition probability matrix is assumed to be completely known. Based on Lyapunov-Krasovskii functional, sufficient conditions for the existence of the controller is expressed in terms of BMIs, which are solved by the proposed iterative algorithm. Chapter 3 presents the new network model where the transition probability matrix is allowed to be partially known. This chapter extends the approach shown in Chapter 2 to a robust $\mathcal{H}_\infty$ state feedback controller design methodology for linear NCSs with partially known transition probability matrix. It is shown that the result shown in Chapter 2 is a special case of the methodology in Chapter 3.

In Chapter 4, robust $\mathcal{H}_\infty$ filter design problem is presented. $\mathcal{H}_\infty$ filter is particularly useful since the exact knowledge of noise is not required to design a filter. Sufficient conditions for the existence of the filter is derived based on Lyapunov-Krasovskii functional. The effectiveness of the proposed methodology is illustrated using a numerical example.

In Chapter 5, robust $\mathcal{H}_\infty$ dynamic output feedback controller design problem is introduced. Sufficient conditions are presented and numerical example is provided. Output feedback control is more practical than state feedback control since it is not always possible to measure all the states in the real world.

Chapter 6 provides an overview of T-S fuzzy model and sum-of-squares (SOS) decomposition. It also presents how sum-of-squares decomposition is used in this thesis to incorporate membership functions into the controller design. By using polynomial approximation, the membership functions, which are nonlinear, can be incorporated into the controller design such that existing numerical tools can be used to obtain the controller.

Chapter 7 presents a robust fuzzy $\mathcal{H}_\infty$ state feedback controller design methodology for nonlinear NCSs described by T-S fuzzy model. The premise variables of the controller is the time delayed version of the premise variables of the plant to acknowledge the presence of the network between the plant and the controller. Sufficient conditions for the existence of a robust $\mathcal{H}_\infty$ state feedback controller design is presented in terms of SOS based on Lyapunov-Krasovskii functional. Using numerical examples, it is shown that incorporating membership functions yield larger stabilization region and the presented approach achieves stability and performance criteria. In Chapter 7, the transition probability matrix is assumed to be completely known.

Chapter 8 presents a robust fuzzy $\mathcal{H}_\infty$ filter design where the plant is modelled by T-S fuzzy model and the network is modelled by a finite state Markov chain with partially known transition probability matrix. This approach provides a way to estimate the controlled output of the plant based on the measured output of the plant. As shown in Chapter 7, the premise variables of the plant and the filter are allowed to be different to acknowledge the time delay between the system components. The membership functions of the plant and the filter are incorporated into the filter design. Similarly, robust fuzzy $\mathcal{H}_\infty$ dynamic output feedback controller design for nonlinear NCSs is presented in Chapter 9. In this approach, it is assumed that the premise variables of the plant are unmeasurable or unavailable to the controller. The premise variables and membership functions of the controller are allowed to be different to those of the plant. Membership functions are incorporated into the controller design by approximating them using polynomial functions. Numerical examples are provided to illustrate the effectiveness of the proposed methodologies. It is shown in Chapter 9 that incorporating membership functions yields larger stabilization region.

Finally, the summary of the thesis and future research direction are discussed in Chapter 10.

2

Robust $\mathcal{H}_\infty$ State Feedback Control of Discrete-Time Networked Control Systems With Completely Known Transition Probability Matrix

Abstract

The aim of this chapter is to study stability analysis and controller design for a robust mode delay-dependent $\mathcal{H}_\infty$ controller design for discrete-time networked control systems. Network-induced delays between sensors and controllers are modelled by a finite state Markov process whose transition probability matrix is assumed to be completely known. Based on Lyapunov-Krasovskii functional, a novel methodology for designing a mode delay-dependent state feedback controller has been presented. It is also shown that the existing delay-dependent approach is a special case of the mode delay-dependent approach proposed in this study. The mode delay-dependent controller is obtained by solving linear matrix inequality optimisation problems using the cone complementarity linearisation algorithm. The effectiveness of the proposed design methodology is verified by a numerical example.

2.1 Introduction

Recent advances in communication networks have introduced a new field in control systems called networked control systems (NCSs) where the spatially distributed system components, such as sensors, actuators and controllers, are connected via network. This new development has fulfilled many requirements that have not been able to be met with the traditional point-to-point architecture. NCSs have achieved modularity, quick and easy maintenance and low cost because of the absence of wire connections between the system components [1–8].

However, as already address in Chapter 1, NCSs are challenged by numerous network constraints, arising from limited bandwidth. The network-induced delay could potentially deteriorate the stability and control performance of the system. Since the network-induced delays are usually time varying and non-deterministic, the traditional control methodologies for delay systems [10, 12, 52, 102–104] may not gain satisfactory performance for the control of NCSs. Recently, stochastic approaches are generally adopted to cope with network packet dropout and packet delays. In [43, 105], the stability robustness of NCSs is addressed, where the packet losses are modelled according to an independent and identically distributed Bernoulli distribution and the control input becomes zero when the data are lost (the so-called zero-control strategy). In [43–45], the delay is considered as white in nature with known probability distributions. Recently, Markovian jump systems (MJSs) proposed by Krasovskii and Lidskii in [106] has been a popular approach in modelling changes in the system. This class of systems is normally used to model stochastic process where transition from one mode to another occurs based on some probabilities called transition probabilities. Extensive work on Markovian jump linear systems has been carried out, such as controllability, stabilization, observability and optimal control as shown in [34, 107–111]. By incorporating Markov chain to model the network, the controller becomes delay mode dependent, where the controller gain is dependent on the delay in the network. At each time instance, appropriate controller will be “switched on” based on the current delay of the network.

This chapter aims to consider a class of uncertain discrete-time linear systems with random communication delays that exist between sensors and controllers. Markov process is used to model the communication channel where each mode in the Markov chain corresponds to the possible delays in the channel. The transition probability matrix for the Markov chain is assumed to be completely known. Based on the LyapunovKrasovskii functional, a mode delay-dependent state feedback controller is proposed to stabilise a class of systems. This mode delay-dependent controller is obtained by solving Bilinear matrix inequalities (BMIs) using the cone complementarity linearisation algorithm. Here, our approach depends on each mode delay, hence, as it is expected, the mode delay-dependent approach will yield a less conservative result as compared to the delay-dependent approach [110, 112–114].

The main contributions of this chapter is as follows.

- A mode delay-dependent state feedback controller is proposed to stabilise a class of networked control systems
- Based on the approach presented in this chapter, robust $\mathcal{H}_\infty$ state feedback, robust $\mathcal{H}_\infty$ filter and robust $\mathcal{H}_\infty$ dynamic output feedback controller design approaches for linear NCSs with partially known transition probability matrix in the Markov chain is derived

The rest of the chapter is organised as follows. Section 2.2 presents the system description, modelling of network-induced delays as well as necessary lemma and problem formulation. In Section 2.3, stability analysis and the robust $\mathcal{H}_\infty$ state feedback controller design are presented in terms of BMIs. A cone complementarity algorithm to convert the BMIs to quasi-convex LMIs is also presented. Section 2.4 provide a numerical example to illustrate the effectiveness of the controller design. Finally, conclusions are drawn in Section 2.5.

2.2 System Description and Definitions

Consider the NCSs setup shown in Figure 2.1. The sensor measures the states variables, which is then sent to the controller via a network at each time instance. A class of uncertain discrete-time linear systems under consideration is described by the following model:

$$\begin{aligned} x(k+1) &= [A+\Delta A(k)]x(k) + [B_1+\Delta B_1(k)]w(k) + [B_2+\Delta B_2(k)]u(k), \quad x(0)=0 \\ z(k) &= [C_1+\Delta C_1(k)]x(k) + [D_{11}+\Delta D_{11}(k)]w(k) + [D_{12}+\Delta D_{12}(k)]u(k) \end{aligned} \tag{2.2.1}$$

where $x(k) \in \Re^n$ is the state vector, $z(k) \in \Re^p$ is the controlled output and $w(k) \in \Re^q$ is the disturbance which belong to $\mathcal{L}_2[0,\infty)$, the space of square summable vector sequence over $[0,\infty]$. The matrices A, B_1, B_2, C_1, D_{11} and D_{12} are of appropriate dimensions. The matrix functions $\Delta A(k)$, $\Delta B_1(k)$, $\Delta B_2(k)$, $\Delta C_1(k)$, $\Delta D_{11}(k)$ and $\Delta D_{12}(k)$ represent the time-varying uncertainties in the system and satisfy the following assumption.

Assumption 2.2.1

$$\begin{bmatrix} \Delta A(k) & \Delta B_1(k) & \Delta B_2(k) \\ \Delta C_1(k) & \Delta D_{11}(k) & \Delta D_{12}(k) \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} F(k) \begin{bmatrix} H_1 & H_2 & H_3 \end{bmatrix}$$

where H_i and E_i are known matrices which characterize the structure of the uncertainties. Furthermore, there exists a positive constant $\mathcal{W}$ such that the following inequality holds:

$$F^T(k)\mathcal{W}F(k) \leq \mathcal{W} \tag{2.2.2}$$

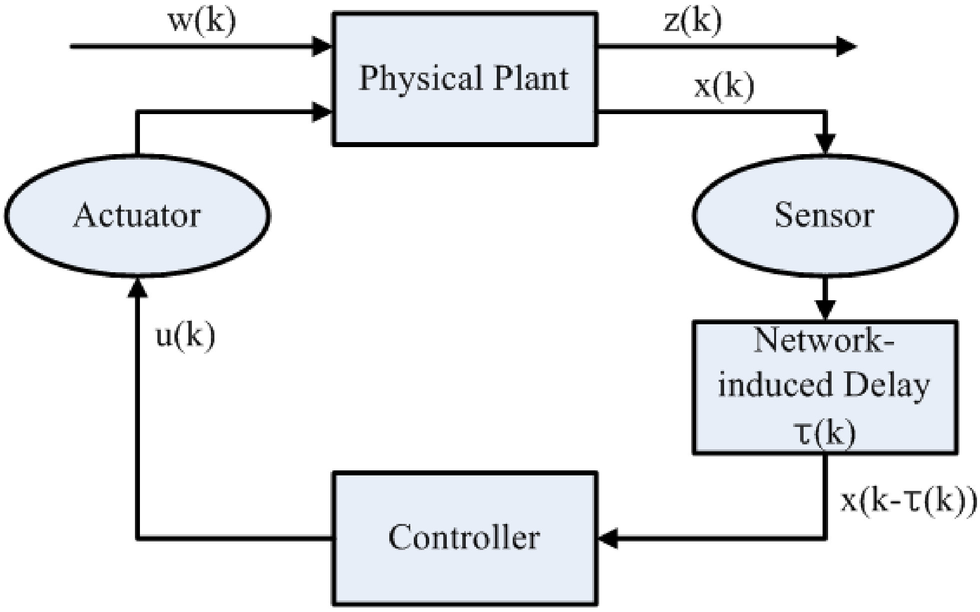

Figure 2.1: A networked control system with sensor-to-controller delay

Controller design methodologies derived based on this model can be extended to NCSs with both sensor-to-controller delays and controller-to-actuator delays. Refer to the works in [34, 58] for more information with respect to two separate delays in the system.

Let $\{r_k\}$ be a discrete homogeneous Markov chain taking values in a finite set $\mathcal{S} = \{1, 2, \cdots, s\}$, with the following transition probability from mode i at k to mode j at time $k+1$

$$p_{ij} := \mathbf{Prob}\{r_{k+1} = j | r_k = i\}$$

where $i, j \in \mathcal{S}$.

The random delays τ_k is modeled by a finite state Markov process as $\tau_k = \tau(r_k)$ with $0 \leq \tau(1) < \tau(2) < \cdots < \tau(s) \leq \infty$. Throughout this thesis, it is assumed that the proposed controller will always use the most recent data for feedback. That means that if there is no new information available at step $k+1$, due to either packet loss or delay longer than one sampling period, $x(k-\tau_k)$ will be used for feedback, that is, the delay can only increase at most by 1 at each time instance. Therefore we have

$$\mathbf{Prob}\{\tau_{k+1} > \tau_k + 1\} = 0$$

Based on the above, the transition probability matrix for the Markov chain above is of the following form

$$P_\tau = \begin{bmatrix} p_{11} & p_{12} & 0 & 0 & \dots & 0 \\ p_{21} & p_{22} & p_{23} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & p_{(s-1)s} \\ p_{s1} & p_{s2} & p_{s3} & p_{s4} & \dots & p_{ss} \end{bmatrix} \tag{2.2.3}$$

Note $0 \leq p_{ij} \leq 1$ and $\sum_{j=1}^{i+1} p_{ij} = 1$ where $p_{s(s+1)} = 0$. These transition probabilities may be obtained by, for example, carrying out experiments to send packets and obtaining the delay. Larger number of packets sent in the experiments would provide more accurate delay distribution of the stochastic process.

It is noteworthy that, in this thesis, the packet loss is treated as a network-induced delay longer than one sampling period as shown in Figure 1.6 in Chapter 1. Consequently, it is shown that a Markov chain may be used to model packet dropout as well as the network-induced delays with time-driven components.

The mode-dependent switching state feedback control law is

$$u(k) = K(r_k)x(k-\tau_k) \tag{2.2.4}$$

The states measurement, $x(k)$ is delayed due to the network between the plant and the controller. The control law is mode-dependent as the control gain, $K(r_k)$, depends on the delay at each time instance. The magnitude of the delay in each packet, can be obtained by using time stamp in each message. Unlike the traditional system where no delay between components are assumed, the data received by the receiver in NCSs contains the past information sent by the transmitter. Since the data used to compute the control signal are not the current information, the controller designed without considering the effects of the delay degrades the performance of the system.

The problem of the robust $\mathcal{H}_\infty$ control for such system is formulated as follows.

Problem Formulation: Given a prescribed $\gamma > 0$, design a state feedback controller of the form (2.2.4) such that

1. The system (2.2.1) with (2.2.4) and $w(k) = 0$ is stochastically stable, i.e., there exists a constant $0 < \alpha < \infty$ such that

$$E\left\{\sum_{\ell=0}^{\infty} x^T(\ell)x(\ell)\right\} < \alpha \tag{2.2.5}$$

   for all $x(0), r_0$.

2. Under the zero-initial condition, the controlled output $z(k)$ satisfies

$$E\left\{\sum_{k=0}^{\infty} z^T(k)z(k)|r_0\right\} < \gamma \sum_{k=0}^{\infty} w^T(k)w(k) \tag{2.2.6}$$

   for all nonzero $w(k)$.

The following lemma which will play a vital role in deriving our main results in this thesis is shown below.

Lemma 2.2.1 *Let* $y(k) = x(k+1) - x(k)$ *and* $\tilde{x}(k) = \Big[x^T(k) \;\; x^T(k-\tau_k) \;\; w^T(k)$ $x^T(k)H_1^T F^T(k) \;\; x^T(k-\tau_k)K^T(r_k)H_3^T F^T(k) \;\; w^T(k)H_2^T F^T(k)\Big]^T \in \Re^l$, *then for any matrices* $R \in \Re^{n\times n}$, $M \in \Re^{n\times l}$ *and* $Z \in \Re^{l\times l}$ *satisfying*

$$\begin{bmatrix} R & M \\ M^T & Z \end{bmatrix} \geq 0 \tag{2.2.7}$$

the following inequality holds

$$-\sum_{i=k-\tau_k}^{k-1} y^T(i)Ry(i) \leq \tilde{x}^T(k)\Big\{\Upsilon_1 + \Upsilon_1^T + \tau_k Z\Big\}\tilde{x}(k) \tag{2.2.8}$$

where $\Upsilon_1 = M^T[I \quad -I \;\; 0 \;\; 0 \;\; 0 \;\; 0]$.

Proof: From (2.2.7), the following holds

$$\sum_{i=k-\tau_k}^{k-1} \begin{bmatrix} y(i) \\ \tilde{x}(k) \end{bmatrix}^T \begin{bmatrix} R & M \\ M^T & Z \end{bmatrix} \begin{bmatrix} y(i) \\ \tilde{x}(k) \end{bmatrix} \geq 0 \tag{2.2.9}$$

By expanding (2.2.9), it becomes

$$\sum_{i=k-\tau_k}^{k-1} y^T(i)Ry(i) + \tilde{x}^T(k)M^T y(i) + y^T(i)M\tilde{x}(k) + \tilde{x}^T(k)Z\tilde{x}(k) \geq 0$$

Rearranging the above results

$$-\sum_{i=k-\tau_k}^{k-1} y^T(i)Ry(i) \quad \leq \sum_{i=k-\tau_k}^{k-1} [\tilde{x}^T(k)M^T y(i) + y^T(i)M\tilde{x}(k) + \tilde{x}^T(k)Z\tilde{x}(i)]$$

Since $y(k) = x(k+1) - x(k)$ the right hand side of the equation becomes

$$\sum_{i=k-\tau_k}^{k-1} \Big[(\tilde{x}^T(k)M^T[x(i+1) - x(i)] + [x^T(i+1) - x^T(i)]M\tilde{x}(k) + \tilde{x}^T(k)Z\tilde{x}(k)\Big]$$

Expanding the summation results cancellation of terms. Then the equation above becomes

$$\begin{aligned} & \tilde{x}^T(k)M^T[x(k) - x(k-\tau_k)] + [x^T(k) - x^T(k-\tau_k)]M\tilde{x}(k) + \tilde{x}^T(k)\tau_k Z\tilde{x}(k) \\ = \; & \tilde{x}^T(k)M^T[I \;\; -I \;\; 0]\tilde{x}(k) + \tilde{x}^T(k)[I \;\; -I \;\; 0]^T M\tilde{x}(k) + \tilde{x}^T(k)\tau_k Z\tilde{x}(k) \end{aligned}$$

where $I \in \Re^{n\times n}$ and $0 \in \Re^{n\times l-2n}$. ∇∇∇

2.3 Main Results

In the previous section, the problem of robust $\mathcal{H}_\infty$ control for a class of discrete-time NCSs with a completely known transition probability matrix is presented. This section proposes the stability criteria and the controller design for uncertain discrete-time systems with random communication delays with completely known transition probability matrix.

Theorem 2.3.1 *For given controller gains $K(i)$, $i = 1, \cdots, s$ and $\gamma > 0$, if there exist* *sets of positive-definite matrices $P(i)$, $R_1(i)$, R_1, $R_2(i)$, R_2, $W_1(i)$, $W_2(i)$, $W_3(i)$, Q,$Z(i)$ and matrices $M(i)$ for $i = 1, 2, \cdots, s$ satisfying the following inequalities*

$$R_1 > R_1(i), \quad R_2 > R_2(i) \tag{2.3.1}$$

$$\begin{array}{c}\Lambda(i) + \Gamma_1^T(i)\tilde{P}(i)\Gamma_1(i) + \Gamma_2^T(i)\left[\tilde{\tau}(i)R_1 + \tau(s)R_2\right]\Gamma_2(i) + \Upsilon_1(i) + \Upsilon_1^T(i) \\ +\tau(i)Z(i) + \Xi^T(i)\Xi(i) < 0\end{array} \tag{2.3.2}$$

and

$$\begin{bmatrix} (1 - p_{i(i+1)})R_1(i) + R_2(i) & M(i) \\ M^T(i) & Z(i) \end{bmatrix} \geq 0 \tag{2.3.3}$$

where

$$\begin{array}{rcl}\tilde{P}(i) & = & \sum_{j=1}^{i+1} p_{ij}P(j) \\ \tilde{\tau}(i) & = & \sum_{j=1}^{i+1} p_{ij}\tau(j) \\ \Gamma_1(i) & = & \begin{bmatrix} A & B_2K(i) & B_1 & E_1 & E_1 & E_1 \end{bmatrix} \\ \Xi(i) & = & \begin{bmatrix} C_1 & D_{12}K(i) & D_{11} & E_2 & E_2 & E_2 \end{bmatrix} \\ \Gamma_2(i) & = & \begin{bmatrix} A - I & B_2K(i) & B_1 & E_1 & E_1 & E_1 \end{bmatrix} \\ \Lambda(i) & = & diag\Big\{\Big((\tau(s) - \tau(1) + 1)Q + H_1^TW_1(i)H_1 - P(i)\Big), \\ & & \Big(K^T(i)H_3^TW_2(i)H_3K(i) - Q\Big), \Big(H_2^TW_3(i)H_2 - \gamma I\Big), -W_1(i), \\ & & -W_2(i), -W_3(i)\Big\} \\ \Upsilon_1(i) & = & M^T(i)[I \quad -I \quad 0 \quad 0 \quad 0 \quad 0].\end{array} \tag{2.3.4}$$

Then the closed-loop system is stochastically stable with the prescribed $\mathcal{H}_\infty$ performance.

Proof: The system (2.2.1) with (2.2.4) can be rewritten as

$$\begin{array}{rcl} x_{k+1} & = & \Gamma_1(r_k)\tilde{x}_k \\ z_k & = & \Xi(r_k)\tilde{x}_k \end{array} \tag{2.3.5}$$

where $x_\ell = x(\ell)$, $z_\ell = z(\ell)$, $\tilde{x}_\ell = \tilde{x}(\ell)$, $\Gamma_1(r_k)$ is given in (2.3.4), and $\tilde{x}_k$ is defined in Lemma 2.2.1.

In order to study the stability criteria of NCSs, Lyapunov-Krasovskii candidate functional is used. Consider the following Lyapunov-Krasovskii candidate functional:

$$V(x_k, r_k) = V_1(x_k, r_k) + V_2(x_k, r_k) + V_3(x_k, r_k) \tag{2.3.6}$$

with

$$V_1(x_k, r_k) = x_k^T P(r_k) x_k \tag{2.3.7}$$

$$V_2(x_k, r_k) = \sum_{\ell=-\tau_k}^{-1} \sum_{j=k+\ell}^{k-1} y_j^T R_1 y_j + \sum_{\ell=-\tau(s)}^{-1} \sum_{j=k+\ell}^{k-1} y_j^T R_2 y_j \tag{2.3.8}$$

$$V_3(x_k, r_k) = \sum_{\ell=k-\tau(k)}^{k-1} x_\ell^T Q x_\ell + \sum_{\ell=-\tau(s)+2}^{-\tau(1)+1} \sum_{j=k+\ell-1}^{k-1} x_j^T Q x_j \tag{2.3.9}$$

Along any trajectory of the closed-loop system, the expectation value of the first forward difference of $V(x_k, r_k)$ is given as follows:

$$\Delta V(x_k, r_k) = \Delta V_1(x_k, r_k) + \Delta V_2(x_k, r_k) + \Delta V_3(x_k, r_k) \tag{2.3.10}$$

with

$$\begin{aligned} \Delta V_1(x_k, r_k) &= x_{k+1}^T \tilde{P}(r_k) x_{k+1} - x_k^T P(r_k) x_k \\ &= \tilde{x}_k^T \Gamma_1^T(r_k) \tilde{P}(r_k) \Gamma_1(r_k) \tilde{x}_k - x_k^T P(r_k) x_k \end{aligned} \tag{2.3.11}$$

$$\begin{aligned} \Delta V_2(x_k, r_k) &= \sum_{i=1}^{s} p_{r_k i} \sum_{\ell=-\tau(i)}^{-1} \sum_{j=k+1+\ell}^{k} y_j^T R_1 y_j + \sum_{\ell=-\tau(s)}^{-1} \sum_{j=k+\ell+1}^{k} y_j^T R_2 y_j \\ &\quad - \sum_{\ell=-\tau_k}^{-1} \sum_{j=k+\ell}^{k-1} y_j^T R_1 y_j - \sum_{\ell=-\tau(s)}^{-1} \sum_{j=k+\ell}^{k-1} y_j^T R_2 y_j \\ &= \sum_{i=1}^{s} p_{r_k i} \left\{ \sum_{\ell=-\tau(i)}^{-1} y_k^T R_1 y_k + \sum_{\ell=-\tau(i)}^{-1} \sum_{j=k+\ell+1}^{k-1} y_j^T R_1 y_j - \sum_{\ell=-\tau_k}^{-1} \sum_{j=k+\ell+1}^{k-1} y_j^T R_1 y_j \right. \\ &\quad \left. - \sum_{\ell=-\tau_k}^{-1} y_{k+\ell}^T R_1 y_{k+\ell} \right\} + \sum_{\ell=-\tau(s)}^{-1} \left\{ y_k^T R_2 y_k - y_{k+\ell}^T R_2 y_{k+\ell} \right\} \\ &= \sum_{i=1}^{s} p_{r_k i} \left\{ \sum_{\ell=-\tau(i)}^{-1} \sum_{j=k+\ell+1}^{k-1} y_j^T R_1 y_j - \sum_{\ell=-\tau_k}^{-1} \sum_{j=k+\ell+1}^{k-1} y_j^T R_1 y_j \right. \\ &\quad \left. - \sum_{j=k-\tau_k}^{k-1} y_j^T R_1 y_j \right\} - \sum_{j=k-\tau(s)}^{k-1} y_j^T R_2 y_j + y_k^T \Big[\tilde{\tau}_k R_1 + \tau(s) R_2 \Big] y_k \end{aligned} \tag{2.3.12}$$

and

$$
\begin{aligned}
\Delta V_3(x_k, r_k) &= \sum_{i=1}^{s} p_{r_k i} \sum_{\ell=k-\tau(i)+1}^{k} x_\ell^T Q x_\ell + \sum_{\ell=-\tau(s)+2}^{-\tau(1)+1} \sum_{j=k+\ell}^{k} x_j^T Q x_j - \sum_{\ell=k-\tau_k}^{k-1} x_\ell^T Q x_\ell \\
&\quad - \sum_{\ell=-\tau(s)+2}^{-\tau(1)+1} \sum_{j=k+\ell-1}^{k-1} x_j^T Q x_j \\
&= \sum_{i=1}^{s} p_{r_k i} \left\{ x_k^T Q x_k + \sum_{\ell=k-\tau(i)+1}^{k-1} x_\ell^T Q x_\ell - \sum_{\ell=k-\tau_k+1}^{k-1} x_\ell^T Q x_\ell - x_{k-\tau_k}^T Q x_{k-\tau_k} \right\} \\
&\quad + \sum_{\ell=-\tau(s)+2}^{-\tau(1)+1} \left\{ x_k^T Q x_k - x_{k+\ell-1}^T Q x_{k+\ell-1} \right\} \\
&= \sum_{i=1}^{s} p_{r_k i} \left\{ \sum_{\ell=k-\tau(i)+1}^{k-1} x_\ell^T Q x_\ell - \sum_{\ell=k-\tau_k+1}^{k-1} x_\ell^T Q x_\ell - x_{k-\tau_k}^T Q x_{k-\tau_k} \right\} \\
&\quad - \sum_{\ell=-\tau(s)+2}^{-\tau(1)+1} x_{k+\ell-1}^T Q x_{k+\ell-1} + (\tau(s) - \tau(1) + 1) x_k^T Q x_k \\
&= \sum_{i=1}^{s} p_{r_k i} \left\{ \sum_{\ell=k-\tau(i)+1}^{k-\tau(1)} x_\ell^T Q x_\ell + \sum_{\ell=k-\tau(1)+1}^{k-1} x_\ell^T Q x_\ell - \sum_{\ell=k-\tau_k+1}^{k-1} x_\ell^T Q x_\ell \right\} \\
&\quad - \sum_{\ell=k-\tau(s)+1}^{k-\tau(1)} x_\ell^T Q x_\ell + (\tau(s) - \tau(1) + 1) x_k^T Q x_k - x_{k-\tau_k}^T Q x_{k-\tau_k} \qquad (2.3.13)
\end{aligned}
$$

with $\tilde{P}(r_k) = \sum_{j=1}^{r_k+1} p_{r_k j} P(j)$.

Knowing that $\text{Prob}\{\tau_{k+1} > \tau_k + 1\} = 0$, $\tau(1) \leq \tau_{k+1} \leq \tau_k + 1 \leq \tau(s)$ and $\tau(1) \leq \tau_k \leq \tau(s)$, the terms $\Delta V_2(x_k, r_k)$ and $\Delta V_3(x_k, r_k)$ above can be upper bounded as

$$
\Delta V_2(x_k, r_k) \leq y_k^T \Big[\tilde{\tau}(i) R_1 + \tau(s) R_2 \Big] y_k - \sum_{\ell=k-\tau_k}^{k-1} y_\ell^T \Big[(1 - p_{r_k(r_k+1)}) R_1 + R_2 \Big] y_\ell \qquad (2.3.14)
$$

and

$$
\Delta V_3(x_k, r_k) \leq (\tau(s) - \tau(1) + 1) x_k^T Q x_k - x_{k-\tau_k}^T Q x_{k-\tau_k}. \qquad (2.3.15)
$$

Using Lemma 2.2.1 and $y_k = x_{k+1} - x_k$ we have

$$
\Delta V_2(x_k, r_k) \leq \tilde{x}_k^T \Big\{ \Gamma_2^T(r_k) \left[\tilde{\tau}(r_k) R_1 + \tau(s) R_2 \right] \Gamma_2(r_k) + \Upsilon_1(r_k) + \Upsilon_1^T(r_k) + \tau_k Z(r_k) \Big\} \tilde{x}_k \qquad (2.3.16)
$$

where $\Gamma_2(r_k)$ is given in (2.3.4).

Therefore,

$$\begin{aligned}\Delta V(x_k, r_k) \;\; \leq \;\; & -x_k^T\Big(P(r_k) - (\tau(s) - \tau(1) + 1)Q\Big)x_k - x_{k-\tau_k}^T Q x_{k-\tau_k} + \\ & \tilde{x}_k^T\Big\{\Gamma_1^T(r_k)\tilde{P}(r_k)\Gamma_1(r_k) + \Gamma_2^T(r_k)\left[\tilde{\tau}(r_k)R_1 + \tau(s)R_2\right]\Gamma_2(r_k) \\ & + \Upsilon_1(r_k) + \Upsilon_1^T(r_k) + \tau_k Z(r_k)\Big\}\tilde{x}_k \qquad (2.3.17)\end{aligned}$$

Using Assumption 2.2.1, and adding and subtracting $x_k^T H_1^T F_k^T W_1(r_k) F_k H_1 x_k$, $w_k^T H_2^T F_k^T W_3(r_k) F_k H_2 w_k$, $z_k^T z_k$, $x_{(k-\tau_k)}^T K^T(r_k) H_3^T F_k^T W_2(r_k) F_k H_3 K(r_k) x_{(k-\tau_k)}$ and $\gamma w_k^T w_k$ to and from (2.3.17), the following is obtained

$$\begin{aligned}\Delta V(x_k, r_k) \;\; \leq \;\; & -x_k^T\Big(P(r_k) - (\tau(s) - \tau(1) + 1)Q - H_1^T W_1(r_k) H_1\Big)x_k \\ & -x_{k-\tau_k}^T\Big(Q - K^T(r_k) H_3^T W_2(r_k) H_3 K(r_k)\Big)x_{k-\tau_k} \\ & +\tilde{x}_k^T\Big\{\Gamma_1^T(r_k)\tilde{P}(r_k)\Gamma_1(r_k) + \Gamma_2^T(r_k)\left[\tilde{\tau}(r_k)R_1 + \tau(s)R_2\right]\Gamma_2(r_k) \\ & + \Upsilon_1(r_k) + \Upsilon_1^T(r_k) + \tau_k Z(r_k) + \Xi^T\Xi\Big\}\tilde{x}_k \\ & - z_k^T z_k + \gamma w_k^T w_k - w_k^T\Big(\gamma I - H_2^T W_3(r_k) H_2\Big)w_k \\ & - x_{(k-\tau_k)}^T K^T(r_k) H_3^T F_k^T W_2(r_k) F_k H_3 K(r_k) x_{(k-\tau_k)} \\ & - x_k^T H_1^T F_k^T W_1(r_k) F_k H_1 x_k - w_k^T H_2^T F_k^T W_3(r_k) F_k H_2 w_k \qquad (2.3.18)\end{aligned}$$

Using (2.3.4), (2.3.18) can be rewritten as

$$\begin{aligned}\Delta V(x_k, r_k) \;\; \leq \;\; & \tilde{x}_k^T\Big\{\Lambda(r_k) + \Gamma_1^T(r_k)\tilde{P}(r_k)\Gamma_1(r_k) + \Gamma_2^T(r_k)\left[\tilde{\tau}(r_k)R_1 + \tau(s)R_2\right]\Gamma_2(r_k) \\ & + \Upsilon_1(r_k) + \Upsilon_1^T(r_k) + \tau_k Z(r_k) + \Xi^T(r_k)\Xi(r_k)\Big\}\tilde{x}_k - z_k^T z_k + \gamma w_k^T w_k \\ & \qquad (2.3.19)\end{aligned}$$

Using (2.3.2),

$$\Delta V(x_k, r_k) \leq -z_k^T z_k + \gamma w_k^T w_k \qquad (2.3.20)$$

Taking expectation and sum from 0 to ∞ on both sides of (2.3.20) yields

$$E\{V(x_\infty, r_\infty)\} - E\{V(x_0, r_0)\} \leq -E\Big\{\sum_{k=0}^{\infty} z_\ell^T z_\ell\Big\} + \gamma \sum_{k=0}^{\infty} w_\ell^T w_\ell \qquad (2.3.21)$$

Under zero initial condition, $V(x_0, r_0) = 0$,

$$E\Big\{\sum_{k=0}^{\infty} z_\ell^T z_\ell\Big\} \leq \gamma \sum_{k=0}^{\infty} w_\ell^T w_\ell \qquad (2.3.22)$$

That is, (2.2.6) holds.

In order to show that the closed-loop system is stochastically stable under $w(k) = 0, \forall k \geq 0$, the following is obtained from (2.3.19) and (2.3.2).

$$V(x_{(k+1)}, r_{(k+1)}) - V(x_k, r_k) \leq -\beta \tilde{x}_k^T \tilde{x}_k \qquad (2.3.23)$$

where $\beta = \inf\{\lambda_{\min}[-\mathcal{M}(i)], i \in S\}$ with

$$\begin{aligned}\mathcal{M}(i) = \quad & \Lambda(i) + \Gamma_1^T(i)\tilde{P}(i)\Gamma_1(i) + \Gamma_2^T(i)\left[\tilde{\tau}(i)R_1 + \tau(s)R_2\right]\Gamma_2(i) + \Upsilon_1(i) + \Upsilon_1^T(i) \\ & + \tau(i)Z(i) + \Xi^T(i)\Xi(i) \qquad (2.3.24)\end{aligned}$$

Taking expectation and sum from 0 to ∞ on both sides of (2.3.23) yields

$$\begin{aligned}E\{V(x_\infty, r_\infty)\} - E\{V(x_0, r_0)\} \;&\leq\; -\beta E\Big\{\sum_{k=0}^{\infty} \tilde{x}_\ell^T \tilde{x}_\ell\Big\} \\ &\leq\; -\beta E\Big\{\sum_{k=0}^{\infty} x_\ell^T x_\ell\Big\} \qquad (2.3.25)\end{aligned}$$

Re-arranging (2.3.25), the following is obtained:

$$\begin{aligned}E\Big\{\sum_{k=0}^{\infty} x_\ell^T x_\ell\Big\} \;&\leq\; \frac{1}{\beta}E\{V(x_0, r_0)\} - \frac{1}{\beta}E\{V(x_\infty, r_\infty)\} \\ &\leq\; \alpha \qquad (2.3.26)\end{aligned}$$

where $\alpha = \frac{1}{\beta}E\{V(x_0, r_0)\} < \infty$. Hence, it is concluded that the closed-loop system is stochastically stable.

Sufficient conditions for the existence of a robust $\mathcal{H}_\infty$ state feedback controller for the system (2.2.1) with completely known transition probabilities are provided by the following theorem.

Theorem 2.3.2 *For a given $\gamma > 0$, if there exist sets of positive-definite matrices $X(i)$, $\tilde{R}_1(i)$, $\tilde{R}_1$, $\tilde{R}_2(i)$, $\tilde{R}_2$, $W_1(i)$, $W_2(i)$, $W_3(i)$, $\tilde{Q}$, $\tilde{W}_1(i)$, $\tilde{W}_2(i)$, N_1, N_2, $\tilde{Z}(i)$ and matrices $\tilde{M}(i)$ and $Y(i)$ for $i = 1, 2, \cdots, s$ satisfying the following inequalities*

$$\tilde{R}_1 > \tilde{R}_1(i), \;\; \tilde{R}_2 > \tilde{R}_2(i) \tag{2.3.27}$$

$$\begin{bmatrix} \tilde{\Lambda}(i) + \tilde{\Upsilon}_1(i) + \tilde{\Upsilon}_1^T(i) + \tau(i)\tilde{Z}(i) & \tilde{\Gamma_1}^T(i) & \tilde{\Gamma_2}^T(i) & \tilde{\Xi}^T(i) & \mathcal{H}^T(i) \\ * & -\mathcal{X} & 0 & 0 & 0 \\ * & * & -\mathcal{R} & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -\mathcal{W} \end{bmatrix} \tag{2.3.28}$$

$$\begin{bmatrix} (1 - p_{i(i+1)})\tilde{R}_1(i) + \tilde{R}_2(i) & \tilde{M}(i) \\ * & \tilde{Z}(i) \end{bmatrix} \geq 0 \tag{2.3.29}$$

$$\begin{bmatrix} S(i,j) & J^T(i) \\ * & X(j) \end{bmatrix} > 0 \tag{2.3.30}$$

$$N_1\tilde{R}_1 = I, \;\; N_2\tilde{R}_2 = I, \tilde{W}_1(i)W_1(i) = I \;\; \textit{and} \;\; \tilde{W}_2(i)W_2(i) = I, \tag{2.3.31}$$

where

$$\begin{aligned}
\mathcal{X} &= -\sum_{j=1}^{i+1} p_{ij}S(i,j) + J^T(i) + J(i) \\
\mathcal{R} &= diag\Big\{N_1, N_2\Big\} \\
\mathcal{W} &= diag\Big\{\tilde{W}_1(i), \tilde{W}_2(i)\Big\} \\
\tilde{\Gamma}_1(i) &= \Big[AX(i) \;\; B_2Y(i) \;\; B_1 \;\; E_1 \;\; E_1 \;\; E_1\Big] \\
\tilde{\Xi}(i) &= \Big[\; C_1X(i) \;\; D_{12}Y(i) \;\; D_{11} \;\; E_2 \;\; E_2 \;\; E_2 \;\Big] \\
\mathcal{H}(i) &= \begin{bmatrix} H_1X(i) & 0 & 0 & 0 & 0 & 0 \\ 0 & H_3Y(i) & 0 & 0 & 0 & 0 \end{bmatrix} \\
\tilde{\Gamma}_2(i) &= \left[\sqrt{\sum_{j=1}^{i+1} p_{ij}\tau(j)} \;\;\; \sqrt{\tau(s)}\right]^T \Big[AX(i) - X(i) \;\; B_2Y(i) \;\; B_1 \;\; E_1 \;\; E_1 \;\; E_1\Big]
\end{aligned}$$

$$\begin{aligned}\tilde{\Lambda}(i) &= diag\Big\{\Big((\tau(s)-\tau(1)+1)\tilde{Q}-X(i)\Big),\Big(-X^T(i)-X(i)+\tilde{Q}\Big),\\ &\quad \Big(H_2^T W_3(i)H_2-\gamma I\Big),-W_1(i),-W_2(i),-W_3(i)\Big\}\\ \tilde{\Upsilon}_1(i) &= \tilde{M}^T(i)[I \quad -I \quad 0 \quad 0 \quad 0 \quad 0].\end{aligned}$$

Then the closed-loop system is stochastically stable with the prescribed $\mathcal{H}_\infty$ performance. Furthermore, the controller gains are given as follows:

$$K(i) = Y(i)X^{-1}(i) \tag{2.3.32}$$

Proof: Rearranging (2.3.2) as

$$\begin{aligned}&\tilde{\Lambda}(i)+\Upsilon_1(i)+\Upsilon_1^T(i)+\tau(i)Z(i)+\Gamma_1^T(i)\tilde{P}(i)\Gamma_1(i)+\Gamma_2^T(i)\left[\tilde{\tau}(i)R_1+\tau(s)R_2\right]\Gamma_2(i)+\\ &\Xi^T(i)\Xi(i)+\text{diag}\Big\{H_1^T W_1(i)H_1,K^T(i)H_3^T W_2(i)H_3K(i),0,0,0,0\Big\}<0\end{aligned} \tag{2.3.33}$$

where

$$\Lambda(i) = \tilde{\Lambda}(i)+\text{diag}\Big\{H_1^T W_1(i)H_1,K^T(i)H_3^T W_2(i)H_3K(i),0,0,0,0\Big\}$$

Apply Schur complement on (2.3.33):

$$\begin{bmatrix}\tilde{\Lambda}(i)+\Upsilon_1(i)+\Upsilon_1^T(i)+\tau(i)Z(i) & \Gamma_1^T(i) & \Gamma_2^T(i) & \Xi^T(i) & \mathcal{H}_1^T(i)\\ * & -\mathcal{X} & 0 & 0 & 0\\ * & * & -\mathcal{R}_1 & 0 & 0\\ * & * & * & -I & 0\\ * & * & * & * & -\mathcal{W}_1\end{bmatrix}<0 \tag{2.3.34}$$

where

$$\mathcal{H}_1^T(i) = \begin{bmatrix}H_1 & 0 & 0 & 0 & 0 & 0\\ 0 & H_3K(i) & 0 & 0 & 0 & 0\end{bmatrix},$$

$$\mathcal{X}_1 = \tilde{P}^{-1}(i),$$

$$\mathcal{R}_1 = \text{diag}\Big\{\tilde{\tau}(i)R_1,\tau(s)R_2\Big\}^{-1}$$

and

$$\mathcal{W}_1 = \text{diag}\Big\{W_1(i),W_2(i)\Big\}^{-1}$$

Multiply (2.3.34) with diag$\Big\{X(i), X(i), I, I, I, I\Big\}$ on both sides and use the identity below yields (2.3.28) where

$$P^{-1}(i) = X(i)$$

Note that directly applying Schur complement on (2.3.28) results $-\tilde{P}_{\mathcal{K}}^{-1}(i)$ instead of $-\sum_{j=1}^{i+1} p_{ij} S(i,j) + J^T(i) + J(i)$. Applying Schur complement on (2.3.30) and consequently multiplying these inequalities by p_{ij} and summing up for all $j \in \mathcal{S}_{\mathcal{K}}^i$, we obtain

$$\begin{aligned}
\sum_{j=1}^{i+1} p_{ij} S(i,j) - J^T(i) - J(i) &= -J^T(i) - J(i) + \sum_{j=1}^{i+1} p_{ij} S(i,j) \\
&\leq -J^T(i) - J(i) + J^T(i)\tilde{P}_{\mathcal{K}}(i)J(i) \\
&= \tilde{P}_{\mathcal{K}}^{-1}(i) - \Big(J(i) - \tilde{P}_{\mathcal{K}}^{-1}(i)\Big)^T \tilde{P}_{\mathcal{K}}(i)\Big(J(i) - \tilde{P}_{\mathcal{K}}^{-1}(i)\Big) \\
&\leq \tilde{P}_{\mathcal{K}}^{-1}(i), \quad \forall j \in \mathcal{S}_{\mathcal{K}}^i
\end{aligned} \tag{2.3.35}$$

which implies that (2.3.2) remains valid even if $\sum_{j=1}^{i+1} p_{ij} S(i,j) - J^T(i) - J(i)$ is replaced by $\tilde{P}_{\mathcal{K}}^{-1}(i)$.

Furthermore, the multiplication of diag$\Big\{$diag$\Big\{X(i), X(i), I, I, I\Big\}, I, I, I\Big\}$ and its transpose on (2.3.34) creates two terms $X(i)^T Q X(i)$ and $-X(i)^T Q X(i)$ in $\tilde{\Lambda}(i)$. Using Schur complement on $X(i)^T Q X(i)$ and the identity shown below, (2.3.28) is obtained.

Note that

$$\begin{aligned}
&-\Big(X(i) - Q^{-1}\Big)^T Q\Big(X(i) - Q^{-1}\Big) < 0 \qquad (2.3.36) \\
\Rightarrow\ &-X(i)^T Q X(i) + X^T(i) + X(i) - Q^{-1} < 0 \\
\Rightarrow\ &-X(i)^T Q X(i) < -X^T(i) - X(i) + Q^{-1}
\end{aligned}$$

holds true since Q is a positive matrix. Therefore $-X(i)^T Q X(i)$ term can be replaced with $-X^T(i) - X(i) + Q^{-1}$. Then it is obvious that $\tilde{Q} = Q^{-1}$. $\nabla\nabla\nabla$

Remark 2.3.1 *The conditions given in Theorem 2.3.2 are not strictly LMI conditions due to equality constraints in (2.3.31). However, this problem can be converted into a nonlinear minimization problem subject to LMIs by the cone complementarity linearization algorithm proposed in [115].*

In accordance with the cone complementary algorithm, the nonconvex feasibility problem formulated by (2.3.27)-(2.3.29) can be converted into the following nonlinear minimisation problem subject to LMIs:

$$\text{Minimize } Tr\Big(N_1\tilde{R}_1 + N_2\tilde{R}_2 + \tilde{W}_1(i)W_1(i) + \tilde{W}_2(i)W_2(i)\Big)$$

Subject to (2.3.27)-(2.3.29) and

$$\begin{bmatrix} N_1 & I \\ I & \tilde{R}_1 \end{bmatrix} \geq 0, \begin{bmatrix} N_2 & I \\ I & \tilde{R}_2 \end{bmatrix} \geq 0, \begin{bmatrix} \tilde{W}_1(i) & I \\ I & W_1(i) \end{bmatrix} \geq 0, \begin{bmatrix} \tilde{W}_2(i) & I \\ I & W_2(i) \end{bmatrix} \geq 0 \tag{2.3.37}$$

To solve this optimisation problem, the following algorithm can be used:

Algorithm :

Step 1: Set $\jmath = 0$ and solve (2.3.27)-(2.3.29) and (2.3.37) to obtain the initial conditions,

$$\Big[X(i), \tilde{R}_1(i), \tilde{R}_1, \tilde{R}_2(i), \tilde{R}_2, W_1(i), W_2(i), W_3(i), \tilde{W}_1(i), \tilde{W}_2(i), \tilde{Q}, N_1, N_2, \tilde{Z}(i), Y(i)\Big]^0$$

Step 2: Solve the LMI problem

$$\text{Minimize } Tr\Big(N_1^\jmath\tilde{R}_1 + N_1\tilde{R}_1^\jmath + N_2^\jmath\tilde{R} + N_2\tilde{R}_2^\jmath + \tilde{W}_1(i)^\jmath W_1(i) + \tilde{W}_1(i)W_1(i)^\jmath + \tilde{W}_2(i)^\jmath W_2(i) + \tilde{W}_2(i)W_2(i)^\jmath\Big)$$

Subject to (2.3.27)-(2.3.29) and (2.3.37)

The obtained solutions are denoted as

$$\Big[X(i), \tilde{R}_1(i), \tilde{R}_1, \tilde{R}_2(i), \tilde{R}_2, W_1(i), W_2(i), W_3(i), \tilde{W}_1(i), \tilde{W}_2(i), \tilde{Q}, N_1, N_2, \tilde{Z}(i), Y(i)\Big]^{\jmath+1}$$

Step 3: Solve Theorem 2.3.1 with $K(i)^{\jmath+1} = Y^{\jmath+1}(i)X^{-1}(i)^{\jmath+1}$, if there exist solutions, then $K(i)^{\jmath+1}$ are the desired controller gains and EXIT. Otherwise, set $\jmath = \jmath + 1$ and return to Step 2.

2.4 Example

In order to illustrate the effectiveness of the proposed methodology, the following numerical example is used where the plant is described in (2.2.1) form with the following matrices

$$\begin{aligned} A &= \begin{bmatrix} 0.2802 & -0.0273 \\ 1 & 0 \end{bmatrix} & B_1 &= \begin{bmatrix} 0.02 \\ 0.05 \end{bmatrix} & B_2 &= \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \\ C_1 &= \begin{bmatrix} 0.8875 & -0.1404 \end{bmatrix} & D_{11} &= 0.01 & D_{12} &= 0.5 \end{aligned} \tag{2.4.1}$$

and the uncertainties are characterised by matrices below:

$$\begin{aligned} E_1 &= \begin{bmatrix} 0.001 \\ 0.002 \end{bmatrix} & E_2 &= 0.005 \\ H_1 &= \begin{bmatrix} 0.005 & 0.002 \end{bmatrix} & H_2 &= 0.004 \quad H_3 = 0.007 \end{aligned} \tag{2.4.2}$$

The sampling period of this example is $0.01s$ and the delays are modelled by a Markov chain taking values in a finite set $\mathcal{S} = \{1, 2, 3\}$, which correspond to 0.2, 0.3, 0.4 seconds delays, respectively. The transition probability matrix is given by;

$$P_\tau = \begin{bmatrix} 0.3701 & 0.6298 & 0 \\ 0.3701 & 0.6157 & 0.0142 \\ 0.3701 & 0.6157 & 0.0142 \end{bmatrix} \tag{2.4.3}$$

Using Theorem 2.3.2 and the algorithm shown, the following Ks are obtained:

$$\begin{aligned} K(1) &= \begin{bmatrix} -0.1227 & 0.0273 \end{bmatrix} \\ K(2) &= \begin{bmatrix} -0.1229 & 0.0274 \end{bmatrix} \\ K(3) &= \begin{bmatrix} -0.0067 & 0.0011 \end{bmatrix} \end{aligned} \tag{2.4.4}$$

The state response of the plant with the proposed controller is shown in Figure 2.2 and 2.3 with $w = 0$. The initial states are chosen to be $x(0) = [1.0 \quad 0]^T$. It can be seen that the state feedback controller stabilizes the system, demonstrating the validity of the proposed controller. Figure 2.4 shows the ratio of energy of the controlled output to the energy of the disturbance ($w(k) = e^{-0.1k}\sin(0.5k)$). From Figure 2.4, the ratio is approximately equal to 4.4×10^{-4}, which is less than the prescribed level $\gamma = 0.5$. Figure 2.5 shows the mode transitions in the Markov chain.

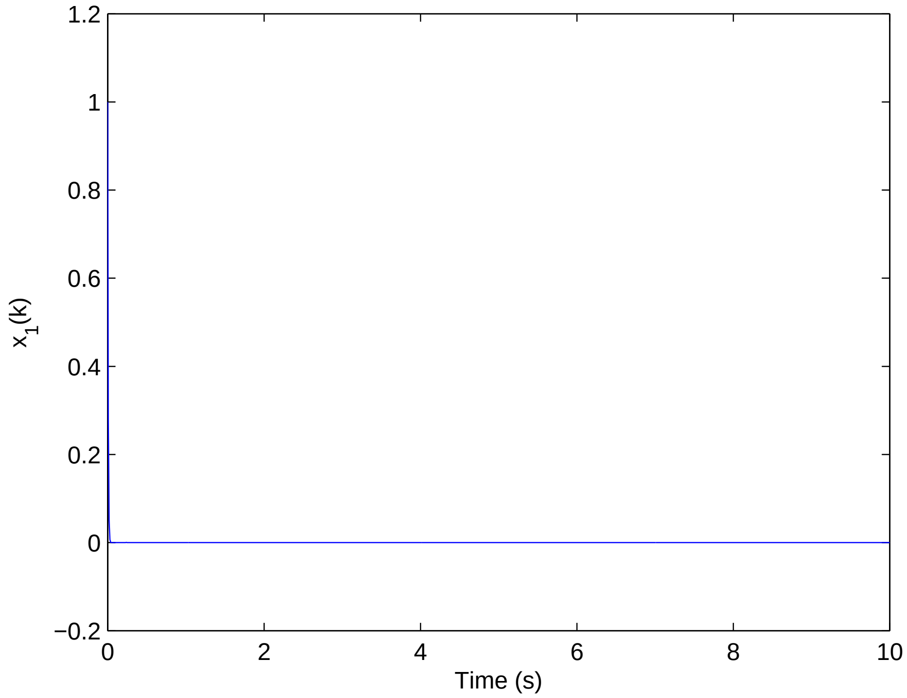

Figure 2.2: Response of $x_1(k)$

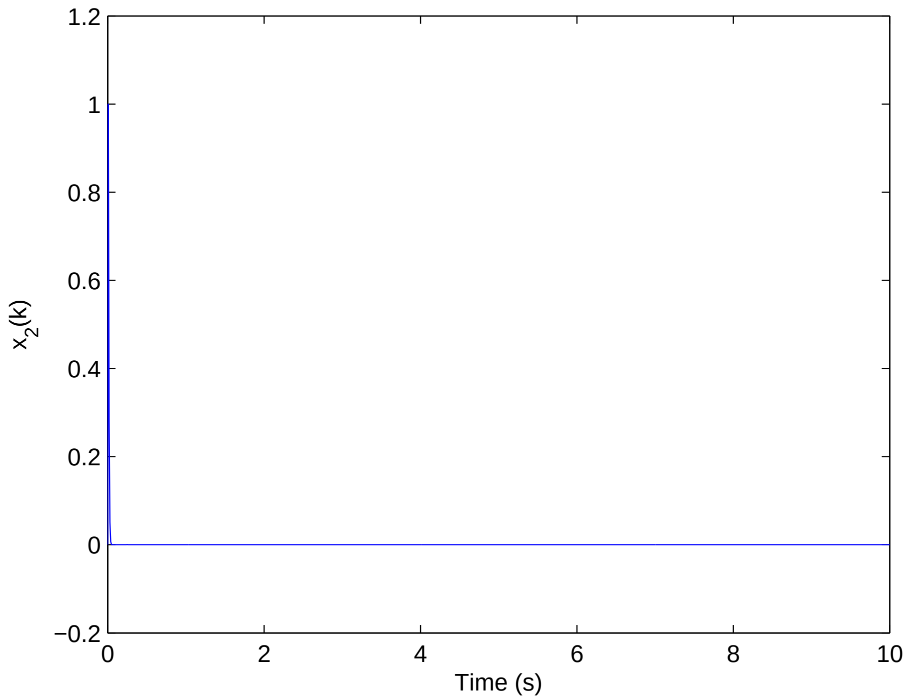

Figure 2.3: Response of $x_2(k)$

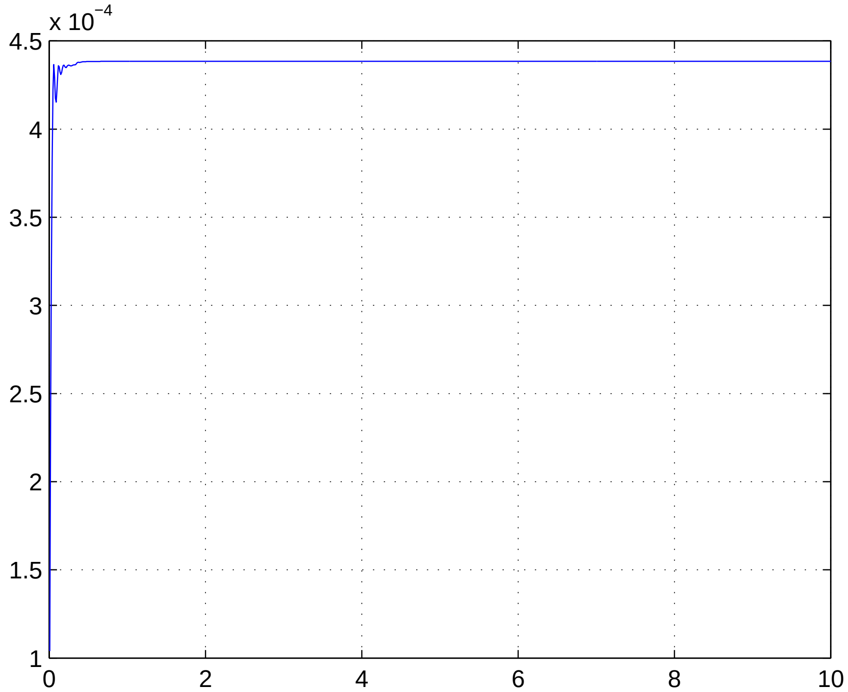

Figure 2.4: Ratio of energy of the controlled output to the energy of the disturbance $(\gamma = 0.5)$

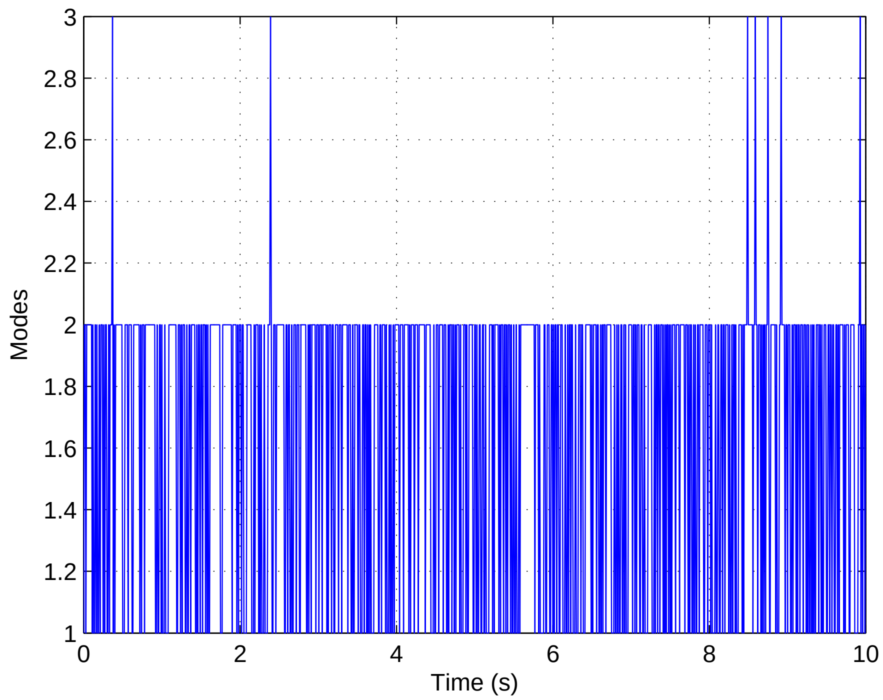

Figure 2.5: Change of modes in Markov chain

2.5 Conclusions

In this chapter, stability criteria and mode delay dependent $\mathcal{H}_\infty$ state feedback controller design are developed for linear NCSs. The network modelled by Markov chain where the transition probability matrix is assume to be completely known. Based on Lyapunov-Krasovskii functional, the conditions for the existence of the controller is obtained in terms of BMIs. An iterative algorithm is presented to change the nonconvex problem into quasi-convex optimization problems, which can be solved effectively by available mathematical tools. The validity of the methodology is verified by a numerical example.

