BEST REPLY DYNAMICS FOR SCORING RULES

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ABSTRACT. We consider best-reply dynamics for voting games in which all players are strategic and no coalitions are formed. We study the class of scoring rules, show convergence of a suitably restricted version for the plurality and veto rules, and failure of convergence for other rules including k-approval and Borda. In particular, for 3 candidates convergence fails for all rules other than plurality and veto. We give a unified proof for the convergence of these two rules. Our proofs in the case of plurality improve the known bound on convergence, and the other convergence results are new.

1. INTRODUCTION

Strategic misrepresentation of a voter's true preferences, as a way of obtaining an outcome preferable to that which would be expected by voting sincerely, dates back thousands of years. The amount of information available to voters and their ability to communicate influence voter behaviour greatly. Here we consider the case in which all players behave strategically, but coalitions are not formed. The natural setting then is that of a normal form game with ordinal preferences, or more generally a game form.

Voting games of this type have enormously many Nash equilibria and are not necessarily dominance solvable [2]. Eliminating dominated strategies is not also helpful because typically far too many equilibria remain for the Nash equilibrium to be a credible prediction. Other refinements such as strong and coalition-proof Nash equilibria may not always exist [8]. One natural direction of enquiry is to consider best-reply dynamics, where players take turns in moving myopically in response to previous moves by other players (these moves are pure strategies of the associated game). For many games this process leads to convergence (necessarily at a pure Nash equilibrium). It can also be interpreted in the voting context as a method of reaching consensus, and is in fact used in this way in some applications such as Doodle (for scheduling). According to Fudenberg and Levine [4], in some cases, most learning models do not converge to any equilibrium and just coincide with the notion of rationalizability, but if best-reply dynamics converges, it necessarily finds a NE. Therefore, the question that arises here is in which cases these best-reply dynamics converge for voting games. To our knowledge, in the voting context the first paper to discuss best-reply dynamics is [7], which concentrated on the plurality rule. The authors considered the effect of initial state, tie-breaking rule, the players' strategy and weights on convergence. The results show that this definition of best reply, even for such a rule which restricts voter expression severely, is too general to guarantee convergence. Sequential and simultaneous voting games for plurality with abstention have been discussed in [1]. For the sequential case, they provide a complete analysis of the setting with two candidates, and show that for three or more candidates the equilibria of sequential voting may behave in a counterintuitive manner. The strategy of each voter depends strongly on the information he has about the other players' preference orders.

1.1. **Our Contribution.** A natural extension of [7] is to consider general positional scoring rules, which we do. We find that non-convergence occurs much more often in this case, as might be expected because of the much larger strategy spaces involved. For the antiplurality (veto) rule, which restricts strategy spaces as much as plurality, we give a complete analysis and show convergence under rather general conditions. We also give unified simple proofs for plurality and antiplurality and give more details on the boundary between convergence and nonconvergence when tiebreaking methods are considered. We study cycles in the scoring rules between plurality and antiplurality. For a general scoring rule, the order in which players respond in the best reply dynamics influences the convergence considerably. Our results show that some tightening of the definition of best reply is indeed required for convergence for plurality and antiplurality. However, a natural extension of this tighter definition to general scoring rules fails to guarantee convergence.

2. PROBLEM DESCRIPTION

2.1. **Voting Setup.** There is a set C of alternatives (candidates) and a set V of players (voters), with $m := |C|$, $n := |V|$. Each voter has a strict total order on candidates, the preference order of that voter, denoted σ_v. This defines the set $\mathcal{T}$ of types of voters, and $|\mathcal{T}| = m!$. The function mapping $v \mapsto \sigma_v$ is the profile. A voting rule (or social choice correspondence) that maps each profile to a nonempty subset of C (the winner set).

For a voting rule R, we study the game $G(V,C,R)$ where each voter v submits a permutation π_v of the candidates as an action. The set of pure strategies available to voter i, S_i, consists of the $m!$ possible types. In other words, a voter can report a preference order, which may not be his sincere one. We denote the sincere profile and the profile at time t respectively by p_0 and p_t. We order the types lexicographically, based on a fixed order of candidates.

A *voting situation* is a multi-set from T with total weight n. For anonymous rules (those invariant under permutations of the voters), the voting situation gives a more compact description than the full profile, with no loss of information. For example, if we have 3 candidates a, b and c, and 4 voters with preference orders abc, bca, cab and bca, the voting situation coinciding with that profile is $(1,0,0,2,1,0)$.

