Robust Control for Boiler-Turbine Systems and Control Synthesis for Polynomial Systems

Matthias Krug

A Thesis Submitted in Fulfillment of
the Requirements for the Degree of
Doctor of Philosophy

Department of Electrical and Computer Engineering
The University of Auckland
New Zealand

December 2013

Abstract

In this thesis, a direct approach to improve the control of highly nonlinear, strongly coupled boiler-turbine systems that are commonly found in the power generation environment is introduced. Following the direct approach, more generalized concepts of controlling polynomial systems, a class of nonlinear systems that is superior to linear systems in its adaptability to real life systems in terms of system modelling or approximation of other nonlinearities, is discussed in detail.

The motivation for this research stems from its usefulness for a variety of power generating facilities used around the world. In particular, the implementation of an online model predictive control scheme based on evolutionary computation will be introduced, including an extension to a switching control regime to further increase the overall performance.

The discussions on polynomial system control is based on the lack of a natural extension of linear control strategies to polynomial systems, a difficult problem that cannot be directly addressed by standard convex optimization tools like semidefinite programming. However, new methodologies will be introduced for a variety of polynomial control problems, including H_∞ control for systems with and without polytropic or norm-bounded uncertainties, which lead to an overall less conservative control design. The discussion will include robust H_∞ control procedures for near real-world control problems that are subjects to polytropic and norm-bounded uncertainties for systems with the state and output feedback.

Finally, to demonstrate the effectiveness and advantages of the proposed design methodologies in this thesis, numerical examples are given in each chapter. The simulation results show that the proposed design methodologies can achieve the prescribed performance requirements.

Acknowledgements

Frist and foremost, I would like to take this opportunity to express my most grateful appreciation to my supervisor, Prof. Sing Kiong Nguang, for his enthusism, vision, patience, wisdom and guidance throughout my research journey at the University of Auckland. His experience as an outstanding and well renown researcher make him an ideal role model for any Ph.D student. I was particularly grateful for his continued full and extensive support, encouragement and motivation throughout my Ph.D. Without Prof. Nguang's help, this work could not have been accomplished. I am especially grateful for his refinements, patience and guidance in the publication processes throughout my time at the University of Auckland. For this, I owe him my sincere gratitude.

I would also like to acknowledge the continuous support I have received from my co-supervisor Dr. Akshya Swain, as well as all the advisory committee members, and am grateful for their precious time and insightful comments.

Furthermore, I would like to thank all the support staff at the Department of Electrical and Computer Engineering at the University of Auckland. They have provided me with valuable support throughout my studies, and this thesis would not have been possible without their excellent support.

I would also like to extend my gratitude towards my family in distant Germany. My parents, Martina and Rigobert Krug, offered me the moral and financial support neccessary to undertaken this journey. I would also like to thank my sister Christine Krug, her partner Michael Starzonek, as well as their children Jakob and Svenja for their love and support.

Finally, I would also like to acknowledge the great support I have received from my partner, Bonnie K. Latour, throughout my thesis. It was her love, understanding for long nights and odd work hours, as well as her support in so many other ways that kept me motivated. I could not have done it without you.

Thank you all.

Contents

List of Tables

List of Figures

Notations

The notations used throughout this thesis are in accordance with the research field's standard. $\mathbb{R}$ and $\mathbb{R}^{n\times n}$ denote the set of $n\times 1$ vectors and $n\times n$ matrices, respectively. The superscript $(\cdot)^T$ denotes the transpose of a vector or matrix, and $(*)$ is used to represent the transposed symmetric entries in matrix inequalities. Further, I denotes the identy matrix of appropriate dimensions and $L_2[0,\infty]$ is the space of square summable vector sequences over $[0,\infty]$. The $\|\cdot\|_{[0,\infty]}$ denotes the $L_2[0,\infty]$ norm over $[0,\infty)$ defined as $\|f(x)\|^2_{[0,\infty]} = \int_0^\infty \|f(x)\|^2 dx$. For any matrix Q, the relationships $Q\succ 0$ $(Q\succeq 0)$ are used to describe positive (semi-)definteness of Q, respectively. For simplicity, the time variant expressions for system states, system outputs and system inputs are denoted as x,y,u rather than $x(t),y(t),u(t)$, respectively. The term degree of a polynomial refers to the highest integer exponent of a polynomial in x. For example, the degrees of a linear, quadratic and cubic equations are $1,2,$ and 3, respectively.

THE UNIVERSITY OF AUCKLAND
SCHOOL OF GRADUATE STUDIES

Co-Authorship Form

Graduate Centre
ClockTower – East Wing
22 Princes Street, Auckland
Phone: +64 9 373 7599 ext 81321
Fax: +64 9 373 7610
Email: postgraduate@auckland.ac.nz
www.postgrad.auckland.ac.nz

This form is to accompany the submission of any PhD that contains research reported in published or unpublished co-authored work. **Please include one copy of this form for each co-authored work.** Completed forms should be included in all copies of your thesis submitted for examination and library deposit (including digital deposit), following your thesis Acknowledgements.

Please indicate the chapter/section/pages of this thesis that are extracted from a co-authored work and give the title and publication details or details of submission of the co-authored work.

Chapter 2, based on: J. Wu and M. Krug, and S.K. Nguang and J. Shen and Y.G. Li, H_infinity fuzzy tracking control for boiler-turbine systems, IEEE International Conference on Control and Automation, 2009, pp. 1980-1985.

Nature of contribution by PhD candidate	derived theories, examples, editing
Extent of contribution by PhD candidate (%)	80

CO-AUTHORS

Name	Nature of Contribution
Jie Wu	discussions on theories, writing
S.K. Nguang	PhD supervisor, minor direct contributions
J. Shen	PhD supervisor, minor direct contributions
Y.G. Li	PhD supervisor, minor direct contributions

Certification by Co-Authors

The undersigned hereby certify that:

- the above statement correctly reflects the nature and extent of the PhD candidate's contribution to this work, and the nature of the contribution of each of the co-authors; and
- in cases where the PhD candidate was the lead author of the work that the candidate wrote the text.

Name	Signature	Date
Jie Wu	Jiè Wu	30/05/2013
S.K. Nguang	[illegible]	30/05/2013
J. Shen	Jing Shen	30/05/2013
Y.G. Li	Yiguo Li	30/05/2013
		Click here
		Click here

THE UNIVERSITY OF AUCKLAND
SCHOOL OF GRADUATE STUDIES

Co-Authorship Form

Graduate Centre
ClockTower – East Wing
22 Princes Street, Auckland
Phone: +64 9 373 7599 ext 81321
Fax: +64 9 373 7610
Email: postgraduate@auckland.ac.nz
www.postgrad.auckland.ac.nz

This form is to accompany the submission of any PhD that contains research reported in published or unpublished co-authored work. **Please include one copy of this form for each co-authored work.** Completed forms should be included in all copies of your thesis submitted for examination and library deposit (including digital deposit), following your thesis Acknowledgements.

Please indicate the chapter/section/pages of this thesis that are extracted from a co-authored work and give the title and publication details or details of submission of the co-authored work.

Chapter 2, based on: M. Krug and S.K. Nguang and J. Wu and J. Shen, GA-based model predictive control of boiler-turbine systems, International Journal of Innovative Computing, Information and Control, vol. 6, no. 11, pp. 5237-5248, 2010.

Nature of contribution by PhD candidate	derived theories, examples, lead-author
Extent of contribution by PhD candidate (%)	90

CO-AUTHORS

Name	Nature of Contribution
J. Wu	discussions on theories, editing
S.K. Nguang	PhD supervisor, minor direct contributions
J. Shen	PhD supervisor, minor direct contributions

Certification by Co-Authors

The undersigned hereby certify that:

- the above statement correctly reflects the nature and extent of the PhD candidate's contribution to this work, and the nature of the contribution of each of the co-authors; and
- in cases where the PhD candidate was the lead author of the work that the candidate wrote the text.

Name	Signature	Date
J. Wu	Jiè Wu	30/05/2013
S.K. Nguang	[illegible]	30/05/2013
J. Shen	Jing Shen	30/05/2013
		Click here
		Click here
		Click here

Last updated: 25 March 2013

THE UNIVERSITY OF AUCKLAND
SCHOOL OF GRADUATE STUDIES

Co-Authorship Form

Graduate Centre
ClockTower – East Wing
22 Princes Street, Auckland
Phone: +64 9 373 7599 ext 81321
Fax: +64 9 373 7610
Email: postgraduate@auckland.ac.nz
www.postgrad.auckland.ac.nz

This form is to accompany the submission of any PhD that contains research reported in published or unpublished co-authored work. **Please include one copy of this form for each co-authored work.** Completed forms should be included in all copies of your thesis submitted for examination and library deposit (including digital deposit), following your thesis Acknowledgements.

Please indicate the chapter/section/pages of this thesis that are extracted from a co-authored work and give the title and publication details or details of submission of the co-authored work.

Chapter 2, based on: J. Wu and J. Shen and M. Krug and S.K. Nguang and Y.G. Li, GA-based nonlinear predicitve switching control for a boiler-turbine system, Journal of Control Theory and Applications, vol. 10, no. 1, pp. 100-106, 2012.

Nature of contribution by PhD candidate	derived theories, examples, editing
Extent of contribution by PhD candidate (%)	80

CO-AUTHORS

Name	Nature of Contribution
J. Wu	discussions on theories, writing
J. Shen	PhD supervisor, minor direct contributions
S.K. Nguang	PhD supervisor, minor direct contributions
Y.G. Li	PhD supervisor, minor direct contributions

Certification by Co-Authors

The undersigned hereby certify that:

- the above statement correctly reflects the nature and extent of the PhD candidate's contribution to this work, and the nature of the contribution of each of the co-authors; and
- in cases where the PhD candidate was the lead author of the work that the candidate wrote the text.

Name	Signature	Date
J. Wu	Jiè Wu	30/05/2013
J. Shen	Jing Shen	30/05/2013
S.K. Nguang	[illegible]	30/05/2013
Y.G. Li	Yiguo Li	30/05/2013
		Click here
		Click here

THE UNIVERSITY OF AUCKLAND
SCHOOL OF GRADUATE STUDIES

Co-Authorship Form

Graduate Centre
ClockTower – East Wing
22 Princes Street, Auckland
Phone: +64 9 373 7599 ext 81321
Fax: +64 9 373 7610
Email: postgraduate@auckland.ac.nz
www.postgrad.auckland.ac.nz

This form is to accompany the submission of any PhD that contains research reported in published or unpublished co-authored work. **Please include one copy of this form for each co-authored work.** Completed forms should be included in all copies of your thesis submitted for examination and library deposit (including digital deposit), following your thesis Acknowledgements.

Please indicate the chapter/section/pages of this thesis that are extracted from a co-authored work and give the title and publication details or details of submission of the co-authored work.

Part 2 of the thesis, in particular Chapter 7: S.K. Nguang and S. Saat and M. Krug, Static output feedback controller design for uncertain polynomial systems: An iterative sums of squares approach, IET Control Theory and Applications, vol. 5, no. 9, pp. 1079-1084, 2011

Nature of contribution by PhD candidate	derived theories, examples, editing
Extent of contribution by PhD candidate (%)	80

CO-AUTHORS

Name	Nature of Contribution
S.K. Nguang	PhD supervisor, lead-author
S. Saat	discussion on theories, editing

Certification by Co-Authors

The undersigned hereby certify that:

- the above statement correctly reflects the nature and extent of the PhD candidate’s contribution to this work, and the nature of the contribution of each of the co-authors; and
- in cases where the PhD candidate was the lead author of the work that the candidate wrote the text.

Name	Signature	Date
S. K. Nguang	[signature]	30/05/2013
S. Saat	[signature]	30/05/2013
		Click here
		Click here
		Click here
		Click here

THE UNIVERSITY OF AUCKLAND
SCHOOL OF GRADUATE STUDIES

Co-Authorship Form

Graduate Centre
ClockTower – East Wing
22 Princes Street, Auckland
Phone: +64 9 373 7599 ext 81321
Fax: +64 9 373 7610
Email: postgraduate@auckland.ac.nz
www.postgrad.auckland.ac.nz

This form is to accompany the submission of any PhD that contains research reported in published or unpublished co-authored work. **Please include one copy of this form for each co-authored work**. Completed forms should be included in all copies of your thesis submitted for examination and library deposit (including digital deposit), following your thesis Acknowledgements.

Please indicate the chapter/section/pages of this thesis that are extracted from a co-authored work and give the title and publication details or details of submission of the co-authored work.

Part 2 of the thesis: S. Saat and S.K. Nguang and M.Krug, A nonlinear static output controller design for polynomial systems: An iterative sums of squares approach, Proceedings of International Conference on Mechatronics, 2011, pp. 1-6.

Nature of contribution by PhD candidate	derived theories, examples, editing
Extent of contribution by PhD candidate (%)	80

CO-AUTHORS

Name	Nature of Contribution
S. Saat	discussions on theory, lead author
S.K. Nguang	PhD supervisor, minor direct contributions

Certification by Co-Authors

The undersigned hereby certify that:

- the above statement correctly reflects the nature and extent of the PhD candidate's contribution to this work, and the nature of the contribution of each of the co-authors; and
- in cases where the PhD candidate was the lead author of the work that the candidate wrote the text.

