DELTA METHODS IN ENVELOPING ALGEBRAS OF LIE COLOUR ALGEBRAS

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Abstract. In recent papers J. Bergen and D. S. Passman have applied so-called 'Delta methods' to enveloping algebras of Lie superalgebras. This paper generalizes their results to the class of Lie colour algebras. The methods and results in this paper are very similar to those of Bergen and Passman, and many of their proofs generalize easily. However, at some points there are serious difficulties to overcome.

The results obtained show that if L is a Lie colour algebra then the join of all finite- dimensional ideals of L, denoted Δ_L, controls certain properties of the universal enveloping algebra $U(L)$. Specifically, we consider primeness, semiprimeness, constants, semi-invariants, almost constants, faithfulness of the adjoint action, the centre, almost centralizers and the central closure.

1. Introduction

This paper generalizes the results of [BP3, BP4] on enveloping algebras of Lie superalgebras to the case of Lie colour algebras. As in those papers, we first prove theorems reducing certain identities in $U(L)$ to corresponding identities in certain subalgebras.

The chief obstacles to extending the results of [BP1, BP2] on Lie algebras to Lie superalgebras as in [BP3, BP4] were the fact that the natural degree function on the universal enveloping algebra $U(L)$ is no longer additive for superalgebras, and the more complicated relationship between the adjoint action and multiplication in $U(L)$. The second problem proves much more troublesome in the Lie colour algebra case. Our degree arguments are very similar to those in [BP4] but it is not possible for us to finesse the adjoint map complications as easily as was done in that paper.

Many of the arguments of Bergen and Passman made essential use of the fact that the grading group had only 1 or 2 elements, and consequently these arguments require major modification. Several of their proofs using degree arguments proved more than was stated, and we have chosen to make these details more explicit here.

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The reader is strongly advised not to attempt a detailed reading of this paper without either being familiar with or having close by the papers [BP3, BP4], as many proofs from those papers are cited in lieu of giving full proofs here.

All ring-theoretic applications are found in Section 5. In Section 2 the basic definitions and elementary properties of our objects of study are discussed. In Section 3 we reduce both derivation and linear identities in $U(L)$ to identities in $U(\Delta)$ and in Section 4 we continue the reduction to $U(\Delta_L)$. The brief final section contains some comments on unsolved problems.

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2. Preliminaries

2.1. Graded spaces. Let G be a finite group. A vector space V is said to be *G-graded* if

$$V = \bigoplus_{a \in G} V_a$$

with each V_a a subspace of V. An element of some V_a is called *homogeneous* of *parity* a. We shall write $p(v) = a$ to mean that v has parity a.

We shall mostly be dealing with infinite-dimensional vector spaces. Call a subspace *large* if it has finite codimension.

Proposition 2.1. *Let V be a G-graded vector space and let W be a subspace of V.*

(i) *If W is finite-dimensional then W is contained in a finite-dimensional graded subspace.*
(ii) *If W is large then W contains a large graded subspace.*

Proof. (i) If $\{w_1, \dots, w_r\}$ is a basis for W then the set $\{(w_i)_a | a \in G, 1 \leq i \leq r\}$ spans the required subspace.
(ii) For each $a \in G$, $|V_a \colon W \cap V_a| = |V_a + W \colon W| \leq |V \colon W| < \infty$. Thus $\sum_a (W \cap V_a)$ is the required subspace. □

From now on G will be assumed abelian. Let $\exp G$ be the exponent of G. Suppose that K contains a primitive $\exp G$-th root of 1. Then V is G-graded if and only if it is stable under the action of the character group $\mathrm{Hom}(G, K^*)$ given for homogeneous v by $v^\chi = \chi(p(v))v$ (see e.g. [Berg, Section 1]). Let $\varphi(\chi)$ denote the linear automorphism by which χ acts.

Proposition 2.2. *Suppose that K contains a primitive $\exp G$-th root of 1. Then a subspace of V is graded if and only if it is stable under all $\varphi(\chi)$.* □

2.2. Lie colour algebras. A general reference for this subsection is [BMPZ]; for superalgebras, [S2] is the standard. Let G be a finite abelian group, written multiplicatively. Let $\varepsilon : G \times G \to K^*$ be an alternating bilinear form, i.e. a function satisfying:

$$\varepsilon(a,b) = \varepsilon(b,a)^{-1} \tag{1}$$

$$\varepsilon(a,bc) = \varepsilon(a,b)\varepsilon(a,c) \tag{2}$$

for all $a, b, c \in G$.

Some obvious consequences of the above definition which we shall frequently use are:

$$\varepsilon(ab,c) = \varepsilon(a,c)\varepsilon(b,c) \tag{3}$$

$$\varepsilon(a,b^n) = \varepsilon(a,b)^n \tag{4}$$

$$\varepsilon(a,a)^2 = 1 \tag{5}$$

$$\varepsilon(1,a) = \varepsilon(a,1) = 1 \tag{6}$$

for all $a, b, c \in G$ and all $n \in \mathbb{Z}$.

By (2) and (3) the map $a \mapsto \varepsilon(a, .)$ is a homomorphism from G into the character group $\mathrm{Hom}(G, K^*)$, and so (5) implies that $a \mapsto \varepsilon(a,a)$ is a homomorphism of G into $\{\pm 1\}$. Its kernel is denoted by G_+. Thus G_+ is a (normal) subgroup of G of index 1 or 2. Let $G_- = \{a \in G \,|\, \varepsilon(a,a) \neq 1\}$. Then either $G_- = \emptyset$ or else G_- is the other coset of G_+.

For some choices of G and K no nontrivial maps ε exist. By (4) and (6) each $\varepsilon(a,b)$ is an $\exp G$-th root of 1. Thus if $\mathit{char} K = p > 0$ and G is a p-group the only possibility for ε is $\varepsilon \equiv 1$. If G has odd order then $G = G_+$. If further G is cyclic and generated by a then from (4) and $\varepsilon(a,a) = 1$ we obtain $\varepsilon \equiv 1$. Nontrivial forms ε can exist on noncyclic groups of odd order.

For a G-graded vector space V and a form ε as above define $V_+ = \sum_{a \in G_+} V_a$ and $V_- = \sum_{a \in G_-} V_a$.

Definition. A G-graded (nonassociative) algebra L is called a *Lie colour algebra* if G has a map ε as above and L has a bilinear product $[\ ,\ ] : L \times L \to L$ such that

(ε-antisymmetry) $\quad [x,y] = -\varepsilon(a,b)[y,x]$

and

(ε-Jacobi identity)

$$\varepsilon(c,a)[x,[y,z]] + \varepsilon(b,c)[z,[x,y]] + \varepsilon(a,b)[y,[z,x]] = 0$$

for all $a, b, c \in G$ and all $x \in L_a, y \in L_b, z \in L_c$. Also, if $\mathrm{char}\, K = 2$, then we require $[x,x] = 0$ for all homogeneous $x \in L_-$, and if $\mathrm{char}\, K = 3$ then we require $[x,[x,x]] = 0$ for all homogeneous $x \in L_-$.

Note that the extra assumptions for characteristic 2 and 3 are provable from ε-antisymmetry and the ε-Jacobi identity in other characteristics.

Examples of Lie colour algebras are:

(i) Lie algebras: here $G = 1$, $\varepsilon \equiv 1$.
(ii) G-graded Lie algebras: $\varepsilon \equiv 1$.
(iii) Lie superalgebras: $G = \{-1, 1\}$, $\varepsilon(a,b) = (-1)^{(a-1)(b-1)/4}$.
(iv) G-graded Lie superalgebras: $\varepsilon(a,b) = 1$ unless both a and b belong to G_-, in which case $\varepsilon(a,b) = -1$.

We shall frequently write $\varepsilon(x,y)$ or even $\varepsilon(a,y)$ instead of $\varepsilon(a,b)$ if $x \in L_a$ and $y \in L_b$. The group elements a and b are not well-defined if x or y is zero. Thus we shall always choose $a = 1$ if $x = 0$, so that $\varepsilon(x,y) = 1$ whenever x or y is zero.

Now L_+ is a Lie colour algebra with grading group G_+ and defining form ε, and L_- is a graded L_+-module under the adjoint action. We shall sometimes call elements of L_+ *even* and those of L_- *odd*. Note that since we use multiplicative notation the component L_1 lies in L_+ and this differs from [BP3, BP4].

One possible cause of later difficulty is that distinct elements of G may induce the same characters of G, i.e. the map $a \mapsto \varepsilon(a,\cdot)$ may not be 1-1. However if $\bar{G}$ is the quotient of G modulo the kernel of this map then L can be naturally viewed as a Lie colour algebra with grading group $\bar{G}$ and associated form $\bar{\varepsilon}$ such that the map $\bar{a} \mapsto \bar{\varepsilon}(\bar{a},\cdot)$ is 1-1. Note for later that if $\operatorname{char} K = p > 0$ then the kernel above contains the Sylow p-subgroup of G, so $\bar{G}$ has the property that $p \nmid |\bar{G}|$ and Proposition 2.2 applies.

The *universal enveloping algebra* $U(L)$ of L is defined as usual by its universal property in the appropriate class of algebras. It is the G-graded associative K-algebra generated by L subject to the relations $xy - \varepsilon(x,y)yx = [x,y]$ for all homogeneous $x, y \in L$. The analogue of the Poincare-Birkhoff-Witt theorem holds for Lie colour algebras.

PBW Theorem. *Let $\mathcal{X}$ be a totally ordered homogeneous basis for L. Then a basis for $U(L)$ is given by the set of all monomials $x_1^{m_1} \cdots x_n^{m_n}$ such that $x_i \in \mathcal{X}$, $x_1 < \cdots < x_m$, m_i are nonnegative integers and $m_i \leq 1$ if $x_i \in L_-$.* □

If $\operatorname{char} K = p$ then we shall consider only *restricted* Lie colour algebras, i.e. Lie colour algebras with a linear map $x \mapsto x^{[p]}$ on L_+ which satisfies:

$$(kx)^{[p]} = k^p x^{[p]} \quad \text{for all } k \in K \text{ and homogeneous } x \in L_+,$$
$$[x^{[p]}, y] = (ad x)^p(y) \quad \text{for all homogeneous } x \in L_+ \text{ and } y \in L,$$

and

$$(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y) \quad \text{for all } x, y \in L_+ \text{ with } p(x) = p(y)$$

where $is_i(x,y)$ is the coefficient of λ^{i-1} in $(ad(\lambda x + y))^{p-1}(x)$. For each restricted Lie colour algebra L, the restricted enveloping algebra (usually denoted $u(L)$) exists and the analogue of Jacobson's theorem holds — a basis for $u(L)$ is given by the monomials in the PBW theorem which satisfy the further property that $m_i \leq p-1$ if $x_i \in L_+$.

The basis monomials described above are called *straightened*. The *support* supp α of an element α with respect to a given homogeneous basis for L is the set of distinct x_i occurring when α is (uniquely) written in terms of straightened monomials with nonzero coefficients.

If Y is a subset of L, s a nonnegative integer, then we let Y^s be the linear span of all monomials of length at most s all of whose factors lie in Y. Let $Y^\infty = \bigcup_{s\geq 0} Y^s$. If Y is a graded subalgebra of L then the set of straightened monomials of length at most s, with all factors coming from a fixed ordered homogeneous basis for Y, is a basis for Y^s, and $Y^\infty = U(Y)$.

Notation. From now on L denotes either a Lie colour algebra in characteristic zero or a restricted Lie colour algebra in prime characteristic, with grading group G and alternating bilinear map ε. We assume that distinct elements of G induce distinct characters via $a \mapsto \varepsilon(a,\cdot)$. By $U(L)$ we shall mean either the universal enveloping algebra of L in characteristic zero or the restricted enveloping algebra in prime characteristic p. The adjective 'graded' shall always mean 'G-graded' unless otherwise stated. Subspaces and subalgebras will not be assumed to be graded unless explicitly stated.

The following very useful proposition is an immediate consequence of the PBW (or Jacobson's) theorem.

Proposition 2.3. *Let H be a (restricted) graded subalgebra of L. If $\mathcal{Y}$ is an ordered homogeneous basis for a graded vector space complement of H in L, then the straightened $\mathcal{Y}$-monomials form a basis for $U(L)$ as a left and right $U(H)$-module. Explicitly, every $\alpha \in U(L)$ can be uniquely written $\alpha = \sum \alpha_\eta \eta$ and $\alpha = \sum \eta\alpha'_\eta$ where the η are straightened $\mathcal{Y}$-monomials and $\alpha_\eta, \alpha'_\eta \in U(H)$.* $\square$

Note that if α is homogeneous then so are all α_η and all α'_η. In the above situation we say that the elements of $U(L)$ are written *based on H*. An ordered homogeneous basis for a complement of H will be called a *complementary basis* for H in L. Complementary bases will always be homogeneous in this paper.