A *voting rule* (or social choice correspondence) is a mapping taking each profile to a nonempty subset of C (the *winners*). A voting rule is *resolute* (or a social choice function) if the set of winners always has size 1.

The *scoring rule* determined by a weight vector w with

$$1 = w_1 \geq w_2 \geq \cdots \geq w_{m-1} \geq w_m = 0$$

assigns the score

$$s(c) := \sum_{t \in \mathcal{T}} |\{v \in V | \pi_v = t\}| \, w_{\pi_v{}^{-1}(c)} \tag{1}$$

to each candidate. For example, several well-known scoring rules are:

- Plurality: $w = (1,0,\ldots,0,0)$ in which each voter in effect votes for one candidate.
- Antiplurality (veto): $w = (1,1,\ldots,1,0)$ in which each voter in effect votes against one candidate.
- Borda: $w = (m-1, m-2, \ldots, 1, 0)$.

The winners are the candidates with the highest score. These rules allow ties in scores and to make them resolute, we need to use a deterministic tie-breaking rule. However, for neutrality (symmetry between candidates) we need to consider randomized tie-breaking.

2.2. **Improvement Step.** Let p be a profile. Suppose that voter v changes his vote. We say this is an *improvement step* if p' (the new profile) is *preferred* to p by voter v. The fundamental results on strategic manipulation initiated by Gibbard [5] and Satterthwaite [10] imply that, provided the voting rule is resolute, under very mild additional conditions (such as not being dictatorial), and provided that $m \geq 3$ and $n \geq 2$, some agent in some sincere voting situation has an improvement step.

In order to describe improvement steps in more detail, we need to discuss outcomes and payoffs (at least ordinal, if not cardinal). The obvious way to do this in the case of resolute voting rules is to declare that the outcome in which the winner is a is preferred by voter v to the outcome in which the winner is b if and only if a is higher than b in v's sincere preference order.

Example 2.1. *(alphabetical tie-breaking) Consider the Borda rule, given by the weight vector* $(2,1,0)$, *and the voting situation with 2 abc, 2 bac, 2 bca, 3 cab voters. The current winner is b. If one of the cab voters changes as acb, then a wins. The new outcome is preferred by that voter because he prefers a to b.*

Stochastic dominance. In the case of multiple winners (or randomized tiebreaking), more assumptions are needed. We unify the two cases by using the idea of stochastic dominance as in [9]. This corresponds to a rather risk-averse model of manipulation, as we now describe. It can be described in probabilistic language as follows. For each winner set constructed by the voting rule, we have a uniform distribution on the candidates in that set, and other candidates have probability zero associated to them. Voter v prefers an outcome with winner set W to an outcome with winner set W' if and only if the following condition holds. List the candidates in decreasing order of preference for voter v, and consider the probability distributions as described above. We say that W is preferred to W' if and only if for each $k = 1 \cdots m$ the probability of electing one of the first k candidates given outcome W should be no less than given W'. (If $W' \neq W$ the condition implies that this probability will be strictly greater for some k).

Our definition of improvement step implies that, for example, a vote by a voter with preference bac which changes the winner set from a to $\{b, c\}$ is not an improvement. Of course, if we assigned cardinal utilities to outcomes, there might be some voters for which such a move increases expected utility. In fact, it is easily shown that our definition above says that the probability distribution associated with W first order stochastically dominates the distribution associated with W'. It is well known [3] that this is equivalent to requiring that W is preferred to W' in terms of expected utility, for all cardinal utilities consistent with the preference order of the voter.

Example 2.2. *(random tie-breaking) Suppose that in profile p the outcome is that a and c tie as the winner, in profile p' the outcome is that b is the absolute winner, and in p'' the outcome is that a and b tie as the winner. The probability distribution of winning on (a,b,c) is $(1/2,0,1/2)$ for p, $(0,1,0)$ for p' and $(1/2,1/2,0)$ for p''. Thus, taking $k = 1$ in the definition, we see that p' is not preferred to p by a voter with sincere opinion abc. Also, taking $k = 2$ shows that p is not preferred to p' either. However, p'' is preferred to both p and p'.*

Other possibilities. For example, [7] has considered the case where voters have fixed but arbitrary cardinal utilities. This allows for situations in which more moves are considered to be improvement steps than in our stochastic dominance model above.

3. BEST REPLY DYNAMICS

We make the following assumptions in our analysis of best reply dynamics for scoring rules.