Name	Signature	Date
S. Saat		30/05/2013
S.K. Nguang		30/05/2013
		Click here
		Click here
		Click here
		Click here

THE UNIVERSITY OF AUCKLAND
SCHOOL OF GRADUATE STUDIES

Co-Authorship Form

Graduate Centre
ClockTower – East Wing
22 Princes Street, Auckland
Phone: +64 9 373 7599 ext 81321
Fax: +64 9 373 7610
Email: postgraduate@auckland.ac.nz
www.postgrad.auckland.ac.nz

This form is to accompany the submission of any PhD that contains research reported in published or unpublished co-authored work. **Please include one copy of this form for each co-authored work**. Completed forms should be included in all copies of your thesis submitted for examination and library deposit (including digital deposit), following your thesis Acknowledgements.

Please indicate the chapter/section/pages of this thesis that are extracted from a co-authored work and give the title and publication details or details of submission of the co-authored work.

Part 2 of the thesis: S. Saat and M. Krug and S.K. Nguang, Nonlinear H_infinity static output feedback controller design for polynomial systems: An iterative sums of squares approach, Proceedings of the IEEE Conference on Industrial Electronics and Applications, 2011, pp. 985 - 990.

Nature of contribution by PhD candidate	derived theories, examples, editing
Extent of contribution by PhD candidate (%)	80

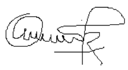
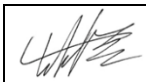
CO-AUTHORS

Name	Nature of Contribution
S. Saat	discussions on theory, lead author
S.K. Nguang	PhD supervisor, minor direct contributions

Certification by Co-Authors

The undersigned hereby certify that:

- the above statement correctly reflects the nature and extent of the PhD candidate's contribution to this work, and the nature of the contribution of each of the co-authors; and
- in cases where the PhD candidate was the lead author of the work that the candidate wrote the text.

Name	Signature	Date
S. Saat		30/05/2013
S.K. Nguang		30/05/2013
		Click here
		Click here
		Click here
		Click here

Last updated: 25 March 2013

THE UNIVERSITY OF AUCKLAND
SCHOOL OF GRADUATE STUDIES

Co-Authorship Form

Graduate Centre
ClockTower – East Wing
22 Princes Street, Auckland
Phone: +64 9 373 7599 ext 81321
Fax: +64 9 373 7610
Email: postgraduate@auckland.ac.nz
www.postgrad.auckland.ac.nz

This form is to accompany the submission of any PhD that contains research reported in published or unpublished co-authored work. **Please include one copy of this form for each co-authored work**. Completed forms should be included in all copies of your thesis submitted for examination and library deposit (including digital deposit), following your thesis Acknowledgements.

Please indicate the chapter/section/pages of this thesis that are extracted from a co-authored work and give the title and publication details or details of submission of the co-authored work.

Part 2 of the thesis: S.K. Nguang and M. Krug and S. Saat, Nonlinear static output feedback control design for polynomial systems: An iterative sums of squares approach, Proceedings of the IEEE Conference on Undustrial Electronics and Applications, 2011, pp. 979 - 984.

Nature of contribution by PhD candidate	derived theories, examples, editing
Extent of contribution by PhD candidate (%)	80

CO-AUTHORS

Name	Nature of Contribution
S.K. Nguang	PhD supervisor, lead author
S. Saat	discussions on theory

Certification by Co-Authors

The undersigned hereby certify that:

- the above statement correctly reflects the nature and extent of the PhD candidate's contribution to this work, and the nature of the contribution of each of the co-authors; and
- in cases where the PhD candidate was the lead author of the work that the candidate wrote the text.

Name	Signature	Date
S.K. Nguang	[signature]	30/05/2013
S. Saat	[signature]	30/05/2013
		Click here
		Click here
		Click here
		Click here

THE UNIVERSITY OF AUCKLAND
SCHOOL OF GRADUATE STUDIES

Co-Authorship Form

Graduate Centre
ClockTower – East Wing
22 Princes Street, Auckland
Phone: +64 9 373 7599 ext 81321
Fax: +64 9 373 7610
Email: postgraduate@auckland.ac.nz
www.postgrad.auckland.ac.nz

This form is to accompany the submission of any PhD that contains research reported in published or unpublished co-authored work. **Please include one copy of this form for each co-authored work.** Completed forms should be included in all copies of your thesis submitted for examination and library deposit (including digital deposit), following your thesis Acknowledgements.

Please indicate the chapter/section/pages of this thesis that are extracted from a co-authored work and give the title and publication details or details of submission of the co-authored work.

Part 2 of the thesis: M. Krug and S. Saat and S.K. Nguang, Nonlinear robust H_infinity static output feedback controller design for parameter dependent polynomial systems: An iterative sum of squares approach, Proceedings of the IEEE Conference on Decision and Control, 2011, pp. 3502 - 3507

Nature of contribution by PhD candidate	derived theories, examples, lead author
Extent of contribution by PhD candidate (%)	90

CO-AUTHORS

Name	Nature of Contribution
S. Saat	discussions on theory, editing
S.K. Nguang	PhD supervisor, minor direct contributions

Certification by Co-Authors

The undersigned hereby certify that:

- the above statement correctly reflects the nature and extent of the PhD candidate's contribution to this work, and the nature of the contribution of each of the co-authors; and
- in cases where the PhD candidate was the lead author of the work that the candidate wrote the text.

Name	Signature	Date
S. Saat		30/05/2013
S.K. Nguang		30/05/2013
		Click here
		Click here
		Click here
		Click here

THE UNIVERSITY OF AUCKLAND
SCHOOL OF GRADUATE STUDIES

Co-Authorship Form

Graduate Centre
ClockTower – East Wing
22 Princes Street, Auckland
Phone: +64 9 373 7599 ext 81321
Fax: +64 9 373 7610
Email: postgraduate@auckland.ac.nz
www.postgrad.auckland.ac.nz

This form is to accompany the submission of any PhD that contains research reported in published or unpublished co-authored work. **Please include one copy of this form for each co-authored work**. Completed forms should be included in all copies of your thesis submitted for examination and library deposit (including digital deposit), following your thesis Acknowledgements.

Please indicate the chapter/section/pages of this thesis that are extracted from a co-authored work and give the title and publication details or details of submission of the co-authored work.

Part 2 of the thesis: M. Krug and S. Saat and S.K. Nguang, Robust H_infinity static output feedback controller design for parameter dependent polynomial systems: An iterative sums of squares approach, Journal of the Franklin Institute, vol. 350, no. 2, pp. 318 - 330, 2013.

Nature of contribution by PhD candidate	derived theories, examples, lead author
Extent of contribution by PhD candidate (%)	85

CO-AUTHORS

Name	Nature of Contribution
S. Saat	discussions on theory, editing
S.K. Nguang	PhD supervisor, editing

Certification by Co-Authors

The undersigned hereby certify that:

- the above statement correctly reflects the nature and extent of the PhD candidate's contribution to this work, and the nature of the contribution of each of the co-authors; and
- in cases where the PhD candidate was the lead author of the work that the candidate wrote the text.

Name	Signature	Date
S. Saat		30/05/2013
S.K. Nguang		30/05/2013
		Click here
		Click here
		Click here
		Click here

Chapter 1

Introduction

In this chapter, an introduction to this thesis on robust control for boiler-turbine systems and the control synthesis for polynomial systems will be provided. After a short discussion on general nonlinear systems, the two main parts of the thesis will be introduced: In the first part, an introduction to the control problem of boiler-turbine systems will be provided; the second part of the introduction discusses basic concepts on controlling polynomial systems.

Both parts include an overview of existing approaches in the literature dealing with the particular difficulties of both problems. This discussion leads to the motivation for this thesis and its contributions to the research community. The Introduction to this thesis is concluded with the outline of the remainder of this thesis, highlighting the main contributions of each chapter.

1.1 Nonlinear System Control

Control engineering deals with the control of systems. A system in this context is traditionally associated with a model of a real physical environment that can be of linear or nonlinear nature, expressed in the form of differential or difference equations that are based on physical laws governing the dynamics or measurements. In the past, major research was undertaken in the field of linear control.

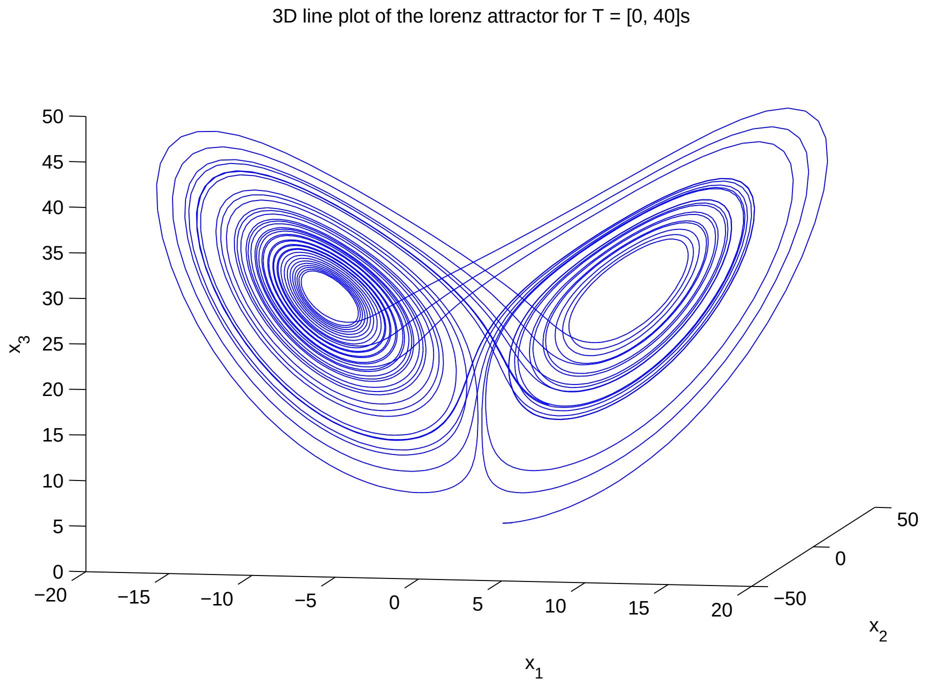


Figure 1.1: Line plot of the Lorenz Attractor

However, physical systems are inherently nonlinear [1]. Linear control techniques can only be applied within a very limited range of operation at best, and it has not been possible to generalize linear control theory to nonlinear systems [2]. Furthermore, extensive tests are needed to verify the adaptability of linear controllers to nonlinear systems and often result in tedious redesigns to meet the control objectives.

In general, the system description of a nonlinear system can be described as

$$\left.\begin{aligned} \dot{x}(t) &= f(x(t),u(t)), \\ y(t) &= g(x(t),u(t)), \end{aligned}\right\} \tag{1.1}$$

where $f(\cdot)$ and $g(\cdot)$ are nonlinear function of the state $x(t)$ and the input $u(t)$ and $y(t)$ is the measured output. For example, the system equation for a simplified mathematical model for atmospheric convection derived by Edward Lorenz in 1963 in the form of

$$\left.\begin{aligned}\dot{x} &= \sigma(y-x),\\ \dot{y} &= x\rho - xz - y,\\ \dot{z} &= xy - \beta z,\end{aligned}\right\} \tag{1.2}$$

where σ is the Prandtl number, ρ is the Rayleigh number, and β is a geometric factor [3]. The system is nonlinear due to the product of its states in the terms xz and xy and cannot be controlled directly using linear control theory. Further, the system exhibits chaotic behaviour for a range of parameters and is often referred to as the *Lorenz Attractor*. A 3D line plot depicts this behaviour in Figure 1.1 for $\sigma = 10, \rho = 28, \beta = \frac{8}{3}$.

A common approach to deal with these difficulties and to find a controller for a wider range of operation is the application of advanced approximation techniques such as fuzzy control or neural networks [2], as well as the implementation of online model predictive control procedures. The complexity of controllers based on such techniques increases rapidly with an increase in nonlinearity or an increase in the range of operation, and often results in very complex controller designs. Furthermore, these controllers aim at controlling an approximation of the system rather than a system itself, and it is thus not possible to guarantee performance requirements such as stability for the nonlinear system.

Due to the complexity and wide variety of nonlinear systems, it is customary for research to aim at solving a specific control problem rather than taking a more general approach.

1.2 Boiler-Turbine Systems

Boiler-turbine systems are commonly used energy conversion devices that consist of a steam boiler and a turbine [4]. It's purpose is to transform chemical energy to thermal energy, which in turn can be used to generate electricity [5]. Their popularity in the power generation field is due to their capability to meet varying power demands much faster than traditional header systems [6].

Traditionally, the following requirements are posed on a typical boiler-turbine control system [7, 8]:

1. The electric power output must meet the load demand
2. The drum pressure must be maintained within some system tolerances despite the load variations
3. The water level in the steam drum of the boiler must be maintained at a desired level to prevent overheating or flooding
4. The steam temperature must be maintained at a desired level to prevent overheating or leakage of wet steam to the turbines
5. Input and system constraints have to be met at all times

Boiler-turbine systems can be modeled as a strongly coupled multiple-input multiple-output (MIMO) nonlinear system. To capture the system performance better, various constraints on inputs, slew rates of the inputs and the system outputs have to be considered. This strong coupling and the constraints on the system inputs lead to an overall moderately slow system response compared to many other control systems. Therefore, classical control schemes can only be applied in a very limited manner with a very high degree of customization as precautions have to be undertaken to ensure the overall system stability within the given operating parameters.