3

Robust $\mathcal{H}_\infty$ State Feedback Control of Discrete-Time Networked Control Systems With Partially Known Transition Probability Matrix

Abstract

This chapter proposes stability analysis and a methodology for designing a partially mode delay dependent $\mathcal{H}_\infty$ controller design for discrete-time networked control systems. Network-induced delays between the plant and the controller are modelled by a finite state Markov chain where the transition probability matrix is partially known. Stability criteria are obtained based on LyapunovKrasovskii functional and a novel methodology for designing a partially mode delay-dependent state feedback controller has been proposed. The controller is obtained by solving linear matrix inequality optimisation problems using cone complimentarity linearisation algorithm. A numerical example is provided to illustrate the effectiveness of the proposed controller.

3.1 Introduction

As already addressed in the Chapter 1, many existing literature use Markov chain to model the network-induced delays, making the overall system as Markovian jump systems [8, 11, 27, 34, 51–60]. While Markovian jump systems provide adequate solution to NCSs, the majority of literature assume that the transition probabilities are *a priori*. These probabilities must be known as they are incorporated into the stability analysis and the controller design. However, in complex systems, obtaining a completely known transition probability matrix is either impossible or costly. In light of this, several attempts have been made to design a controller or a filter when transition probabilities are partially known. In [61, 63], a state feedback controller is designed where the controller depends on the upper bound of the delay and the delay range. Using a similar approach, [62] presents a filter design for Markovian jump linear system with partially known transition probability matrix. In the aforementioned papers, the stability conditions are separated into parts corresponding to known and unknown transition probabilities. Then the unknown transition probabilities are simply discarded. Refer to Chapter 1 for more information about the approach presented in the aforementioned papers. In this chapter, the unknown probabilities are bounded using probability theory and the summation of known and unknown parts are considered.

The aim of this chapter is to establish a novel state feedback controller design methodology for discrete-time NCSs where the transition probability matrix is partially known. Based on Lyapunov-Krasovskii functional, the controller design for the discrete-time NCSs is derived. The unknown probabilities are upper bounded and included in the stability analysis and the controller design. By doing so, the summation of known and unknown parts is considered, instead of separating them shown in [61–63]. Sufficient conditions for the existence of the controller is given in terms of the solvability of BMIs and an algorithm to solve the BMIs are also presented in this chapter.

The main contributions of this chapter can be summarised as follows:

- This chapter acknowledges the difficulties in obtaining a completely known transition probability matrix in the real world. In this chapter, a state feedback controller is obtained for a class of discrete-time NCSs with partially known transition probability matrix. It is shown that the existing results for completely known presented in Chapter 2 and completely unknown transition probabilities [110, 112–114] can be viewed as a special case of the presented results.
- The existing methods in [61–63] obtain the controller/filter without using the tran-

sition probability information of unknown elements. However, we know that the sum of unknown transition probabilities is equal to one minus the sum of known transition probabilities. Information of the unknown transition probabilities is included in the controller design. Furthermore, the stability criteria are formed such that the summation of known and unknown parts is less than zero in this chapter.

The rest of the chapter is organised as follows. System descriptions and definitions including modelling of the network-induced delays using a finite state Markov chain with partially known transition probability matrix are presented in Section 3.2. Necessary lemma and the mode-dependent switching state feedback control law are also presented in this section. Section 3.3 extends the theorem in Chapter 2 to derive a theorem for stability analysis and a robust $\mathcal{H}_\infty$ state feedback controller with partially known transition probability matrix. An iterative algorithm to solve BMIs in order to obtain the controller gain for systems with partially known transition probability matrix is also presented. Section 3.4 illustrates the effectiveness of the proposed design methodology using a numerical example. Conclusions are presented in Section 3.5.

3.2 System Description and Definitions

Consider the NCSs setup shown in Figure 2.1. A class of uncertain discrete-time linear systems under consideration is described by the following model:

$$\begin{aligned} x(k+1) &= [A+\Delta A(k)]x(k) + [B_1+\Delta B_1(k)]w(k) + [B_2+\Delta B_2(k)]u(k), \quad x(0)=0 \\ z(k) &= [C_1+\Delta C_1(k)]x(k) + [D_{11}+\Delta D_{11}(k)]w(k) + [D_{12}+\Delta D_{12}(k)]u(k) \end{aligned} \tag{3.2.1}$$

where $x(k) \in \Re^n$ is the state vector, $z(k) \in \Re^p$ is the controlled output and $w(k) \in \Re^q$ is the disturbance which belong to $\mathcal{L}_2[0,\infty)$, the space of square summable vector sequence over $[0,\infty]$. The matrices A, B_1, B_2, C_1, D_{11} and D_{12} are of appropriate dimensions. The matrix functions $\Delta A(k)$, $\Delta B_1(k)$, $\Delta B_2(k)$, $\Delta C_1(k)$, $\Delta D_{11}(k)$ and $\Delta D_{12}(k)$ represent the time-varying uncertainties in the system and satisfy the following assumption.

Assumption 3.2.1

$$\begin{bmatrix} \Delta A(k) & \Delta B_1(k) & \Delta B_2(k) \\ \Delta C_1(k) & \Delta D_{11}(k) & \Delta D_{12}(k) \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} F(k) \begin{bmatrix} H_1 & H_2 & H_3 \end{bmatrix}$$

where H_i and E_i are known matrices which characterize the structure of the uncertainties. Furthermore, there exists a positive-definite matrix $\mathcal{W}$ such that the following inequality holds:

$$F^T(k)\mathcal{W}F(k) \leq \mathcal{W} \tag{3.2.2}$$

Unlike the previous chapter, the transition probability matrix of the Markov chain is allowed to be partially known. This model is more practical as it is usually expensive or difficult to obtain a completely known transition probability matrix in the real world. Furthermore, as it will be shown later on, the case of either completely known or unknown matrix is a special case of the approach presented in this chapter.

The transition probability matrix of the new network model, which allows some of the transition probabilities to be unknown, is of the following form

$$P_\tau = \begin{bmatrix} p_{11} & p_{12} & 0 & 0 & \dots & 0 \\ ? & ? & p_{23} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & p_{(s-1)s} \\ p_{s1} & ? & ? & p_{s4} & \dots & p_{ss} \end{bmatrix} \tag{3.2.3}$$

where "?" represents the unknown, but time-invariant probabilities. The transition probability matrix may be made of entirely unknown transition probabilities, resulting a completely unknown transition probability matrix. Note $0 \leq p_{ij} \leq 1$ and $\sum_{j=1}^{i+1} p_{ij} = 1$ where $p_{s(s+1)} = 0$.

The mode-dependent switching state feedback control law is

$$u(k) = K(r_k)x(k - \tau_k) \tag{3.2.4}$$

The same robust $\mathcal{H}_\infty$ control problem is considered in this chapter, except the fact that the transition probability matrix is partially known. The following information is presented again for reading convenience.

As shown in Chapter 2, the closed-loop system is to achieve stochastic stability, as shown in (2.2.5), and the $\mathcal{H}_\infty$ performance condition, as shown in (2.2.6).

Before moving any further, the following lemma is presented, which will be used throughout this thesis to create an upper bound for unknown probabilities in the transition probability matrix.

Lemma 3.2.1 *For given scalars $a_i \geq 0$ and $b_i \geq 0$,$i = 1, 2, \ldots, N$ we have*

$$\sum_{i=1}^{N} a_i b_i \leq \sum_{i=1}^{N} a_i \sum_{i=1}^{N} b_i \tag{3.2.5}$$

Proof. *We use mathematical induction to prove the above lemma. When $N = 1$, it is obvious that $a_1 b_1 = a_1 b_1$, thus the above holds. When $N = k + 1$ where $k \geq 1$ we have*

$$\begin{aligned} \sum_{i=1}^{k+1} a_i b_i &= (a_1 b_1 + \cdots + a_{k+1} b_{k+1}) \\ &\leq \sum_{i=1}^{k} a_i \sum_{i=1}^{k} b_i + a_{k+1} b_{k+1} \end{aligned} \tag{3.2.6}$$

The right hand side of (3.2.5) is

$$\sum_{i=1}^{k+1} a_i \sum_{i=1}^{k+1} b_i = \Big(\sum_{i=1}^{k} a_i + a_{k+1}\Big)\Big(\sum_{i=1}^{k} b_i + b_{k+1}\Big) \tag{3.2.7}$$

Since a_i and b_i are positive, (3.2.6) and (3.2.7) implies that (3.2.5) is valid. $\nabla\nabla\nabla$

Lemma 3.2.2 *For given scalars $\lambda_i \geq 0$ and matrices $P_i \geq 0$, $i = 1, 2, \ldots, N$, we have*

$$\sum_{i=1}^{N} \lambda_i P_i \leq \sum_{i=1}^{N} \lambda_i \sum_{i=1}^{N} P_i \tag{3.2.8}$$

Proof: *We pre- and post-multiply a nonzero vector $x(k)$ to (3.2.8) then we have*

$$x^T(k)\Big(\sum_{i=1}^{N} \lambda_i P_i\Big)x(k) \leq x^T(k)\Big(\sum_{i=1}^{N} \lambda_i \sum_{i=1}^{N} P_i\Big)x(k) \tag{3.2.9}$$

Since λ_i are scalar the above is equivalent to

$$\sum_{i=1}^{N} \lambda_i x^T(k) P_i x(k) \leq \sum_{i=1}^{N} \lambda_i \sum_{i=1}^{N} x^T(k) P_i x(k) \tag{3.2.10}$$

From Lemma 3.2.1, by letting $a_i = \lambda_i$ and $b_i = x^T(k) P_i x(k)$, it can be shown that (3.2.8) holds. $\nabla\nabla\nabla$

Based on Lemma 3.2.2, the unknown probabilities and the matrices they are multiplied to are separated. Then by using the fact that the summation of all probabilities in each row of transition probability matrix must equal to one, as presented in Chapter 1, the upper bound for sum of unknown probabilities are created.

3.3 Main Results

In the previous section, the problem of robust $\mathcal{H}_\infty$ control for a class of discrete-time NCSs with a partially known transition probability matrix is introduced. This section presents the stability criteria and the $\mathcal{H}_\infty$ state feedback controller design for the NCSs where the transition probability matrix is allowed to be partially known.

The following theorem proposes stability criteria for the system shown in (3.2.1) with partially known transition probabilities.

Theorem 3.3.1 *For given controller gains $K(i)$, $i \in \mathcal{S}$, and $\gamma > 0$, if there exist sets of positive-definite matrices $P(i)$, $R_1(i)$, R_1, $R_2(i)$, R_2, $W_1(i)$, $W_2(i)$, $W_3(i)$, Q, $Z(i)$ and matrices $M(i)$, $\Omega_1(i)$, $\Omega_2(i)$, $\forall i \in \mathcal{S}$, satisfying the following inequalities*

$$R_1 > R_1(i), \quad R_2 > R_2(i) \tag{3.3.1}$$

$$\Lambda(i)+\Gamma_2^T(i)\tau(s)R_2\Gamma_2(i)+\Upsilon_1(i)+\Upsilon_1^T(i)+\tau(i)Z(i)+\Xi^T(i)\Xi(i)+\Omega_1(i)+\Omega_2(i) < 0 \tag{3.3.2}$$

$$\Gamma_1^T(i)\tilde{P}_{\mathcal{K}}(i)\Gamma_1(i) + \Gamma_2^T(i)\tilde{\tau}_{\mathcal{K}}R_1\Gamma_2(i) - \Omega_1(i) < 0, \quad \forall j \in \mathcal{S}_{\mathcal{K}}^i \tag{3.3.3}$$

$$\Gamma_1^T(i)(1-p_{\mathcal{K}}^i)\sum_{j=1}^{i+1} P(j)\Gamma_1(i)+\Gamma_2^T(i)(1-p_{\mathcal{K}}^i)\sum_{j=1}^{i+1}\tau(j)R_1\Gamma_2(i)-\Omega_2(i) < 0, \ \forall j \in \mathcal{S}_{\mathcal{UK}}^i \tag{3.3.4}$$

and

$$\begin{bmatrix} (1-p_{i(i+1)})R_1(i)+R_2(i) & M(i) \\ M^T(i) & Z(i) \end{bmatrix} \geq 0 \tag{3.3.5}$$

$$\begin{bmatrix} p_{\mathcal{K}}^i R_1(i)+R_2(i) & M(i) \\ M^T(i) & Z(i) \end{bmatrix} \geq 0, \quad \forall (i+1) \in \mathcal{S}_{\mathcal{UK}}^i \tag{3.3.6}$$

where

$$\begin{aligned}
\Gamma_1(i) &= \begin{bmatrix} A & B_2K(i) & B_1 & E_1 & E_1 & E_1 \end{bmatrix} \\
\Xi(i) &= \begin{bmatrix} C_1 & D_{12}K(i) & D_{11} & E_2 & E_2 & E_2 \end{bmatrix} \\
\Gamma_2(i) &= \begin{bmatrix} A-I & B_2K(i) & B_1 & E_1 & E_1 & E_1 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\Lambda(i) &= diag\Big\{\Big((\tau(s)-\tau(1)+1)Q + H_1^T W_1(i) H_1 - P(i)\Big), \Big(K^T(i) H_3^T W_2(i) H_3 K(i) \\
&\quad -Q\Big), \Big(H_2^T W_3(i) H_2 - \gamma I\Big), -W_1(i), -W_2(i), -W_3(i)\Big\} \\
\Upsilon_1(i) &= M^T(i)[I \quad -I \quad 0 \quad 0 \quad 0 \quad 0] \\
\tilde{P}_{\mathcal{K}}(i) &= \textstyle\sum_{j\in\mathcal{S}_{\mathcal{K}}^i}^{i+1} p_{ij} P(j) \\
\tilde{\tau}_{\mathcal{K}}(i) &= \textstyle\sum_{j\in\mathcal{S}_{\mathcal{K}}^i}^{i+1} p_{ij} \tau(j)
\end{aligned}$$

Then the closed-loop system is stochastically stable with the prescribed $\mathcal{H}_\infty$ performance.

Proof: The system (3.2.1) with (3.2.4) can be rewritten as

$$\begin{aligned} x_{k+1} &= \Gamma_1(r_k)\tilde{x}_k \\ z_k &= \Xi(r_k)\tilde{x}_k \end{aligned} \tag{3.3.7}$$

where $x_\ell = x(\ell)$, $z_\ell = z(\ell)$, $\tilde{x}_\ell = \tilde{x}(\ell)$, and $\Gamma_1(r_k)$ is given in Theorem 3.3.1, and $\tilde{x}_k$ is defined in Lemma 2.2.1.

Let us consider the following Lyapunov-Krasovskii candidate functional:

$$V(x_k, r_k) = V_1(x_k, r_k) + V_2(x_k, r_k) + V_3(x_k, r_k) \tag{3.3.8}$$

with

$$V_1(x_k, r_k) = x_k^T P(r_k) x_k \tag{3.3.9}$$

$$V_2(x_k, r_k) = \sum_{\ell=-\tau_k}^{-1} \sum_{j=k+\ell}^{k-1} y_j^T R_1 y_j + \sum_{\ell=-\tau(s)}^{-1} \sum_{j=k+\ell}^{k-1} y_j^T R_2 y_j \tag{3.3.10}$$

$$V_3(x_k, r_k) = \sum_{\ell=k-\tau_k}^{k-1} x_\ell^T Q x_\ell + \sum_{\ell=-\tau(s)+2}^{-\tau(1)+1} \sum_{j=k+\ell-1}^{k-1} x_j^T Q x_j \tag{3.3.11}$$

Using the same approach to the proof of the stability analysis in Chapter 2, the following is obtained.

$$\begin{aligned}
\Delta V(x_k, r_k) \;\le\; & \tilde{x}_k^T \Big\{\Lambda(r_k) + \Gamma_1^T \tilde{P}(r_k)\Gamma_1 + \Gamma_2^T \left[\tilde{\tau}_k R_1 + \tau(s) R_2\right]\Gamma_2 + \Upsilon_1(r_k) + \Upsilon_1^T(r_k) \\
& + \tau_k Z(r_k) + \Xi^T \Xi\Big\}\tilde{x}_k - z_k^T z_k + \gamma w_k^T w_k
\end{aligned} \tag{3.3.12}$$

The following procedure shows how the terms containing unknown transition probabilities are handled. Note that using Lemma 3.2.2, $\Gamma_1^T \tilde{P}(r_k)\Gamma_1 + \Gamma_2^T(i)\tilde{\tau}(i) R_1 \Gamma_2(i)$ can be rewritten as

$$\begin{aligned}&\Gamma_1^T(i)\tilde{P}_{\mathcal{K}}(i)\Gamma_1(i)+\Gamma_1^T(i)\sum_{j\in\mathcal{S}_{\mathcal{UK}}^i}^{i+1}p_{ij}P(j)\Gamma_1(i)+\Gamma_2^T(i)\tilde{\tau}_{\mathcal{K}}(i)R_1\Gamma_2(i)\\&+\Gamma_2^T(i)\sum_{j\in\mathcal{S}_{\mathcal{UK}}}^{i+1}p_{ij}\tau(j)R_1\Gamma_2(i)\leq\Gamma_1^T(i)\tilde{P}_{\mathcal{K}}(i)\Gamma_1(i)+\Gamma_1^T(i)(1-p_{\mathcal{K}}^i)\sum_{j\in\mathcal{S}_{\mathcal{UK}}^i}^{i+1}P(j)\Gamma_1(i)\\&+\Gamma_2^T(i)\tilde{\tau}_{\mathcal{K}}(i)R_1\Gamma_2(i)+\Gamma_2^T(i)(1-p_{\mathcal{K}}^i)\sum_{j\in\mathcal{S}_{\mathcal{UK}}}^{i+1}\tau(j)R_1\Gamma_2(i)\end{aligned}\tag{3.3.13}$$

Using (3.3.3) and (3.3.4), the following holds

$$\begin{aligned}&\Gamma_1^T(i)\tilde{P}_{\mathcal{K}}(i)\Gamma_1(i)+\Gamma_1^T(i)(1-p_{\mathcal{K}}^i)\sum_{j\in\mathcal{S}_{\mathcal{UK}}^i}^{i+1}P(j)\Gamma_1(i)+\Gamma_2^T(i)\tilde{\tau}_{\mathcal{K}}(i)R_1\Gamma_2(i)\\&+\Gamma_2^T(i)(1-p_{\mathcal{K}}^i)\sum_{j\in\mathcal{S}_{\mathcal{UK}}}^{i+1}\tau(j)R_1\Gamma_2(i)\leq\Omega_1(i)+\Omega_2(i)\end{aligned}\tag{3.3.14}$$

where $(1-p_{\mathcal{K}}^i)$ is one minus the summation of all known probabilities.

Using (3.3.2)-(3.3.4) and (3.3.14), (3.3.12) becomes

$$\Delta V(x_k,r_k)\leq -z_k^Tz_k+\gamma w_k^Tw_k \tag{3.3.15}$$

Taking expectation and sum from 0 to ∞ on both sides of (3.3.15) yields

$$E\{V(x_\infty,r_\infty)\}-E\{V(x_0,r_0)\}\leq -E\Big\{\sum_{\ell=0}^{\infty}z_\ell^Tz_\ell\Big\}+\gamma\sum_{\ell=0}^{\infty}w_\ell^Tw_\ell \tag{3.3.16}$$

Under zero initial condition, $V(x_0,r_0)=0$, we have

$$E\Big\{\sum_{\ell=0}^{\infty}z_\ell^Tz_\ell\Big\}\leq\gamma\sum_{\ell=0}^{\infty}w_\ell^Tw_\ell \tag{3.3.17}$$

That is, the second criteria in the problem formulation, as shown in Chapter 2, holds.

Next, under $w(k)=0,\forall k\geq 0$ we need to show that the closed-loop system is stochastically stable. From (3.3.12) and (3.3.2)-(3.3.4) we learn that

$$V(x_{(k+1)}, r_{(k+1)}) - V(x_k, r_k) \leq -\beta \tilde{x}_k^T \tilde{x}_k \tag{3.3.18}$$

where $\beta = \inf\{\lambda_{\min}[-\mathcal{M}(i)], i \in S\}$ with

$$\begin{aligned} \mathcal{M}(i) &= \Lambda(i) + \Gamma_1^T(i)\tilde{P}(i)\Gamma_1(i) + \Gamma_2^T(i)\left[\tilde{\tau}(i)R_1 + \tau(s)R_2\right]\Gamma_2(i) + \Upsilon_1(i) + \Upsilon_1^T(i) \\ &\quad + \tau(i)Z(i) + \Xi^T(i)\Xi(i) \end{aligned} \tag{3.3.19}$$

Taking expectation and sum from 0 to ∞ on both sides of (3.3.18) yields

$$\begin{aligned} E\{V(x_\infty, r_\infty)\} - E\{V(x_0, r_0)\} &\leq -\beta E\Big\{\sum_{k=0}^{\infty} \tilde{x}_\ell^T \tilde{x}_\ell\Big\} \\ &\leq -\beta E\Big\{\sum_{k=0}^{\infty} x_\ell^T x_\ell\Big\} \end{aligned} \tag{3.3.20}$$

Re-arranging (3.3.20), we get

$$\begin{aligned} E\Big\{\sum_{k=0}^{\infty} x_\ell^T x_\ell\Big\} &\leq \frac{1}{\beta}E\{V(x_0, r_0)\} - \frac{1}{\beta}E\{V(x_\infty, r_\infty)\} \\ &\leq \alpha \end{aligned} \tag{3.3.21}$$

where $\alpha = \frac{1}{\beta}E\{V(x_0, r_0)\} < \infty$. $\nabla\nabla\nabla$

Sufficient conditions for the existence of a robust $\mathcal{H}_\infty$ state feedback controller for the system (3.2.1) with partially known transition probabilities are provided by the following theorem.

Theorem 3.3.2 *For a given $\gamma > 0$, if there exist sets of positive-definite matrices $X(i)$, $\tilde{R}_1(i)$, $\tilde{R}_1$, $\tilde{R}_2(i)$, $\tilde{R}_2$, $W_1(i)$, $W_2(i)$, $W_3(i)$, $\tilde{Q}$, $\tilde{W}_1(i)$, $\tilde{W}_2(i)$, N_1, N_2, $\tilde{Z}(i)$ and matrices $\tilde{M}(i)$, $\tilde{\Omega}_1(i)$, $\tilde{\Omega}_2(i)$ and $Y(i)$ for $i = 1, 2, \cdots, s$ satisfying the following inequalities*

$$\tilde{R}_1 > \tilde{R}_1(i), \quad \tilde{R}_2 > \tilde{R}_2(i) \tag{3.3.22}$$

$$\begin{bmatrix} \hat{\Lambda}(i) & \sqrt{\tau(s)}\tilde{\Gamma_2}^T(i) & \tilde{\Gamma_3}^T(i) & \tilde{\Xi}^T(i) & \mathcal{H}^T(i) \\ * & -N_2 & 0 & 0 & 0 \\ * & * & -\mathcal{Q} & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -\mathcal{W} \end{bmatrix} < 0 \tag{3.3.23}$$

$$\begin{bmatrix} -\tilde{\Omega}_1(i) & \tilde{\Gamma_1}^T(i) & \sqrt{\tilde{\tau}_{\mathcal{K}}(i)}\tilde{\Gamma_2}^T(i) \\ * & -\sum_{j=1}^{i+1} p_{ij} S(i,j) + J^T(i) + J(i) & 0 \\ * & * & -N_1 \end{bmatrix} < 0, \ \ \forall j \in \mathcal{S}_{\mathcal{K}}^i \quad (3.3.24)$$

$$\begin{bmatrix} -\tilde{\Omega}_2(i) & \tilde{\Gamma_1}^T(i) & \sqrt{\sum_{j=1}^{i+1}\tau(j)}\tilde{\Gamma_2}^T(i) \\ * & (1-p_{\mathcal{K}}^i)^{-1}\sum_{j=1}^{i+1} S(i,j) + J^T(i) + J(i) & 0 \\ * & * & -N_1 \end{bmatrix} < 0, \ \ \forall j \in \mathcal{S}_{\mathcal{UK}}^i \quad (3.3.25)$$

$$\begin{bmatrix} (1-p_{i(i+1)})\tilde{R}_1(i) + \tilde{R}_2(i) & \tilde{M}(i) \\ * & \tilde{Z}(i) \end{bmatrix} \geq 0 \quad (3.3.26)$$

$$\begin{bmatrix} p_{\mathcal{K}}^i \tilde{R}_1(i) + \tilde{R}_2(i) & \tilde{M}(i) \\ * & \tilde{Z}(i) \end{bmatrix} \geq 0, \quad \forall (i+1) \in \mathcal{S}_{\mathcal{UK}}^i \quad (3.3.27)$$

$$\begin{bmatrix} S(i,j) & J^T(i) \\ * & X(j) \end{bmatrix} > 0, \ \ \forall j \in \mathcal{S}_{\mathcal{K}}^i \quad (3.3.28)$$

and

$$N_1\tilde{R}_1 = I, \ N_2\tilde{R}_2 = I, \tilde{W}_1(i)W_1(i) = I \ \ and \ \tilde{W}_2(i)W_2(i) = I, \quad (3.3.29)$$

where

$$\begin{aligned}
\hat{\Lambda}(i) &= \tilde{\Lambda}(i) + \tilde{\Upsilon}_1(i) + \tilde{\Upsilon}_1^T(i) + \tau(i)\tilde{Z}(i) + \tilde{\Omega}_1(i) + \tilde{\Omega}_2(i) \\
\tilde{\Gamma}_1(i) &= \begin{bmatrix} AX(i) & B_2Y(i) & B_1 & E_1 & E_1 & E_1 \end{bmatrix} \\
\tilde{\Gamma}_2(i) &= \begin{bmatrix} AX(i) - X(i) & B_2Y(i) & B_1 & E_1 & E_1 & E_1 \end{bmatrix} \\
\tilde{\Gamma}_3(i) &= \begin{bmatrix} X(i) & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
\mathcal{X} &= \sum_{j=\mathcal{S}_{\mathcal{K}}^i} p_{ij} S(i,j) - J^T(i) - J(i) \\
\mathcal{Q} &= \Big(\tau(s) - \tau(1) + 1\Big)\tilde{Q} \\
\mathcal{W} &= diag\Big\{\tilde{W}_1(i), \tilde{W}_2(i)\Big\} \\
\tilde{\Xi}(i) &= \begin{bmatrix} C_1X(i) & D_{12}Y(i) & D_{11} & E_2 & E_2 & E_2 \end{bmatrix} \\
\mathcal{H} &= \begin{bmatrix} H_1X(i) & 0 & 0 & 0 & 0 & 0 \\ 0 & H_3Y(i) & 0 & 0 & 0 & 0 \end{bmatrix} \\
\tilde{\Lambda}(i) &= diag\Big\{ - X(i), \Big(- X^T(i) - X(i) + \tilde{Q}\Big), \Big(H_2^T W_3(i) H_2 - \gamma I\Big), \\
&\quad -W_1(i), -W_2(i), -W_3(i)\Big\} \\
\tilde{\Upsilon}_1(i) &= \tilde{M}^T(i)[I \ \ -I \ \ 0 \ \ 0 \ \ 0 \ \ 0]
\end{aligned}$$

Then the closed-loop system is stochastically stable with the prescribed $\mathcal{H}_\infty$ performance. Furthermore, the controller gains are given as follows:

$$K(i) = Y(i)X^{-1}(i) \tag{3.3.30}$$

Proof: Rearranging (3.3.2) as

$$\begin{aligned}&\tilde{\Lambda}(i) + \Upsilon_1(i) + \Upsilon_1^T(i) + \tau(i)Z(i) + \Gamma_2^T(i)\tau(s)R_2\Gamma_2(i) + \Xi^T(i)\Xi(i) + \Omega_1(i) + \Omega_2(i)\\&+\text{diag}\Big\{H_1^T W_1(i)H_1, K^T(i)H_3^T W_2(i)H_3K(i), 0, 0, 0, 0\Big\} < 0\end{aligned} \tag{3.3.31}$$

where

$$\Lambda(i) = \tilde{\Lambda}(i) + \text{diag}\Big\{H_1^T W_1(i)H_1, K^T(i)H_3^T W_2(i)H_3K(i), 0, 0, 0, 0\Big\}$$

Applying Schur complement on (3.3.31) we get

$$\begin{bmatrix} \tilde{\Lambda}(i) + \Upsilon_1(i) + \Upsilon_1^T(i) + \tau(i)Z(i) + \Omega_1(i) + \Omega_2(i) & \bar{\Gamma}_2^T(i) & \bar{\Xi}^T(i) & \bar{\mathcal{H}}^T(i) \\ * & -(\tau(s)R_2)^{-1} & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -\mathcal{W}_1 \end{bmatrix} < 0 \tag{3.3.32}$$

where

$$\bar{\Gamma}_1(i) = \Big[A \;\; B_2K(i) \;\; B_1 \;\; E_1 \;\; E_1 \;\; E_1\Big]$$

$$\bar{\Gamma}_2(i) = \Big[\; A - I \;\; B_2K(i) \;\; B_1 \;\; E_1 \;\; E_1 \;\; E_1 \;\Big]$$

$$\bar{\Xi}(i) = \Big[\; C_1 \;\; D_{12}K(i) \;\; D_{11} \;\; E_2 \;\; E_2 \;\; E_2 \;\Big]$$

$$\bar{\mathcal{H}}^T(i) = \begin{bmatrix} H_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & H_3K(i) & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$\mathcal{W}_1 = \text{diag}\Big\{W_1(i), W_2(i)\Big\}^{-1}$$

Multiplying (3.3.32) with $\text{diag}\Big\{\text{diag}\Big\{X(i), X(i), I, I, I, I\Big\}, I, I, I\Big\}$ on the right hand side and its transpose on the left we obtain (3.3.23).

Applying Schur complement on (3.3.3) results the following.

$$\begin{bmatrix} -\bar{\Omega}_1(i) & \bar{\Gamma}_1^T(i) & \bar{\Gamma}_2^T(i) \\ * & -\tilde{P}_{\mathcal{K}}^{-1}(i) & 0 \\ * & * & -\big(\tilde{\tau}_{\mathcal{K}}(i)R_1\big)^{-1} \end{bmatrix} < 0, \quad \forall j \in \mathcal{S}_{\mathcal{K}}^i \tag{3.3.33}$$

Multiplying $\text{diag}\Big\{\text{diag}\Big\{X(i), X(i), I, I, I, I\Big\}, I, I\Big\}$ on the right and its transpose on the left we obtain (3.3.24) where $\tilde{\Omega}_1(i) = X(i)^T\Omega_1(i)X(i)$.

Similarly applying Schur complement on (3.3.4);

$$\begin{bmatrix} -\bar{\Omega}_2(i) & \bar{\Gamma}_1^T(i) & \bar{\Gamma}_2^T(i) \\ * & -\big((1-p_{\mathcal{K}}^i)\sum_{j=1}^{i+1} P(j)\big)^{-1} & 0 \\ * & * & -\big((1-p_{\mathcal{K}}^i)(\sum_{j=1}^{i+1}\tau(j)R_1)\big)^{-1} \end{bmatrix} < 0, \quad \forall j \in \mathcal{S}_{\mathcal{UK}}^i \tag{3.3.34}$$

Multiplying the above with $\text{diag}\Big\{\text{diag}\Big\{X(i), X(i), I, I, I, I\Big\}, I, I\Big\}$ on the right and its transpose on the left we obtain (3.3.25) where $\tilde{\Omega}_2(i) = X(i)^T\Omega_2(i)X(i)$.

Note that directly applying Schur complement on (3.3.24) results $-\tilde{P}_{\mathcal{K}}^{-1}(i)$ instead of $-\sum_{j=1}^{i+1} p_{ij}S(i,j) + J^T(i) + J(i)$ and similarly $\sum_{j=1}^{i+1} S(i,j) + J^T(i) + J(i)$ in (3.3.25). Applying Schur complement on (3.3.28) and consequently multiplying these inequalities by p_{ij} and summing up for all $j \in \mathcal{S}_{\mathcal{K}}^i$, we obtain

$$\begin{aligned} \sum_{j=1}^{i+1} p_{ij}S(i,j) - J^T(i) - J(i) &= -J^T(i) - J(i) + \sum_{j=1}^{i+1} p_{ij}S(i,j) \\ &\leq -J^T(i) - J(i) + J^T(i)\tilde{P}_{\mathcal{K}}(i)J(i) \\ &= \tilde{P}_{\mathcal{K}}^{-1}(i) - \Big(J(i) - \tilde{P}_{\mathcal{K}}^{-1}(i)\Big)^T \tilde{P}_{\mathcal{K}}(i)\Big(J(i) - \tilde{P}_{\mathcal{K}}^{-1}(i)\Big) \\ &\leq \tilde{P}_{\mathcal{K}}^{-1}(i), \quad \forall j \in \mathcal{S}_{\mathcal{K}}^i \end{aligned} \tag{3.3.35}$$

which implies that (3.3.3) remains valid even if $\sum_{j=1}^{i+1} p_{ij}S(i,j) - J^T(i) - J(i)$ is replaced by $\tilde{P}_{\mathcal{K}}^{-1}(i)$ for all $j \in \mathcal{S}_{\mathcal{K}}^i$. Similarly, by applying Schur complement on (3.3.28) and consequently summing up for all $j \in \mathcal{S}_{\mathcal{UK}}^i$, we can show that (3.3.4) is valid.

Furthermore, the multiplication of $\text{diag}\Big\{\text{diag}\Big\{X(i), X(i), I, I, I\Big\}, I, I, I\Big\}$ and its transpose on (3.3.32) creates two terms $X(i)^TQX(i)$ and $-X(i)^TQX(i)$ in $\tilde{\Lambda}$. Using Schur complement on $X(i)^TQX(i)$ and the identity shown below, (3.3.23) is obtained.

Note that

$$
\begin{aligned}
& -\Big(X(i)-Q^{-1}\Big)^T Q\Big(X(i)-Q^{-1}\Big)<0 \qquad (3.3.36)\\
\Rightarrow\ & -X(i)^T Q X(i)+X^T(i)+X(i)-Q^{-1}<0\\
\Rightarrow\ & -X(i)^T Q X(i)<-X^T(i)-X(i)+Q^{-1}
\end{aligned}
$$

holds true since Q is a positive matrix. Therefore $-X(i)^T QX(i)$ term can be replaced with $-X^T(i) - X(i) + Q^{-1}$. Then it is obvious that $\tilde{Q} = Q^{-1}$. $\nabla\nabla\nabla$

The same iterative algorithm presented in Chapter 2 can be used to convert the nonconvex feasibility problem into the nonlinear minimization problem subject to LMIs.

Remark 3.3.1 *Note that in [61–63], the unknown part does not contain any transition probability information. However, we know that the sum of unknown transition probabilities is equal to one minus the sum of known transition probabilities. Therefore, in this chapter, this information is incorporated to yield less conservative results. In this chapter, new slack matrices $\Omega_1(i)$ and $\Omega_2(i)$ and transition probabilities information are incorporated into the design to relax the results. The stability criteria is derived based on the summation of known and unknown parts is less than zero whereas in [61–63] known and unknown parts have been considered separately. Hence, the proposed results are less conservative than [61–63]. From the above derivation, it is clear that if the transition probabilities are completely known, we can set $\Omega_2(i) = 0$ and it reduces to the results given in Chapter 2. When the transition probabilities are completely unknown, we can set $\Omega_1(i) = 0$ and it reduces to [110, 112–114] which are independent on delay's modes. Therefore the cases where the transition probability matrix is either completely known or unknown is a special case, meaning that the presented approach is more general.*

3.4 Example

In order to illustrate the effectiveness of the methodology, the same plant presented in Chapter 2 is considered.

The delays are modelled by a Markov chain taking values in a finite set $\mathcal{S} = \{1, 2\}$, which correspond to 0.2, 0.3 seconds delays, respectively. The transition probability matrix is given by;

$$
P_\tau = \begin{bmatrix} 0.4 & 0.6 \\ ? & ? \end{bmatrix} \qquad (3.4.1)
$$

Applying the results given in [61], no controller can be found. Using Theorem 3.3.2, the following controller gains are obtained:

$$\begin{aligned} K(1) &= \begin{bmatrix} -10.7152 & 4.0472 \end{bmatrix} \\ K(2) &= \begin{bmatrix} -10.0302 & 3.6856 \end{bmatrix} \end{aligned} \tag{3.4.2}$$

In order to show the effectiveness of the controller with partially known transition probability matrix, the following two cases are considered.

Case 1: Let us say that the transition probability matrix is given by;

$$P_{\tau 1} = \begin{bmatrix} 0.4 & 0.6 \\ (0.3) & (0.7) \end{bmatrix} \tag{3.4.3}$$

Case 2: The transition probability is now given by;

$$P_{\tau 2} = \begin{bmatrix} 0.4 & 0.6 \\ (0.9) & (0.1) \end{bmatrix} \tag{3.4.4}$$

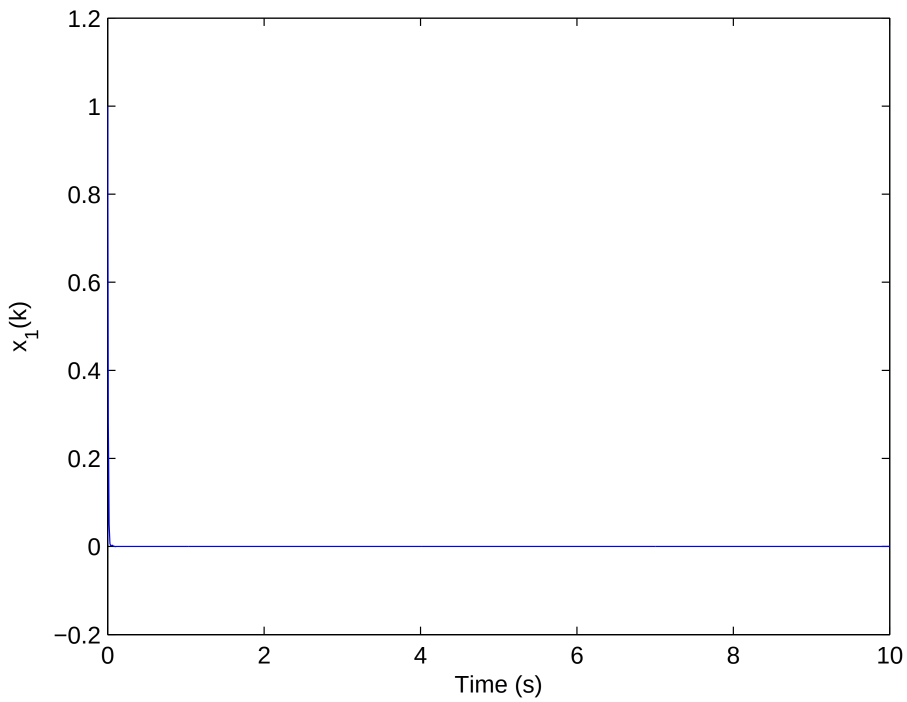

Figure 3.1: Response of $x_1(k)$, $P_{\tau 1}$

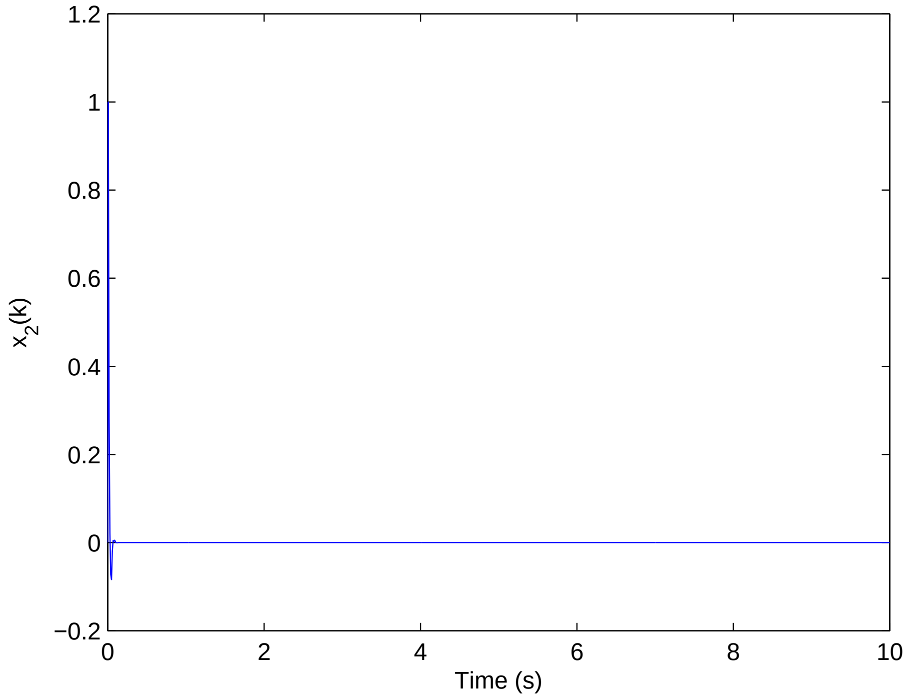

Figure 3.2: Response of $x_2(k)$, $P_{\tau 1}$

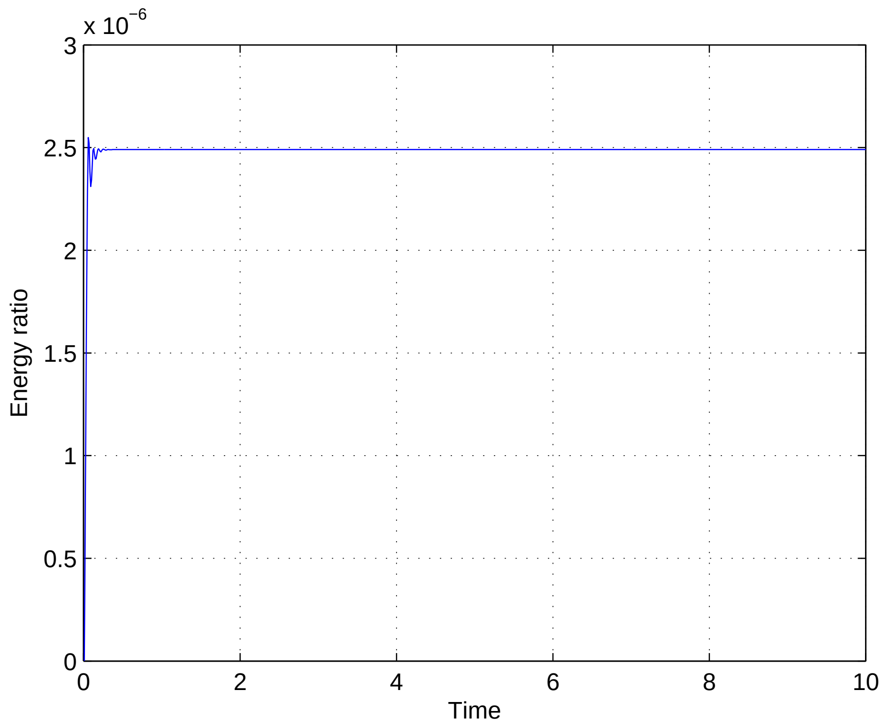

Figure 3.3: Ratio of energy of the controlled output to the energy of the disturbance $(\gamma = 0.5)$, $P_{\tau 1}$

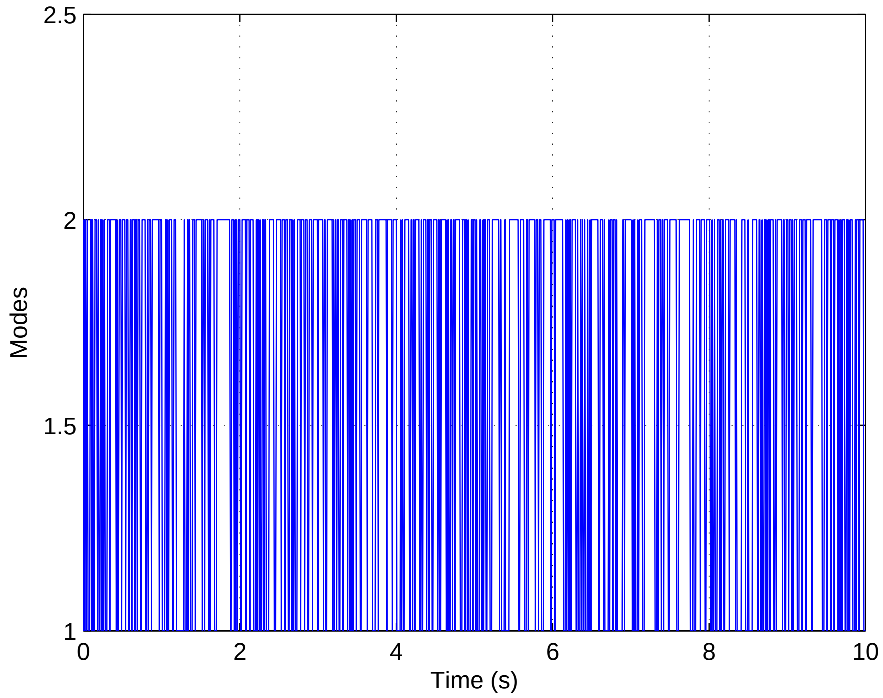

Figure 3.4: Change of modes in Markov chain, $P_{\tau 1}$

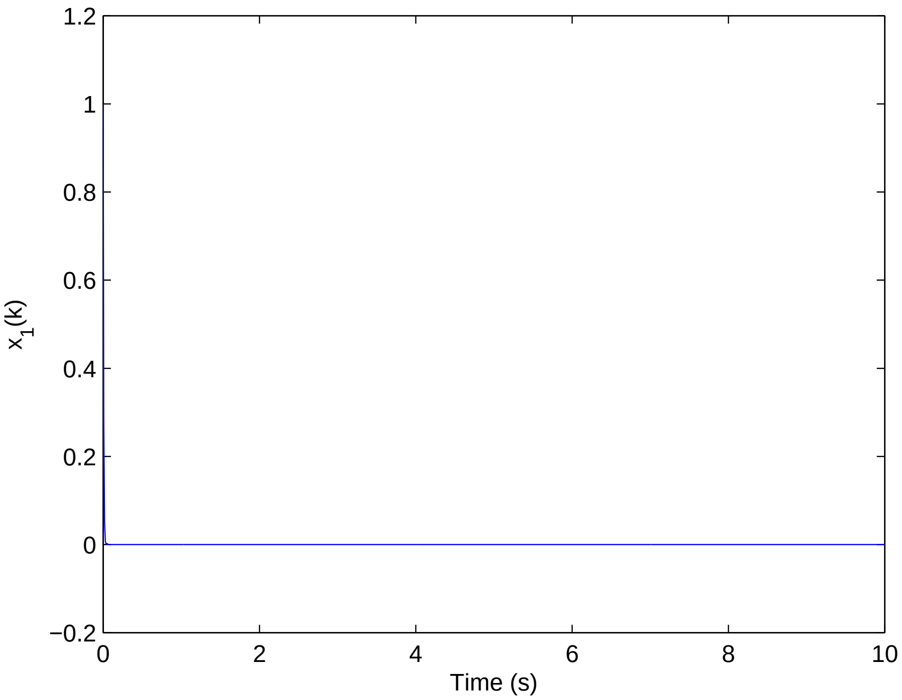

Figure 3.5: Response of $x_1(k)$, $P_{\tau 2}$

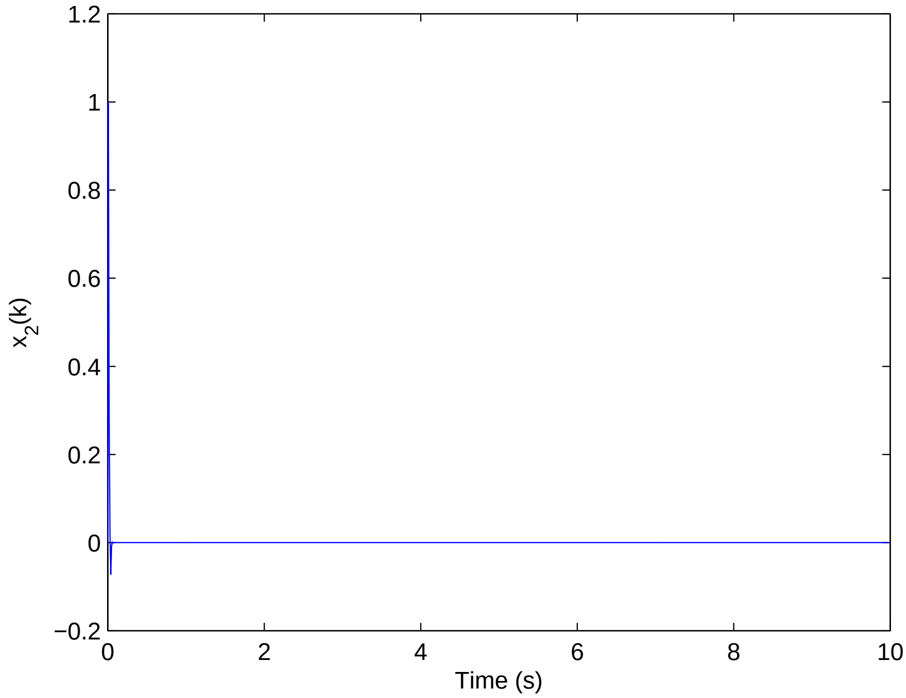

Figure 3.6: Response of $x_2(k)$, $P_{\tau 2}$

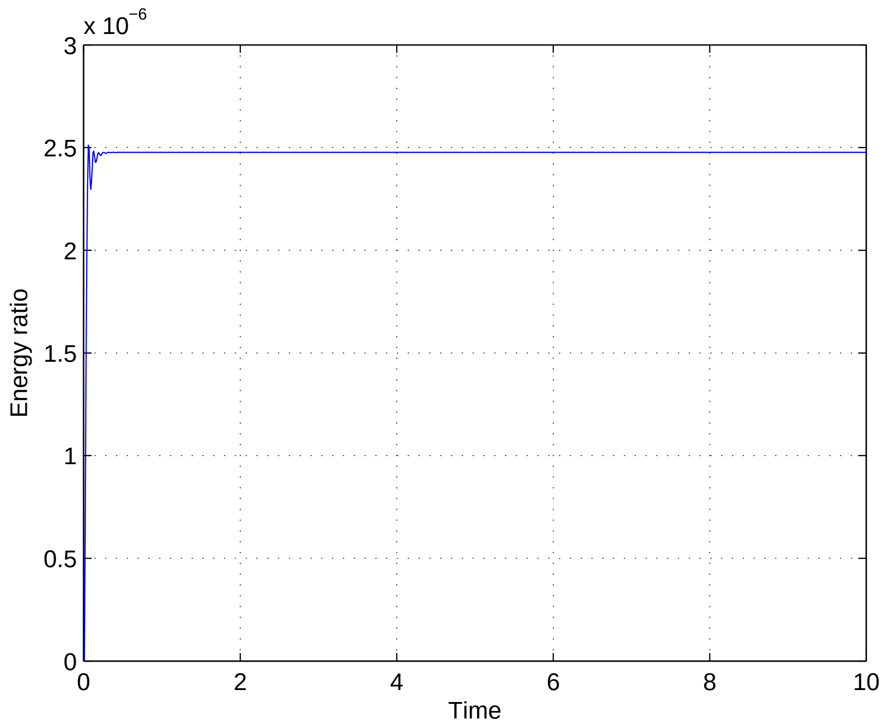

Figure 3.7: Ratio of energy of the controlled output to the energy of the disturbance $(\gamma = 0.5)$, $P_{\tau 2}$

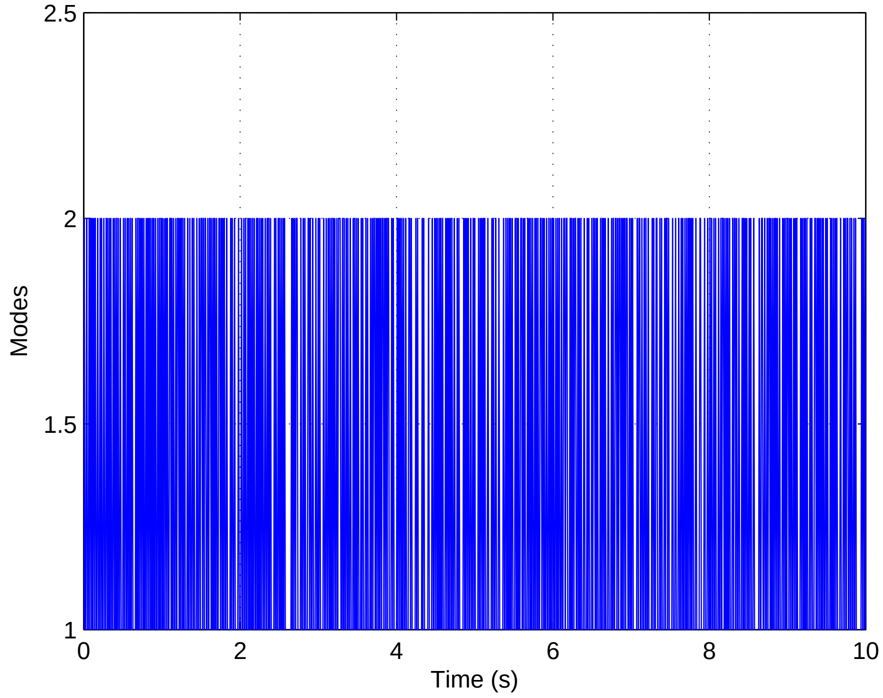

Figure 3.8: Change of modes in Markov chain, $P_{\tau 2}$

Figure 3.1, 3.2, 3.5 and 3.6 show the state response of Case 1 and Case 2 respectively with $w = 0$. The initial states are chosen to be $x(0) = [1.0 \;\; 0]^T$. It is shown in these figures that the state feedback controller stabilizes the system with partially known transition probability matrix. Figure 3.3 and 3.7, show the ratio of energy of the controlled output to the energy of the disturbance ($w(k) = e^{-0.1k}\sin(0.5k)$) for Case 1 and Case 2. In Figure 3.3 and 3.7, the attenuation levels are approximately equals to 2.5×10^{-6}, which is less than the prescribed level $\gamma = 0.5$. Figure 3.4 and 3.8 show the change of modes in the Markov chain. State transition is made according to the transition probability matrices, P_τ. The same controller gains in (3.4.2), obtained from the theorem, control systems with two different transition probability matrix.