- No fixed order for players' turns: in fact, whichever voter has an improvement step can move next.
- Myopic moves: Voters act as though each move is their only chance for improving the result, regardless of considering any chance of changing in the future.
- Costly voting: if there would no change in the winner set, no move is made.
- Restricted best reply (RBR): we may have several improvement steps which give the same outcome, in which case we choose the one that maximizes the winning score margin of the new winner.
- Stochastic dominance-based improvement step for non-resolute rules.

All the assumptions except the last one are consistent with those in [7]. The fourth applies only for scoring rules, but the others make sense for all voting rules.

Example 3.1. *Consider the antiplurality rule with 2 voters $V = \{1,2\}$ and 4 candidates $C = \{a,b,c,d\}$, alphabetically tie-breaking. The sincere profile is $p_0 = (acbd, bacd)$. Vetoing candidate c is represented by $-c$ in the strategy profile of voters. The number above the arrow represents the player who moves, and the candidate in braces shows the winner. If voters start from sincere state, we have:*

$(-d,-d)\{a\} \xrightarrow{2} (-d,-a)\{b\} \xrightarrow{1} (-b,-a)\{c\} \xrightarrow{2} (-b,-c)\{a\}$

As you can see in the example, best reply is not unique, for example, the last move by the second player can instead be $-d$. However, $-c$ (vetoing the current winner) is what we call RBR for antiplurality .

4. BEST REPLY DYNAMICS FOR ANTIPLURALITY

In this section we show convergence of best reply dynamics under rather general conditions, for a very special scoring rule, namely the antiplurality rule.

For the game $G(V,C,A)$, since $w=(1,1,\dots,1,0)$, we can without loss of generality assume that $S_i=\{-c|c\in C\}$ (because subtracting the vector $(1,1,\dots,1)$ from the weight vector makes no difference to the outcome of the game or to the differences in scores). In fact, there are $(m-1)!$ possible orders that give the same score. Thus, each improvement step can be written $-a\to -b$ where $b\neq a$.

Remark. *We define o_t as the winner set after the move of player i at time t. For alphabetical tie-breaking this set is a singleton.*

Analogous to the case for plurality [7], there are 3 types of improvement steps.

Type 1:: $a\notin o_t$ and $b\in o_{t-1}$

Type 2:: $a\in o_t$ and $b\notin o_{t-1}$

Type 3:: $a\in o_t$ and $b\in o_{t-1}$

Remark. *It can easily be shown that if $a\notin o_t$ and $b\notin o_{t-1}$, this move does not change the winner set. Therefore, it is not an improvement step.*

Example 4.1. *Suppose we have 2 voters and 3 candidates using antiplurality rule with alphabetical tie-breaking. The sincere profile is $p_0=(abc,bac)$. If voters start from the sincere state, the current winner is a. If the second player changes his vote from $-c$ to $-a$, the winner switches to b. According to our definition, it is a type 1 move.*

Some notations. We define some notations that we use through the rest of the paper.

- We write $c\rhd c'$ if c has a lower index (higher priority) than c' in alphabetical tie-breaking.
- We write $s(c')\curlyeqprec s(c)$ if either $s(c')<s(c)$ or $s(c)=s(c')$ and $c\rhd c'$(note that it is not a logical notation, and we just use it for simplicity).
- We use the symbol $a\succ_i b$ when voter i prefers candidate a to b.
- We denote the score of candidate a after the improvement step at time t by $s_t(a)$.
- We use the notation $x\xrightarrow{i} y$ when voter i changes his vote from x to y.

Theorem 4.2. *Suppose that $-a\to -c$ is a type 2 improvement step at time t, and let $b\in o_{t-1}$. Then $-a\to -b$ is a type 3 improvement step leading to the same set o_t. Furthermore, in this case the margin of victory of the new winner will be more than in the original case.*

Proof. After the improvement step $-a\to -c$ at time t, we have

$$s_t(a)=s_{t-1}(a)+1$$
$$s_t(c)=s_{t-1}(c)-1.$$

Since $a\in o_t$ (according to the definition of type 2) and $b\in o_{t-1}$ and $s_{t-1}(b)=s_t(b)$, in alphabetical tie-breaking, we have

$$s_t(a)\curlyeqsucc s_t(b)\curlyeqsucc s_t(c) \quad\text{and}\quad s_t(a)\curlyeqsucc s_t(y)\ y\in C\setminus\{a,b\} \tag{2}$$

If we had the improvement step $-a\to -b$ at time t instead, (we denote the score in this case with s'_t)

$$s'_t(a)=s_t(a) \quad\text{and}\quad s'_t(b)=s_t(b)-1;$$
$$s'_t(c)=s_t(c)+1 \quad\text{and}\quad s'_t(y)=s_t(y).$$