Now $U(L)$ has a degree function such that the degree of any straightened monomial is the sum of the exponents appearing in that monomial. The degree is independent of the homogeneous basis chosen. The presence of odd elements (and the p-mapping in characteristic p) means that we have only $\deg(\alpha\beta) \leq \deg\alpha + \deg\beta$ in general. However degree is additive in certain cases. If char $K = 0$ and $\alpha \in U(L_+)$ then $\deg(\alpha\beta) = \deg\alpha + \deg\beta$ for any $\beta \in U(L)$. Also it is easily seen that $\deg(\alpha\beta) = \deg\alpha + \deg\beta$ whenever α and β are elements of $U(L)$ with disjoint supports. As in [BP4] we shall write $\alpha \equiv \beta\ (n)$ if and only if $\deg(\alpha - \beta) \leq n$. Here of course $\deg 0 = -\infty$.

The next proposition will be used frequently in our reduction results of sections 3 and 4.

Proposition 2.4. *Let D be a (restricted) graded subalgebra of L and let α_i, β_i for $1 \leq i \leq r$ be elements of $U(L)$. Expand $\alpha_i = \sum_\eta \eta\alpha_{\eta i}$ and $\beta_i = \sum_\mu \beta_{i\mu}\mu$ with respect* *to arbitrary complementary bases for D in L. Fix an integer m and suppose that either*

$$\sum_i \alpha_{\eta i}\beta_i \equiv 0 \ (m - \deg \eta) \quad \textit{for all } \eta$$

or

$$\sum_i \alpha_i \beta_{i\mu} \equiv 0 \ (m - \deg \mu) \quad \textit{for all } \mu.$$

Then if we write $\alpha_i = \sum_\xi \xi \alpha'_{\xi i}$ and $\beta_i = \sum_\nu \beta'_{i\nu}\nu$ with respect to any complementary bases for D in L, we have

$$\sum_i \alpha_i \beta'_{i\nu} \equiv 0 \quad (m - \deg \nu) \qquad \textit{for all } \nu,$$

$$\sum_i \alpha'_{\xi i}\beta_i \equiv 0 \quad (m - \deg \xi) \qquad \textit{for all } \xi$$

$$\textit{and} \quad \sum_i \alpha'_{\xi i}\beta'_{i\nu} \equiv 0 \quad (m - \deg \xi - \deg \nu) \quad \textit{for all } \xi \textit{ and all } \nu.$$

Note that by subadditivity of degree, the conclusion implies that each of the two congruences in the hypothesis holds, for any choice of complementary bases.

Proof. We prove one case as the other is completely analogous. Suppose that

$$\sum \alpha_{\eta i}\beta_i \equiv 0 \quad (m - \deg \eta) \qquad \text{for all } \eta.$$

Substitute the expression for β_i using the monomials ν to obtain

$$\sum_\nu \Big(\sum_i \alpha_{\eta i}\beta'_{i\nu} \Big) \nu \equiv 0 \quad (m - \deg \eta) \qquad \text{for all } \eta.$$

This expression is written based on D, and so by disjointness of supports the degree of the ν summand is equal to $\deg \nu + \deg \sum_i \alpha_{\eta i}\beta'_{i\nu}$ and must be at most $m - \deg \eta$. Thus we have

$$\sum_i \alpha_{\eta i}\beta'_{i\nu} \equiv 0 \quad (m - \deg \eta - \deg \nu) \qquad \text{for all } \eta \text{ and } \nu.$$

Multiply on the left by η and sum over η to get

$$\sum_i \alpha_i \beta'_{i\nu} \equiv 0 \quad (m - \deg \nu) \qquad \text{for all } \nu.$$

Now expand each α_i with respect to the basis elements ξ. Repeating the argument above on the left instead of the right yields the result. □

The *left adjoint* map ad: $L \to End_K(L)$ is given by $\operatorname{ad} x(y) = [x,y]$ for all homogeneous $x, y \in L$. If ad is the zero map, that is if $[L,L] = 0$, then L is *abelian.* If $l \in L$ is homogeneous of parity a then $\operatorname{ad} l$ extends to a linear map $\partial = \partial(l)$ on $U(L)$ satisfying

$$(yz)^\partial = y^\partial z + \varepsilon(a,y) y z^\partial$$

for all homogeneous $y, z \in U(L)$, i.e. ∂ is a *derivation of parity* a. Thus we have

$$\alpha^\partial = \alpha^l = [l, \alpha] = l\alpha - \varepsilon(l,\alpha)\alpha l$$

for all homogeneous $l \in L$ and $\alpha \in U(L)$. This action then extends to an action of the algebra $U(L)$ on the K-module $U(L)$, also denoted ad, so that a straightened monomial $\rho = x_1^{m_1} \cdots x_r^{m_r}$ acts on $U(L)$ as $\partial(x_1)^{m_1} \circ \cdots \circ \partial(x_r)^{m_r}$. Thus we have

$$\deg \operatorname{ad} \rho(\alpha) = \deg \alpha^\rho \le \deg \alpha \qquad \text{for all } \alpha, \rho \in U(L).$$

For each graded subspace H of L, and each nonnegative integer r let $\partial^r(H)$ be the set of all multiple derivations $\partial(x_1) \cdots \partial(x_r)$ induced by homogeneous elements of H (equivalently, the set of all transformations induced by the action of homogeneous H-monomials with r factors). Let $\partial^\infty(H) = \bigcup_{r \ge 0} \partial^r(H)$.

The next proposition will be used repeatedly in §4 in order to switch derivations from one side to the other of an expression.

Proposition 2.5. (c.f. [BP4, Lemma 2.5]) *Let α_i, β_i be homogeneous elements of $U(L)$ for $1 \le i \le v$ and let $m \ge -1$. If H is a graded subspace of L then the following statements are equivalent:*

(i) $\sum_i \alpha_i^\rho \beta_i \equiv 0 \ (m)$ *for all* $\rho \in \partial^\infty(H)$
(ii) $\sum_i \alpha_i^\rho \beta_i^{\rho'} \varepsilon(\rho', \alpha_i^\rho) \equiv 0 \ (m)$ *for all* $\rho', \rho \in \partial^\infty(H)$
(iii) $\sum_i \alpha_i^\rho \beta_i^{\rho'} \varepsilon(\rho', \alpha_i) \equiv 0 \ (m)$ *for all* $\rho', \rho \in \partial^\infty(H)$
(iv) $\sum_i \alpha_i \beta_i^{\rho'} \varepsilon(\rho', \alpha_i) \equiv 0 \ (m)$ *for all* $\rho' \in \partial^\infty(H)$.

Proof. We prove only the implication (i) $\Leftrightarrow$ (iii) as the others are similar. Given (i), let $h \in H$ be homogeneous. Then applying $\partial(h)$ to the congruence in (i) yields

$$\sum_i \alpha_i^{h\rho} \beta_i + \sum_i \alpha_i^\rho \beta_i^h \varepsilon(h, \alpha_i)\varepsilon(h,\rho) \equiv 0 \quad (m) \quad \text{for all } \rho \in \partial^\infty(H).$$

Now each $h\rho \in \partial^\infty(H)$ so by hypothesis, since $\varepsilon(h, \alpha_i^\rho) = \varepsilon(h,\alpha_i)\varepsilon(h,\rho)$, the first sum above has degree at most m, and thus so does the second. The factor $\varepsilon(h,\rho)$ is nonzero and independent of i so we may cancel it. This yields (iii) in the special case $\rho' = \partial(h)$ and induction gives the general case. Clearly (iii) $\Rightarrow$ (i) if we choose $\rho' = 1$. □

2.3. The discoloration functor. Just as the relation between the adjoint action and the multiplication in $U(L)$ becomes more complicated in going from Lie algebras to Lie superalgebras, many of the arguments of [BP3, BP4] which deal with multiplication need nontrivial changes in the Lie colour algebra case. However many of the proofs of [BP3, BP4] do carry over to the Lie colour algebra case *almost* verbatim. They are usually those concerned with the adjoint action of L on $U(L)$. Rather than arguing that the proofs are "just the same" we shall at times employ the *discoloration functor* discussed below, and use the appropriate results of [BP3, BP4] rather than their proofs.

In [S1] it is shown that one can find a 2-cocycle $\sigma\colon G \times G \to K^*$ and a crossed product $S = (K, G, \sigma)$ with the following properties. If s_a is the basis element of S corresponding to $a \in G$, and we define $\hat{L}_a = L_a \otimes_K s_a$, then $\hat{L} = \sum_a \hat{L}_a$ becomes a G-graded Lie superalgebra under the multiplication defined below. For homogeneous elements x and y of parity a and b respectively, write $\hat{x} = x \otimes s_a$ and $\hat{y} = y \otimes s_b$. Then define

$$[\hat{x}, \hat{y}] = [x, y] \otimes s_a s_b = \sigma(a, b)[x, y]\hat{\ }$$

Here $\hat{L}_1 = (L_+)\hat{\ }$ and $\hat{L}_{-1} = (L_-)\hat{\ }$. The correspondence $L \to \hat{L}$ is the object map of a functor. If $f\colon L \to M$ is a morphism of Lie colour algebras then defining $\hat{f}\colon \hat{L} \to \widehat{M}$ by $\hat{f}(x) = \widehat{f(x)}$ for all homogeneous x gives us a functor from the Lie colour algebra category (with fixed G and ε) to the category of G-graded Lie superalgebras and maps. This functor is bijective, its inverse being given by the analogous construction with σ replaced by σ^{-1}.

This functor can be extended to the category $\mathbf{C}$ of G-graded nonassociative algebras and maps by defining $\hat{x}\hat{y} = (x \otimes s_a)(y \otimes s_b) = xy \otimes s_a s_b = \sigma(a, b)xy \otimes s_{ab}$ and treating morphisms as above. Then $\hat{A}$ is associative if and only if A is and the extended functor is again bijective with the analogous inverse construction on both $\mathbf{C}$ and the subcategory of graded associative algebras and maps. The functor respects graded subalgebras, graded ideals and homogeneous nilpotent elements in each of the relevant above categories.

If we temporarily denote by $U_G(\hat{L})$ the universal enveloping algebra of $\hat{L}$ considered as a G-graded Lie superalgebra (defined as usual by its universal mapping property in the appropriate category), it follows from above that $U_G(\hat{L}) = U(L)\hat{\ }$. But now $U_G(\hat{L}) = U(\hat{L})$, the universal envelope of $\hat{L}$ as an ungraded Lie superalgebra, as can easily be seen by noting that $U(\hat{L})$ is naturally G-graded and completes the defining diagram for $U_G(\hat{L})$. Thus we have

$$U(L)\hat{\ } = U(\hat{L}).$$

2.4. Δ-sets. This section collects many basic properties we shall need later. Since the proofs are all either routine and/or essentially contained in [BP3, BP4] they will mostly be omitted.

Let H be a graded subspace of L. Define

$$\mathbb{D}_L(H) = \{l \in L \mid \dim_K[H, l] < \infty\}.$$

We shall frequently without explicit mention use the fact that if H' is a large subspace of H then $\mathbb{D}_L(H) = \mathbb{D}_L(H')$. An element l of L belongs to $\mathbb{D}_L(H)$ if and only if $\operatorname{ad} l$ annihilates a large subspace of H. By Proposition 2.1 and the above observation we may assume this subspace to be graded, which shows that $\mathbb{D}_L(H)$ is graded. The following proposition collects some basic properties of $\mathbb{D}_L(H)$.

Proposition 2.6. *Let H be a graded subspace of L and let $D = \mathbb{D}_L(H)$. Then*

(i) $D = \{x \in L \mid \dim_K \operatorname{ad} H \cdot x < \infty\}$.
(ii) *D is a graded subalgebra of L.*
(iii) *If L is restricted then so is D.*
(iv) *If $H \lhd L$ then $D \lhd L$.*
(v) $\widehat{D} = \mathbb{D}_{\widehat{L}}(\widehat{H})$. □

The *delta ideal* of L is defined as $\Delta(L) = \mathbb{D}_L(L)$. It is a graded characteristic ideal of L. The structure of $\Delta(L)$ is nicer than that of an arbitrary infinite-dimensional algebra.

Proposition 2.7. *Let X be a finite subset of Δ. Then*

(i) $[X, H] = 0$ *for some large subspace H of L.*
(ii) *X generates a finite-dimensional ideal of Δ.* □

For each graded subalgebra H of L, define Δ_H to be the sum (equivalently, the union) of all finite-dimensional $\operatorname{ad} H$-stable graded ideals of Δ. Here are some structural results about Δ_H.

Proposition 2.8. *Let H be a graded subalgebra of L. Then*

(i) *Δ_H is the join of all finite-dimensional* $\operatorname{ad} H$*-stable subspaces of Δ. Thus*

$$\Delta_H = \{x \in \Delta \mid \dim_K \operatorname{ad} U(H) \cdot x < \infty\}.$$

(ii) *Δ_H is a graded ideal of L.*
(iii) *Δ_L is the join of all finite-dimensional graded ideals of L, and hence is a graded characteristic ideal of L.* □

In general Δ_H is properly smaller than Δ. Another important point is that in general if H' is large in H then $\Delta_{H'}$ does not equal Δ_H. Both phenomena are illustrated in the split extension $A \rtimes Kl$ of Lie algebras, where A is abelian with countably infinite basis $\{x_i : i \geq 1\}$ and $[l, x_i] = x_{i+1}$ for all i.

It will be essential to our arguments to be able to restrict our considerations to nonzero homogeneous elements of $H_+ \setminus \Delta$. For each graded 1-dimensional subalgebra $V = Kl$ of L, write Δ_l for Δ_V.