3.5 Conclusions

In this chapter, stability criteria and mode delay dependent $\mathcal{H}_\infty$ state feedback controller is developed for a class of networked control systems. Random network-induced delays are modelled by a Markov process where the transition probability matrix is allowed to be partially known. Conditions for stochastic stability with a given attenuation gain is derived by using Lyapunov-Krasovskii functional. The controller design technique is given in terms of the solvability of bilinear matrix inequalities. An iterative algorithm is proposed to change this non-convex problem into quasi-convex optimization problems, which can be solved effectively by available mathematical tools. It has been pointed out that systems with completely known or unknown transition probability matrix is a special case of the approach presented in this paper. Finally, the effectiveness of the proposed design methodology is illustrated by a numerical example.

4

Robust $\mathcal{H}_\infty$ Filtering for Discrete-time Networked Control Systems With Partially Known Transition Probability Matrix

Abstract

In this chapter, stability analysis and a methodology for designing a robust $\mathcal{H}_\infty$ filter for discrete-time networked control systems is presented. Network-induced delays between sensors and controllers are modelled by a finite state Markov chain. The transition probability matrix of the Markov chain is allowed to be partially known. Based on Lyapunov-Krasovskii functional, stability criteria are derived and a novel methodology of designing a partially mode delay-dependent filter design is presented. The filter gains are obtained by solving linear matrix inequality optimisation problems using the cone complementarity linearisation algorithm. A DC motor servo system simulation is used to illustrate the validity of the presented methodology.

4.1 Introduction

Objective of a filtering problem is to estimate information of the plant. This type of problem is especially important in signal processing applications as filters provide ways to estimate information of the plant such as the states from the measurable output of the plant. Unlike the traditional Kalman filter approach, $\mathcal{H}_\infty$ filtering does not require the exact knowledge of the external noise. For this reason, study of $\mathcal{H}_\infty$ filtering approach has been very popular in recent years and several papers presented their results in NCSs [42, 59, 97–100, 116, 117]. In [97], $\mathcal{H}_\infty$ filter is used for fault detection with larger transfer delays. In this paper, multiple sampling method along with augmented state matrix method is used to model the delays. Robust $\mathcal{H}_\infty$ filter design for NCSs are presented in [98, 99] where the objective of the filter is to estimate a signal from the plant. In [100], event-based $\mathcal{H}_\infty$ filtering for NCSs with communication delay is presented. It presents a novel event-triggering scheme where the sensor data is transmitted only when a specified event condition is violated.

In existing literature, Markovian jump systems have been proven to be very effective with NCSs [8, 11, 27, 34, 51–60]. In these approaches, the network-induced delays are modelled by a finite state Markov chain where each mode in the Markov chain corresponds to possible delays in the network. However, most of existing literature in Markovian jump systems require a completely known transition probability matrix. In complex network, it is often expensive or practically impossible to obtain a completely known transition probability matrix. Motivated by this, several papers have been published to incorporate partially known transition probability matrix to Markovian jump systems [61–63]. In these aforementioned papers, state feedback controller [61, 63] and filter design [62] are considered. These approaches however leave room for improvement as the terms corresponding to known and unknown probabilities are separated. Furthermore, these approaches discard the unknown probabilities. These drawbacks may lead to severe conservatism.

In this chapter, $\mathcal{H}_\infty$ filtering problem for a discrete-time networked control systems is investigated where the network-induced delays are modelled by a Markov chain. The transition probability matrix of the Markov chain is allowed to be partially known. Based on Lyapunov-Krasovskii functional, sufficient conditions for the existence of the filter is presented in terms of solvability of BMIs. An iterative algorithm is presented to solve these BMIs using existing mathematical toolbox.

The rest of the chapter is organised as follows. Section 4.2 presents the system description and definitions. The $\mathcal{H}_\infty$ filtering problem is formulated in this section. Stability criteria and the robust $\mathcal{H}_\infty$ filter design theorem are presented in Section 4.3. An iterative algorithm is presented to solve BMIs to obtain the filter gain for NCSs with partially known transition probability matrix is also presented. A DC motor servo example is used in Section 4.4 to illustrate the validity of the presented methodology. Conclusions are drawn in Section 4.5.

4.2 System Description and Definitions

The aim of this chapter is to design a robust $\mathcal{H}_\infty$ filter for NCSs where the network is modelled by a Markov chain with partially known transition probability matrix. In this chapter, it is assumed that only measured output is available from the sensor. Consider a class of uncertain discrete-time linear systems described by the following model:

$$\begin{aligned} x(k+1) &= [A+\Delta A(k)]x(k)+[B+\Delta B(k)]w(k) \\ z(k) &= [C_1+\Delta C_1(k)]x(k)+[D+\Delta D(k)]w(k) \\ y(k) &= C_2x(k) \end{aligned} \tag{4.2.1}$$

where $x(k) \in \Re^n, y(k) \in \Re^{m_2}, z(k) \in \Re^{m_1}$ are the state, measured output and the objective signal to be estimated, respectively and $w(k) \in \Re^{m_3}$ is the disturbance which belongs to $\mathcal{L}_2[0,\infty)$, the space of square summable vector sequence over $[0,\infty]$. The matrices A, B, C_1, D and C_2 are of known dimensions. As in the previous chapter, the matrix functions $\Delta A(k)$, $\Delta B(k)$, $\Delta C_1(k)$ and $\Delta D(k)$ represent the time-varying uncertainties in the system which satisfy the following assumption.

Assumption 4.2.1

$$\begin{bmatrix} \Delta A(k) & \Delta B(k) \\ \Delta C_1(k) & \Delta D(k) \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} F(k) \begin{bmatrix} H_1 & H_2 \end{bmatrix}$$

where H_i and E_i are known matrices which characterize the structure of the uncertainties. Furthermore, there exists a positive-definite matrix $\mathcal{W}$ such that the following inequality holds:

$$F^T(k)\mathcal{W}F(k) \le \mathcal{W} \tag{4.2.2}$$

The network is modelled by a finite state Markov chain with partially known transition probability matrix. Refer to Chapter 3 for more information about the modelling procedure.

The aim of this chapter is to design a full-order filter of the following form

$$\begin{aligned} x_f(k+1) &= A_f(i)x_f(k) + B_f(i)y(k-\tau_k) \\ z_f(k) &= C_f(i)x_f(k) \end{aligned} \tag{4.2.3}$$

where $x_f(k)$ and $z_f(k)$ are the state and output of the filter respectively. $A_f(i), B_f(i)$ and $C_f(i)$ are the filter gains to be determined.

The augmented filtering error system of (4.2.1) with (4.2.3) is given as follows:

$$\begin{aligned} \tilde{x}(k+1) &= [\tilde{A}(i) + \bar{E}_1 F(k)\bar{H}_1(i)]\tilde{x}(k) + \tilde{B}(i)\bar{C}_2\tilde{x}(k-\tau_k) + [\bar{B} + \bar{E}_1 F(k) H_2]w(k) \\ e(k) &= [\tilde{C}(i) + E_2 F(k)\bar{H}_1(i)]\tilde{x}(k) + [D + E_2 F(k) H_2]w(k) \end{aligned} \tag{4.2.4}$$

where $\tilde{x}(k)^T = \begin{bmatrix} x(k)^T & x_f(k)^T \end{bmatrix}$, $e(k) = z(k) - z_f(k)$,

$$\tilde{A}(i) = \begin{bmatrix} A & 0 \\ 0 & A_f(i) \end{bmatrix}, \tilde{B}(i) = \begin{bmatrix} 0 \\ B_f(i) \end{bmatrix}, \bar{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \bar{C}_2 = \begin{bmatrix} C_2 & 0 \end{bmatrix},$$

$$\bar{E}_1 = \begin{bmatrix} E_1 \\ 0 \end{bmatrix}, \bar{H}_1 = \begin{bmatrix} H_1 & 0 \end{bmatrix}, \tilde{C}(i) = \begin{bmatrix} C_1 & -C_f(i) \end{bmatrix}.$$

As shown in Chapter 2, the closed-loop system is to achieve stochastic stability, as shown in (2.2.5), and the $\mathcal{H}_\infty$ performance condition, as shown in (2.2.6), except that $e(k)$ is used in this chapter instead of $z(k)$ in Chapter 2.

4.3 Main Results

This section presents the stability criteria and the robust $\mathcal{H}_\infty$ filter design, which incorporate partially known transition probability matrix into the theorem. Based on Lemma 3.2.2, the unknown transition probabilities are upper bounded and incorporated.

The following theorem proposes stability analysis of the filtering error system (4.2.4) with partially known transition probabilities.

Theorem 4.3.1 *For given filter gains $A_f(i)$, $B_f(i)$, $C_f(i)$, $i \in \mathcal{S}$, and $\gamma > 0$, if there exist sets of positive-definite matrices $P(i)$, $R_1(i)$, R_1, $R_2(i)$, R_2, $W_1(i)$, $W_2(i)$, Q, $Z(i)$*

and matrices $M(i)$, $\Omega_1(i)$, $\Omega_2(i)$, $\forall i \in \mathcal{S}$, satisfying the following inequalities

$$R_1 > R_1(i), \quad R_2 > R_2(i) \tag{4.3.1}$$

$$\Lambda(i) + \Gamma_2^T(i)\tau(s)R_2\Gamma_2(i) + \Upsilon_1(i) + \Upsilon_1^T(i) + \tau(i)Z(i) + \Xi^T(i)\Xi(i) + \Omega_1(i) + \big(1 - p_{\mathcal{K}}^i\big)\Omega_2(i) < 0 \tag{4.3.2}$$

$$\Gamma_1^T(i)\tilde{P}_{\mathcal{K}}(i)\Gamma_1(i) + \Gamma_2^T(i)\tilde{\tau}_{\mathcal{K}}R_1\Gamma_2(i) - \Omega_1(i) < 0, \quad \forall j \in \mathcal{S}_{\mathcal{K}}^i \tag{4.3.3}$$

$$\Big(1 - p_{\mathcal{K}}^i\Big)\left[\Gamma_1^T(i)\sum_{j=1}^{i+1}P(j)\Gamma_1(i) + \Gamma_2^T(i)\sum_{j=1}^{i+1}\tau(j)R_1\Gamma_2(i)\right] - \big(1 - p_{\mathcal{K}}^i\big)\Omega_2(i) < 0, \ \forall j \in \mathcal{S}_{\mathcal{UK}}^i \tag{4.3.4}$$

and

$$\begin{bmatrix} (1 - p_{i(i+1)})R_1(i) + R_2(i) & M(i) \\ M^T(i) & Z(i) \end{bmatrix} \geq 0 \tag{4.3.5}$$

$$\begin{bmatrix} p_{\mathcal{K}}^i R_1(i) + R_2(i) & M(i) \\ M^T(i) & Z(i) \end{bmatrix} \geq 0, \quad p_{i(i+1)} \in \mathcal{S}_{\mathcal{UK}}^i \tag{4.3.6}$$

where

$$\begin{aligned}
\Gamma_1(i) &= \Big[\tilde{A}(i) \ \ \tilde{B}(i)\bar{C}_2 \ \ \bar{B} \ \ \bar{E}_1 \ \ \bar{E}_1\Big] \\
\Xi(i) &= \Big[\tilde{C}(i) \ \ 0 \ \ D \ \ E_2 \ \ E_2\Big] \\
\Gamma_2(i) &= \Big[\bar{A} \ \ 0 \ \ \bar{B} \ \ \bar{E}_1 \ \ \bar{E}_1\Big] \\
\Lambda(i) &= diag\Big\{\Big((\tau(s) - \tau(1) + 1)Q + \bar{H}_1^T(i)W_1(i)\bar{H}_1(i) - P(i)\Big), \\
&\quad -Q, \Big(H_2^T W_2(i)H_2 - \gamma I\Big), -W_1(i), -W_2(i)\Big\} \\
\Upsilon_1 &= M^T[diag\{I, 0\} \ \ diag\{-I, 0\} \ \ 0 \ \ 0 \ \ 0] \\
\tilde{P}_{\mathcal{K}}(i) &= \textstyle\sum_{j\in\mathcal{S}_{\mathcal{K}}^i}^{i+1} p_{ij}P(j) \\
\tilde{\tau}_{\mathcal{K}}(i) &= \textstyle\sum_{j\in\mathcal{S}_{\mathcal{UK}}^i}^{i+1} p_{ij}\tau(j) \\
\bar{A} &= \begin{bmatrix} A - I & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned} \tag{4.3.7}$$

Then the filtering error system (4.2.4) is stochastically stable with the prescribed $\mathcal{H}_\infty$ performance.

Proof: The filtering error system (4.2.4) can be rewritten as

$$\begin{aligned}
\tilde{x}_{k+1} &= \Gamma_1(r_k)\zeta_k \\
z_k &= \Xi(r_k)\zeta_k
\end{aligned} \tag{4.3.8}$$

where $\zeta_\ell = \zeta(\ell)$, $z_\ell = z(\ell)$, $\tilde{x}_\ell = \tilde{x}(\ell)$, and $\Gamma_1(r_k) = \begin{bmatrix}\tilde{A}(i) & \tilde{B}(i)\bar{C}_2 & \bar{B} & \bar{E}_1 & \bar{E}_1\end{bmatrix}$, and $\zeta(k)^T = \begin{bmatrix}\tilde{x}^T(k) & \tilde{x}^T(k-\tau_k) & w^T(k) & \tilde{x}^T(k)\bar{H}_1^T(i)F^T(k) & w^T(k)H_2^T F^T(k)\end{bmatrix} \in \Re^l$.

The following Lyapunov-Krasovskii candidate functional is considered

$$V(\tilde{x}_k, r_k) = V_1(\tilde{x}_k, r_k) + V_2(\tilde{x}_k, r_k) + V_3(\tilde{x}_k, r_k) \tag{4.3.9}$$

with

$$V_1(\tilde{x}_k, r_k) = \tilde{x}_k^T P(r_k)\tilde{x}_k \tag{4.3.10}$$

$$V_2(\tilde{x}_k, r_k) = \sum_{\ell=-\tau_k}^{-1} \sum_{j=k+\ell}^{k-1} \bar{x}_j^T R_1 \bar{x}_j + \sum_{\ell=-\tau(s)}^{-1} \sum_{j=k+\ell}^{k-1} \bar{x}_j^T R_2 \bar{x}_j \tag{4.3.11}$$

$$V_3(\tilde{x}_k, r_k) = \sum_{\ell=k-\tau_k}^{k-1} \tilde{x}_\ell^T Q \tilde{x}_\ell + \sum_{\ell=-\tau(s)+2}^{-\tau(1)+1} \sum_{j=k+\ell-1}^{k-1} \tilde{x}_j^T Q \tilde{x}_j \tag{4.3.12}$$

where $\bar{x}(k) = \begin{bmatrix} x(k+1) - x(k) \\ 0 \end{bmatrix}$.

Following a similar technique used in the proof of the stability analysis in Chapter 3, substituting $x(k)$ with $\tilde{x}(k)$ and $y(k)$ with $\bar{x}(k)$, we obtain

$$\begin{aligned}\Delta V(\tilde{x}_k, r_k) \;\; &\leq \;\; -\tilde{x}_k^T\Big(P(r_k) - (\tau(s) - \tau(1) + 1)Q\Big)\tilde{x}_k - \tilde{x}_{k-\tau_k}^T Q \tilde{x}_{k-\tau_k} + \zeta_k^T\Big\{\Gamma_1^T \tilde{P}(r_k)\Gamma_1 \\ &\quad + \Gamma_2^T\left[\tilde{\tau}_k R_1 + \tau(s)R_2\right]\Gamma_2 + \Upsilon_1(r_k) + \Upsilon_1^T(r_k) + \tau_k Z(r_k)\Big\}\zeta_k \end{aligned} \tag{4.3.13}$$

Using Assumption 4.2.1, and adding and subtracting $\tilde{x}_k^T \bar{H}_1^T F_k^T W_1(r_k) F_k \bar{H}_1 \tilde{x}_k$, $w_k^T H_2^T F_k^T W_2(r_k) F_k H_2 w_k$, $z_k^T z_k$ and $\gamma w_k^T w_k$ to and from (4.3.13), we obtain

$$\begin{aligned}\Delta V(\tilde{x}_k, r_k) \;\; &\leq \;\; -\tilde{x}_k^T\Big(P(r_k) - (\tau(s) - \tau(1) + 1)Q - H_1^T W_1(r_k) H_1\Big)\tilde{x}_k - \tilde{x}_{k-\tau_k}^T Q \tilde{x}_{k-\tau_k} \\ &\quad + \zeta_k^T\Big\{\Gamma_1^T \tilde{P}(r_k)\Gamma_1 + \Gamma_2^T\left[\tilde{\tau}_k R_1 + \tau(s)R_2\right]\Gamma_2 + \Upsilon_1(r_k) + \Upsilon_1^T(r_k) \\ &\quad + \tau_k Z(r_k) + \Xi^T\Xi\Big\}\zeta_k - z_k^T z_k + \gamma w_k^T w_k - w_k^T\Big(\gamma I - H_2^T W_2(r_k) H_2\Big)w_k \\ &\quad - \tilde{x}_k^T \bar{H}_1^T F_k^T W_1(r_k) F_k \bar{H}_1 \tilde{x}_k - w_k^T H_2^T F_k^T W_2(r_k) F_k H_2 w_k \end{aligned} \tag{4.3.14}$$

Using (4.3.7), (4.3.14) can be rewritten as

$$\begin{aligned}\Delta V(\tilde{x}_k, r_k) \;\; &\leq \;\; \tilde{x}_k^T\Big\{\Lambda(r_k) + \Gamma_1^T \tilde{P}(r_k)\Gamma_1 + \Gamma_2^T\left[\tilde{\tau}_k R_1 + \tau(s)R_2\right]\Gamma_2 + \Upsilon_1(r_k) + \Upsilon_1^T(r_k) \\ &\quad + \tau_k Z(r_k) + \Xi^T(i)\Xi(i)\Big\}\tilde{x}_k - z_k^T z_k + \gamma w_k^T w_k \end{aligned} \tag{4.3.15}$$

which will be referred later on. Following the similar approach of handling the unknown transition probabilities shown in Chapter 3, we show that the filter satisfies the conditions shown in the problem formulation. $\nabla\nabla\nabla$

The following theorem provides sufficient conditions for the existence of a robust $\mathcal{H}_\infty$ filter for the system (4.2.1) with partially known transition probabilities.

Theorem 4.3.2 *For a given $\gamma > 0$, if there exist sets of positive-definite matrices $X(i)$, $Y(i)$, $\mathcal{Y}(i)$, $\tilde{R}_1(i)$, $\tilde{R}_1$, $\tilde{R}_2(i)$, $\tilde{R}_2$, $W_1(i)$, $W_2(i)$, Q, $\mathcal{Q}$, $\tilde{W}_1(i)$, $\tilde{W}_2(i)$, N_1, N_2, $\tilde{Z}(i)$, $S(i,j)$ and matrices $\tilde{M}(i)$, $\tilde{\Omega}_1(i)$, $\tilde{\Omega}_2(i)$, $\mathcal{A}(i)$, $\mathcal{B}(i)$, $\mathcal{C}(i)$ and $J(i)$ for $i = 1, 2, \cdots, s$ satisfying the following inequalities*

$$\tilde{R}_1 > \tilde{R}_1(i), \quad \tilde{R}_2 > \tilde{R}_2(i) \tag{4.3.16}$$

$$\begin{bmatrix} \bar{\Lambda}(i) & \sqrt{\tau(s)}\tilde{\Gamma}_2^T(i) & \tilde{\Gamma}_3^T(i) & \tilde{\Xi}^T(i) & \mathcal{H}^T(i) \\ * & -N_2 & 0 & 0 & 0 \\ * & * & -\mathcal{Q} & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -\tilde{W}_1(i) \end{bmatrix} < 0 \tag{4.3.17}$$

$$\begin{bmatrix} -\Omega_1(i) & p^i_{\mathcal{K}}\tilde{\Gamma}_1^T(i) & \sqrt{\sum_{j\in\mathcal{S}^i_{\mathcal{K}}} p_{ij}\tau(j)}\tilde{\Gamma}_2^T(i) \\ * & -\Phi_{\mathcal{K}}(i) & 0 \\ * & * & -p^i_{\mathcal{K}}N_1 \end{bmatrix} < 0, \quad \forall j \in \mathcal{S}^i_{\mathcal{K}} \tag{4.3.18}$$

$$\begin{bmatrix} -(1-p^i_{\mathcal{K}})\Omega_2(i) & (1-p^i_{\mathcal{K}})\tilde{\Gamma}_1^T(i) & \sqrt{(1-p^i_{\mathcal{K}})\sum_{j\in\mathcal{S}^i_{\mathcal{UK}}} \tau(j)}\tilde{\Gamma}_2^T(i) \\ * & -\Phi_{\mathcal{UK}}(i) & 0 \\ * & * & -(1-p^i_{\mathcal{K}})N_1 \end{bmatrix} < 0, \quad \forall j \in \mathcal{S}^i_{\mathcal{UK}} \tag{4.3.19}$$

$$\begin{bmatrix} S(i,j) & J^T(i) \\ * & Y(j) \end{bmatrix} > 0 \tag{4.3.20}$$

$$\begin{bmatrix} (1-p_{i(i+1)})\tilde{R}_1(i) + \tilde{R}_2(i) & \tilde{M}(i) \\ * & \tilde{Z}(i) \end{bmatrix} \geq 0 \tag{4.3.21}$$

$$\begin{bmatrix} p^i_{\mathcal{K}}\tilde{R}_1(i) + \tilde{R}_2(i) & \tilde{M}(i) \\ * & \tilde{Z}(i) \end{bmatrix} \geq 0, \quad \forall (i+1) \in \mathcal{S}^i_{\mathcal{UK}} \tag{4.3.22}$$

and

$$N_1\tilde{R}_1 = I, \; N_2\tilde{R}_2 = I, \; \tilde{W}_1(i)W_1(i) = I \text{ and } \mathcal{Q}Q = I, \tag{4.3.23}$$

where

$$\begin{aligned}
\bar{\Lambda}(i) &= \tilde{\Lambda}(i) + \tilde{\Upsilon}_1(i) + \tilde{\Upsilon}_1^T(i) + \tau(i)\tilde{Z}(i) + \tilde{\Omega}_1(i) + (1-p^i_{\mathcal{K}})\tilde{\Omega}_2(i) \\
\tilde{\Lambda}(i) &= diag\Big\{ -\begin{bmatrix} Y(i) & I \\ I & X(i) \end{bmatrix}, -Q, \Big(H_2^T W_2(i) H_2 - \gamma I\Big), -W_1(i), -W_2(i)\Big\} \\
\tilde{\Gamma}_1(i) &= \begin{bmatrix} \check{A}(i) & \hat{B}(i)\bar{C}_2(i) & \check{B}(i) & \check{E}_1 & \check{E}_1 \end{bmatrix} \\
\tilde{\Gamma}_2(i) &= \begin{bmatrix} \check{A}(i) & 0 & \bar{B} & \bar{E}_1 & \bar{E}_1 \end{bmatrix} \\
\tilde{\Gamma}_3(i) &= \begin{bmatrix} \sqrt{(\tau(s)-\tau(1)+1)}T(i) & 0 & 0 & 0 & 0 \end{bmatrix} \\
\Phi_{\mathcal{K}} &= \begin{bmatrix} -\sum_{j=1}^{i+1} p_{ij}S(i,j) - J(i) - J^T(i) & p^i_{\mathcal{K}} I \\ p^i_{\mathcal{K}} I & \sum_{j=1}^{i+1} p_{ij}X(j) \end{bmatrix} \\
\Phi_{\mathcal{UK}} &= \begin{bmatrix} -\big(1-p^i_{\mathcal{K}}\big)^{-1}\big(\sum_{j=1}^{i+1} S(i,j) - J(i) - J^T(i)\big) & (1-p^i_{\mathcal{K}})I \\ (1-p^i_{\mathcal{K}})I & (1-p^i_{\mathcal{K}})\sum_{j=1}^{i+1} X(j) \end{bmatrix} \\
\tilde{\Xi}(i) &= \begin{bmatrix} \check{C}(i) & 0 & D & E_2 & E_2 \end{bmatrix} \\
\mathcal{H} &= \begin{bmatrix} \check{H}_1(i) & 0 & 0 & 0 & 0 \end{bmatrix} \\
\tilde{\Upsilon}_1(i) &= \tilde{M}^T(i)[I \quad -I \ 0 \ 0 \ 0] \\
\check{A}(i) &= \begin{bmatrix} AY(i) & A \\ \mathcal{A}(i) & \sum_{j=1}^{i+1} \bar{p}_{ij}X(j)A \end{bmatrix}, \ \check{B}(i) = \begin{bmatrix} B \\ \sum_{j=1}^{i+1} \bar{p}_{ij}X(j)B \end{bmatrix} \\
\check{A}(i) &= \begin{bmatrix} (A-I)Y(i) & A-I \\ 0 & 0 \end{bmatrix} \ \check{C}(i) = \begin{bmatrix} C_1Y(i) - \mathcal{C}(i) & C_1 \end{bmatrix}, \\
\check{H}_1(i) &= \begin{bmatrix} H_1Y(i) & H_1 \end{bmatrix}, \ \check{E}_1(i) = \begin{bmatrix} E_1 \\ \sum_{j=1}^{i+1} \bar{p}_{ij}X(j)E_1 \end{bmatrix}, \\
\hat{B}(i) &= \begin{bmatrix} 0 \\ \mathcal{B}(i) \end{bmatrix}, \ T(i) = \begin{bmatrix} Y(i) & I \\ Y(i) & 0 \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=1}^{i+1} \bar{p}_{ij}X(i) &= \sum_{j\in\mathcal{S}^i_{\mathcal{K}}}^{i+1} p_{ij}X(j) + (1-p^i_{\mathcal{K}}) \sum_{j\in\mathcal{S}^i_{\mathcal{UK}}}^{i+1} X(j) \\
\sum_{j=1}^{i+1} \bar{p}_{ij}Y^{-1}(j) &= \sum_{j\in\mathcal{S}^i_{\mathcal{K}}}^{i+1} p_{ij}Y^{-1}(j) + (1-p^i_{\mathcal{K}}) \sum_{j\in\mathcal{S}^i_{\mathcal{UK}}}^{i+1} Y^{-1}(j).
\end{aligned}$$

Then the filtering error is stochastically stable with the prescribed $\mathcal{H}_\infty$ performance.

Furthermore, the filter gains as shown in (4.2.3) are given as follows:

$$\begin{aligned} A_f(i) &= \Big(\textstyle\sum_{j=1}^{i+1}\bar{p}_{ij}Y^{-1}(j) - \sum_{j=1}^{i+1}\bar{p}_{ij}X(j)\Big)^{-1}\Big(\mathcal{A}(i) - \sum_{j=1}^{i+1}\bar{p}_{ij}X(j)AY(i)\Big)Y^{-1}(i) \\ B_f(i) &= \Big(\textstyle\sum_{j=1}^{i+1}\bar{p}_{ij}Y^{-1}(j) - \sum_{j=1}^{i+1}\bar{p}_{ij}X(j)\Big)^{-1}\mathcal{B}(i) \\ C_f(i) &= \mathcal{C}(i)Y^{-1}(i). \end{aligned} \tag{4.3.24}$$

Proof: We obtain the filter design by applying Schur complement on the terms within the curly bracket of (4.3.15). We then have

$$\begin{bmatrix} \hat{\Lambda}(i) & \bar{\Gamma}_1^T(i) & \bar{\Gamma}_2^T & \bar{\Gamma}_3^T & \bar{\Xi}^T(i) & \bar{\mathcal{H}}^T \\ * & -\tilde{P}^{-1}(i) & 0 & 0 & 0 & 0 \\ * & * & -(\tilde{\tau}_k R_1 + \tau(s)R_2)^{-1} & 0 & 0 & 0 \\ * & * & * & -Q^{-1} & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -W_1^{-1} \end{bmatrix} < 0 \tag{4.3.25}$$

where

$$\begin{aligned} \hat{\Lambda}(i) &= \mathrm{diag}\Big\{ - P(i), -Q, (H_2^T W_3(i) H_2 - \gamma I), -W_1(i), -W_2(i)\Big\} \\ &\quad + \Upsilon_1(i) + \Upsilon_1^T(i) + \tau(i)Z(i) \\ \bar{\Gamma}_1(i) &= \begin{bmatrix} \tilde{A}(i) & \tilde{B}(i)\bar{C}_2 & \bar{B} & \bar{E}_1 & \bar{E}_1 \end{bmatrix} \\ \bar{\Gamma}_2 &= \begin{bmatrix} \bar{A} & 0 & \bar{B} & \bar{E}_1 & \bar{E}_1 \end{bmatrix} \\ \bar{\Gamma}_3 &= \begin{bmatrix} \sqrt{(\tau(s) - \tau(1) + 1)} & 0 & 0 & 0 & 0 \end{bmatrix} \\ \bar{\Xi}(i) &= \begin{bmatrix} \tilde{C}(i) & 0 & D & E_2 & E_2 \end{bmatrix} \\ \bar{\mathcal{H}}^T &= \begin{bmatrix} \bar{H}_1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Following from [118], with loss of generality, $P(i)$ and $\tilde{P}(i)$ are, respectively, partitioned as

$$P(i) = \begin{bmatrix} X(i) & Y^{-1}(i) - X(i) \\ Y^{-1}(i) - X(i) & X(i) - Y^{-1}(i) \end{bmatrix} \tag{4.3.26}$$

and

$$\tilde{P}(i) = \begin{bmatrix} \sum_{j=1}^{i+1} p_{ij}X(j) & \sum_{j=1}^{i+1} p_{ij}Y^{-1}(j) - \sum_{j=1}^{i+1} p_{ij}X(j) \\ \sum_{j=1}^{i+1} p_{ij}Y^{-1}(j) - \sum_{j=1}^{i+1} p_{ij}X(j) & \sum_{j=1}^{i+1} p_{ij}X(j) - \sum_{j=1}^{i+1} p_{ij}Y^{-1}(j) \end{bmatrix}. \tag{4.3.27}$$

Note that each summation $\sum_{j=1}^{i+1} p_{ij}X(j)$ and $\sum_{j=1}^{i+1} p_{ij}Y^{-1}(j)$ can be divided into known and unknown terms, $\sum_{j\in\mathcal{S}_{\mathcal{K}}^i} p_{ij}X(j) + \sum_{j\in\mathcal{S}_{\mathcal{UK}}^i} p_{ij}X(j)$ and $\sum_{j\in\mathcal{S}_{\mathcal{K}}^i} p_{ij}Y^{-1}(j) + \sum_{j\in\mathcal{S}_{\mathcal{UK}}^i} p_{ij}Y^{-1}(j)$ respectively. Now using Lemma 3.2.2 and introducing new terms, $\sum_{j=1}^{i+1} \bar{p}_{ij}X(i)$ and $\sum_{j=1}^{i+1} \bar{p}_{ij}Y^{-1}(j)$, the above can be expressed as

$$\tilde{P}(i) = \begin{bmatrix} \sum_{j=1}^{i+1} \bar{p}_{ij}X(j) & \sum_{j=1}^{i+1} \bar{p}_{ij}Y^{-1}(j) - \sum_{j=1}^{i+1} \bar{p}_{ij}X(j) \\ \sum_{j=1}^{i+1} \bar{p}_{ij}Y^{-1}(j) - \sum_{j=1}^{i+1} \bar{p}_{ij}X(j) & \sum_{j=1}^{i+1} \bar{p}_{ij}X(j) - \sum_{j=1}^{i+1} \bar{p}_{ij}Y^{-1}(j) \end{bmatrix} \tag{4.3.28}$$

Note that $\sum_{j=1}^{i+1} \bar{p}_{ij} \neq 1$.

Now define the following

$$T_2(i) = \begin{bmatrix} I & \sum_{j=1}^{i+1} \bar{p}_{ij}X(j) \\ 0 & \sum_{j=1}^{i+1} \bar{p}_{ij}Y^{-1}(j) - \sum_{j=1}^{i+1} \bar{p}_{ij}X(j) \end{bmatrix} \tag{4.3.29}$$

Multiplying (4.3.25) to the right by the matrix $\mathrm{diag}\Big\{\mathrm{diag}\Big\{T(i), I, I, I, I\Big\}, T_2(i), I, I, I, I\Big\}$ and the left by its transpose, we obtain

$$\begin{bmatrix} \bar{\Lambda}(i) & \tilde{\Gamma}_1^T(i) & \tilde{\Gamma}_2^T(i) & \tilde{\Gamma}_3^T & \tilde{\Xi}^T(i) & \tilde{\mathcal{H}}^T(i) \\ * & -\Phi & 0 & 0 & 0 & 0 \\ * & * & -(\tilde{\tau}_k R_1 + \tau(s)R_2)^{-1} & 0 & 0 & 0 \\ * & * & * & -Q^{-1} & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -\tilde{W}_1(i) \end{bmatrix} < 0 \tag{4.3.30}$$

where

$$\Phi = \begin{bmatrix} \Big(\sum_{j=1}^{i+1} \bar{p}_{ij}Y^{-1}(j)\Big)^{-1} & I \\ I & \sum_{j=1}^{i+1} \bar{p}_{ij}X(j) \end{bmatrix} \tag{4.3.31}$$

Using Schur complement on (4.3.30), we obtain

$$\tilde{\Lambda}(i) + \Upsilon_1(i) + \Upsilon_1^T(i) + \tau(i)Z(i) + \tilde{\Gamma}_1^T(i)\Phi^{-1}\tilde{\Gamma}_1(i) + \tilde{\Gamma}_2^T(i)\Big[(\tilde{\tau}_k R_1 + \tau(s)R_2)\Big]\tilde{\Gamma}_2(i)$$
$$+\tilde{\Gamma}_3^T(i)Q\tilde{\Gamma}_3(i) + \tilde{\Xi}^T(i)\tilde{\Xi}(i) + \tilde{\mathcal{H}}^T(i)\tilde{W}_1(i)\tilde{\mathcal{H}}(i) < 0 \tag{4.3.32}$$

Rearranging the above equation and adding and subtracting $\Omega_1(i)$, $\sum_{j\in\mathcal{S}_{\mathcal{UK}}^i}^{i+1} p_{ij}\Omega_2(i)$

results

$$\tilde{\Lambda}(i) + \Upsilon_1(i) + \Upsilon_1^T(i) + \tau(i)Z(i) + \tilde{\Gamma}_1^T(i)\Phi^{-1}\tilde{\Gamma}_1(i) + \tilde{\Gamma}_2^T(i)\tilde{\tau}_k R_1 \tilde{\Gamma}_2(i)$$
$$+\tilde{\Gamma}_2^T(i)\tau(s)R_2\tilde{\Gamma}_2(i) + \tilde{\Gamma}_3^T(i)Q\tilde{\Gamma}_3(i) + \tilde{\Xi}^T(i)\tilde{\Xi}(i) + \tilde{\mathcal{H}}^T(i)\tilde{W}_1(i)\tilde{\mathcal{H}}(i)$$
$$+\Omega_1(i) + \sum_{j\in\mathcal{S}^i_{\mathcal{UK}}}^{i+1} p_{ij}\Omega_2(i) - \Omega_1(i) - \sum_{j\in\mathcal{S}^i_{\mathcal{UK}}}^{i+1} p_{ij}\Omega_2(i) < 0 \tag{4.3.33}$$

The above terms can be re-expressed as

$$\tilde{\Lambda}(i) + \Upsilon_1(i) + \Upsilon_1^T(i) + \tau(i)Z(i) + \tilde{\Gamma}_2^T(i)\tau(s)R_2\tilde{\Gamma}_2(i) + \tilde{\Gamma}_3^T(i)Q\tilde{\Gamma}_3(i) + \tilde{\Xi}^T(i)\tilde{\Xi}(i)$$
$$+\tilde{\mathcal{H}}^T(i)\tilde{W}_1(i)\tilde{\mathcal{H}}(i) + \Omega_1(i) + (1-p^i_{\mathcal{K}})\Omega_2(i) < 0 \tag{4.3.34}$$

and

$$\tilde{\Gamma}_1^T(i)\Phi^{-1}\tilde{\Gamma}_1(i) + \tilde{\Gamma}_2^T(i)\tilde{\tau}_k R_1\tilde{\Gamma}_2(i) - \Omega_1(i) - (1-p^i_{\mathcal{K}})\Omega_2(i) < 0. \tag{4.3.35}$$

Applying Schur complement on (4.3.34) results (4.3.17).

Now we apply Schur complement to (4.3.35). Then the following is obtained.

$$\begin{bmatrix} -\Omega_1(i) - (1-p^i_{\mathcal{K}})\Omega_2(i) & \tilde{\Gamma}_1^T(i) & \sqrt{\tilde{\tau}}\tilde{\Gamma}_2^T(i) \\ * & -\Phi & 0 \\ * & * & -R_1^{-1} \end{bmatrix} < 0 \tag{4.3.36}$$

The above can also be separated into terms related to known and unknown transition probabilities, with the help of Lemma 3.2.2, as shown below.

$$\begin{bmatrix} -\Omega_1(i) & p^i_{\mathcal{K}}\tilde{\Gamma}_1^T(i) & \sqrt{\sum_{j=1}^{i+1} p_{ij}\tau(j)}\tilde{\Gamma}_2^T(i) \\ * & -\Phi_{\mathcal{K}} & 0 \\ * & * & -p^i_{\mathcal{K}}R_1^{-1} \end{bmatrix} < 0 \ \ \forall j \in \mathcal{S}^i_{\mathcal{K}} \tag{4.3.37}$$

and

$$\begin{bmatrix} -(1-p^i_{\mathcal{K}})\Omega_2(i) & (1-p^i_{\mathcal{K}})\tilde{\Gamma}_1^T(i) & \sqrt{(1-p^i_{\mathcal{K}})\sum_{j=1}^{i+1}\tau(j)}\tilde{\Gamma}_2^T(i) \\ * & -\Phi_{\mathcal{UK}} & 0 \\ * & * & -(1-p^i_{\mathcal{K}})R_1^{-1} \end{bmatrix} < 0 \ \ \forall j \in \mathcal{S}^i_{\mathcal{UK}} \tag{4.3.38}$$

where

$$\Phi_{\mathcal{K}} = \begin{bmatrix} \left(\sum_{j=1}^{i+1} p_{ij} Y^{-1}(j)\right)^{-1} & p_k^i I \\ p_k^i I & \sum_{j=1}^{i+1} p_{ij} X(j) \end{bmatrix} \quad \forall j \in \mathcal{S}_{\mathcal{K}}^i \tag{4.3.39}$$

and

$$\Phi_{\mathcal{UK}} = \begin{bmatrix} \left((1-p_{\mathcal{K}}^i)\sum_{j=1}^{i+1} Y^{-1}(j)\right)^{-1} & (1-p_{\mathcal{K}}^i)I \\ (1-p_k^i)I & (1-p_{\mathcal{K}}^i)\sum_{j=1}^{i+1} X(j) \end{bmatrix} \quad \forall j \in \mathcal{S}_{\mathcal{UK}}^i \tag{4.3.40}$$

Using the similar approach shown in (3.3.35) in the previous chapter, we can show that that (4.3.18) remains valid even if $\sum_{j=1}^{i+1} p_{ij} S(i,j) - J(i) - J^T(i)$ replaces $\left(\sum_{j=1}^{i+1} p_{ij} Y^{-1}(j)\right)^{-1}$ for all $j \in \mathcal{S}_{\mathcal{K}}^i$ and similarly even if $\sum_{j=1}^{i+1} S(i,j) - J(i) - J^T(i)$ replaces $\left(\sum_{j=1}^{i+1} Y^{-1}(j)\right)^{-1}$ for all $j \in \mathcal{S}_{\mathcal{UK}}^i$. $\nabla\nabla\nabla$

The nonconvex feasibility problem formulated by (4.3.16)-(4.3.21) can be converted into the following nonlinear minimisation problem subject to LMIs:

Minimize $Tr\Big(N_1\tilde{R}_1 + N_2\tilde{R}_2 + \tilde{W}_1(i)W_1(i) + Q\mathcal{Q}\Big)$

Subject to (4.3.16)-(4.3.21) and

$$\begin{bmatrix} N_1 & I \\ I & \tilde{R}_1 \end{bmatrix} \geq 0, \quad \begin{bmatrix} N_2 & I \\ I & \tilde{R}_2 \end{bmatrix} \geq 0, \quad \begin{bmatrix} \tilde{W}_1(i) & I \\ I & W_1(i) \end{bmatrix} \geq 0, \quad \begin{bmatrix} Q & I \\ I & \mathcal{Q} \end{bmatrix} \geq 0 \tag{4.3.41}$$

To solve this optimisation problem, the following algorithm can be used:

Algorithm :

Step 1: Set $\jmath = 0$ and solve (4.3.16)-(4.3.21) and (4.3.41) to obtain the initial conditions,

$$\Big[\mathcal{A}(i), \mathcal{C}(i), B_c(i), X(i), Y(i), \tilde{R}_1(i), \tilde{R}_1, \tilde{R}_2(i), \tilde{R}_2, W_1(i), W_2(i), \tilde{W}_1(i), \tilde{W}_2(i), Q, \mathcal{Q}, N_1, N_2, \tilde{Z}(i), Y(i)\Big]^0$$

Step 2: Solve the LMI problem

Minimize $Tr\Big(N_1^\jmath\tilde{R}_1 + N_1\tilde{R}_1^\jmath + N_2^\jmath\tilde{R} + N_2\tilde{R}_2^\jmath + \tilde{W}_1(i)^\jmath W_1(i) + \tilde{W}_1(i)W_1(i)^\jmath + Q^\jmath\mathcal{Q} + Q\mathcal{Q}^\jmath\Big)$

Subject to (4.3.16)-(4.3.21) and (4.3.41)

The obtained solutions are denoted as

$$\Big[\mathcal{A}(i), \mathcal{C}(i), B_c(i), X(i), Y(i), \tilde{R}_1(i), \tilde{R}_1, \tilde{R}_2(i), \tilde{R}_2, W_1(i), W_2(i), \tilde{W}_1(i), \\ Q, \mathcal{Q}, N_1, N_2, \tilde{Z}(i), Y(i)\Big]^{\jmath+1}$$

Step 3: Solve Theorem 4.3.1 with $A_f(i)^{\jmath+1}$, $B_f(i)^{\jmath+1}$, $C_f(i)^{\jmath+1}$, if there exist solutions, then $A_f(i)^{\jmath+1}$, $B_f(i)^{\jmath+1}$, $C_f(i)^{\jmath+1}$ are the desired filter gains and EXIT. Otherwise, set $\jmath = \jmath + 1$ and return to Step 2.

4.4 Example

A DC motor servo system simulation is used to illustrate the effectiveness of the proposed filter. A discrete state space representation of the DC motor sampled at 0.01s is given as follows:

$$\begin{aligned} x(k+1) &= [A + \Delta A(k)]x(k) + [B + \Delta B(k)]w(k) \\ z(k) &= [C_1 + \Delta C_1(k)]x(k) + [D + \Delta D(k)]w(k) \\ y(k) &= C_2 x(k) \end{aligned} \tag{4.4.1}$$

where

$$\begin{aligned} x(k) &= \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \quad A = \begin{bmatrix} 0.9532 & 0.1924 \\ -0.02477 & -0.0005 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ C_1 &= \begin{bmatrix} 1 & 0 \end{bmatrix} \quad C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D = 0.01 \end{aligned} \tag{4.4.2}$$

and $x_1(k)$ and $x_2(k)$ are the motor angular velocity and the armature current, respectively.

The uncertainties in this example are assumed to be characterized by matrices below:

$$\begin{aligned} E_1 &= \begin{bmatrix} 0.02 \\ 0.01 \end{bmatrix} \quad E_2 = 0.01 \\ H_1 &= \begin{bmatrix} 0.05 & 0.02 \end{bmatrix} \quad H_2 = 0.01 \quad H_3 = 0.01 \end{aligned} \tag{4.4.3}$$

We use a Markov chain taking values in a finite set $\mathcal{S} = \{1, 2, 3, 4\}$, which correspond to 0.1, 0.2, 0.3, 0.4 seconds delays, respectively. In this particular example, it is assumed that only the transition probabilities to itself are determined through example to reduce the cost. The transition probability matrix is shown as follows:

$$P_\tau = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ (0.5) & 0.3 & (0.2) & 0 \\ (0.5) & (0.3) & 0.1 & (0.1) \\ (0.5) & (0.3) & (0.1) & 0.1 \end{bmatrix} \tag{4.4.4}$$

where probabilities inside bracket are the unknown transition probabilities.

The attenuation level, γ, is selected as 1.0. Applying Theorem 4.3.2 and the iterative algorithm, the following filter gains are obtained:

$$\begin{aligned}
A_c(1) &= \begin{bmatrix} 0.9532 & 0.1924 \\ -0.0251 & -0.0057 \end{bmatrix} B_c(1) = \begin{bmatrix} -8.0982 \times 10^{-6} \\ -0.0123 \end{bmatrix} \\
C_c(1) &= \begin{bmatrix} 1.0000 & -2.4604 \times 10^{-5} \end{bmatrix} \\
A_c(2) &= \begin{bmatrix} 0.9532 & 0.1924 \\ -0.0252 & -0.0051 \end{bmatrix} B_c(2) = \begin{bmatrix} -9.2835 \times 10^{-6} \\ -0.0018 \end{bmatrix} \\
C_c(2) &= \begin{bmatrix} 1.0000 & -2.1569 \times 10^{-5} \end{bmatrix} \\
A_c(3) &= \begin{bmatrix} 0.9532 & 0.1924 \\ -0.0251 & -0.0037 \end{bmatrix} B_c(3) = \begin{bmatrix} -8.6253 \times 10^{-6} \\ -7.3684 \times 10^{-4} \end{bmatrix} \\
C_c(3) &= \begin{bmatrix} 1.0000 & -1.6136 \times 10^{-5} \end{bmatrix} \\
A_c(4) &= \begin{bmatrix} 0.9532 & 0.1924 \\ -0.0251 & -0.0039 \end{bmatrix} B_c(4) = \begin{bmatrix} -9.8149 \times 10^{-6} \\ -9.1035 \times 10^{-4} \end{bmatrix} \\
C_c(4) &= \begin{bmatrix} 1.0000 & -1.4970 \times 10^{-5} \end{bmatrix}
\end{aligned}$$

Figure 4.1 shows the response of filtering error system. The initial condition is $x(0) = [1 \quad 0]^T$. The external disturbance, $w(k)$, is given as $e^{-0.1k}\sin(0.5k)$. Figure 4.3 shows the mode transitions for this example. The ratio of the energy of the error to the energy of the disturbance is shown in Figure 4.2. It is shown that the ratio is approximately 0.68, which is less than the prescribed $\gamma = 1.0$, illustrating the validity of the filter.

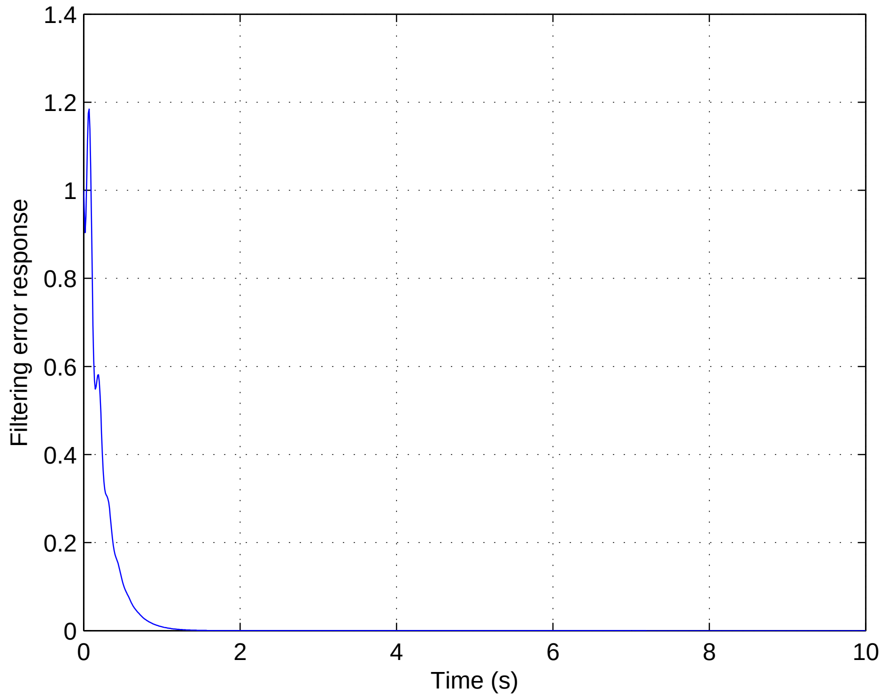

Figure 4.1: Response of filtering error

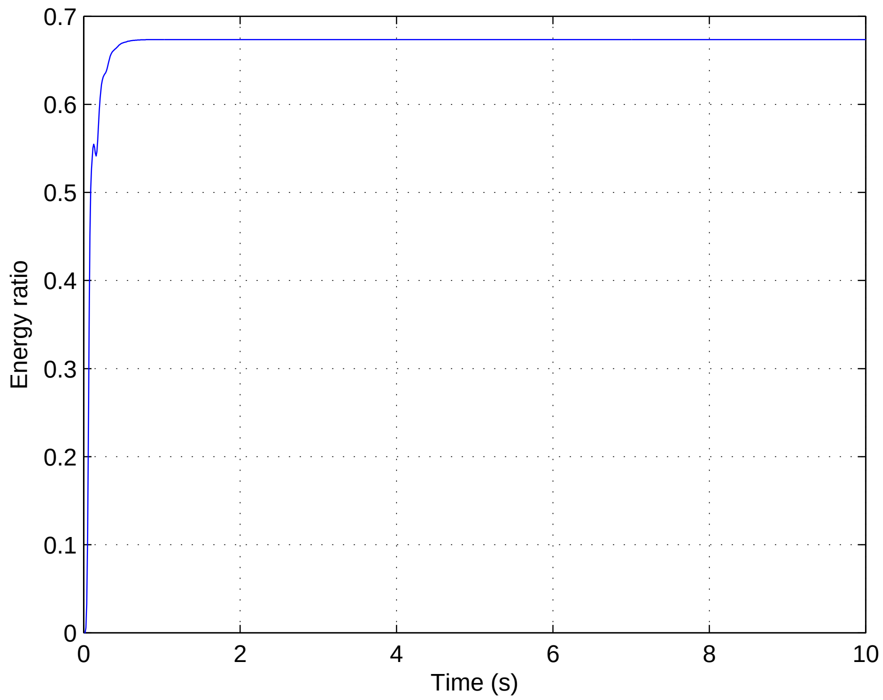

Figure 4.2: Ratio of energy of the filtering error to the energy of the disturbance ($\gamma = 1.0$)

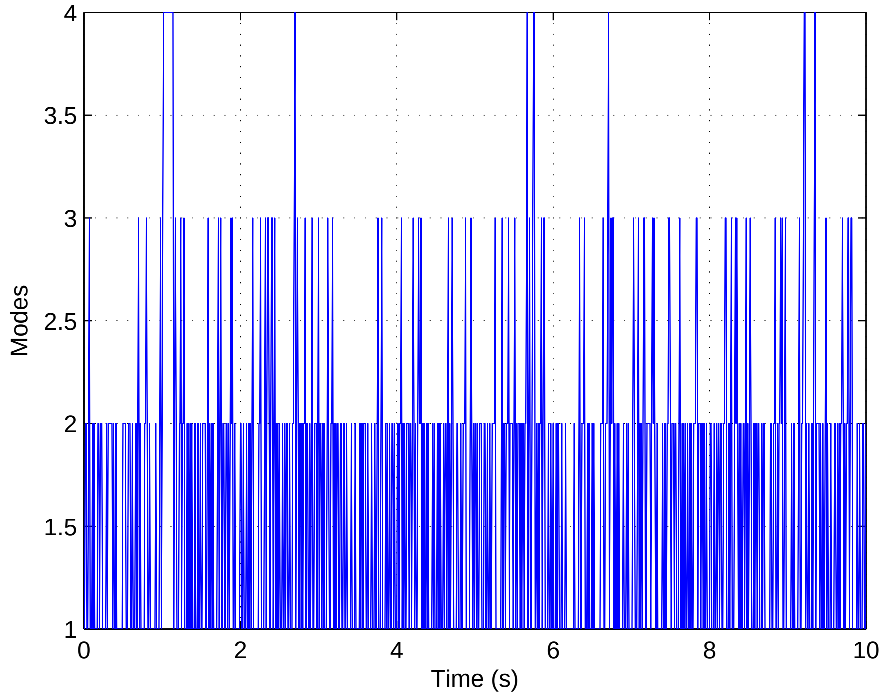

Figure 4.3: Transitions of the modes in Markov chain

4.5 Conclusions

In this chapter, $\mathcal{H}_\infty$ filter design approach is presented for a discrete-time networked control systems. The network-induced delays are modelled by a finite state Markov chain whose transition probability matrix is allowed to be partially known. The existence of the filter is given in terms of solvability of BMIs and an algorithm is presented to solve these BMIs. A DC servo example is used to illustrate the efficiency of the proposed methodology.

