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# GROUP ALGEBRAS AND ENVELOPING ALGEBRAS WITH NONMATRIX AND SEMIGROUP IDENTITIES

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ABSTRACT. Let  $K$  be a field of characteristic  $p > 0$ . Denote by  $\omega(R)$  the augmentation ideal of either a group algebra  $R = K[G]$  or a restricted enveloping algebra  $R = u(L)$  over  $K$ . We first characterize those  $R$  for which  $\omega(R)$  satisfies a polynomial identity not satisfied by the algebra of all  $2 \times 2$  matrices over  $K$ . Then, we examine those  $R$  for which  $\omega(R)$  satisfies a semigroup identity (that is, a polynomial identity which can be written as the difference of two monomials).

## 1. INTRODUCTION AND STATEMENT OF RESULTS

The structure of group algebras and restricted enveloping algebras, over a field  $K$  of characteristic  $p > 0$ , that are Lie nilpotent, Lie solvable ( $p \neq 2$ ) or satisfy the Engel condition has been completely determined in [PPS73, RS93, Seh78]. Integral to the proof of these results was the fact that each of these conditions corresponds to a particular *nonmatrix identity*, which is to say, a polynomial identity not also satisfied by the algebra  $M_2(K)$  of all  $2 \times 2$  matrices.

Nonmatrix identities in characteristic zero were studied by Kemer in [Kem81], who showed, in particular, that nonmatrix varieties are Lie solvable. The situation in positive characteristic is more complicated. More recently, the existence of an arbitrary nonmatrix identity was shown in [BRT97] to be intimately related to the existence of a group identity in the group of units of a certain class of associative algebras, which includes group algebras of periodic groups and restricted enveloping algebras of  $p$ -nil restricted Lie algebras.

In this first half of this paper we completely describe enveloping algebras and group algebras which satisfy a nonmatrix identity. In the results below,  $K[G]$  and  $u(L)$  denote, respectively, the group algebra of a group  $G$  and the restricted universal enveloping algebra of the restricted Lie algebra  $L$ . In addition,  $\omega(R)$  represents the augmentation ideal of  $R = u(L)$  or  $K[G]$ ,

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while  $\gamma^2(R)$  denotes its (associative) commutator ideal. Finally,  $G'$  and  $L'$  are the usual commutator subgroup and Lie subalgebra, respectively.

**Theorem 1.1.** *The following statements are equivalent for a restricted Lie algebra  $L$  over a field of characteristic  $p > 0$ :*

- (i)  $u(L)$  satisfies a nonmatrix identity;
- (ii)  $\omega(u(L))$  satisfies a nonmatrix identity;
- (iii)  $L$  contains a restricted ideal  $A$  such that both  $L/A$  and  $A'$  are finite-dimensional, and  $L'$  is  $p$ -nil of bounded index;
- (iv)  $\gamma^2(u(L))$  is nil of bounded index.

**Theorem 1.2.** *The following statements are equivalent for a group  $G$  and a field  $K$  of characteristic  $p > 0$ :*

- (i)  $K[G]$  satisfies a nonmatrix identity;
- (ii)  $\omega(K[G])$  satisfies a nonmatrix identity;
- (iii)  $G$  has a normal subgroup  $A$  such that both  $A'$  and  $G/A$  are finite, and  $G'$  is a  $p$ -group of finite exponent;
- (iv)  $\gamma^2(K[G])$  is nil of bounded index.

It was conjectured in [Ril97] and [Ril96], respectively, that if  $R$  and  $S$  are arbitrary algebras over a field of characteristic  $p > 0$  satisfying nonmatrix identities, then  $\gamma^2(R)$  is nil of bounded index and the tensor product  $R \otimes_K S$  also satisfies some nonmatrix identity. The corresponding results were shown to fail in characteristic 0. Theorems 1.1 and 1.2 verify both these conjectures for the class of restricted enveloping algebras, since  $u(L) \otimes_K u(M) \cong u(L \times M)$ , and the class of group algebras, since  $K[G] \otimes_K K[H] \cong K[G \times H]$ .

It will become apparent in the next section (see the paragraph after Proposition 2.1) that if  $R$  is either an ordinary enveloping algebra over an arbitrary field, or a group algebra over a field of characteristic zero, and  $R$  satisfies a nonmatrix identity, then  $R$  must be commutative.