Proposition 2.9. *Let H be a graded subalgebra of L, and let $\mathcal{B}$ be a complementary basis for $H_+ \cap \Delta$ in H_+. Then*

(i) *If L is restricted then $\Delta_L = \Delta$.*
(ii) *If $H \subseteq \Delta$ then $\Delta_H = \Delta$.*
(iii) $\Delta_H = \Delta_{H_+}$.
(iv) $\Delta_H = \bigcap\{\Delta_l \mid l \in \mathcal{B}\}$.

(v) *If* $x \in \Delta$ *and* $l \in L$*, then* $x \in \Delta_l$ *if and only if* $\operatorname{ad} l$ *acts algebraically on* x*.*
(vi) $\widehat{\Delta_H} = \hat{\Delta}_{\hat{H}}$. □

Example. As an illustration of some of the above, let V be the vector space of smooth complex-valued functions on the real line and H the 3-dimensional Heisenberg Lie algebra with basis elements x, y, z, $[z, H] = 0$ and $[y, x] = z$. Then V is an H-module where $(x.f)(t) = tf(t)$, y acts as the ordinary derivative and z fixes each element of V. Let $L = V \rtimes H$. Then $\Delta = V$, $\Delta_z = \Delta$, Δ_y consists of all functions satisfying a linear ODE with constant coefficients and $\Delta_x = 0$.

3. Reduction to Δ

3.1. Derivation identities. As in [BP3] we use a special kind of complementary basis for the first reduction result. However it is not clear, if $|G| > 2$, that a basis with all the properties described in that paper need exist. Thus a more roundabout method is required. The next two lemmas provide a basis which is sufficient to prove the key Lemma 3.4.

Lemma 3.1. *Let* V *be a finite-dimensional graded vector space and let* $W_1, \dots, W_n$ *be graded subspaces with* $\bigcap_j W_j = 0$*. Then there is a homogeneous basis* $\mathcal{Y}$ *for* V *such that the following property holds*:

$$\text{for all } y \in \mathcal{Y}, \text{ there is } j \text{ with } W_j \subseteq K(\mathcal{Y} \setminus \{y\}).$$

Proof. First consider the ungraded ($G = 1$) case. In the dual V^* we obtain $\sum_j W_j^{\perp} = V^*$. Thus there is a basis $\mathcal{F}$ of V^* such that each $f \in \mathcal{F}$ belongs to some $W_j^{\perp}$. Let $\mathcal{Y}$ be the dual basis in V to $\mathcal{F}$. If $y \in \mathcal{Y}$ is dual to $f \in \mathcal{F}$ and $f \in W_j^{\perp}$ then since the span of $\mathcal{Y} \setminus \{y\}$ is precisely the annihilator $f^{\perp}$ of f, we obtain $W_j \subseteq K(\mathcal{Y} \setminus \{y\})$.

In the general case, for each $a \in G$ apply the previous paragraph to the space V_a and subspaces $(W_j)_a$. This yields homogeneous bases $\mathcal{Y}_a$ for V_a. Let $\mathcal{Y} = \dot{\bigcup} \mathcal{Y}_a$. Then if $y \in \mathcal{Y}$, there is $a \in G$ and j with $(W_j)_a \subseteq K(\mathcal{Y}_a \setminus \{y\})$. Each other component $(W_j)_b$ is spanned by $\mathcal{Y}_b$ and so W_j is the required subspace. □

In the next lemma we shall index the W's by superscripts in order to avoid confusion with the homogeneous components W_a.

Lemma 3.2. *Let* H *be a graded subspace of* L *and let* V *be a finite-dimensional graded subspace of* L *such that* $V \cap \mathbb{D}_L(H) = 0$*. Let* X *be a finite-dimensional subspace of* L *and let* N *be a positive integer. Then there exists a homogeneous basis* $\{y_1, \dots, y_q\}$ *of* V *such that for each* i *with* $1 \le i \le q$ *there exist homogeneous elements* $x_{i1}, \dots x_{iN} \in H$*, a subset* S_i *of* $\{1, \dots, q\}$ *containing* i*, and homogeneous elements* $t_{ijk} \in L$ *for all* $j \in S_i, 1 \le k \le N$*. These satisfy the conditions*

$$\text{The set } \{t_{ijk} | 1 \le i \le q\} \text{ is linearly independent modulo } X. \tag{7}$$

$$\begin{aligned} &[x_{ik}, y_j] = t_{ijk}, \text{ if } j \in S_i, \\ &[x_{ik}, y_j] \text{ is a linear combination of } t_{ij'k} \text{ with } j' \in S_i \setminus \{i\}, \text{ if } j \notin S_i. \end{aligned} \tag{8}$$

Proof. Define $W^a = \mathbb{D}_L(H_a) \cap V$, and note that each W^a is homogeneous and that $\bigcap_a W^a = 0$. By passing to a large subspace of H_a we may assume that H_a kills W^a for all a. Let $\mathcal{Y} = \{y_1, \ldots y_q\}$ be as in the previous lemma. We prove inductively that for each m with $1 \leq m \leq q$, there exist appropriate subsets S_i and homogeneous elements x_{ik} and t_{ijk} for $1 \leq i \leq m$, $1 \leq j \leq q$, $1 \leq k \leq N$ such that properties (7) and (8) above hold with q replaced by m. Taking $m = q$ gives the desired result.

Suppose $m = 1$. By the last lemma, there is $a \in G$ such that W^a is contained in V_1, the linear span of $\{y_2, \ldots, y_q\}$. Thus there is a subset $S_1^{\#}$ of $\{y_2, \ldots, y_q\}$ such that $\{y_j | j \in S_1^{\#}\}$ is a complementary basis for W^a in V_1. Let $S_1 = S_1^{\#} \cup \{1\}$ and note that $1 \in S_1$. Then $\{y_j | j \in S_1\}$ is a complementary basis for W^a in V. By [BP1, Proposition 2.2] there is a subspace A of H_a of dimension N such that if $x_{11}, \ldots, x_{1N}$ is a basis for A, the commutators $t_{1jk} = [x_{1k}, y_j]$ for $1 \leq k \leq N$ and $j \in S_1$ are linearly independent modulo X. Then all the x_{1k} and t_{1jk} are homogeneous, and (7) is satisfied for these elements. Note that if $j \notin S_1$ then since y_j is contained in $W^a + < \{y_{j'} | j' \in S_1^{\#}\} >$ and x_{1k} kills W^a we see that $[x_{1k}, y_j]$ is a linear combination of the $[x_{1k}, y_{j'}]$ which gives (8).

Suppose $m > 1$ and let T be the linear span of all t_{ijk} so far constructed, for $1 \leq i \leq m-1$. We repeat the above argument with y_m in place of y_1, except that now the t_{mjk} are required to be linearly independent modulo $X + T$. This produces the x_{mk} and t_{mjk} and ensures that property (7) then holds for all the t_{ijk} so far constructed. Also property (8) holds as above. □

For a given V, a basis $\mathcal{Y}$ as above, and ordered so the even elements come first, will be called *special*. Straightened monomials on a basis $\mathcal{Y}$ as constructed above will also be called special.

Lemma 3.3. (c.f. [BP3, Lemma 3.4]) *Under the hypotheses of the last lemma, let* $\xi = y_1^{m_1} y_2^{m_2} \cdots y_q^{m_q}$ *be a special straightened monomial in* $U(L)$ *of degree* $m = \sum_i m_i$. *Let* T *be the linear span of all* t_{ijk}.

Now let $\rho = x_{11} \cdots x_{1l_1} x_{21} \cdots x_{2l_2} \cdots x_{q1} \cdots x_{ql_q}$, *with* $l_i \geq 0$ *and* $\sum_i l_i = m$. *Then* $\xi^\rho = \xi_1 + \xi_2 + \xi_3$, *where* $\xi_1 \in \sum_{s=0}^{m-1} L^s \mathcal{Y}^1 L^{m-s-1}$, ξ_2 *is a* K*-linear combination of straightened monomials of degree* m *on* T *having at least one factor* t_{ijk} *with* $i \neq j$, *and*

$$\xi_3 = \theta l_1! l_2! \cdots l_q! \, t_{111} \cdots t_{11l_1} t_{221} \cdots t_{22l_2} \cdots t_{qq1} \cdots t_{qql_q} \quad \textit{for some } \theta \in K^*$$

if $l_i = m_i$ *for all* i, *but* $\xi_3 = 0$ *otherwise.*

Proof. The existence of such a decomposition as in [BP3, Lemma 3.4] is straightforward. The only nontrivial assertion is that about the form of ξ_3 when $l_i = m_i$ for all i. We apply the discoloration functor. The "discoloured" versions of the the x_{ik}, the y_j, and the t_{iik} are amenable to the calculation of ξ_3 in [BP3, Lemma 3.4]. Pulling back the formula obtained there via the discoloration functor yields the desired result, since each $[\hat{x}_{ik}, \hat{y}_j]$ is a nonzero scalar multiple of $[x_{ik}, y_j]\hat{}$. □

The following result will also be used in the next subsection on linear identities.

Lemma 3.4. *Let H be a graded subspace of L and let H' be a large graded subspace of H. Set $D = \mathbb{D}_L(H)$, fix a complementary basis $\mathcal{X}$ for D in L and suppose that all η's below are straightened monomials on $\mathcal{X}$. For each of these finitely many distinct η, let $\gamma_\eta \in U(L)$. Suppose that $\deg \eta \leq r$ for each η and that $\deg \sum_\eta \eta^\rho \gamma_\eta \leq s$ for all $\rho \in \partial^r(H')$. Then for all η we have $\deg \eta + \deg \gamma_\eta \leq s$.*

Proof. We may clearly assume that $H = H'$ and that not all γ_η are zero. Define the integer m by $m + 1 = \max_\eta(\deg \eta + \deg \gamma_\eta)$. Assume by way of contradiction that $m \geq s$. Then we have

$$\sum_\eta \eta^\rho \gamma_\eta \equiv 0 \quad (m) \quad \text{for all } \rho \in \partial^r(H).$$

We first reduce to considering monomials on a special basis $\mathcal{Y}$. By choice of m we may assume that for all η we have $\deg \eta + \deg \gamma_\eta = m + 1$. Let V be the linear span of all the elements which belong to the support of some η. Then V is finite-dimensional and graded and $V \cap D = 0$, so by Lemma 3.2 there is a homogeneous basis $\mathcal{Y}$ for V with the properties specified in that lemma.

For each t with $0 \leq t \leq r$ and each straightened monomial ξ of degree t on $\mathcal{X}$ there is an expression $\xi \equiv \sum_\mu c_{\xi\mu}\mu$ $(\deg \xi - 1)$ where the μ are straightened monomials on $\mathcal{Y}$ and the $c_{\xi\mu}$ belong to K. It is important to note that the $c_{\xi\mu}$ may be chosen so that for a given ξ, if $\deg \mu \neq \deg \xi$ then $c_{\xi\mu} = 0$. Now for each t with $0 \leq t \leq r$, the images of the straightened monomials ξ of degree t on $\mathcal{X}$ and the images of the straightened monomials μ of degree t on $\mathcal{Y}$ each form a basis for V^t/V^{t-1}. Thus the matrix $(c_{\xi\mu})$ represents an invertible map of $\bigoplus_{t=0}^r V^t/V^{t-1}$.

For each μ with $\deg \mu \leq r$ define $\delta_\mu = \sum_\eta c_{\eta\mu}\gamma_\eta$. Now by hypothesis and the last paragraph we have that

$$\sum_\mu \mu^\rho \delta_\mu = \sum_\mu \mu^\rho \Big(\sum_\eta c_{\eta\mu}\gamma_\eta\Big) \equiv \sum_\eta \Big(\sum_\mu c_{\eta\mu}\mu\Big)^\rho \gamma_\eta \equiv 0 \quad (m) \quad \text{for all } \rho \in \partial^r(H).$$

Also $\deg \mu \leq r$ for all μ, and so the hypotheses of this lemma apply to the "special" monomials μ. The proof of [BP3, Lemma 3.5] now adapts easily to prove that $\deg \mu + \deg \delta_\mu \leq m$ for all μ. Note that the crucial property of the t_{ijk} used in that proof of that lemma was that the "diagonal" elements $[x_{ik}, y_i]$ are all linearly independent modulo the linear span of all the "off-diagonal" $[x_{ik}, y_j]$, and that this property holds in our situation by Lemma 3.2.

Now if η is one of our original monomials and $\deg \eta = t$ then since the matrix $(c_{\xi\mu})$ is invertible we may express η in terms of the μ. However as above $\deg \mu = t$ for all μ occurring with nonzero coefficient and we have that $\deg \delta_\mu \leq m - t$ for such μ. But then γ_η is a linear combination of these δ_μ. This yields $\deg \gamma_\eta + \deg \eta \leq m$ and this contradiction proves the lemma. □

This last result can also be interpreted in the following way. Clearly the expression $\sum_\eta \eta^\rho \gamma_\eta$ can have degree no more than $\max_\eta(\deg \eta + \deg \gamma_\eta)$. The conclusion of the proposition is that equality is attained for some $\rho \in \partial^r(H')$, i.e. only r differentiations are required. An analogous statement is not true for Δ_L as will be seen in Section 4.