1.2.1 Boiler-Turbine Model

Throughout the discussions on the control of boiler-turbine systems, the model of a 160MW oil-fired electrical power plant model of a drum type boiler and a turbine will be considered. The model is based on the P16/G16 at the Sydvenska Kraft AB plant in Malmö, Sweden [9].

The boiler dynamic model as in (1.3) is the result of both physical and empirical methods based on data acquired from a series of experiments and identifications which capture all the relevant characteristics of the process.

$$
\begin{aligned}
\dot{x_1}(t) &= -0.0018u_2(t)x_1^{9/8}(t) + 0.9u_1(t) - 0.15u_3(t) + 0.01w_1(t),\\
\dot{x_2}(t) &= (0.073u_2(t) - 0.016)x_1^{9/8}(t) - 0.1x_2(t) + 0.01w_2(t),\\
\dot{x_3}(t) &= [141u_3(t) - (1.1u_2(t) - 0.19)x_1(t)]/85 + 0.01w_3(t),\\
y_1(t) &= x_1(t),\\
y_2(t) &= x_2(t),\\
y_3(t) &= 0.05(0.1307x_3(t) + 100a_{cs} + q_e/9 - 67.975).
\end{aligned} \tag{1.3}
$$

Here, the inputs $u_1(t)$, $u_2(t)$ and $u_3(t)$ are the valve positions for fuel flow, steam control and feedwater flow, respectively. The state variables $x_1(t), x_2(t)$ and $x_3(t)$ are the drum pressure (kg/cm^2), electric output (MW) and fluid density (kg/cm^3), respectively. w_1, w_2 and w_3 are used to capture process disturbances and uncertainties. The output $y_3(t)$ is the drum water level deviation (m). a_{cs} and q_e are steam quantity and evaporation rate (kg/s), respectively, and are given as follows:

$$
\begin{aligned}
a_{cs} &= \frac{(1 - 0.001538x_3(t))(0.8x_1(t) - 25.6)}{x_3(t)(1.0394 - 0.0012304x_1(t))},\\
q_e &= (0.854u_2(t) - 0.147)x_1(t) + 45.59u_1(t) - 2.514u_3(t) - 2.096.
\end{aligned} \tag{1.4}
$$

The control inputs are subject to magnitude and rate saturations as follows:

$$
\begin{aligned}
0 \le u_1(t), u_2(t), u_3(t) &\le 1,\\
-0.007 \le \dot{u_1}(t) &\le 0.007,\\
-2 \le \dot{u_2}(t) &\le 0.02,\\
-0.05 \le \dot{u_3}(t) &\le 0.05.
\end{aligned} \tag{1.5}
$$

Table 1.1: Typical operating points of a boiler-turbine system[10]

	#1	#2	#3	#4	#5	#6	#7
x_1^0	75.60	86.40	97.20	108	118.8	129.6	140.4
x_2^0	15.27	36.65	50.52	66.65	85.06	105.8	128.9
x_3^0	299.6	342.4	385.2	428	470.8	513.6	556.4
u_1^0	0.156	0.209	0.271	0.34	0.418	0.505	0.6
u_2^0	0.483	0.552	0.621	0.69	0.759	0.828	0.897
u_3^0	0.183	0.256	0.340	0.433	0.543	0.663	0.793
y_3^0	-0.97	-0.65	-0.32	0	0.32	0.64	0.98

Some typical operating points of the boiler-turbine model (1.3) are tabulated in Table 1.1.

1.2.2 Recent Work on Boiler-Turbine Systems

This thesis presents work on the implementation of refined traditional as well as modern and alternative control techniques for boiler-turbine systems. The novel approach and its subsequent extension are presented in Chapter 2 and are based on online model predictive control (MPC) that use Genetic Algorithms (GAs) to optimize the complex control problem subject to a variety of nonlinear constraints.

As boiler-turbine units are popular modules in modern power generation, considerable research has been undertaken, see for example [10] [7, 8, 11, 12, 13, 14, 15, 16, 17, 18, 19] and references therein. The main approaches and techniques used can be summarized as follows

1. **Approaches Based on Linear Control Theory:** Linear control approaches show an overall good system response for the boiler-turbine units as long as the change of operation mode is sufficiently small [10, 15, 16]. The authors assume that only limited changes in the operation of the boiler-turbine unit are to be expected in normal operations and approximate the nonlinear system around appropriate operating points. Unfortunately, these restrictions are very limiting and violations result in a

slow and oscillatory system response. Also, the direct implementation of the system constraints (1.5) is generally not possible.

2. **Gain Scheduling:** In [11], the authors propose a gain scheduling ℓ_1 optimal controller. The design approach relies on a transformation of the nonlinear plant dynamics to a linear parameter varying form. The resulting control structure is augmented to address the common problems encountered with large changes in the reference signals. In [19], a design for a fuzzy gain-scheduling model predictive controller is presented. To address the nonlinearities of the system, a global fuzzy model for the boiler-turbine unit is derived that is consequently used to control the overall system.

3. **Autoregressive Moving Average Control:** The implementation of self-organizing fuzzy logic controllers has been presented in [13, 17]. The autoregressive moving average control approach is based on an online implementation without the use of a mathematical system model. The approach is based on an online generation of the plant rules that are stored and updated on a regular basis. Therefore and in contrast to traditional fuzzy control schemes, no expert knowledge is required to derive a suitable fuzzy rule set. Unfortunately, this approach also suffers from similar problems when the changes in the reference signal are too severe.

4. **Other Artificial Intelligence Approaches:** As the control problem of boiler-turbine systems is highly nonlinear and a hard problem for traditional control approaches, recent research suggests the use of artificial intelligence approaches. For example, [14] suggests to use Genetic Algorithms to design a more suitable PI controller compared to previous results. In [18], an approach using neural network inverse control to realize a decoupling of the nonlinear system. A single neuron PID controller is designed for the decoupled system that performs well as long as the change in reference signals is limited.

By far the most common underlying approach is to use linearizations of the nonlinear system and the assumption that the change of reference signal is always performed gradually. However, this assumption cannot be guaranteed in practice and thus is a major drawback.

1.3 Polynomial Systems

1.3.1 Polynomial Systems

A variety of nonlinear systems can be exactly represented by polynomial systems, a superset to linear systems. One such example is the Lorenz Attractor [3]. Compared to linear systems, polynomial systems offer superior approximations characteristics of other nonlinear system behaviour. It is therefore understandable that this class of nonlinearity has attracted considerable attention from researchers around the world, in particular in the areas of stability analysis and controller synthesis [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34].

In a polynomial system, the functions $f(x,u)$ and $g(x,u)$ in (1.1) are of polynomial nature in x and u. More precisely, the systems under consideration in this thesis can be described in terms of a state-depended linear-like form as

$$\left.\begin{aligned} \dot{x} =& A(x)x + B(x)u, \\ y =& C(x)x, \end{aligned}\right\} \tag{1.6}$$

with $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector and y is the vector of measured output. $A(x), B(x)$ and $C(x)$ are the polynomial system matrices of appropriate dimensions. It should be noted that a nonlinear system in the form of (1.1) may have more than one representation in the state-depended linear-like form (1.6). For example, the Lorenz System

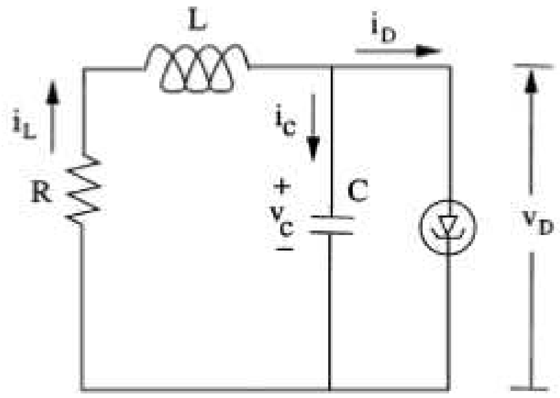


Figure 1.2: Tunnel diode [35]

(1.2) with an input in $\dot{x}_1$ can be represented as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & -x_1 \\ x_2 & 0 & -\beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u, \tag{1.7}$$

as well as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho - x_3 & -1 & 0 \\ 0 & x_1 & -\beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u. \tag{1.8}$$

Another example of a polynomial system is the differential equation of a tunnel diode circuit in Figure 1.2 with polynomial differential equation

$$\begin{bmatrix} C\dot{x}_1 \\ L\dot{x}_2 \end{bmatrix} = \begin{bmatrix} -0.002 - 0.01x_1^2 & 1 \\ -1 & R \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{1.9}$$

where C is the value for the capacitor, L is the value for the inductance, and R is the resistor value.

1.3.2 Recent Work on Polynomial Systems

This thesis presents work on the controller synthesis of polynomial systems. The work is build on Linear Matrix Inequalities (LMIs) and Sum of Squares (SOS) decompositions. As the field of approaches to nonlinear control is almost as broad as the research topic itself, the overview presented here will focuses on related research. A brief description on some of the underlying techniques used for the control of polynomial systems will be presented in a later part of this chapter.

There has been considerable research on controller synthesis and stabilization of polynomial systems in the past, for example see [20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 36, 33] and references therein. The main approaches and techniques used can be summarized as follows

1. **Moments and Sum of Squares (SOS):** The moment problem is the dual to the problem of non-negative polynomials and Hilbert's 17th problem on the representation of non-negative polynomials [37]. This approach has been used in [33] to find a less conservative solution to the global primal/dual problem to show non-negativity using semi-definite programming (SDP). For an in depth introduction to SDP, see [38]. To obtain a convex problem, this algorithm is based on the Hermite stability criterion rather than the Lyapunov stability theorem. This allows the decoupling of the controller and Lyapunov function to obtain convex solvability conditions through a hierarchy of convex LMI relaxations. The authors have pointed out that this approach is prone to numerical errors from the large number of imposed constraints in the resulting formulation.

2. **Dissipation inequalities and SOS:** Control theories for nonlinear systems based on the theory of dissipative energy are known to be one of the successful methods of analysing nonlinear systems [39]. It is based on the mathematical formulation of dissipation inequalities, an approach that reduces the possibly large number of differ-

ential equations that describe the system to a reasonably small number of algebraic inequalities, resulting in a less complex problem. This approach has been used in combination with SOS programming in [32]. The system of interests is represented as a descriptor that is based on polynomial equations. Affine dissipation inequalities are obtained that are in turn solved by SOS. Unfortunately, there is no unique process to obtain the necessary affine dissipation inequalities, which ultimately limits this approach to a subset of problems.

3. **Semi-tensor products:** This approach allows the consideration of general polynomial systems without any homogeneous assumption. It is based on the theory of semi-tensors, which represent an extension of the conventional matrix product that has a matching rows/columns requirement. A brief overview can be obtained in [31] and the references therein. The presented algorithm is based on a positive definite Lyapunov function and it's negative definite derivative along the system trajectories. These conditions are presented as linear algebraic equations and are suitable to verify a candidate solution. Unfortunately, it seems that the sufficient condition is a very loose condition and is not clear how a candidate solution is to be obtained.

4. **Kronecker products and LMIs:** A sufficient condition for the existence of a controller is given in the form of LMIs based on a Kronecker product decomposition of the system equations, see for example [36, 27]. The proposed algorithm can be applied to higher order polynomial systems, which is the main advantage of this approach and makes it stand out from the other common approaches that are limited in their usability for higher order system.

5. **Fuzzy control methods and SOS:** Linear TS fuzzy control approaches have been shown to be an effective tool to control nonlinear systems within a predefined space. In [28], a SOS based approach that allows for higher order Lyapunov functions has been presented, hence representing a less restrictive and generally less conserva-

tive approach than previously available TS fuzzy methods that were solely based on quadratic Lyapunov functions. Sufficient conditions for the existence of the Lyapunov function as well as the controller are given in the form of polynomial matrix inequalities that can be implemented using SOS. In [22], a polynomial fuzzy model has been paired with a polynomial fuzzy controller. The authors investigate how (im)prefect premise matching, i.e. fuzzy model and fuzzy controller (do not) share the same premise variable membership function, influence the Lyapunov stability test as well as the control synthesis, respectively. Yet another extension is presented in [24], where polynomial fuzzy control is investigated for static output feedback. The application of polynomial fuzzy control for a two-link robot arm has been outlined in [23].

6. **Localized control:** Similarly, to the research in global control, improvements have been made in the field of localized control. Generally speaking, local controllers often provide better solutions than global controllers for the same system. For example, [29] proposes a rational Lyapunov function approach that shows that it is possible to embed the domain of attraction into the region outlined by the nonlinear vector field as long as the variation in the states is bounded, resulting in polynomial matrix inequalities. Even though an extension to rational Lyapunov functions has been introduced, the results still have to be considered rather conservatively due to the coupling between the system and Lyapunov matrices. To reduce this effect, a slack variable matrix that decouples the Lyapunov and system matrices has been introduced in [25], resulting in the parametrization of the resulting controller. They show that the results can be readily extended to robust control of uncertain polynomial system.

By far the most common underlying principal for the control of polynomial systems is a stability criterion based on Lyapunov's second method for stability. This will be discussed in detail in the next section.