In the second half of this article we look at restricted enveloping algebras and group algebras which satisfy a semigroup identity. A *semigroup identity* is a polynomial identity of the form

$$f(x_1, \dots, x_n) = w_1(x_1, \dots, x_n) - w_2(x_1, \dots, x_n) = 0,$$

where  $w_1$  and  $w_2$  are monomials. It follows from well-known results that a semigroup identity is a nonmatrix identity precisely when the base field  $K$  is infinite. In fact, by a result of Jones [Jon74], every proper variety of groups can contain only finitely many simple groups. Hence the general linear group  $GL_2(K)$  does not satisfy any group identity unless  $K$  is finite. This last fact is also a special case of [BRT97, Theorem 1.2]. The present authors studied arbitrary algebras  $R$  over an infinite field that satisfy a semigroup identity in [RW]. In particular, we proved that if  $R$  is unital then  $R$  satisfies a semigroup identity if and only if  $R$  satisfies the Engel condition:

$$[x, \underbrace{y, \dots, y}_m] = 0,$$

for some  $m$ . Hence, using the characterization of the Engel condition in restricted enveloping algebras and group algebras in [RS93] and [Seh78], respectively, we immediately obtain the equivalence of the first three conditions in each of the following two theorems:

**Theorem 1.3.** *The following statements are equivalent for a restricted Lie algebra  $L$  over an infinite field of characteristic  $p > 0$ :*

- (i)  $u(L)$  satisfies a semigroup identity;
- (ii)  $u(L)$  satisfies the Engel condition;
- (iii)  $L$  is nilpotent,  $L'$  is  $p$ -nil of bounded index, and  $L$  contains a restricted ideal  $A$  such that both  $L/A$  and  $A'$  are finite-dimensional;
- (iv)  $L'$  and  $L/Z(L)$  are  $p$ -nil of bounded index and  $L$  contains a restricted ideal  $A$  such that both  $L/A$  and  $A'$  are finite-dimensional.

**Theorem 1.4.** *The following statements are equivalent for a group  $G$  and an infinite field  $K$  of characteristic  $p > 0$ :*

- (i)  $K[G]$  satisfies a semigroup identity;
- (ii)  $K[G]$  satisfies the Engel condition;
- (iii)  $G$  is nilpotent and contains a normal subgroup  $A$  such that both  $G/A$  and  $A'$  are finite  $p$ -groups;
- (iv)  $G'$  and  $G/Z(G)$  are  $p$ -groups of finite exponent and  $G$  contains a normal subgroup  $A$  such that both  $G/A$  and  $A'$  are finite.

Note that since every finite ring satisfies an identity of the form  $x^k - x^l = 0$ , these results do not hold when  $K$  is finite. Observe also that the corresponding conditions (iv) provide an exact analogue between the group algebra and restricted enveloping algebra cases. These alternate characterizations will prove useful below. The proof of their equivalence requires some work and will be given later.

For nonunital algebras, semigroup identities are more difficult to classify. In particular, the existence of a semigroup identity does not generally imply the Engel condition. In fact, this implication does not even hold for augmentation ideals:

**Example 1.5.** Let  $K$  be a field of characteristic 2, and let  $L$  be the 2-dimensional restricted Lie algebra over  $K$  generated by  $a, b$  with relations  $[a, b] = a$ ,  $a^2 = 0$ ,  $b^2 = b$ . Then  $\omega(u(L))$  satisfies the semigroup identity

$$wxyz - wyxz = 0,$$

despite the fact that  $\omega(u(L))$  does not satisfy the Engel condition.

**Example 1.6.** Let  $K$  be a field of characteristic 3, and let  $G$  be the dihedral group of order 6. Then  $\omega(K[G])$  satisfies the semigroup identity

$$yxy^2 - y^2xy = 0,$$

and yet  $\omega(K[G])$  does not satisfy the Engel condition.

We shall see in Section 3 though, that these two examples are in some sense canonical.

**Theorem 1.7.** *Let  $L$  be a restricted Lie algebra over an infinite field of characteristic  $p > 2$ . Then the following statements are equivalent:*

- (i)  $\omega(u(L))$  satisfies a semigroup identity;
- (ii)  $\omega(u(L))$  satisfies an identity of the form  $y^m[x,{}_m y]y^m = 0$ ;
- (iii)  $L'$  and  $L/Z(L)$  are both  $p$ -nil of bounded index and  $L$  contains a restricted ideal  $A$  of finite codimension such that  $A'$  is finite-dimensional;
- (iv)  $u(L)$  satisfies the Engel condition.