Here is the main theorem on reduction of derivation identities to $U(\Delta)$.

Theorem 3.5. *Let H be a graded subspace of L, let $\alpha_i, \beta_i \in U(L)$ for $1 \le i \le r$ and suppose that*

$$\sum_i \alpha_i^\rho \beta_i \equiv 0 \quad (m) \qquad \text{for all } \rho \in \partial^\infty(H')$$

where H' is a large graded subspace of H. Choose two arbitrary complementary bases for $D = \mathbb{D}_L(H)$ in L, and use them to write each $\alpha_i = \sum_\eta \eta\alpha_{\eta i}$ and each $\beta_i = \sum_\mu \beta_{i\mu}\mu$ based on D. Then for all $\rho \in \partial^\infty(H')$ and all η and μ,

$$\sum_i \alpha_{\eta i}^\rho \beta_i \equiv 0 \quad (m - \deg \eta),$$

$$\sum_i \alpha_i^\rho \beta_{i\mu} \equiv 0 \quad (m - \deg \mu),$$

and

$$\sum_i \alpha_{\eta i}^\rho \beta_{i\mu} \equiv 0 \quad (m - \deg \eta - \deg \mu).$$

Proof. Without loss of generality all α_i are homogeneous. We first prove that the conclusion holds with $\rho = 1$, as in [BP3, Theorem 3.6]. By passing to a large graded subspace of H' we may assume that $[H, \alpha_{\eta i}] = 0$ for all η and i. Then $(\eta\alpha_{\eta i})^\rho = \eta^\rho \alpha_{\eta i}$ for all η, ρ and i. Letting $\gamma_\eta = \sum_i \alpha_{\eta i}\beta_i$ we obtain from the given congruence

$$\sum_\eta \eta^\rho \gamma_\eta \equiv 0 \quad (m) \qquad \text{for all } \rho \in \partial^\infty(H).$$

By Lemma 3.4 with $r = \max_\eta \deg \eta$ and $s = m$ we obtain $\deg \gamma_\eta + \deg \eta \le m$ for all η, so that

$$\sum_i \alpha_{\eta i}\beta_i \equiv 0 \quad (m - \deg \eta) \qquad \text{for all } \eta.$$

Now Proposition 2.4 concludes the proof of the $\rho = 1$ case.

We now prove the general result. From our hypothesis and Proposition 2.5 we obtain

$$\sum_i \alpha_i^\rho \beta_i^{\rho'} \varepsilon(\rho', \alpha_i) \equiv 0 \quad (m) \qquad \text{for all } \rho', \rho \in \partial^\infty(H).$$

Thus if we fix ρ' and define $\tilde\beta_i = \beta_i^{\rho'}\varepsilon(\rho', \alpha_i)$, then $\tilde\beta_i$ satisfies the hypotheses on β_i. Thus by the $\rho = 1$ case above we have, on writing out $\tilde\beta_i$ in full,

$$\sum_i \alpha_{\eta i}\beta_i^{\rho'}\varepsilon(\rho', \alpha_{\eta i})\varepsilon(\rho', \eta) \equiv 0 \quad (m - \deg \eta) \qquad \text{for all } \rho' \in \partial^\infty(H) \text{ and for all } \eta$$

where we have used the fact that $\varepsilon(\rho', \alpha_i) = \varepsilon(\rho', \alpha_{\eta i})\varepsilon(\rho', \eta)$. Cancelling the nonzero common factor $\varepsilon(\rho', \eta)$ and applying Proposition 2.5 again yields $\sum_i \alpha_{\eta i}^\rho \beta_i \equiv 0$ $(m - \deg \eta)$ for all η and all $\rho \in \partial^\infty(H)$, proving the first conclusion above. A similar procedure (define $\tilde{\alpha}_i = \alpha_i^\rho$) proves the second. The third conclusion follows from the second by the same argument (with $\alpha_{\eta i}$ in place of α_i) used to prove the first. □

From the $\rho = 1$ case of the theorem we obtain the following corollary.

Corollary. *Suppose $\sum_i \alpha_i^\rho \beta_i = 0$ for all $\rho \in \partial^\infty(H)$. If some α_i (respectively some β_i) is nonzero then the distinct β_i (resp. distinct α_i) are left (resp. right) linearly dependent over $U(D)$.* □

We note that Lemma 3.4 is a special case of this theorem. Also we see that the conclusion of Theorem 3.5 implies the congruence in the hypothesis of the theorem. For the congruence

$$\sum_i \alpha_{\eta i}^\rho \beta_i \equiv 0 \quad (m - \deg \eta) \qquad \text{for all } \eta \text{ and all } \rho \in \partial^\infty(H)$$

implies that $\sum_i \alpha_i^\rho \beta_i = \sum_{\eta,i} \eta^\rho \alpha_{\eta i} \beta_i + \sum_{\eta,i} \eta \varepsilon(\rho, \eta) \alpha_{\eta i}^\rho \beta_i$ is the sum of two subsums of degree at most m. Our subsequent reduction theorems will share this property of conclusion implying hypothesis. They will also have corollaries similar to the one above. We shall not explicitly point these out.

3.2. Linear identities. Linear identities are of course more interesting in a ring-theoretic context than derivation identities. If L is just a Lie algebra then a linear identity immediately yields a corresponding derivation identity. However the relation between the adjoint action and the multiplication in $U(L)$ is more complicated in the general case. The following somewhat unpleasant lemma follows the idea of [BP4, Theorem 5.1] and uses Lemma 3.4 rather than Theorem 3.5. It is the major step in the proof of, and is implied by, the theorem following it. Note that if all the α_i below are homogeneous of the same parity or if H_1 is large in H, then the argument of [BP3, Theorem 4.1] carries over almost verbatim.

Lemma 3.6. *Let H be a graded subspace of L and let $\alpha_i, \beta_i \in U(L)$ for $1 \le i \le r$ with each α_i homogeneous. Suppose that*

$$\sum_i \alpha_i x \beta_i = 0 \qquad \textit{for all } x \in (H')^\infty$$

for some large subspace H' of H. If each α_i is written $\alpha_i = \sum_\eta \eta \alpha_{\eta i}$ based on $D = \mathbb{D}_L(H)$, then

$$\sum_i \alpha_{\eta i} \beta_i = 0 \qquad \textit{for all } \eta.$$

Proof. If H is finite-dimensional then $D = L$ and the result follows by taking $x = 1$. Thus we shall suppose that H is infinite-dimensional. Let H_{inf} be the sum of all infinite-dimensional components of H. Then H_{inf} is large in H. By passing to a large graded subspace of $H' \cap H_{\text{inf}}$ we can assume that $H = H' = H_{\text{inf}}$ and $[H, \alpha_{\eta i}] = 0$ for all η and i. By taking homogeneous components we can further assume that each β_i is homogeneous and that there is $c \in G$ such that the parities satisfy $p(\alpha_i)p(\beta_i) = p(\eta)p(\alpha_{\eta i})p(\beta_i) = c$ for all η and i.

Since each $[\alpha_{\eta i}, x] = 0$, the given identity is equivalent to

$$\sum_{\eta} \eta x \sum_{a \in G} \varepsilon(a, x)\gamma_{\eta a} = 0 \qquad \text{for all homogeneous } x \in H^{\infty}$$

where $\gamma_{\eta a}$ is the sum of all $\alpha_{\eta i}\beta_i$ such that $p(\alpha_{\eta i}) = a$. Note that each $\gamma_{\eta a}$ is homogeneous and that in fact $p(\gamma_{\eta a}) = c(p(\eta))^{-1}$ depends only on η and not on a.

Our desired conclusion is that $\sum_a \gamma_{\eta a} = 0$ for all η. We will in fact show that each $\gamma_{\eta a} = 0$.

Suppose on the contrary that $\gamma_{\xi b}$ is nonzero, and define

$$N = \max_{\eta, a}\{\deg \gamma_{\eta a}, \deg \eta\}.$$

Let M denote the set of monomials η we are considering. Define $S \subseteq M \times G$ by

$$S = \{(\eta, a) \,|\, \varepsilon(a, h)\varepsilon(h, \gamma_{\eta a}) = \varepsilon(b, h)\varepsilon(h, \gamma_{\xi b}) \quad \text{for all homogeneous } h \in H.\}$$

Note that $(\xi, b) \in S$. Let T be the complement $M \times G \setminus S$ and let $n = \max\{1, |T|\}$.

Claim. For all integers $j \geq 0$ there are elements $\gamma_{\eta a}^{(j)}$ belonging to $U(L)$ such that

(9) $$\sum_{\eta, a} \eta x \varepsilon(a, x)\gamma_{\eta a}^{(j)} = 0 \quad \text{for all homogeneous } x \in H^{\infty}$$

(10) $$\deg \gamma_{\eta a}^{(j)} \geq jn \qquad \text{if } (\eta, a) \in S$$

(11) $$\deg \gamma_{\eta a}^{(j)} \leq N + j(n-1) \qquad \text{if } (\eta, a) \in T$$

(12) $$\varepsilon(a, \ell)\varepsilon(\ell, \gamma_{\eta a}^{(j)}) = \varepsilon(b, \ell)\varepsilon(\ell, \gamma_{\xi b}^{(j)}) \qquad \text{for all homogeneous } \ell \in H$$
$$\text{if and only if } (\eta, a) \in S.$$

(13) $$p(\eta)p(\gamma_{\eta a}^{(j)}) = p(\xi)p(\gamma_{\xi b}^{(j)})$$

If $j = 0$ the five conditions hold with $\gamma_{\eta a}^{(j)} = \gamma_{\eta a}$. Suppose the claim is true for some $j \geq 0$. We consider the cases (i) $T = \emptyset$ and (ii) $T \neq \emptyset$ separately.

Suppose $T = \emptyset$. By our assumptions above every component of H is infinite-dimensional. Thus we can choose a nonzero homogeneous $h \in H$ so that h lies outside the finite-dimensional subspace of L spanned by the supports of all $\gamma_{\eta a}$ with $(\eta, a) \in S$. Let θ be any element of K not equal to $\varepsilon(b, h)\varepsilon(h, \gamma_{\xi b})$. We note that for this case $\theta = 0$ suffices; we shall use the hypothesis on θ more strongly in case (ii) below. We claim that the elements

$$\gamma_{\eta a}^{(j+1)} = \varepsilon(a, h)h\gamma_{\eta a}^{(j)} - \theta\gamma_{\eta a}^{(j)}h$$

meet our requirements. To see this, replace x with xh in (9) and multiply (9) on the right by $-\theta h$. Adding the two resulting equations yields

$$\sum_{\eta, a} \eta x \varepsilon(a, x)\{\varepsilon(a, h)h\gamma_{\eta a}^{(j)} - \theta\gamma_{\eta a}^{(j)}h\} = 0$$

so that (9) holds with $\gamma_{\eta a}^{(j)}$ replaced by $\gamma_{\eta a}^{(j+1)}$. Also since $p(\gamma_{\eta a}^{(j+1)}) = p(\gamma_{\eta a}^{(j)})p(h)$, (12) and (13) remain true under this replacement. By (12), for each $(\eta, a) \in S$ we have

$$\varepsilon(h,a)\theta \neq \varepsilon(h, \gamma_{\eta a}^{(j)}).$$

Now $\deg(\gamma_{\eta a}^{(j)})^h \leq \deg \gamma_{\eta a}^{(j)}$, and since h is not in the support of $\gamma_{\eta a}^{(j)}$ we have $\deg \gamma_{\eta a}^{(j)} h = 1 + \deg \gamma_{\eta a}^{(j)}$. Thus

$$\gamma_{\eta a}^{(j+1)} = \varepsilon(a,h)\{(\gamma_{\eta a}^{(j)})^h + (\varepsilon(h,\gamma_{\eta a}) - \varepsilon(h,a)\theta)\gamma_{\eta a}^{(j)} h\}$$

has degree 1 more than that of $\gamma_{\eta a}^{(j)}$. Since $n = 1$ here, (10) and (11) are clearly satisfied, (11) vacuously.

Now suppose that T is nonempty. List the elements of T in some order and let (η_1, a_1) be the first. Choose $h \in H$ so that h is homogeneous and $\varepsilon(a_1,h)\varepsilon(h,\gamma_{\eta_1 a_1}^{(j)}) \neq \varepsilon(b,h)\varepsilon(h,\gamma_{\xi b}^{(j)})$, which is possible by definition of S and T and by (12). Since every component of H is infinite-dimensional, we can further restrict h so that h misses the supports of all $\gamma_{\eta a}^{(j)}$ with $(\eta, a) \in S$. Define $\theta = \varepsilon(a_1,h)\varepsilon(h,\gamma_{\eta_1 a_1}^{(j)})$. Proceed as in case (i), letting

$$\gamma_{\eta a}^{'} = \varepsilon(a,h)h\gamma_{\eta a}^{(j)} - \theta\gamma_{\eta a}^{(j)} h.$$

Note that (9) is satisfied with $\gamma_{\eta a}^{(j)}$ replaced by $\gamma_{\eta a}^{'}$, and since $p(\gamma_{\eta a}^{'}) = p(\gamma_{\eta a}^{(j)})p(h)$, (12) and (13) are also satisfied. Again if $(\eta, a) \in S$ we have increased the degree of $\gamma_{\eta a}^{(j)}$ by 1. If $(\eta, a) = (\eta_1, a_1)$ then by choice of θ we have $\gamma_{\eta a}^{'} = \varepsilon(a,h)\gamma_{\eta a}^h$ so the degree has not increased. For the other pairs (η, a) in T the degree has increased at most by 1. Now repeat this procedure for each other element of T, running through them all exactly once. After n iterations let $\gamma_{\eta a}^{(j+1)}$ be the $\gamma_{\eta a}^{'}$ obtained in the last iteration. Since the degree has always increased for pairs $(\eta, a) \in S$, (10) holds. Since for each element $(\eta, a) \in T$ the degree of the corresponding $\gamma_{\eta a}^{'}$ increased by at most 1 at each step and at one step its degree did not increase, we obtain (11). This completes the inductive step and establishes the claim.