By substituting in Equation (2), we have $s'_t(a)\curlyeqsucc s'_t(y)$ for each $y\in C$. Therefore, a is the new winner. For randomized tie-breaking, we can substitute $\curlyeqsucc$ by $\geq$. Also, the margin of victory with a type 3 improvement step would be $s'_t(a)-s'_t(b)=s_t(a)-s_t(b)+1$ which is more than the original margin $s_t(a)-s_t(b)$. □ □

We now make a key definition of the allowed moves. Allowing type 2 moves can lead to a cycle. An example for plurality has been presented in [7] (Proposition 4). We have a similar example for antiplurality with 7 candidates and 10 voters that we omit because of space constraints.

Definition 4.3. *(**RBR**) A* restricted best reply *is any improvement step of type 1 or type 3, in which the player making the step vetoes his least preferred member of o_{t-1}, denoted β_{t-1}.*

From now on, we consider only improvement steps using restricted best replies. It is also clear from the definition that no two consecutive improvement steps can be made by the same voter.

Example 4.4. *When voters start from the sincere initial state, and the sincere scoreboard is a tie among all candidates, all improvement steps would be type 3 ones. Therefore, no improvement step can occur, as voters have already voted against their least desirable candidate, and any change will allow that candidate to win.*

Definition 4.5. *(**set of potential winners**) The set of potential winners at time t, W_t consists of those candidates who have a chance of winning at the next step (time $t+1$), depending on the different RBR of voters.*

Remark. *If candidate c can win by type 1, it can also win by type 3 because when a candidate can win without increasing its score, it is obviously still a winner when its score is increased by 1. Therefore,*

$$W_t = \{c \mid \text{if some player moves } -c \to -b \text{ at time } t+1, \text{ then } c \in o_{t+1}\} \tag{3}$$

4.1. Alphabetical Tie-breaking.

Lemma 4.6. *If $t < t'$ then $W_t \subseteq W'_t$.*

Proof. Consider an improvement step $-a \to -b$ at time t. According to Definition 4.3, $o_{t-1} = b$. Let $c \in W_{t-1}$ and $y \in C \setminus \{a,b\}$. Then, by considering that the scores of c and y, $\forall y \in C$; $y \neq a,b$ don't change at time t, we have:

$$s_t(c)+1 = s_{t-1}(c)+1 \succeq s_{t-1}(b)-1 = s_t(b) \tag{4}$$

$$s_t(c)+1 = s_{t-1}(c)+1 \succeq s_{t-1}(y) = s_t(y) \tag{5}$$

If the improvement step is of type 3, then best reply $-c \to -b$ at time t gives the same scores as the best reply $-a \to -b$ followed by $-c \to -a$ at time $t+1$. Therefore, $c \in W_t$.

If the improvement step is of type 1, let $b' = o_t$. Note that $b' \notin \{a,b\}$.

According to equation (5), for $y = b'$,

$$s_t(c)+1 \succeq s_t(b') > s_t(b')-1 \tag{6}$$

According to the definition of winner,

$$s_t(b') \succeq s_t(y); \forall y \in C \tag{7}$$

In particular for $y = a$,

$$s_t(c)+1 \succeq s_t(b') \succeq s_t(a) \tag{8}$$

Thus, by transitivity of $\succeq$ (which follows from the underlying transitive lexicographic order on C), $c \in W_t$. □ □

A counter-example for an arbitrary deterministic tie-breaking rule. Consider a situation with candidates a,b,c and x under the antiplurality rule. Suppose the set of candidates with the highest score after round $t-1$ is $\{b,x\}$ and $s_{t-1}(a) = s_{t-1}(c) = s_{t-1}(b)-1$. Suppose further that the order of candidates in tie-breaking is as follows: $b \rhd x$ and $c \rhd x$ and $x \rhd a$ and $a \rhd c$. Based on Definition 4.3, $c \in W_{t-1}$. Consider a best reply $-a \to -b$ at time t. If it is a type 3 move then $o_t = a$ and c is still in W_t, as $-c \to -a$ makes c winner. Suppose the move is of type 1 and $o_t = x$. According to the tie-breaking rule, $b \rhd x$ and $c \rhd x \rhd a$ but, $a \rhd c$. Thus, c is not in W_t because $-c \to -x$ does not make c win.