**Theorem 1.8.** *Let  $K$  be an infinite field of characteristic  $p > 0$ , and suppose that  $p = 2$  or  $G$  is a group with no 2-torsion. Then the following statements are equivalent:*

- (i)  $\omega(K[G])$  satisfies a semigroup identity;
- (ii)  $\omega(K[G])$  satisfies an identity of the form  $y^m[x,{}_m y]y^m = 0$ ;
- (iii)  $G'$  and  $G/Z(G)$  are  $p$ -groups of finite exponent and  $G$  contains a normal subgroup  $A$  of finite index such that  $A'$  is finite;
- (iv)  $K[G]$  satisfies the Engel condition.

## 2. NONMATRIX IDENTITIES

In this section we prove Theorems 1.1 and 1.2. Throughout,  $K$  is a field,  $G$  a group and  $L$  a restricted Lie  $K$ -algebra. For each (ordinary) subalgebra  $H$  of  $L$ ,  $H_p$  denotes its  $p$ -hull (that is, the smallest restricted Lie subalgebra containing it) inside  $L$ .

We begin by collecting some important facts about algebras satisfying a nonmatrix identity.

**Proposition 2.1.** *Let  $R$  be an algebra over a field  $K$ . If  $R$  satisfies a nonmatrix identity then the following statements hold:*

- (i) *The unital hull of  $R$  also satisfies a nonmatrix identity.*
- (ii)  *$\gamma^2(R)$  is nil, and is nilpotent if  $R$  is finitely generated over  $K$ .*
- (iii) *If  $\text{char } K = p > 0$  then  $R$  satisfies an identity of the form  $([x, y]z)^{p^t} = 0$ , so that  $\gamma^2(R)$  is generated by nilpotent elements of bounded index.*
- (iv) *If  $\text{char } K = p > 0$  and  $R = \omega(K[G])$  then  $G'$  is a  $p$ -group generated by elements of bounded  $p$ -power order.*
- (v) *If  $\text{char } K = p > 0$  and  $R = \omega(u(L))$  then  $(L')_p$  is a  $p$ -nil restricted Lie algebra generated by  $p$ -nilpotent elements of bounded index.*

*Proof.* Let  $S$  be the unital hull of  $R$ , that is,  $S = R$  if  $R$  is unital and  $S = R \oplus K \cdot 1$  otherwise. Then  $R$  is an associative ideal of  $S$  such that  $[S, S]S \subseteq R$ . Let  $f(x_1, \dots, x_n) = 0$  be a nonmatrix identity satisfied by  $R$ . It follows that  $S$  satisfies the identity

$$f([x_1, y_1]z_1, \dots, [x_n, y_n]z_n) = 0.$$

Now this last identity is also nonmatrix, since

$$a = [e_{12}, e_{21}](e_{11} - e_{22})a$$

for every  $a \in M_2(K)$ , and this proves (i).

Henceforth, we assume that  $R$  is unital.

Let  $f$  be the given nonmatrix identity, and  $J(R)$  the Jacobson radical of  $R$ . We first show that  $R/J(R)$  is commutative. Now  $R/J(R)$  is a semiprimitive PI-algebra, and hence a subdirect product of primitive PI-algebras, each of which satisfies  $f$ . By a theorem of Kaplansky, a primitive PI-algebra is simple and finite-dimensional over its centre, and thus has the form  $M_n(D)$  where  $D$  is a division algebra finite-dimensional over its centre  $F$ . Since  $M_2(F)$  does not satisfy  $f$ ,  $n = 1$ . If  $D$  is finite then  $D = F$ , while if  $D$  is infinite then  $D$  satisfies the same identities as some  $M_m(F)$ . The nonmatrix condition gives  $m = 1$  and so either way  $D = F$ . Thus  $R/J(R)$  is a subdirect product of fields and hence commutative. It follows that  $\gamma^2(R) \subseteq J(R)$ .