Now take any $j > 3N$, and write $\delta_{\eta a}$ for $\gamma_{\eta a}^{(j)}$. Then by (9) and (11) we have

$$\sum_{(\eta,a)\in S} \eta x \varepsilon(a,x)\delta_{\eta a} \equiv 0 \quad (2N + j(n-1) + \deg x)$$

for all $x \in H^\infty$. Recall that $\deg \eta \leq N$ for all η.

Every element of $\partial^\infty(H)$ is a linear combination of linear transformations of the form $\rho = \operatorname{ad} u$ where $u = h_1 \cdots h_q \in H^\infty$ is a homogeneous monomial. Call such a ρ a *monomial*. Define the length of ρ by letting $\operatorname{len}\rho$ be the minimum value of q taken over all such u. We now claim by induction on $\operatorname{len}\rho$ that if $\rho \in \partial^\infty(H)$, then

(14)
$$\sum_{(\eta,a)\in S} \eta^\rho x \varepsilon(a,x)\delta_{\eta a} \equiv 0 \ (2N + j(n-1) + \deg x + \operatorname{len}\rho) \qquad \text{for all } x \in H^\infty.$$

We already know that this holds if $\operatorname{len}\rho = 0$. Let $\rho' = \partial(h)\rho$ belong to $\partial^\infty(H)$, where $h \in H$ is homogeneous. Replacing x by $-\varepsilon(h,b)\varepsilon(h,\xi^\rho)hx$ in (14) and multiplying (14) on the left by h (each of which increases the degree by at most 1) yields, on using the inductive hypothesis and adding the two resulting congruences, the new congruence

$$\sum_{(\eta,a)\in S} [h\eta^\rho - \varepsilon(h,\xi^\rho)\varepsilon(a,h)\varepsilon(h,b)\eta^\rho h]x\varepsilon(a,x)\delta_{\eta a} \equiv 0 \quad (2N+j(n-1)+\deg x+1+\operatorname{len}\rho).$$

If $(\eta, a) \in S$ then it is clear from (13) that $p(\eta^\rho)p(\delta_{\eta a}) = p(\xi^\rho)p(\delta_{\xi b})$. Combining this with (12) shows that

$$\begin{aligned}\varepsilon(h,\xi^\rho)\varepsilon(a,h)\varepsilon(h,b) &= \varepsilon(h,\xi^\rho)\varepsilon(h,\delta_{\xi b})\varepsilon(\delta_{\eta a},h)\\ &= \varepsilon(h,\eta^\rho)\end{aligned}$$

Thus each term in square brackets in the left side of the last congruence equals $\eta^{\rho'}$, and this establishes our claim, since $\operatorname{len}\rho' \leq 1 + \operatorname{len}\rho$.

In particular (14) holds with $x = 1$ for all $\rho \in \partial^N(H)$. We obtain

$$\sum_{(\eta,a)\in S} \eta^\rho\delta_{\eta a} \equiv 0 \quad (3N + j(n-1)) \qquad \text{for all } \rho \in \partial^N(H).$$

We now apply Lemma 3.4 with $r = N$, to obtain $\deg\xi + \deg\delta_{\xi b} \leq 3N + j(n-1)$. But since $j > 3N$, $\deg\delta_{\xi b} \geq jn > 3N + j(n-1)$, yielding a contradiction to our original assumption that $\gamma_{\xi b} \neq 0$. Thus $\gamma_{\eta a} = 0$ for all pairs (η, a), proving the result. □

Here is the main theorem on the reduction of linear identities to $U(\Delta)$.

Theorem 3.7. *Let H be a graded subspace of L and let $\alpha_i, \beta_i \in U(L)$ for $1 \leq i \leq r$. Suppose that*

$$\sum_i \alpha_i x\beta_i = 0 \quad \textit{for all } x \in (H')^\infty$$

for some large subspace H' of H. Choose two arbitrary complementary bases for $D = \mathbb{D}_L(H)$ in L and use them to write $\alpha_i = \sum_\eta \eta\alpha_{\eta i}$ and $\beta_i = \sum_\mu \beta_{i\mu}\mu$ based on D. Then for all $z \in (H')^\infty$ and all η and μ,

$$\sum_i \alpha_i z\beta_{i\mu} = 0, \qquad \sum_i \alpha_{\eta i} z\beta_i = 0 \qquad \textit{and} \qquad \sum_i \alpha_{\eta i} z\beta_{i\mu} = 0.$$

Proof. By passing to a large graded subspace of H', we may assume that $H = H'$ is a graded subspace. Write each α_i as a sum of homogeneous components, write each component based on D and apply the previous result to each component. This yields $\sum_i \alpha_{\eta i}\beta_i = 0$ and now Proposition 2.4 gives us the result for $z = 1$.

Now let $y \in H^\infty$. Then for all $x \in H^\infty$, we have $xy \in H^\infty$ and so the identity $\sum_i \alpha_i x(y\beta_i) = 0$ is satisfied. By the $z = 1$ case above we have $\sum_i \alpha_{\eta i} y\beta_i = 0$. This is true for all $y \in H^\infty$, so for each $z \in H^\infty$, replacing y by zy and applying the above procedure on the other side yields

$$\sum_i \alpha_{\eta i} z\beta_{i\mu} = 0 \qquad \text{for all } z \in H^\infty \text{ and all } \eta \text{ and } \mu$$

as required. □

4. Reduction to Δ_L

In view of Proposition 2.9(i) the results of this section yield nothing new in positive characteristic and so we shall assume that char $K = 0$ throughout.

4.1. Derivation identities. Throughout most of this subsection the following hypothesis will be in force.

Hypothesis. *Suppose that H is a graded subalgebra of L, let $\overline{H}$ be a graded complement for $H_+ \cap \Delta$ in H_+ and let $\alpha_j, \beta_i \in U(\Delta)$ and $\gamma_{ji} \in U(\Delta_H)$ be homogeneous elements for $1 \leq i \leq u$ and $1 \leq j \leq v$. For a fixed integer $n \geq -1$, assume that*

$$\sum_{j,i} (\alpha_j \gamma_{ji})^\rho \beta_i \equiv 0 \quad (n) \quad \text{for all } \rho \in \partial^\infty(\overline{H}) \text{ and that}$$

$$\deg \alpha_j + \max\{\deg \sum_i \gamma_{ji}^\rho \beta_i | \rho \in \partial^\infty(\overline{H})\} \leq n + 1 \qquad \text{for all } j.$$

Furthermore, suppose that for all i and j the product $p(\alpha_j)p(\gamma_{ji})p(\beta_i)$ is equal to a fixed group element c.

The reason that this hypothesis looks different from that in [BP4] is because we use the left adjoint but act on the left rather than on the right factor. This is apparently forced by the key Proposition 2.5 which seems to fail if we consider identities of the form $\sum_i \alpha_i \beta_i^\rho = 0$ as in [BP4].

The proofs of the following lemmas are almost exactly the same as those in [BP4]. The only difficulties lie in the repeated applications of Proposition 2.5. We shall only go into detail in the cases where these applications are required. As in [BP4] we call a K-multiple of a straightened monomial a *K-straightened monomial.*

Lemma 4.1. (c.f. [BP4, Lemma 3.2]) *Let $l \in \overline{H}$ be homogeneous and let $\mathcal{D} = \mathcal{W} \dot\cup \mathcal{X} \dot\cup \mathcal{Y} \dot\cup \mathcal{Z}$ be a homogeneous basis for Δ with $\mathcal{W} < \mathcal{X} < \mathcal{Y} < \mathcal{Z}$. Assume that $K\mathcal{Y}$ is $\operatorname{ad} l$-stable and $\mathcal{X} = \{x_{-1}, x_{-2}, \ldots\}$ with $x_t > x_{t-1} = [l, x_t]$ for all $t \geq 1$. Suppose that each α_j is a K-straightened monomial in $\mathcal{X} \dot\cup \mathcal{Y}$ and that for a fixed integer k, each α_j is given by*

$$\alpha_j = x_{-k}^{a_{jk}} \cdots x_{-1}^{a_{j1}} \bar\alpha_j$$

with $\bar\alpha_j$ a K-straightened monomial in $\mathcal{Y}$. For convenience, write $\alpha_j = x_{-k}^{a_{jk}} \alpha'_j$. If $a = \max\{a_{jk} | 1 \leq j \leq v\}$ then

$$\sum{}' \alpha'_j \left(\sum_i \gamma_{ji} \beta_i \right) \equiv 0 \quad (n - a)$$

where $\sum'$ denotes the sum over all j with $a_{jk} = a$. In particular

$$\sum{}' \alpha_j \left(\sum_i \gamma_{ji} \beta_i \right) \equiv 0 \quad (n).$$

Proof. The proof of [BP4, Lemma 3.2] generalizes in a straightforward manner. However we shall outline the argument.

We may clearly suppose that $a \geq 1$. It follows from the definitions that the set

$$\left\{\sum_i \gamma_{ji}^{\rho}\beta_i | \rho \in \partial^{\infty}(H), 1 \leq j \leq v\right\}$$

is contained in a finite-dimensional subspace of $U(\Delta)$. Thus we may choose an integer t such that $k+t > s$ where $\{x_{-s}, \dots, x_{-1}\}$ are all the elements of $\mathcal{X}$ required when we express the elements of the displayed set above in terms of $\mathcal{D}$. Letting $q = at$ and $\rho = \partial(l)^q$ we proceed for each j to consider the straightened $\mathcal{D}$-monomials of degree $n+1$ which occur in $\sum_i (\alpha_j\gamma_{ji})^{\rho}\beta_i$ and contain a factor of $x_{-(k+t)}^a$. Note that since $h \in H_+$ the x_i are either all even or all odd and thus have the same restriction on their exponents. Hence if x_{-k}^a occurs in a straightened monomial, so too can $x_{-(k+t)}^a$ occur in one.

Now

$$\sum_i (\alpha_j\gamma_{ji})^{\rho}\beta_i = \sum_{\rho_1,\rho_2} \theta(\rho_1,\rho_2,\alpha_j)\alpha_j^{\rho_1}\sum_i \gamma_{ji}^{\rho_2}\beta_i$$

where $\theta(\rho_1,\rho_2,\alpha_j)$ is a product of values of ε (and is therefore nonzero) and ρ_1,ρ_2 partition ρ. It is clear from the Hypothesis that the degree of the displayed sum is no more than $n+1$. Thus the supports of any degree $n+1$ monomials occurring are not augmented by the straightening process. Now it follows (as in [BP4]) from our choice of q and the fact that $K\mathcal{Y}$ is $\operatorname{ad} l$-stable that the monomials we require can arise only from the sum $\sum_i \alpha_j^{\rho}\gamma_{ji}\beta_i$ and then only when $a_{jk} = a$ and $\deg \sum_i \alpha_j'\gamma_{ji}\beta_i = n+1-a$. Applying the discoloration functor to the "left" version of the relevant formula of [BP4, Lemma 3.2] shows that we have

$$\sum_i (\alpha_j\gamma_{ji})^{\rho}\beta_i \equiv (\theta m)x_{-(k+t)}^a x_{-(k-1)}^{a_{k-1}}\cdots x_{-1}^{a_1}\bar{\alpha}_j \sum_i \gamma_{ji}\beta_i$$
$$+ \text{ terms not divisible by } x_{-(k+t)}^a \ (n)$$

where $\theta \in K^*$ and $m = q!/(t!)^a$. The proof concludes from this key formula as in [BP4, Lemma 3.2] with the obvious minor notational changes. □

Lemma 4.2. (c.f. [BP4, Lemma 3.3]) *Let $l \in \overline{H}$ be homogeneous and let $\mathcal{D} = \mathcal{A} \dot\cup \mathcal{B} \dot\cup \mathcal{C}$ be a homogeneous basis for Δ with $\mathcal{A} < \mathcal{B} < \mathcal{C}$. Suppose $K\mathcal{B}$ is an* $\operatorname{ad} l$*-stable subspace of Δ_l and that $\mathcal{A} = \mathcal{A}_{-m} \dot\cup \cdots \dot\cup \mathcal{A}_{-1}$ is a finite union of* $\operatorname{ad} l$*-stable subsets. Assume that $\mathcal{A}_{-1} > \cdots > \mathcal{A}_{-m}$ and that* $\operatorname{ad} l$ *acts on each $\mathcal{A}_s$ as a shift, decreasing by 1 the order of each element. Further suppose that each α_j is a K-straightened monomial in $\mathcal{A} \cup \mathcal{B}$. Now choose a complementary basis for Δ_l in Δ and write $\beta_i = \sum_i \beta_{i\mu}\mu$ based on Δ_l. Then for all μ,*

$$\sum_{i,j} (\alpha_j\gamma_{ji})\beta_{i\mu} \equiv 0 \quad (n - \deg \mu).$$

Proof. We proceed as in [BP4, Lemma 3.3]. Just as in that lemma, if all α_j are K-straightened monomials on $\mathcal{B}$ the result follows easily. Thus we assume that some β_j requires an element of $\mathcal{A}$ in its representation, and define $\mathcal{W}, \mathcal{X}, \mathcal{X}', \mathcal{Y}$ and$\mathcal{Z}$ in the analogous way to [BP4, Lemma 3.3]. Then for each j,

$$\alpha_j = x_{-k}^{a_{jk}} \cdots x_{-1}^{a_{j1}} \bar{\alpha}_j$$

where $\bar{\alpha}_j$ is a K-straightened monomial on $\mathcal{Y}$ and some $a_{jk} \neq 0$. Define a by $a = \max_j a_{jk} > 0$.