5

Robust $\mathcal{H}_\infty$ Dynamic Output Feedback Control of Discrete-time Networked Control Systems With Partially Known Transition Probability Matrix

Abstract

In this chapter, a methodology for designing an $\mathcal{H}_\infty$ dynamic output feedback controller for discrete-time networked control systems has been considered. Markov chain is used to model the communication delays between the sensor and the controller and it is assumed that the transition probability matrix is partially known. Based on Lyapunov-Krasovskii functional the stability criteria has been developed. By considering a dynamic output feedback controller, this chapter provides a way to control NCSs when not all state variables are measurable or available. In conjunction with partially known transition probability matrix, this approach provides more practical controller design in the real world. The proposed design methodology is verified by using a DC servo motor example.

5.1 Introduction

This chapter investigates designing a robust $\mathcal{H}_\infty$ dynamic feedback controller for a class of uncertain NCSs with random network-induced delays. As in the previous chapter, the network-induced delays are modelled by a finite state Markov chain with partially known transition probability matrix. The plant is modelled by a discrete-time linear model where the output of the plant is measured and transmitted via a network. State feedback control requires that all the state variables of the plant are measurable. However, in real world, not all state variables may be measurable. In the previous chapter, $\mathcal{H}_\infty$ filter design is presented where the controlled output of the plant is estimated based on the measured output. In control systems, the main interest is to stabilize and maintain the performance of the overall system. By utilizing an output feedback control, the system can still be stabilized based on the measured output of the system, providing more practical approach to controlling a system than a state feedback control.

Discrete linear Markovian jump systems with partially known transition probabilities have been studied in [61–63] as it is either impossible or costly to obtain a completely known transition probability matrix for complex systems. In the aforementioned papers, the unknown transition probabilities are separated from known transition probabilities and then discarded. Note that the aforementioned literatures [61–63], do not consider NCSs where the Markov chain is used to model the network-induced delays.

In handling the unknown transition probability matrix, instead of using a similar approach to [61–63], where the unknown part is handled without using any probability information, the fact that the summation of all probabilities in each row of transition probability matrix is used throughout the thesis. Furthermore, instead of considering the known and the unknown parts separately, we consider the summation of both parts, yielding less conservative results and decreasing computational burden.

The organisation of this chapter is as follows. Description of the system, definitions and problem formulation are presented in Section 5.2. Stability analysis and $\mathcal{H}_\infty$ dynamic output feedback controller design for a class of discrete-time NCSs with partially known transition probability matrix are presented in Section 5.3. Section 5.4 uses a DC motor servo example to illustrate the effectiveness of the presented methodology. Finally, conclusions of this chapter are presented in Section 5.5.

5.2 System Description and Definitions

Once again, the NCSs setup shown in Figure 2.1 shown in Chapter 2 is considered. In this chapter, it is assumed that not all state variables are measurable and available to the controller; and the measured output, $y(k)$, is transmitted via the network. The measured output, $y(k)$, is measured by the sensor and passed to the controller via the network where the signal will experience network-induced delays. A class of uncertain discrete-time linear systems under consideration is described by the following model:

$$
\begin{aligned}
x(k+1) &= [A+\Delta A(k)]x(k) + [B_1+\Delta B_1(k)]w(k) + [B_2+\Delta B_2(k)]u(k), \quad x(0)=0 \\
z(k) &= [C_1+\Delta C_1(k)]x(k) + [D_{11}+\Delta D_{11}(k)]w(k) + [D_{12}+\Delta D_{12}(k)]u(k) \\
y(k) &= C_2 x(k)
\end{aligned}
\tag{5.2.1}
$$

where $x(k) \in \Re^n, u(k) \in \Re^m, z(k) \in \Re^{m_1}, y(k) \in \Re^{m_2}$ are the state, input, controlled output and measured output, respectively and $w(k) \in \Re^{m_3}$ is the disturbance which belongs to $\mathcal{L}_2[0,\infty)$, the space of square summable vector sequence over $[0,\infty]$. The matrices A, B_1, B_2, C_1, D_{11}, D_{12} and C_2 are of known dimensions. As in the previous chapter, the matrix functions $\Delta A(k)$, $\Delta B_1(k)$, $\Delta B_2(k)$, $\Delta C_1(k)$, $\Delta D_{11}(k)$ and $\Delta D_{12}(k)$ represent the time-varying uncertainties in the system which satisfy the following assumption.

Assumption 5.2.1

$$
\begin{bmatrix} \Delta A(k) & \Delta B_1(k) & \Delta B_2(k) \\ \Delta C_1(k) & \Delta D_{11}(k) & \Delta D_{12}(k) \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} F(k) \begin{bmatrix} H_1 & H_2 & H_3 \end{bmatrix}
$$

where H_i and E_i are known matrices which characterize the structure of the uncertainties. Furthermore, there exists a positive-definite matrix $\mathcal{W}$ such that the following inequality holds:

$$
F^T(k)\mathcal{W}F(k) \leq \mathcal{W} \tag{5.2.2}
$$

The network is modelled by a finite state Markov chain with partially known transition probability matrix. Refer to the previous chapter for more information about the modelling procedure.

The aim of this chapter is to design a dynamic output feedback controller of the following form

$$
\begin{aligned}
\hat{x}(k+1) &= A_c(i)\hat{x}(k) + B_c(i)y(k-\tau_k) \\
u(k) &= C_c(i)\hat{x}(k)
\end{aligned}
\tag{5.2.3}
$$

where $\hat{x}(k)$ is the controller's state, $A_c(i), B_c(i)$ and $C_c(i)$ are the controller matrices to be determined.

The closed loop system of (5.2.1) with (5.2.3) is given as follows:

$$\begin{aligned} \zeta(k+1) &= [A_{cl}(i) + \bar{E}_1 F(k)\bar{H}_1(i)]\zeta(k) + B_{cl}(i)\bar{C}_2\zeta(k-\tau_k) + [\bar{B}_1 + \bar{E}_1 F(k) H_2]w(k) \\ z(k) &= [C_{cl}(i) + E_2 F(k)\bar{H}_1(i)]\zeta(k) + [D_{11} + E_2 F(k) H_2]w(k). \end{aligned} \tag{5.2.4}$$

where $\zeta(k)^T = \begin{bmatrix} x(k)^T & \hat{x}(k)^T \end{bmatrix}$,

$$A_{cl}(i) = \begin{bmatrix} A & B_2 C_c(i) \\ 0 & A_c(i) \end{bmatrix}, B_{cl}(i) = \begin{bmatrix} 0 \\ B_c(i) \end{bmatrix}, \bar{B}_1 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \bar{C}_2 = \begin{bmatrix} C_2 & 0 \end{bmatrix},$$
$$\bar{E}_1 = \begin{bmatrix} E_1 \\ 0 \end{bmatrix}, \bar{H}_1(i) = \begin{bmatrix} H_1 & H_3 C_c(i) \end{bmatrix}, C_{cl}(i) = \begin{bmatrix} C_1 & D_{12} C_c(i) \end{bmatrix}.$$

The $\mathcal{H}_\infty$ dynamic output feedback control problem of the NCSs is formulated as follows.

As shown in Chapter 2, the closed-loop system is to achieve stochastic stability, as shown in (2.2.5), and the $\mathcal{H}_\infty$ performance condition, as shown in (2.2.6).

5.3 Main Results

This section presents the stability criteria and the robust $\mathcal{H}_\infty$ dynamic output feedback controller design for a class of linear NCSs. The network is modelled by a finite state Markov chain whose transition probability matrix is allowed to be partially known. The controller computes the control signal based on the measured output of the plant, which experience delays and is subject to packet dropout as it is transmitted via the network.

The following theorem proposes stability criteria for the system shown in (5.2.1) with the controller in the form of (5.2.3) with partially known transition probabilities in the Markov chain.

Theorem 5.3.1 *For given controller gains $A_c(i)$, $B_c(i)$, $C_c(i)$, $i \in \mathcal{S}$, and $\gamma > 0$, if there exist sets of positive-definite matrices $P(i)$, $R_1(i)$, R_1, $R_2(i)$, R_2, $W_1(i)$, $W_2(i)$, Q, $Z(i)$*

and matrices $M(i)$, $\Omega_1(i)$, $\Omega_2(i)$, $\forall i \in \mathcal{S}$, satisfying the following inequalities

$$R_1 > R_1(i), \quad R_2 > R_2(i) \tag{5.3.1}$$

$$\Lambda(i)+\Gamma_2^T(i)\tau(s)R_2\Gamma_2(i)+\Upsilon_1(i)+\Upsilon_1^T(i)+\tau(i)Z(i)+\Xi^T(i)\Xi(i)+\Omega_1(i)+\big(1-p_{\mathcal{K}}^i\big)\Omega_2(i) < 0 \tag{5.3.2}$$

$$\Gamma_1^T(i)\tilde{P}_{\mathcal{K}}(i)\Gamma_1(i) + \Gamma_2^T(i)\tilde{\tau}_{\mathcal{K}}R_1\Gamma_2(i) - \Omega_1(i) < 0, \quad \forall j \in \mathcal{S}_{\mathcal{K}}^i \tag{5.3.3}$$

$$\Big(1 - p_{\mathcal{K}}^i\Big)\left[\Gamma_1^T(i)\sum_{j=1}^{i+1} P(j)\Gamma_1(i) + \Gamma_2^T(i)\sum_{j=1}^{i+1}\tau(j)R_1\Gamma_2(i)\right] - \big(1 - p_{\mathcal{K}}^i\big)\Omega_2(i) < 0, \ \forall j \in \mathcal{S}_{\mathcal{UK}}^i \tag{5.3.4}$$

and

$$\begin{bmatrix} (1 - p_{i(i+1)})R_1(i) + R_2(i) & M(i) \\ M^T(i) & Z(i) \end{bmatrix} \geq 0 \tag{5.3.5}$$

$$\begin{bmatrix} p_{\mathcal{K}}^i R_1(i) + R_2(i) & M(i) \\ M^T(i) & Z(i) \end{bmatrix} \geq 0, \quad p_{i(i+1)} \in \mathcal{S}_{\mathcal{UK}}^i \tag{5.3.6}$$

where

$$\begin{aligned}
\Gamma_1(i) &= \begin{bmatrix} A_{cl}(i) & B_{cl}(i)\bar{C}_2 & \bar{B}_1 & \bar{E}_1 & \bar{E}_1 \end{bmatrix} \\
\Xi(i) &= \begin{bmatrix} C_{cl} & 0 & D_{11} & E_2 & E_2 \end{bmatrix} \\
\Gamma_2(i) &= \begin{bmatrix} \bar{A} & 0 & \bar{B}_1 & \bar{E}_1 & \bar{E}_1 \end{bmatrix} \\
\Lambda(i) &= diag\Big\{\Big((\tau(s) - \tau(1) + 1)Q + \bar{H}_1^T(i)W_1(i)\bar{H}_1(i) - P(i)\Big), \\
&\qquad -Q, \Big(H_2^T W_2(i)H_2 - \gamma I\Big), -W_1(i), -W_2(i)\Big\} \\
\Upsilon_1 &= M^T[diag\{I, 0\} \ \ diag\{-I, 0\} \ \ 0 \ \ 0 \ \ 0] \\
\tilde{P}_{\mathcal{K}}(i) &= \textstyle\sum_{j\in\mathcal{S}_{\mathcal{K}}^i}^{i+1} p_{ij}P(j) \\
\tilde{\tau}_{\mathcal{K}}(i) &= \textstyle\sum_{j\in\mathcal{S}_{\mathcal{UK}}^i}^{i+1} p_{ij}\tau(j) \\
\bar{A} &= \begin{bmatrix} A - I & B_2C_c(i) \\ 0 & 0 \end{bmatrix}.
\end{aligned}$$

Then the closed-loop system is stochastically stable with the prescribed $\mathcal{H}_\infty$ performance.

Proof: The system (5.2.4) can be rewritten as

$$\begin{aligned} \zeta_{k+1} &= \Gamma_1(r_k)\tilde{\zeta}_k \\ z_k &= \Xi(r_k)\tilde{\zeta}_k \end{aligned} \tag{5.3.7}$$

where $\zeta_\ell = \zeta(\ell)$, $z_\ell = z(\ell)$, $\tilde{\zeta}_\ell = \tilde{\zeta}(\ell)$, and $\Gamma_1(r_k) = \Big[A_{cl}(i) \ \ B_{cl}(i)\bar{C}_2 \ \ \bar{B}_1 \ \ \bar{E}_1 \ \ \bar{E}_1\Big]$, and $\tilde{\zeta}(k)^T = \Big[\zeta^T(k) \ \ \zeta^T(k - \tau(r_k)) \ \ w^T(k) \ \ \zeta^T(k)\bar{H}_1^T(i)F^T(k) \ \ w^T(k)H_2^T F^T(k)\Big] \in \Re^l$.

The following Lyapunov-Krasovskii candidate functional is considered

$$V(\zeta_k, r_k) = V_1(\zeta_k, r_k) + V_2(\zeta_k, r_k) + V_3(\zeta_k, r_k) \tag{5.3.8}$$

with

$$V_1(\zeta_k, r_k) = \zeta_k^T P(r_k)\zeta_k \tag{5.3.9}$$

$$V_2(\zeta_k, r_k) = \sum_{\ell=-\tau_k}^{-1} \sum_{j=k+\ell}^{k-1} \bar{x}_j^T R_1 \bar{x}_j + \sum_{\ell=-\tau(s)}^{-1} \sum_{j=k+\ell}^{k-1} \bar{x}_j^T R_2 \bar{x}_j \tag{5.3.10}$$

$$V_3(\zeta_k, r_k) = \sum_{\ell=k-\tau_k}^{k-1} \zeta_\ell^T Q \zeta_\ell + \sum_{\ell=-\tau(s)+2}^{-\tau(1)+1} \sum_{j=k+\ell-1}^{k-1} \zeta_j^T Q \zeta_j \tag{5.3.11}$$

where $\bar{x}(k) = \begin{bmatrix} x(k+1) - x(k) \\ 0 \end{bmatrix}$.

Following a similar technique used in the proof of the stability analysis in Chapter 3 and 4, we obtain

$$\begin{aligned} \Delta V(x_k, r_k) \;\; \leq \;\; & \tilde{\zeta}_k^T \Big\{ \Lambda(r_k) + \Gamma_1^T \tilde{P}(r_k)\Gamma_1 + \Gamma_2^T \left[\tilde{\tau}_k R_1 + \tau(s) R_2\right] \Gamma_2 + \Upsilon_1(r_k) + \Upsilon_1^T(r_k) \\ & + \tau_k Z(r_k) + \Xi^T(i)\Xi(i) \Big\} \tilde{\zeta}_k - z_k^T z_k + \gamma w_k^T w_k \end{aligned} \tag{5.3.12}$$

which will be referred later on. Following the similar approach of handling the unknown transition probabilities shown in Chapter 3, we show that the controller satisfies the conditions shown in the problem formulation. ∇∇∇

The following theorem provides sufficient conditions for the existence of a robust partially mode delay-dependent $\mathcal{H}_\infty$ output feedback controller for the system (5.2.1) with partially known transition probabilities.

Theorem 5.3.2 *For a given $\gamma > 0$, if there exist sets of positive-definite matrices $X(i)$, $Y(i)$, $\mathcal{Y}(i)$, $\tilde{R}_1(i)$, $\tilde{R}_1$, $\tilde{R}_2(i)$, $\tilde{R}_2$, $W_1(i)$, $W_2(i)$, Q, $\mathcal{Q}$, $\tilde{W}_1(i)$, $\tilde{W}_2(i)$, N_1, N_2, $\tilde{Z}(i)$, $S(i,j)$ and matrices $\tilde{M}(i)$, $\tilde{\Omega}_1(i)$, $\tilde{\Omega}_2(i)$, $\mathcal{A}(i)$, $\mathcal{B}(i)$, $\mathcal{C}(i)$ and $J(i)$ for $i = 1, 2, \cdots, s$ satisfying the following inequalities*

$$\tilde{R}_1 > \tilde{R}_1(i), \;\; \tilde{R}_2 > \tilde{R}_2(i) \tag{5.3.13}$$

$$\begin{bmatrix} \bar{\Lambda}(i) & \sqrt{\tau(s)}\tilde{\Gamma}_2^T(i) & \tilde{\Gamma}_3^T(i) & \tilde{\Xi}^T(i) & \mathcal{H}^T(i) \\ * & -N_2 & 0 & 0 & 0 \\ * & * & -\mathcal{Q} & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -\tilde{W}_1(i) \end{bmatrix} < 0 \tag{5.3.14}$$

$$\begin{bmatrix} -\Omega_1(i) & p^i_{\mathcal{K}}\tilde{\Gamma}^T_1(i) & \sqrt{\sum_{j\in\mathcal{S}^i_{\mathcal{K}}} p_{ij}\tau(j)}\tilde{\Gamma}^T_2(i) \\ * & -\Phi_{\mathcal{K}}(i) & 0 \\ * & * & -p^i_{\mathcal{K}}N_1 \end{bmatrix} < 0, \quad \forall j\in\mathcal{S}^i_{\mathcal{K}} \tag{5.3.15}$$

$$\begin{bmatrix} -(1-p^i_{\mathcal{K}})\Omega_2(i) & (1-p^i_{\mathcal{K}})\tilde{\Gamma}^T_1(i) & \sqrt{(1-p^i_{\mathcal{K}})\sum_{j\in\mathcal{S}^i_{\mathcal{UK}}}\tau(j)}\tilde{\Gamma}^T_2(i) \\ * & -\Phi_{\mathcal{UK}}(i) & 0 \\ * & * & -(1-p^i_{\mathcal{K}})N_1 \end{bmatrix} < 0, \quad \forall j\in\mathcal{S}^i_{\mathcal{UK}} \tag{5.3.16}$$

$$\begin{bmatrix} S(i,j) & J^T(i) \\ * & Y(j) \end{bmatrix} > 0 \tag{5.3.17}$$

$$\begin{bmatrix} (1-p_{i(i+1)})\tilde{R}_1(i)+\tilde{R}_2(i) & \tilde{M}(i) \\ * & \tilde{Z}(i) \end{bmatrix} \geq 0 \tag{5.3.18}$$

$$\begin{bmatrix} p^i_{\mathcal{K}}\tilde{R}_1(i)+\tilde{R}_2(i) & \tilde{M}(i) \\ * & \tilde{Z}(i) \end{bmatrix} \geq 0, \quad \forall(i+1)\in\mathcal{S}^i_{\mathcal{UK}} \tag{5.3.19}$$

and

$$N_1\tilde{R}_1 = I,\ N_2\tilde{R}_2 = I,\ \tilde{W}_1(i)W_1(i) = I \ \text{and}\ \mathcal{Q}Q = I, \tag{5.3.20}$$

where

$$\begin{aligned}
\bar{\Lambda}(i) &= \tilde{\Lambda}(i)+\tilde{\Upsilon}_1(i)+\tilde{\Upsilon}^T_1(i)+\tau(i)\tilde{Z}(i)+\tilde{\Omega}_1(i)+(1-p^i_{\mathcal{K}})\tilde{\Omega}_2(i) \\
\tilde{\Lambda}(i) &= diag\Big\{-\begin{bmatrix} Y(i) & I \\ I & X(i)\end{bmatrix}, -Q, \Big(H^T_2W_2(i)H_2-\gamma I\Big), -W_1(i), -W_2(i)\Big\} \\
\tilde{\Gamma}_1(i) &= \begin{bmatrix} \check{A}_{cl}(i) & \check{B}_{cl}(i)\bar{C}_2(i) & \check{B}_1 & \check{E}_1 & \check{E}_1 \end{bmatrix} \\
\tilde{\Gamma}_2(i) &= \begin{bmatrix} \check{A}(i) & 0 & \bar{B}_1 & \bar{E}_1 & \bar{E}_1 \end{bmatrix} \\
\tilde{\Gamma}_3(i) &= \begin{bmatrix} \sqrt{(\tau(s)-\tau(1)+1)}T(i) & 0 & 0 & 0 & 0 \end{bmatrix} \\
\Phi_{\mathcal{K}} &= \begin{bmatrix} -\sum_{j=1}^{i+1}p_{ij}S(i,j)-J(i)-J^T(i) & p^i_{\mathcal{K}}I \\ p^i_{\mathcal{K}}I & \sum_{j=1}^{i+1}p_{ij}X(j) \end{bmatrix} \\
\Phi_{\mathcal{UK}} &= \begin{bmatrix} -\big(1-p^i_{\mathcal{K}}\big)^{-1}\big(\sum_{j=1}^{i+1}S(i,j)-J(i)-J^T(i)\big) & (1-p^i_{\mathcal{K}})I \\ (1-p^i_{\mathcal{K}})I & (1-p^i_{\mathcal{K}})\sum_{j=1}^{i+1}X(j) \end{bmatrix} \\
\tilde{\Xi}(i) &= \begin{bmatrix} \check{C}_{cl}(i) & 0 & D_{11} & E_2 & E_2 \end{bmatrix} \\
\mathcal{H} &= \begin{bmatrix} \check{H}_1(i) & 0 & 0 & 0 & 0 \end{bmatrix} \\
\tilde{\Upsilon}_1(i) &= \tilde{M}^T(i)[I \quad -I \ \ 0 \ \ 0 \ \ 0]
\end{aligned}$$

$$\breve{A}_{cl}(i) = \begin{bmatrix} AY(i) + B_2\mathcal{C}(i) & A \\ \mathcal{A}(i) & \sum_{j=1}^{i+1} \bar{p}_{ij} X(j) A \end{bmatrix}, \quad \breve{B}_1(i) = \begin{bmatrix} B_1 \\ \sum_{j=1}^{i+1} \bar{p}_{ij} X(j) B_1 \end{bmatrix}$$

$$\breve{A}(i) = \begin{bmatrix} (A-I)Y(i) + B_2\mathcal{C}(i) & A - I \\ 0 & 0 \end{bmatrix} \quad \breve{C}_{cl}(i) = \begin{bmatrix} C_1 Y(i) + D_{12}\mathcal{C}(i) & C_1 \end{bmatrix},$$

$$\breve{H}_1(i) = \begin{bmatrix} H_1 Y(i) + H_3\mathcal{C}(i) & H_1 \end{bmatrix}, \quad \breve{E}_1(i) = \begin{bmatrix} E_1 \\ \sum_{j=1}^{i+1} \bar{p}_{ij} X(j) E_1 \end{bmatrix},$$

$$\breve{B}_{cl}(i) = \begin{bmatrix} 0 \\ \mathcal{B}(i) \end{bmatrix}, \quad T(i) = \begin{bmatrix} Y(i) & I \\ Y(i) & 0 \end{bmatrix}$$

and

$$\begin{aligned} \sum_{j=1}^{i+1} \bar{p}_{ij} X(i) &= \sum_{j \in \mathcal{S}_{\mathcal{K}}^i}^{i+1} p_{ij} X(j) + (1 - p_{\mathcal{K}}^i) \sum_{j \in \mathcal{S}_{\mathcal{UK}}^i}^{i+1} X(j) \\ \sum_{j=1}^{i+1} \bar{p}_{ij} Y^{-1}(j) &= \sum_{j \in \mathcal{S}_{\mathcal{K}}^i}^{i+1} p_{ij} Y^{-1}(j) + (1 - p_{\mathcal{K}}^i) \sum_{j \in \mathcal{S}_{\mathcal{UK}}^i}^{i+1} Y^{-1}(j). \end{aligned}$$

Then the closed-loop system is stochastically stable with the prescribed $\mathcal{H}_\infty$ performance. Furthermore, the controller is given as follows:

$$\begin{aligned} A_c(i) &= \Big(\sum_{j=1}^{i+1} \bar{p}_{ij} Y^{-1}(j) - \sum_{j=1}^{i+1} \bar{p}_{ij} X(j) \Big)^{-1} \Big(\mathcal{A}(i) - \sum_{j=1}^{i+1} \bar{p}_{ij} X(j) \\ &\quad (AY(i) + B_2\mathcal{C}(i)) \Big) Y^{-1}(i) \\ B_c(i) &= \Big(\sum_{j=1}^{i+1} \bar{p}_{ij} Y^{-1}(j) - \sum_{j=1}^{i+1} \bar{p}_{ij} X(j) \Big)^{-1} \mathcal{B}(i) \\ C_c(i) &= \mathcal{C}(i) Y^{-1}(i). \end{aligned} \tag{5.3.21}$$

Proof: We obtain the controller design by applying Schur complement on the terms within the curly bracket of (5.3.12). We then have

$$\begin{bmatrix} \hat{\Lambda}(i) & \bar{\Gamma}_1^T(i) & \bar{\Gamma}_2^T(i) & \bar{\Gamma}_3^T & \bar{\Xi}^T(i) & \bar{\mathcal{H}}^T(i) \\ * & -\tilde{P}^{-1}(i) & 0 & 0 & 0 & 0 \\ * & * & -(\tilde{\tau}_k R_1 + \tau(s) R_2)^{-1} & 0 & 0 & 0 \\ * & * & * & -Q^{-1} & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -W_1^{-1} \end{bmatrix} < 0 \tag{5.3.22}$$

where

$$\begin{aligned}
\hat{\Lambda}(i) &= \text{diag}\Big\{-P(i), -Q, (H_2^T W_3(i) H_2 - \gamma I), -W_1(i), -W_2(i)\Big\} \\
&\quad + \Upsilon_1(i) + \Upsilon_1^T(i) + \tau(i) Z(i) \\
\bar{\Gamma}_1(i) &= \begin{bmatrix} A_{cl}(i) & B_{cl}(i)\bar{C}_2 & \bar{B}_1 & \bar{E}_1 & \bar{E}_1 \end{bmatrix} \\
\bar{\Gamma}_2(i) &= \begin{bmatrix} \bar{A} & 0 & \bar{B}_1 & \bar{E}_1 & \bar{E}_1 \end{bmatrix} \\
\bar{\Gamma}_3 &= \begin{bmatrix} \sqrt{(\tau(s) - \tau(1) + 1)} & 0 & 0 & 0 & 0 \end{bmatrix} \\
\bar{\Xi}(i) &= \begin{bmatrix} C_{cl}(i) & 0 & D_{11} & E_2 & E_2 \end{bmatrix} \\
\bar{\mathcal{H}}^T(i) &= \begin{bmatrix} \bar{H}_1(i) & 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

Similar approach shown in the proof section of Chapter 4, can be used to prove the rest of the theorem. ∇∇∇

A similar iterative algorithm to what is shown in Chapter 4 can be used to obtain the dynamic output feedback controller.

5.4 Example

The DC motor servo system presented in Chapter 4 is used to illustrate the effectiveness of the proposed methodology.

The attenuation level, γ, is selected as 1.0. Applying Theorem 5.3.2 and the algorithm in the previous section, we obtain the controller matrices as follows:

$$\begin{aligned}
A_c(1) &= \begin{bmatrix} -0.7614 & 0.0633 \\ -0.1202 & -0.1269 \end{bmatrix} \quad B_c(1) = \begin{bmatrix} 0.0074 \\ -0.0216 \end{bmatrix} \\
C_c(1) &= \begin{bmatrix} -0.8794 & -0.0663 \end{bmatrix} \\
A_c(2) &= \begin{bmatrix} -0.2395 & 0.0614 \\ -0.1057 & -0.1985 \end{bmatrix} \quad B_c(2) = \begin{bmatrix} 0.0052 \\ -0.0333 \end{bmatrix} \\
C_c(2) &= \begin{bmatrix} -0.6123 & -0.0672 \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
A_c(3) &= \begin{bmatrix} 0.0788 & 0.0587 \\ -0.0941 & -0.1545 \end{bmatrix} \quad B_c(3) = \begin{bmatrix} 0.0052 \\ -0.0121 \end{bmatrix} \\
C_c(3) &= \begin{bmatrix} -0.4491 & -0.0687 \end{bmatrix} \\
A_c(4) &= \begin{bmatrix} 0.0769 & 0.0584 \\ -0.0913 & -0.1802 \end{bmatrix} \quad B_c(4) = \begin{bmatrix} 0.0055 \\ -0.0127 \end{bmatrix} \\
C_c(4) &= \begin{bmatrix} -0.4503 & -0.0688 \end{bmatrix}
\end{aligned}
$$

State response of this example is shown in Figure 5.1 with $w = 0$. The initial states are chosen to be $x(0) = [1.0 \quad 0]^T$. It is shown in Figure 5.1 that the controller obtained above stabilize the system, thus demonstrating the validity of the proposed methodology. Figure 5.2 shows the ratio of energy of the controlled output to the energy of the disturbance ($w(k) = e^{-0.1k}\sin(0.5k)$). The ratio is approximately equal to 0.65, which is less than the prescribed level $\gamma = 1.0$. The change of modes in this example is shown in Figure 5.3

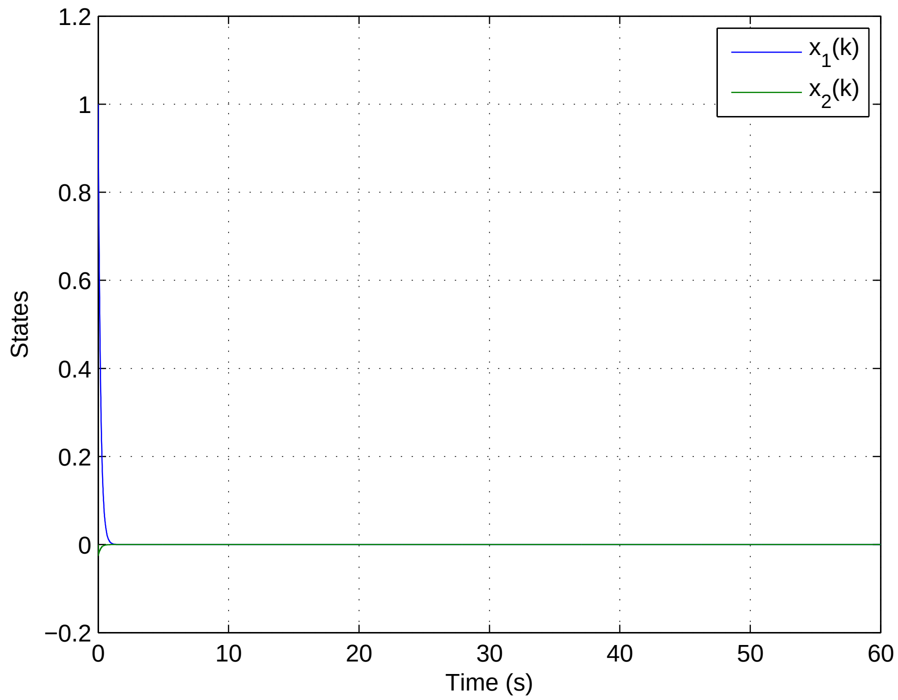

Figure 5.1: State Response

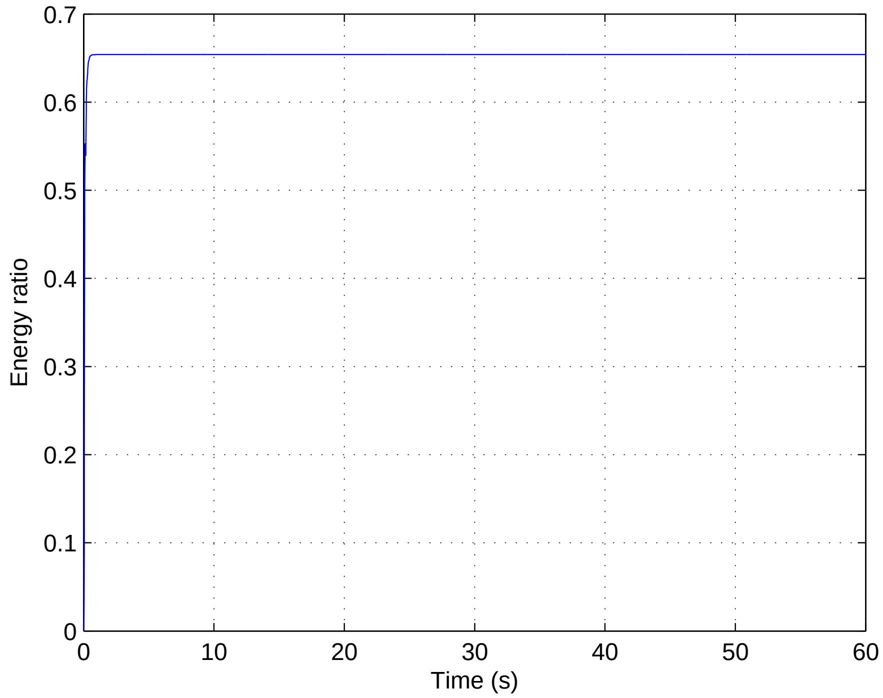

Figure 5.2: Ratio of energy of the controlled output to the energy of the disturbance $(\gamma = 1.0)$

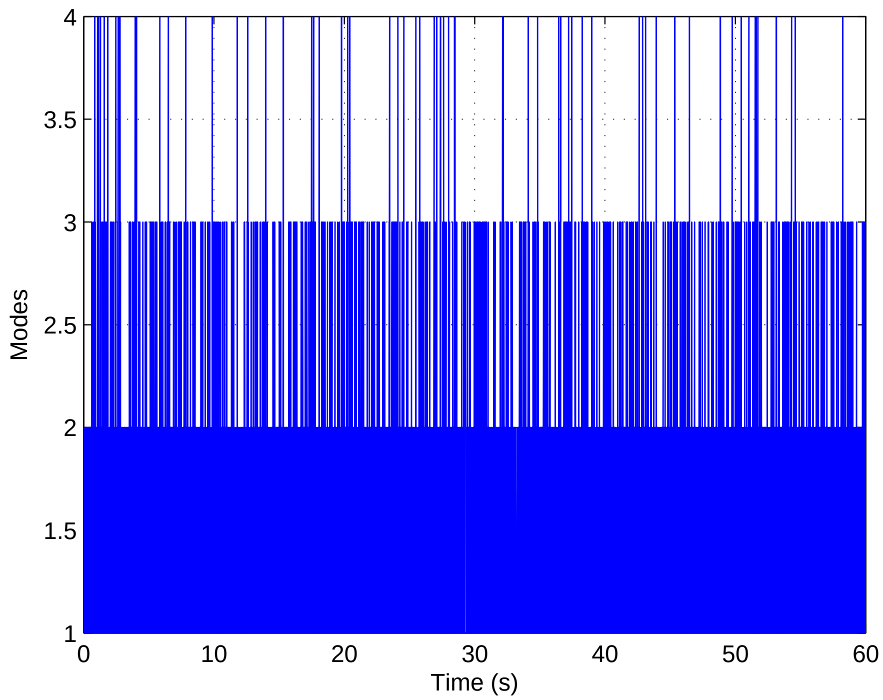

Figure 5.3: Mode transitions

5.5 Conclusions

In this chapter, stability criteria and partially mode delay dependent $\mathcal{H}_\infty$ dynamic output feedback controller are developed for a class of networked control systems. Random network-induced delays are modelled by Markov processes with partially known transition probability matrix. Conditions for stochastic stability with a given attenuation gain are derived by using Lyapunov-Krasovskii functional. By using a DC servo simulation example, it is shown that the proposed methodology meets the performance conditions.

6

Takagi-Sugeno Fuzzy Model and Sum-of-squares Decomposition

Abstract

This chapter provides an overview of Takagi-Sugeno fuzzy model, which will be used in the remainder of this thesis to model nonlinear networked control systems. Modelling procedure of a single-link rigid robot using a T-S fuzzy model is presented, which will be used as a numerical example in the subsequent chapters of this thesis. A brief overview of sum-of-squares decomposition is also presented in this section, which will be adopted to incorporate membership functions into the fuzzy controller/filter design.

6.1 Introduction

Study of nonlinear system is crucial as systems in the real world contain nonlinearities to a certain degree. However, due to its mathematical complexities, studies on nonlinear systems is not as comprehensive to its linear counterpart. The traditional approach to designing a controller for nonlinear systems is to linearize the plant around its point of operation and design a controller based on the linear plant from numerous design methodologies. While this approach has been proven to be useful in ideal condition due to its mathematical simplicity, the accuracy of the model degrades as the system moves further from the point of operation. Therefore, the system may become unstable as the system deviates from the region of operation. Furthermore, this approach is not possible if the system model is not linearizable. These drawbacks motivate researchers to investigate designing a controller for nonlinear systems where one no longer is restricted to the point of operation.

Takagi-Sugeno (T-S) fuzzy model has been widely used to model nonlinear plants in control systems ever since its proposition in [101]. The strength of this approach lies in the fact that the local dynamics of each fuzzy implication (rule) is expressed by a linear system model. The overall fuzzy model of the nonlinear system is obtained by fuzzy "blending" of the linear system models. By doing so, one may take an advantage of abundance of linear control theory and apply them to nonlinear control systems. Even though the nonlinear plant is modelled by local linear system models, it has been proven that T-S fuzzy model provides universal approximators of any smooth nonlinear system [119, 120].

Section 6.2 describes how a nonlinear plant is modelled using T-S fuzzy model. A T-S fuzzy model of a single-link rigid robot is also presented in this section. An overview of T-S fuzzy controller is given in Section 6.3. Sum-of-squares decomposition is briefly explained in Section 6.4 to provide the fundamental knowledge to understand how it is used to incorporate membership functions into the controller/filter design methodology in later chapters.

6.2 Takagi-Sugeno Fuzzy Model Construction

A fuzzy model uses fuzzy rules, which are linguistic IF-THEN statements involving fuzzy sets, fuzzy logic and fuzzy inference. This plays the key role in linking the input variables of the models to the output variables using linguistic information between these two. The gth rule of the T-S fuzzy models are of the following form:

Plant Rule g:
IF $\theta_1(x(k))$ is J_1^g AND $\cdots$ AND $\theta_p(x(k))$ is J_p^g,

THEN

$$\begin{aligned} x(k+1) &= A_g x(k) + B_g u(k) \\ y(k) &= C_g x(k) + D_g u(k) \end{aligned} \tag{6.2.1}$$

where g denotes the g^{th} fuzzy inference rule; $g = 1, \dots, r$; r is the number of inference rules; $\theta_1(x(k)), \dots, \theta_p(x(k))$ are the premise variables; p is the number of premise variables and $J_1^g, \dots, J_p^g$ are the fuzzy terms. Furthermore $x(k) \in \Re^n$, $u(k) \in \Re^{m_1}$, $y(k) \in \Re^{m_2}$ are the state, input and output respectively. The matrices A_g, B_g, C_g, and D_g are of appropriate dimensions.

These local linear system models are then integrated into a global nonlinear model by using a center-average defuzzifier, product inference and singleton fuzzifier:

$$\begin{aligned} x(k+1) &= \sum_{g=1}^{r} \mu_g(\theta(x(k)))\big\{A_g x(k) + B_g u(k)\big\} \\ y(k) &= \sum_{g=1}^{r} \mu_g(\theta(x(k)))\big\{C_g x(k) + D_g u(k)\big\} \end{aligned} \tag{6.2.2}$$

where

$$\theta(x(k)) = [\theta_1(x(k)), \dots, \theta_p(x(k))],$$

$$\chi_g(\theta(x(k))) = \prod_{t=1}^{p} J_t^g(\theta_t(x(k))),$$

$$\mu_g(\theta(x(k)) = \frac{\chi_g(\theta(x(k)))}{\sum_{\ell=1}^{r} \chi_\ell(\theta(x(k)))} \in [0,1],$$

$$\sum_{g=1}^{r} \mu_g(\theta(x(k))) = 1$$

Figure 6.1 shows illustrates the T-S fuzzy model using an illustration. It is shown that the global input-output nonlinear relationship is expressed in terms of weighted summation of linear system model.

The fuzzy "blending" of linear system models are achieved by $\mu_g(\theta(x(k)))$ terms or the "weighted" blocks in Figure 6.1. These terms are called membership functions and they play a vital role in the overall nonlinear model. Even though the local linear system models are the same, different membership functions result different input-output relationship of the global nonlinear model. However, many of the existing literature

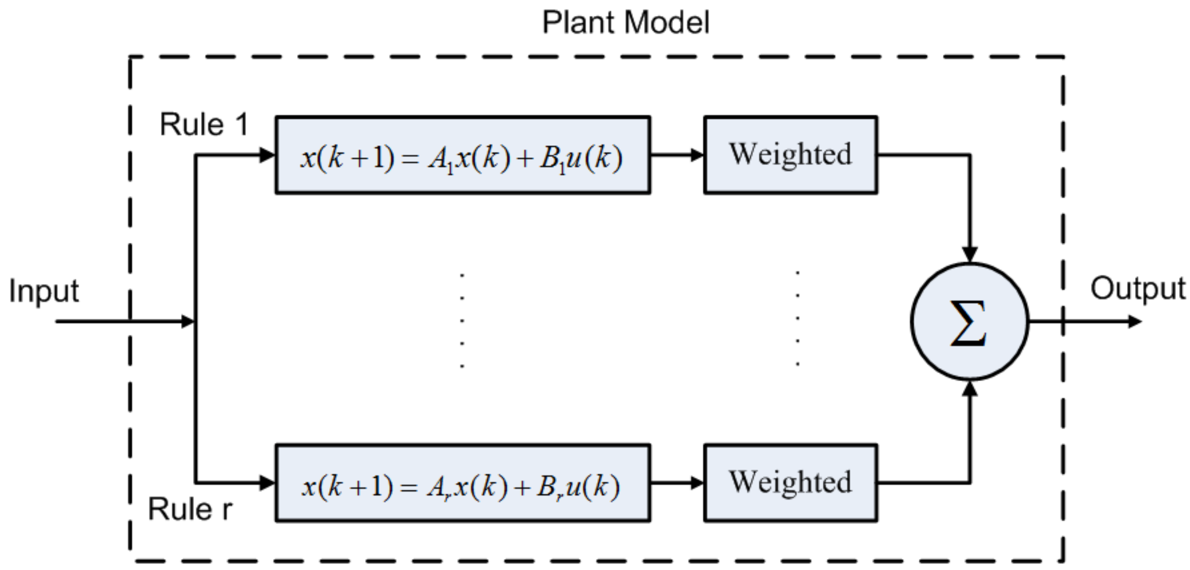

Figure 6.1: Illustration of a T-S fuzzy model

on nonlinear control systems based on T-S fuzzy model, such as [36, 41, 58, 66, 67, 84–91], do not consider membership functions in the controller design. This is because membership functions are nonlinear functions in $x(k)$ therefore linear matrix inequalities (LMIs) approach cannot be used. This results in the controller being valid for any shape of membership functions, thus leading to conservatism.

There are two approaches of constructing T-S fuzzy models:

1. Fuzzy modelling (identification) using input-output data
2. Derivation from given nonlinear system equations.

The latter uses the idea of sector nonlinearity and this allows to obtain T-S fuzzy model easily when nonlinear dynamical models for the systems can be obtained. In this section, the second approach is used to obtain a T-S fuzzy model of a single-link rigid robot. The following is the modelling procedure where the T-S model will be used as a numerical example in the following chapters of this thesis.

The motion equation of the single-link rigid robot system is shown as

$$J\ddot{\theta} = -(0.5mgl + Mgl)sin(\theta) + u \tag{6.2.3}$$

where θ denote the joint rotation angle in radians, m is the mass of the load, M is the mass of the rigid link, $g = 9.8m/s^2$ is the gravity constant, l is the length of the robot link, $J = Ml^2 + (1/3)ml^2$ is the moment of inertia, and u is the control torque applied at the joint in Nm. $\theta = 0$ denotes the lowest vertical equilibrium position under zero control torque. The control task is to move the robot arm from any initial state $\theta \in [-(\pi/2),\ (\pi/2)]$ to the equilibrium position defined by $\theta = 0$, $\dot{\theta} = 0$, and $\ddot{\theta} = 0$ despite of perturbations of plant parameters.

In order to obtain a T-S fuzzy model, new state variables are defined as $x_1 = \theta$, $x_2 = \dot{\theta}$.

When x_1 is near zero, the nonlinear equation above can be simplified with the new state variables as

$$\dot{x_1} = x_2 \tag{6.2.4}$$

$$\dot{x_2} = -\frac{(0.5mgl + Mgl)}{J} + \frac{1}{J}u \tag{6.2.5}$$

When x_1 is near $\pm\pi/2$, the nonlinear equations are simplified as

$$\dot{x_1} = x_2 \tag{6.2.6}$$

$$\dot{x_2} = -\frac{2(0.5mgl + Mgl)}{\pi J} + \frac{1}{J}u \tag{6.2.7}$$

Note that (6.2.4)-(6.2.7) are now linear system models. Based on the above, the following T-S fuzzy model is obtained.

Model Rule 1:
IF x_1 is about 0, THEN $\dot{x} = A_1x + B_1u$

Model Rule 2:
IF x_1 is about $\pm\pi/2$, THEN $\dot{x} = A_2x + B_2u$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ -\frac{(0.5mgl+Mgl)}{J} & 0 \end{bmatrix},\quad A_2 = \begin{bmatrix} 0 & 1 \\ -\frac{2(0.5mgl+Mgl)}{\pi J} & 0 \end{bmatrix}$$
$$B_1 = B_2 = \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix} \tag{6.2.8}$$

The membership functions for Rule 1 and Rule 2 are shown in Figure 6.2.

Discrete-time T-S fuzzy model of the above with $m = 1.5kg$, $M = 3kg$, $l = 0.5m$,

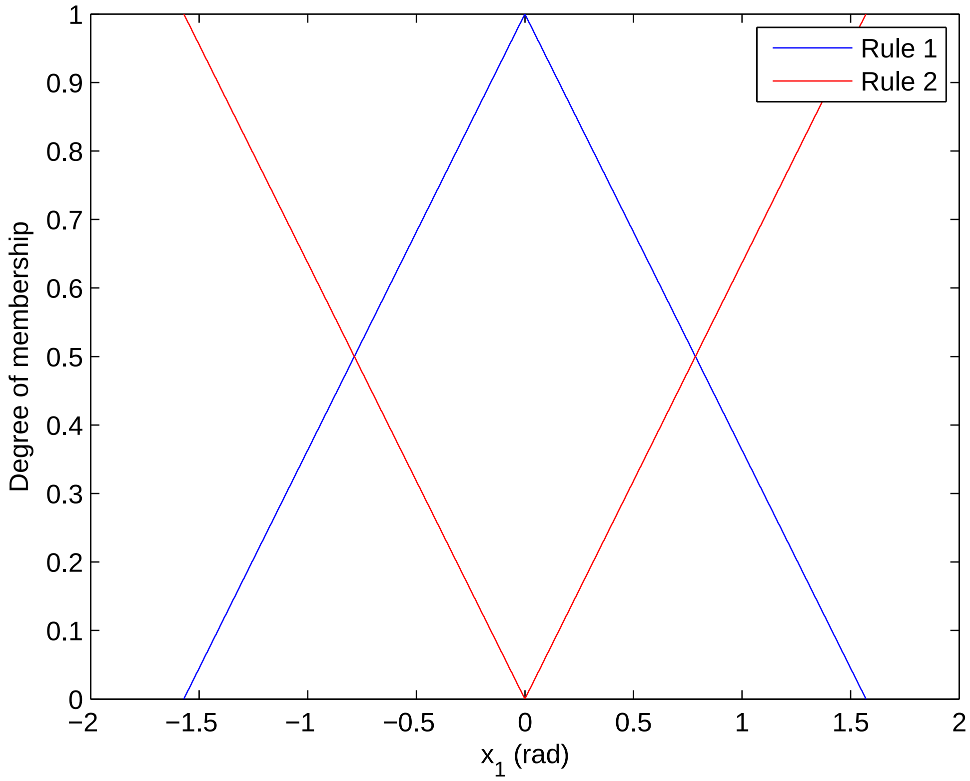

Figure 6.2: Membership functions of single-link rigid robot example

and $J = 0.875kgm^2$ with $T_s = 0.01s$ is obtained as follows

$$A_1 = \begin{bmatrix} 0.0990 & 0.0100 \\ -0.2099 & 0.9990 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.0993 & 0.0100 \\ -0.1337 & 0.9993 \end{bmatrix}$$
$$B_1 = B_2 = \begin{bmatrix} 5.7133 \times 10^{-5} \\ 0.0114 \end{bmatrix} \tag{6.2.9}$$

where the state variables are $x^T(k) = [x_1^T(k) \quad x_2^T(k)]$ and other variables are also in discrete-time domain.

6.3 Takagi-Sugeno Fuzzy Controller Construction

The concept of T-S fuzzy controller is very similar to T-S fuzzy model as shown in Figure 6.3. Figure 6.3 illustrates a T-S fuzzy state feedback controller where the state variables of the plant is multiplied by controller gains. A nonlinear controller is created through fuzzy blending of local linear controllers.

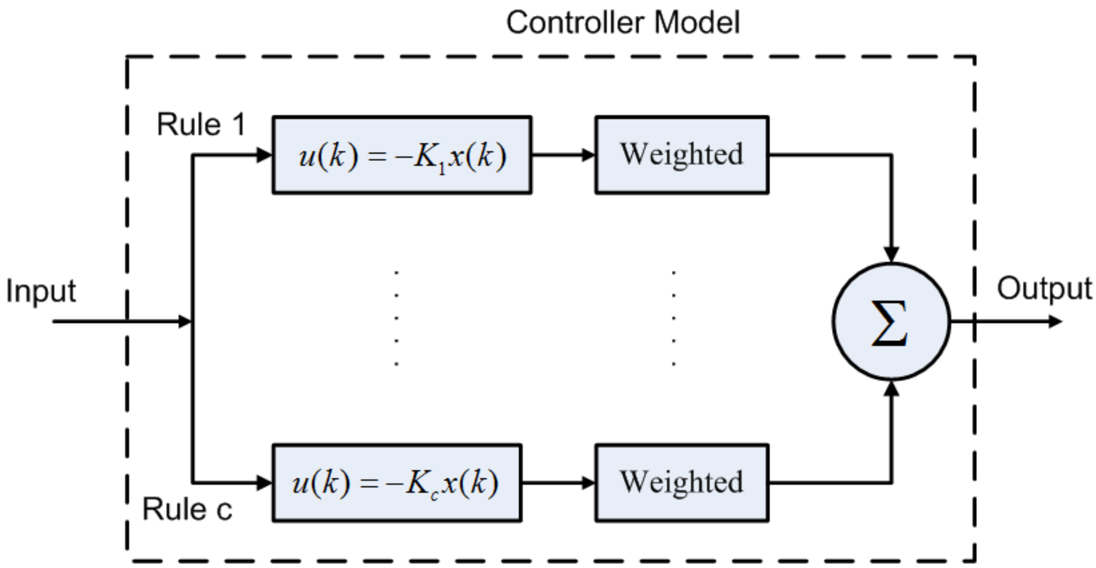

Figure 6.3: Illustration of a T-S fuzzy controller

In NCSs, information within a system is transmitted via a network and inevitably experience network-induced delays as shown in Figure 6.4. This means that when the state variables are measured by the sensor and transmitted to the controller, the signal will experience network-induced delays, τ_k. Hence the input of the fuzzy controller is the time delayed version of the premise variables, which often are the state variables, of the plant. Most of existing literature on nonlinear NCSs based on T-S fuzzy model neglects this and formulate the problem incorrectly [36, 67, 85, 87, 88]. Based on this information, the fuzzy state feedback controller of NCSs is shown as follows:

Control Rule h:
IF $\sigma_1(x(k-\tau_k))$ is N_1^h AND $\cdots$ AND $\sigma_q(x(k-\tau_k))$ is N_q^h,
THEN

$$u(k) = F_i x(k-\tau_k) \tag{6.3.1}$$

where h denotes the h^{th} fuzzy inference rule; $h = 1, \ldots, c$; c is the number of inference rules; $\sigma_1(x(k-\tau_k)), \ldots, \sigma_q(x(k-\tau_k))$, are the premise variables; q is the number of premise variables and $N_1^h, \ldots, N_q^h$ are the fuzzy terms.

Similar to the plant, the fuzzy controller is inferred as shown below.

$$u(k) = \sum_{h=1}^{c} \lambda_h(\sigma(x(k-\tau_k)))F_i x(k-\tau_k) \tag{6.3.2}$$

where

$$\sigma(x(k-\tau_k)) = [\sigma_1(x(k-\tau_k)), \ldots, \sigma_q(x(k-\tau_k))],$$

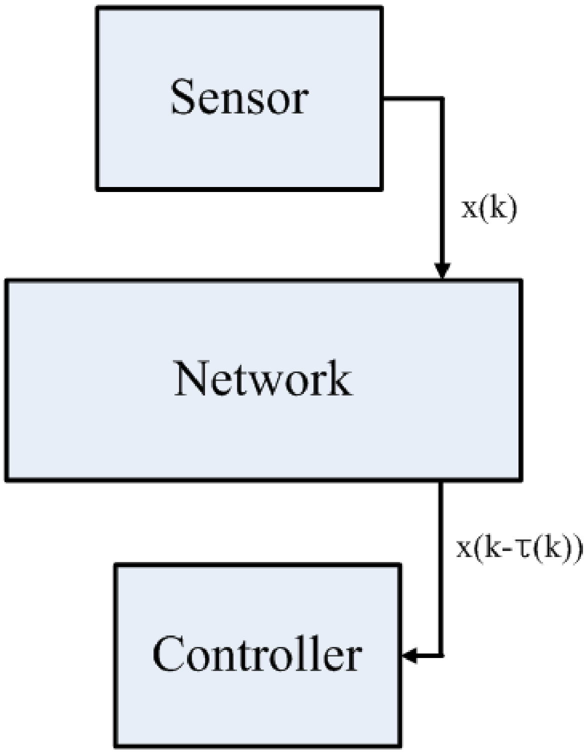

Figure 6.4: Illustration of premise variables of T-S fuzzy model in NCSs

$$\phi_h(\sigma(x(k-\tau_k))) = \prod_{t=1}^{q} N_t^h(\sigma_t(x(k-\tau_k))),$$

$$\lambda_h(\sigma(x(k-\tau_k)) = \frac{\phi_h(\sigma(x(k-\tau_k)))}{\sum_{\ell=1}^{c}\phi_\ell(\sigma(x(k-\tau_k)))} \in [0,1],$$

$$\sum_{h=1}^{c}\lambda_h(\sigma(x(k-\tau_k))) = 1$$

Note that the control action is based on the past information of the plant, in this case $x(k-\tau_k)$. Through the fuzzy blending, the overall controller is nonlinear.

6.4 Overview of Sum-of-squares Decomposition

As explained earlier, many existing literature on nonlinear control based on T-S fuzzy model discards membership functions when designing a controller [36, 67, 85, 87, 88]. This is because the membership functions are nonlinear functions in x therefore LMI approach cannot be used. In order to circumvent this limitation, sum-of-square decomposition approach is exploited in this thesis. This section provides a brief overview of sum-of-squares decomposition and explains how this approach can be used to incorporate membership functions into the controller design.

Sum-of-squares decomposition of multivariate polynomials using semidefinite programming has been attracting interest due to the fact that it provides convex relaxations for many hard problems [121–125]. Due to this recent interest and the advantages of the approach, several toolbox such as YALMIP [126] and SOSTOOLS [127] have been developed. In this section, a brief overview of sum-of-squares problem is presented.

As the name suggests, a multivariate polynomial $p(x_1, \ldots, x_n) \triangleq p(x)$ is a sum-of-squares (SOS), if there exist polynomials $f_1(x), \ldots, f_m(x)$ such that

$$p(x) = \sum_{i=1}^{m} f_i^2(x) \tag{6.4.1}$$

The above implies that $f(x) \geq 0$ for all $x \in \Re^n$. Above is equivalent to the existence of a positive semidefinite matrix Q such that

$$p(x) = v^T(x)Qv(x) \tag{6.4.2}$$

Therefore if one can find a vector of monomials $v(x)$ and a positive semidefinite matrix Q, nonnegativity is ensured. This turns the sum-of-squares programme into a semidefinite programme. By using this, SOS approach has been widely used for checking nonnegativity condition. The above process shown in (6.4.2) is known as sum-of-squares decomposition. This approach has been widely used as sum-of-squares decomposition of a multivariate polynomial has less computational burden than guarantying nonnegativity of polynomials.