By a theorem of Braun [Bra84], if  $A$  is a finitely generated PI-algebra then  $J(A)$  is nilpotent. Let  $A$  be the relatively free algebra on 3 generators in the variety defined by the given nonmatrix identity  $f$ . Then  $\gamma^2(A) \subseteq J(A)$  is nilpotent and so  $A$  satisfies an identity of the form  $([x, y]z)^m = 0$ . It follows that  $R$  also satisfies this same identity. This yields (ii) and (iii). If  $R = u(L)$  then  $L' \subseteq \gamma^2(R)$ , whereas if  $R = K[G]$  then  $G' \subseteq 1 + \gamma^2(R)$ , yielding (iv) and (v). □

Both ordinary enveloping algebras  $U(L)$  and group algebras in characteristic zero are known to be semiprime, so by Proposition 2.1 (ii), if they satisfy a nonmatrix identity then they are necessarily commutative. Thus we shall focus on modular group algebras and restricted enveloping algebras from now on. Because of the strong parallels between the group algebra and restricted enveloping algebra cases, we prove only the restricted enveloping algebra case in detail.

For the rest of this section,  $K$  will be assumed to be of characteristic  $p > 0$ .

**Enveloping algebras.** We adapt a lemma of Passman [Pas97, Lemma 3.2] to the enveloping algebra situation.

**Lemma 2.2.** *Let  $A$  be an abelian restricted ideal of a restricted Lie algebra  $L$  such that  $\dim L/A$  is finite. If  $I$  is an  $L$ -stable ideal of  $u(A)$  then the ideal  $J = Iu(L) = u(L)I$  of  $u(L)$  generated by  $I$  is nil of bounded index if (and only if)  $I$  is.*

*Proof.* Let  $n = \dim L/A$ . Jacobson's theorem (the analogue of the PBW theorem for restricted Lie algebras) implies that as a left  $u(A)$ -module,  $u(L)$  is free of rank  $q = p^n$ . Indeed, if  $\{x_1, \dots, x_n\}$  is an ordered basis for a vector space complement for  $A$  in  $L$ , the monomials  $x_1^{i_1} \cdots x_n^{i_n}$  form a basis for  $u(L)$  over  $u(A)$ .

The right regular representation of  $u(L)$  embeds  $u(L)$  into  $M_q(u(A))$  in such a way that for all basis monomials  $\nu$ ,  $\nu x = \sum_{\mu} x_{\nu\mu}\mu$  for  $x \in u(L)$ . If  $x \in J$  then also  $\nu x \in J$ . Since  $J = \bigoplus_{\mu} I\mu$  we have  $\sum_{\mu} x_{\nu\mu}\mu \in \bigoplus_{\mu} I\mu$ . Since all  $x_{\nu\mu} \in u(A)$ , freeness yields  $x_{\nu\mu} \in I$  and so  $J$  embeds in  $M_q(I)$ .

If  $I$  is nil of bounded degree  $p^t$ , then since every element of  $M_q(I)$  is algebraic of degree at most  $q$  over the central subalgebra  $I$ , it follows easily that  $M_q(I)$  is nil of bounded index at most  $qp^t$ .  $\square$

*Proof of Theorem 1.1.* Clearly (i) is equivalent to (ii) by Proposition 2.1, while the fact that (iv) implies (i) is immediate.

Recall that for an (ordinary) ideal  $J$  of  $L$ , the ideal generated by  $J$  in  $u(L)$  is  $\omega(u(J_p))u(L) = u(L)\omega(u(J_p))$ . By Jacobson's theorem, this ideal is nilpotent if and only if  $J_p$  is finite-dimensional and  $p$ -nil.

Results of Passman [Pas90] and Petrogradski [Pet91], yield that  $u(L)$  satisfies a polynomial identity if and only if  $L$  contains a restricted ideal  $A$  such that  $\dim L/A < \infty$ ,  $\dim A' < \infty$  and  $(A')_p$  is  $p$ -nil. Suppose that condition (i) holds, so that  $L$  has a restricted ideal  $A$  with these basic properties. In particular,  $A$  is solvable since by Engel's theorem it is nilpotent-by-abelian. To show (i) implies (iii), it remains to show that  $(L')_p$  is  $p$ -nil of bounded index. Certainly  $(L')_p$  is  $p$ -nil and generated by  $p$ -nilpotent elements of bounded index by Proposition 2.1. Thus  $(L')_p/(A \cap (L')_p)$  is finite-dimensional and  $p$ -nil, so that  $(L')_p$  is solvable. Consequently,  $L$  is solvable. Now arguing by induction on the derived length of  $L$  enables us to reduce to the case when  $L'$  is abelian. But in this case  $(L')_p$  is clearly  $p$ -nil of bounded index, since it is generated by elements with that property.