By the Hypothesis and Proposition 2.5 we have

$$\sum_{j,i} (\alpha_j \gamma_{ji})^\rho \beta_i^{\rho'} \varepsilon(\rho', \alpha_j \gamma_{ji}) \equiv 0 \quad (n) \qquad \text{for all } \rho', \rho \in \partial^\infty(\overline{H}).$$

Note that it follows from the Hypothesis that the factors $\varepsilon(\rho', \alpha_j \gamma_{ji})$ depend only on ρ' and i. It follows from Proposition 2.5 and the Hypothesis that if we fix ρ' and define $\tilde{\beta}_i = \beta_i^{\rho'} \varepsilon(\rho', \alpha_j \gamma_{ji})$ that we have

$$\deg \alpha_j + \deg \sum_i \gamma_{ji}^\rho \tilde{\beta}_i \leq n + 1 \qquad \text{for all } j \text{ and all } \rho \in \partial^\infty(\overline{H}).$$

Apply the last lemma with $\tilde{\beta}_i$ in place of β_i and write out $\tilde{\beta}_i$ in full to obtain

$$\sum{}' \alpha_j' \Big(\sum_i \gamma_{ji} \beta_i^{\rho'} \varepsilon(\rho', x_{-k}^a) \varepsilon(\rho', \alpha_j' \gamma_{ji}) \Big) \equiv 0 \quad (n - a).$$

This holds for all $\rho' \in \partial^\infty(\overline{H})$ so we obtain, using Proposition 2.5 and cancelling the common factor $\varepsilon(\rho', x_{-k}^a)$,

$$\sum{}' \sum_i (\alpha_j' \gamma_{ji})^\rho \beta_i \equiv 0 \quad (n - a) \qquad \text{for all } \rho \in \partial^\infty(\overline{H}).$$

The proof concludes as in [BP4, Lemma 3.3], with the obvious notational changes. □

Lemma 4.3. (c.f. [BP4, Lemma 3.4]) *Let $l \in \overline{H}$ be homogeneous, choose a complementary basis for Δ_l in Δ and write each $\beta_i = \sum_\mu \beta_{i\mu} \mu$ based on Δ_l. Then for all μ,*

$$\sum_{i,j} (\alpha_j \gamma_{ji})^\rho \beta_{i\mu} \equiv 0 \quad (n - \deg \mu) \quad \textit{for all } \rho \in \partial^\infty(\overline{H}).$$

Proof. As in [BP4, Lemma 3.4], first prove the result for the special case $\rho = 1$. The first 3 paragraphs of that proof carry over with only the required notational changes. In particular we have $\alpha_j \equiv \sum_k \alpha_{kj}$ $(\deg \alpha_j - 1)$, where the α_{kj} are K-straightened monomials. Note that for all k we have $p(\alpha_j) = p(\alpha_{kj})$. Now the Hypothesis and Proposition 2.5 imply that

$$\sum_j \alpha_j \gamma_{ji} \beta_i^\rho \varepsilon(\rho, \alpha_j \gamma_{ji}) \equiv 0 \quad (n) \quad \text{for all } \rho \in \partial^\infty(\overline{H}).$$

Also for each j we have

$$\deg \alpha_j + \deg \sum_i \gamma_i \beta_i^\rho \varepsilon(\rho, \alpha_j \gamma_j i) \leq n + 1 \quad \text{for all } \rho \in \partial^\infty(\overline{H}).$$

To verify this last formula, cancel the common factor $\varepsilon(\rho, \alpha_j)$. Now the remainder of the proof of this case follows as in [BP4, Lemma 3.4].

For the general case, fix $\tau \in \partial^\infty(\overline{H})$. Note that as in Lemma 4.1 we have

$$(\alpha_j \gamma_{ji})^\tau = \sum_k \theta(\tau_k, \tau_k', \alpha_j) \alpha_j^{\tau_k} \gamma_{ji}^{\tau_k'}$$

where τ_k, τ_k' partition τ. Define $\gamma_{kji} = \theta(\tau_k, \tau_k', \alpha_j)\gamma_{ji}^{\tau_k'}$ and $\alpha_{kj} = \alpha_j^{\tau_k}$. We observe that the Hypothesis is satisfied with α_{kj} and γ_{kji} in place of α_j and γ_{ji} and the double index kj in place of the index j. To check the last part of the Hypothesis, recall that τ is fixed and thus we have

$$p(\alpha_{kj})p(\gamma_{kji})p(\beta_i) = p(\alpha_j)p(\tau_k)p(\tau_k')p(\gamma_{ji})p(\beta_i) = c \cdot p(\tau),$$

a constant. The proof now concludes as in [BP4, Lemma 3.4]. □

Lemma 4.4. (c.f. [BP4, Lemma 3.5]) *Let I be a finite-dimensional graded ideal of Δ. Then there exists a complementary basis $\mathcal{X}$ for $I \cap \Delta_H$ in I with the following property. Suppose $\beta_i \in U(I)$ for all i and using $\mathcal{X}$ and working in $U(I)$, write each $\beta_i = \sum_\mu \beta_{i\mu}\mu$ based on $I \cap \Delta_H$. Then for all monomials μ,*

$$\sum_{i,j} (\alpha_j \gamma_{ji})^\rho \beta_{i\mu} \equiv 0 \quad (n - \deg \mu) \quad \textit{for all } \rho \in \partial^\infty(\overline{H}).$$

Proof. The argument of [BP4, Lemma 3.5] goes over with only the obvious notational changes. □

We now drop the Hypothesis unless explicitly invoked. The next proposition is the penultimate step in our reduction of derivation identities.

Proposition 4.5. (c.f. [BP4, Prop 4.1]) *Let H be a graded subalgebra of L, let $\alpha_i, \beta_i \in U(\Delta)$ for $1 \le i \le r$, let m be an integer and suppose that*

$$\sum_{i=1}^r \alpha_i^\rho \beta_i \equiv 0 \quad (m) \quad \textit{for all } \rho \in \partial^\infty(\overline{H})$$

where $\overline{H}$ is a graded complement for $H_+ \cap \Delta$ in H_+. Choose a complementary basis for Δ_H in Δ and use it to write each $\alpha_i = \sum_\mu \mu \alpha_{\mu i}$ based on Δ_H. Then

$$\sum_{i=1}^r \alpha_{\mu i} \beta_i \equiv 0 \quad (m - \deg \mu) \qquad \textit{for all } \mu.$$

Proof. We may assume that all the α_i and β_i are homogeneous, and that $p(\alpha_i) + p(\beta_i) = c$ for a fixed $c \in G$. Let S denote the (finite) set of monomials used above to write the various α_i. Suppose that for some $\rho \in \partial^\infty(\overline{H})$ and μ, we have $\deg \mu + \deg \sum_i \alpha_{\mu i}^\rho \beta_i > m$. Define the integer n by

$$n + 1 = \max\{\deg \mu + \deg \sum_i \alpha_{\mu i}^\rho \beta_i | \rho \in \partial^\infty(\overline{H}), \mu \in S\}.$$

Now $n \geq m$ so $\sum_i (\mu\alpha_{\mu i})^\rho \beta_i \equiv 0\ (n)$ for all $\rho \in \partial^\infty(\overline{H})$. Thus the Hypothesis is satisfied with μ corresponding to α_j, $\alpha_{\mu i}$ to γ_{ji} and β_i to β_i. If we choose a finite-dimensional graded ideal I of Δ and use the complementary basis $\mathcal{X}$ for $I \cap \Delta_H$ in I as in Lemma 4.4, and expand $\beta_i = \sum_\eta \eta\beta_{\eta i}$ with respect to $\mathcal{X}$ we obtain

$$\sum_i \alpha_i^\rho \beta_{i\eta} \equiv 0 \quad (n - \deg \eta) \quad \text{for all } \rho \in \partial^\infty(\overline{H}) \text{ and all } \eta.$$

Thus by Proposition 2.5 we have

$$\sum_i \alpha_i \beta_{i\eta}^{\rho'} \varepsilon(\rho', \alpha_i) \equiv 0 \quad (n - \deg \eta) \quad \text{for all } \rho' \in \partial^\infty(\overline{H}) \text{ and all } \eta.$$

Substituting the expression for α_i based on Δ_H into the last identity we obtain

$$\sum_{\mu,i} \mu\alpha_{\mu i} \beta_{i\eta}^{\rho'} \varepsilon(\rho', \alpha_{\mu i}) \varepsilon(\rho', \mu) \equiv 0 \quad (n - \deg \eta) \quad \text{for all } \rho' \in \partial^\infty(\overline{H}) \text{ and all } \eta.$$

This last expression is written based on Δ_H, and the factor $\varepsilon(\rho', \mu)$ is nonzero and independent of i, so cancelling it and using freeness we obtain

$$\sum_i \alpha_{\mu i} \beta_{i\eta}^{\rho'} \varepsilon(\rho', \alpha_{\mu i}) \equiv 0 \quad (n - \deg \eta - \deg \mu) \quad \text{for all } \rho' \in \partial^\infty(\overline{H}) \text{ and all } \eta, \mu.$$

Again by Proposition 2.5 this yields

$$\sum_i \alpha_{\mu i}^\rho \beta_{i\eta} \equiv 0 \quad (n - \deg \eta - \deg \mu) \quad \text{for all } \rho \in \partial^\infty(\overline{H}) \text{ and all } \eta, \mu.$$

Multiplying this last identity on the right by η and summing over η gives

$$\sum_i \alpha_{\mu i}^\rho \beta_i \equiv 0 \quad (n - \deg \mu) \quad \text{for all } \rho \in \partial^\infty(\overline{H}) \text{ and all } \mu.$$

Thus

$$\max\{\deg \mu + \deg \sum_i \alpha_{\mu i}^\rho \beta_i | \rho \in \partial^\infty(\overline{H}), \mu \in S\} \leq n,$$

contradicting the definition of n above.

We conclude that $\sum_i \alpha_{\mu i}^\rho \beta_i = 0$ for all $\rho \in \partial^\infty(\overline{H})$. In particular this holds for the choice $\rho = 1$. $\square$

Here is the main theorem on reduction of derivation identities to $U(\Delta_L)$. We use our previous results to reduce first to Δ and then to Δ_L.

Theorem 4.6. *Let H be a large graded subalgebra of L and let $\alpha_i, \beta_i \in U(L)$ for $1 \leq i \leq r$. Let m be an integer and suppose that*

$$\sum_{i=1}^r \alpha_i^\rho \beta_i \equiv 0 \quad (m) \quad \textit{for all } \rho \in \partial^\infty(H).$$

Choose two arbitrary complementary bases for Δ_H in L, and use them to write $\alpha_i = \sum_\eta \eta\alpha_{\eta i}$ and $\beta_i = \sum_\mu \beta_{i\mu}\mu$ based on Δ_H. Then for all $\rho \in \partial^\infty(H)$ and all η and μ,

$$\sum_{i=1}^{r} \alpha_{\eta i}^{\rho}\beta_i \equiv 0 \quad (m - \deg \eta),$$

$$\sum_{i=1}^{r} \alpha_i^{\rho}\beta_{i\mu} \equiv 0 \quad (m - \deg \mu)$$

and

$$\sum_{i=1}^{r} \alpha_{\eta i}^{\rho}\beta_{i\mu} \equiv 0 \quad (m - \deg \eta - \deg \mu).$$

Proof. Again we may clearly assume all α_i and β_i to be homogeneous. We first prove the result holds with $\rho = 1$.

Since H is large in L we have $\mathbb{D}_L(H) = \mathbb{D}_L(L) = \Delta$. Let $\mathcal{Y}_1$ be a complementary basis for Δ in L and let $\mathcal{Y}_2$ be a complementary basis for Δ_H in Δ. Then $\mathcal{Y} = \mathcal{Y}_1 \dot\cup \mathcal{Y}_2$ is a complementary basis for Δ_H in L. Order $\mathcal{Y}$ so that $\mathcal{Y}_1 < \mathcal{Y}_2$.