It is noteworthy to point out that sum-of-squares provides sufficient conditions for nonnegativity. This is because sum-of-squares decomposable polynomial is a subset of nonnegative polynomial. What this means is that all sum-of-squares decomposable polynomials are nonnegative **but not all nonnegative polynomials are sum-of-squares decomposable**. However, as already explained, proving that a polynomial is sum-of-squares is easier than proving a polynomial is nonnegative.

Since SOS deals with polynomial functions, membership functions can be incorporated into the theorem by approximating membership functions with sum-of-squares decomposable polynomial functions so that toolbox such as YALMIP can handle the polynomial functions. Refer to the next chapter for more information of how SOS approach is used to incorporate membership functions into the controller design. Previous approach where membership functions are discarded results a controller based on any shape of membership functions. By incorporating the membership functions, it is ensured that the controller is specific for the membership function given. As it is shown in the next chapters, the incorporation of membership functions results larger stability region than the approaches where membership functions are discarded.

7

Robust Fuzzy $\mathcal{H}_\infty$ State Feedback Control of Nonlinear Networked Control Systems With Completely Known Transition Probability Matrix

Abstract

The chapter presents a methodology for designing a robust fuzzy $\mathcal{H}_\infty$ state feedback controller for nonlinear discrete-time networked control systems (NCSs). The nonlinear systems are modelled by a Takagi-Sugeno fuzzy model, and network induced delays between sensors and controllers are modelled by a finite state Markov process. The fuzzy controller's membership functions are allowed to be different from the plant's membership functions to accommodate the fact that the plant's premise variables are transmitted via the network. The membership functions of the plant and the fuzzy controller are then approximated by polynomial functions and incorporated into the controller design. Based on Lyapunov-Krasovskii functional, sufficient conditions for the existence of the controller are provided. Numerical examples are used to illustrate the effectiveness of the presented methodology and to confirm that incorporating

membership functions yield larger stabilization region.

7.1 Introduction

In a traditional point-to-point architecture, the system components such as the plant and the controllers are connected via physical wires. In this architecture, it is assumed that the signals experience no delay within the system, which simplified stability analysis and controller design methodology. Because the controller is designed with an assumption that the sensor measurements are received as soon as it becomes available, the physical distance between the controller and the plant has to be small. Furthermore the overall system cannot be modularized, which results high installation and maintenance cost. With recent advance in communication systems, hybrid systems whereby the system components are connected via a network, named networked control systems, have been receiving significant attention in the last decade [1–8]. However, the majority of existing literature are based on linear NCSs with a small portion of studies dedicated to nonlinear NCSs. Ever since its proposition in [101] Takagi-Sugeno (T-S) fuzzy model has been proven to be very effective in modelling nonlinear systems. It has been proven that modelling nonlinear NCS with T-S fuzzy model is just as effective [36, 58, 85, 86, 128–130].

In [36],the effects of network induced delay and packet dropout on a class of nonlinear NCSs are investigated. In [58], two separate network induced delays, from the sensor to the controller can from the controller to the actuator, are considered. These delays are modelled by two Markov chains and sufficient conditions for a state feedback fuzzy controller are presented. In [85], the authors propose a general framework of networked control where the zero-order hold can choose the latest control signal when packet is received out of order and consider both network induced delays and packet dropout. [86] considers $\mathcal{H}_\infty$ control of uncertain nonlinear NCSs modelled by T-S fuzzy model where the plant states are quantized. Sufficient conditions for the solvability of the robust $\mathcal{H}_\infty$ control problems without quantizers are also presented. [128] develops a robust control scheme based on a fuzzy estimator, which estimates the plant states. The network is modelled as a sampler between the plant and the controller in their control scheme. In [129, 130] time-driven sensors and event-driven actuators are considered when designing an $\mathcal{H}_\infty$ controller. However the authors assume that the network induced delays are less than one sampling period and disregard the effect of the network induced delays. The major drawback of the aforementioned papers [36, 58, 85, 86, 128–130]. is that the controller's premise variables are assumed to be the same as the plant's premise variables without delays, i.e. the premise variables experience no delay when transmitted through the network. However in NCSs, where there is a network between the system components, such assumption is very impractical. Hence, the existing controller designs in NCSs [36, 58, 85, 86, 128–130], where the controller's premise variables do not contain delays, are not practical.

It is important to point out that none of the aforementioned NCS papers [36, 58, 85, 86, 128–130]. include membership functions in the controller design. The major disadvantage of disregarding membership functions is that the obtained controller is valid for any membership functions, leading to severe conservatism. The importance of membership functions in the controller design has been illustrated in [93–95]. The difficulties of incorporating membership functions into the controller design arises from the fact that membership functions are nonlinear so controller design can no longer be expressed in linear matrix inequality (LMI) conditions.

Motivated by the aforementioned drawbacks, this chapter aims to design a state feedback controller for nonlinear NCSs modelled by T-S fuzzy model, where membership functions and delays in premise variables are incorporated in the controller design. Since membership functions are incorporated into the controller design, formulating the problem with proper premise variables becomes very important. Unlike existing approaches in [36, 58, 85, 86, 128–130]. where impractical assumption of same premise variables in NCSs is made, we acknowledge the issue and formulate the problem where the premise variable of the controller is the delayed version of the premise variable of the plant. To the best of authors' knowledge this is the first attempt in nonlinear NCSs to incorporate membership functions and delayed premise variables into the controller design. The membership functions of the plant and controller are approximated by polynomial functions and incorporated into the controller design. A numerical example is used to compare the proposed methodology with [86], where the controller design is derived without incorporating membership functions and without delays in the controller's premise variables. We show that the proposed methodology has a wider stabilization region than [86].

The network-induced delays between sensors and controllers are modelled by a finite state Markov chain. The current mode of the Markov chain determines the individual local controller, resulting mode dependent controllers. The transition probability matrix of the Markov chain in this chapter is assumed to be fully known. This chapter is the counterpart of Chapter 2 and will be used as the basis for the robust fuzzy $\mathcal{H}_\infty$ filter and the robust fuzzy $\mathcal{H}_\infty$ dynamic output feedback controller design, which will be presented in the next chapter.

The main contributions of this chapter are:

- Membership functions are incorporated into the controller design to reduce the conservatism of the controller design for nonlinear NCSs
- The controller's membership functions are allowed to be different from the plant's membership functions and its premise variables are allowed to have delays

The rest of the chapter is organised as follows. Section 7.2 presents the T-S fuzzy model of the plant and the controller. The $\mathcal{H}_\infty$ control problem is formulated in this section. The approach to approximate the membership functions to incorporate them into the controller design is presented in Section 7.3. Sufficient conditions for the existence of the robust fuzzy $\mathcal{H}_\infty$ state feedback controller are presented in this section as well. Numerical examples are provided in Section 7.4 to show that incorporating membership functions yield larger stabilization region and that the overall system performance meets the requirement. Section 7.5 present the conclusions of this chapter.

7.2 System Description and Definitions

The NCSs setup shown in Figure 2.1 is considered in this chapter. The class of nonlinear discrete systems under consideration is described by the following fuzzy system model:

Plant Rule g:
IF $\theta_1(x(k))$ is J_1^g AND $\cdots$ AND $\theta_p(x(k))$ is J_p^g,
THEN

$$\begin{aligned} x(k+1) &= [A_g + \Delta A_g(k)]x(k) + [B_{1g} + \Delta B_{1g}(k)]w(k) + [B_{2g} + \Delta B_{2g}(k)]u(k) \\ z(k) &= [C_g + \Delta C_g(k)]x(k) + [D_{1g} + \Delta D_{1g}(k)]w(k) + [D_{2g} + \Delta D_{2g}(k)]u(k) \end{aligned} \tag{7.2.1}$$

where g denotes the g^{th} fuzzy inference rule; $g = 1, \ldots, r$; r is the number of inference rules; $\theta_1(x(k)), \ldots, \theta_p(x(k))$ are the premise variables; p is the number of premise variables and $J_1^g, \ldots, J_p^g$ are the fuzzy terms. Furthermore $x(k) \in \Re^n$, $u(k) \in \Re^{m_1}$, $z(k) \in \Re^{m_2}$ are the state, input and output respectively and $w(k) \in \Re^{m_3}$ is the disturbance which belongs to $\mathcal{L}_2[0, \infty)$, the space of square summable vector sequence over $[0, \infty]$. The matrices A_g, B_{1g}, B_{2g}, C_g, D_{1g} and D_{2g} are of appropriate dimensions. The matrix functions $\Delta A_g(k)$, $\Delta B_{1g}(k)$, $\Delta B_{2g}(k)$, $\Delta C_g(k)$, $\Delta D_{1g}(k)$ and $\Delta D_{2g}(k)$ represent the time-varying uncertainties in the system.

By using a center-average defuzzifier, product inference and singleton fuzzifier, the following global nonlinear model is obtained.

$$
\begin{aligned}
x(k+1) &= \sum_{g=1}^{r} \mu_g(\theta(x(k)))\big\{[A_g + \Delta A_g(k)]x(k) + [B_{1g} + \Delta B_{1g}(k)]w(k) \\
&\quad + [B_{2g} + \Delta B_{2g}(k)]u(k)\big\} \\
&= [A(\mu) + \Delta A(\mu,k)]x(k) + [B_1(\mu) + \Delta B_1(\mu,k)]w(k) + [B_2(\mu) + \Delta B_2(\mu,k)]u(k) \\
z(k) &= \sum_{g=1}^{r} \mu_g(\theta(x(k)))\big\{[C_g + \Delta C_g(k)]x(k) + [D_{1g} + \Delta D_{1g}(k)]w(k) + [D_{2g} \\
&\quad + \Delta D_{2g}(k)]u(k)\big\} \\
&= [C(\mu) + \Delta C(\mu,k)]x(k) + [D_1(\mu) + \Delta D_1(\mu,k)]w(k) \\
&\quad + [D_2(\mu) + \Delta D_2(\mu,k)]u(k) \qquad (7.2.2)
\end{aligned}
$$

where

$$\theta(x(k)) = [\theta_1(x(k)), \ldots, \theta_p(x(k))],$$

$$\chi_g(\theta(x(k))) = \prod_{t=1}^{p} J_t^g(\theta_t(x(k))),$$

$$\mu_g(\theta(x(k)) = \frac{\chi_g(\theta(x(k)))}{\sum_{\ell=1}^{r}\chi_\ell(\theta(x(k)))} \in [0,1],$$

$$\sum_{g=1}^{r} \mu_g(\theta(x(k))) = 1$$

$$
\begin{array}{ll}
A(\mu) = \sum_{g=1}^{r}\mu_g(\theta(x(k)))A_g & C(\mu) = \sum_{g=1}^{r}\mu_g(\theta(x(k)))C_g \\
B_1(\mu) = \sum_{g=1}^{r}\mu_g(\theta(x(k)))B_{1g} & B_{2g}(\mu) = \sum_{g=1}^{r}\mu_g(\theta(x(k)))B_{2g} \\
D_1(\mu) = \sum_{g=1}^{r}\mu_g(\theta(x(k)))D_{1g} & D_{2g}(\mu) = \sum_{g=1}^{r}\mu_g(\theta(x(k)))D_{2g}
\end{array}
$$

$$
\begin{array}{ll}
\Delta A(\mu,k) = \sum_{g=1}^{r}\mu_g(\theta(x(k)))\Delta A_g(k) & \Delta C(\mu,k) = \sum_{g=1}^{r}\mu_g(\theta(x(k)))\Delta C_g(k) \\
\Delta B_1(\mu,k) = \sum_{g=1}^{r}\mu_g(\theta(x(k)))\Delta B_{1g}(k) & \Delta B_2(\mu,k) = \sum_{g=1}^{r}\mu_g(\theta(x(k)))\Delta B_{2g}(k) \\
\Delta D_1(\mu,k) = \sum_{g=1}^{r}\mu_g(\theta(x(k)))\Delta D_{1g}(k) & \Delta D_2(\mu,k) = \sum_{g=1}^{r}\mu_g(\theta(x(k)))\Delta D_{2g}(k)
\end{array}
$$

We assume that the uncertainty functions in μ and k are norm-bounded by the following:

Assumption 7.2.1

$$
\begin{bmatrix} \Delta A_g(\mu,k) & \Delta B_1(\mu,k) & \Delta B_2(\mu,k) \\ \Delta C_g(\mu,k) & \Delta D_1(\mu,k) & \Delta D_2(\mu,k) \end{bmatrix} = \begin{bmatrix} E_1(\mu) \\ E_2(\mu) \end{bmatrix} F(k) \begin{bmatrix} H_1 & H_2 & H_3 \end{bmatrix}
$$

where $g = 1, \ldots, r$; r is the number of fuzzy inference rules; $E_1(\mu) = \sum_{g=1}^{r}\mu_g(\theta(x(k)))E_{1g}$,

$E_2(\mu) = \sum_{g=1}^{r} \mu_g(\theta(x(k)))E_{2g}$; H_1, H_2, H_3, E_{1g} and E_{2g} are known matrices which characterize the structure of the uncertainties; $F(k)$ is an unknown matrix function that satisfy $F^T(k)\mathcal{W}F(k) \leq \mathcal{W}$ where $\mathcal{W}$ is a positive-definite matrix.

The model of the network in this chapter is identical to what is shown in Chapter 3 except the fact that the transition probability matrix is completely known.

In this chapter, we consider the following fuzzy state feedback:

Control Rule h:
IF $\sigma_1(x(k-\tau_k))$ is N_1^h AND $\cdots$ AND $\sigma_q(x(k-\tau_k))$ is N_q^h,
THEN

$$u(k) = K_h(r_k)x(k-\tau_k) \tag{7.2.3}$$

where h denotes the h^{th} fuzzy inference rule; $h = 1, \ldots, c$; c is the number of inference rules; $\sigma_1(x(k-\tau_k))$, $\ldots$, $\sigma_q(x(k-\tau_k))$, are the premise variables; q is the number of premise variables and N_1^h, $\ldots$, N_q^h are the fuzzy terms, r_k represents the mode in the Markov chain. $K_h(r_k)$ in each control rule is a local mode delay dependent state feedback controller gain.

Similar to the plant, the fuzzy state feedback controller is inferred as shown below.

$$u(k) = \sum_{h=1}^{c} \lambda_h(\sigma(x(k-\tau_k)))K_h(r_k)x(k-\tau_k) \tag{7.2.4}$$

$$= K(\lambda, r_k)x(k-\tau_k) \tag{7.2.5}$$

where

$$\sigma(x(k-\tau_k)) = [\sigma_1(x(k-\tau_k)), \ldots, \sigma_q(x(k-\tau_k))],$$

$$\phi_h(\sigma(x(k-\tau_k))) = \prod_{t=1}^{q} N_t^h(\sigma_t(x(k-\tau_k))),$$

$$\lambda_h(\sigma(x(k-\tau_k)) = \frac{\phi_h(\sigma(x(k-\tau_k)))}{\sum_{\ell=1}^{c} \phi_\ell(\sigma(x(k-\tau_k)))} \in [0,1],$$

$$\sum_{h=1}^{c} \lambda_h(\sigma(x(k-\tau_k))) = 1$$

$$K(\lambda, r_k) = \sum_{h=1}^{c} \lambda_h(\sigma(x(k-\tau_k)))K_h(r_k)$$

Remark 7.2.1 *Note that, in (7.2.4), the fuzzy controller's membership functions and* *premise variables are allowed to be different from the plant's membership functions, and also delays are allowed in the fuzzy controller's premise variables. The fuzzy controller (7.2.4) is more realistic in NCSs. Furthermore, since the premise variables of the plant and the controller are allowed to be different, resulting in the number of fuzzy rules of the controller no longer restricted by the number of fuzzy rules of the plant. Hence a small number of fuzzy rules may be implemented even though the number of fuzzy rules of the plant is large.*

As shown in Chapter 2, the closed-loop system is to achieve stochastic stability, as shown in (2.2.5), and the $\mathcal{H}_\infty$ performance condition, as shown in (2.2.6).

7.3 Main Result

Considering the fuzzy model (7.2.2) and the fuzzy controller (7.2.4), the closed loop system is obtained as follows:

$$\begin{aligned} x(k+1) &= [A(\mu) + \Delta A(\mu,k)]x(k) + [B_1(\mu) + \Delta B_1(\mu,k)]w(k) \\ &\quad + [B_2(\mu) + \Delta B_2(\mu,k)]K(\lambda, r_k)x(k-\tau_k) \\ &= A(\mu)x(k) + E_1(\mu)F(k)H_1x(k) + B_1(\mu)w(k) + E_1(\mu)F(k)H_2w(k) \\ &\quad + B_2(\mu)K(\lambda, r_k)x(k-\tau_k) + E_1(\mu)F(k)H_3K(\lambda, r_k)x(k-\tau_k) \end{aligned} \tag{7.3.1}$$

$$\begin{aligned} z(k) &= [C(\mu) + \Delta C(\mu,k)]x(k) + [D_1(\mu) + \Delta D_1(\mu,k)]w(k) \\ &\quad + [D_2(\mu) + \Delta D_2(\mu,k)]K(\lambda, r_k)x(k-\tau_k) \\ &= C(\mu)x(k) + E_2(\mu)F(k)H_1x(k) + D_1(\mu)w(k) + E_2(\mu)F(k)H_2w(k) \\ &\quad + D_2(\mu)K(\lambda, r_k)x(k-\tau_k) + E_2(\mu)F(k)H_3K(\lambda, r_k)x(k-\tau_k) \end{aligned} \tag{7.3.2}$$

For brevity, $x(k)$, $x(k-\tau_k)$, $\mu_g(\theta(x(k)))$, $\lambda_h(\sigma(x(k-\tau_k)))$ are denoted as x, x_τ, $\mu_g(x)$, $\lambda_h(x_\tau)$, respectively, throughout this chapter.

MATLAB LMI toolbox is not able to handle nonlinear membership functions, however, third-party MATLAB toolbox such as YALMIP and SOSTOOLS can handle multivariate polynomial functions. For this reason each product term $\mu_g(x)\lambda_h(x_\tau)$ is approximated by polynomial functions. In order to approximate the membership product terms as accurate as possible, they are divided into sub-regions and a polynomial function approximation is found for each sub-region as shown below

$$\mu_g(x)\lambda_h(x_\tau) = \sum_{\kappa=1}^{D} \zeta_\kappa(x, x_\tau)\{\eta_{gh,s_\kappa}(x, x_\tau) + \Delta\eta_{gh,s_\kappa}(x, x_\tau)\} \tag{7.3.3}$$

where $\eta_{gh,s_\kappa}(x, x_\tau)$ are the polynomial function approximations and $\Delta\eta_{gh,s_\kappa}(x, x_\tau)$ are the error terms in each sub-region [96]. $\zeta_\kappa(x, x_\tau)$ is a scalar function which takes 1 if x and x_τ are inside the sub-region, s_κ, and 0 otherwise.

We define lower and upper bounds of the error terms as

$$\alpha_{gh,s_\kappa} \leq \Delta\eta_{gh,s_\kappa}(x, x_\tau) \leq \beta_{gh,s_\kappa} \tag{7.3.4}$$

where α_{gh,s_κ} and β_{gh,s_κ} are known constants [96].

The following theorem provides sufficient conditions for the existence of a mode delay dependent state feedback controller. The theorem incorporates the polynomial function approximations of the product of membership functions into the controller design in order to reduce conservatism.

Theorem 7.3.1 *Given a prescribed $\mathcal{H}_\infty$ performance, $\gamma > 0$, the closed-loop system is stochastically stable with the prescribed $\mathcal{H}_\infty$ performance, if there exist sets of positive-definite matrices $X(i)$, $\tilde{R}_1(i)$, $\tilde{R}_1$, $\tilde{R}_2(i)$, $\tilde{R}_2$, $W_1(i)$, $W_2(i)$, $W_3(i)$, $\tilde{Q}$, $\tilde{W}_1(i)$, $\tilde{W}_2(i)$, N_1, N_2, $\tilde{Z}(i)$, $S(i,j)$, $\tilde{\Psi}_1^{gh}(i)$, $\tilde{\Psi}_2^{gh}(i)$ and matrices $\tilde{M}(i)$, $J(i)$, $Y_h(i)$ for $i = 1, 2, \cdots, s$, $g = 1, 2, \cdots, r$, $h = 1, 2, \cdots, c$ satisfying the following*

$$\tilde{R}_1 > \tilde{R}_1(i) \tag{7.3.5}$$

$$\tilde{R}_2 > \tilde{R}_2(i) \tag{7.3.6}$$

$$-v^T\Big[\sum_{g=1}^{r}\sum_{h=1}^{c}\Big\{(\eta_{gh,s_\kappa}(x, x_\tau) + \frac{1}{2}\alpha_{gh,s_\kappa} + \frac{1}{2}\beta_{gh,s_\kappa})\tilde{T}^{gh}(i) + \tilde{V}^{gh,s_\kappa}(i)\Big\} - \tilde{\epsilon}_{s_\kappa}I\Big]v \text{ is SOS}$$
$$\forall s_\kappa = 1, \ldots, D \tag{7.3.7}$$

$$\frac{1}{2}\tilde{T}^{gh}(i) - \tilde{\Psi}_1^{gh}(i) < 0 \tag{7.3.8}$$

$$-\frac{1}{2}\tilde{T}^{gh}(i) - \tilde{\Psi}_2^{gh}(i) < 0 \tag{7.3.9}$$

$$\begin{bmatrix} (1-p_{i(i+1)})\tilde{R}_1(i)+\tilde{R}_2(i) & \tilde{M}(i) \\ * & \tilde{Z}(i) \end{bmatrix} \geq 0 \tag{7.3.10}$$

$$\begin{bmatrix} S(i,j) & J^T(i) \\ * & X(j) \end{bmatrix} > 0 \tag{7.3.11}$$

and

$$N_1\tilde{R}_1 = I,\ N_2\tilde{R}_2 = I, \tilde{W}_1(i)W_1(i) = I \ \text{and}\ \tilde{W}_2(i)W_2(i) = I, \tag{7.3.12}$$

where v is a real vector of appropriate dimension and independent of x; $\tilde{\epsilon}_{s_\kappa}$ are predefined scalars; $\eta_{gh,s_\kappa}(x,x_\tau)$ are defined in (7.3.3);

$$\begin{aligned}
\tilde{T}^{gh}(i) &= \begin{bmatrix} \tilde{\Lambda}(i) & \big(\tilde{\Gamma}_1^{gh}(i)\big)^T & \big(\tilde{\Gamma}_2^{gh}(i)\big)^T & \big(\tilde{\Gamma}_3(i)\big)^T & \big(\tilde{\Xi}^{gh}(i)\big)^T & \big(\tilde{\mathcal{H}}^h(i)\big)^T \\ * & -\mathcal{X} & 0 & 0 & 0 & 0 \\ * & * & -\mathcal{R} & 0 & 0 & 0 \\ * & * & * & -\mathcal{Q} & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -\mathcal{W} \end{bmatrix} \\
\tilde{V}^{gh,s_\kappa}(i) &= (\beta_{gh,s_\kappa}-\alpha_{gh,s_\kappa})\Big\{\tilde{\Psi}_1^{gh}(i)+\tilde{\Psi}_2^{gh}(i)\Big\} \\
\tilde{\Lambda}(i) &= diag\Big\{-X(i), \Big(-X^T(i)-X(i)+\tilde{Q}\Big), \Big(H_2^TW_3(i)H_2-\gamma I\Big), \\
&\quad -W_1(i), -W_2(i), -W_3(i)\Big\} + \tilde{\Upsilon}_1^T(i)+\tilde{\Upsilon}_1(i)+\tau(i)\tilde{Z}(i) \\
\tilde{\Gamma}_1^{gh}(i) &= \Big[A_gX(i) \quad B_{2g}Y_h(i) \quad B_{1g} \quad E_{1g} \quad E_{1g} \quad E_{1g}\Big] \\
\tilde{\Gamma}_2^{gh}(i) &= \begin{bmatrix} \sqrt{\sum_{j=1}^{i+1} p_{ij}\tau(j)} \\ \sqrt{\tau(s)} \end{bmatrix} \Big[A_gX(i)-X(i) \quad B_{2g}Y_h(i) \quad B_{1g} \quad E_{1g} \quad E_{1g} \quad E_{1g}\Big] \\
\tilde{\Gamma}_3(i) &= \Big[X(i) \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\Big] \\
\mathcal{X} &= -\textstyle\sum_{j=1}^{i+1} p_{ij}S(i,j)+J^T(i)+J(i) \\
\mathcal{R} &= diag\Big\{N_1, N_2\Big\} \\
\mathcal{Q} &= \Big(\tau(s)-\tau(1)+1\Big)\tilde{Q} \\
\mathcal{W} &= diag\Big\{\tilde{W}_1(i), \tilde{W}_2(i)\Big\} \\
\tilde{\Xi}^{gh}(i) &= \Big[\ C_gX(i) \quad D_{2g}Y_h(i) \quad D_{1g} \quad E_{2g} \quad E_{2g} \quad E_{2g}\ \Big] \\
\tilde{\mathcal{H}}^h(i) &= \begin{bmatrix} H_1X(i) & 0 & 0 & 0 & 0 & 0 \\ 0 & H_3Y_h(i) & 0 & 0 & 0 & 0 \end{bmatrix} \\
\tilde{\Upsilon}_1(i) &= \tilde{M}^T(i)[I \quad -I \quad 0 \quad 0 \quad 0 \quad 0]
\end{aligned}$$

and the mode delay dependent fuzzy controller is

$$u(k) = \sum_{h=1}^{c} \lambda_h(\sigma(x(k-\tau_k)))K_h(i)x(k-\tau_k) \quad (7.3.13)$$

where

$$K_h(i) = Y_h(i)X^{-1}(i) \quad (7.3.14)$$

Proof: We introduce the following augmented closed loop system as shown below

$$\begin{aligned} x_{k+1} &= \Gamma_1(\mu,\lambda,r_k)\tilde{x}_k && (7.3.15)\\ z_k &= \Xi(\mu,\lambda,r_k)\tilde{x}_k && (7.3.16) \end{aligned}$$

where $x_\ell = x(\ell)$, $z_\ell = z(\ell)$, $\tilde{x}_\ell = \tilde{x}(\ell)$, and

$$\begin{aligned} \Gamma_1(\mu,\lambda,r_k) &= \begin{bmatrix} A(\mu) & B_2(\mu)K(\lambda,r_k) & B_1(\mu) & E_1(\mu) & E_1(\mu) & E_1(\mu) \end{bmatrix} && (7.3.17)\\ \Xi(\mu,\lambda,r_k) &= \begin{bmatrix} C(\mu) & D_2(\mu)K(\lambda,r_k) & D_1(\mu) & E_2(\mu) & E_2(\mu) & E_2(\mu) \end{bmatrix} && (7.3.18)\\ \tilde{x}(k) &= \Big[x^T(k) \;\; x^T(k-\tau_k) \;\; w^T(k) \;\; x^T(k)H_1^T F^T(k) \\ &\qquad x^T(k-\tau_k)K^T(\lambda,r_k)H_3^T F^T(k) \;\; w^T(k)H_2^T F^T(k) \Big]^T \in \Re^l && (7.3.19) \end{aligned}$$

Consider the following Lyapunov-Krasovskii candidate functional:

$$V(x_k,r_k) = V_1(x_k,r_k) + V_2(x_k,r_k) + V_3(x_k,r_k) \quad (7.3.20)$$

with

$$V_1(x_k,r_k) = x_k^T P(r_k) x_k \quad (7.3.21)$$

$$V_2(x_k,r_k) = \sum_{\ell=-\tau_k}^{-1} \sum_{j=k+\ell}^{k-1} y_j^T R_1 y_j + \sum_{\ell=-\tau(s)}^{-1} \sum_{j=k+\ell}^{k-1} y_j^T R_2 y_j \quad (7.3.22)$$

$$V_3(x_k,r_k) = \sum_{\ell=k-\tau_k}^{k-1} x_\ell^T Q x_\ell + \sum_{\ell=-\tau(s)+2}^{-\tau(1)+1} \sum_{j=k+\ell-1}^{k-1} x_j^T Q x_j \quad (7.3.23)$$

Taking the forward difference of $V(x_k,r_k)$, we have

$$\Delta V(x_k,r_k) = \Delta V_1(x_k,r_k) + \Delta V_2(x_k,r_k) + \Delta V_3(x_k,r_k) \quad (7.3.24)$$

where

$$\begin{aligned}\Delta V_1(x_k, r_k) &= x_{k+1}^T \tilde{P}(r_k) x_{k+1} - x_k^T P(r_k) x_k \\ &= \tilde{x}_k^T \Gamma_1^T(\mu, \lambda, r_k) \tilde{P}(r_k) \Gamma_1(\mu, \lambda, r_k) \tilde{x}_k - x_k^T P(r_k) x_k \qquad (7.3.25)\end{aligned}$$

$$\begin{aligned}\Delta V_2(x_k, r_k) &= \sum_{i=1}^{s} p_{r_k i} \sum_{\ell=-\tau(i)}^{-1} \sum_{j=k+1+\ell}^{k} y_j^T R_1 y_j + \sum_{\ell=-\tau(s)}^{-1} \sum_{j=k+\ell+1}^{k} y_j^T R_2 y_j \\ &\quad - \sum_{\ell=-\tau_k}^{-1} \sum_{j=k+\ell}^{k-1} y_j^T R_1 y_j - \sum_{\ell=-\tau(s)}^{-1} \sum_{j=k+\ell}^{k-1} y_j^T R_2 y_j \\ &= \sum_{i=1}^{s} p_{r_k i} \Bigg\{ \sum_{\ell=-\tau(i)}^{-1} y_k^T R_1 y_k + \sum_{\ell=-\tau(i)}^{-1} \sum_{j=k+\ell+1}^{k-1} y_j^T R_1 y_j \\ &\quad - \sum_{\ell=-\tau_k}^{-1} \sum_{j=k+\ell+1}^{k-1} y_j^T R_1 y_j - \sum_{\ell=-\tau_k}^{-1} y_{k+\ell}^T R_1 y_{k+\ell} \Bigg\} + \\ &\quad \sum_{\ell=-\tau(s)}^{-1} \left\{ y_k^T R_2 y_k - y_{k+\ell}^T R_2 y_{k+\ell} \right\} \\ &= \sum_{i=1}^{s} p_{r_k i} \Bigg\{ \sum_{\ell=-\tau(i)}^{-1} \sum_{j=k+\ell+1}^{k-1} y_j^T R_1 y_j - \sum_{\ell=-\tau_k}^{-1} \sum_{j=k+\ell+1}^{k-1} y_j^T R_1 y_j \\ &\quad - \sum_{j=k-\tau_k}^{k-1} y_j^T R_1 y_j \Bigg\} - \sum_{j=k-\tau(s)}^{k-1} y_j^T R_2 y_j + y_k^T \Big[\tilde{\tau}_k R_1 + \tau(s) R_2 \Big] y_k \qquad (7.3.26)\end{aligned}$$

and

$$\begin{aligned}\Delta V_3(x_k, r_k) &= \sum_{i=1}^{s} p_{r_k i} \sum_{\ell=k-\tau(i)+1}^{k} x_\ell^T Q x_\ell + \sum_{\ell=-\tau(s)+2}^{-\tau(1)+1} \sum_{j=k+\ell}^{k} x_j^T Q x_j - \sum_{\ell=k-\tau_k}^{k-1} x_\ell^T Q x_\ell \\ &\quad - \sum_{\ell=-\tau(s)+2}^{-\tau(1)+1} \sum_{j=k+\ell-1}^{k-1} x_j^T Q x_j \\ &= \sum_{i=1}^{s} p_{r_k i} \left\{ x_k^T Q x_k + \sum_{\ell=k-\tau(i)+1}^{k-1} x_\ell^T Q x_\ell - \sum_{\ell=k-\tau_k+1}^{k-1} x_\ell^T Q x_\ell - x_{k-\tau_k}^T Q x_{k-\tau_k} \right\} \\ &\quad + \sum_{\ell=-\tau(s)+2}^{-\tau(1)+1} \left\{ x_k^T Q x_k - x_{k+\ell-1}^T Q x_{k+\ell-1} \right\}\end{aligned}$$

$$= \sum_{i=1}^{s} p_{r_k i} \left\{ \sum_{\ell=k-\tau(i)+1}^{k-1} x_\ell^T Q x_\ell - \sum_{\ell=k-\tau_k+1}^{k-1} x_\ell^T Q x_\ell - x_{k-\tau_k}^T Q x_{k-\tau_k} \right\}$$
$$- \sum_{\ell=-\tau(s)+2}^{-\tau(1)+1} x_{k+\ell-1}^T Q x_{k+\ell-1} + (\tau(s)-\tau(1)+1) x_k^T Q x_k$$
$$= \sum_{i=1}^{s} p_{r_k i} \left\{ \sum_{\ell=k-\tau(i)+1}^{k-\tau(1)} x_\ell^T Q x_\ell + \sum_{\ell=k-\tau(1)+1}^{k-1} x_\ell^T Q x_\ell - \sum_{\ell=k-\tau_k+1}^{k-1} x_\ell^T Q x_\ell \right\}$$
$$- \sum_{\ell=k-\tau(s)+1}^{k-\tau(1)} x_\ell^T Q x_\ell + (\tau(s)-\tau(1)+1) x_k^T Q x_k - x_{k-\tau_k}^T Q x_{k-\tau_k} \tag{7.3.27}$$

with $\tilde{P}(r_k) = \sum_{j=1}^{r_k+1} p_{r_k j} P(j)$.

Knowing that $\text{Prob}\{\tau_{k+1} > \tau_k + 1\} = 0$, $\tau(1) \leq \tau_{k+1} \leq \tau_k + 1 \leq \tau(s)$ and $\tau(1) \leq \tau_k \leq \tau(s)$, the terms $\Delta V_2(x_k, r_k)$ and $\Delta V_3(x_k, r_k)$ can be upper bounded as

$$\Delta V_2(x_k, r_k) \leq y_k^T \Big[\tilde{\tau}_k R_1 + \tau(s) R_2\Big] y_k - \sum_{\ell=k-\tau_k}^{k-1} y_\ell^T \Big[(1 - p_{r_k(r_k+1)}) R_1 + R_2\Big] y_\ell \tag{7.3.28}$$

and

$$\Delta V_3(x_k, r_k) \leq (\tau(s)-\tau(1)+1) x_k^T Q x_k - x_{k-\tau_k}^T Q x_{k-\tau_k}. \tag{7.3.29}$$

From (7.3.10), we know that

$$\sum_{r_k=k-\tau_k}^{k-1} \begin{bmatrix} y(r_k) \\ \tilde{x}(k) \end{bmatrix}^T \begin{bmatrix} (1-p_{r_k(r_k+1)})\tilde{R}_1(r_k) + \tilde{R}_2(r_k) & \tilde{M}(r_k) \\ * & \tilde{Z}(r_k) \end{bmatrix} \begin{bmatrix} y(r_k) \\ \tilde{x}(k) \end{bmatrix} \geq 0 \tag{7.3.30}$$

Expand and rearranging the above results

$$-\sum_{r_k=k-\tau_k}^{k-1} y^T(r_k)[(1-p_{r_k(r_k+1)})\tilde{R}_1(r_k) + \tilde{R}_2(r_k)]y(r_k) \leq \tilde{x}^T(k)\Big\{\tilde{\Upsilon}_1(r_k) + \tilde{\Upsilon}_1^T(r_k)$$
$$+\tau_k \tilde{Z}(r_k)\Big\}\tilde{x}(k) \tag{7.3.31}$$

where $y(k) = x(k+1) - x(k)$ and $\tilde{\Upsilon}_1(r_k) = \tilde{M}^T(r_k)[I \;\; -I \;\; 0 \;\; 0 \;\; 0 \;\; 0]$.

Multiplying the above with $\text{diag}\Big\{\text{diag}\Big\{X^{-1}(r_k), X^{-1}(r_k), I, I, I, I\Big\}, I, I, I, I\Big\}$ on the right hand side and its transpose on its left hand side of each term with conditions

(7.3.5) and (7.3.6) we have

$$-\sum_{r_k=k-\tau_k}^{k-1} y^T(r_k)[(1-p_{r_k(r_k+1)})R_1+R_2]y(r_k) \le \tilde{x}^T(k)\Big\{\Upsilon_1(r_k)+\Upsilon_1^T(r_k)+\tau_k Z(r_k)\Big\}\tilde{x}(k) \tag{7.3.32}$$

where $\Upsilon_1(r_k) = M^T(r_k)[I \ \ -I \ \ 0 \ \ 0 \ \ 0 \ \ 0]$.

Employing (7.3.32) in (7.3.28), we obtain

$$\begin{aligned}\Delta V_2(x_k,r_k) \ \le \ & \tilde{x}_k^T\Big\{\Gamma_2^T(\mu,\lambda,r_k)\,[\tilde{\tau}_k R_1+\tau(s)R_2]\,\Gamma_2(\mu,\lambda,r_k)+\Upsilon_1(r_k)+\Upsilon_1^T(r_k)\\ &+\tau_k Z(r_k)\Big\}\tilde{x}_k\end{aligned} \tag{7.3.33}$$

where

$$\Gamma_2(\mu,\lambda,r_k) = \big[A(\mu)-I \ \ B_2(\mu)K(\lambda,r_k) \ \ B_1(\mu) \ \ E_1(\mu) \ \ E_1(\mu) \ \ E_1(\mu)\big] \tag{7.3.34}$$

Using (7.3.25), (7.3.33) and (7.3.29), the overall forward difference of the Lyapunov-Krasovskii functional is obtained as

$$\begin{aligned}\Delta V(x_k,r_k) \ \le \ & -x_k^T\Big(P(r_k)-(\tau(s)-\tau(1)+1)Q\Big)x_k - x_{k-\tau_k}^T Q x_{k-\tau_k}\\ &+\tilde{x}_k^T\Big\{\Gamma_1^T(\mu,\lambda,r_k)\tilde{P}(r_k)\Gamma_1(\mu,\lambda,r_k)\\ &+\Gamma_2^T(\mu,\lambda,r_k)\,[\tilde{\tau}_k R_1+\tau(s)R_2]\,\Gamma_2(\mu,\lambda,r_k)\\ &+\Upsilon_1(r_k)+\Upsilon_1^T(r_k)+\tau_k Z(r_k)\Big\}\tilde{x}_k\end{aligned} \tag{7.3.35}$$

Using Assumption 7.2.1, and adding and subtracting $x_k^T H_1^T F^T(k) W_1(r_k) F(k) H_1 x_k$, $w_k^T H_2^T F^T(k) W_3(r_k) F(k) H_2 w_k$, $x_{(k-\tau_k)}^T K^T(\lambda,r_k) H_3^T F^T(k) W_2(r_k) F(k) H_3 K(\lambda,r_k) x_{(k-\tau_k)}$, $z_k^T z_k$, and $\gamma w_k^T w_k$ to and from (7.3.35), we obtain

$$\begin{aligned}&\Delta V(x_k,r_k) \le\\ &-x_k^T\Big(P(r_k)-(\tau(s)-\tau(1)+1)Q-H_1^T W_1(r_k)H_1\Big)x_k\\ &-x_{k-\tau_k}^T\Big(Q-K^T(\lambda,r_k)H_3^T W_2(r_k)H_3K(\lambda,r_k)\Big)x_{k-\tau_k}\\ &+\tilde{x}_k^T\Big\{\Gamma_1^T(\mu,\lambda,r_k)\tilde{P}(r_k)\Gamma_1(\mu,\lambda,r_k)\\ &+\Gamma_2^T(\mu,\lambda,r_k)\,[\tilde{\tau}_k R_1+\tau(s)R_2]\,\Gamma_2(\mu,\lambda,r_k)\\ &+\Upsilon_1(r_k)+\Upsilon_1^T(r_k)+\tau_k Z(r_k)+\Xi^T(\mu,\lambda,r_k)\Xi(\mu,\lambda,r_k)\Big\}\tilde{x}_k - z_k^T z_k\\ &+\gamma w_k^T w_k - w_k^T\Big(\gamma I - H_2^T W_3(r_k)H_2\Big)w_k - x_k^T H_1^T F^T(k)W_1(r_k)F(k)H_1x_k\end{aligned}$$

$$-x^T_{(k-\tau_k)}K^T(\lambda,r_k)H_3^TF^T(r_k)W_2(r_k)F(r_k)H_3K(\lambda,r_k)x_{(k-\tau_k)}$$
$$-w_k^TH_2^TF^T(k)W_3(r_k)F(k)H_2w_k \tag{7.3.36}$$

Re-express (7.3.36) as

$$\begin{aligned}\Delta V(x_k,r_k) \ \leq\ & \tilde{x}_k^T\Big\{\Lambda(\lambda,r_k)+\Gamma_1^T(\mu,\lambda,r_k)\tilde{P}(r_k)\Gamma_1(\mu,\lambda,r_k)\\ &+\Gamma_2^T(\mu,\lambda,r_k)\left[\tilde{\tau}_kR_1+\tau(s)R_2\right]\Gamma_2(\mu,\lambda,r_k)\\ &+\Xi^T(\mu,\lambda,r_k)\Xi(\mu,\lambda,r_k)+\Upsilon_1(r_k)+\Upsilon_1^T(r_k)+\tau_kZ(r_k)\Big\}\tilde{x}_k\\ &-z_k^Tz_k+\gamma w_k^Tw_k\end{aligned} \tag{7.3.37}$$

where

$$\begin{aligned}\Lambda(\lambda,r_k)=&\mathrm{diag}\Big\{\Big((\tau(s)-\tau(1)+1)Q+H_1^TW_1(r_k)H_1-P(r_k)\Big),\\ &\Big(K^T(\lambda,r_k)H_3^TW_2(r_k)H_3K(\lambda,r_k)-Q\Big),\\ &\Big(H_2^TW_3(r_k)H_2-\gamma I\Big),-W_1(r_k),-W_2(r_k),-W_3(r_k)\Big\}\end{aligned}$$

Following from (7.3.7), we learn that for all $s_\kappa=1,\ldots,D$

$$\begin{aligned}\textstyle\sum_{g=1}^r\sum_{h=1}^c\Big[&(\eta_{gh,s_\kappa}(x,x_\tau)+\tfrac{1}{2}\alpha_{gh,s_\kappa}+\tfrac{1}{2}\beta_{gh,s_\kappa})\tilde{T}^{gh}(r_k)+(\beta_{gh,s_\kappa}-\alpha_{gh,s_\kappa})\\ &\{\tilde{\Psi}_1^{gh}(r_k)+\tilde{\Psi}_2^{gh}(r_k)\}\Big]<0\end{aligned} \tag{7.3.38}$$

Since $\tilde{\Psi}_1^{gh}(r_k)$ and $\tilde{\Psi}_2^{gh}(r_k)$ are positive definite matrices, along with (7.3.3), we know that for all $s_\kappa=1,\ldots,D$

$$(\Delta\eta_{gh,s_\kappa}(x,x_\tau)-\alpha_{gh,s_\kappa})\tilde{\Psi}_1^{gh}(r_k)\leq(\beta_{gh,s_\kappa}-\alpha_{gh,s_\kappa})\tilde{\Psi}_1^{gh}(r_k) \tag{7.3.39}$$
$$(\beta_{gh,s_\kappa}-\Delta\eta_{gh,s_\kappa}(x,x_\tau))\tilde{\Psi}_2^{gh}(r_k)\leq(\beta_{gh,s_\kappa}-\alpha_{gh,s_\kappa})\tilde{\Psi}_2^{gh}(r_k) \tag{7.3.40}$$

Therefore, for all $s_\kappa=1,\ldots,D$

$$\begin{aligned}\sum_{g=1}^r\sum_{h=1}^c\Big[&(\eta_{gh,s_\kappa}(x,x_\tau)+\frac{1}{2}\alpha_{gh,s_\kappa}+\frac{1}{2}\beta_{gh,s_\kappa})\tilde{T}^{gh}(r_k)\\ &+(\Delta\eta_{gh,s_\kappa}(x,x_\tau)-\alpha_{gh,s_\kappa})\tilde{\Psi}_1^{gh}(r_k)+(\beta_{gh,s_\kappa}-\Delta\eta_{gh,s_\kappa}(x,x_\tau))\tilde{\Psi}_2^{gh}(r_k)\Big]<0\end{aligned} \tag{7.3.41}$$

Using (7.3.8) and (7.3.9), we have

$$\begin{aligned}&\textstyle\sum_{g=1}^{r}\sum_{h=1}^{c}\Big[(\eta_{gh,s_\kappa}(x,x_\tau)+\frac{1}{2}\alpha_{gh,s_\kappa}+\frac{1}{2}\beta_{gh,s_\kappa})\tilde{T}^{gh}(r_k)\\ &+(\Delta\eta_{gh,s_\kappa}(x,x_\tau)-\alpha_{gh,s_\kappa})\tfrac{1}{2}\tilde{T}^{gh}(r_k)-(\beta_{gh,s_\kappa}-\Delta\eta_{gh,s_\kappa}(x,x_\tau))\tfrac{1}{2}\tilde{T}^{gh}(r_k)\Big]<0\end{aligned}\tag{7.3.42}$$

for all $s_\kappa = 1,\ldots,D$.

Simplifying (7.3.42) as

$$\begin{aligned}&\sum_{g=1}^{r}\sum_{h=1}^{c}\Big[\eta_{gh,s_\kappa}(x,x_\tau)\tilde{T}^{gh}(r_k)+\frac{1}{2}(\Delta\eta_{gh,s_\kappa}(x,x_\tau)+\beta_{gh,s_\kappa}-\beta_{gh,s_\kappa})\tilde{T}^{gh}(r_k)\\ &+\frac{1}{2}(\Delta\eta_{gh,s_\kappa}(x,x_\tau)+\alpha_{gh,s_\kappa}-\alpha_{gh,s_\kappa})\tilde{T}^{gh}(r_k)\\ =\;&\sum_{g=1}^{r}\sum_{h=1}^{c}\Big[\{\eta_{gh,s_\kappa}(x,x_\tau)+\Delta\eta_{gh,s_\kappa}(x,x_\tau)\}\tilde{T}^{gh}(r_k)\Big]<0\end{aligned}\tag{7.3.43}$$

for all $s_\kappa = 1,\ldots,D$.

Employing (7.3.3) and the notation for $\tilde{T}^{gh}(r_k)$, (7.3.43) can be expressed as

$$\sum_{g=1}^{r}\sum_{h=1}^{c}\mu_g(x)\lambda_h(x_\tau)\begin{bmatrix}\tilde{\Lambda}(r_k) & \left(\tilde{\Gamma}_1^{gh}(r_k)\right)^T & \left(\tilde{\Gamma}_2^{gh}(r_k)\right)^T & \left(\tilde{\Gamma}_3(r_k)\right)^T & \left(\tilde{\Xi}^{gh}(r_k)\right)^T & \left(\tilde{\mathcal{H}}^{h}(r_k)\right)^T\\ * & -\mathcal{X} & 0 & 0 & 0 & 0\\ * & * & -\mathcal{R} & 0 & 0 & 0\\ * & * & * & -\mathcal{Q} & 0 & 0\\ * & * & * & * & -I & 0\\ * & * & * & * & * & -\mathcal{W}\end{bmatrix}<0\tag{7.3.44}$$

for all $s_\kappa = 1,\ldots,D$.

Applying Schur complement on (7.3.11) and consequently multiplying these inequalities by p_{ij} and summing up for all j we obtain

$$\begin{aligned}-\sum_{j=1}^{i+1}p_{ij}S(i,j)+J^T(i)+J(i) &= J^T(i)+J(i)-\sum_{j=1}^{i+1}p_{ij}S(i,j)\\ &< J^T(i)+J(i)-J^T(i)\tilde{P}(i)J(i)\\ &= \tilde{P}^{-1}(i)-\Big(J(i)-\tilde{P}^{-1}(i)\Big)^T\tilde{P}(i)\Big(J(i)-\tilde{P}^{-1}(i)\Big)\\ &< \tilde{P}^{-1}(i)\end{aligned}\tag{7.3.45}$$

The following holds true since $\tilde{Q}$ is a positive matrix

$$\begin{aligned} & \left(P(i)-\tilde{Q}^{-1}\right)^T \tilde{Q}\left(P(i)-\tilde{Q}^{-1}\right) > 0 \\ \Rightarrow\ & P(i)^T\tilde{Q}P(i) - P^T(i) - P(i) + \tilde{Q}^{-1} > 0 \\ \Rightarrow\ & -P^T(i) - P(i) + Q > -P(i)^T\tilde{Q}P(i) \end{aligned} \tag{7.3.46}$$

where $Q = \tilde{Q}^{-1}$ and $P(i) = X(i)^{-1}$. Multiplying $X(i)$ to the right hand side and its transpose to the left we have $-X^T(i) - X(i) + \tilde{Q} > -X(i)^T QX(i)$.

Therefore $-\mathcal{X}$ and $-X^T(i) - X(i) + \tilde{Q}$ can be replaced with $-\tilde{P}^{-1}(i)$ and $-X^T(i)QX(i)$ respectively and the condition (7.3.44) still satisfies.

Multiplying with $\text{diag}\Big\{\text{diag}\Big\{X^{-1}(i), X^{-1}(i), I, I, I, I\Big\}, I, I, I, I\Big\}$ on the right hand side and its transpose on the left and applying Schur complement to (7.3.44) with $-\tilde{P}^{-1}(i)$ and $-X(i)^T QX(i)$ yields

$$\begin{aligned} & \Lambda(\lambda, r_k) + \Gamma_1^T(\mu,\lambda,r_k)\tilde{P}(r_k)\Gamma_1(\mu,\lambda,r_k) + \Gamma_2^T(\mu,\lambda,r_k)\left[\tilde{\tau}_k R_1 + \tau(s)R_2\right]\Gamma_2(\mu,\lambda,r_k) \\ & + \Xi^T(\mu,\lambda,r_k)\Xi(\mu,\lambda,r_k) + \Upsilon_1(r_k) + \Upsilon_1^T(r_k) + \tau_k Z(r_k) < 0 \end{aligned} \tag{7.3.47}$$

Therefore if (7.3.7)-(7.3.9) are satisfied we have

$$\Delta V(x_k, r_k) \le -z_k^T z_k + \gamma w_k^T w_k \tag{7.3.48}$$

Taking expectation and sum from 0 to ∞ on both sides of (7.3.48) we obtain

$$E\{V(x_\infty, r_\infty)\} - E\{V(x_0, r_0)\} \le -E\Big\{\sum_{\ell=0}^{\infty} z_\ell^T z_\ell\Big\} + \gamma\sum_{\ell=0}^{\infty} w_\ell^T w_\ell \tag{7.3.49}$$

It is clear that under zero initial condition, $V(x_0, r_0) = 0$, the $\mathcal{H}_\infty$ criteria holds.

Next, under $w(k) = 0, \forall k \ge 0$ we need to show that the closed-loop system is stochastically stable. From (7.3.37) and (7.3.7)-(7.3.9), we learn that

$$V(x_{(k+1)}, r_{(k+1)}) - V(x_k, r_k) \le -\beta\tilde{x}_k^T\tilde{x}_k \tag{7.3.50}$$

where $\beta = \inf\{\lambda_{\min}[-\mathcal{M}(i)], i \in S\}$ with

$$\mathcal{M}(i) = \Lambda(\lambda, r_k) + \Gamma_1^T(\mu, \lambda, r_k)\tilde{P}(r_k)\Gamma_1(\mu, \lambda, r_k) + \Gamma_2^T(\mu, \lambda, r_k)\left[\tilde{\tau}_k R_1 + \tau(s) R_2\right]\Gamma_2(\mu, \lambda, r_k) + \Xi^T(\mu, \lambda, r_k)\Xi(\mu, \lambda, r_k) + \Upsilon_1(r_k) + \Upsilon_1^T(r_k) + \tau_k Z(r_k) < 0 \quad (7.3.51)$$

Similar to the above, taking expectation and sum from 0 to ∞ on both sides of (7.3.50), we have

$$\begin{aligned} E\{V(x_\infty, r_\infty)\} - E\{V(x_0, r_0)\} &\leq -\beta E\Big\{\sum_{\ell=0}^{\infty} \tilde{x}_\ell^T \tilde{x}_\ell\Big\} \\ &\leq -\beta E\Big\{\sum_{\ell=0}^{\infty} x_\ell^T x_\ell\Big\} \end{aligned} \quad (7.3.52)$$

By re-arranging (7.3.52), we can show that the system is stochastically stable, where $\varrho = \frac{1}{\beta}E\{V(x_0, r_0)\} < \infty$. $\nabla\nabla\nabla$

Remark 7.3.1 *When membership functions are not incorporated into the controller design, (7.3.7)-(7.3.9) reduce to $\tilde{T}_{gh}(i) < 0$ which are given in [86]. This is a very conservative approach since such controller is valid for any membership functions. By incorporating the product of membership functions into the design we design a specific controller for the specific membership functions, making the controller less conservative. Note that the premise variables of the controller, x_τ are the delayed version of the plant's premise variables x, since premise variables are transmitted via communication networks. The polynomial approximations of the product of membership functions consist of both x and x_τ. Also note that only (7.3.7) is formulated in SOS conditions, since it is the only condition with polynomial functions, $\eta_{gh,s_\kappa}(x, x_\tau)$. By formulating other conditions in LMIs, the computation burden of the solver such as YALMIP is reduced.*

In accordance with the cone complementarity algorithm [115], the nonconvex feasibility problem formulated by (7.3.5)-(7.3.11) can be converted into the following nonlinear minimisation problem subject to SOS:

Minimize $Tr\Big(N_1\tilde{R}_1 + N_2\tilde{R}_2 + \tilde{W}_1(i)W_1(i) + \tilde{W}_2(i)W_2(i)\Big)$

Subject to (7.3.5)-(7.3.11) and

$$\begin{bmatrix} N_1 & I \\ I & \tilde{R}_1 \end{bmatrix} > 0, \begin{bmatrix} N_2 & I \\ I & \tilde{R}_2 \end{bmatrix} > 0, \begin{bmatrix} \tilde{W}_1(i) & I \\ I & W_1(i) \end{bmatrix} > 0, \begin{bmatrix} \tilde{W}_2(i) & I \\ I & W_2(i) \end{bmatrix} > 0 \quad (7.3.53)$$

To solve this optimisation problem, the following algorithm can be used:

Algorithm :

Step 1: Set $\jmath = 0$ and solve (7.3.5)-(7.3.11) and (7.3.53) to obtain the initial conditions,

$$\left[X(i), \tilde{R}_1(i), \tilde{R}_1, \tilde{R}_2(i), \tilde{R}_2, W_1(i), W_2(i), W_3(i), \tilde{W}_1(i), \tilde{W}_2(i), \tilde{Q}, N_1, N_2, \tilde{Z}(i), Y(i)\right]^0$$

Step 2: Solve the problem

Minimize $Tr\Big(N_1^\jmath \tilde{R}_1 + N_1 \tilde{R}_1^\jmath + N_2^\jmath \tilde{R} + N_2 \tilde{R}_2^\jmath + \tilde{W}_1(i)^\jmath W_1(i) + \tilde{W}_1(i) W_1(i)^\jmath$
$+ \tilde{W}_2(i)^\jmath W_2(i) + \tilde{W}_2(i) W_2(i)^\jmath\Big)$

Subject to (7.3.5)-(7.3.11) and (7.3.53)

The obtained solutions are denoted as

$$\Big[X(i), \tilde{R}_1(i), \tilde{R}_1, \tilde{R}_2(i), \tilde{R}_2, W_1(i), W_2(i), W_3(i), \tilde{W}_1(i), \tilde{W}_2(i), \tilde{Q}, N_1, N_2, \tilde{Z}(i), Y(i)\Big]^{\jmath+1} \quad (7.3.54)$$

Step 3: Check the system stability with $K(i)^{\jmath+1} = Y^{\jmath+1}(i) X^{-1}(i)^{\jmath+1}$, if there exist a stabilizing controller, then $K(i)^{\jmath+1}$ are the desired controller gains and EXIT. Otherwise, set $\jmath = \jmath + 1$ and return to Step 2.