Finally, suppose that (iii) holds. We show that  $\gamma^2(u(L))$  is nil of bounded index. We make a series of reductions based on the fact that the class of algebras which are nil of bounded index is closed under extensions. Because  $A'$  generates a nilpotent ideal in  $u(L)$ , it suffices for us to assume  $A$  is abelian. Now put  $B = [A, L]$ . Then  $B$  is an ideal of  $L$  contained in  $A \cap L'$  and so  $B_p$  is  $p$ -nil of bounded index. Since  $A$  is abelian, it follows that  $B$  generates an ideal  $I$  of  $u(A)$  which is  $L$ -stable and nil of bounded index. By Lemma 2.2 the same is true of the expanded ideal  $Iu(L)$  of  $u(L)$ . Thus we may assume that  $B = 0$  and hence that  $A$  is central in  $L$ . This implies that  $L'$  is finite-dimensional and so  $L'$  generates a nilpotent ideal of  $u(L)$ . Finally, we can assume that  $L$  is abelian and the result follows immediately.  $\square$

**Corollary.**  *$u(L)$  satisfies a nonmatrix identity if and only if it satisfies some polynomial identity and  $L'$  is  $p$ -nil of bounded index.*  $\square$

### Group algebras.

*Proof of Theorem 1.2.* The proof is entirely analogous to the restricted enveloping algebra case, and is in fact simpler since there is no need to trouble ourselves with any analogue of the  $p$ -hull.

The main ingredients are:

- By a result of Passman [Pas77, Corollary 5.3.10],  $K[G]$  satisfies a polynomial identity if and only if  $G$  has a subgroup  $A$  such that  $G/A$  is finite and  $A'$  is a finite  $p$ -group.

- A normal subgroup  $H$  of  $G$  generates a nilpotent ideal of  $K[G]$  if and only if  $H$  is a finite  $p$ -group.
- In place of Lemma 2.2, we use the group algebra analogue [Pas97, Lemma 3.2].

□

**Corollary.**  $K[G]$  satisfies a nonmatrix identity if and only if it satisfies some polynomial identity and  $G'$  is a  $p$ -group of finite exponent. □

### 3. SEMIGROUP IDENTITIES

In this section we prove Theorems 1.3, 1.4, 1.7 and 1.8, and justify Examples 1.5 and 1.6. The same notational hypotheses as in the previous section are in force throughout. In addition, the field  $K$  is assumed to be infinite of characteristic  $p > 0$ .

#### Enveloping algebras.

*Proof of Theorem 1.3.* It remains to prove the equivalence of condition (iv) and condition (iii), say. That (iii) implies (iv) is obvious. Conversely, assume that (iv) holds. Replacing  $A$  by the centralizer of  $A'$  in  $A$ , we may assume that  $A$  is a nilpotent restricted ideal of  $L$  containing  $Z(L)$ . But then  $L$  is nilpotent-by-(finite-dimensional and  $p$ -nil), and hence nilpotent by [Sha93].

□

**Proposition 3.1.** *Let  $p = 2$  and let  $D$  be the 2-dimensional restricted Lie algebra over  $K$  generated by  $a, b$  with relations  $[a, b] = a$ ,  $a^2 = 0$ ,  $b^2 = b$ . Then  $\omega(u(D))$  satisfies the semigroup identity*

$$wxyz - wyxz = 0.$$

*Proof.* By Jacobson's theorem,  $\omega(u(D))$  has  $K$ -basis

$$\{a, b, ab\}.$$

Because the given semigroup identity is multilinear, it suffices to show that it holds on basis elements. Observe

$$[a, b] = a, \quad [a, ab] = a^2 = 0, \quad [ab, b] = ab = (1 + b)a,$$

each of which is annihilated by  $a$  on the right and left. The remaining possibilities are:

$$bab = b(1 + b)a = 2ba = 0, \quad b[(1 + b)a]b = 2bab = 0.$$

□

**Proposition 3.2.** *If  $\omega(u(L))$  satisfies an identity of the form  $y^m[x, {}_m y]y^m = 0$  then either  $u(L)$  satisfies the Engel condition or  $p = 2$  and  $L$  contains a restricted subalgebra isomorphic to  $D$ , where  $D$  is as in Proposition 3.1.*