Controller synthesis for polynomial systems based on Lyapunov's second stability criterion using SOS decompositions

The stability theorems developed by the Russian mathematician Lyapunov are widely regarded as some of the most fundamental in modern control theory. An English translation of his original publication, as well as his biography and bibliography can be found in [40]. Some of the reasons these more than 100 year old theories remain popular in modern control system theory is their general adaptability and simplicity. His famous theorems were originally intended to be used for stability analysis, have however, become equally important for modern controller design synthesis [41]. For a dynamic system $\dot{x} = f(x)$ with an equilibrium at $x = 0$, the theorem can be summarized as follows. Consider a function $V(x) : \mathbb{R}^n \to \mathbb{R}$ such that $V(x)$ is positive definite for $x \neq 0$ and $V(x)|_{x=0} = 0$. If the time derivative of $V(x)$ along the system trajectories of $f(x)$ is negative semidefinite, then $f(x)$ is asymptotically stable:

$$\begin{aligned} &V(x) > 0 \text{ for } \forall x \neq 0, \quad V(x=0) = 0, \\ &\dot{V}(x) = \frac{dV(x)}{dt} = \frac{dV(x)}{dx}\frac{dx}{dt} \leq 0. \end{aligned} \tag{1.10}$$

To check stability for a linear system of the form $\dot{x} = Ax$ amounts to finding a symmetric positive definite matrix P such that $A^T P + PA \preceq 0$, where the Lyapunov function is $V(x) = x^T Px$ [42].

The problem of finding a Lyapunov function candidate and solving (1.10) is in essence a polynomial nonnegativity problem. The idea to to use SOS decomposition of polynomials on control problems like this has been introduced about a decade ago in [43]. It has been shown that this approach allows a more efficient system analysis, and has since been widely adapted in a variety of control applications. The benefits of applying the SOS decomposition algorithm to linear systems with quadratic Lyapunov functions is only marginal, and in many cases experienced researchers can construct a suitable Lyapunov function candidate manually or use traditional methods based on linear matrix inequalities (LMIs). This changes dramatically for polynomial vector fields $f(x)$ or higher order polynomial Lyapunov function candidates $V(x)$. Feasibility of such problems can be NP hard to test[44]. Using a SOS decomposition as a relaxation of (1.10), however, allows for efficient computation using the SOS and SDP framework [43].

Since it's first application to control problems, several natural extension of SOS decomposition have been been presented. In [45, 46], a linear-like form is used for a polynomial system and the construction of a Lyapunov function candidate is proposed in the form of sufficient nonnegativity conditions of polynomial vector fields that can be solved using SOS decompositions and SDPs. To avoid nonconvex terms, some conditions on the way the matrix P is constructed are imposed, in particular that only states which dynamics aren't directly affected by the control input may appear in the Lyapunov matrix, i.e. that the input matrix $B(x)$ has some zero rows. These conditions impose some conservatism in the design process.

To overcome these conditions, the authors of [30] introduce an additional matrix variable that allows the decoupling of the system and Lyapunov matrices. In theory, this allows for a closer approximation of the nonnegativity problem. However, to pose a tractable problem, an upper bound has to be imposed on the matrix variable, which in turn leads to another source of conservatism.

Besides classical state feedback control for polynomial system, a lot of attention has been given to the problem of static output feedback control. In [30], the authors use a state-depended linear-like system description, and, similarly to the state feedback case, assume that the Lyapunov function only depends on the states that are not directly effected by the feedback controller, thus introducing conservatism to the design approach.

Sum of Squares Decomposition

In this section, a brief outline of the concepts of SOS decompositions will be given. For a more elaborate discussion see [43].

Due to the importance of the stability theorem introduced by Lyapunov for a wide variety of control problems, nonnegative multivariate polynomials are of central concern. To show that a multivariate polynomial $F(x)$ is always positive, it is obvious that it needs to be a polynomial of even degree. Formally, one is interested to show that

$$F(x_1, \cdots, x_n) \geq 0, \quad x_1, \cdots, x_n \in \mathbb{R} \tag{1.11}$$

holds for any choice of x.

A simple, yet effective way to show that a polynomial of form (1.11) is always nonnegative is the existence of a SOS decomposition of $F(x)$ as

$$F(x) = \sum_i f_i^2(x). \tag{1.12}$$

If such a decomposition exists, it is clear that each squared polynomial term is nonnegative everywhere, thus their sum must also be nonnegative. The set of SOS polynomials in n variables is a convex cone, and it can be shown that this convex cone is proper [47]. If a decomposition of $F(x)$ in the form above can be obtained, it is clear that $F(x) \geq 0, \forall x \in \mathbb{R}^n$. The converse, however, is generally not true. This problem has been studied by Hilbert more than a century ago.

Proposition 1.3.1 *[43] Let $F(x)$ be a polynomial in $x \in \mathbb{R}^n$ of degree $2d$. Let $Z(x)$ be a column vector which entries are all monomials in x with degree $\leq d$. Then, $F(x)$ is said to be SOS if and only if there exists a positive semidefinite matrix Q such that*

$$F(x) = Z(x)^T Q Z(x), \quad Q \succeq 0, \tag{1.13}$$

Since Q is positive semidefinite, we can apply Cholesky decomposition to (1.13) that yields

$$F(x) = Z(x)^T QZ(x) = Z(x)^T (L^T L)Z(x) = \|LZ(x)\|^2 = \sum_i (LZ(x))_i^2, \tag{1.14}$$

thus certifying nonnegativity. However, since the variables in $Z(x)$ are not independent, (1.14) generally does not yield a unique solution.

In general, determining nonnegativity for $F(x)$ for $\deg(F) \geq 4$ is a NP hard problem [48, 44]. Proposition 1.3.1 provides a relaxation to formulate the nonnegativity conditions on polynomials that is computationally tractable. A more general formulation of this transformation for symmetric polynomial matrices is given in the following proposition.

Proposition 1.3.2 *[45] Let $F(x)$ be an $N \times N$ symmetric polynomial matrix of degree $2d$ in $x \in \mathbb{R}^n$. Furthermore, let $Z(x)$ be a column vector whose entries are all monomials in x with a degree no greater than d, and consider the following conditions*

(1) $F(x) \succeq 0$ for all $x \in \mathbb{R}^n$;

(2) $v^T F(x)v$ is a SOS, where $v \in \mathbb{R}^N$;

(3) There exists a positive semidefinite matrix Q such that $v^T F(x)v = (v \otimes Z(x))^T Q (v \otimes Z(x))$, with $\otimes$ denoting the Kronecker product.

It is clear that from this if follows that $F(x)$ being a SOS implies that $F(x) \geq 0$. The converse, however, is generally not true. Furthermore, statement (2) and (3) are equivalent.

Consider the following example from [43]

Example 1.3.1 *Consider the quartic form in two variables described below, and define* $z_1 := x_1^2, z_2 := x_2^2, z_3 = x_1x_2$*:*

$$F(x_1,x_2) = 2x_1^4 2x_1^3 x_2 - x_1x_2 + 5x_2^4$$
$$= \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix}^T \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix}$$
$$= \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix}^T \begin{bmatrix} 2 & -\lambda & 1 \\ -\lambda & 5 & 0 \\ 1 & 0 & -1+2\lambda \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix}$$

Take for instance $\lambda = 3$*. In this case,*

$$Q = L^T L, \quad L = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix},$$

and therefore we have the sum of squares decomposition

$$F(x_1,x_2) = \frac{1}{2}\left((2x_1^2 - 3x_2^2 + x_1x_2)^2 + (x_2^2 + 3x_1x_2)^2\right).$$

The problem of finding a suitable Q can be cast as a SDP and solved efficiently [43].

It should be noted that it is not always possible to find a SOS decomposition and thus a certificate for nonnegativity for nonnegative polynomials. For instance, a simple counter example is the Motzkin form (here, for $n = 3$)

$$M(x,y,z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2. \tag{1.15}$$

Nonnegativity can be easily shown using the arithmetic-geometric inequality, however there does not exist a SOS decomposition. This can be shown using standard algebraic manipulations and is showcased in [49]. The gap between polynomial nonnegativity and showing that a polynomial has a SOS decomposition cannot be clearly defined, however recent research suggests that the gap is small [34].

Using SOS decomposition relaxes the NP-hard problem of showing nonnegativity of a polynomial $F(x)$ into a computationally tractable problem that can be solved efficiently using SDPs in at worst polynomial time. The term NP-hard is frequently used in computational complexity theory and refers to a class of problems that are non-deterministic polynomial-time hard, i.e. problems that are at least as hard as the hardest problems in NP. This does, however imply that SOS decompositions are inherently limited to reasonably small systems with reasonably small maximum degrees.

There are a variety of toolboxes available that readily transform a SOS problem to a SDP, solve the SDP, and return the results in a form suitable to the original problem. Most available solvers have been developed by research teams around the world and are available free of charge on the Internet. Some of the more common solvers are SOSTOOLS[50], YALMIP [51], CVX [52, 53], and GLoptPoly [54]. It is noteworthy that of the above mentioned software packages only SOSTOOLS is specifically designed for and limited to SOS decompositions, whereas the other packages allow to address a wider variety of optimization problems.

1.4 Research Motivation

The study of highly nonlinear and strongly coupled boiler-turbine systems poses an interesting problem of immediate and direct concern to power facilities around the world. It is of utmost importance to these facilities to guarantee that their systems run within given operating parameters at all times and that they quickly adapt to changes in load demand. Therefore, an improvement over existing control schemes and algorithms is always a concern.

The study of polynomial systems is a natural extension to the study of linear systems. In its most basic form with polynomials of a maximum degree of 1 polynomial systems simplify to linear systems. System control is, generally speaking, concerned with the control of real life dynamic systems. Most of these systems are inherently nonlinear and it has been customary to approximate these nonlinearities with system dynamics that can be easier addressed mathematically. Polynomial approximations of complex system dynamics can be designed to represent the real dynamics closer than standards linear approximations. Further, there is a variety of systems that come naturally in polynomial forms like biological systems, mechatronics or laser physics [55, 56]. Other examples of polynomial systems are Lorenz systems, Brockett integrators, Van der Pol oscillators, Artstein Circle, or MY conjecture, see [32] for further discussion. Even though a lot of research has been undertaken in the field of polynomial control, no general solution has been obtained to date.

Results using higher order Lyapunov function candidates to control polynomial systems has been shown to produce better results than was previously possible with the restriction to quadratic Lyapunov functions, see for example [46, 45, 57, 30]. Unfortunately, it is not possible to directly apply the theory of Lyapunov stability to polynomial systems, as this leads to a nonconvex problem that cannot be solved with SDPs. Therefore, most research relies on restrictions on the design parameters, in particular it is assumed that Lyapunov matrix only depends on the states which rows in the input matrix are zero. This does, however, add some conservatism to the design and opens the door to improve the design process in this respect as will be outlined in the following chapters. In particular, an iterative algorithm will be introduced that allows for general Lyapunov functions.

As polynomial system control is aimed at real life systems or close polynomial approximation of these systems, it is therefore important to investigate how to ensure that the obtained results are robust.

1.5 Contribution of the thesis

The focus of this thesis is to establish new methodologies for boiler-turbine system control as well as polynomial systems control.

The main contributions with respect to the problem of controlling the highly nonlinear and strongly coupled boiler-turbine system is the application of a novel H_∞ fuzzy reference tracking controller that is superior to previous results in terms of tracking a desired system trajectory. Furthermore, the integration of a novel online model predictive control scheme for boiler-turbine systems is presented. By incorporating an evolutionary computation approach like genetic algorithms, it is possible to overcome the inherent difficulties that more traditional control approaches have with respect to highly nonlinear systems. This stochastic artificial intelligence approach does not rely on gradient methods to find an optimal solution to a control problem, nor does it require that the search space is in any way convex. Furthermore, it is also capable to deal with unusual system behaviour and operating points without requiring a complete redesign of the control structure.

The main contributions with respect to the polynomial system control problem is the implementation of a novel iterative sum of squares approach to a variety of control problems. It derivation of convex stability criteria for polynomial systems is a hard problem, an in general requires that certain assumptions on the form of the Lyapunov function, the form of the controller, or the system matrices are met. These restrictions can be overcome with the proposed relaxation of the control problems and thus has to be considered an improvement over previous results.

Within the framework of an iterative sum of squares approach, novel methodologies for designing robust nonlinear controllers in the presence of polytropic or norm-bounded uncertainties are presented.

To demonstrate the effectiveness and problem solving capabilities, some numerical examples are given. Where applicable, the simulation results also outline how the presented methodologies can achieve the prescribed performance indices.

1.6 Thesis outline

The remainder of this thesis is organized as follows.

Part I is concerned with the control design for boiler-turbine systems. In particular,

Chapter 2 describes how online model predictive control can be used to achieve superior tracking performance for the highly nonlinear and strongly coupled boiler-turbine system. In particular, the focus of Chapter 2 is on the design process of a genetic algorithm to solve the optimization problem arising from the model predictive control approach. Furthermore, the extension of the proposed algorithm to a adaptive switching control law to take advantage of the strengths of Receding Horizon Control as well as the fast settling capabilities of H_∞ fuzzy tracking control in the presence of small deviations from the reference signal is presented. Simulation results are provided to showcase the overall performance.