7.4 Examples

In this section, two simulation examples are presented to show that 1. the incorporation of membership functions result a wider stability region and 2. the controller can stabilize a practical system.

Example 1 Consider a T-S fuzzy system with two plant rules ($r = 2$) and two controller rules ($c = 2$). The sub-systems are described as follows

$$\begin{aligned}
A_1 &= \begin{bmatrix} -a & -0.1 \\ 1 & 0 \end{bmatrix} & A_2 &= \begin{bmatrix} -1 & 0.1 \\ 1 & 0 \end{bmatrix} & B_{11} &= B_{12} = \begin{bmatrix} 0.02 \\ 0.03 \end{bmatrix} \\
C_1 &= \begin{bmatrix} 1 & 0 \end{bmatrix} & C_2 &= \begin{bmatrix} -1 & 0 \end{bmatrix} & B_{21} &= \begin{bmatrix} b \\ 1 \end{bmatrix} \quad B_{22} = \begin{bmatrix} 1.9 \\ 1 \end{bmatrix} \\
D_{11} &= 0.01 & D_{12} &= 0.01 & D_{21} &= D_{22} = 0.1
\end{aligned} \quad (7.4.1)$$

and the uncertainties are characterised by matrices below:

$$\begin{aligned} E_{11} &= E_{12} = \begin{bmatrix} 0.02 \\ 0.01 \end{bmatrix} & E_{21} &= E_{22} = 0.01 \\ H_1 &= \begin{bmatrix} 0.01 & 0.02 \end{bmatrix} & H_2 &= 0.01 \\ H_3 &= 0.01 \end{aligned} \tag{7.4.2}$$

The membership functions for the plant, μ_g are as follows

$$\begin{aligned} \mu_1(x_1(k)) &= \begin{cases} 0 & x_1(k) < -0.5 \\ x_1(k) + 0.5 & -0.5 \le x_1(k) < 0.5 \\ 1 & 0.5 \le x_1(k) \end{cases}, \\ \mu_2(x_1(k)) &= 1 - \mu_1(x_1(k)) \end{aligned} \tag{7.4.3}$$

The membership functions for the controller, λ_h are shown below

$$\begin{aligned} \lambda_1(x_1(k-\tau_k)) &= \begin{cases} 1 & x_1(k-\tau_k) < -0.5 \\ -x_1(k-\tau_k) + 0.5 & -0.5 \le x_1(k-\tau_k) < 0.5 \\ 0 & 0.5 \le x_1(k-\tau_k) \end{cases}, \\ \lambda_2(x_1(k-\tau_k)) &= 1 - \lambda_1(x_1(k-\tau_k)) \end{aligned} \tag{7.4.4}$$

Figure 7.1 shows $\mu_g(x_1(k))$ and $\lambda_h(x_1(k-\tau_k))$ with sub-regions in each membership function, $x_1 \in [-\infty, -0.5]$, $x_1 \in [-0.5, 0]$, $x_1 \in [0, 0.5]$ and $x_1 \in [0.5, \infty]$, and similarly for the membership function of the controller. This means that we divide $\mu_g(x)\lambda_h(x_\tau)$ into 16 sub-regions. For every sub-region we use $\eta_{gh,s_\kappa}(x, x_\tau)$ to obtain polynomial approximation as shown in Table 7.1. The upper and lower bounds of the error terms, α_{gh,s_κ} and β_{gh,s_κ}, are obtained numerically and shown in Table 7.2 and 7.3 respectively. Note that $\mu_1(x_1(k)) = \lambda_2(x_1(k-\tau_k))$ and $\mu_2(x_1(k)) = \lambda_1(x_1(k-\tau_k))$ in this particular example are the same.

The delays are characterized by a Markov chain taking values in a finite set $\mathcal{S} = \{1, 2\}$, which correspond to 0.1, 0.2 seconds delays, respectively. The transition probability matrix is given by;

$$P_\tau = \begin{bmatrix} 0.6 & 0.4 \\ 0.7 & 0.3 \end{bmatrix} \tag{7.4.5}$$

The sampling time in this example is $0.01s$ and the prescribed γ is 1.0.

In this thesis, YALMIP [126] is used to obtain the controller. In this example stabi-

	κ	$\eta_{gh,s_\kappa}(x, x_\tau)$
$i=1, j=1$	1,...,4,8,12,13,16	0
	5	$1.4(x_1(k)+0.63)^2$
	6	$1.96(x_1(k)+0.63)^2\{(x_1(k-\tau_k)-0.13)^2+0.5\}$
	7	$1.96(x_1(k)+0.63)^2(x_1(k-\tau_k)-0.63)^2$
	9	$1.4(x_1(k)+0.13)^2+0.5$
	10	$1.96\{(x_1(k)+0.13)^2+0.5\}\{(x_1(k-\tau_k)-0.13)^2+0.5\}$
	11	$1.96\{(x_1(k)+0.13)^2+0.5\}(x_1(k-\tau_k)-0.63)^2$
	14	$1.4(x_1(k-\tau_k)-0.13)^2+0.5$
	15	$1.4(x_1(k-\tau_k)-0.63)^2$
$i=1, j=2$	1,...,5,9,13,16	0
	6	$1.96(x_1(k)+0.63)^2(x_1(k-\tau_k)+0.63)^2$
	7	$1.96(x_1(k)+0.63)^2\{(x_1(k-\tau_k)+0.13)^2+0.5\}$
	8	$1.4(x_1(k)+0.63)^2$
	10	$1.96\{(x_1(k)+0.13)^2+0.5\}(x_1(k-\tau_k)+0.63)^2$
	11	$1.96\{(x_1(k)+0.13)^2+0.5\}\{(x_1(k-\tau_k)+0.13)^2+0.5\}$
	12	$1.4(x_1(k)+0.13)^2+0.5$
	14	$1.4(x_1(k-\tau_k)+0.63)^2$
	15	$1.4(x_1(k-\tau_k)+0.13)^2+0.5$
$i=2, j=1$	1,4,8,12,...,16	0
	2	$1.4(x_1(k-\tau_k)-0.13)^2+0.5$
	3	$1.4(x_1(k-\tau_k)-0.63)^2$
	5	$1.4(x_1(k)-0.13)^2+0.5$
	6	$1.96\{(x_1(k)-0.13)^2+0.5\}\{(x_1(k-\tau_k)-0.13)^2+0.5\}$
	7	$1.96\{(x_1(k)-0.13)^2+0.5\}(x_1(k-\tau_k)-0.63)^2$
	9	$1.4(x_1(k)-0.63)^2$
	10	$1.96(x_1(k)-0.63)^2\{(x_1(k-\tau_k)-0.13)^2+0.5\}$
	11	$1.96(x_1(k)-0.63)^2(x_1(k-\tau_k)-0.63)^2$
$i=2, j=2$	1,4,5,9,13,...,16	0
	2	$1.4(x_1(k-\tau_k)+0.63)^2$
	3	$1.4(x_1(k-\tau_k)+0.13)^2+0.5$
	6	$1.96\{(x_1(k)-0.13)^2+0.5\}(x_1(k-\tau_k)+0.63)^2$
	7	$1.96\{(x_1(k)-0.13)^2+0.5\}\{(x_1(k-\tau_k)+0.13)^2+0.5\}$
	8	$1.4(x_1(k)-0.13)^2+0.5$
	10	$1.4(x_1(k)-0.63)^2(x_1(k-\tau_k)+0.63)^2$
	11	$1.4(x_1(k)-0.63)^2\{(x_1(k-\tau_k)+0.13)^2+0.5\}$
	12	$1.4(x_1(k)-0.63)^2$

Table 7.1: Polynomial approximation of membership functions for Example 1

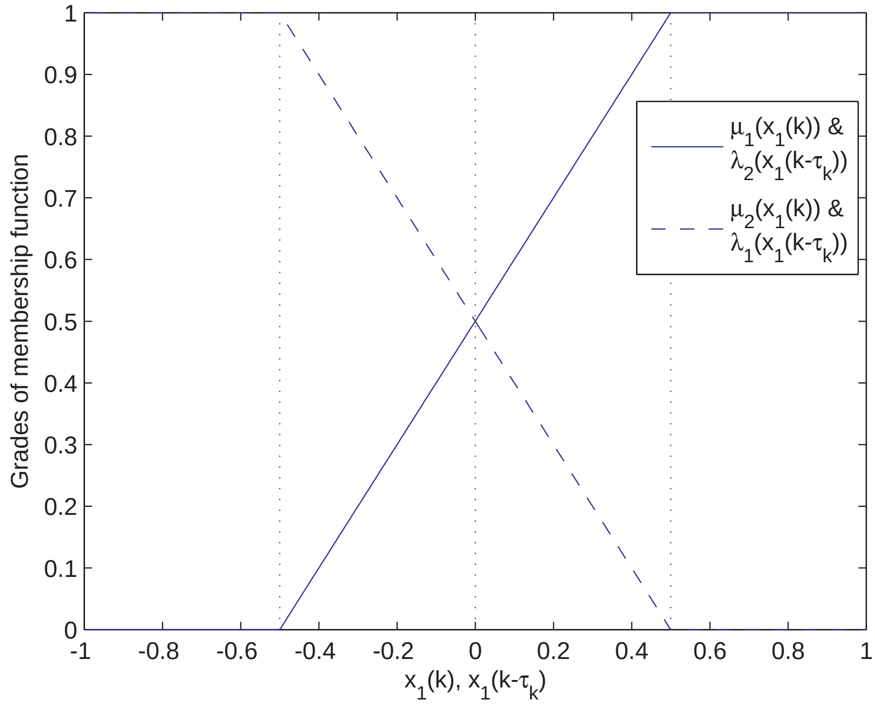

Figure 7.1: Membership functions of the plant and the controller in Example 1

	κ	β_{gh,s_κ}
$i=1, j=1$	1,...,4,8,12,13,16	0
	5,9,14,15	0.049
	6,11	0.047
	7	0.023
	10	0.070
$i=1, j=2$	1,...,5,9,13,16	0
	6	0.023
	7,10	0.047
	8,12,14,15	0.049
	11	0.070
$i=2, j=1$	1,4,8,12,...,16	0
	2,3,5,9	0.049
	6	0.070
	7,10	0.047
	11	0.023
$i=2, j=2$	1,4,5,9,13,...,16	0
	2,3,8,12	0.049
	6,11	0.047
	7	0.070
	10	0.023

Table 7.2: Upper bounds of the error terms for Example 1, β_{gh,s_κ}

	κ	α_{gh,s_κ}
$i=1, j=1$	1,...,4,8,12,13,16	0
	5,9,14,15	-0.056
	6,11	-0.087
	7	-0.059
	10	-0.114
$i=1, j=2$	1,...,5,9,13,16	0
	6	-0.059
	7,10	-0.087
	8,12,14,15	-0.056
	11	-0.114
$i=2, j=1$	1,4,8,12,...,16	0
	2,3,5,9	-0.056
	6	-0.114
	7,10	-0.087
	11	-0.059
$i=2, j=2$	1,4,5,9,13,...,16	0
	2,3,8,12	-0.056
	6,11	-0.087
	7	-0.114
	10	-0.059

Table 7.3: Lower bounds of the error terms for Example 1, α_{gh,s_κ}

lization regions are obtained for Theorem 7.3.1 and the approach without incorporating membership functions as shown in [86]. Figure 7.2 shows the comparison of stabilization regions where $0.6 \leq a \leq 1.2$ and $0 \leq b \leq 3$.

Remark 7.4.1 *It is shown in Figure 7.2 that the existing approach in [86] where membership functions are not considered has smaller stabilization region. For example, when* $a = 0.6$ *the largest value of* b *without considering membership functions is* 0.8 *whereas Theorem 7.3.1 provides significantly larger values, 2.8. It is noteworthy that when* $a = 1.1$ *only Theorem 7.3.1 provides a feasible controller.*

Example 2 Consider the single-link rigid robot connected through a joint to the basement and whose plane of motion is vertical. The motion equation of this mechanical system is given by [131]

$$J\ddot{\theta} = -(0.5mgl + Mgl)sin(\theta) + u \tag{7.4.6}$$

where θ denote the joint rotation angle in radians, $m = 1.5kg$ is the mass of the load, $M = 3kg$ is the mass of the rigid link, $g = 9.8m/s^2$ is the gravity constant, $l = 0.5m$ is the length of the robot link, $J = Ml^2 + (1/3)ml^2 = 0.875$ is the moment of inertia, and u is the control torque applied at the joint in Nm. $\theta = 0$ denotes the lowest vertical

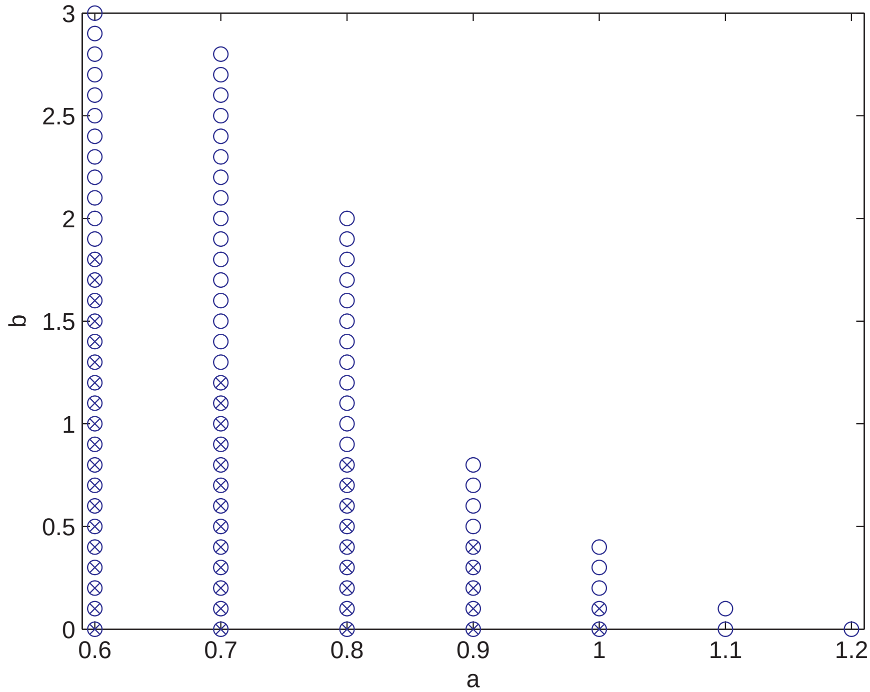

Figure 7.2: Stability region from Theorem 7.3.1 (o) and without considering membership functions (x)

equilibrium position under zero control torque. The control task is to move the robot arm from any initial state $\theta \in [-(\pi/2),\ (\pi/2)]$ to the equilibrium position defined by $\theta = 0$, $\dot{\theta} = 0$, and $\ddot{\theta} = 0$ despite of perturbations of plant parameters.

As shown in Chapter 6, the discrete-time T-S fuzzy model of the plant above is obtained with two plant rules ($r = 2$) and two controller rules ($c = 2$). The sub-systems are described as follows

$$\begin{aligned}
A_1 &= \begin{bmatrix} 0.0990 & 0.0100 \\ -0.2099 & 0.9990 \end{bmatrix} & A_2 &= \begin{bmatrix} 0.0993 & 0.0100 \\ -0.1337 & 0.9993 \end{bmatrix} \\
B_{11} &= \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} & B_{12} &= \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} \\
C_1 &= \begin{bmatrix} 1 & 0 \end{bmatrix} & C_2 &= \begin{bmatrix} 1 & 0 \end{bmatrix} \\
B_{21} &= \begin{bmatrix} 5.7133 \times 10^{-5} \\ 0.0114 \end{bmatrix} & B_{22} &= \begin{bmatrix} 5.7133 \times 10^{-5} \\ 0.0114 \end{bmatrix} \\
D_{11} &= D_{12} = 0.01 & D_{21} &= D_{22} = 0.1
\end{aligned} \tag{7.4.7}$$

The membership functions for Rule 1 and Rule 2 are shown in Figure 6.2 in the previous Chapter. The transition probability matrix for the network is identical as the previous example. The sub-regions in each membership function are defined as $x_1 \in [-\pi/2, 0]$ and $x_1 \in [0, \pi/2]$, and similarly for the membership function of the

	κ	$\eta_{gh,s_\kappa}(x, x_\tau)$
$i=1, j=1$	1	$0.04\{(x_1(k)+\frac{1.45\pi}{2})^2\}\{(x_1(k-\tau_k)+\frac{1.45\pi}{2})^2\}$
	2	$0.04\{(x_1(k)+\frac{1.45\pi}{2})^2\}\{(x_1(k-\tau_k)-\frac{1.45\pi}{2})^2\}$
	3	$0.04\{(x_1(k)-\frac{1.45\pi}{2})^2\}\{(x_1(k-\tau_k)+\frac{1.45\pi}{2})^2\}$
	4	$0.04\{(x_1(k)-\frac{1.45\pi}{2})^2\}\{(x_1(k-\tau_k)-\frac{1.45\pi}{2})^2\}$
$i=1, j=2$	1	$0.04\{(x_1(k)+\frac{1.45\pi}{2})^2\}\{(x_1(k-\tau_k)-\frac{0.45\pi}{2})^2\}$
	2	$0.04\{(x_1(k)+\frac{1.45\pi}{2})^2\}\{(x_1(k-\tau_k)+\frac{0.45\pi}{2})^2\}$
	3	$0.04\{(x_1(k)-\frac{1.45\pi}{2})^2\}\{(x_1(k-\tau_k)-\frac{0.45\pi}{2})^2\}$
	4	$0.04\{(x_1(k)-\frac{1.45\pi}{2})^2\}\{(x_1(k-\tau_k)+\frac{0.45\pi}{2})^2\}$
$i=2, j=1$	1	$0.04\{(x_1(k)-\frac{1.45\pi}{2})^2\}\{(x_1(k-\tau_k)+\frac{1.45\pi}{2})^2\}$
	2	$0.04\{(x_1(k)-\frac{1.45\pi}{2})^2\}\{(x_1(k-\tau_k)-\frac{1.45\pi}{2})^2\}$
	3	$0.04\{(x_1(k)+\frac{1.45\pi}{2})^2\}\{(x_1(k-\tau_k)+\frac{1.45\pi}{2})^2\}$
	4	$0.04\{(x_1(k)+\frac{1.45\pi}{2})^2\}\{(x_1(k-\tau_k)-\frac{1.45\pi}{2})^2\}$
$i=2, j=2$	1	$0.04\{(x_1(k)-\frac{1.45\pi}{2})^2\}\{(x_1(k-\tau_k)-\frac{0.45\pi}{2})^2\}$
	2	$0.04\{(x_1(k)-\frac{1.45\pi}{2})^2\}\{(x_1(k-\tau_k)+\frac{0.45\pi}{2})^2\}$
	3	$0.04\{(x_1(k)+\frac{1.45\pi}{2})^2\}\{(x_1(k-\tau_k)-\frac{0.45\pi}{2})^2\}$
	4	$0.04\{(x_1(k)+\frac{1.45\pi}{2})^2\}\{(x_1(k-\tau_k)+\frac{0.45\pi}{2})^2\}$

Table 7.4: Polynomial approximation of membership functions for Example 2

	κ	β_{gh,s_κ}	α_{gh,s_κ}
$i=1, j=1$	1,...,4	0.660	-0.1037
$i=1, j=2$	1,...,4	0.660	-0.1037
$i=2, j=1$	1,...,4	0.660	-0.1037
$i=2, j=2$	1,...,4	0.660	-0.1037

Table 7.5: Upper and lower bounds of the error terms, β_{gh,s_κ} and α_{gh,s_κ}, of Example 2

controller. This means that we divide $\mu_g(x)\lambda_h(x_\tau)$ into 4 sub-regions. For every sub-region polynomial approximation, $\eta_{gh,s_\kappa}(x, x_\tau)$ is obtained. For example, in the sub-region when $x_1 \in [-\pi/2, 0]$ and $x_{1\tau} = x_1(k-\tau) \in [-\pi/2, 0]$, $\mu_1(x_1) = (\pi/2 - |x_1|)/(\pi/2)$ and $\lambda_1(x_{1\tau}) = 1 - \big((\pi/2 - |x_\tau|)/(\pi/2)\big)$ can be approximated as $0.2(x_1 + 1.45\pi/2)^2$ and $0.2(x_{1\tau} - 1.45\pi/2 + \pi/2)^2$, respectively. Hence, $\mu_1(x_1)\lambda_1(x_{1\tau})$ can be approximated as $(0.2(x_1 + 1.45\pi/2)^2)(0.2(x_{1\tau} - 1.45\pi/2 + \pi/2)^2)$ in that sub-region. Table 7.4 shows the polynomial approximations of the membership functions for other sub-regions.

The upper and lower bounds of the error terms, α_{gh,s_κ} and β_{gh,s_κ}, are obtained by finding the maximum and minimum values of the difference between the actual membership functions and the polynomial approximations. For example, in the sub-region when $x_1 \in [-\pi/2, 0]$ and $x_{1\tau} \in [-\pi/2, 0]$, the upper and lower bounds of the error are obtained, respectively, by searching for the maximum and minimum value of $\Big\{\big((\pi/2 - |x_1|)/(\pi/2)\big)\{1 - \big((\pi/2 - |x_{1\tau}|)/(\pi/2)\big)\}\Big\} - \Big\{(0.2(x_1 + 1.45\pi/2)^2)(0.2(x_{1\tau} - 1.45\pi/2 + \pi/2)^2)\Big\}$. Table 7.5 shows the upper and lower bounds for the sub-regions.

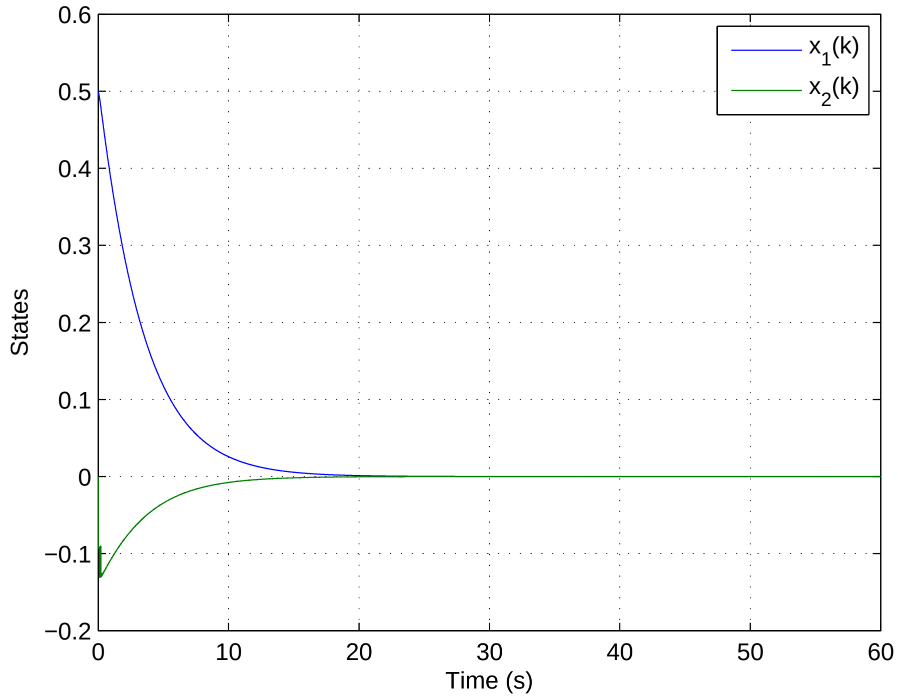

Figure 7.3: State response of the robot

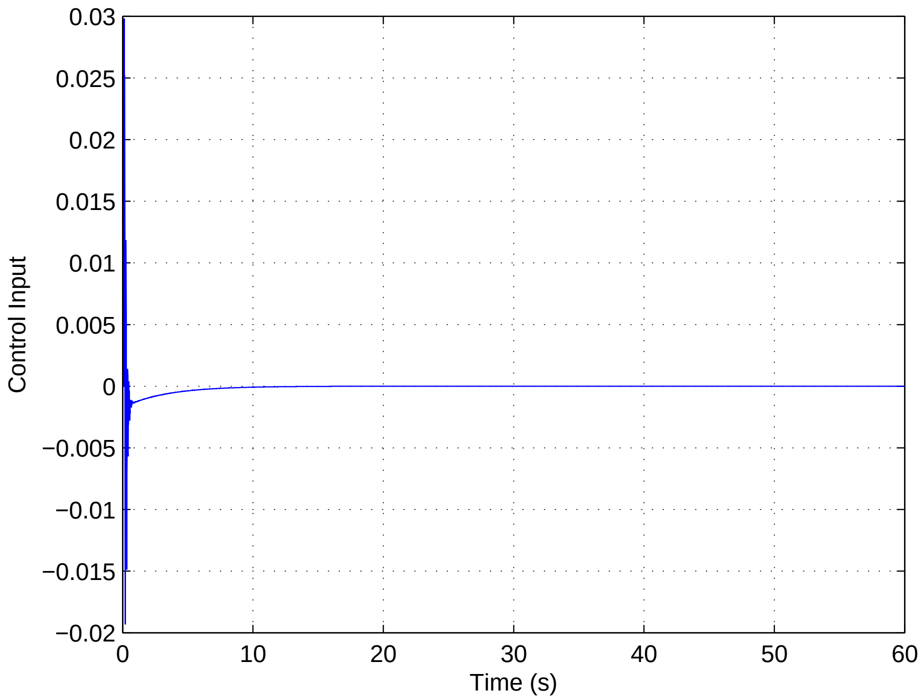

Figure 7.4: Control input of the system

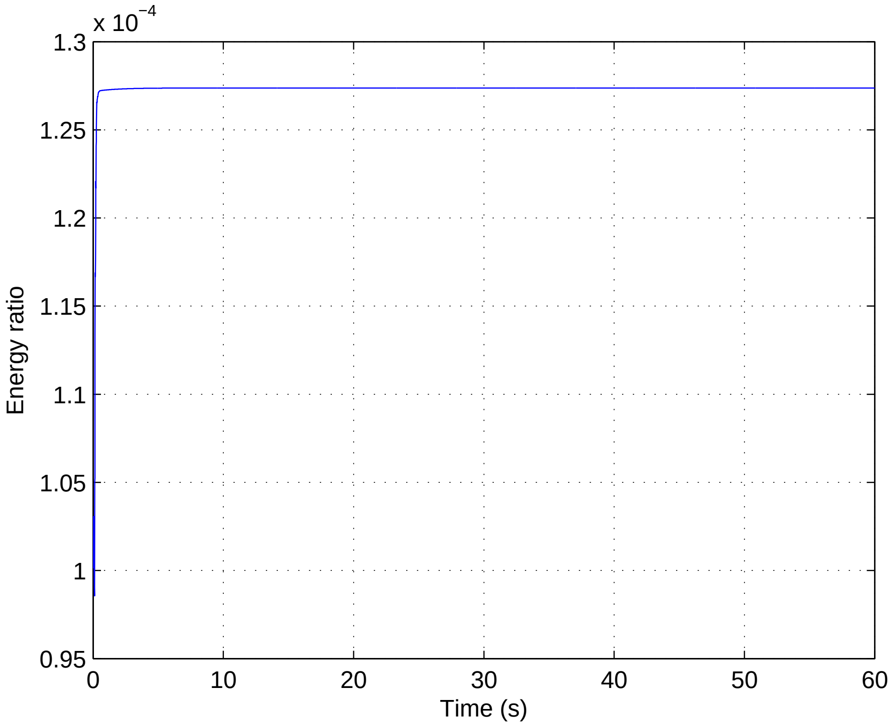

Figure 7.5: Ratio of the energy of the controlled output to the energy of the disturbance ($\gamma = 1.0$)

The network-induced delays, the transition probability matrix and the prescribed γ are the same as Example 1.

Using Theorem 7.3.1 and the algorithm, a controller of the form (7.2.4) with the following gains are obtained.

$$K_1(1) = \begin{bmatrix} -6.1974 & 4.3743 \end{bmatrix}, \quad K_1(2) = \begin{bmatrix} -6.0786 & 3.7676 \end{bmatrix}$$
$$K_2(1) = \begin{bmatrix} -6.1974 & 4.3743 \end{bmatrix}, \quad K_2(2) = \begin{bmatrix} -6.0786 & 3.7676 \end{bmatrix}$$

Remark 7.4.2 *The state response of the closed-loop system is shown in Figure 7.3 and the control input is shown in Figure 7.4. The initial states are chosen to be* $x(0) = [0.5 \quad 0]^T$*. It can be seen that the system is stochastically stable. Figure 7.5 shows the ratio of the energy of the output to the energy of the disturbance (*$w(k) = e^{-0.1k}\sin(0.5k)$*). From Figure 7.5, one can see that the ratio tends to roughly* 1.27×10^{-4}*, which is less than the prescribed* $\gamma = 1$*. It is shown that the ratio is well under the prescribed value of 1.0, illustrating the validity of the controller.*

7.5 Conclusions

This chapter presents a mode delay dependent fuzzy state feedback controller design for a class of nonlinear discrete-time networked control systems, where the plant is modelled by Takagi-Sugeno fuzzy model, and random network induced delays by finite state Markov chain. The fuzzy controller's premise variables are allowed to have delays, to cater for network induced delays in the premise variables of the plant while transmitting through communication networks. Membership functions of the plant and controller are then approximated by polynomial functions and incorporated in the design. Through numerical examples, we show that the proposed methodology yields a much wider stabilization region and demonstrate the validity of the proposed methodology.

8

Robust Fuzzy $\mathcal{H}_\infty$ Filtering of Nonlinear Networked Control Systems With Partially Known Transition Probability Matrix

Abstract

In this chapter, a robust fuzzy $\mathcal{H}_\infty$ filtering problem for nonlinear NCSs is discussed where the plant is modelled by T-S fuzzy model and the network is modelled by a Markov chain whose transition probability matrix is allowed to be partially known. As shown in the previous chapter, the membership functions of the plant and the filter are incorporated into the filter design using SOS approach. Furthermore, the fact that the premise variables of the plant, which is the measured output in this case, experience delays as they are transmitted to the filter is acknowledged. Sufficient conditions for the existence of a robust fuzzy $\mathcal{H}_\infty$ filter is obtained based on Lyapunov-Krasovskii functional. A numerical example is provided to illustrate the effectiveness of the proposed methodology.

8.1 Introduction

As already discussed in Chapter 4, filtering problem is an important area of study, particularly in signal processing, as it provides a way to estimate information of the plant based on the measured output. The major advantage of an $\mathcal{H}_\infty$ filter over the traditional Kalman filter is that the former does not require the exact information of the external disturbance. For this reason there has been several literatures in recent years on study of $\mathcal{H}_\infty$ filtering and this trend progressed to NCSs where the objective of the filter is to estimate information of the plant based on the signal transmitted over a network [42, 97–100, 116, 117]. In [42], more specific approach to modelling the constraints is presented where the pack loss is modelled by Bernoulli distributed white sequence. However, this paper assumes that the network-induced delays are constant. In [116, 117] $\mathcal{H}_\infty$ filter design is considered for nonlinear NCSs described by T-S fuzzy model. In this paper, the network-induced delays and packet loss which are induced by the limited bandwidth of communication networks, are considered.

In NCSs, where the system components are connected via a network, any signal between the system components will experience network-induced delays and be subject to packet loss. In filtering problems, the signal passed to the filter will inevitably experience these constraints. This means that the signal the filter receives will be the time delayed version of the signal produced at the plant. Since the signal received by the filter will become the premise variable of the fuzzy filter, it is very important to acknowledge this issue. However, none of the aforementioned literature on fuzzy filtering problem of nonlinear NCSs described by T-S fuzzy model [42, 97–100, 116, 117] acknowledge this.

The traditional approach to fuzzy filter design is to discard the membership functions of the plant and the filter in order to derive the conditions for the filter in terms of LMIs. Since the membership functions are nonlinear functions in x, LMI solvers cannot handle the membership functions. Discarding the membership functions in the filter design means that the filer is valid for any shape of membership functions and this may lead to severe conservatism.

Motivated the the aforementioned drawbacks, membership functions are incorporated in the design methodology of a robust fuzzy $\mathcal{H}_\infty$ filter for nonlinear NCSs using SOS approach. The plant is modelled by T-S fuzzy model and the network is modelled by a Markov chain whose transition probability matrix is allowed to be partially known. The premise variables of the filter is allowed to be different to the premise variables of the plant to acknowledge the network-induced delays between the plant and the filter.

Sufficient conditions for the existence of a robust fuzzy $\mathcal{H}_\infty$ filter is obtained based on Lyapunov-Krasovskii functional, which can be solved by existing tools such as YALMIP and SOSTOOLS.

The rest of the chapter is organised as follows. Section 8.2 provides the description of the system and the filter. The $\mathcal{H}_\infty$ filtering problem is formulated and presented in this this section. Sufficient conditions for the existence of the fuzzy filter is presented in Section 8.3. An algorithm to obtain the filter gains is also presented. A numerical example of single-link rigid robot is shown in Section 8.4 to illustrate the validity of the presented approach. Finally, conclusions are presented in Section 8.5.

8.2 System Description and Definitions

The nonlinear plant is described by the following T-S fuzzy model:

Plant Rule g:
IF $\theta_1(x(k))$ is J_1^g AND $\cdots$ AND $\theta_p(x(k))$ is J_p^g,
THEN

$$\begin{aligned} x(k+1) &= [A_g + \Delta A_g(k)]x(k) + [B_{1g} + \Delta B_{1g}(k)]w(k) \\ z(k) &= [C_{1g} + \Delta C_g(k)]x(k) + [D_{1g} + \Delta D_{1g}(k)]w(k) \\ y(k) &= C_{2g}x(k) \end{aligned} \tag{8.2.1}$$

where g denotes the g^{th} fuzzy inference rule; $g = 1, \ldots, r$; r is the number of inference rules; $\theta_1(x(k)), \ldots, \theta_p(x(k))$ are the premise variables; p is the number of premise variables and $J_1^g, \ldots, J_p^g$ are the fuzzy terms. Furthermore $x(k) \in \Re^n$, $u(k) \in \Re^{m_1}$, $z(k) \in \Re^{m_2}$, $y(k) \in \Re^{m_3}$ are the state, control input, objective signal to be estimated and measurable output respectively and $w(k) \in \Re^{m_3}$ is the disturbance which belongs to $\mathcal{L}_2[0, \infty)$, the space of square summable vector sequence over $[0, \infty]$. The matrices A_g, B_{1g}, B_{2g}, C_{1g}, C_{2g}, D_{1g} and D_{2g} are of appropriate dimensions. The matrix functions $\Delta A_g(k)$, $\Delta B_{1g}(k)$, $\Delta B_{2g}(k)$, $\Delta C_{1g}(k)$, $\Delta D_{1g}(k)$ and $\Delta D_{2g}(k)$ represent the time-varying uncertainties in the system.

By using a center-average defuzzifier, product inference and singleton fuzzifier, the following global nonlinear model is obtained.

$$
\begin{aligned}
x(k+1) &= \sum_{g=1}^{r} \mu_g(\theta(x(k)))\big\{[A_g + \Delta A_g(k)]x(k) + [B_g + \Delta B_g(k)]w(k)\big\} \\
&= [A(\mu) + \Delta A(\mu,k)]x(k) + [B(\mu) + \Delta B(\mu,k)]w(k) \\
z(k) &= \sum_{g=1}^{r} \mu_g(\theta(x(k)))\big\{[C_{1g} + \Delta C_g(k)]x(k) + [D_g + \Delta D_g(k)]w(k)\big\} \\
&= [C_1(\mu) + \Delta C(\mu,k)]x(k) + [D(\mu) + \Delta D(\mu,k)]w(k) \\
y(k) &= \sum_{g=1}^{r} \mu_g(\theta(x(k)))C_{2g}x(k) = C_2(\mu)x(k)
\end{aligned}
\tag{8.2.2}
$$

where

$$
\theta(x(k)) = [\theta_1(x(k)), \ldots, \theta_p(x(k))],
$$

$$
\chi_g(\theta(x(k))) = \prod_{t=1}^{p} J_t^g(\theta_t(x(k))),
$$

$$
\mu_g(\theta(x(k)) = \frac{\chi_g(\theta(x(k)))}{\sum_{\ell=1}^{r} \chi_\ell(\theta(x(k)))} \in [0,1],
$$

$$
\sum_{g=1}^{r} \mu_g(\theta(x(k))) = 1
$$

$$
\begin{array}{ll}
A(\mu) = \sum_{g=1}^{r} \mu_g(\theta(x(k)))A_g & C_1(\mu) = \sum_{g=1}^{r} \mu_g(\theta(x(k)))C_{1g} \\
B(\mu) = \sum_{g=1}^{r} \mu_g(\theta(x(k)))B_g & D(\mu) = \sum_{g=1}^{r} \mu_g(\theta(x(k)))D_g \\
C_2(\mu) = \sum_{g=1}^{r} \mu_g(\theta(x(k)))C_{2g} &
\end{array}
$$

$$
\begin{array}{ll}
\Delta A(\mu,k) = \sum_{g=1}^{r} \mu_g(\theta(x(k)))\Delta A_g(k) & \Delta C_1(\mu,k) = \sum_{g=1}^{r} \mu_g(\theta(x(k)))\Delta C_{1g}(k) \\
\Delta B(\mu,k) = \sum_{g=1}^{r} \mu_g(\theta(x(k)))\Delta B_g(k) & \Delta D(\mu,k) = \sum_{g=1}^{r} \mu_g(\theta(x(k)))\Delta D_g(k)
\end{array}
$$

We assume that the uncertainty functions in k are norm-bounded by the following:

Assumption 8.2.1

$$
\begin{bmatrix} \Delta A_g(k) & \Delta B(k) \\ \Delta C_{1g}(k) & \Delta D(k) \end{bmatrix} = \begin{bmatrix} E_{1g} \\ E_{2g} \end{bmatrix} F(k) \begin{bmatrix} H_1 & H_2 \end{bmatrix}
$$

where $g = 1, \ldots, r$; r is the number of fuzzy inference rules; H_1, H_2, E_{1g} and E_{2g} are known matrices which characterize the structure of the uncertainties; $F(k)$ is an unknown matrix function that satisfy $F^T(k)\mathcal{W}F(k) \le \mathcal{W}$ where $\mathcal{W}$ is a positive-definite matrix.

The network is modelled by a finite state homogeneous Markov process as shown in Chapter 3. The transition probability matrix is allowed to be partially known.

In this chapter, we consider the following fuzzy full-order filter:

Filter Rule h:
IF $\sigma_1(x_f(k))$ is N_1^h AND $\cdots$ AND $\sigma_q(x_f(k))$ is N_q^h,
THEN

$$\begin{aligned} x_f(k+1) &= \hat{A}_h(i)x_f(k) + \hat{B}_h(i)y(k-\tau_k) \\ z_f(k) &= \hat{C}_h(i)x_f(k) \end{aligned} \tag{8.2.3}$$

where $x_f(k)$ is the filter's state; $\hat{A}_h(i)$, $\hat{B}_h(i)$, $\hat{C}_h(i)$ are the filter gain matrices; h denotes the h^{th} fuzzy inference rule; $h = 1, \ldots, c$; c is the number of inference rules; $\sigma_1(x_f(k))$, $\ldots$, $\sigma_q(x_f(k))$, are the premise variables; q is the number of premise variables and N_1^h, $\ldots$, N_q^h are the fuzzy terms.

Similar to the plant, the fuzzy filter is inferred as shown below.

$$\begin{aligned} x_f(k+1) &= \sum_{h=1}^{c} \lambda_h(\sigma(x_f(k))\big\{\hat{A}_h(r_k)x_f(k) + \hat{B}_h(r_k)y(k-\tau_k)\big\} \\ &= \hat{A}(\lambda, r_k)x_f(k) + \hat{B}(\lambda, r_k)y(k-\tau_k) \\ z_f(k) &= \sum_{h=1}^{c} \lambda_h(\sigma(x_f(k))\hat{C}_h(r_k)x_f(k) \\ &= \hat{C}(\lambda, r_k)x_f(k) \end{aligned} \tag{8.2.4}$$

where

$$\sigma(x_f(k)) = [\sigma_1(x_f(k)), \ldots, \sigma_q(x_f(k))],$$

$$\phi_h(\sigma(x_f(k)) = \prod_{t=1}^{q} N_t^h(\sigma_t(x_f(k))),$$

$$\lambda_h(\sigma(x_f(k)) = \frac{\phi_h(\sigma(x_f(k)))}{\sum_{\ell=1}^{c} \phi_\ell(\sigma(x_f(k)))} \in [0, 1],$$

$$\sum_{h=1}^{c} \lambda_h(\sigma(x_f(k))) = 1$$

The augmented filtering error system with the fuzzy model (8.2.2) and the fuzzy filter (8.2.4) is

$$\begin{aligned}\tilde{x}(k+1) &= [\tilde{A}(\mu,\lambda,r_k)+\bar{E}_1(\mu)F(k)\bar{H}_1]\tilde{x}(k)+\tilde{B}(\lambda,r_k)\bar{C}_2(\mu)\tilde{x}(k-\tau_k)\\ &\quad +[\bar{B}(\mu)+\bar{E}_1(\mu)F(k)H_2]w(k)\\ e(k) &= [\tilde{C}(\mu,\lambda,r_k)+E_2(\mu)F(k)\bar{H}_1]\tilde{x}(k)+[D(\mu)\\ &\quad +E_2(\mu)F(k)H_2]w(k) \qquad (8.2.5)\end{aligned}$$

where $\tilde{x}^T(k)=\begin{bmatrix}x^T(k) & x_f^T(k),\end{bmatrix}$, $e(k)=z(k)-z_f(k)$ and

$$\begin{aligned}\tilde{A}(\mu,\lambda,r_k) &= \begin{bmatrix}A(\mu) & 0\\ 0 & \hat{A}(\lambda,r_k)\end{bmatrix}, \tilde{B}(\lambda,r_k)=\begin{bmatrix}0\\ \hat{B}(\lambda,r_k)\end{bmatrix},\\ \bar{B}(\mu) &= \begin{bmatrix}B_1(\mu)\\ 0\end{bmatrix}, \bar{C}_2(\mu)=\begin{bmatrix}C_2(\mu) & 0\end{bmatrix},\\ \bar{E}_1(\mu) &= \begin{bmatrix}E_1(\mu)\\ 0\end{bmatrix}, \bar{H}_1=\begin{bmatrix}H_1 & 0\end{bmatrix},\\ C_{cl}(\mu,\lambda,r_k) &= \begin{bmatrix}C_1(\mu) & -\hat{C}(\lambda,r_k)\end{bmatrix}.\end{aligned}$$

For brevity, $x(k)$, $x_f(k)$, $\mu_g(\theta(x(k)))$, $\lambda_h(\sigma(x_f(k)))$ are denoted as x, x_f, $\mu_g(x)$, $\lambda_h(x_f)$, respectively, throughout this chapter.

As shown in Chapter 2, the closed-loop system is to achieve stochastic stability, as shown in (2.2.5), and the $\mathcal{H}_\infty$ performance condition, as shown in (2.2.6), except that $e(k)$ is used in this chapter as opposed to $z(k)$ in Chapter 2.

In order to incorporate the unknown transition probabilities, Lemma 3.2.2 introduced in Chapter 3 is used.

Similar to Chapter 7, the products of membership functions are approximated by polynomial functions shown as below

$$\mu_g(x)\lambda_h(x_f)=\sum_{\kappa=1}^{D}\zeta_\kappa(x,x_f)\{\eta_{gh,s_\kappa}(x,x_f)+\Delta\eta_{gh,s_\kappa}(x,x_f)\} \qquad (8.2.6)$$

where $\eta_{gh,s_\kappa}(x,x_f)$ are the polynomial function approximations and $\Delta\eta_{gh,s_\kappa}(x,x_f)$ are the error terms in each sub-region. $\zeta_\kappa(x,x_f)$ is a scalar function which takes 1 if x and x_f are inside the sub-region, s_κ, and 0 otherwise.

The upper and lower bounds of the error terms are introduced as follows

$$\alpha_{gh,s_\kappa} \leq \Delta\eta_{gh,s_\kappa}(x, x_f) \leq \beta_{gh,s_\kappa} \tag{8.2.7}$$

where α_{gh,s_κ} and β_{gh,s_κ} are known constants.

8.3 Main Result

In this section, sufficient conditions for the existence of a robust fuzzy $\mathcal{H}_\infty$ filter for a class of discrete-time nonlinear NCSs where the network is modelled by a Markov chain with partially known transition probability matrix.

Theorem 8.3.1 *Given a prescribed $\mathcal{H}_\infty$ performance, $\gamma > 0$, the closed-loop system is stochastically stable with the prescribed $\mathcal{H}_\infty$ performance, if there exist sets of positive-definite matrices $P(i)$, $X(i)$, $\tilde{R}_1(i)$, $\tilde{R}_1$, $\tilde{R}_2(i)$, $\tilde{R}_2$, $W_1(i)$, $W_2(i)$, $\tilde{W}_1(i)$, Q, $\tilde{Q}$, N_1, N_2, $\tilde{Z}(i)$, $S(i,j)$, $\tilde{\Psi}_1^{gh}(i)$, $\tilde{\Psi}_2^{gh}(i)$ and matrices $K_h(i)$, $\hat{C}_h(i)$, $\tilde{M}(i)$, $J(i)$ for $i = 1, 2, \cdots, s$, $g = 1, 2, \cdots, r$, $h = 1, 2, \cdots, c$ satisfying the following*

$$\tilde{R}_1 > \tilde{R}_1(i) \tag{8.3.1}$$

$$\tilde{R}_2 > \tilde{R}_2(i) \tag{8.3.2}$$

$$-v^T\Big[\sum_{g=1}^{r}\sum_{h=1}^{c}\Big\{(\eta_{gh,s_\kappa}(x, x_f) + \frac{1}{2}\alpha_{gh,s_\kappa} + \frac{1}{2}\beta_{gh,s_\kappa})\tilde{T}^{gh}(i) + \tilde{V}^{gh,s_\kappa}(i)\Big\} - \tilde{\epsilon}_{s_\kappa} I\Big]v \text{ is SOS}$$
$$\forall s_\kappa = 1, \ldots, D \tag{8.3.3}$$

$$\frac{1}{2}\tilde{T}^{gh}(i) - \tilde{\Psi}_1^{gh}(i) < 0 \tag{8.3.4}$$

$$-\frac{1}{2}\tilde{T}^{gh}(i) - \tilde{\Psi}_2^{gh}(i) < 0 \tag{8.3.5}$$

$$\begin{bmatrix} (1 - p_{i(i+1)})\tilde{R}_1(i) + \tilde{R}_2(i) & \tilde{M}(i) \\ * & \tilde{Z}(i) \end{bmatrix} \geq 0 \tag{8.3.6}$$

$$\begin{bmatrix} p^i_{known}\tilde{R}_1(i) + \tilde{R}_2(i) & \tilde{M}(i) \\ * & \tilde{Z}(i) \end{bmatrix} \geq 0, \quad \forall (i+1) \in \mathcal{S}^i_{unknown} \tag{8.3.7}$$

$$\begin{bmatrix} S(i,j) & J^T(i) \\ * & X(j) \end{bmatrix} > 0 \tag{8.3.8}$$

and

$$N_1\tilde{R}_1 = I,\ N_2\tilde{R}_2 = I, \tilde{W}_1(i)W_1(i) = I, Q\tilde{Q} = I \text{ and } P(i)X(i) = I, \tag{8.3.9}$$

where v *is a real vector of appropriate dimension and independent of* x*;* $\tilde{\epsilon}_{s_\kappa}$ *are predefined scalars;* $\eta_{gh,s_\kappa}(x, x_f)$ *are defined in (9.2.6);*

$$\begin{aligned}
\tilde{T}^{gh}(i) &= \begin{bmatrix} \tilde{\Lambda}(i) & \big(\tilde{\Gamma}_1^{gh}(i)\big)^T & \big(\tilde{\Gamma}_2^{gh}(i)\big)^T & \big(\tilde{\Gamma}_3\big)^T & \big(\tilde{\Xi}^{gh}(i)\big)^T & \big(\tilde{\mathcal{H}}\big)^T \\ * & -\mathcal{X} & 0 & 0 & 0 & 0 \\ * & * & -\mathcal{R} & 0 & 0 & 0 \\ * & * & * & -\tilde{Q} & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -\tilde{W}_1(i) \end{bmatrix} \\
\tilde{V}^{gh,s_\kappa}(i) &= (\beta_{gh,s_\kappa} - \alpha_{gh,s_\kappa})\Big\{\tilde{\Psi}_1^{gh}(i) + \tilde{\Psi}_2^{gh}(i)\Big\} \\
\tilde{\Lambda}(i) &= diag\Big\{ - P(i), -Q, \Big(H_2^T W_2(i) H_2 - \gamma I\Big), -W_1(i), -W_2(i)\Big\} \\
&\quad + \tilde{\Upsilon}_1^T(i) + \tilde{\Upsilon}_1(i) + \tau(i)\tilde{Z}(i) \\
\tilde{\Gamma}_1^{gh}(i) &= \Big[\big(\tilde{A}_g + \bar{E}K_h(i)\bar{R}\big) \quad \bar{E}K_h(i)\hat{E}\bar{C}_{2g} \quad \bar{B}_{1g} \quad \bar{E}_{1g} \quad \bar{E}_{1g}\Big] \\
\tilde{\Gamma}_2^{gh}(i) &= \Big[\sqrt{\tilde{\tau}(i)} \quad \sqrt{\tau(s)}\Big]^T \Big[\big(\tilde{A}_g - diag\{I,0\}\big) \quad 0 \quad \bar{B}_{1g} \quad \bar{E}_{1g} \quad \bar{E}_{1g}\Big] \\
\tilde{\Gamma}_3 &= \Big[\sqrt{\tau(s) - \tau(1) + 1} \quad 0 \quad 0 \quad 0 \quad 0\Big] \\
\tilde{\tau}(i) &= \textstyle\sum_{j\in\mathcal{S}^i_{known}}^{i+1} p_{ij}\tau(j) + (1 - p^i_{known}) \sum_{j\in\mathcal{S}^i_{unknown}}^{i+1} \tau(j) \\
\mathcal{X} &= -\{\textstyle\sum_{j\in\mathcal{S}^i_{known}}^{i+1} p_{ij}S(i,j) + (1 - p^i_{known}) \sum_{j\in\mathcal{S}^i_{unknown}}^{i+1} S(i,j)\} + J^T(i) + J(i) \\
\mathcal{R} &= diag\Big\{N_1, N_2\Big\} \\
\tilde{\Xi}^{gh}(i) &= \Big[\ \bar{C}_1^{gh}(i) \quad 0 \quad D_{1g} \quad E_{2g} \quad E_{2g}\ \Big] \\
\tilde{\mathcal{H}} &= \Big[\ \bar{H}_1 \quad 0 \quad 0 \quad 0 \quad 0\ \Big] \\
\tilde{\Upsilon}_1(i) &= \tilde{M}^T(i)[diag\{I,0\} \quad diag\{-I,0\} \quad 0 \quad 0 \quad 0] \\
\tilde{A}_g &= \begin{bmatrix} A_g & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{E} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \bar{R} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \quad \hat{E} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \\
\bar{B}_{2g} &= \begin{bmatrix} B_{2g} \\ 0 \end{bmatrix}, \quad \bar{C}_1^{gh}(i) = \Big[C_{1g} \quad -\hat{C}_h(i)\Big], \quad \bar{C}_{2g} = \Big[C_{2g} \quad 0\Big]
\end{aligned}$$

Note that $\bar{E}$ *and* $\hat{E}$ *have different dimension. Moreover, the robust fuzzy* $\mathcal{H}_\infty$ *filter* *is given as*

$$x_f(k+1) = \sum_{h=1}^{c} \lambda_h(\sigma(x_f(k))\{\hat{A}_h(i)x_f(k) + \hat{B}_h(i)y(k-\tau_k)\}$$
$$z_f(k) = \sum_{h=1}^{c} \lambda_h(\sigma(x_f(k))\hat{C}_h(i)x_f(k) \tag{8.3.10}$$

where the filter gain matrices, $\hat{A}_h(i)$ *and* $\hat{B}_h(i)$ *are given by*

$$\begin{bmatrix} \hat{A}_h(i) & \hat{B}_h(i) \end{bmatrix} = K_h(i)$$

and $\hat{C}_h(i)$ *directly from the theorem.*

Proof: We introduce the following augmented closed system from (8.2.5)

$$\begin{aligned} \tilde{x}_{k+1} &= \Gamma_1(r_k)\zeta_k \\ e_k &= \Xi(r_k)\zeta_k \end{aligned} \tag{8.3.11}$$

where $\tilde{x}_\ell = \tilde{x}(\ell)$, $z_\ell = z(\ell)$, $\zeta_\ell = \zeta(\ell)$, and

$$\zeta(k)^T = \begin{bmatrix} \tilde{x}^T(k) & \tilde{x}^T(k-\tau(r_k)) & w^T(k) & \tilde{x}^T(k)\bar{H}_1^T F^T(k) & w^T(k)H_2^T F^T(k) \end{bmatrix} \in \Re^l$$
$$\Gamma_1(r_k) = \begin{bmatrix} \tilde{A}(\mu,\lambda,r_k) & \tilde{B}(\lambda,r_k)\bar{C}_2(\mu) & \bar{B}(\mu) & \bar{E}_1(\mu) & \bar{E}_1(\mu) \end{bmatrix}$$
$$\Xi(r_k) = \begin{bmatrix} \tilde{C}(\mu,\lambda,r_k) & 0 & D(\mu) & E_2(\mu) & E_2(\mu) \end{bmatrix}$$

Introducing the following Lyapunov-Krasovskii functional

$$V(\tilde{x}_k, r_k) = V_1(\tilde{x}_k, r_k) + V_2(\tilde{x}_k, r_k) + V_3(\tilde{x}_k, r_k) \tag{8.3.12}$$

where

$$V_1(\tilde{x}_k, r_k) = \tilde{x}_k^T P(r_k)\tilde{x}_k \tag{8.3.13}$$

$$V_2(\tilde{x}_k, r_k) = \sum_{\ell=-\tau_k}^{-1} \sum_{j=k+\ell}^{k-1} \bar{x}_j^T R_1 \bar{x}_j + \sum_{\ell=-\tau(s)}^{-1} \sum_{j=k+\ell}^{k-1} \bar{x}_j^T R_2 \bar{x}_j \tag{8.3.14}$$

$$V_3(\tilde{x}_k, r_k) = \sum_{\ell=k-\tau_k}^{k-1} \tilde{x}_\ell^T Q \tilde{x}_\ell + \sum_{\ell=-\tau(s)+2}^{-\tau(1)+1} \sum_{j=k+\ell-1}^{k-1} \tilde{x}_j^T Q \tilde{x}_j \tag{8.3.15}$$

where $\bar{x}(k) = \begin{bmatrix} x(k+1) - x(k) \\ 0 \end{bmatrix}$.