*Proof.* Since  $y^m[x, {}_m y]y^m = 0$  is a nonmatrix identity, it is clear from Theorems 1.1 and 1.3 that  $u(L)$  satisfies the Engel condition as soon as  $L/Z(L)$  is  $p$ -nil of bounded exponent. Because

$$y^m[x, {}_{m+1} y]y^m = [y^m[x, {}_m y]y^m, y] = 0$$

is also an identity for  $\omega(u(L))$ , we may assume that  $m = p^t$  for some  $t$ . Then  $\omega(u(L))$  satisfies the identity

$$y^{p^t}[x, y^{p^t}]y^{p^t} = y^{p^t}[x, {}_{p^t} y]y^{p^t} = 0.$$

Suppose now that  $a, c$  are arbitrary elements in  $L$  and put  $b = c^{p^t}$ . Then

$$[a, b]b^2 - [a, b, b]b = b[a, b]b = 0.$$

Assume, for the moment, that  $p > 2$ . Then it follows easily using Jacobson's theorem that  $[a, b, b] = 0$ ; hence,  $L^{p^{t+1}} \subseteq Z(L)$ , as required.

Assume, finally, that  $p = 2$  and, without loss of generality,  $[a, {}_3 b] \neq 0$  for some  $a, c$  as above. Then, by Jacobson's theorem, there exist  $\lambda, \mu, \nu \in K$  such that  $[a, b, b] = \lambda[a, b] + \mu b + \nu b^2$ . It follows that  $[a, b, b, b] = \lambda[a, b, b]$ , where  $\lambda \neq 0$ . Replacing  $a$  by  $[a, b, b]$  and subsequently  $b$  by  $\lambda^{-1}b$ , we can assume  $[a, b] = a$ . Therefore

$$a(b^2 - b) = [a, b]b^2 - [a, b, b]b = 0,$$

so that  $b^2 = b$ . Clearly now,  $a$  and  $b$  generate a restricted subalgebra of  $L$  isomorphic to  $D$ , as required.  $\square$

*Proof of Theorem 1.7.* According to [RW], if an algebra over an infinite field satisfies a semigroup identity then it satisfies an identity of the form

$$y^m[x, {}_m y]y^m = 0,$$

for some  $m$ . Now combining the preceding proposition with Theorem 1.3 readily yields the result.  $\square$

### Group algebras.

*Proof of Theorem 1.4.* It remains to prove the equivalence of condition (iv) and conditions (i)-(iii). That condition (ii) implies (iv) is clear from Theorem 1.2 since if  $g, h \in G$  then

$$gh^{p^t} - h^{p^t}g = [g, h^{p^t}] = [g, {}_{p^t} h].$$

Assume now that only (iv) holds. Replacing  $A$  by the centralizer of  $A'$  in  $A$ , we may assume that  $A$  is a nilpotent normal subgroup of  $G$  containing  $Z(G)$ . But then  $G/Z(G)$  is a  $p$ -group of finite exponent that is an extension of a nilpotent group by a finite  $p$ -group. Such a group is nilpotent by [Bau59]. Hence,  $G$  itself is nilpotent and (iii) holds.  $\square$

**Proposition 3.3.** *Let  $p = 3$  and let  $G$  be the dihedral group of order 6. Then  $\omega(K[G])$  satisfies the identity  $xyx^2 = y^2xy$ .*

*Proof.* Write  $R = \omega(K[G])$ . We use the presentation  $G = \langle a, b | a^2 = b^3 = abab = 1 \rangle$ . Let  $d = b - b^{-1} \in R$  and let  $e = (1 - a)/2 \in R$ . It is easy to see that  $e^2 = e, d^3 = 0$  and  $d^2$  is central in  $R$ . Furthermore,  $ed = d(1 - e)$ . A  $K$ -basis for  $R$  is  $\{e, d^2, ed^2, d, ed\}$ . Let  $R_0$  be the span of  $e, d^2, ed^2$  and let  $R_1$  be the span of  $d, ed$ . Then  $R = R_0 \dot{+} R_1$  is a superalgebra and the following basic properties hold:

- $R_0$  is commutative;
- $(R_1)^3 = 0$ ;
- $R_0 R_1 R_0 = 0$ .

Consider the given identity in the form  $y[x, y]y = 0$ . Since the identity is linear in  $x$  it suffices to assume that  $x \in R_0$  or  $x \in R_1$ . Fix  $y = s_0 + s_1 \in R_0 \dot{+} R_1$ .