Part II is concerned with the control design for polynomial systems. In particular,

Chapter 3 describes a nonlinear feedback controller for polynomial systems. In this chapter, a problem relaxation in terms of solvability conditions of polynomial matrix inequality is introduced and solved by an interactive sum of squares decomposition algorithm. The direct extension of the results to systems with polytropic uncertainty is outlined before the chapter concludes with a numerical example.

Chapter 4 outlines how the control problem of polynomial systems with norm-bounded uncertainties can be addressed using a iterative sum of squares approach. This is achieved by using an upper bound technique for on the uncertainties. A numerical example is given to illustrate the approach.

Chapter 5 presents the design of a robust H_∞ state feedback control for polynomial systems. The requirements for this control problem are twofold: The stability of the system has to be insured while also having to guarantee that the H_∞ performance criterion is guaranteed. The implementation of the iterative sum of squares approach is first outlined for a single system and then subsequently extended to polynomial systems with polytropic uncertainties. A numerical example is provided to showcase the validity of the approach.

Chapter 6 deals with the problem of robust nonlinear H_∞ state feedback control for polynomial systems in the presence of norm-bounded uncertainties. This can be seen as the superposition of the design requirements of the nonlinear H_∞ control problem described in Chapter 5 and the robust control problem from Chapter 4. The effectiveness of the approach is showcased in a numerical example

The previous chapters have assumed that all system states are available for a state feedback law. This assumption is, however, not true in many cases. Therefore, **Chapter 7** discusses the extension of the robust control problem to the generalized output feedback case. The results are immediately extended to the application to output feedback cases with polytropic uncertainties. A numerical example is provided to show the effectiveness of the presented methodology.

Chapter 7 is the extension of the results from Chapter 6 to the output feedback case, and the iterative sum of squares approach is applied to the problem of robust H_∞ output feedback control in the presence of norm-bounded uncertainties. A numerical examples showcases the effectiveness of the approach.

In **Chapter 8**, the problem of robust nonlinear H_∞ output feedback control for polynomial systems in the presence of norm-bounded uncertainties is presented. This approach represents the superposition of the design requirements of the nonlinear H_∞ output feedback problem described in Chapter 7 and the robust output feedback control problem outlined in Chapter 7. To show the validity of the presented approach, a numerical example is discussed.

Concluding remarks on the presented work as well as on outlook on suggested future work is given in **Chapter 9**.

Lastly, some background information on Schur Complements that is used throughout this research is presented in **Appendix A**.

Part I

Robust Control for Boiler-Turbine Systems

Chapter 2

Genetic Algorithms in Model Predictive Control for Boiler-Turbine Systems

2.1 Introduction

The control of boiler-turbine systems is a hard problem, see for example [7, 10, 15, 58, 12]. In this chapter, an online model predictive control approach is introduced. In particular, a receding horizon control (RHC) scheme that relies on artificial intelligence to optimize the highly nonlinear and coupled boiler-turbine system control problem is used. Genetic algorithms have been shown to be effective in handling a variety of nonlinear control problems, see for example [59, 58, 60, 61, 62, 63] and references therein.

The proposed online RHC approach uses a discretized version of the nonlinear boiler-turbine characteristics and directly implements all nonlinear characteristics of the system, including all input and input slew rate constraints.

The remainder of this chapter is organized as follows. In section 2.2 the online RHC approach with GAs for the boiler-turbine system (1.3) is introduced. Simulation results for the boiler-turbine system are presented in section 2.3, before the chapter concludes with some final remarks in section 2.4.

2.2 Main Results

The RHC approach discussed in this section is based on the input-output information of (1.3) with system constraints (1.5) only and does not imply explicit knowledge of all system states. To evaluate the quality of the solution candidates, Genetic Algorithms (GAs) are used. They are a form of artificial intelligence (AI) algorithm that is based on evolutionary search techniques that mimic the phenomenon of natural selection and the idea of survival of the fittest, the phenomenon that drives biological evolution [64] [65] [66, 67]. It is based on stochastic methods and is inherently driven by randomness and is thus strictly non-deterministic [68]. GAs are a natural fit for computational problems that require a search of a huge number of possibilities to find the best solution [69]. Artificial intelligence approaches made their first appearance in modern computational methods for control problems as early as the 1960s and marked the beginning of a new era of control [70], [71].

The terminology used to describe GAs in computational optimization is adopted from their biological role model. The following is a brief summary of the terminology used in GA literature with a focus on its biological origin.

The blueprint of each organism can be found in their DNA. It is made up of *chromosomes*, that can be divided into functional blocks called *genes*. Genes can be thought of as an encoded *trait* that can have multiple settings (e.g. blue, brown, green eyes) called *alleles*. Each gene has a particular *locus* or position on the chromosome. The *genome* is the complete collection of genetic materials of an organism.

Organisms that are considered are *diploid*, i.e. their chromosomes are in a paired array. During reproduction, a phenomenon called *crossover* takes place. Genes of the parent generation are exchanged to form a *gamete* (a single chromosome) in the offspring. Furthermore, all offspring are subject to *mutation*, a process in which single *nucleotides* are subject to changes resulting from copying errors.

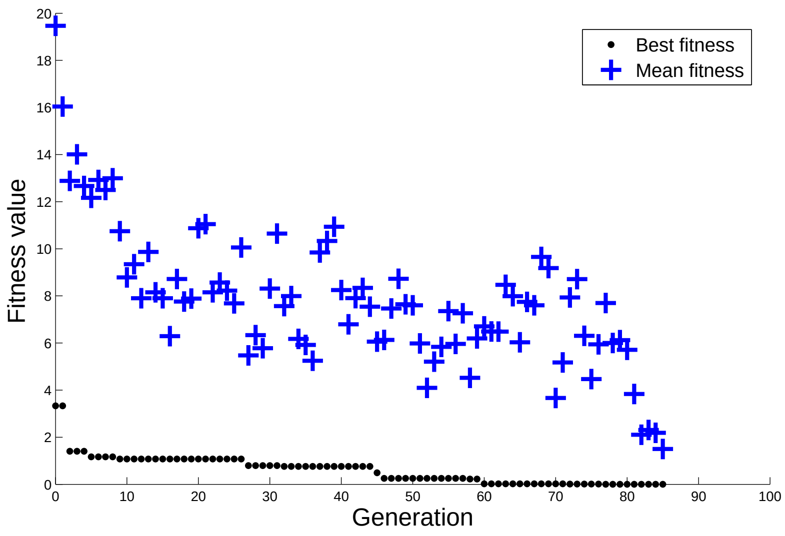


Figure 2.1: Typical progression of genetic algorithms

The *fitness* of an individual is a measure for the probability that an organism will live to reproduce, called *viability*. Individuals with a higher fitness compared to others have a higher chance of being selected for reproduction in a process called *selection*.

The computational implementation of this evolutionary process is as follows. A chromosome typically refers to a candidate solution that is often coded in bitstrings. Several of these individuals form a population of solution candidates. These candidates are evaluated using a cost function or inverse fitness function. Based on their fitness (or cost), the next generation is generated through a combination of the stochastic processes of crossover and mutation. Once a population of children has been generated, their fitness is assessed and the process is repeated until a solution satisfying the requirements is found or the algorithm stalls for several generations. A typical progression of the best and mean costs in a GA are depicted in 2.1. Here, one can see two things. First, the best fitness is maintained for several iterations before an more suitable candidate can be found. This is a common occurrence to most GAs. Second, there is a certain degree of variance in the mean fitness value of each generation, that - on average - converges slowly towards the best fitness value. This is due to the stochastic nature of the algorithm based on crossover and mutation.

One of the key benefits of GAs is their great versatility. They can be adapted to a wide range of problems including clustering analysis, optimization, machine learning, parameter estimation, economics and control [72, 73, 74, 75, 76]. Further, they are able to find suboptimal or optimal solutions in large or complex search spaces and do not depend on gradient search directions. This great adaptability does, however, pose a high cost on the computational burden and requires that thousands or even millions of solutions are evaluated to obtain a good final solution. Therefore, their application is mostly restricted to offline computations.

However, if the system response is slow enough, it is possible to implement GAs in an online RHC control scheme, it is necessary to discretize the plant model with respect to a suitable sampling time first. As the boiler-turbine system is a highly coupled and slow system, a sampling time of $T_s = 10s$ is chosen. The input signals are chosen to be constant for $T_{N_u} = 30s$. To use evolutionary algorithms involves evaluating the cost of thousands if not millions of solution candidates and, in general, the results improve if more samples can be evaluated. As this approach aims for an online adaption of this optimization problem, the benefit of more computation time and potentially better results have to carefully weighted carefully against potential deviations of the actual system response to the predicted response as well as disturbance rejection properties. Therefore, a new input sequence is chosen every 3 sampling times.

Another necessary consideration in online RHC approaches is the length of the prediction horizon. In general, the best results for a given problem can be obtained for an infinite horizon. This is, however, in impractical in practice, as it would involve literally evaluating an infinite number of candidate solutions. Therefore, the prediction horizon is usually limited to only a few sampling periods. If only a very small horizon is chosen, the demands on the computational complexity are significantly reduced, however an optimal short term solution may actually drive the system to an unstable state and the algorithm may fail altogether. A longer horizon generally leads to a more balanced and smoother solution, but also increases the computational demand exponentially.

For the boiler-turbine system, simulations with a prediction horizon of 3 of input sequences $N_u = 3$ have shown a robust performance and could in general guarantee that the system stays within its physical limitations. This does, however, imply an optimization in 9 variables and extensive computations are necessary. In general, this leads to the termination of the GA optimization process due to time constraint and before the minimum cost floor has been found and maintained over several GA generations. Therefore, a more favorable approach in terms of computational complexity can be achieved with a horizon $N_u = 2$, which does however not guarantee a robust performance by itself. This problem can be overcome by carefully setting additional constraints to the optimization problem, and thus a reduction of the overall required computational time in an optimization problem with only 6 parameters can be achieved.

The additional constraints of the boiler-turbine system are set up as follows. Consider the system states of the boiler-turbine system x_1, x_2, x_3. They are modeled such that they represent the drum pressure (kg/cm^2), the electrical power output (MW), and the fluid density (kg/m^3) of the system, respectively. Neither of these values should reach a negative value in normal operations, thus we can use the following additional constraints

$$0 \leq x_i \qquad (i = 1, 2, 3), \tag{2.1}$$

and therefore limit the search space for the optimal input sequences notably.

Furthermore, the steady convergence to the reference signal is monitored for each output signal. The signal development from the known last output to the end of the prediction horizon allows to penalize unfavorable signal responses such as dips and peaks as well as oscillatory behaviour and leads to a smoother output signal that prevents the algorithm from leading the system to an unstable region. In particular, none of the future outputs is allowed to move away from the trajectory towards the reference signal by more than 5 units for each prediction horizon.

To evaluate the fitness of the solutions that do not violate any of the constraints, the following cost function is used

$$J = \left.\begin{aligned} &\sum_{i=1}^{3N_u} \left(2^{i-1}(y[i]-y_{ref})^T Q(y[i]-y_{ref})\right) \\ &+ \sum_{j=0}^{N_u-1} \left(2^j(u[j+1]-u[j])^T R(u[j+1]-u[j])\right) \end{aligned}\right\} \tag{2.2}$$

with penalty design matrices Q and R. The first term of (2.2) penalizes the deviation of the measured output from the reference output. The choice of an appropriate weighting matrix Q needs to be based on the system dynamics and can be further improved by considering several numerical simulation results.

Furthermore, a progressive penalty factor 2^{i-1} has been used to pose a higher penalty on the extrapolated outputs beyond the current prediction horizon, which helps a quick convergence to the reference signal and an elimination of the steady state error. For this extrapolation, it is assumed that the last output in the prediction horizon does not change and is applied to consecutive sampling instances.

Moreover, it should be noted that the sum is over the length of the outputs at each system sampling instance rather than the length of the actually applied system inputs. Since the plant is discretized with a sampling time of $10s$, there are 2 intermediate output results available between the sampling times of constant input. Exploiting this additional information helps to speed up the search for the optimal solution and generally a smoother transition towards the reference signal can be obtained.

The second term of (2.2) captures the penalties on the change of the input signals in R. To guarantee a fast convergence towards a steady state, the change in the input signals should be as small as possible once the output signals approach the reference signals. In general, the impact of R should be small compared to the penalty on deviations from the reference signal, but large enough to allow quick settling in the vicinity of the steady state.

The setup of the GAs is discussed next. GAs are based on testing a large number of possible solutions before eventually converging to the optimum. The solutions under consideration are dependent on the population size and the number of generations before the algorithm terminates. Both parameters must be sufficiently large to ensure that on the one side the whole solution space is explored, and on the other side the algorithm can converge to the real optimum.

A larger population size usually helps to identify new regions that are far from the best solution so far with new local optimal solutions faster. However, the time needed to evaluate all candidates for each generation increases linearly with an increase in population size. For off-line computation these parameters are usually chosen rather large as computation time is not a main concern. This changes however, if the time to termination becomes an important design objective.

The optimization problem is a problem in 6 parameters that represent the inputs and uses a population size of 40. The initial population is initialized with the inputs of the previous time step as well as some random alternations within the constraints (1.5) and (2.1). This is done to promote the continuation of a good input sequences from previous prediction cycles as well as to promote diversity of the population to avoid premature convergence to a local optimum.