Lemma 4.7. *Each voter has at most one type 1 move and at most $m-1$ moves of type 3.*

Proof. Suppose a step $-a \xrightarrow{i} -b$ is a type 1 move by voter i at time t. We claim this improvement step is the first improvement step of voter i. If it is not his first improvement step, according to Definition 4.3, a has been a winner before. Therefore, a has been in the winner set in the past.. In other words, $\exists t' : t' < t \ \ a = o_{t'}$ and therefore, $a \in W_{t'}$. According to Lemma 4.6, $a \in W_{t-1}$ which means after improvement step $-a \to -b$ at time t, a is a winner. However, this has contradiction with the assumption of improvement step of type 1. Therefore, voter i has at most one type 1 move. According to the definition of improvement step, at every step $-a \xrightarrow{i} -b$ of type 3, it must hold that $a \succ_i b$. Therefore, each voter has at most $m-1$ steps of type 3. □ □

Theorem 4.8. *Restricted Best Reply Dynamics (RBRD) for $G(V,C,A)$ with alphabetical tie-breaking will converge to a NE from any state in at most mn steps.*

Proof. If we have n voters, Lemma 4.7 implies that each voter makes at most m moves. □ □

4.2. **Randomized Tie-breaking.**

Lemma 4.9. *If $t < t'$ then $W_t \subseteq W'_t$.*

Proof. The proof is very similar to the alphabetical case (Lemma 4.6). Except, we do not need to deal with tie-breaking. Therefore, we can substitute the notation $\succcurlyeq$ by $\geq$. For the second part of the proof where we consider a type 1 improvement step, we can always find such a b'. To see this, note that according to the definition of improvement step, the winner set should be changed and the score of b decreases. Therefore, b cannot be the unique winner at time t as it results in b being the unique winner at time $t-1$, contradicting the definition of improvement step. □ □

Lemma 4.10. *Each voter has at most one type 1 move and at most $m-1$ moves of type 3.*

Proof. The first part can be proved in a similar way to Lemma 4.7. For the second part, similarly, we show that $a \succ_i b$ if voter i makes the type 3 improvement step $-a \to -b$. According to the definition of type 3 improvement step, $b \in o_{t-1}$ and $a \in o_t$. We define $p(a)$ as the probability of winning of a. Two cases can occur.

Case 1: $a \in o_{t-1}$

$p(a)$ increases to 1 and $p(b)$ decreases to 0. The probability of winning of candidates in the set o_{t-1} decreases and for other candidates stay 0.

In this case, a becomes the unique winner at time t. Therefore, according to the definition of stochastic dominance improvement step, a should be preferred to all other elements of o_{t-1}.

Case 2: $a \notin o_{t-1}$

$i)$ $b = o_{t-1}$ In this case, $p(a)$ and $p(c)$ increases to $\frac{1}{k+2}$ and $p(b)$ decreases from 1 to $\frac{1}{k+2}$ (assuming the number of candidates (c) whose score is 1 point behind b is k) and for other candidates it remains the same.

$ii)$ $b \in o_{t-1}$ therefore, $p(a)$ increases and $p(b)$ decreases and $p(c)$ stays the same. Therefore $a \succ_i b$, otherwise, it is not an improvement step. □ □

The analogue of Theorem 4.8 now follows.

Theorem 4.11. *RBRD for $G(V,C,A)$ with randomized tie-breaking, will converge to a NE from any state in at most mn steps.*

Remark. *The only part in the proof for randomized tie-breaking, where we used stochastic dominance assumption of improvement step is for the bound on type 3 moves. An example of cycle is already shown in [7] for a fixed utility case.*

4.3. **Who Can Win?** In this part, we describe W_t in more detail.

$$W_t = W_t^0 \cup W_t^1 \cup W_t^2 \tag{9}$$

where W^0 is the level of winner set which includes the candidates who are tied with the winner, W^1 contains the candidates who can win by a type 1 move and W^2 those who can win by a type 3 move and not a type 1 move. Let $M_t = s_t(o_t)$ and $d_t(c) = M_t - s_t(c)$. In fact $d_t(c)$ represents the score difference of candidate c and the winner after move t. Therefore, $W^0 = \{c \mid d(c) = 0\}$. The description of the other two subsets is straightforward.