Suppose that  $0 \neq x = r_0 \in R_0$ . Then using the above properties we obtain

$$y[x, y]y = s_0[r_0, s_1]s_1 + s_1[r_0, s_1]s_0.$$

Since  $d^2$  kills all basis elements except  $e$ , and the last expression is homogeneous in  $r_0$  and  $s_0$ , without loss of generality we may assume that  $r_0 = s_0 = e$ . Then the above simplifies to

$$y[x, y]y = e[e, s_1]s_1 + s_1[e, s_1]e = es_1^2 - s_1^2e = 0.$$

Suppose that  $0 \neq x = r_1 \in R_1$ . Then the basic properties above yield

$$y[x, y]y = [r_1, s_1]s_0^2 + s_0[r_1, s_0]s_1 + s_1[r_1, s_0]s_0.$$

Since  $x$  has degree 1 in  $d$  we may again assume that  $s_0 = e$ , which yields

$$y[x, y]y = [r_1, s_1]e + e[r_1, e]s_1 + s_1[r_1, e]e = r_1s_1e - s_1r_1e - er_1s_1 + s_1r_1e = 0.$$

□

**Proposition 3.4.** *If  $\omega(K[G])$  satisfies an identity of the form  $y^m[x, y]y^m = 0$  then either  $K[G]$  satisfies the Engel condition or  $p > 2$  and  $G$  contains a subgroup isomorphic to the dihedral group  $D_{2p}$  of order  $2p$ .*

*Proof.* As above, if  $G/Z(G)$  is a  $p$ -group of bounded exponent, then the assumption on  $\omega(K[G])$  forces  $K[G]$  to satisfy the Engel condition.

Assume  $m = p^t$ . We claim that given  $h \in G$ , either  $h^{p^{t+1}} \in Z(G)$  or  $h^{2p^t} = 1$ . Let  $g, h \in G$ . Substituting  $x = 1 - g, y = 1 - h$  into the polynomial identity yields

$$(1 - h^{p^t})[h^{p^t}, g](1 - h^{p^t}) = (1 - h)^{p^t}[g, (1 - h)^{p^t}](1 - h)^{p^t} = 0.$$

Expanding, we obtain

$$h^{p^t}g - gh^{p^t} - h^{2p^t}g + gh^{2p^t} + h^{2p^t}gh^{p^t} - h^{p^t}gh^{2p^t} = 0.$$

Consider first the case when  $p > 5$ . Then by the linear independence of group elements we have:

$$\{h^{p^t}g, gh^{2p^t}, h^{2p^t}gh^{p^t}\} = \{gh^{p^t}, h^{2p^t}g, h^{p^t}gh^{2p^t}\}.$$

By inspection, it follows that either  $h^{p^t}$  commutes with  $g$  or  $h^{2p^t} = 1$ , as claimed.

Next assume  $p = 3$ . One additional possibility can occur:

$$h^{3^t}g = gh^{2 \cdot 3^t} = h^{2 \cdot 3^t}gh^{3^t}, \quad gh^{3^t} = h^{2 \cdot 3^t}g = h^{3^t}gh^{2 \cdot 3^t}.$$

But then  $h^{3^t}g = h^{2 \cdot 3^t}gh^{3^t}$  implies  $g = h^{3^t}gh^{3^t}$ , so that  $g = (h^{3^t}g)h^{3^t} = (gh^{2 \cdot 3^t})h^{3^t}$  implies that  $h^{3^{t+1}} = 1$ .

Finally, in the  $p = 2$  case the fact that the group element terms above must cancel pairwise eventually shows that  $h^{2^{t+1}}$  commutes with  $g$ , thus establishing the claim for all characteristics. This also completes the proof in the case  $p = 2$ ; henceforth, we assume  $p > 2$ .

Now let  $P$  denote the set of  $p$ -elements in  $G$ . Since  $\omega(K[G])$  satisfies a nonmatrix identity,  $G'$  is a  $p$ -group by Theorem 1.2. It follows immediately that  $P$  is a normal subgroup of  $G$ .