The maximum number of generations before the algorithm terminates is another important parameter. A reasonably large number of generation is required to ensure that the algorithm converges to the optimal solution. If it is chosen too large, only negligible progress in the quality of the final solution will be made in the last generations. However, if it is chosen too small, the algorithm might converge to a local optima that may be much worse than the global optimum before better solutions in other regions of the search space can be obtained.

To avoid premature convergence and the termination of the algorithm with a suboptimal solution, a multistage strategy is implemented as follows. Instead of calling the GA once with a large number of generations, it is called twice with a lower overall number of generations. Both calls start with a high mutation variance, which helps to identify optima that are far away from the so far best solution. In the first GA call, the initial population is based on the previous inputs, whereas in the second call, it is based on the results of the first run. Therefore, a reasonably good result is obtained from the first run, which most likely reflects a local optima. Using this information to set up the second run generally results in a better search direction, which in turn might be able to converge to an even better solution due to the high mutation rate in the beginning of the algorithm. Experimental results for the boiler-turbine system suggest that at least 300 generations are necessary for a single run before the algorithm converges towards a stable solution and only negligible improvements can be made. This is due to the complex structure of the search space, in particular for the output y_3. Using the suggested multistage approach, a similar stable final result can be obtained using 2 runs with 100 generations each, thus reducing the computational burden by around 33%.

Moreover, several parameters have to be tuned properly so that they don't increase the computation time, the rate of convergence to a final value and the quality of the best solution found before the algorithm terminates. GAs make extensive use of random numbers, so that multiple runs of the algorithm produce different results. These differences can be rather big after only a couple of generations, but become negligible in the long-run. However, this behaviour makes it necessary to consider this characteristic if only a limited computation time is available. As the novel controller approach presented here aims at providing a real-time control, a sensible choice of parameters is necessary.

The presented GA uses a real valued coding to avoid extensive encoding and decoding of the parameters in otherwise commonly used bit sequences. New generations are created using stochastic uniform selection with a crossover rate of 0.8 and a mutation with a

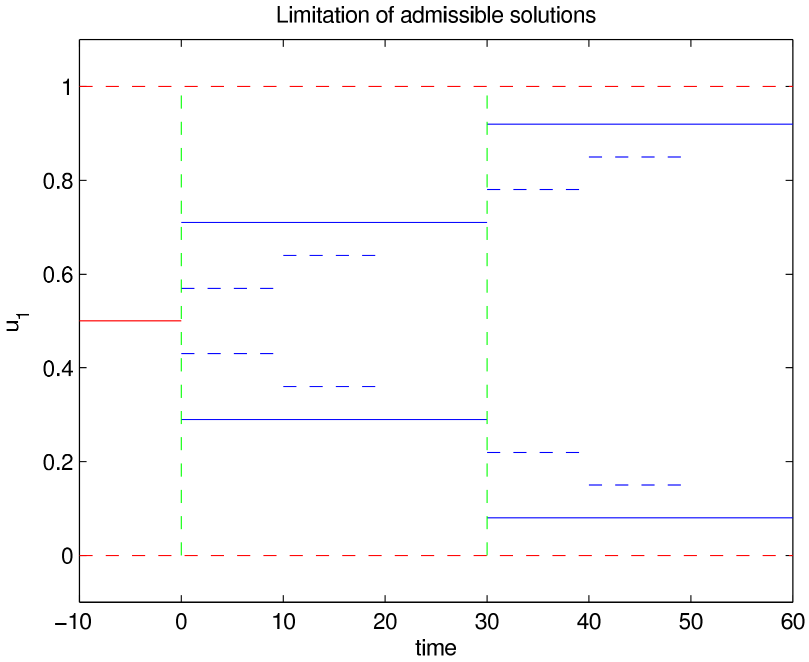


Figure 2.2: Limitation of admissible solutions for input u_1

variance of 10 in the first generation that decreases linearly to 0 by the time the algorithm arrives at the last generation. This approach helps to identify optima that are far away from the so far best solution in the early stages of the algorithm, as well as optimizing the localized search for a better solution at the end in the vicinity of the best solution. This has shown to be very effective in the proposed multistage strategy. Moreover, an elite survival strategy was implemented that guarantees the unaltered survival of the best solution to the next generation.

Besides the consideration of the additional constraints in (2.2), the search space can be further reduced by taking the slew rate constraints from (1.5) into consideration and thus restricting the search space to admissible solution candidates with respect to the previous inputs only. This approach is illustrated for the input u_1 in Figure 2.2. The initial input is shown as 0.5. Limiting the input with respect to the slew rate constraints in (1.5) for the prediction horizon results in the solid blue lines and thus limit the search space for the u_1 to be $\begin{bmatrix} 0.29 & 0.71 \end{bmatrix}$ and $\begin{bmatrix} 0.08 & 0.92 \end{bmatrix}$ for the first and second constant input, respectively. Moreover, the dashed blue lines indicate the limits for the discretized plant model with respect to the sampling time and therefore allow to get a more accurate model of the system behaviour and overall performance. It should be noted that the slew rate constraints of the other inputs are not as constrictive as for u_1 and do not provide the means to limit the search plane as much as has been shown for u_1. Nonetheless, they can be efficiently used for the discretized model to support more accurate intermediate solutions.

After the optimal input sequence is obtained, only the first input is applied to the plant. Based on this input, the final conditions of the plant at the end of the next period of constant input can be predicted and used as initial conditions for the next RHC cycle. The later inputs obtained are used as the initial conditions for the next RHC cycle, where besides a constant progression also random perturbations within the boundaries of the input constraints are used to create a diverse initial population for the next RHC cycle. This preserves valuable computation time that would otherwise be spend recovering already available information, as well as also promoting a steady state input at the end of the horizon.

2.3 Numerical Example

2.3.1 Receding Horizon Control with Genetic Algorithms

The following weighting matrices for the cost function (2.2) are chosen based on several test runs:

$$Q = \begin{bmatrix} 10^1 & 0 & 0 \\ 0 & 10^0 & 0 \\ 0 & 0 & 10^5 \end{bmatrix}, \quad R = \begin{bmatrix} 10^2 & 0 & 0 \\ 0 & 10^2 & 0 \\ 0 & 0 & 10^2 \end{bmatrix} \tag{2.3}$$

The reason for the high penalty on y_3 is its highly nonlinear and coupled nature.

In the examples that follow, a step change in the reference signal occurs at $t = 100$ and the RHC with GAs is started. Since the output y_3 represents the drum water level, we assume that it ideally stays constant and set the reference signal for this output equal to zero

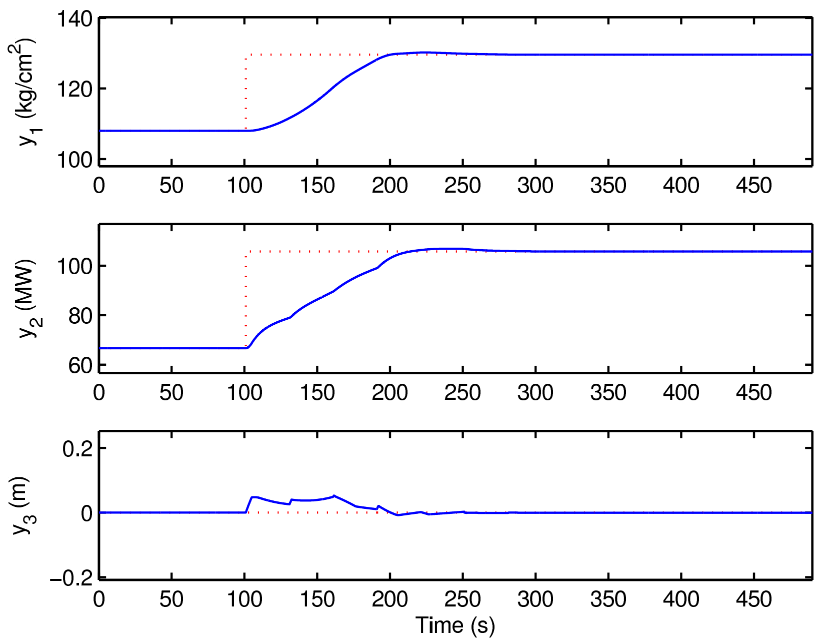


Figure 2.3: System response for a change to a close operating point.

for all simulations that follow, unless otherwise stated. The observed steady state settling refers to a quasi steady state cost function value $J < 10^{-3}$, where deviations deviations from the reference signal become negligible.

First, the results for a small change in the reference signal are considered. This is a control task that can be efficiently handled by many controllers that use linearization techniques, see for example [7]. Consider y_1 is increased from 108 to 129.6, y_2 changes from 66.65 to 105.8. The output responses are shown in Figure 2.3. The final operating point is approached quickly and a virtually steady state is achieved with almost no overshoot. It should be noted that it takes some time for the system to finally settle to the steady state using the proposed RHC GA approach. This can be explained by the discrete inputs with a length of 30 seconds each. However, the overall time to reach the steady state is comparable to the one achieved in [7], whereas the drum water level could be kept closer to the desired level.

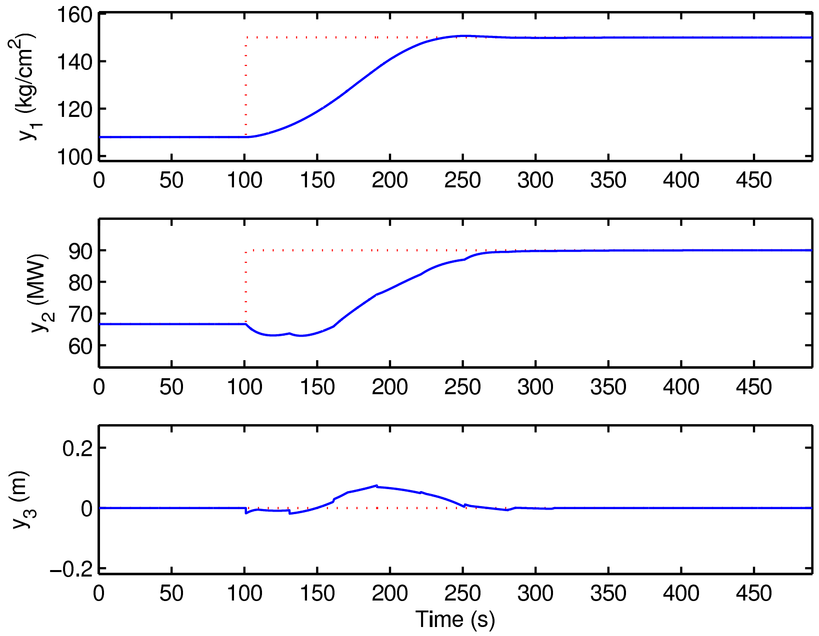


Figure 2.4: System response from a nominal operating point to a far operating point.

Next, the efficiency of the presented control approach to transfer the system to a operating point that can be considered far away is investigated [7]. This is usually problematic for controllers that are designed using only a linearized model. We consider the changes in the reference for y_1 from 108 to 150 and y_2 from 66.65 to 90. As can be seen in Figure 2.4, control using RHC in combination with GAs allows a fast transition to the new reference and settles to the final value quickly. The linear controller proposed in [7] was not able to control this transition and the plant became quickly unstable.

To show that the proposed controller can operate well over a wide range of operation, a change in the reference signal from operating point 1 to 7 is examined, see table 1.1. Figures 2.5 and 2.6 show the output and input response, respectively. The transition towards the new operating point requires about 500s, which is faster than [7]. There is, however, an overshoot in the signals y_1 and y_2 before they settle towards their steady state values. This can be explained by the choice of the weighting matrices Q and R, which are optimized to enforce a stable output of y_3 rather than a transition of y_3.