Proposition 4.12. *For alphabetical tie-breaking,*

$$W_t^1 = \{c \mid d(c) = 1, c \rhd c'; \forall c' \in W_t^0\} \tag{10}$$

$$W_t^2 = \{c \mid d(c) = 2 \text{ and unique winner and } c \rhd c'; \forall c' \in W_t^1 \cup W_t^0\}. \tag{11}$$

For the case of randomized tie-breaking,

$$W_t = \{c \mid d_t(c) \leq 1 \text{ or } d_t(c) = 2 \text{ and there is a unique winner }\}. \tag{12}$$

□

To obtain a better idea about who is really winning in practice at equilibrium, we ran several simulation experiments with different initial profiles (sincere, random). The numerical results suggest that in the cases with sincere initial state, the winner set of equilibrium is contained in W_0. However, this is not true when we start from an arbitrary state.

5. PLURALITY

The results in this section are completely analogous to those in Section 4, and are quite similar to [7] but with easier proofs. We remove some details of proofs as they are similar to previous section.

Definition 5.1. *(**RBR**) For plurality rule, a* restricted best reply *is any improvement step of type 1 or type 3, in which*

- **Type 1::** $a \notin o_{t-1}$ *and* $b \in o_t$
- **Type 3::** $a \in o_{t-1}$ *and* $b \in o_t$

The restricted best replies defined above are similar to the best replies in [7], where the phrase “better reply" is used for non-restricted best replies.

Remark. *(**set of potential winners**) For plurality also, we just consider the candidates who can win by type 3 moves because of the same argument as antiplurality. Therefore, the set of potential winners is*

$$W_t = \{c \mid \text{if some player moves } a \to c \text{ and } a \in o_t \text{ then } c \in o_{t+1}\} \tag{13}$$

5.1. Alphabetical Tie-breaking.

Lemma 5.2. *If $t < t'$ then $W_t' \subseteq W_t$.*

Proof. Consider an improvement step $a \to b$ at time t. By the definition of best reply in Definition 5.1, $b = o_t$. Let $c \in W_t$. Considering the new scores of b, c and y, $\forall y \in C$; $y \neq a, b$ we have:

$$s_{t-1}(c) + 1 = s_t(c) + 1 \succcurlyeq s_t(b) - 1 = s_{t-1}(b) \tag{14}$$

$$s_{t-1}(c) + 1 = s_t(c) + 1 \succcurlyeq s_t(y) = s_{t-1}(y) \tag{15}$$

If the improvement step $a \to b$ is of type 3, then best reply $a \to b$ followed by $b \to c$ at time $t + 1$ give the same scores as best reply $a \to c$ at time t. Therefore, $c \in W_{t-1}$.

If the improvement step is of type 1, let $a' = o_{t-1}$; Note that $a' \notin \{a, b\}$.

According to Equation (15), for $y = a'$,

$$s_{t-1}(c) + 1 \succcurlyeq s_{t-1}(a') \tag{16}$$

According to the definition of winner,

$$s_{t-1}(a') \succcurlyeq s_{t-1}(y); \forall y \in C \tag{17}$$

In particular for $y = a$,

$$s_{t-1}(c) + 1 \succcurlyeq s_{t-1}(a') \succcurlyeq s_{t-1}(a) \tag{18}$$

Thus, by transitivity of $\succcurlyeq$ (which follows from the underlying transitive lexicographic order on C), $c \in W_{t-1}$. □ □

Lemma 5.3. *The number of type 1 moves is at most m and each voter has at most $m-1$ moves of type 3.*

Proof. Suppose a step $a \to b$ is a type 1 move at time t. We claim $a \notin W_t$. If $a \in W_t$ then $b \to a$ makes a winner but we know $b \to a$ makes a' win (the two consecutive moves have cancelled out each other). Therefore, $a \notin W_t$. According to Lemma 5.2, $a \notin W_{t'}$; $\forall t' > t$. Therefore, the number of type 1 moves is limited and equals the maximal set of potential winners which at most can have m elements. Also, as at every step $a \xrightarrow{i} b$ of type 3, it must hold that $b \succ_i a$ because of the definition of improvement step, each voter has at most $m-1$ moves of type 3. □ □

Theorem 5.4. *RBRD for $G(V,C,P)$ with alphabetical tie-breaking will converge to a NE from any state in at most $m+(m-1)n$ steps.*

Proof. If we have n voters, Lemma 5.3 implies that convergence must occur with at most $m+(m-1)n$ steps. □ □

5.2. **Randomized Tie-breaking.**

Lemma 5.5. *If $t < t'$ then $W_t' \subseteq W_t$.*

Proof. The proof is very similar to the alphabetical case (Lemma 5.2). Except, we do not need to deal with tie-breaking. Therefore, we can substitute the notation $\succcurlyeq$ by $\geq$. For the second part of the proof where we consider a type 1 improvement step, we can always find such a a' by similar reasoning as in proof of Lemma 4.9. □ □