According to the claim above, either  $G/Z(G)$  is a  $p$ -group of finite exponent or  $G$  contains an element  $\sigma$  of order 2. Consider the case that  $\sigma$  centralizes  $P$ . We claim that  $\sigma$  must be central. Indeed, let  $\tau \in G$ . Then since  $G'$  is a  $p$ -group,  $\sigma^\tau = \sigma g$  for some  $p$ -element  $g$ . Thus

$$g^2 = \sigma^2 g^2 = (\sigma g)^2 = (\sigma^\tau)^2 = 1,$$

so that  $g = 1$  since  $p$  is odd.

Now consider the case when  $\sigma$  does not centralize  $P$ . Then there is some  $p$ -element  $g$  in  $G$  such that  $g^\sigma \neq g$ . Put  $h = g^{-1}g^\sigma$ , another  $p$ -element in  $G$ . Then

$$h^\sigma h = (g^{-1}g^\sigma)^\sigma g^{-1}g^\sigma = (g^\sigma)^{-1}gg^{-1}g^\sigma = 1.$$

If the order of  $h$  is  $p^n$  and  $n \geq 2$ , then replacing  $h$  by  $h^{p^{n-1}}$  we may assume  $h$  has order exactly  $p$  and  $h^\sigma = h^{-1}$ . Consequently, the subgroup  $G$  generated by  $\sigma$  and  $h$  is isomorphic to  $D_{2p}$ .  $\square$

*Proof of Theorem 1.8.* According to [RW], if an algebra over an infinite field satisfies a semigroup identity then it satisfies an identity of the form

$$y^m[x, {}_m y]y^m = 0,$$

for some  $m$ . Now combining the preceding proposition with Theorem 1.4 readily yields the result.  $\square$

#### 4. FURTHER COMMENTS

The complete analogy between Theorems 1.1 and 1.2 suggests that a more general result may be possible, perhaps in some class of Hopf algebras containing both examples.

We strongly suspect that  $\omega(K[D_{2p}])$  satisfies a semigroup identity for fields  $K$  of arbitrary prime characteristic  $p$ , but do not have a general proof. The difficulties encountered in even this seemingly simple case lead us to believe that the full characterization of augmentation ideals satisfying a semigroup identity will require considerable additional effort.

## REFERENCES

- [Bau59] G. Baumslag, *Wreath products and  $p$ -groups*, Proc. Camb. Phil. Soc. **55** (1959), 224–231.
- [Bra84] A. Braun, *The nilpotency of the radical in a finitely generated PI ring*, J. Algebra **89** (1984), 375–396.
- [BRT97] Yu. Billig, D. M. Riley, and V. Tasić, *Non-matrix varieties and nil-generated algebras whose units satisfy a group identity*, J. Algebra **190** (1997), 241–252.
- [Jon74] G. A. Jones, *Varieties and simple groups*, J. Austral. Math. Soc. **17** (1974), 163–173.
- [Kem81] A. Kemer, *Nonmatrix varieties*, Algebra and Logic **19** (1981), 157–178.
- [Pas77] D. S. Passman, *The Algebraic Structure of Group Rings*, John Wiley & Sons, 1977.
- [Pas90] D. S. Passman, *Enveloping algebras satisfying a polynomial identity*, J. Algebra **134** (1990), 469–490.
- [Pas97] D. S. Passman, *Group algebras whose units satisfy a group identity, II*, Proc. Amer. Math. Soc. **125** (1997), 657–662.
- [Pet91] V. M. Petrogradski, *On the existence of identity in restricted enveloping algebras*, Mat. Zametki **49** (1991), 84–93.
- [PPS73] I. B. S. Passi, D. S. Passman, and S. K. Sehgal, *Lie solvable group rings*, Canad. J. Math. **25** (1973), 748–757.
- [Ril96] D. M. Riley, *The tensor product of Lie soluble algebras*, Arch. Math. (Basel) **66** (1996), 372–377.
- [Ril97] D. M. Riley, *PI-algebras generated by nilpotent elements of bounded index*, J. Algebra **192** (1997), 1–13.
- [RS93] D. M. Riley and A. Shalev, *The Lie structure of enveloping algebras*, J. Algebra **162** (1993), 46–61.
- [RW] D. M. Riley and M. C. Wilson, *Associative algebras satisfying a semigroup identity*, preprint, 1997.
- [Seh78] S. K. Sehgal, *Topics in Group Rings*, Marcel Dekker, Inc, 1978.
- [Sha93] A. Shalev, *Polynomial identities in graded group rings, restricted Lie algebras and  $p$ -adic analytic groups*, Trans. Amer. Math. Soc. **337** (1993), 451–462.

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