Following a similar approach shown in the proof section of Chapter 7, we have

$$\begin{aligned}\Delta V(\tilde{x}_k, r_k) \;\le\; & -\tilde{x}_k^T\Big(P(r_k) - (\tau(s) - \tau(1) + 1)Q\Big)\tilde{x}_k - \tilde{x}_{k-\tau_k}^T Q\tilde{x}_{k-\tau_k} + \\ & \zeta_k^T\Big\{\Gamma_1^T(\mu,\lambda,r_k)\tilde{P}(r_k)\Gamma_1(\mu,\lambda,r_k) \\ & +\Gamma_2^T(\mu,\lambda,r_k)\left[\tilde{\tau}_k R_1 + \tau(s)R_2\right]\Gamma_2(\mu,\lambda,r_k) \\ & +\tilde{\Upsilon}_1(r_k) + \tilde{\Upsilon}_1^T(r_k) + \tau_k\tilde{Z}(r_k)\Big\}\zeta_k \qquad (8.3.16)\end{aligned}$$

Using Assumption 8.2.1, and adding and subtracting $\tilde{x}_k^T\bar{H}_1^T F_k^T W_1(r_k)F_k\bar{H}_1\tilde{x}_k$, $w_k^T H_2^T F_k^T W_2(r_k)F_k H_2 w_k$, $z_k^T z_k$ and $\gamma w_k^T w_k$ to and from (8.3.16), we obtain the following

$$\begin{aligned}\Delta V(\tilde{x}_k, r_k) \;\le\; & -\tilde{x}_k^T\Big(P(r_k) - (\tau(s) - \tau(1) + 1)Q - \bar{H}_1^T F_k^T W_1(r_k)F_k\bar{H}_1\Big)\tilde{x}_k \\ & -\tilde{x}_{k-\tau_k}^T Q\tilde{x}_{k-\tau_k} + \zeta_k^T\Big\{\Gamma_1^T(\mu,\lambda,r_k)\tilde{P}(r_k)\Gamma_1(\mu,\lambda,r_k) \\ & +\Gamma_2^T(\mu,\lambda,r_k)\left[\tilde{\tau}_k R_1 + \tau(s)R_2\right]\Gamma_2(\mu,\lambda,r_k) \\ & +\tilde{\Upsilon}_1(r_k) + \tilde{\Upsilon}_1^T(r_k) + \tau_k\tilde{Z}(r_k) + \Xi^T(\mu,\lambda,r_k)\Xi(\mu,\lambda,r_k)\Big\}\zeta_k \\ & -w_k^T\Big(\gamma I - H_2^T W_3(r_k)H_2\Big)w_k - \tilde{x}_k^T\bar{H}_1^T F_k^T W_1(r_k)F_k\bar{H}_1\tilde{x}_k \\ & -w_k^T H_2^T F_k^T W_3(r_k)F_k H_2 w_k - z_k^T z_k + \gamma w_k^T w_k \qquad (8.3.17)\end{aligned}$$

The above now becomes

$$\begin{aligned}\Delta V(\tilde{x}_k, r_k) \;\le\; & \zeta_k^T\Big\{\Lambda(\lambda,r_k) + \Gamma_1^T(\mu,\lambda,r_k)\tilde{P}(r_k)\Gamma_1(\mu,\lambda,r_k) \\ & +\Gamma_2^T(\mu,\lambda,r_k)\left[\tilde{\tau}_k R_1 + \tau(s)R_2\right]\Gamma_2(\mu,\lambda,r_k) + \Xi^T(\mu,\lambda,r_k)\Xi(\mu,\lambda,r_k) \\ & +\tilde{\Upsilon}_1(r_k) + \tilde{\Upsilon}_1^T(r_k) + \tau_k\tilde{Z}(r_k)\Big\}\zeta_k - z_k^T z_k + \gamma w_k^T w_k \qquad (8.3.18)\end{aligned}$$

where

$$\begin{aligned}\Lambda(\lambda,r_k) = &\text{diag}\Big\{\Big((\tau(s) - \tau(1) + 1)Q + \bar{H}_1^T W_1(r_k)\bar{H}_1 - P(r_k)\Big), -Q, \\ & \Big(H_2^T W_3(r_k)H_2 - \gamma I\Big), -W_1(r_k), -W_2(r_k)\Big\}\end{aligned}$$

The SOS condition in (8.3.3) implies

$$\begin{aligned}& \sum_{g=1}^{r}\sum_{h=1}^{c}\Big[(\eta_{gh,s_\kappa}(x,x_f) + \frac{1}{2}\alpha_{gh,s_\kappa} + \frac{1}{2}\beta_{gh,s_\kappa})\tilde{T}^{gh}(r_k) \\ & +(\beta_{gh,s_\kappa} - \alpha_{gh,s_\kappa})\{\tilde{\Psi}_1^{gh}(r_k) + \tilde{\Psi}_2^{gh}(r_k)\}\Big] < 0 \qquad (8.3.19)\end{aligned}$$

Since $\tilde{\Psi}_1^{gh}(r_k)$ and $\tilde{\Psi}_2^{gh}(r_k)$ are positive definite matrices, along with (8.2.6), we have

$$(\Delta\eta_{gh,s_\kappa}(x,x_f) - \alpha_{gh,s_\kappa})\tilde{\Psi}_1^{gh}(r_k) \le (\beta_{gh,s_\kappa} - \alpha_{gh,s_\kappa})\tilde{\Psi}_1^{gh}(r_k) \tag{8.3.20}$$

$$(\beta_{gh,s_\kappa} - \Delta\eta_{gh,s_\kappa}(x,x_f))\tilde{\Psi}_2^{gh}(r_k) \le (\beta_{gh,s_\kappa} - \alpha_{gh,s_\kappa})\tilde{\Psi}_2^{gh}(r_k) \tag{8.3.21}$$

Using the above, (8.3.19) becomes

$$\sum_{g=1}^{r}\sum_{h=1}^{c}\Big[(\eta_{gh,s_\kappa}(x,x_f) + \frac{1}{2}\alpha_{gh,s_\kappa} + \frac{1}{2}\beta_{gh,s_\kappa})\tilde{T}^{gh}(r_k) + (\Delta\eta_{gh,s_\kappa}(x,x_f) - \alpha_{gh,s_\kappa})\tilde{\Psi}_1^{gh}(r_k)$$
$$+(\beta_{gh,s_\kappa} - \Delta\eta_{gh,s_\kappa}(x,x_f))\tilde{\Psi}_2^{gh}(r_k)\Big] < 0 \tag{8.3.22}$$

Combining the above with conditions (8.3.4) and (8.3.5) we have

$$\sum_{g=1}^{r}\sum_{h=1}^{c}\Big[(\eta_{gh,s_\kappa}(x,x_f) + \frac{1}{2}\alpha_{gh,s_\kappa} + \frac{1}{2}\beta_{gh,s_\kappa})\tilde{T}^{gh}(r_k) + (\Delta\eta_{gh,s_\kappa}(x,x_f) - \alpha_{gh,s_\kappa})\frac{1}{2}\tilde{T}^{gh}(r_k)$$
$$-(\beta_{gh,s_\kappa} - \Delta\eta_{gh,s_\kappa}(x,x_f))\frac{1}{2}\tilde{T}^{gh}(r_k)\Big] < 0 \tag{8.3.23}$$

By rearranging the above we have

$$\begin{aligned}&\sum_{g=1}^{r}\sum_{h=1}^{c}\Big[\eta_{gh,s_\kappa}(x,x_f)\tilde{T}^{gh}(r_k) + \frac{1}{2}(\Delta\eta_{gh,s_\kappa}(x,x_f) + \beta_{gh,s_\kappa} - \beta_{gh,s_\kappa})\tilde{T}^{gh}(r_k)\\ &+\frac{1}{2}(\Delta\eta_{gh,s_\kappa}(x,x_f) + \alpha_{gh,s_\kappa} - \alpha_{gh,s_\kappa})\tilde{T}^{gh}(r_k)\Big]\\ =\;&\sum_{g=1}^{r}\sum_{h=1}^{c}\{\eta_{gh,s_\kappa}(x,x_f) + \Delta\eta_{gh,s_\kappa}(x,x_f)\}\tilde{T}^{gh}(r_k) < 0\end{aligned} \tag{8.3.24}$$

Using (8.2.6), we now have

$$\sum_{g=1}^{r}\sum_{h=1}^{c}\mu_g(x)\lambda_h(x_f)\begin{bmatrix}\tilde{\Lambda}(r_k) & \left(\tilde{\Gamma}_1^{gh}(r_k)\right)^T & \left(\tilde{\Gamma}_2^{gh}(r_k)\right)^T & \left(\tilde{\Gamma}_3\right)^T & \left(\tilde{\Xi}^{gh}(r_k)\right)^T & \left(\tilde{\mathcal{H}}\right)^T\\ * & -\mathcal{X} & 0 & 0 & 0 & 0\\ * & * & -\mathcal{R} & 0 & 0 & 0\\ * & * & * & -\tilde{Q} & 0 & 0\\ * & * & * & * & -I & 0\\ * & * & * & * & * & -\tilde{W}_1(r_k)\end{bmatrix} < 0 \tag{8.3.25}$$

Note that $\mathcal{X}$ and $\tilde{\Gamma}_2^{gh}(r_k)$ terms contain transition probabilities. We know that the summation of all probabilities in each row of transition probability matrix is one. This means that $(1-p^i_{known}) = \sum_{j\in unknown}^{i+1} p_{ij}$. By using Lemma 3.2.2, we can replace $\tilde{\tau}(i)$ and $\mathcal{X}$ with $\sum_{j=1}^{i+1} p_{ij}\tau(j)$ and $-\sum_{j=1}^{i+1} p_{ij}S(i,j) + J^T(i) + J(i)$ while still satisfying the above condition.

Applying Schur complement on (8.3.8) and consequently multiplying these inequalities by p_{ij} and summing up for all j we obtain

$$\begin{aligned} -\sum_{j=1}^{i+1} p_{ij} S(i,j) + J^T(i) + J(i) &= J^T(i) + J(i) - \sum_{j=1}^{i+1} p_{ij} S(i,j) \\ &< J^T(i) + J(i) - J^T(i)\tilde{P}(i)J(i) \\ &= \tilde{P}^{-1}(i) - \Big(J(i) - \tilde{P}^{-1}(i)\Big)^T \tilde{P}(i)\Big(J(i) - \tilde{P}^{-1}(i)\Big) \\ &< \tilde{P}^{-1}(i) \end{aligned} \tag{8.3.26}$$

where $P(i) = X^{-1}(i)$ and $\tilde{P}(i) = \sum_{j=1}^{i+1} p_{ij} P(j)$. This means that (8.3.25) still satisfies even if $-\sum_{j=1}^{i+1} p_{ij} S(i,j) + J^T(i) + J(i)$ is replaced with $\tilde{P}^{-1}(i)$.

Taking summations and membership functions inside the matrix, along with the above, (8.3.25) becomes

$$\begin{bmatrix} \bar{\Lambda}(r_k)+\tilde{\Upsilon}_1(r_k)+\tilde{\Upsilon}_1^T(r_k)+\tau(r_k)\tilde{Z}(r_k) & \check{\Gamma}_1^T(\mu,\lambda,r_k) & \Gamma_2^T(\mu,\lambda,r_k) & \Big(\Gamma_3\Big)^T & \Xi^T(\mu,\lambda,r_k) & \Big(\tilde{\mathcal{H}}\Big)^T \\ * & -\tilde{P}^{-1}(r_k) & 0 & 0 & 0 & 0 \\ * & * & -(\tau_k R_1+\tau(s)R_2)^{-1} & 0 & 0 & 0 \\ * & * & * & -Q^{-1} & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -W_1^{-1}(r_k) \end{bmatrix} < 0 \tag{8.3.27}$$

Note that expanding $\big(\tilde{A}_g + \bar{E}K_h(i)\bar{R}\big)$, $\big(\bar{E}K_h(i)\hat{E}\bar{C}_{2g}\big)$ and $\big(\tilde{A}_g - \text{diag}\{I,0\}\big)$ and taking summations and membership function inside, we obtain $\tilde{A}(\mu,\lambda,r_k)$, $\tilde{B}(\lambda,r_k)$ and $\bar{A}(\mu,\lambda,r_k)$ respectively in the above.

By following from the proof section in Chapter 7 we can show that the filter satisfies the conditions in the problem formulation, thus completing the proof. $\nabla\nabla\nabla$

A similar algorithm to the one in Chapter 7 is applied to this chapter to solve Theorem 8.3.1.

8.4 Example

The same single-link rigid robot shown in the previous chapter is considered. The transition probability matrix for this example is identical as Example 1 in the previous chapter with the probabilities in the second row being the unknown probabilities. The membership functions for Rule 1 and Rule 2 are shown in Figure 6.2 in Chapter 6. The polynomial approximations and corresponding bounds are presented in Table 7.4 and 7.5 in Chapter 7 respectively. Note that the premise variables of this example are x and x_f, instead of x and x_τ. Refer to Example 2 of Chapter 7 for more information on the membership function approximation procedure.

From Theorem 8.3.1 and the algorithm, a robust fuzzy $\mathcal{H}_\infty$ filter of the form (8.2.4) with the following gains are obtained.

$$\hat{A}_1(1) = 1\times10^{-3}\times\begin{bmatrix}5.6836 & 7.2156\\ -0.0064 & 0.9556\end{bmatrix},\ \hat{A}_1(2) = 1\times10^{-3}\times\begin{bmatrix}7.7136 & 6.9134\\ 11.5303 & -12.1558\end{bmatrix},$$
$$\hat{A}_2(1) = 1\times10^{-3}\times\begin{bmatrix}5.6568 & -7.1665\\ 0.0274 & 0.8845\end{bmatrix},\ \hat{A}_2(2) = 1\times10^{-3}\times\begin{bmatrix}1.8988 & 8.6864\\ 14.6451 & -14.4914\end{bmatrix},$$
$$\hat{B}_1(1) = 1\times10^{-3}\times\begin{bmatrix}10.2731\\ 0.4252\end{bmatrix},\ \hat{B}_1(2) = 1\times10^{-3}\times\begin{bmatrix}-0.8381\\ 0.1223\end{bmatrix}$$
$$\hat{B}_2(1) = 1\times10^{-3}\times\begin{bmatrix}10.2071\\ 0.3872\end{bmatrix},\ \hat{B}_2(2) = 1\times10^{-3}\times\begin{bmatrix}0.0262\\ 0.5207\end{bmatrix}$$
$$\hat{C}_1(1) = 1\times10^{-3}\times\begin{bmatrix}0.9445 & -0.6180\end{bmatrix},\ \hat{C}_1(2) = 1\times10^{-3}\times\begin{bmatrix}-0.1522 & 0.1675\end{bmatrix}$$
$$\hat{C}_2(1) = 1\times10^{-3}\times\begin{bmatrix}0.9452 & -0.6223\end{bmatrix},\ \hat{C}_2(2) = 1\times10^{-3}\times\begin{bmatrix}0.2858 & -0.0448\end{bmatrix}$$

Figure 8.1 shows the response of filtering error system with the prescribed $\gamma = 1.0$. It is shown that the filtering error converges to zero. The initial condition is $x(0) = [0.5\ \ 0]^T$ and the external disturbance, $w(k)$, is given as $e^{-0.01k}sin(0.5k)$. Figure 8.2 shows the ratio of the energy of the error to the energy of the disturbance, which needs to be less than $\gamma = 1.0$. As shown in the figure, the actual ratio is approximately 5.5×10^{-3}, which is well under the prescribed attenuation level.

8.5 Conclusions

A methodology to design a robust fuzzy $\mathcal{H}_\infty$ filter for a class of discrete-time nonlinear NCSs is presented in this chapter. The plant is modelled by T-S fuzzy model and the network is modelled by a finite state Markov chain whose transition probability matrix is allowed to be partially known. The premise variables of the filter is the time delayed version of the measured output to incorporate the network-induced delays between the plant and the filter. The membership functions of the plant and the filter are approximated

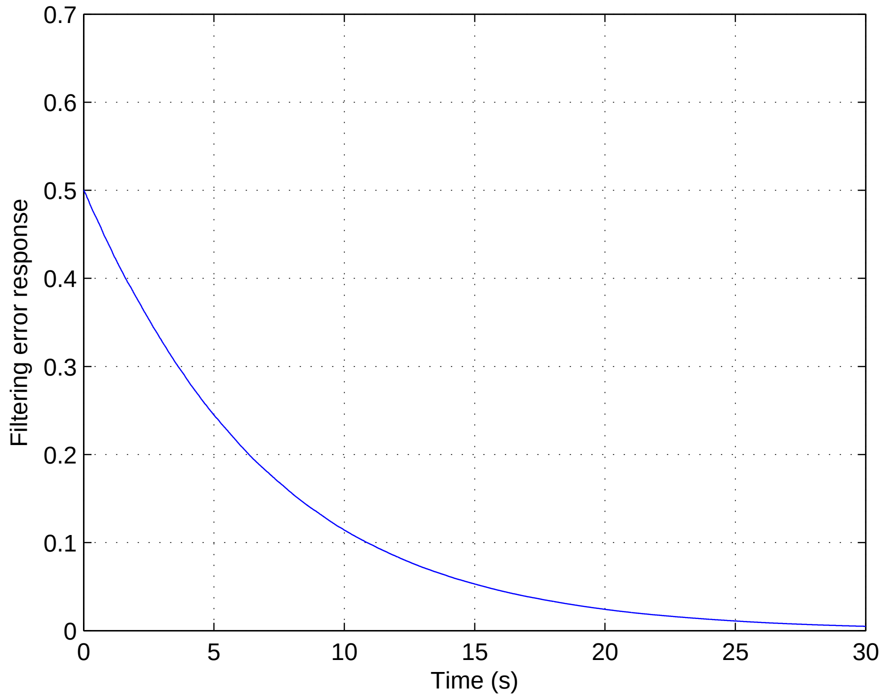

Figure 8.1: Response of filtering error

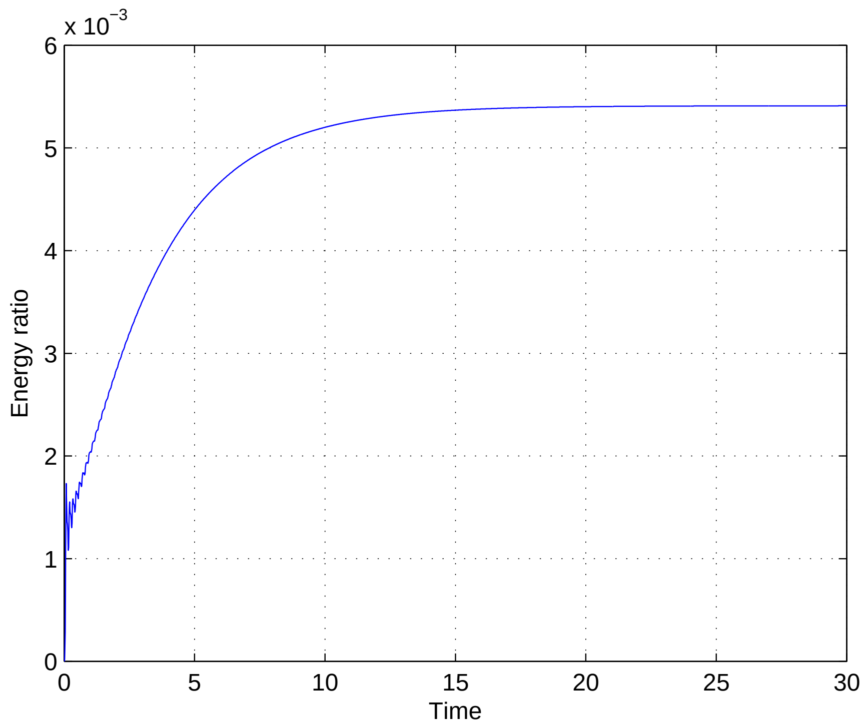

Figure 8.2: Ratio of the energy of the filtering error to the energy of the disturbance ($\gamma = 1.0$)

by polynomial functions and incorporated into the filter design. Sufficient conditions for the existence of the filter is given in terms of sum-of-squares inequalities, which can be solved using YALMIP. A numerical example is provided to demonstrate the validity of the proposed methodology.

9

Robust Fuzzy $\mathcal{H}_\infty$ Dynamic Output Feedback Control of Nonlinear Networked Control Systems With Partially Known Transition Probability Matrix

Abstract

In this paper, a methodology for designing a fuzzy dynamic output feedback controller for discrete-time nonlinear networked control systems is presented where the nonlinear plant is modelled by a Takagi-Sugeno fuzzy model and the network-induced delays by a finite state Markov process. The transition probability matrix for the Markov process is allowed to be partially known, providing a more practical consideration of the real world. Furthermore, the fuzzy controller's membership functions and premise variables are not assumed to be the same as the plant's membership functions and premise variables, that is, the proposed approach can handle the case, when the premise of the plant are not measurable or delayed. The membership functions of the plant and the controller are approximated as polynomial functions, then incorporated into the controller design. Sufficient conditions for the existence of the controller are derived in terms of sum of square inequalities, which are then solved by YALMIP. Finally, a numerical example is used to demonstrate the validity of the proposed methodology.

9.1 Introduction

Most of the existing studies on NCSs are based on linear NCSs and the study of nonlinear NCSs is very limited due to mathematical complexities. However, it is a fact that existing real world systems are nonlinear, making it important to study nonlinear systems. Ever since its proposition almost three decades ago [101], Takagi-Sugeno (T-S) fuzzy model has been very effective in modelling nonlinear NCSs as shown in [36, 58, 85, 86, 128–130]. Many of the existing literatures in NCSs consider state feedback control. In many T-S fuzzy models, the premise variables often are the plant state variables and these are not always completely measurable. Due to this reason, designing a dynamic output feedback controller is more attractive in practice as not all plant state variables are needed to be measurable. However, dynamic output feedback control of nonlinear NCSs has not been explored extensively yet [89], [132]. In [89] a dynamic output feedback controller is designed for a continuous-time nonlinear NCSs. This paper considers distributed delays and disturbance in the system. In [132], Bernoulli distribution is used to describe multiple probabilistic communication delays and packet dropouts. A dynamic output feedback controller is obtained for a discrete-time nonlinear NCSs. However, these aforementioned approaches in [36, 58, 85, 86, 89, 128–130, 132], for both state and output feedback control, assume that the controller's premise variables are the same as that of the plant's. In NCSs, signals are transmitted via network therefore will inevitably experience delays. In [36, 58, 85, 86, 89, 128–130, 132], the premise variables of the plant, which are the state variables of the plant, are assumed to be measurable and the same premise variables are used by the fuzzy controller. However in NCSs the premise will inevitably experience network-induced delays, that is, if the premise variables of the controller are selected to be the same as the plant, then the fuzzy controller has be based on the delayed premise of the plant. Therefore, the fuzzy controller based on the current premise of the plant as in [36, 58, 85, 86, 89, 128–130, 132], may be impractical in NCSs.

As already explained in Chapter 7, many existing literature on T-S fuzzy model neglects the membership functions in the controller design. Due to the fact, in [36, 58, 85, 86, 89, 128–130, 132], the membership functions are discarded in the controller design, hence the obtained controller is valid for any membership functions and it may lead to severe conservatism. It has been shown in Chapter 7 that incorporating membership functions yield larger stabilization region in robust fuzzy $\mathcal{H}_\infty$ state feedback controller design.

Aforementioned practical drawbacks and lack of existing studies in dynamic output feedback control of nonlinear NCSs motivated to investigate a methodology for designing a fuzzy dynamic output feedback controller for nonlinear NCSs modelled by T-S fuzzy model, where the network-induced delays are modelled by a finite state Markov process with a partially known transition probability matrix. Furthermore, membership functions into the controller design are incorporated by approximating them using polynomial functions, ensuring that the controller obtained is valid for the particular membership functions. It is shown that, by using a numerical example, that this approach results wider stabilization region. In this chapter, unlike many existing literatures in nonlinear NCSs [36, 58, 85, 86, 89, 128–130, 132], the premise variables of the plant are allowed to be unmeasurable or unavailable. The premise variables and membership functions of the controller are allowed to be different to those of the plant. Hence, the number of fuzzy rules of the controller is no longer restricted by the number of fuzzy rules of the plant. This allows simple fuzzy controller even when the fuzzy model of the plant is complicated. This design flexibility, and subsequent low implementation cost, is an advantage of the proposal approach over traditional parallel distributed compensation (PDC) approach. This consideration, along with the partially known transition probability matrix, is more practical in the real NCSs, that is, the proposal approach can handle the case, when not all the premise variables are measurable and also when the premise experiencing networked-induced delays. By using numerical examples it is shown that this incorporation yields larger stability region in fuzzy dynamic output feedback control as well as demonstrating the effectiveness of the controller.

The organisation of this chapter is as follows. In Section 9.2, system description and problem formulation are provided. The controller design methodology for the nonlinear NCSs is presented in Section 9.3. A numerical example is provided in Section 9.4 where two approaches, one that incorporates membership functions in the controller design and one that does not, are compared and their stability regions are presented. Conclusions are drawn in Section 9.5.

9.2 System Description and Definitions

The nonlinear plant is described by the following T-S fuzzy model:

Plant Rule g:
IF $\theta_1(x(k))$ is J_1^g AND $\cdots$ AND $\theta_p(x(k))$ is J_p^g,
THEN

$$\begin{aligned} x(k+1) &= [A_g + \Delta A_g(k)]x(k) + [B_{1g} + \Delta B_{1g}(k)]w(k) + [B_{2g} + \Delta B_{2g}(k)]u(k) \\ z(k) &= [C_{1g} + \Delta C_g(k)]x(k) + [D_{1g} + \Delta D_{1g}(k)]w(k) + [D_{2g} + \Delta D_{2g}(k)]u(k) \\ y(k) &= C_{2g}x(k) \end{aligned} \tag{9.2.1}$$

where g denotes the g^{th} fuzzy inference rule; $g = 1, \ldots, r$; r is the number of inference rules; $\theta_1(x(k)), \ldots, \theta_p(x(k))$ are the premise variables; p is the number of premise variables and $J_1^g, \ldots, J_p^g$ are the fuzzy terms. Furthermore $x(k) \in \Re^n$, $u(k) \in \Re^{m_1}$, $z(k) \in \Re^{m_2}$, $y(k) \in \Re^{m_3}$ are the state, control input, controlled output and measurable output respectively and $w(k) \in \Re^{m_3}$ is the disturbance which belongs to $\mathcal{L}_2[0, \infty)$, the space of square summable vector sequence over $[0, \infty]$. The matrices A_g, B_{1g}, B_{2g}, C_{1g}, C_{2g}, D_{1g} and D_{2g} are of appropriate dimensions. The matrix functions $\Delta A_g(k)$, $\Delta B_{1g}(k)$, $\Delta B_{2g}(k)$, $\Delta C_{1g}(k)$, $\Delta D_{1g}(k)$ and $\Delta D_{2g}(k)$ represent the time-varying uncertainties in the system.

By using a center-average defuzzifier, product inference and singleton fuzzifier, the following global nonlinear model is obtained.

$$\begin{aligned} x(k+1) &= \sum_{g=1}^{r} \mu_g(\theta(x(k)))\big\{[A_g + \Delta A_g(k)]x(k) + [B_{1g} + \Delta B_{1g}(k)]w(k) \\ &\quad + [B_{2g} + \Delta B_{2g}(k)]u(k)\big\} \\ &= [A(\mu) + \Delta A(\mu, k)]x(k) + [B_1(\mu) + \Delta B_1(\mu, k)]w(k) + [B_2(\mu) + \Delta B_2(\mu, k)]u(k) \\ z(k) &= \sum_{g=1}^{r} \mu_g(\theta(x(k)))\big\{[C_{1g} + \Delta C_g(k)]x(k) + [D_{1g} + \Delta D_{1g}(k)]w(k) \\ &\quad + [D_{2g} + \Delta D_{2g}(k)]u(k)\big\} \\ &= [C_1(\mu) + \Delta C(\mu, k)]x(k) + [D_1(\mu) + \Delta D_1(\mu, k)]w(k) \\ &\quad + [D_2(\mu) + \Delta D_2(\mu, k)]u(k) \\ y(k) &= \sum_{g=1}^{r} \mu_g(\theta(x(k)))C_{2g}x(k) = C_2(\mu)x(k) \end{aligned} \tag{9.2.2}$$

where

$$\theta(x(k)) = [\theta_1(x(k)), \ldots, \theta_p(x(k))],$$

$$\chi_g(\theta(x(k))) = \prod_{t=1}^{p} J_t^g(\theta_t(x(k))),$$

$$\mu_g(\theta(x(k)) = \frac{\chi_g(\theta(x(k)))}{\sum_{\ell=1}^{r} \chi_\ell(\theta(x(k)))} \in [0, 1],$$

$$\sum_{g=1}^{r} \mu_g(\theta(x(k))) = 1$$

$$\begin{array}{ll} A(\mu) = \sum_{g=1}^{r} \mu_g(\theta(x(k)))A_g & C_1(\mu) = \sum_{g=1}^{r} \mu_g(\theta(x(k)))C_{1g} \\ B_1(\mu) = \sum_{g=1}^{r} \mu_g(\theta(x(k)))B_{1g} & B_2(\mu) = \sum_{g=1}^{r} \mu_g(\theta(x(k)))B_{2g} \\ D_1(\mu) = \sum_{g=1}^{r} \mu_g(\theta(x(k)))D_{1g} & D_2(\mu) = \sum_{g=1}^{r} \mu_g(\theta(x(k)))D_{2g} \\ C_2(\mu) = \sum_{g=1}^{r} \mu_g(\theta(x(k)))C_{2g} & \end{array}$$

$$\begin{array}{ll} \Delta A(\mu,k) = \sum_{g=1}^{r} \mu_g(\theta(x(k)))\Delta A_g(k) & \Delta C_1(\mu,k) = \sum_{g=1}^{r} \mu_g(\theta(x(k)))\Delta C_{1g}(k) \\ \Delta B_1(\mu,k) = \sum_{g=1}^{r} \mu_g(\theta(x(k)))\Delta B_{1g}(k) & \Delta B_2(\mu,k) = \sum_{g=1}^{r} \mu_g(\theta(x(k)))\Delta B_{2g}(k) \\ \Delta D_1(\mu,k) = \sum_{g=1}^{r} \mu_g(\theta(x(k)))\Delta D_{1g}(k) & \Delta D_2(\mu,k) = \sum_{g=1}^{r} \mu_g(\theta(x(k)))\Delta D_{2g}(k) \end{array}$$

We assume that the uncertainty functions in k are norm-bounded by the following:

Assumption 9.2.1

$$\begin{bmatrix} \Delta A_g(k) & \Delta B_1(k) & \Delta B_2(k) \\ \Delta C_{1g}(k) & \Delta D_1(k) & \Delta D_2(k) \end{bmatrix} = \begin{bmatrix} E_{1g} \\ E_{2g} \end{bmatrix} F(k) \begin{bmatrix} H_1 & H_2 & H_3 \end{bmatrix}$$

where $g = 1, \ldots, r$; r is the number of fuzzy inference rules; H_1, H_2, H_3, E_{1g} and E_{2g} are known matrices which characterize the structure of the uncertainties; $F(k)$ is an unknown matrix function that satisfy $F^T(k)\mathcal{W}F(k) \leq \mathcal{W}$ where $\mathcal{W}$ is a positive-definite matrix.

The network is modelled by a finite state homogeneous Markov process as shown in Chapter 3. The transition probability matrix is allowed to be partially known.

In this chapter, we consider the following fuzzy dynamic output feedback controller:

Control Rule h:
IF $\sigma_1(\hat{x}(k))$ is N_1^h AND $\cdots$ AND $\sigma_q(\hat{x}(k))$ is N_q^h,
THEN

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}_h(i)\hat{x}(k) + \hat{B}_h(i)y(k-\tau_k) \\ u(k) &= \hat{C}_h(i)\hat{x}(k) \end{aligned} \tag{9.2.3}$$

where $\hat{x}(k)$ is the controller's state; $\hat{A}_h(i)$, $\hat{B}_h(i)$, $\hat{C}_h(i)$ are the controller matrices; h denotes the h^{th} fuzzy inference rule; $h = 1, \ldots, c$; c is the number of inference rules; $\sigma_1(\hat{x}(k))$, $\ldots$, $\sigma_q(\hat{x}(k))$, are the premise variables; q is the number of premise variables and N_1^h, $\ldots$, N_q^h are the fuzzy terms.

Similar to the plant, the fuzzy output feedback controller is inferred as shown below.

$$
\begin{aligned}
\hat{x}(k+1) &= \sum_{h=1}^{c} \lambda_h(\sigma(\hat{x}(k))\big\{\hat{A}_h(r_k)\hat{x}(k) + \hat{B}_h(r_k)y(k-\tau_k)\big\} \\
&= \hat{A}(\lambda, r_k)\hat{x}(k) + \hat{B}(\lambda, r_k)y(k-\tau_k) \\
u(k) &= \sum_{h=1}^{c} \lambda_h(\sigma(\hat{x}(k))\hat{C}_h(r_k)\hat{x}(k) \\
&= \hat{C}(\lambda, r_k)\hat{x}(k) \qquad (9.2.4)
\end{aligned}
$$

where

$$
\sigma(\hat{x}(k)) = [\sigma_1(\hat{x}(k)), \ldots, \sigma_q(\hat{x}(k))],
$$

$$
\phi_h(\sigma(\hat{x}(k)) = \prod_{t=1}^{q} N_t^h(\sigma_t(\hat{x}(k))),
$$

$$
\lambda_h(\sigma(\hat{x}(k)) = \frac{\phi_h(\sigma(\hat{x}(k)))}{\sum_{\ell=1}^{c}\phi_\ell(\sigma(\hat{x}(k)))} \in [0,1],
$$

$$
\sum_{h=1}^{c} \lambda_h(\sigma(\hat{x}(k))) = 1
$$

Remark 9.2.1 *Note that in the premise variables of the controller (9.2.4) are different to that of the plant. By considering different premise variables and membership functions, the controller design is more flexible compared to existing approaches shown in [36, 58, 85, 86, 89, 128–130, 132]. Therefore, the number of fuzzy rules of the controller is no longer restricted by the number of fuzzy rules of the plant, hence, a small number of fuzzy rules may be implemented even though the number of fuzzy rules of the plant is large.*

The overall closed loop system with the fuzzy model (9.2.2) and the fuzzy controller (9.2.4) is

$$
\begin{aligned}
\zeta(k+1) &= [A_{cl}(\mu, \lambda, r_k) + \bar{E}_1(\mu)F(k)\bar{H}_1(\lambda, r_k)]\zeta(k) + B_{cl}(\lambda, r_k)\bar{C}_2(\mu)\zeta(k-\tau_k) \\
&\quad +[\bar{B}_1(\mu) + \bar{E}_1(\mu)F(k)H_2]w(k) \\
z(k) &= [C_{cl}(\mu, \lambda, r_k) + E_2(\mu)F(k)\bar{H}_1(\lambda, r_k)]\zeta(k) + [D_1(\mu) \\
&\quad +E_2(\mu)F(k)H_2]w(k) \qquad (9.2.5)
\end{aligned}
$$

where $\zeta^T(k) = \begin{bmatrix} x^T(k) & \hat{x}^T(k) \end{bmatrix}$ and

$$\begin{aligned} A_{cl}(\mu,\lambda,r_k) &= \begin{bmatrix} A(\mu) & B_2(\mu)\hat{C}(\lambda,r_k) \\ 0 & \hat{A}(\lambda,r_k) \end{bmatrix}, B_{cl}(\lambda,r_k) = \begin{bmatrix} 0 \\ \hat{B}(\lambda,r_k) \end{bmatrix}, \\ \bar{B}_1(\mu) &= \begin{bmatrix} B_1(\mu) \\ 0 \end{bmatrix}, \bar{C}_2(\mu) = \begin{bmatrix} C_2(\mu) & 0 \end{bmatrix}, \\ \bar{E}_1(\mu) &= \begin{bmatrix} E_1(\mu) \\ 0 \end{bmatrix}, \bar{H}_1(\lambda,r_k) = \begin{bmatrix} H_1 & H_3\hat{C}(\lambda,r_k) \end{bmatrix}, \\ C_{cl}(\mu,\lambda,r_k) &= \begin{bmatrix} C_1(\mu) & D_2(\mu)\hat{C}(\lambda,r_k) \end{bmatrix}. \end{aligned}$$

For brevity, $x(k)$, $\hat{x}(k)$, $\mu_g(\theta(x(k)))$, $\lambda_h(\sigma(\hat{x}(k)))$ are denoted as x, $\hat{x}$, $\mu_g(x)$, $\lambda_h(\hat{x})$, respectively, throughout this chapter.

As shown in Chapter 2, the closed-loop system is to achieve stochastic stability, as shown in (2.2.5), and the $\mathcal{H}_\infty$ performance condition, as shown in (2.2.6).

Lemma 3.2.2 introduced in Chapter 3 is used in this chapter to handle unknown transition probabilities.

Similar to Chapter 7, the products of membership functions are approximated by polynomial functions shown as below

$$\mu_g(x)\lambda_h(\hat{x}) = \sum_{\kappa=1}^{D} \zeta_\kappa(x,\hat{x})\{\eta_{gh,s_\kappa}(x,\hat{x}) + \Delta\eta_{gh,s_\kappa}(x,\hat{x})\} \tag{9.2.6}$$

where $\eta_{gh,s_\kappa}(x,\hat{x})$ are the polynomial function approximations and $\Delta\eta_{gh,s_\kappa}(x,\hat{x})$ are the error terms in each sub-region. $\zeta_\kappa(x,\hat{x})$ is a scalar function which takes 1 if x and $\hat{x}$ are inside the sub-region, s_κ, and 0 otherwise.

The following lower and upper bounds of the error terms are introduced similar to the previous chapter, which will help deriving the theorem later.

$$\alpha_{gh,s_\kappa} \le \Delta\eta_{gh,s_\kappa}(x,\hat{x}) \le \beta_{gh,s_\kappa} \tag{9.2.7}$$

where α_{gh,s_κ} and β_{gh,s_κ} are known constants.

9.3 Main Result

The following theorem provides sufficient conditions for the existence of a partially mode delay-dependent output feedback controller.

Theorem 9.3.1 *Given a prescribed $\mathcal{H}_\infty$ performance, $\gamma > 0$, the closed-loop system is stochastically stable with the prescribed $\mathcal{H}_\infty$ performance, if there exist sets of positive-definite matrices $P(i)$, $X(i)$, $\tilde{R}_1(i)$, $\tilde{R}_1$, $\tilde{R}_2(i)$, $\tilde{R}_2$, $W_1(i)$, $W_2(i)$, $\tilde{W}_1(i)$, Q, $\tilde{Q}$, N_1, N_2, $\tilde{Z}(i)$, $S(i,j)$, $\tilde{\Psi}_1^{gh}(i)$, $\tilde{\Psi}_2^{gh}(i)$ and matrices $K_h(i)$, $\hat{C}_h(i)$, $\tilde{M}(i)$, $J(i)$ for $i = 1, 2, \cdots, s$, $g = 1, 2, \cdots, r$, $h = 1, 2, \cdots, c$ satisfying the following*

$$\tilde{R}_1 > \tilde{R}_1(i) \tag{9.3.1}$$

$$\tilde{R}_2 > \tilde{R}_2(i) \tag{9.3.2}$$

$$-v^T\Big[\sum_{g=1}^{r}\sum_{h=1}^{c}\Big\{(\eta_{gh,s_\kappa}(x,\hat{x}) + \frac{1}{2}\alpha_{gh,s_\kappa} + \frac{1}{2}\beta_{gh,s_\kappa})\tilde{T}^{gh}(i) + \tilde{V}^{gh,s_\kappa}(i)\Big\} - \tilde{\epsilon}_{s_\kappa} I\Big]v \text{ is SOS}$$
$$\forall s_\kappa = 1, \ldots, D \tag{9.3.3}$$

$$\frac{1}{2}\tilde{T}^{gh}(i) - \tilde{\Psi}_1^{gh}(i) < 0 \tag{9.3.4}$$

$$-\frac{1}{2}\tilde{T}^{gh}(i) - \tilde{\Psi}_2^{gh}(i) < 0 \tag{9.3.5}$$

$$\begin{bmatrix} (1 - p_{i(i+1)})\tilde{R}_1(i) + \tilde{R}_2(i) & \tilde{M}(i) \\ * & \tilde{Z}(i) \end{bmatrix} \geq 0 \tag{9.3.6}$$

$$\begin{bmatrix} p^i_{known}\tilde{R}_1(i) + \tilde{R}_2(i) & \tilde{M}(i) \\ * & \tilde{Z}(i) \end{bmatrix} \geq 0, \;\; \forall (i+1) \in \mathcal{S}^i_{unknown} \tag{9.3.7}$$

$$\begin{bmatrix} S(i,j) & J^T(i) \\ * & X(j) \end{bmatrix} > 0 \tag{9.3.8}$$

and

$$N_1\tilde{R}_1 = I, \; N_2\tilde{R}_2 = I, \tilde{W}_1(i)W_1(i) = I, Q\tilde{Q} = I \text{ and } P(i)X(i) = I, \tag{9.3.9}$$

where v is a real vector of appropriate dimension and independent of x; $\tilde{\epsilon}_{s_\kappa}$ are predefined *scalars;* $\eta_{gh,s_\kappa}(x,\hat{x})$ *are defined in (9.2.6);*

$$
\begin{aligned}
\tilde{T}^{gh}(i) &= \begin{bmatrix} \tilde{\Lambda}(i) & \big(\tilde{\Gamma}_1^{gh}(i)\big)^T & \big(\tilde{\Gamma}_2^{gh}(i)\big)^T & \big(\tilde{\Gamma}_3\big)^T & \big(\tilde{\Xi}^{gh}(i)\big)^T & \big(\tilde{\mathcal{H}}^h(i)\big)^T \\ * & -\mathcal{X} & 0 & 0 & 0 & 0 \\ * & * & -\mathcal{R} & 0 & 0 & 0 \\ * & * & * & -\tilde{Q} & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -\tilde{W}_1(i) \end{bmatrix} \\
\tilde{V}^{gh,s_\kappa}(i) &= (\beta_{gh,s_\kappa} - \alpha_{gh,s_\kappa})\Big\{\tilde{\Psi}_1^{gh}(i) + \tilde{\Psi}_2^{gh}(i)\Big\} \\
\tilde{\Lambda}(i) &= diag\Big\{-P(i), -Q, \Big(H_2^T W_2(i) H_2 - \gamma I\Big), -W_1(i), -W_2(i)\Big\} \\
&\quad + \tilde{\Upsilon}_1^T(i) + \tilde{\Upsilon}_1(i) + \tau(i)\tilde{Z}(i) \\
\tilde{\Gamma}_1^{gh}(i) &= \Big[\big(\tilde{A}_g + \bar{E}K_h(i)\bar{R} + \bar{B}_{2g}\tilde{C}_h(i)\big) \quad \bar{E}K_h(i)\hat{E}\bar{C}_{2g} \quad \bar{B}_{1g} \quad \bar{E}_{1g} \quad \bar{E}_{1g}\Big] \\
\tilde{\Gamma}_2^{gh}(i) &= \Big[\sqrt{\tilde{\tau}(i)} \quad \sqrt{\tau(s)}\Big]^T \Big[\big(\tilde{A}_g - diag\{I,0\} + \bar{B}_{2g}\tilde{C}_h(i)\big) \quad 0 \quad \bar{B}_{1g} \quad \bar{E}_{1g} \quad \bar{E}_{1g}\Big] \\
\tilde{\Gamma}_3 &= \Big[\sqrt{\tau(s) - \tau(1) + 1} \quad 0 \quad 0 \quad 0 \quad 0\Big]
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\tau}(i) &= \textstyle\sum_{j\in\mathcal{S}_{known}^i}^{i+1} p_{ij}\tau(j) + (1 - p_{known}^i)\sum_{j\in\mathcal{S}_{unknown}^i}^{i+1} \tau(j) \\
\mathcal{X} &= -\{\textstyle\sum_{j\in\mathcal{S}_{known}^i}^{i+1} p_{ij}S(i,j) + (1 - p_{known}^i)\sum_{j\in\mathcal{S}_{unknown}^i}^{i+1} S(i,j)\} + J^T(i) + J(i) \\
\mathcal{R} &= diag\Big\{N_1, N_2\Big\} \\
\tilde{\Xi}^{gh}(i) &= \Big[\ \bar{C}_{1g} + D_{2g}\tilde{C}_h(i) \quad 0 \quad D_{1g} \quad E_{2g} \quad E_{2g}\ \Big] \\
\tilde{\mathcal{H}}^h(i) &= \Big[\ \tilde{H}_1 + H_3\tilde{C}_h(i) \quad 0 \quad 0 \quad 0 \quad 0\ \Big] \\
\tilde{\Upsilon}_1(i) &= \tilde{M}^T(i)[diag\{I,0\} \quad diag\{-I,0\} \quad 0 \quad 0 \quad 0] \\
\tilde{A}_g &= \begin{bmatrix} A_g & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{E} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \bar{R} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \quad \hat{E} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \\
\bar{B}_{2g} &= \begin{bmatrix} B_{2g} \\ 0 \end{bmatrix}, \quad \bar{C}_{1g} = \begin{bmatrix} C_{1g} & 0 \end{bmatrix}, \quad \bar{C}_{2g} = \begin{bmatrix} C_{2g} & 0 \end{bmatrix}, \\
\tilde{H}_1 &= \begin{bmatrix} H_1 & 0 \end{bmatrix}, \tilde{C}_h(i) = \begin{bmatrix} 0 & \hat{C}_h(i) \end{bmatrix}
\end{aligned}
$$

Note that $\bar{E}$ *and* $\hat{E}$ *have different dimension. Moreover, the mode delay dependent fuzzy output feedback controller is given as*

$$
\begin{aligned}
\hat{x}(k+1) &= \sum_{h=1}^{c} \lambda_h(\sigma(\hat{x}(k))\big\{\hat{A}_h(i)\hat{x}(k) + \hat{B}_h(i)y(k-\tau_k)\big\} \\
u(k) &= \sum_{h=1}^{c} \lambda_h(\sigma(\hat{x}(k))\hat{C}_h(i)\hat{x}(k)
\end{aligned} \tag{9.3.10}
$$

where the controller matrices, $\hat{A}_h(i)$ and $\hat{B}_h(i)$ are given by

$$\begin{bmatrix} \hat{A}_h(i) & \hat{B}_h(i) \end{bmatrix} = K_h(i)$$

Proof: We introduce the following augmented closed system from (9.2.5)

$$\begin{aligned} \zeta_{k+1} &= \Gamma_1(r_k)\tilde{\zeta}_k \\ z_k &= \Xi(r_k)\tilde{\zeta}_k \end{aligned} \tag{9.3.11}$$

where $\zeta_\ell = \zeta(\ell)$, $z_\ell = z(\ell)$, $\tilde{\zeta}_\ell = \tilde{\zeta}(\ell)$, and

$$\begin{aligned} \tilde{\zeta}(k)^T &= \begin{bmatrix} \zeta^T(k) & \zeta^T(k-\tau(r_k)) & w^T(k) & \zeta^T(k)\bar{H}_1^T(\lambda, r_k)F^T(k) & w^T(k)H_2^T F^T(k) \end{bmatrix} \in \Re^l \\ \Gamma_1(r_k) &= \begin{bmatrix} A_{cl}(\mu, \lambda, r_k) & B_{cl}(\lambda, r_k)\bar{C}_2(\mu) & \bar{B}_1(\mu) & \bar{E}_1(\mu) & \bar{E}_1(\mu) \end{bmatrix} \\ \Xi(r_k) &= \begin{bmatrix} C_{cl}(\mu, \lambda, r_k) & 0 & D_1(\mu) & E_2(\mu) & E_2(\mu) \end{bmatrix} \end{aligned}$$

Introducing the following Lyapunov-Krasovskii functional

$$V(\zeta_k, r_k) = V_1(\zeta_k, r_k) + V_2(\zeta_k, r_k) + V_3(\zeta_k, r_k) \tag{9.3.12}$$

where

$$V_1(\zeta_k, r_k) = \zeta_k^T P(r_k)\zeta_k \tag{9.3.13}$$

$$V_2(\zeta_k, r_k) = \sum_{\ell=-\tau_k}^{-1} \sum_{j=k+\ell}^{k-1} \bar{x}_j^T R_1 \bar{x}_j + \sum_{\ell=-\tau(s)}^{-1} \sum_{j=k+\ell}^{k-1} \bar{x}_j^T R_2 \bar{x}_j \tag{9.3.14}$$

$$V_3(\zeta_k, r_k) = \sum_{\ell=k-\tau_k}^{k-1} \zeta_\ell^T Q \zeta_\ell + \sum_{\ell=-\tau(s)+2}^{-\tau(1)+1} \sum_{j=k+\ell-1}^{k-1} \zeta_j^T Q \zeta_j \tag{9.3.15}$$

where $\bar{x}(k) = \begin{bmatrix} x(k+1) - x(k) \\ 0 \end{bmatrix}$.

Following a similar approach shown in the proof section of Chapter 8, we have

$$\sum_{g=1}^{r} \sum_{h=1}^{c} \mu_g(x)\lambda_h(x_\tau) \begin{bmatrix} \tilde{\Lambda}(r_k) & \left(\tilde{\Gamma}_1^{gh}(r_k)\right)^T & \left(\tilde{\Gamma}_2^{gh}(r_k)\right)^T & \left(\tilde{\Gamma}_3\right)^T & \left(\tilde{\Xi}^{gh}(r_k)\right)^T & \left(\tilde{\mathcal{H}}^h(r_k)\right)^T \\ * & -\mathcal{X} & 0 & 0 & 0 & 0 \\ * & * & -\mathcal{R} & 0 & 0 & 0 \\ * & * & * & -\tilde{Q} & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -\tilde{W}_1(r_k) \end{bmatrix} < 0 \tag{9.3.16}$$

Note that $\mathcal{X}$ and $\tilde{\Gamma}_2^{gh}(r_k)$ terms contain transition probabilities. We know that the summation of all probabilities in each row of transition probability matrix is one. This means that $(1-p^i_{known}) = \sum_{j\in unknown}^{i+1} p_{ij}$. By using Lemma 3.2.2, we can replace $\tilde{\tau}(i)$ and $\mathcal{X}$ with $\sum_{j=1}^{i+1} p_{ij}\tau(j)$ and $-\sum_{j=1}^{i+1} p_{ij}S(i,j) + J^T(i) + J(i)$ while still satisfying the above condition.

Applying Schur complement on (9.3.8) and consequently multiplying these inequalities by p_{ij} and summing up for all j we obtain

$$\begin{aligned} -\sum_{j=1}^{i+1} p_{ij}S(i,j) + J^T(i) + J(i) &= J^T(i) + J(i) - \sum_{j=1}^{i+1} p_{ij}S(i,j) \\ &< J^T(i) + J(i) - J^T(i)\tilde{P}(i)J(i) \\ &= \tilde{P}^{-1}(i) - \Big(J(i) - \tilde{P}^{-1}(i)\Big)^T \tilde{P}(i)\Big(J(i) - \tilde{P}^{-1}(i)\Big) \\ &< \tilde{P}^{-1}(i) \end{aligned} \tag{9.3.17}$$

where $P(i) = X^{-1}(i)$ and $\tilde{P}(i) = \sum_{j=1}^{i+1} p_{ij}P(j)$. This means that (9.3.16) still satisfies even if $-\sum_{j=1}^{i+1} p_{ij}S(i,j) + J^T(i) + J(i)$ is replaced with $\tilde{P}^{-1}(i)$.

Taking summations and membership functions inside the matrix, along with the above, (9.3.16) becomes

$$\begin{bmatrix} \bar{\Lambda}(r_k)+\tilde{\Upsilon}_1(r_k)+\tilde{\Upsilon}_1^T(r_k)+\tau(r_k)\tilde{Z}(r_k) & \check{\Gamma}_1^T(\mu,\lambda,r_k) & \Gamma_2^T(\mu,\lambda,r_k) & \Big(\Gamma_3\Big)^T & \Xi^T(\mu,\lambda,r_k) & \Big(\check{\mathcal{H}}^h(\lambda,r_k)\Big)^T \\ * & -\tilde{P}^{-1}(r_k) & 0 & 0 & 0 & 0 \\ * & * & -(\tau_k R_1+\tau(s)R_2)^{-1} & 0 & 0 & 0 \\ * & * & * & -Q^{-1} & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -W_1^{-1}(r_k) \end{bmatrix} < 0 \tag{9.3.18}$$

Note that expanding $\big(\tilde{A}_g + \bar{E}K_h(i)\bar{R} + \bar{B}_{2g}\tilde{C}_h(i)\big)$, $\big(\bar{E}K_h(i)\hat{E}\bar{C}_{2g}\big)$ and $\big(\tilde{A}_g - \mathrm{diag}\{I,0\} + \bar{B}_{2g}\tilde{C}_h(i)\big)$ and taking summations and membership function inside, we obtain $A_{cl}(\mu,\lambda,r_k)$, $B_{cl}(\lambda,r_k)$ and $\bar{A}(\mu,\lambda,r_k)$ respectively in the above.

By following from the proof section in Chapter 7 we can show that the controller satisfies the conditions in the problem formulation, thus completing the proof. $\nabla\nabla\nabla$

Remark 9.3.1 *By disregarding the summations and membership functions in (9.3.16), we obtain an approach that does not consider membership functions in the design. Therefore by stating $\tilde{T}^{gh}(i) < 0$, we have a methodology that has been widely used in fuzzy systems where membership functions are disregarded when designing a controller. This approach can now be expressed in term of LMI conditions as the membership functions,* *which are nonlinear in x are discarded. By including the summations we state that the sum of $\tilde{T}^{gh}(i)$, along with the membership functions, is less than zero, instead of them individually being zero. Furthermore incorporating membership functions ensure that the controller is valid for the particular membership functions. These ensure that our approach is less conservative than discarding membership functions. In the next section, we compare our result with the approach where $\tilde{T}^{gh}(i) < 0$, and hence discarding membership functions, to show that our method results wider stability region.*

A similar algorithm to the one in Chapter 7 is applied to this chapter to solve Theorem 9.3.1.

9.4 Examples

In this section, two simulation examples are presented to show that 1. the incorporation of membership functions result a wider stability region and 2. the controller can stabilize a practical system.