Lemma 5.6. *The number of type 1 moves is at most m and each voter has at most $m-1$ moves of type 3.*

Proof. The proof is very similar to Lemma 4.10 by considering the differences of Lemma 5.3 and 4.7. □ □

Theorem 5.7. *RBRD for $G(V,C,P)$ with randomized tie-breaking will converge to a NE from any state in at most $m+(m-1)n$ steps.*

Proof. If we have n voters, Lemma 5.6 implies that convergence must occur with at most $m+(m-1)n$ steps. □
□

Remark. *The only part in the proof for randomized tie-breaking where we used the assumption of stochastic dominance is for the bound on type 3 moves. Note that an example is given in [7] showing that if we use fixed utility function, and improvement is defined by expected utility increase, a cycle can occur. The stronger definition of improvement step using stochastic dominance allows us a general convergence result.*

6. COUNTEREXAMPLES AND INTERESTING PHENOMENA

Best reply dynamics for scoring rules other than plurality and antiplurality does not necessarily converge (symbol ♢ shows the stage from which the cycle becomes apparent). Each of the examples in this section starts from the sincere initial state.

Example 6.1. *(Cycle for Borda) Consider the sincere profile $p_0 = (abc, bca)$ and voting rule Borda and alphabetical tie-breaking.*

$(abc, bca)\{b\} \xrightarrow{1} (acb, bca)\{a\} \xrightarrow{2} (acb, cba)\{c\} \xrightarrow{1} (abc, cba)\{a\} \xrightarrow{2} (abc, bca)\{b\}$ ♢.

Remark. *The allowed moves in the previous example are reasonable for restricted best replies with 3 candidates. Putting the desirable candidate (the new winner) at the top and the current winner at the bottom maximizes the winning score margin of the new winner.*

Cycle for scoring rules “close to Plurality":

- Suppose we have 3 candidates a, b and c and $p_0 = (abc, bca)$. The scoring rule is $w = (1, \alpha, 0)$; $0<\alpha \leq \frac{1}{2}$ and we use alphabetical tie-breaking.
  $(abc, bca)\{b\} \xrightarrow{1} (acb, bca)\{a\} \xrightarrow{2} (acb, cba)\{c\} \xrightarrow{1} (abc, cba)\{a\} \xrightarrow{2} (abc, bca)\{b\}$ ♢
- general m and $n = 2$
  $(ab\cdots c, bc\cdots a)\{b\} \xrightarrow{1} (a\cdots cb, bc\cdots a)\{a\} \xrightarrow{2} (a\cdots cb, cb\cdots a)\{c\} \xrightarrow{1} (ab\cdots c, cb\cdots a)\{a\} \xrightarrow{2} (ab\cdots c, bc\cdots a)\{b\}$ ♢

Cycle for scoring rules “close to antiplurality":$m = 3, n = 4$. Suppose we have 3 candidates a, b and c. The sincere profile is $p_0 = (abc, bac, cab, bca)$. Our scoring rule is $(1, \alpha, 0)$; $\frac{1}{2} \leq \alpha < 1$ with alphabetical tie-breaking.

$(abc, bac, cab, bca)\{b\} \xrightarrow{1} (acb, bac, cab, bca)\{a\} \xrightarrow{4} (acb, bac, cab, cba)\{c\} \xrightarrow{1} (abc, bac, cab, cba)\{a\} \xrightarrow{4} (abc, bac, cab, bca)\{b\}$ ♢

Example 6.2. *(Order of players matters) To understand the impact of the order of players on the dynamics, we consider Borda rule with 4 voters and 3 candidates. Suppose $p_0 = (acb, acb, cab, cba)$ and players start from the sincere state. The winner is c. The first player is not satisfied with the result and changes his vote to abc to make a the sole winner. For simplicity, we show the moves of players as below:*

$(acb, acb, cab, cba)\{c\} \xrightarrow{1} (abc, acb, cab, cba)\{a\} \xrightarrow{3} (abc, acb, cba, cba)\{c\} \xrightarrow{2} (abc, abc, cba, cba)\{a\} \xrightarrow{4} (abc, abc, cba, bca)\{b\} \xrightarrow{1} (acb, abc, cba, bca)\{1\} \xrightarrow{4} (acb, abc, cba, cba)\{c\} \xrightarrow{1} (abc, abc, cba, bca)\{b\}$ ♢

Note $p_4 = p_7$ and we have a cycle.