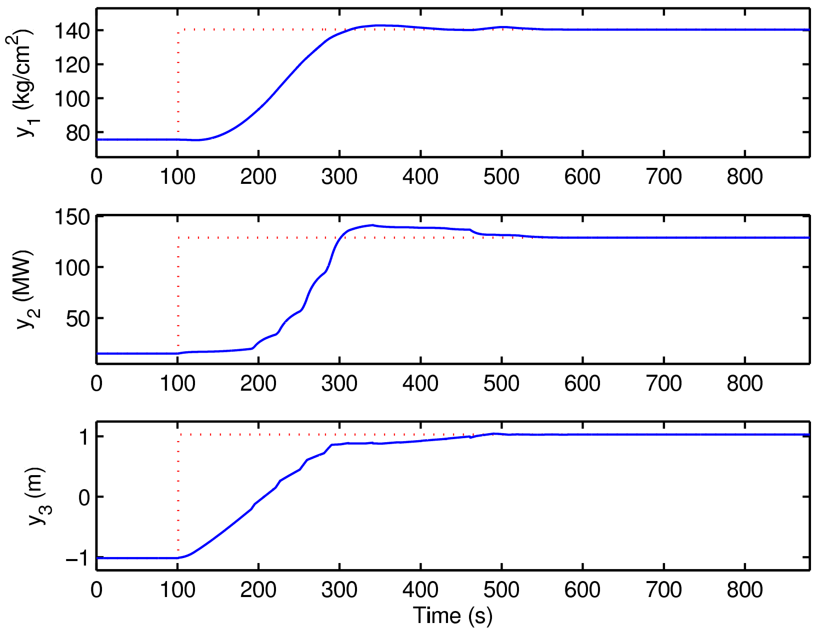


Figure 2.5: System response for change form operating point 1 to 7. Outputs

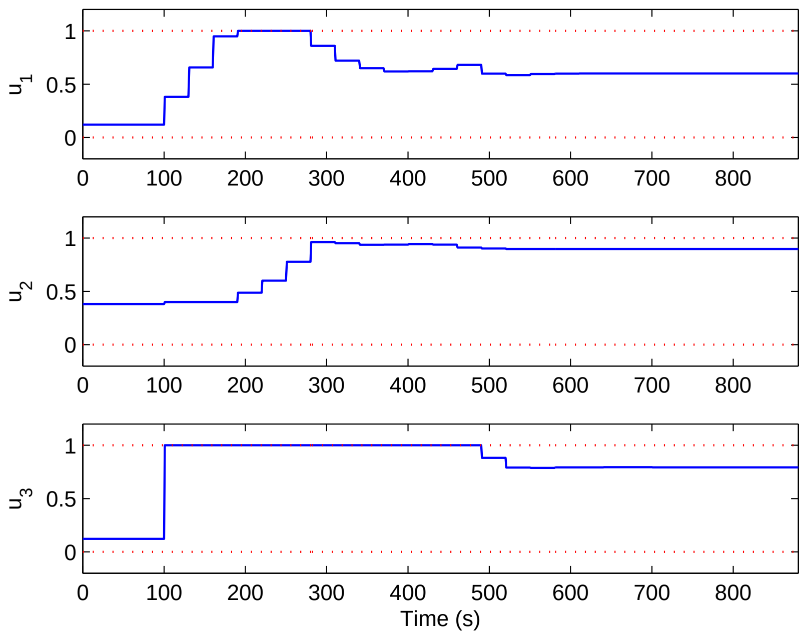


Figure 2.6: System response for change form operating point 1 to 7. Inputs

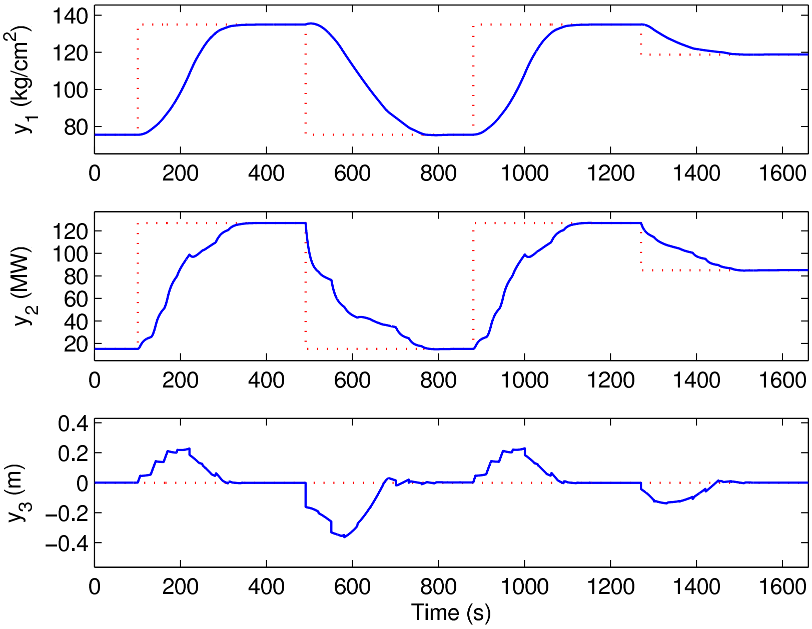


Figure 2.7: System response for a series of changes in the reference signal. Outputs

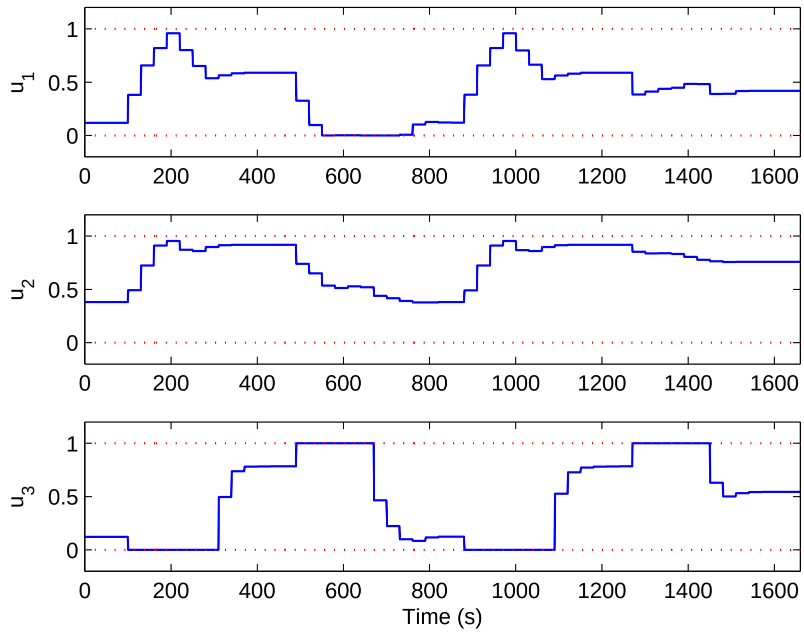


Figure 2.8: System response for a series of changes in the reference signal. Inputs

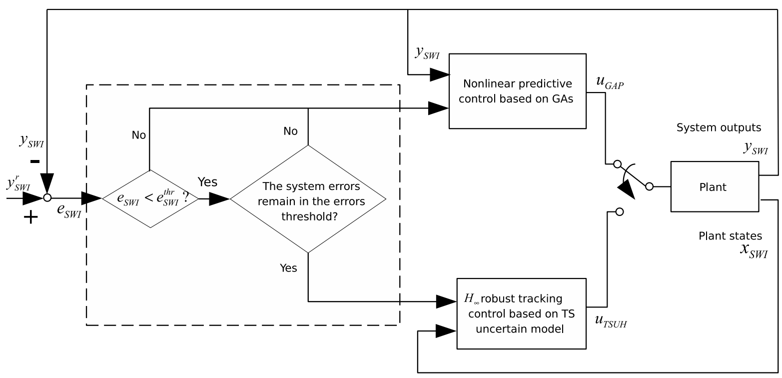


Figure 2.9: Switching principle

In a final simulation, multiple changes in the reference signal are considered to show the tracking capabilities of the proposed controller. The reference signal for y_3 is once again set to zero for the whole simulation, while y_1 is changed from $y_1^{(1)} = 75.6$ to $y_1^{(2)} = 135.0, y_1^{(3)} = 75.6, y_1^{(4)} = 135, y_1^{(5)} = 118.8$ and y_2 is changed from $y_2^{(1)} = 15.27$ $y_2^{(2)} = 127.0, y_2^{(3)} = 15.27, y_2^{(4)} = 127.0, y_2^{(5)} = 85.06$ at time instances $t_1 = 100, t_2 = 490, t_3 = 880, t_4 = 1270$, respectively. It can be observed in Figure 2.7 that the controller is capable of tracking all changes in the reference signal. Other controllers are not capable of tracking this reference trajectory at all or fail to do so within the given time frame, see the designs proposed in[16, 11, 13, 7, 77, 78, 79]. Figure 2.8 shows the input sequences for this setup.

2.3.2 Robust Switching Control for Boiler-Turbine Systems

It is possible to further improve the overall performance by introducing an adaptive switching regime as shown in Figure 2.9.

First, the switching control cycle determines whether the error signal e_{SWI} is inside a set error threshold e_{SWI}^{thr}. If this condition is violated, i.e. there has been a change in the reference signal, the predictive control scheme is used. Otherwise, the next step of the switching control is to evaluate if the error signal is persistently within the error threshold.

If this is the case, the H_∞ fuzzy tracking controller is used. Otherwise, there might be a bigger issue and the predictive control scheme takes over.

The cost function that forms the basis for the RHC control has to be adapted to reflect the new task of the RHC control scheme: Fast settling to a steady state is no longer a concern of the RHC part of the control cycle, as the H_∞ fuzzy tracking controller will be used to track small changes. Thus, no penalties need to be introduced in for a slew rate change,

$$J = \sum_{k=1}^{3N_u} \left(f(k)(y_{ref}(k) - y(k))^T Q (y_{ref}(k) - y(k)) \right). \tag{2.4}$$

Here, N_u is the length of the prediction horizon, Q is the tracking error weighing matrix, and $f(k)$ is a penalty function defined as

$$f(k) = a^k, \quad a \geq 1. \tag{2.5}$$

The penalty function (2.5) has a similar function as the penalty factor in (2.2): Deviation in the tracking performance for later outputs face a heavier penalty.

The H_∞ fuzzy tracking controller is obtained by applying the results of [35]. The design process can be summarized as follows. Consider a nonlinear system in which the space of operation can be partitioned into several regimes with respect to some premise variables. Then, the i-th plant local linear model in Takagi-Sugeno (TS) fuzzy from is [19, 35]

Plant Rule i: IF $v_1(t)$ is M_{i1} and $\cdots$ and $v_\vartheta(t)$ is $M_{i\vartheta}$, THEN

$$\begin{aligned} \dot{x}(t) &= A_i x(t) + B_i u(t) + B_w w(t), \\ z(t) &= C_{zi} x(t) + D_{zi} u(t) \\ y(t) &= x(t) \end{aligned} \tag{2.6}$$

where $i = 1, 2, \cdots, r$, r is the number of rules, M_{ik} $(k = 1, 2, \cdots, \vartheta)$ are fuzzy sets, $v_i(t)$ are premise variables, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input, $w(t) \in \mathbb{R}^p$ is the disturbance signal, $z(t) \in \mathbb{R}^s$ is the controlled output, $y(t) \in \mathbb{R}^s$ is the measurement, the matrices A_i, B_i, B_w, C_{zi} and D_{zi} are of appropriate dimensions.

By using a center-average defuzzifer, product inference and singleton fuzzifer, the local models can be integrated into a global nonlinear model:

$$\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{r} \mu_i(\nu(t))A_i x(t) + \sum_{i=1}^{r} \mu_i(\nu(t))B_i u(t) \\
&\quad + B_w w(t), \\
z(t) &= \sum_{i=1}^{r} \mu_i(\nu(t))[C_{zi}x(t) + D_{zi}u(t)] \\
y(t) &= \sum_{i=1}^{r} \mu_i(\nu(t))C_{yi}x(t)
\end{aligned} \tag{2.7}$$

where

$$\nu(t) = [\nu_1(t), \nu_2(t), \cdots, \nu_\vartheta(t)]^T, \tag{2.8}$$

and

$$\begin{aligned}
\omega_i(\nu(t)) &= \prod_{k=1}^{p} M_{ik}(\nu_k(t)), \quad \omega_i(\nu(t)) \geq 0, \quad \sum_{i=1}^{r} \omega_i(\nu(t)) > 0, \\
\mu_i(\nu(t)) &= \frac{\omega_i(\nu(t))}{\sum_{i=1}^{r} \omega_i(\nu(t))}, \quad \mu_i(\nu(t)) \geq 0, \quad \sum_{i=1}^{r} \mu_i(\nu(t)) = 1.
\end{aligned}$$

Here, $M_{ik}(\nu_k(t))$ denote the grade of membership of $\nu_k(t)$ in M_{ik}.

H_∞ performance is fulfilled if the gain from the disturbance input to the controlled output is less than a prescribed value $\gamma > 0$. In detail, the following condition must hold [35]

$$\int_0^T z^T(t)z(t)dt \leq \gamma^2 \int_0^T w^T(t)w(t)dt. \tag{2.9}$$

For the nonlinear plant represented by (2.7), the fuzzy state feedback controller is inferred as follows:

$$u(t) = \sum_{i=1}^{r} \mu_i(\nu(t))K_i x(t). \tag{2.10}$$

where K_i is the local controller gain for each plant rule.

The closed-loop system (2.7) with (2.10) can be written as:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i(\nu(t))\mu_j(\nu(t))[A_i + B_iK_j]x(t) + B_w w(t) \\ z(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i(\nu(t))\mu_j(\nu(t))[C_{zi} + D_{zi}K_j]x(t) \\ y(t) &= \sum_{i=1}^{r}\mu_i(\nu(t))C_{yi}x(t) \end{aligned} \tag{2.11}$$

Theorem 2.3.1 *If there exit a symmetric positive definite matrix P and a matrix Y_j such that the following condition holds*

$$\begin{bmatrix} A_iP + PA_i^T + B_iY_j + Y_j^TB_i^T & B_w & PC_{z_i}^T + Y_j^TD_{z_i}^T \\ * & -\gamma^2 I & 0 \\ * & * & -I \end{bmatrix} < 0 \tag{2.12}$$

for $i < j \leq r$, then the (2.9) *holds. Moreover, a suitable choice of the fuzzy controller is*

$$u(t) = \sum_{j=1}^{r}\mu_j(x)K_jx(t) \tag{2.13}$$

with

$$K_j = Y_jP^{-1} \tag{2.14}$$

Proof: See [35]. ■

Theorem 2.3.2 *The problem of designing a tracking controller for* (2.7) *with reference tracking error*

$$e(t) = y(t) - y_{ref}(t) \tag{2.15}$$

can be formulated as

$$\begin{aligned} \dot{\tilde{x}}(t) &= \sum_{i=1}^{r}\mu_i(\nu(t))\left[\widetilde{A}_i\widetilde{x}(t) + \widetilde{B}_iu(t) + \widetilde{B_w}w(t) + \widetilde{d}(t)\right], \\ z(t) &= \sum_{i=1}^{r}\mu_i(\nu(t))[C_{zi}x(t) + D_{zi}u(t)], \end{aligned} \tag{2.16}$$

with $E(t) = \int_0^\tau e(t)d\tau$ *and*

$$\begin{aligned} \tilde{x}(t) &= \begin{bmatrix} x(t) \\ E(t) \end{bmatrix}, \quad \tilde{d}(t) = \begin{bmatrix} 0 \\ -y_{ref}(t) \end{bmatrix}, \\ \tilde{A}_i &= \begin{bmatrix} A_i & 0 \\ C_{yi} & 0 \end{bmatrix}, \quad \tilde{B}_i = \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \quad \tilde{B}_\omega = \begin{bmatrix} B_\omega \\ 0 \end{bmatrix}. \end{aligned} \tag{2.17}$$

.