Using $\mathcal{Y}_1$, write each $\beta_i = \sum_\nu \beta_{i\nu}\nu$ and $\alpha_i = \sum_\xi \xi\alpha_{\xi i}$ based on Δ. Let $\tau \in \partial^\infty(H)$. Then for each $\rho \in \partial^\infty(H)$, we also have $\tau\rho \in \partial^\infty(H)$ and hence

$$\sum_{i=1}^{r} (\alpha_i^{\rho})^{\tau}\beta_i \equiv 0 \quad (m) \quad \text{for all } \tau, \rho \in \partial^\infty(H).$$

Then by Lemma 3.4

$$\sum_i \alpha_i^{\rho}\beta_{i\nu} \equiv 0 \quad (m - \deg \nu) \quad \text{for all } \rho \in \partial^\infty(H) \text{ and all } \nu.$$

By Proposition 2.5 we have

$$\sum_i \alpha_i\beta_{i\nu}^{\rho'}\varepsilon(\rho', \alpha_i) \equiv 0 \quad (m - \deg \nu) \quad \text{for all } \rho' \in \partial^\infty(H) \text{ and all } \nu.$$

Substituting the expression for α_i based on Δ into this identity yields

$$\sum_{i,\xi} \xi\alpha_{\xi i}\beta_{i\nu}^{\rho'}\varepsilon(\rho', \xi)\varepsilon(\rho', \alpha_{\xi i}) \equiv 0 \quad (m - \deg \nu) \quad \text{for all } \rho' \in \partial^\infty(H) \text{ and all } \nu.$$

The above expression is written based on Δ. Using freeness and cancelling the nonzero common factor $\varepsilon(\rho', \xi)$ we obtain

$$\sum_i \alpha_{\xi i}\beta_{i\nu}^{\rho'}\varepsilon(\rho', \alpha_{\xi i}) \equiv 0 \quad (m - \deg \nu - \deg \xi) \quad \text{for all } \rho' \in \partial^\infty(H) \text{ and all } \nu \text{ and } \xi.$$

Another application of Proposition 2.5 gives

$$\sum_i \alpha_{\xi i}^{\rho}\beta_{i\nu} \equiv 0 \quad (m - \deg \nu - \deg \xi) \quad \text{for all } \rho \in \partial^\infty(H) \text{ and all } \nu \text{ and } \xi.$$

The last congruence above holds in particular for all $\rho \in \partial^\infty(\overline{H})$. We are now in the situation of Proposition 4.5, with $\alpha_{\xi i}$ in place of α_i and $\beta_{i\nu}$ in place of β_i. Specifically, use $\mathcal{Y}_2$ to write

$$\alpha_{\xi i} = \sum_\lambda \lambda \alpha_{\lambda \xi i}$$

based on Δ_H. Note that each $\xi\lambda$ is a straightened monomial on $\mathcal{Y}$, and that therefore the equation $\alpha_i = \sum_{\lambda,\xi} \xi\lambda\alpha_{\xi\lambda i}$ describes α_i based on Δ_H. Furthermore $\deg(\xi\lambda) = \deg\lambda + \deg\xi$. Now Proposition 4.5 yields the congruence

$$\sum_i \alpha_{\xi\lambda i}\beta_{i\nu} \equiv 0 \quad (m - \deg\nu - \deg(\xi\lambda))$$

and Proposition 2.4 concludes the proof of the $\rho = 1$ case.

For general ρ, use the same argument as that used in the proof of Theorem 3.5. □

Here we note that the obvious analogue of Lemma 3.4 fails for Δ_L. If char $K \neq 2$ and $L = A \rtimes Kl$ is a split extension as in the example preceding Proposition 2.9, except that A is odd and l even, then $\Delta_L = 0$, yet $x_1^\rho x_2 x_3 + x_2^\rho x_1 x_3 = 0$ for all $\rho \in \partial^1(L)$ (and for $\rho = 1$).

4.2. Linear identities. In [BP4] the reduction results for derivation identities immediately led to reductions for linear identities, since in the superalgebra case the elements of H_+ act as ordinary derivations. In the Lie colour algebra case more is needed — the next lemma supplies the missing link.

Lemma 4.7. *Let H be a graded subalgebra of L and let $\alpha_i, \beta_i \in U(\Delta)$ for $1 \leq i \leq r$. Suppose that*

$$\sum_i \alpha_i x \beta_i = 0 \qquad \text{for all } x \in U(H_+).$$

Let $\mathcal{Z}$ be a complementary basis for $(H_+ \cap \Delta)$ in H_+ and let $\overline{H} = K\mathcal{Z}$. Then

$$\sum_i \alpha_i^\rho \beta_i = 0 \qquad \text{for all } \rho \in \partial^\infty(\overline{H}).$$

Proof. It suffices to prove the result for those ρ which come from monomials on $\mathcal{Z}$. We use induction on the length of ρ, as defined in Lemma 3.6, to prove that $\sum_i \alpha_i^\rho x \beta_i = 0$ for all $\rho \in \partial^\infty(\overline{H})$ and all $x \in U(H_+)$. Putting $x = 1$ yields the result of the lemma.

The result clearly holds for $\rho = 1$, so suppose $\rho \neq 1$ and write $\rho = \partial(h)\rho'$ where $h \in \mathcal{Z}$ and $\operatorname{len}\rho' < \operatorname{len}\rho$. Inductively suppose the result holds for ρ' and for convenience write γ_i for $\alpha_i^{\rho'}$.

The inductive hypothesis yields

$$\sum_i \gamma_i x \beta_i = 0 \qquad \text{for all } x \in U(H_+).$$

We wish to prove that $\sum_i \gamma_i^h x \beta_i = 0$ for all $x \in U(H_+)$. It suffices to do this for those x which are straightened monomials on the ordered basis $\mathcal{Y}$ for H_+ defined below. Let $\mathcal{Z}' = \mathcal{Z} \setminus \{h\}$, let $\mathcal{Y}'$ be a homogeneous basis for $(H_+ \cap \Delta)$ and let $\mathcal{Y} = \mathcal{Z} \,\dot\cup\, \mathcal{Y}'$. Order $\mathcal{Y}$ so that $h < \mathcal{Z}' < \mathcal{Y}'$.

Fix a straightened monomial x on $\mathcal{Y}$ and write $x = h^s x'$ where x' is a straightened monomial on $\mathcal{Z}' \dot\cup \mathcal{Y}'$ whose support does not contain h. We may assume that all the γ_i are distinct and homogeneous by changing r if necessary. Divide the γ_i into equivalence classes with respect to h, where $\gamma_i \sim \gamma_j$ if and only if $\varepsilon(\gamma_i, h) = \varepsilon(\gamma_j, h)$. Let $C_1, \ldots, C_t$ be the equivalence classes. For each class C_j define $c_j = \varepsilon(C_j, h) = \varepsilon(\gamma, h)$ where γ is any element of C_j.

Let $n \geq 1$ be an integer. Since $h^n x' \in U(H_+)$ we have the equation

$$\sum_j \sum_{\gamma_i \in C_j} (\gamma_i h^n) x' \beta_i = 0.$$

Note that since h is even $\varepsilon(\gamma_i^h, h) = \varepsilon(\gamma_i, h)\varepsilon(h, h) = \varepsilon(\gamma_i, h)$ for each i. Thus an easy induction shows that if we rewrite the last equation by using the straightening relation $h\gamma_i - \varepsilon(h, \gamma_i)\gamma_i h = \gamma_i^h$ we obtain

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} h^k \Big(\sum_j c_j^n \sum_{\gamma_i \in C_j} \gamma_i^{h^{n-k}} x' \beta_i \Big) = 0.$$

Now note that since Δ is an ideal of L, the coefficient of each power of h on the left side of this last equation can be written based on Δ as $\sum_\xi \xi a_\xi$, where the ξ are straightened submonomials on $\mathcal{Z}'$. Since char $K = 0$ all the elements $(-1)^k \binom{n}{k} h^k$ are linearly independent modulo the span of the coefficients by the PBW theorem. It follows that all the coefficients in the last equation are zero. Since n was arbitrary this yields for each n and each k with $0 \leq k \leq n$ the equation

$$\sum_j c_j^n \sum_{\gamma_i \in C_j} \gamma_i^{h^k} x' \beta_i = 0.$$

Thus for each $k \geq 0$ we obtain the system of t equations

$$\sum_{j=1}^{t} c_j^k \Big(\sum_{\gamma_i \in C_j} \gamma_i^{h^k} x' \beta_i \Big) = \cdots = \sum_{j=1}^{t} c_j^{k+t-1} \Big(\sum_{\gamma_i \in C_j} \gamma_i^{h^k} x' \beta_i \Big) = 0.$$

The (Vandermonde type) determinant of this system is nonzero, since all the c_j are nonzero and distinct by definition. Thus we obtain for each j and for each $k \geq 0$,

E(k): $$\sum_{\gamma_i \in C_j} \gamma_i^{h^k} x' \beta_i = 0.$$

We now restore the h factors to these equations. Multiplying E(k) on the left by h and subtracting E($k+1$) gives, on cancelling the common factor $\varepsilon(h, \gamma_i^{h^k}) = \varepsilon(h, \gamma_i)$, the equation

$$\sum_{\gamma_i \in C_j} \gamma_i^{h^k} h x' \beta_i = 0 \qquad \text{for all } k \geq 0.$$

Iterating this procedure gives in particular at the s-th stage

$$\sum_{\gamma_i \in C_j} \gamma_i^h h^s x' \beta_i = \sum_{\gamma_i \in C_j} \gamma_i^h x \beta_i = 0.$$

This equation holds for all $x \in U(H_+)$ and all j, so summing over j gives

$$\sum_i \gamma_i^h x\beta_i = 0 \qquad \text{for all } x \in U(H_+)$$

which in view of our notational convention is the desired result. □

Here is the main theorem on reduction of linear identities to $U(\Delta_L)$.

Theorem 4.8. *Let H be a large graded subalgebra of L and let $\alpha_i, \beta_i \in U(L)$ for $1 \leq i \leq r$. Suppose that*

$$\sum_i \alpha_i x\beta_i = 0 \qquad \textit{for all } x \in U(H).$$

Choose two arbitrary complementary bases for Δ_H in L, and use them to write each $\alpha_i = \sum_\eta \eta\alpha_{\eta i}$ and $\beta_i = \sum_\mu \beta_{i\mu}\mu$ based on Δ_H. Then for all $z \in U(H)$ and all η and μ,

$$\sum_i \alpha_i z\beta_{i\mu} = 0, \qquad \sum_i \alpha_{\eta i} z\beta_i = 0, \qquad \textit{and} \qquad \sum_i \alpha_{\eta i} z\beta_{i\mu} = 0$$

.

Proof. It remains only to put together our previous results in a (by now) routine way. Choose a complementary basis for Δ in L and one for Δ_H in Δ. Since H is large in L we have $\mathbb{D}_L(H) = \Delta$. Applying Theorem 3.7 we reduce to a linear identity over $U(\Delta)$. The last lemma yields a corresponding derivation identity. Now order bases consistently as in the proof of Theorem 4.6 and apply Theorem 4.6 with $m = -1$. This along with Proposition 2.4 yields the result in the $z = 1$ case on specializing $\rho = 1$. For general z the same argument as in the proof of Theorem 3.7 concludes the proof. □

5. Applications

Having proved the reduction theorems above we can proceed directly to some ring-theoretic applications as in [BP4]. They are mostly rather routine applications of the above results and nearly all the arguments are to be found in [BP3] and [BP4], so we omit most of the proofs. We indicate briefly the connection between our previous work and these results. A ring is not prime if and only if it contains nonzero elements α and β such that the identity $\alpha x\beta = 0$ holds. Similarly semiprimeness, centrality and the other properties considered here have characterizations in terms of existence or nonexistence of linear identities.

If H is a graded subalgebra of L, the *almost constants* for the action of H on $U(L)$ are those elements $\alpha \in U(L)$ for which

$$\dim_K \operatorname{ad} H \cdot \alpha < \infty.$$

Similarly, the almost constants for the action of $U(H)$ on $U(L)$ are those $\alpha \in U(L)$ for which $\dim_K \operatorname{ad} U(H) \cdot \alpha < \infty$. In each case the almost constants form a graded subalgebra of $U(L)$.

Part (i) of the next theorem follows easily from the results of Section 3.1 as in [BP3, Corollary 5.1]. Part (ii) requires a nontrivial additional argument by these means. However a close examination of the proof of [BP4, Theorem 6.2] shows that the result of (ii) below holds for Lie colour algebras. Part (iii) follows from [BP4, Theorem 6.8] via discoloration, since an element belongs to the kernel of ad if and only if all its homogeneous components do.

Theorem 5.1. *Let H be a graded subalgebra of L, and set $D = \mathbb{D}_L(H)$. Then*

(i) *The subalgebra of almost constants for the action of H on $U(L)$ equals $U(D)$.*
(ii) *If H is large in L, then the subalgebra of almost constants for the action of $U(H)$ on $U(L)$ equals $U(\Delta_H)$.*
(iii) *If $\Delta_L = 0$ then the adjoint action of $U(L)$ on $U(L)$ is faithful.* □

In particular the constants and semi-invariants for the action of L on $U(L)$ lie in $U(\Delta_L)$.