Now let's consider another order for the players. We start with another profile coinciding with $V = (0, 2, 0, 0, 1, 1)$.

$(acb, acb, cba, cab)\{c\} \xrightarrow{1} (abc, acb, cba, cab)\{a\} \xrightarrow{4} (abc, acb, cba, cba)\{c\} \xrightarrow{2} (abc, abc, cba, cba)\{a\} \xrightarrow{3} (abc, abc, bca, cba)\{b\} \xrightarrow{4} (abc, abc, bca, cab)\{a\}$ *(equilibrium)*

Thus, in contrast with previous order, we reach an equilibrium with this order of players. 8 of 12 profiles coinciding with this voting situation do not converge.

Example 6.3 (an example of cycle for 2-approval voting)**.** *Consider 4 candidates $C = \{a, b, c, d\}$ and 2 voters with $p_0 = \{acdb, dbca\}$ under 2-approval voting rule with weight vector $w = (1, 1, 0, 0)$. Players start from sincere state and we use alphabetical tie-breaking. Therefore, the sincere winner is c. As voters need to to approve two candidates we show the dynamic process as below:*

$(ac, db)\{a\} \xrightarrow{2} (ac, dc)\{c\} \xrightarrow{1} (ab, dc)\{a\} \xrightarrow{2} (ab, db)\{b\} \xrightarrow{1} (ac, db)\{a\}$ ♢

7. CONCLUSION AND FUTURE DIRECTIONS

A summary of results:

- The upper bound of convergence for plurality in our paper is $m + (m-1)n$. However, it is m^2n^2 in paper [7]. Our upper bound for antiplurality is mn.
- The possibility of winning of a candidate depends on the type of improvement step and also candidate's priority in tie-breaking.
- The number of type 2 moves is not bounded, so we need to use RBR for convergence.
- We need to use stochastic dominance RBR for randomized tie-breaking for plurality and antiplurality. Without this assumption we can have cycles, as shown in [7] and [6].
- Convergence fails for some deterministic tie-breaking rules.
- The order of players influences convergence, equilibrium result and also speed of convergence.
- We have examples of cycling for 2-approval.

During the writing of this paper, we noticed that Lev and Rosenschein have also considered similar questions and have obtained quite similar results [6]. However, our paper is completely independent from their work and has a different approach. We now give a brief discussion of the similarities and differences between these papers.

Both papers give convergence results for antiplurality under alphabetical tiebreaking: our Theorem 4.8 corresponds to [6, Theorem 13]. Both show nonconvergence for k-approval (Example 6.3 vs Theorem 19) and Borda (Example 6.1 vs Theorem 11). The counterexample for Borda in [6] works for any tiebreaking rule, and for $m \geq 4$, whereas ours works for $m \geq 3$ but uses a specific tiebreaking rule. In addition, [6] gives a counterexample for the maximin rule with a non-lexicographic deterministic tiebreaking rule, while we consider only scoring rules.

[6] deals only with deterministic tiebreaking, while we discuss randomized tiebreaking in some detail and show that stochastic dominance is the correct condition for ensuring convergence. Furthermore, we consider plurality and show how the proofs for antiplurality and plurality are essentially dual to each other. Our convergence proofs are shorter and, in our view, simpler. The upper bound in [6, Lemma 17] for antiplurality is $(m-2)n$ which can be contradicted by considering $p_0 = (bac, cab)$. If voters start from $(-b,-c)\{a\} \xrightarrow{1} (-a,-c)\{b\} \xrightarrow{2} (-a,-b)\{c\} \xrightarrow{1} (-c,-b)\{a\}$ $\diamondsuit$ where first voter has $m-1$ improvement steps.

As far as future directions go, the key issue in extending to other voting rules is to properly define a notion of restricted best reply which is general enough to encompass all “reasonable" moves by rational agents seeking to maximize their payoff at each step, yet doesn't allow cycles. Already Example 6.1 shows that this will be difficult for Borda. Our proof skeleton for plurality and antiplurality could be adopted provided this difficulty is overcome. However for this approach to work easily, we would need the composition of two improvement steps to yield the same situation as a single improvement step (as in the discussion of type 3 moves in the proof of Lemma 4.6). One possible way of overcoming this problem would be to impose a domain restriction (do not allow all possible preference profiles to occur). Conceivably this might even allow type 2 moves as defined above to be reinstated as allowable improvement steps, while still maintaining convergence.

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