Example 1 Consider a T-S fuzzy system with two plant rules ($r = 2$) and two controller rules ($c = 2$). The sub-systems are described as follows

$$\begin{aligned}
A_1 &= \begin{bmatrix} -a & -0.1 \\ 1 & 0 \end{bmatrix} & A_2 &= \begin{bmatrix} -1 & 0.1 \\ 1 & 0 \end{bmatrix} & B_{11} &= \begin{bmatrix} -b \\ 0 \end{bmatrix} & B_{12} &= \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \\
C_{11} &= \begin{bmatrix} 1 & 0 \end{bmatrix} & C_{12} &= \begin{bmatrix} -1 & 0 \end{bmatrix} & B_{21} &= \begin{bmatrix} 1.0 \\ 0.5 \end{bmatrix} & B_{22} &= \begin{bmatrix} 1.0 \\ 0.5 \end{bmatrix} \\
D_{11} &= 0.01 & D_{12} &= 0.01 & D_{21} &= 0.1 & D_{22} &= 0.1 \\
C_2 &= \begin{bmatrix} 0.6 & 0 \end{bmatrix}
\end{aligned} \tag{9.4.1}$$

and the uncertainties are characterised by matrices below:

$$\begin{aligned}
E_{11} &= E_{12} = \begin{bmatrix} 0.05 \\ 0.1 \end{bmatrix} & E_{21} &= E_{22} = 0.1 \\
H_1 &= \begin{bmatrix} 0.2 & 0 \end{bmatrix} & H_2 &= 0.1 \\
H_3 &= 0.1
\end{aligned} \tag{9.4.2}$$

The membership functions for the plant, μ_g are as follows

$$\begin{aligned} \mu_1(x_1(k)) &= \begin{cases} 1 & x_1(k) < -0.5 \\ x_1(k) - 0.5 & -0.5 \le x_1(k) < 0.5 \\ 0 & 0.5 \le x_1(k) \end{cases}, \\ \mu_2(x_1(k)) &= 1 - \mu_1(x_1(k)) \end{aligned} \qquad (9.4.3)$$

The membership functions for the controller, λ_h are shown below

$$\begin{aligned} \lambda_1(\hat{x}_1(k)) &= \begin{cases} 0 & \hat{x}_1(k) < -0.5 \\ \hat{x}_1(k) + 0.5 & -0.5 \le \hat{x}_1(k) < 0.5 \\ 1 & 0.5 \le \hat{x}_1(k) \end{cases}, \\ \lambda_2(\hat{x}_1(k)) &= 1 - \lambda_1(x_1(k - \tau_k)) \end{aligned} \qquad (9.4.4)$$

Figure 9.1 shows $\mu_g(x_1(k))$ and $\lambda_h(\hat{x}_1(k))$ with sub-regions in each membership function, $x_1 \in [-\infty, -0.5]$, $x_1 \in [-0.5, 0]$, $x_1 \in [0, 0.5]$ and $x_1 \in [0.5, \infty]$, and similarly for the membership function of the controller. This means that we divide $\mu_g(x)\lambda_h(\hat{x})$ into 16 sub-regions. For every sub-region we use $\eta_{gh,s_\kappa}(x, \hat{x})$ to obtain polynomial approximation. The upper and lower bounds of the error terms, α_{gh,s_κ} and β_{gh,s_κ}, are obtained numerically. Note that the shape of membership functions for $\mu_1(x_1(k))$, $\lambda_2(\hat{x}_1(k))$ and $\mu_2(x_1(k))$ $\lambda_1(\hat{x}_1(k))$ in this particular example are the same. Refer to Table 7.1, 7.2 and 7.3 in Chapter 7 for $\eta_{gh,s_\kappa}(x, \hat{x})$, β_{gh,s_κ} and α_{gh,s_κ} used in this example. Note that the premise variables of the controller in this example is $\hat{x}$ compared to x_τ in Chapter 7. The approximation procedure is described in Example 2 of Chapter 7.

A Markov chain with two modes are used to model the network-induced delays of $0.1s$ and $0.2s$ respectively. The underlying transition probability matrix for the Markov chain is as shown below

$$P_\tau = \begin{bmatrix} 0.6 & 0.4 \\ (0.7) & (0.3) \end{bmatrix} \qquad (9.4.5)$$

with sampling time of $0.01s$ and (0.7) and (0.3) are unknown to the controller design.

By using YALMIP [126], with prescribed $\gamma = 1.0$, we obtain stability region for Theorem 9.3.1 and the approach without incorporating membership functions as mentioned in Remark 9.3.1. Figure 9.2 shows the stability region of the system above where $0.5 \le a \le 0.9$ and $0 \le b \le 1.0$. It shows that Theorem 9.3.1 results wider stability region.

Remark 9.4.1 *The stability region of approach without membership functions is obtained*

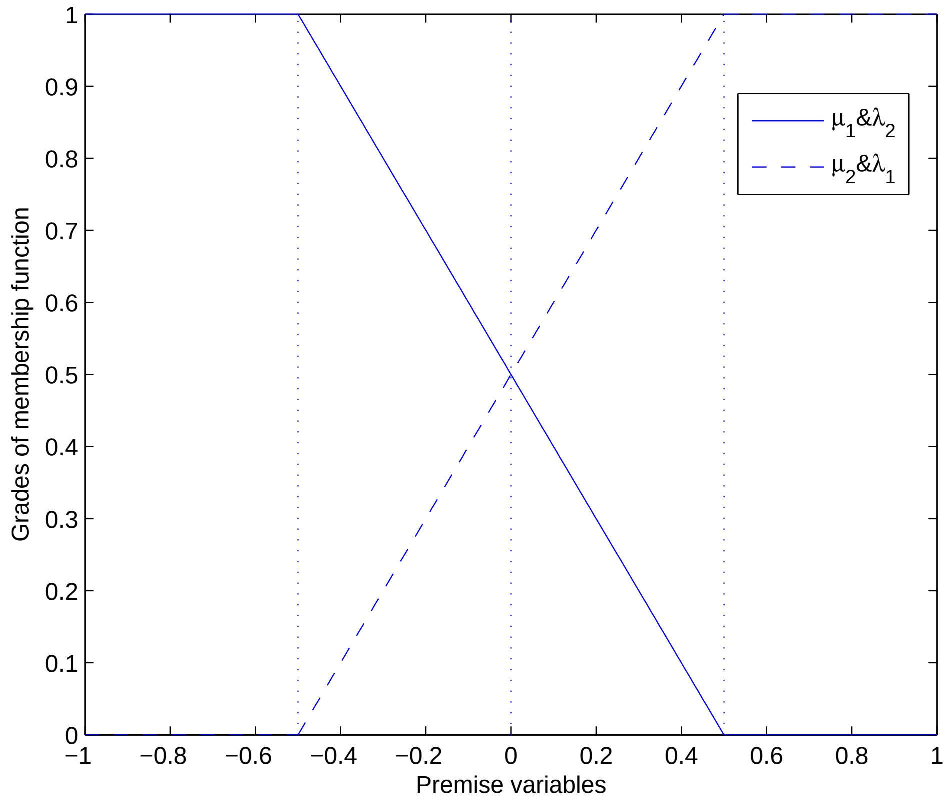

Figure 9.1: Membership functions of the plant and the controller

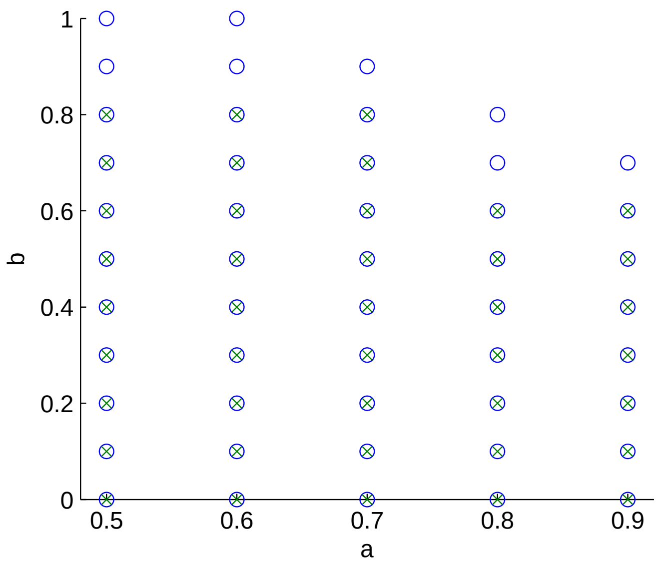

Figure 9.2: Stability region from Theorem 9.3.1 (o) and without considering membership functions (x)

by solving LMI conditions, $\tilde{T}^{gh}(i) < 0$. It is clear that incorporating membership functions into the design yields larger stability region. By incorporating membership functions, we ensure that the controller designed is specific for the membership functions of both the plant and the controller.

Example 2 The same plant used in Example 2 of Chapter 7, single-link rigid robot, is considered. The network-induced delays, the transition probability matrix and the prescribed γ are the same as Example 1. The membership functions for Rule 1 and Rule 2 are shown in Figure 6.2 in Chapter 6. The polynomial approximations and the corresponding bounds are presented in Table 7.4 and 7.5 in Chapter 7. Note that the premise variable of the controller in this example is $\hat{x}$, as opposed to x_τ in Chapter 7. Refer to Example 2 of Chapter 7 for more information on the approximation procedure.

Using Theorem 9.3.1 and the algorithm, a controller of the form (9.2.4) with the following gains are obtained.

$$\hat{A}_1(1) = 1 \times 10^{-7} \times \begin{bmatrix} 3.8061 & -0.5374 \\ 7.2679 & -2.4919 \end{bmatrix}, \quad \hat{A}_1(2) = 1 \times 10^{-7} \times \begin{bmatrix} 2.2516 & 1.7894 \\ -0.6884 & 9.5003 \end{bmatrix},$$
$$\hat{A}_2(1) = 1 \times 10^{-7} \times \begin{bmatrix} 3.8027 & -0.5609 \\ 7.2920 & -2.5131 \end{bmatrix}, \quad \hat{A}_2(2) = 1 \times 10^{-7} \times \begin{bmatrix} 2.2462 & 1.7872 \\ -0.6807 & 9.5003 \end{bmatrix},$$
$$\hat{B}_1(1) = 1 \times 10^{-7} \times \begin{bmatrix} 0.3034 \\ -1.6402 \end{bmatrix}, \quad \hat{B}_1(2) = 1 \times 10^{-7} \times \begin{bmatrix} 5.4335 \\ 3.0203 \end{bmatrix}$$
$$\hat{B}_2(1) = 1 \times 10^{-7} \times \begin{bmatrix} 0.3042 \\ -1.6641 \end{bmatrix}, \quad \hat{B}_2(2) = 1 \times 10^{-7} \times \begin{bmatrix} 5.4145 \\ 3.0199 \end{bmatrix}$$
$$\hat{C}_1(1) = 1 \times 10^{-8} \times \begin{bmatrix} -7.9402 & 3.4307 \end{bmatrix}, \quad \hat{C}_1(2) = 1 \times 10^{-7} \times \begin{bmatrix} -0.2759 & -1.0254 \end{bmatrix}$$
$$\hat{C}_2(1) = 1 \times 10^{-8} \times \begin{bmatrix} -7.9145 & 3.4457 \end{bmatrix}, \quad \hat{C}_2(2) = 1 \times 10^{-7} \times \begin{bmatrix} -0.2773 & -1.0239 \end{bmatrix}$$

Remark 9.4.2 *The state response of the plant with the proposed controller is shown in Figure 9.3 with $w = 0$. The initial states are chosen to be $x(0) = [0.5 \;\; 0]^T$. It can be seen that the dynamic fuzzy output controller stabilizes the system, demonstrating the validity of the proposed controller. Figure 9.5 shows the ratio of the energy of the output to the energy of the disturbance ($w(k) = e^{-0.1k} \sin(0.5k)$). From Figure 9.5, one can see that the ratio tends to roughly 0.135, which is less than the prescribed $\gamma = 1$. Note that no feasible solution exists for this particular example when the theorem is reduced to $\tilde{T}^{gh}(i) < 0$.*

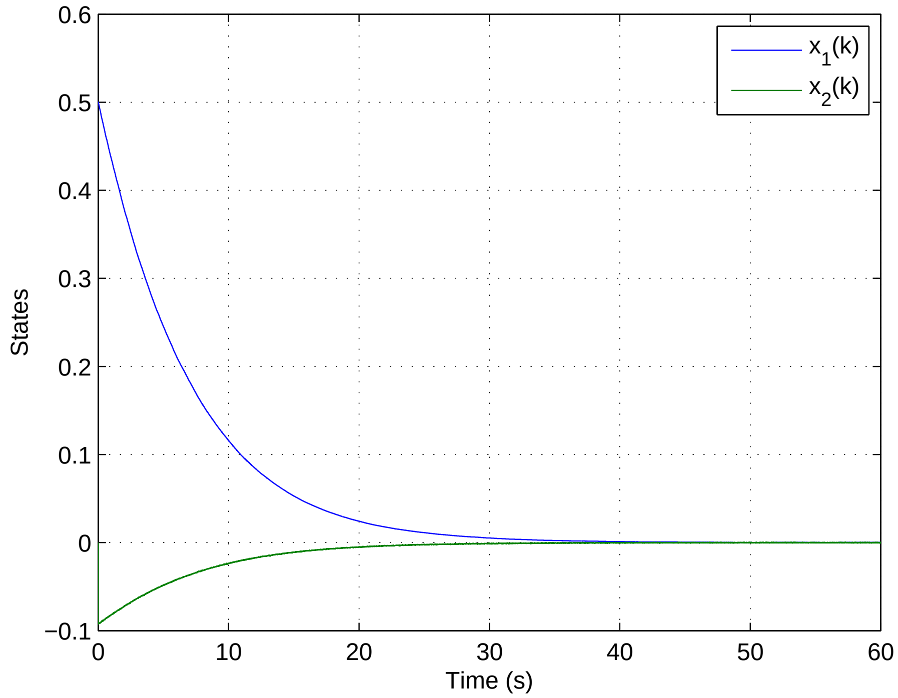

Figure 9.3: State response of the robot

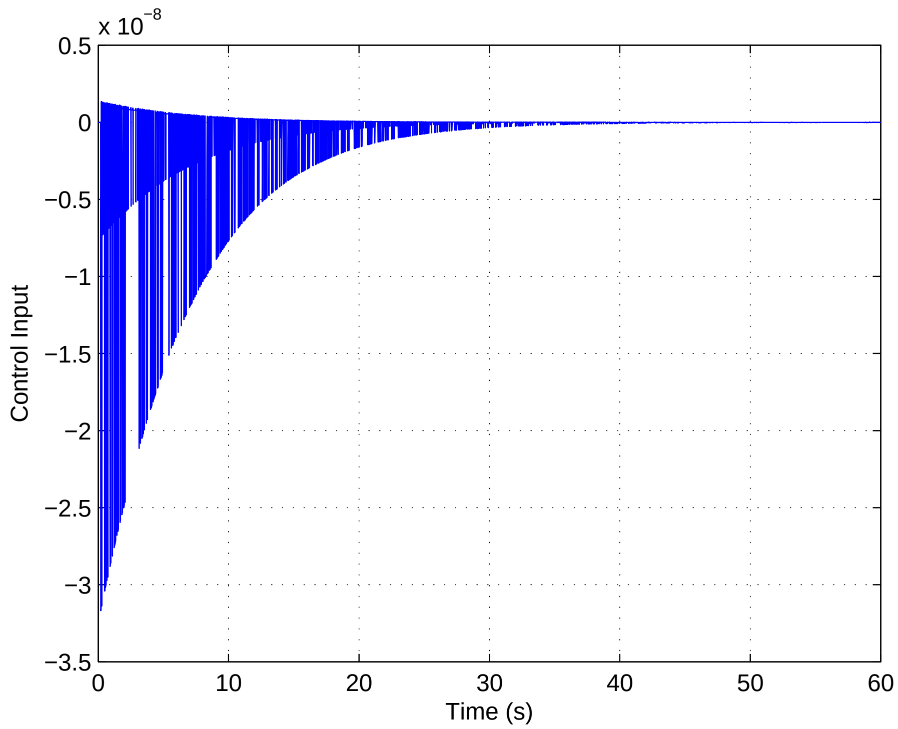

Figure 9.4: Control input of the system

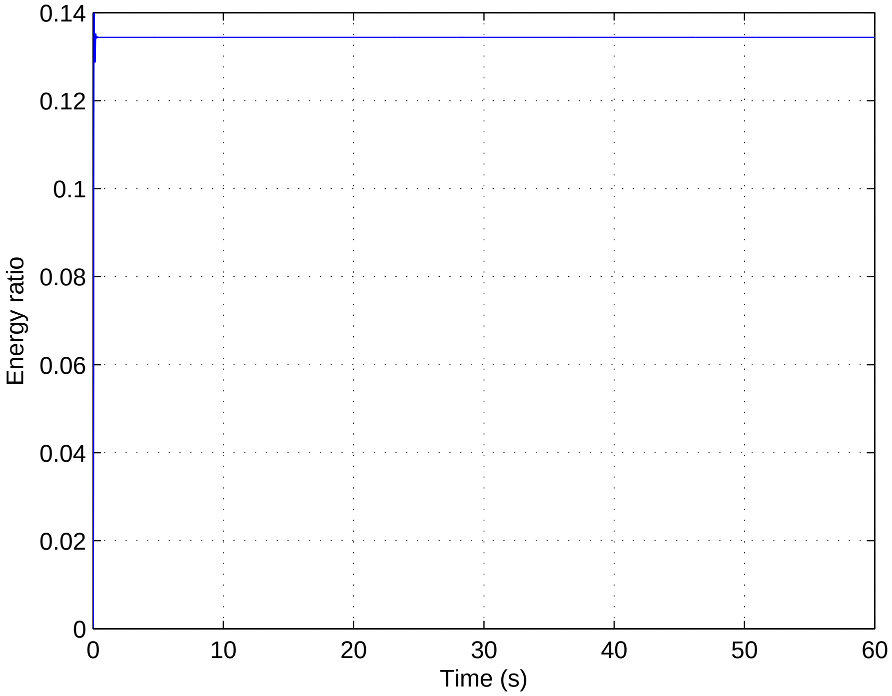

Figure 9.5: Ratio of the energy of the controlled output to the energy of the disturbance ($\gamma = 1.0$)

9.5 Conclusions

This chapter presents a design methodology for partially mode delay-dependent fuzzy dynamic output feedback controller for a class of nonlinear discrete-time networked control system. The nonlinear plant is modelled by Takagi-Sugeno fuzzy model and the random network-induced delays are modelled by a finite state Markov chain with partially known transition probability matrix. The controller's premise variables and its membership functions are allowed to be different from the plant's premise variables and its membership functions. The membership functions of both the plant and the controller are approximated by polynomial functions and incorporated into the design. Sufficient conditions for the existence of the controller are derived in terms of the sum-of -squares inequalities which are then solved by the YALMIP. Numerical examples are used to show that incorporating membership functions into the controller design yields a larger stability region and the effectiveness of the proposed methodology.

10

Conclusions

10.1 Summary of Thesis

In this thesis, novel methodologies for designing a robust $\mathcal{H}_\infty$ state feedback controller, robust $\mathcal{H}_\infty$ filter and robust $\mathcal{H}_\infty$ dynamic output feedback controller for a class of linear and nonlinear uncertain NCSs are proposed where the network is modelled by a finite state Markov chain. The main interest of the Markov chain in this thesis is that the transition probability matrix is allowed to be partially known. This provides more practical consideration as it is often costly or even impossible to obtain a completely known transition probability matrix in the real world. The fact that the summation of all probabilities equal to one has been used to create an upper bound for unknown transition probabilities. Based on Lyapunov-Krasovskii functional, sufficient conditions for the existence of the controller is given in terms of BMIs for linear systems or SOS for nonlinear systems. T-S fuzzy model has been used to model nonlinear NCSs and corresponding fuzzy controller design methodologies are presented where membership functions are incorporated into the controller design. By using numerical examples, it is shown that this incorporation results wider stabilization region. Furthermore, the proper fuzzy formulation of NCSs has been presented where the controller's premise variables are the time delayed version of the plant's premise variable. The majority of existing literature on NCSs neglect this fact, resulting in unrealistic formation of nonlinear NCSs. It is noteworthy that the methodologies with partially known transition probability matrix is more general as the case of either completely known or completely unknown transition probability matrix can be seen as a special case of the proposed approach.

The controller in NCSs is to stabilize the system despite the presence of the network. In recent years, the researchers have been trying to achieve this goal by modelling the network constraints such as the network-induced delays and/or packet dropouts. Following the introductory review in Chapter 1, Chapter 2 presents a methodology to designing a robust $\mathcal{H}_\infty$ state feedback controller for a class of discrete-time linear NCSs is presented. Chapter 3 introduces the problem of partially known transition probability matrix and a lemma that plays a vital role in creating an upper bound of the unknown probabilities is presented. It has been shown that unlike previous attempts where unknown transition probabilities are discarded, upper bound of the unknown probabilities can be used. This is demonstrated by presenting a robust $\mathcal{H}_\infty$ state feedback controller design where the network is modelled by a Markov chain with partially known transition probability matrix. In Chapter 4 and 5, methodologies to design a robust $\mathcal{H}_\infty$ filter and a robust $\mathcal{H}_\infty$ dynamic output feedback controller for linear NCSs are presented. In these aforementioned methodologies, the summation of known and unknown parts are considered whereas known and unknown parts are separated and the unknown transition probabilities are discarded in existing methodologies. It has been shown that considering partially known transition probability matrix provides more general solution as these methodologies can be reduced to those of either completely known or completely unknown transition probabilities. By considering a dynamic output feedback controller design in Chapter 5, more practical solution of controller design is presented where not all state variables of the plant needs to be measurable.

In Chapter 6, brief overview of Takagi-Sugeno fuzzy model is presented. It illustrates how a fuzzy plant model and controller is constructed to model nonlinear NCSs. A brief overview of sum-of-squares decomposition is presented to provide preliminary knowledge, which will be helpful to understand how membership functions are incorporated in the controller design.

In the nonlinear NCSs part of this thesis, presented in Chapter 7, 8 and 9, a robust fuzzy $\mathcal{H}_\infty$ state feedback controller, a robust fuzzy $\mathcal{H}_\infty$ filter and a robust fuzzy $\mathcal{H}_\infty$ dynamic output feedback controller design are presented respectively. Unlike previous approaches to discard membership functions so that LMI approaches can be used, polynomial functions are used to approximate the products of membership functions so that existing tools such as YALMIP or SOSTOOLS can be used to evaluate the solution. In Chapter 7, the first attempt to incorporate membership functions in the controller design of nonlinear NCSs has been made where the transition probability matrix is assumed to be completely known. By using a numerical example, it has been shown that incorporating membership functions yields larger stabilization region, emphasizing the importance of membership functions in the controller design. In Chapter 8 and 9, a robust fuzzy $\mathcal{H}_\infty$ filter and a robust fuzzy $\mathcal{H}_\infty$ dynamic output feedback controller design are presented where the transition probability matrix is assumed to be partially known. In Chapter 7, the premise variables of the controller are different to the plant's premise variables; the controller uses the delayed version of the signal of the plant. In Chapter 9, premise variables of the plant are allowed to be unmeasurable. In this chapter, the states of the dynamic output feedback controller are used as the premise variables of the controller. Most of the existing literature on nonlinear NCSs modelled by T-S fuzzy model assume no delay between the plant and the controller, creating an unrealistic T-S fuzzy model of the nonlinear NCSs.

The main contributions of the systems are as follows:

- Network-induced delays are modelled by a finite state Markov chain with partially known transition probability matrix. This provides more practical consideration as it is often costly or impossible to obtain a completely known transition probability matrix in the real world. Furthermore it has been shown that existing methodologies with either completely known or completely unknown transition probability matrix is a special case of the presented methodology.
- $\mathcal{H}_\infty$ state feedback, $\mathcal{H}_\infty$ filter and $\mathcal{H}_\infty$ dynamic output feedback controller design methodologies are developed in linear NCSs with partially known transition probability matrix. The unknown probabilities are not discarded in these methodologies; they are upper bounded instead and summation of known and unknown parts are considered as opposed to considering them separately.
- T-S fuzzy model is used to model the nonlinear NCSs. Based on this T-S fuzzy model, robust $\mathcal{H}_\infty$ fuzzy state feedback, robust $\mathcal{H}_\infty$ filter and robust $\mathcal{H}_\infty$ fuzzy dynamic output feedback controller design methodologies are developed where membership functions are incorporated into the controller design. It has been shown, by using numerical examples, that incorporating membership functions yield larger stabilization region. Furthermore, the controller's premise variables are different to the plant's premise variables to provide more realistic consideration of NCSs.

10.2 Future Work

Networked control systems still provide a wide range of research that are worth carrying out. While the results of this research provides study of controller design in both linear and nonlinear NCSs, it is nevertheless complete. Further research work may be carried out in the following areas:

- A new model of network to incorporate network constraints may be carried out. It has been shown that Markov chain, while it has been the most researched model, is not the only way the network may be modelled.
- A network-induced delay between the controller and the actuator as well as sensor and controller may be considered. In this thesis, only sensor-to-controller delays are considered. Some researchers have studied both sensor-to-controller delays and controller-to-actuator delays but fail to acknowledge that when the controller gain is to be chosen, no information of the controller-to-actuator delays are available. The controller may be selected based on the current sensor-to-controller delay and the controller-to-actuator delay of the previous time instance.
- A different model of nonlinear NCSs and its respective controller design may be studied. T-S fuzzy model has been proven to be very effective in nonlinear NCSs but other nonlinear models may provide satisfactory result in nonlinear NCSs.
- Controller/filter design methodologies presented in this thesis may be implemented on real systems where the plant and the controller are connected via a communication network. Experiments are carried out to obtain the transition probability matrix of the communication network where some of the elements in the transition probability matrix may be unknown.

References

[1] L. Bushnell, "Networks and control [guest editorial]," *Control Systems Magazine, IEEE*, vol. 21, no. 1, pp. 22–23, 2001.

[2] W. Zhang, M. Branicky, and S. Phillips, "Stability of networked control systems," *Control Systems Magazine, IEEE*, vol. 21, no. 1, pp. 84–99, 2001.

[3] J. Baillieul and P. Antsaklis, "Control and communication challenges in networked real-time systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 9–28, 2007.

[4] W. Ling, X. D. yu, and E. D. zhi, "Some basic issues in networked control systems," in *Proceedings of the 2nd IEEE Conference on Industrial Electronics and Applications*, 2007, pp. 2098–2102.

[5] Y. Zhang, Q. Zhong, and L. Wei, "Stability of networked control systems with communication constraints," in *Chinese Control and Decision Conference*, 2008, pp. 335–339.

[6] A. Ray, "Distributed data communication networks for real-time process control," *Chemical Engineering Communications*, vol. 65, no. 1, pp. 139–154, Mar 1988.

[7] U. Ozguner, H. Goktas, H. Chan, J. Winkelman, M. Liubakka, and R. Krtolica, "Automotive suspension control through a computer communication network," in *First IEEE Conference on Control Applications*, 1992, pp. 895–900 vol.2.

[8] D. Huang and S. K. Nguang, "State feedback control of uncertain networked control systems with random time delays," *IEEE Transactions on Automatic Control*, vol. 53, no. 3, pp. 829–834, 2008.

[9] Y. Tipsuwan and M.-Y. Chow, "Control methodologies in networked control systems," *Control Engineering Practice*, vol. 11, no. 10, pp. 1099–1111, Oct 2003.

[10] H. Gao, T. Chen, and J. Lam, "A new model for time-delay systems with application to network based control," in *Chinese Control Conference*, 2006, pp. 56–61.

[11] J. Nilsson, "Real-time control systems with delays," Ph.D. dissertation, Lund: Lund Institute of Technology, 1998.

[12] Z. Wang and F. Yang, "Robust filtering for uncertain linear systems with delayed states and outputs," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 49, no. 1, pp. 125–130, 2002.

[13] A. S. Tanenbaum, *Computer Networks*, 4th ed. Prentice Hall, Upper Saddle River, NJ, 2003.

[14] M. Tangemann and K. Sauer, "Performance analysis of the timed token protocol of fddi and fddi-ii," *IEEE Journal on Selected Areas in Communications*, vol. 9, pp. 271–278, Feb 1991.

[15] D. Anderson, *FireWire System Architecture*. Addison-Wesley, Reading, MA, 1998.

[16] S. H. Hong, "Scheduling algorithm of data sampling times in the integrated communication and control systems," *IEEE Transactions on Control Systems Technology*, vol. 3, pp. 225–230, 1995.

[17] J. Nilsson, B. Bernhardsson, and B. Wittenmark, "Stochastic analysis and control of real-time systems with random time delays," *Automatica*, vol. 34, no. 1, pp. 57–64, Jan. 1998.

[18] D. Vivanco and A. Jayasumana, "A measurement-based modeling approach for network-induced packet delay," in *32nd IEEE Conference on Local Computer Networks*, 2007, pp. 175–182.

[19] Y. Y. Cao, Y. X. Sun, and J. Lam, "Delay-dependent robust $\mathcal{H}_\infty$ control for uncertain systems with time-varying delays," *IEE Proceedings of Control Theory and Applications*, vol. 145, no. 3, pp. 338–344, 1998.

[20] C. E. de Souza and X. Li, "Delay-dependent robust $\mathcal{H}_\infty$ control of uncertain linear state-delayed systems," *Automatica*, vol. 35, no. 7, pp. 1313–1321, Jul. 1999.

[21] X. Xiao, L. Zhou, Z. Zhang, and G. Lu, "Delay-dependent robust $\mathcal{L}_2$ - $\mathcal{L}_\infty$ filter design for uncertain delta-operator time delay systems," in *27th Chinese Control Conference, 2008.*, 2008, pp. 653–657.

[22] D. Yue, E. Tian, Y. Zhang, and C. Peng, "Delay-distribution-dependent robust stability of uncertain systems with time-varying delay," *International Journal of Robust and Nonlinear Control*, vol. 19, no. r, pp. 377–393, 2009.

[23] O. Santos and S. Mondie, “On the optimal control of time delay systems: a complete type functionals approach,” in *Proceedings of the 46th IEEE Conference on Decision and Control*, 2007, pp. 6035–6040.

[24] Z. Shuqian, Z. Cheng, and J. Feng, “Robust output feedback stabilization for uncertain singular time delay systems,” in *Proceedings of the 2004 American Control Conference*, vol. 6, 2004, pp. 5416–5421 vol.6.

[25] A. Kojima, K. Uchida, and E. Shimemura, “Robust stabilization of uncertain time delay systems via combined internal-external approach,” *IEEE Transactions on Automatic Control*, vol. 38, no. 2, pp. 373–378, 1993.

[26] M. Wang, E. Lee, and D. Boley, “Stabilization of linear time delay systems by hybrid controllers,” in *Proceedings of the 32nd IEEE Conference on Decision and Control*, 1993, pp. 3815–3820 vol.4.

[27] Q. Zhu, G. Lu, J. Cao, and S. Hu, “Stability analysis of networked control systems with markov delay,” in *International Conference on Control and Automation*, vol. 2, 2005, pp. 720–724 Vol. 2.

[28] G. Liu and D. Rees, “Stability criteria of networked predictive control systems with random network delay,” in *Proceedings of the 44th IEEE Conference on Decision and Control and 2005 European Control Conference*, 2005, pp. 203–208.

[29] R. Krtolica, U. Özgüner, H. Chan, H. Goktas, J. Winkelman, and M. Liubakka, “Stability of linear feedback systems with random communication delays,” *International Journal of Control*, vol. 59, no. 4, pp. 925–953, 1994.

[30] M. Tabbara, D. Nesic, and A. Teel, “Stability of wireless and wireline networked control systems,” *IEEE Transactions on Automatic Control*, vol. 52, no. 9, pp. 1615–1630, 2007.

[31] D. Nesic and A. Teel, “Input-output stability properties of networked control systems,” *IEEE Transactions on Automatic Control*, vol. 49, no. 10, pp. 1650–1667, 2004.

[32] ——, “Input-to-state stability of networked control systems,” *Automatica*, vol. 40, no. 12, pp. 2121–2128, Dec 2004.

[33] G. Walsh, H. Ye, and L. Bushnell, “Stability analysis of networked control systems,” *IEEE Transactions on Control Systems Technology*, vol. 10, no. 3, pp. 438–446, 2002.

[34] L. Zhang, Y. Shi, T. Chen, and B. Huang, "A new method for stabilization of networked control systems with random delays," *IEEE Transactions on Automatic Control*, vol. 50, no. 8, pp. 1177–1181, 2005.

[35] J. Sun and G. Liu, "Robust stabilization of a class of nonlinear networked control systems," in *Chinese Control Conference*, 2006, pp. 2035–2040.

[36] X. Jiang and Q.-L. Han, "On designing fuzzy controllers for a class of nonlinear networked control systems," *IEEE Transactions on Fuzzy Systems*, vol. 16, no. 4, pp. 1050–1060, 2008.

[37] J. Wang, W. Wang, F. Yang, and J. Zhang, "Fault detection for descriptor networked systems with multiple packet dropouts," in *Proceedings of the 48th IEEE Conference on Decision and Control and Chinese Control Conference*, 2009, pp. 6976–6981.

[38] D. Wu, J. Wu, and S. Chen, "On separation principle for a class of networked control systems," in *Proceedings of the 48th IEEE Conference on Decision and Control and Chinese Control Conference*, 2009, pp. 5827–5831.

[39] Z. Wang, F. Yang, D. Ho, and X. Liu, "Robust $\mathcal{H}_\infty$ filtering for stochastic time-delay systems with missing measurements," *IEEE Transactions on Signal Processing*, vol. 54, no. 7, pp. 2579–2587, 2006.

[40] Z. Wang, D. Ho, and X. Liu, "Variance-constrained control for uncertain stochastic systems with missing measurements," *IEEE Transactions on Systems, Man and Cybernetics, Part A: Systems and Humans*, vol. 35, no. 5, pp. 746–753, 2005.

[41] J. Qiu, G. Feng, and H. Gao, "Asynchronous output-feedback control of networked nonlinear systems with multiple packet dropouts: T-s fuzzy affine model-based approach," *IEEE Transactions on Fuzzy Systems*, vol. 19, no. 6, pp. 1014–1030, 2011.

[42] B. Zhang and W. X. Zheng, "$\mathcal{H}_\infty$ filter design for nonlinear networked control systems with uncertain packet-loss probability," *Signal Processing*, vol. 92, no. 6, pp. 1499–1507, Jun 2012.

[43] F. Yang, Z. Wang, Y. Hung, and M. Gani, "$\mathcal{H}_\infty$ control for networked systems with random communication delays," *IEEE Transactions on Automatic Control*, vol. 51, no. 3, pp. 511–518, 2006.

[44] W. Wang, Q. Lin, F. Cai, and F. Yang, "l_2 - l_∞ output feedback control for a class of networked system with random delay," in *IEEE International Conference on Control and Automation*, 2007, pp. 63–67.

[45] D. Srinivasagupta, H. Schttler, and B. Joseph, "Time-stamped model predictive control: an algorithm for control of processes with random delays," *Computers & Chemical Engineering*, vol. 28, no. 8, pp. 1337–1346, Jul 2004.

[46] P. Seiler and R. Sengupta, "An $\mathcal{H}_\infty$ approach to networked control," *IEEE Transactions on Automatic Control*, vol. 50, no. 3, pp. 356–364, 2005.

[47] J. Wu and T. Chen, "Design of networked control systems with packet dropouts," *IEEE Transactions on Automatic Control*, vol. 52, no. 7, pp. 1314–1319, 2007.

[48] Q. Ling and M. Lemmon, "Soft real-time scheduling of networked control systems with dropouts governed by a markov chain," in *Proceedings of the 2003 American Control Conference*, vol. 6, 2003, pp. 4845–4850.

[49] Y. Tipsuwan and M.-Y. Chow, "Gain scheduler middleware: a methodology to enable existing controllers for networked control and teleoperation - part i: networked control," *IEEE Transactions on Industrial Electronics*, vol. 51, no. 6, pp. 1218–1227, 2004.

[50] Q. Li and D. Mills, "Jitter-based delay-boundary prediction of wide-area networks," *IEEE/ACM Transactions on Networking*, vol. 9, no. 5, pp. 578–590, 2001.

[51] J. F. Hayes, *Modeling and Analysis of Computer Communications networks.* Plenum Press, New York and London, 1984.

[52] Y. Wang, W. Changhong, and X. Huang, "Guaranteed cost control with random communication delays via jump linear system approach," in *8th Control, Automation, Robotics and Vision Conference*, vol. 1, 2004, pp. 298–303.

[53] I. V. Kolmanovsky and T. L. Maizenberg, "Optimal control of continuous-time linear systems with a time-varying, random delay," *Systems & Control Letters*, vol. 44, no. 2, pp. 119–126, Oct. 2001.

[54] L. M. Liu, C. N. Tong, and H. J. Zhang, "Analysis and design of networked control systems with long delays based on markovian jump model," in *Proceedings of 2005 International Conference on Machine Learning and Cybernetics*, vol. 2, 2005, pp. 953–959 Vol. 2.

[55] J. Hayes, *Modeling and analysis of telecommunications networks.* Wiley, NJ, 2004.

[56] F. C. Liu and Y. Yao, "Modeling and analysis of networked control systems using hidden markov models," in *Proceedings of 2005 International Conference on Machine Learning and Cybernetics*, vol. 2, 2005, pp. 928–931 Vol. 2.

[57] Y. Yang, D. Xu, M. Tan, and X. Dai, "Stochastic stability analysis and control of networked control systems with randomly varying long time-delays," in *Fifth World Congress on Intelligent Control and Automation*, vol. 2, 2004, pp. 1391–1395 Vol.2.

[58] D. Huang, S. K. Nguang, and N. Patel, "Fuzzy control of uncertain nonlinear networked control systems with random time-delays," in *Proceedings of the 2007 American Control Conference*, 2007, pp. 299–304.

[59] D. Huang, S. Nguang, and F. Rasool, "Robust fuzzy filter design for uncertain nonlinear networked control systems with random time-delays," in *IEEE International Conference on Control and Automation*, 2009, pp. 1152–1157.

[60] Y. Zhang, S. Xu, and B. Zhang, "Robust output feedback stabilization for uncertain discrete-time fuzzy markovian jump systems with time-varying delays," *IEEE Transactions on Fuzzy Systems*, vol. 17, no. 2, pp. 411–420, 2009.

[61] L. Zhang, E.-K. Boukas, and J. Lam, "Analysis and synthesis of markov jump linear systems with time-varying delays and partially known transition probabilities," *IEEE Transactions on Automatic Control*, vol. 53, no. 10, pp. 2458–2464, 2008.

[62] L. Zhang and E.-K. Boukas, "Mode-dependent filtering for discrete-time markovian jump linear systems with partly unknown transition probabilities," *Automatica*, vol. 45, no. 6, pp. 1462–1467, Jun. 2009. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0005109809000612

[63] ——, "Stability and stabilization of markovian jump linear systems with partly unknown transition probabilities," *Automatica*, vol. 45, no. 2, pp. 463–468, Feb. 2009. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0005109808004512

[64] L. Xie, W. Zhao, and Z. Ji, "Lqgcontrol of networked control system with long time delays using delta -operator," in *Sixth International Conference on Intelligent Systems Design and Applications*, vol. 2, 2006, pp. 183–187.

[65] G. Walsh, O. Beldiman, and L. Bushnell, "Asymptotic behavior of nonlinear networked control systems," *IEEE Transactions on Automatic Control*, vol. 46, no. 7, pp. 1093–1097, 2001.

[66] X. Jia, X. Chi, and Q.-L. Han, "A membership function deviation approach to network-based $\mathcal{H}_\infty$ fuzzy output feedback control for takagi-sugeno fuzzy systems," in *37th Annual Conference on IEEE Industrial Electronics Society*, 2011, pp. 2288–2293.

[67] X. Jia, D. Zhang, X. Hao, and N. Zheng, "Fuzzy $\mathcal{H}_\infty$ tracking control for nonlinear networked control systems in t-s fuzzy model," *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 39, no. 4, pp. 1073–1079, 2009.

[68] I. G. Polushin, P. X. Liu, and C.-H. Lung, "On the model-based approach to nonlinear networked control systems," *Automatica*, vol. 44, no. 9, pp. 2409–2414, Sep. 2008.

[69] S. H. Hong and W. H. Kim, "Bandwidth allocation scheme in can protocol," *IEEE Transactions on Control Theory and Applications*, vol. 147, pp. 37–44, 2000.

[70] Y. Kim, H. S. Park, and W. H. Kwon, "Stability and a scheduling method for network-based control systems," *The 22nd Annual Conference of the IEEE Industrial Electronics Society*, vol. 2, pp. 934–939, 1996.

[71] ——, "A scheduling method for network-based control systems," *Proceedings of the 1998 American Control Conference*, vol. 2, pp. 718–722, 1998.

[72] G. C. Walsh, O. Beldiman, and L. Bushnell, "Asymptotic behavior of networked control systems," *Proceedings of the 1999 IEEE International Conference on Control Applications*, pp. 1448–1453, Aug 1999.

[73] G. Walsh, H. Ye, and L. Bushnell, "Stability analysis of networked control systems," in *Proceedings of the 1999 American Control Conference*, 1999, pp. 2876–2880.

[74] G. Walsh, O. Beldiman, and L. Bushnell, "Error encoding algorithms for networked control systems," in *Proceedings of the 38th IEEE Conference on Decision and Control*, 1999, pp. 4933–4938.

[75] J. Wu, F. Q. Deng, and J. G. Gao, "Modeling and stability of long random delay networked control systems," in *Proceedings of 2005 International Conference on Machine Learning and Cybernetics*, vol. 2, 2005, pp. 947–952 Vol. 2.

[76] H. Lin, G. Zhai, and P. J. Antsaklis, "Robust stability and disturbance attenuation analysis of a class of networked control systems," in *Proceedings of 42nd IEEE Conference on Decision and Control*, 2003, pp. 1182–1187.

[77] R. A. Decarlo, M. S. Branicky, S. Pettersson, and B. Lennartson, "Perspectives and results on the stability and stabilizability of hybrid systems," *Proceedings of the IEEE*, vol. 88, pp. 1069–1082, 2000.

[78] D. Liberzon and A. S. Morse, "Basic problems in stability and design of switched systems," *IEEE Control Systems Magazine,*, vol. 19, pp. 59–70, 1999.

[79] M. S. Branicky, "Multiple lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Transactions on Automatic Control*, vol. 43, pp. 475–482, 1998.

[80] R. Luck and A. Ray, "An observer-based compensator for distributed delays," *Automatica*, vol. 26, no. 5, pp. 903–908, 1990.

[81] ——, "Experimental verification of a delay compensation algorithm for integrated communication and control systems," *International Journal of Control*, vol. 59, no. 6, pp. 1357–1372, 1994.

[82] H. Chan and U. Özgüner, "Closed-loopcontrol of systems over a communications network with queues," *International Journal of Control*, vol. 62, no. 3, pp. 493–510, 1995.

[83] A. Tzes and G. Nikolakopoulos, "Lqr-output feedback gain scheduling of mobile networked controlled systems," in *American Control Conference, 2004. Proceedings of the 2004*, vol. 5, 2004, pp. 4325–4329 vol.5.

[84] Y. Zheng, H. Fang, and H. Wang, "Takagi-sugeno fuzzy-model-based fault detection for networked control systems with markov delays," *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 36, no. 4, pp. 924–929, 2006.

[85] C. Peng and T. C. Yang, "Communication-delay-distribution-dependent networked control for a class of t-s fuzzy systems," *IEEE Transactions on Fuzzy Systems*, vol. 18, no. 2, pp. 326–335, 2010.

[86] M. S. Mahmoud, "$\mathcal{H}_\infty$ control of uncertain fuzzy networked control systems with state quantization," *Intelligent Control and Automation*, vol. 3, pp. 59–70, 2012.

[87] J. Dong and G.-H. Yang, "$\mathcal{H}_\infty$ controller synthesis via switched pdc scheme for discrete-time t-s fuzzy systems," *IEEE Transactions on Fuzzy Systems*, vol. 17, no. 3, pp. 544–555, 2009.

[88] W. Deng, M. Fei, and K. Li, "T-s fuzzy control of uncertain nonlinear networked control systems," in *International Conference on Advanced Computer Control*, 2009, pp. 36–40.

[89] W. Rui, H. Jiang, and L. Liu, "Continuously dynamic output feedback control for t-s fuzzy nonlinear networked control systems," in *International Conference on System Science and Engineering*, 2012, pp. 454–458.

[90] X. Chi, X. Jia, L. Li, and B. Liu, "$\mathcal{H}_\infty$ output tracking control for nonlinear discrete-time systems in random network environment," in *8th World Congress on Intelligent Control and Automation*, 2010, pp. 4413–4418.

[91] D. Zhang, Q.-L. Han, and X. Jia, "Network-based static output feedback tracking control for fuzzy-model-based nonlinear systems," in *IEEE International Conference on Fuzzy Systems*, 2012, pp. 1–8.

[92] H. K. Lam and F. H. F. Leung, "Stability analysis of fuzzy control systems subject to uncertain grades of membership," *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 35, no. 6, pp. 1322–1325, 2005.

[93] C. Ario and A. Sala, "Extensions to "stability analysis of fuzzy control systems subject to uncertain grades of membership"," *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 38, no. 2, pp. 558–563, Apr 2008.

[94] H. Lam and M. Narimani, "Quadratic-stability analysis of fuzzy-model-based control systems using staircase membership functions," *IEEE Transactions on Fuzzy Systems*, vol. 18, no. 1, pp. 125–137, 2010.

[95] H. Lam, M. Narimani, and F. Leung, "Stability analysis and stabilization of polynomial fuzzy-model-based control systems using piecewise linear membership functions," in *IEEE International Conference on Fuzzy Systems*, 2010, pp. 1–8.

[96] M. Narimani and H. K. Lam, "Sos-based stability analysis of takagi-sugeno fuzzy control systems via polynomial membership functions," in *IEEE International Conference on Fuzzy Systems*, 2009, pp. 203–208.

[97] Z. Mao, B. Jiang, and P. Shi, "$\mathcal{H}_\infty$ fault detection filter design for networked control systems modelled by discrete markovian jump systems," *IET Control Theory & Applications*, vol. 1, no. 5, pp. 1336–1343, 2007.

[98] Z. Huo and H. Fang, "Robust $\mathcal{H}_\infty$ filter design for networked control system with random time delays," in *Proceedings of the 10th IEEE International Conference on Engineering of Complex Computer Systems*, 2005, pp. 333–340.

[99] C. Jiang, D. Zou, and Q. Zhang, "$\mathcal{H}_\infty$ filtering for networked control systems with quantized signals," in *Proceedings of the 8th IEEE International Conference on Control and Automation*, 2010, pp. 634–639.

[100] S. Hu and D. Yue, "Event-based $\mathcal{H}_\infty$ filtering for networked system with communication delay," *Signal Processing*, vol. 92, no. 9, pp. 2029–2039, Sep 2012.

[101] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its applications to modeling and control," *IEEE Transactions on Systems, Man and Cybernetics*, vol. 15, no. 1, pp. 116–132, 1985.

[102] Y.-Y. Cao, J. Lam, and L. Hu, "Delay-dependent stochastic stability and $\mathcal{H}_\infty$ analysis for time-delay systems with markovian jumping parameters," *Journal of the Franklin Institute*, vol. 340, no. 6, pp. 423–434, Sep 2003.

[103] Z. Wang, J. Lam, and X. Liu, "Exponential filtering for uncertain markovian jump time-delay systems with nonlinear disturbances," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 51, no. 5, pp. 262–268, 2004.

[104] S. Xu, J. Lam, and X. Mao, "Delay-dependent $\mathcal{H}_\infty$ control and filtering for uncertain markovian jump systems with time-varying delays," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 54, no. 9, pp. 2070–2077, 2007.

[105] S. Hu and W.-Y. Yan, "Stability robustness of networked control systems with respect to packet loss," *Automatica*, vol. 43, no. 7, pp. 1243–1248, Jul 2007.

[106] N. N. Krasovskii and E. A. Lidskii, "Analytical design of controllers in systems with random attributes i, ii, iii," *Automatic Remote Control*, vol. 22, pp. 1021–1025, 1141–1146, 1289–1294, 1961.

[107] Y. Ji and H. J. Chizeck, "Controllability, stabilizability, and continuous-time markovian jump linear quadratic control," *IEEE Transactions on Automatic Control*, vol. 35, no. 7, pp. 777–788, July 1990.

[108] S. K. Nguang, W. Assawinchaichote, P. Shi, and Y. Shi, "Robust $\mathcal{H}_\infty$ control design for uncertain fuzzy systems with markovian jumps: an lmi approach," in *Proceedings of the American Control Conference*, June 2005, pp. 1805–1810.

[109] S. Xu, T. Chen, and J. Lam, "Robust $\mathcal{H}_\infty$ filtering for uncertain markovian jump systems with mode-dependent time delays," *IEEE Transactions on Automatic Control*, vol. 48, pp. 900–907, May 2003.

[110] E. K. Boukas and Z. K. Liu, "Robust stability and stabilizability of markov jump linear uncertain systems with mode-dependent time delays," *Journal of Optimal Theory and Application*, vol. 109, no. 3, pp. 587–600, 2001.

[111] Y. Y. Cao and L. J., "Robust $\mathcal{H}_\infty$ control of uncertain markov jump systems with time-delay," *IEEE Transactions on Automatic Control*, vol. 45, pp. 77–83, January 2000.

[112] E. K. Boukas, P. Shi, M. Karan, and C. Y. Kaya, "Linear discrete-time systems with markovian jumps and mode dependent time-delay: Stability and stabilizability," *Mathematical Problems in Engineering*, vol. 8, no. 2, pp. 123–133, 2002.

[113] Z. H. Guan, W. H. Chen, and J. X. Xu, "Delay-dependent stability and stabilizability of uncertain jump bilinear stochastic systems with mode-dependent time-delays," *International Journal of Systems Science*, vol. 36, no. 5, pp. 275–285, 2005.

[114] W.-H. Chen and W. X. Zheng, "Robust stabilization of delayed markovian jump systems subject to parametric uncertainties," in *Proceedings of the 46th IEEE Conference on Decision and Control*, 2007, pp. 3054–3059.

[115] L. El Ghaoui, F. Oustry, and M. AitRami, "A cone complementarity linearization algorithm for static output-feedback and related problems," *Automatic Control, IEEE Transactions on*, vol. 42, no. 8, pp. 1171–1176, 1997.

[116] Z. Mao, B. Jiang, and Y. Xu, "$\mathcal{H}_\infty$ filter design for a class of networked control systems via t-s fuzzy model approach," in *Proceedings of the IEEE International Conference on Fuzzy Systems*, 2010, pp. 1–8.

[117] B. Jiang, Z. Mao, and P. Shi, "$\mathcal{H}_\infty$ filter design for a class of networked control systems via t-s fuzzy-model approach," *Fuzzy Systems, IEEE Transactions on*, vol. 18, no. 1, pp. 201–208, 2010.

[118] A. Goncalves, A. Fioravanti, and J. Geromel, "Dynamic output feedback $\mathcal{H}_\infty$ control of discrete-time markov jump linear systems through linear matrix inequalities," in *Decision and Control, 2008. CDC 2008. 47th IEEE Conference on*, 2008, pp. 4787–4792.

[119] C. Fantuzzi and R. Rovatti, "On the approximation capabilities of the homogeneous takagi-sugeno model," in *Proceedings of the Fifth IEEE International Conference on Fuzzy Systems*, vol. 2, 1996, pp. 1067–1072.

[120] J. Buckley, "Universal fuzzy controllers," *Automatica*, vol. 28, no. 6, pp. 1245–1248, 1992.

[121] S. Prajna, A. Papachristodoulou, P. Seiler, and P. Parrilo, "Sostools: control applications and new developments," in *IEEE International Symposium on Computer Aided Control Systems Design*, 2004, pp. 315–320.

[122] S. Prajna, A. Papachristodoulou, and F. Wu, "Nonlinear control synthesis by sum of squares optimization: a lyapunov-based approach," in *5th Asian Control Conference*, vol. 1, 2004, pp. 157–165 Vol.1.

[123] X. Chen, C. Li, and N. Li, "A new development of sum of squares optimization in control application," in *International Conference on Intelligent Computation Technology and Automation (ICICTA)*, vol. 1, 2008, pp. 1200–1204.

[124] K. Tanaka, H. Yoshida, H. Ohtake, and H. Wang, "A sum-of-squares approach to modeling and control of nonlinear dynamical systems with polynomial fuzzy systems," *IEEE Transactions on Fuzzy Systems*, vol. 17, no. 4, pp. 911–922, 2009.

[125] ——, "A sum of squares approach to stability analysis of polynomial fuzzy systems," in *Proceedings of the American Control Conference*, 2007, pp. 4071–4076.

[126] J. Löfberg, "Pre- and post-processing sum-of-squares programs in practice," *IEEE Transactions on Automatic Control*, vol. 54, no. 5, pp. 1007–1011, 2009.

[127] S. Prajna, A. Papachristodoulou, and P. Parrilo, "Introducing sostools: a general purpose sum of squares programming solver," in *Decision and Control, 2002, Proceedings of the 41st IEEE Conference on*, vol. 1, 2002, pp. 741–746 vol.1.

[128] H. Zhang, M. Li, J. Yang, and D. Yang, "Fuzzy model-based robust networked control for a class of nonlinear systems," *IEEE Transactions on Systems, Man and Cybernetics, Part A: Systems and Humans*, vol. 39, no. 2, pp. 437–447, 2009.

[129] Y. Wang, W. Hu, and W. Fan, "Fuzzy model and control of nonlinear networked control systems," in *Proceedings of the 2005 International Conference on Wireless Communications, Networking and Mobile Computing*, vol. 2, 2005, pp. 1106–1109.

[130] ——, "Fuzzy robust $\mathcal{H}_\infty$ control for networked control systems," in *Intelligent Control and Automation, 2006. WCICA 2006. The Sixth World Congress on*, vol. 1, 2006, pp. 4448–4452.

[131] Y.-J. Pan, H. Marquez, and T. Chen, "Sampled-data iterative learning control for a class of nonlinear networked control systems," in *Proceedings of the American Control Conference*, Jun 2006, pp. 3494–3499.

[132] H. Dong, Z. Wang, D. Ho, and H. Gao, "Robust $\mathcal{H}_\infty$ fuzzy output-feedback control with multiple probabilistic delays and multiple missing measurements," *IEEE Transactions on Fuzzy Systems*, vol. 18, no. 4, pp. 712–725, 2010.

List of Author's Publications

Journals

1. S. Chae, D. Huang and S. K. Nguang, Robust partially mode delay dependent $\mathcal{H}_\infty$ control of discrete-time networked control systems, in *International Journal of Systems Science*, vol. 43, no. 9, pp. 1764-1773, 2012

2. S. Chae, F. Rasool, S. K Nguang and A. Swain, "Robust mode delay-dependent $\mathcal{H}_\infty$ control of discrete-time systems with random communication delays," in *IET Control Theory & Applications*, vol. 4, no. 6, pp. 936-944, June 2010

3. S. Chae, D. Huang and S. K. Nguang, Robust partially mode delay-dependent $\mathcal{H}_\infty$ output feedback control of discrete-time networked control system, in *Asian Journal of Control*, In review

4. S. Chae, S. K. Nguang and W. Wang, Robust $\mathcal{H}_\infty$ fuzzy control of discrete nonlinear networked control systems: a SOS approach, in *Journal of Franklin Institutue*, In review

5. S. Chae and S. K. Nguang, SOS based robust $\mathcal{H}_\infty$ fuzzy dynamic output feedback control of nonlinear networked control systems, in *IEEE Transactions on Systems, Man and Cybernetics Part B: Cybernetics*, Accepted

Conferences

1. S. Chae and S. K. Nguang, "Robust Partially Mode Delay Dependent $\mathcal{H}_\infty$ Control of Discrete-Time Networked Control Systems," in *Proceedings of the Conference on Decision and Control*, Atlanta, GA, USA, pp. 1798-1803, 2010

2. S. Chae, D. Huang and S. K Nguang, Robust partially mode delay-dependent $\mathcal{H}_\infty$ output feedback control of discrete-time networked control systems, in *Proceedings of the 2011 American Control Conference, San Francisco*, CA, USA, pp. 1680-1685, 2011