Suppose that $U(L)$ is semiprime. Then it has a symmetric Martindale ring of quotients Q. We assume that K contains a primitive $\exp G$-th root of unity. Then the linear automorphisms $\varphi(\chi)$ of $U(L)$ as in Proposition 2.2 are in fact algebra automorphisms and extend uniquely to Q. Thus Q is graded. The argument of [BP4, Theorem 6.3] goes over directly and so all constants for the action of L on Q are even. In particular all constants for the action of L on $U(L)$ are even. This last observation is obvious, since an odd constant generates a nilpotent ideal in $U(L)$.

However even with our hypothesis on ε, it is not necessarily the case that the constants lie in the component of the identity. For example, suppose that char $K \neq 2$ and let $G =< a > \times < b >$ be the noncyclic group of order 4 with $G = G_+$ and $\varepsilon(a,b) = -1$. Let $L = Kx$ be one-dimensional with x having parity a. Then $U(L) = K[x]$ has constants of parity $a \neq 1$.

Moving to linear identities, we can define the *almost centralizer* of H in $U(L)$ as the set of all $\alpha \in U(L)$ such that $x\alpha - \alpha x = 0$ for all x in some large subspace of H, and similarly the almost centralizer of $U(H)$ in $U(L)$. Each is a graded subalgebra of $U(L)$. Part (i) of the next result follows as in [BP3, Corollary 5.2]. Part (ii) is proved just as in [BP4, Theorem 6.1].

Theorem 5.2. *Let H be a graded subalgebra of L, and set $D = \mathbb{D}_L(H)$. Then*

(i) *The almost centralizer of H in $U(L)$ is contained in $U(D)$.*
(ii) *If H is large in L, then the centralizer of $U(H)$ in $U(L)$ is contained in $U(\Delta_H)$. In particular the centre of $U(L)$ lies in $U(\Delta_L)$.* □

The centre of $U(L)$ need not lie in the component of the identity, as the above example shows.

We now move on to consider annihilator ideals and primeness. The next theorem follows from our reduction theorems as in [BP4, Theorem 6.1]. It deals with ungraded ideals and seems to require the full machinery of the Δ-methods.

Theorem 5.3. *Let $A, B \lhd U(L)$.*

(i) *If $A = \mathrm{l.ann}\, B$, then $A = U(L)(A \cap U(\Delta_L)) = U(L)(A \cap U(\Delta))$, and if $B = \mathrm{r.ann}\, A$, then $B = (B \cap U(\Delta_L))U(L) = (B \cap U(\Delta))U(L)$.*
(ii) *If $U(\Delta)$ is prime or $U(\Delta_L)$ is prime then so is $U(L)$.* □

Note that $U(L)$ can be prime even if $U(\Delta)$ or $U(\Delta_L)$ is not, as the examples below show. Note however that in those examples these subalgebras are L-prime, that is they do not contain nonzero ideals I, J stable under the action of L such that $IJ = 0$. In the first example below $U(\Delta_L)$ is prime while $U(\Delta)$ is not. I do not know whether $U(\Delta)$ can be prime without $U(\Delta_L)$ being prime.

Let D denote either Δ or Δ_L. As in [BP4], for any subset A of $U(L)$ and complementary basis for D in L we can consider the K-subspace $\pi(A)$ spanned by all the *leading coefficients* in D (i.e. those α_μ corresponding to μ of maximal degree) of elements $\alpha = \sum_\mu \mu\alpha_\mu$ of A. Then $\pi(A)$ is independent of the complementary basis chosen for D in L, is graded and right-left symmetric if A is graded, and is an L-stable ideal of $U(D)$ if A is an ideal of $U(L)$.

Our next result is the analogue of [BP4, Theorem 6.5], and the proof is just as in that theorem. Part (iii) could also be obtained directly from [BP4, Theorem 6.5(iii)] by using the discoloration functor, once we note that semiprimeness is automatically a graded property. By our conventions $\operatorname{char} K \nmid |G|$, so some extension F of K contains a primitive $\exp G$-th root of 1. If I is a nonzero nilpotent ideal of $U(L)$, then $\overline{I} = I \otimes F$ is a nonzero nilpotent ideal of $U(L) \otimes F$. Letting $J = \sum \overline{I}^\chi$, where the sum runs over all characters χ of G, it follows from Proposition 2.2 that J is a nonzero nilpotent graded ideal, and intersecting with $U(L)$ gives a nilpotent graded ideal of $U(L)$ containing I. Thus $U(L)$ is semiprime if and only if it is graded semiprime.

Theorem 5.4. *Let L be a Lie colour algebra. Then*

(i) *Let I and J be graded ideals of $U(L)$. Then $IJ = 0$ implies that $\pi(I)\pi(J) = 0$.*
(ii) *$U(L)$ is graded prime if and only if $U(\Delta)$ is graded L-prime, if and only if $U(\Delta_L)$ is graded L-prime.*
(iii) *$U(L)$ is semiprime if and only if $U(\Delta)$ is L-semiprime, if and only if $U(\Delta_L)$ is L-semiprime.* □

It is the lack of an ungraded version of (i) above which leaves us with an incomplete answer to the question of when $U(L)$ is prime.

The examples below show that being L-semiprime (respectively graded L-prime) is weaker than being semiprime (respectively graded prime) for $U(\Delta)$ and $U(\Delta_L)$, so we cannot drop the 'L-'s in the above theorem.

Example. Let L be the Lie superalgebra in the example following Theorem 4.6. In terms of generators and relations, L is the Lie superalgebra generated by l and the x_i subject to the relations $[x_i, x_j] = 0$ and $[l, x_i] = x_{i+1}$ and the requirement that l be even and the x_i odd. Then $\Delta = A$, $\Delta_L = 0$ and $U(A)$ is L-prime, but not semiprime since it is an exterior algebra.

Example. Suppose $\operatorname{char} K \neq 2$ and let $S = Ku \dotplus Kv$ be a 2-dimensional abelian Lie superalgebra with u even and v odd. Then S admits an odd outer derivation ∂ sending u to 0 and v to u. Note that $[\partial, \partial] = 2\partial^2 = 0$ and we have an action of the one-dimensional Lie superalgebra $K\partial$ on S. Now let T be any Lie superalgebra which has a homomorphism onto $K\partial$ and has no nonzero finite-dimensional ideals. Form the semidirect product $L = S \rtimes T$ with T acting on S via the action of $K\partial$. Then it is easily seen that $\Delta_L = S$. The ring $R = U(S)$ is isomorphic to the truncated polynomial ring $K[u, v|v^2 = 0]$. Its nilradical is $N = vR$, and $N \cap N^\partial = 0$. In fact any two nonzero ideals I, J of R with $IJ = 0$ are contained in N, so that R is L-prime but not semiprime.

An example of such a superalgebra T is the superalgebra L of the previous example. The map from L to S given by $l \mapsto 0$, $x_1 \mapsto \partial$ and $x_i \mapsto 0$ for all $i \geq 2$ is well-defined since the relators defining L all have an obvious factor mapping to zero except for $[x_1, x_1]$ which maps to $[\partial, \partial] = 0$.

It is necessary to use odd outer derivations in order to construct such an example, at least in characteristic zero. If $\operatorname{char} K = 0$ and L is a finite-dimensional Lie superalgebra then $U(L)$ is noetherian and its nilradical is a nilpotent graded ideal. It is easily seen that even derivations preserve the nilradical (apply the discoloration functor along with [R, Proposition 2.6.28]).

In the remaining theorems, the hypotheses imply that $U(L)$ is prime and that Q, the Martindale symmetric ring of quotients, exists and is graded. Let $C = C_Q(U(L))$ be the extended centroid of $U(L)$. The next theorem follows as in the analogous results [BP3, Corollary 5.8] and [BP4, Theorem 6.6].

Theorem 5.5. *Suppose that K contains a primitive* $\exp G$*-th root of unity, and let D denote either Δ or Δ_L.*

- (i) *Suppose $U(D)$ is prime. Then the extended centroid of $U(L)$ embeds naturally into the extended centroid of $U(D)$. Thus if $D = 0$ then $C = K$, and $U(L)$ is centrally closed.*
- (ii) *If $D = 0$, then the almost centralizer of L in Q is equal to K.* □

If S is a subset of G and $L_a = 0$ when $a \notin S$ we say L is *concentrated* in S. Recall Proposition 2.2 and let $\varphi(\chi)$ be the automorphism of Q induced by the action of the character χ.

Theorem 5.6. (*c.f.*[BP4, Theorem 6.7]) *Suppose that K contains a primitive* $\exp G$*-th root of* 1*, and that $\Delta_L = 0$. If $\varphi(\chi)$ is X-inner, then L is concentrated in* $\ker\chi$*. Thus if L is not a Lie algebra, then some $\varphi(\chi)$ is X-outer.*

Proof. Suppose that $\varphi(\chi)$ is X-inner. Then there is a nonzero $q \in Q$ with $xq = qx^\chi$ for all $x \in U(L)$. We may assume q to be homogeneous (consider homogeneous components). Also, there exists a nonzero ideal I of $U(L)$ with $0 \neq Iq \subseteq U(L)$. Replacing I by the nonzero finite intersection $\bigcap I^\psi$, where ψ runs over $\operatorname{Hom}(G, K^*)$, we can assume that I is graded. Thus there is a homogeneous nonzero $\alpha \in I$ with $\alpha q = \beta$, a nonzero homogeneous element of $U(L)$.

The identity $xq = qx^\chi$ yields, on multiplying each side on the left and right by α and rearranging, the identity

$$(*) \qquad \chi^{-1}(p(\alpha))\alpha x\beta = \beta x^\chi \alpha \qquad \text{for all } x \in U(L).$$

Recall we are assuming that distinct elements of G give rise to distinct characters via $a \mapsto \varepsilon(a, \cdot)$. Thus since G is finite this map is onto the character group and we can choose $a \in G$ with $\varepsilon(a, \cdot) = \chi^{-1}$. We can form the larger Lie colour algebra $L' = L \oplus T$ where $T = Kt$ is 1-dimensional, $[t, t] = 0$ and $p(t) = a$. We then have $xt = tx^\chi$ for all $x \in U(L)$, and if x is homogeneous then $x^\chi = \varepsilon(x, t)x$. Furthermore, note that $\Delta_{L'} = T$.

Now working in the larger ring $U(L')$, multiply equation (*) on the left by t and use the above relations to obtain

$$\varepsilon(t, \alpha)^2 \varepsilon(\beta, t)\alpha t x \beta = \beta x t \alpha \qquad \text{for all } x \in U(L).$$

Write $\alpha = \sum_\mu \mu\alpha_\mu$ and $\beta = \sum_\mu \mu\beta_\mu$ based on $\Delta_L = 0$. Then

$$\varepsilon(t, \alpha)^2 \varepsilon(\beta, t)\alpha t = \sum_\mu \mu(\varepsilon(t, \alpha)^2 \varepsilon(\beta, t)\alpha_\mu t)$$

describes $\varepsilon(t, \alpha)^2 \varepsilon(\beta, t)\alpha t$ based on $\Delta_{L'} = T$ and since $\alpha_\mu \in K$, $t\alpha = \sum_\mu \alpha_\mu t \mu$ likewise describes $t\alpha$. Now since L is large in L', Theorem 4.8 gives the reduction

$$\varepsilon(t, \alpha)^2 \varepsilon(\beta, t)\alpha_\mu t x \beta = \beta_\mu x t \alpha \qquad \text{for all } x \in U(L) \text{ and all } \mu.$$

If we choose μ so that $\beta_\mu \neq 0$ then it follows from above that also $\alpha_\mu \neq 0$. Then Theorem 4.8 also gives

$$\varepsilon(t, \alpha)^2 \varepsilon(\beta, t)\alpha_\mu t x \beta_\mu = \beta_\mu x t \alpha_\mu \qquad \text{for all } x \in U(L).$$

Cancelling the nonzero field elements α_μ and β_μ gives $\varepsilon(t, \alpha)^2 \varepsilon(\beta, t)tx = xt$ for all $x \in U(L)$ and setting $x = 1$ implies that $\varepsilon(t, \alpha)^2 \varepsilon(\beta, t) = 1$. Thus in particular $tx = xt$ for all $x \in L$. Since also $tx = x^\chi t$ this implies that L is concentrated in $\ker \chi$. □

This result fails if we remove the hypothesis $\Delta_L = 0$ and only require $U(L)$ to be prime, as was shown in [BP4].

6. Comments

The original group ring delta methods (see [P]) yielded similar results, for example $K[G]$ is prime if and only if $K[\Delta^+]$ is prime. However the finite situation for group rings is much easier since if G is finite and nontrivial then KG is never prime. By contrast, it is not even known for which finite-dimensional Lie superalgebras L the algebra $U(L)$ is prime. The best available positive result seems to be that of [B].

There is also the problem of obtaining a better characterization of primeness in the characteristic p case. We know that this property is controlled by Δ but an answer in terms of finite-dimensional restricted ideals is apparently ruled out by an example of [BP1]. We close with a list of questions which (some not for the first time) came up in the course of this work.

(i) Is 'graded prime' equivalent to 'prime' for algebras $U(L)$?
(ii) Is 'semiprime' equivalent to 'prime' for $U(L)$?
(iii) Is $U(L)$ prime if and only if $U(\Delta_L)$ is prime?

(iv) If $U(\Delta)$ is semiprime, is $U(\Delta_L)$ necessarily semiprime?

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