Proof: The proof is obvious. ■

The benefit of the augmented problem formulation of Theorem 2.3.2 is that the augmented system (2.16) can be directly addressed using Theorem 2.3.1. Thus, the H_∞ fuzzy reference tracking problem with integral action can be treated just as the standard H_∞ problem.

For the adaptive switching controller, the error threshold is set to be 10% of the initially observed error. If the tracking error stays within this threshold for 2 consecutive sampling periods, the control is switched to H_∞ fuzzy tracking control, which then efficiently stabilizes the system. Once that error threshold is violated again, the control switches back to RHC control and a new error threshold is set with respect to the new initial deviation.

The following design parameters have been chosen for the RHC scheme. The penalty factor a in the penalty function (2.5) is set to $a = 2$. This results in a high penalty for tracking errors at the end of the prediction horizon. The prediction horizon is chose as $N_u = 3$ for discrete inputs of $10s$ each. This change from the design in Chapter 2 is possible due to the less complex cost function and a relaxed approach with respect to steady state settling behaviour of the RHC controller. The error weighting function in (2.4) is chosen as

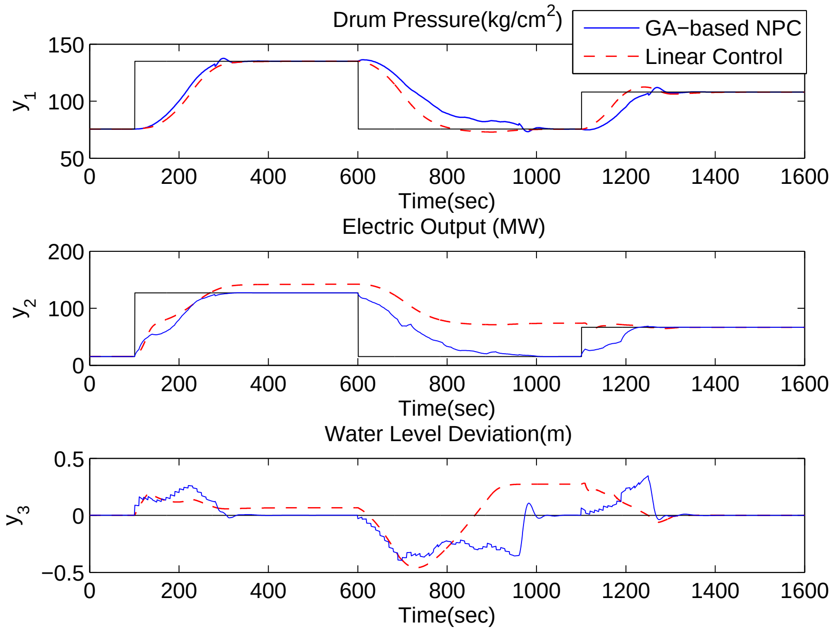


Figure 2.10: Switching control system outputs

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10^4 \end{bmatrix}. \tag{2.18}$$

Based on the new requirements for the RHC procedure, the choice of Q is slightly different than in (2.3), but still imposes a much higher penalty on tracking errors in y_3 based on its high degree of nonlinearity and coupling.

The population size for the GA is set to 50 individuals and the GA terminates after at most 300 iterations. The crossover ratio is set to 0.8, and an elite count of 2 is enforced to preserve the two fittest of each generation unaltered.

To prevent integrator windup, an anti windup strategy as outlined in [80] has been implemented for the H_∞ fuzzy tracking controller.

The simulation results are compared to a linear H_∞ control approach for a linearized system model, see [7] for example.

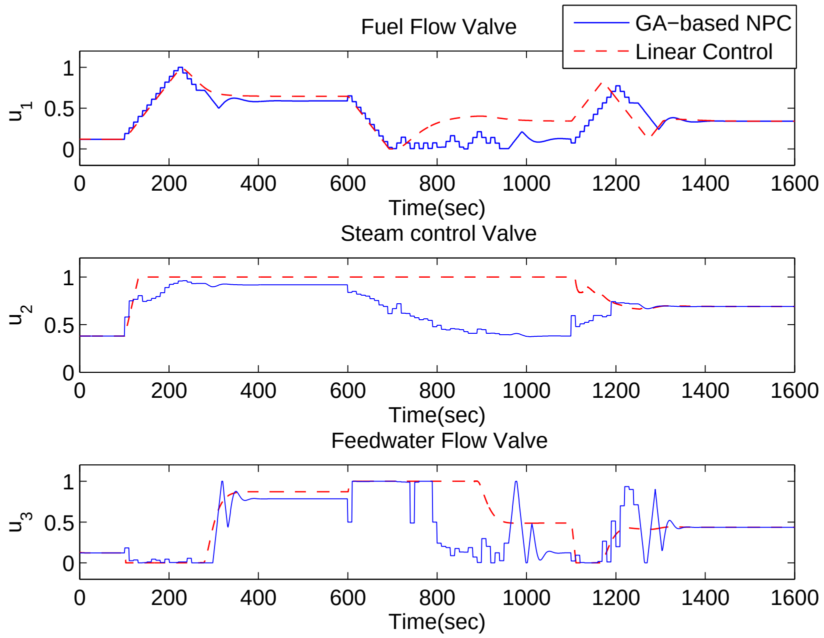


Figure 2.11: Switching control system inputs

To show the advantage of the propose adaptive control scheme, a change between distant operating points is of central interest. Therefore, a reference trajectory with respect to Table 1.1 is chosen as: transition from operating point 1 to 7 at $t = 100s$; from 7 to 1 at $t = 600s$; and from 1 to 4 at $t = 1100s$. We set the reference for $y_3^{ref} = 0$ for the whole simulation to underline our desire to keep the drum water level as stable as possible. The outputs and inputs for this tracking problem are shown in Figure 2.10 and Figure 2.11, respectively. It can be observed that the linear controller has problems to stabilize the system for large changes in the reference signal. There is a large derivation from the reference value for y_2 present which may result in a complete control failure. Due to the linearization, there is a constant steady state error present for y_2 and y_3 after the first transition, which increases significantly after the second transition. It incidentally manages to stabilize the system after the last change of reference.

This section focuses on the design of an H_∞ tracking control for the boiler-turbine modelled by a TS fuzzy model. Defining

$$u_2^*(t) = u_2(t)x_1(t), \tag{2.19}$$

the dynamics of the boiler-turbine system (1.3) can be rewritten as

$$\begin{aligned}
\dot{x_1}(t) &= -0.0018u_2^*(t)x_1^{1/8}(t) + 0.9u_1(t) - 0.15u_3(t) + 0.01w_1(t),\\
\dot{x_2}(t) &= 0.073u_2^*(t)x_1^{1/8}(t) - 0.016x_1(t)x_1^{1/8}(t) - 0.1x_2(t) + 0.01w_2(t),\\
\dot{x_3}(t) &= (141u_3(t) - 1.1u_2^*(t) + 0.19x_1(t))/85 + 0.01w_3(t),\\
y_1 &= x_1(t),\\
y_2 &= x_2(t),\\
y_3 &= 0.05(0.1307x_3(t) + 100a_{cs} + q_e/9 - 67.975).
\end{aligned} \tag{2.20}$$

From the typical operation points given in Table 1.1, we can assume that $x_1(t) \in \begin{bmatrix} 50 & 150 \end{bmatrix}$. Hence, the nonlinear term $x_1^{1/8}(t)$ in (2.20) can be expressed as

$$x_1^{1/8}(t) = \mu_1(x_1(t)) \times 50 + \mu_2(x_1(t)) \times 150 \tag{2.21}$$

where

$$\begin{aligned}
\mu_1(x_1(t)) &= \frac{150^{\frac{1}{8}} - x_1^{\frac{1}{8}}(t)}{150^{\frac{1}{8}} - 50^{\frac{1}{8}}},\\
\mu_2(x_1(t)) &= 1 - \mu_1(x_1(t)).
\end{aligned} \tag{2.22}$$

The plots for $\mu_1(x_1(t))$ and $\mu_1(x_1(t))$ are given in Figure 2.12. Now, using the above membership functions, the boiler-turbine system (2.20) can be exactly represented by the following TS fuzzy model

$$\dot{x}(t) = \sum_{i=1}^{2} \mu_i(x_1(t))\left[A_i x(t) + B_i u(t)\right] + B_w w(t) \tag{2.23}$$

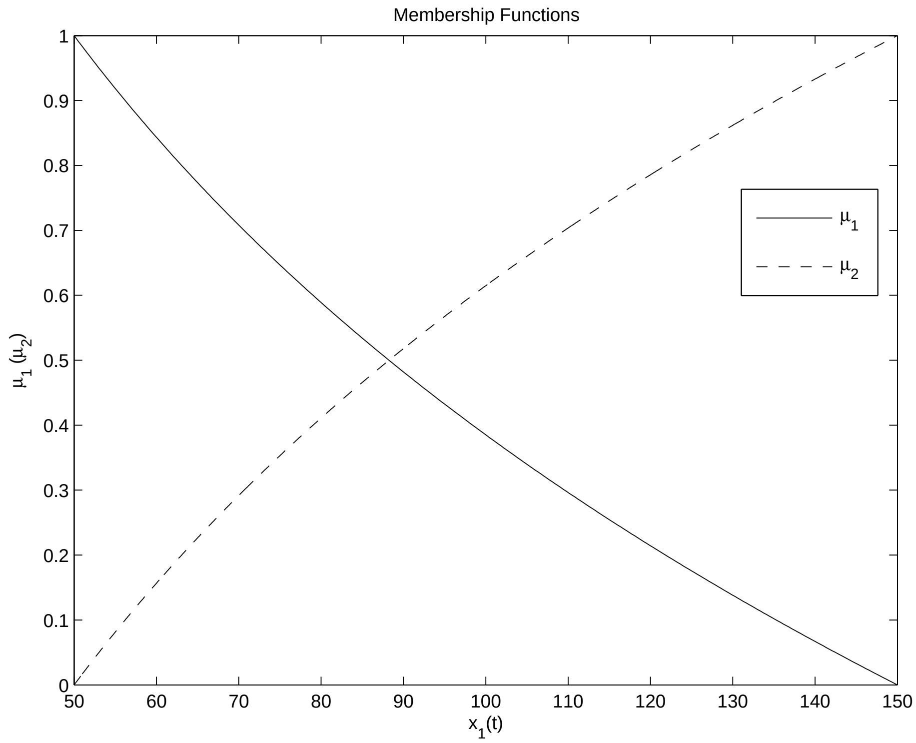


Figure 2.12: The membership functions for H_∞ fuzzy tracking control

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \quad w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{bmatrix}, \quad u(t) = \begin{bmatrix} u_1(t) \\ u_2^*(t) \\ u_3(t) \end{bmatrix}, \tag{2.24}$$

and

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 0 & 0 \\ -0.016 \times 150^{\frac{1}{8}} & -0.1 & 0 \\ \frac{0.19}{85} & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.9 & -0.0018 \times 150^{\frac{1}{8}} & -0.15 \\ 0 & 0.073 \times 150^{\frac{1}{8}} & 0 \\ 0 & -\frac{1.1}{85} & \frac{141}{85} \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 0 & 0 & 0 \\ -0.016 \times 50^{\frac{1}{8}} & -0.1 & 0 \\ \frac{0.19}{85} & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.9 & -0.0018 \times 50^{\frac{1}{8}} & -0.15 \\ 0 & 0.073 \times 50^{\frac{1}{8}} & 0 \\ 0 & -\frac{1.1}{85} & \frac{141}{85} \end{bmatrix}, \end{aligned} \tag{2.25}$$

and

$$B_w = 0.01 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The controlled output $z(t)$ is chosen to be

$$z(t) = 10^{-4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t) + 10^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u(t). \tag{2.26}$$

By applying the Theorem 2.3.2 with the integral action, the corresponding solutions are obtained.

$$P = 10^7 \begin{bmatrix} 0.0003 & 0.0003 & -0.0001 & 0.0074 & 0.0013 & -0.0010 \\ 0.0003 & 0.0210 & -0.0001 & 0.0180 & 0.2044 & -0.0010 \\ -0.0001 & -0.0001 & 0.0010 & -0.0013 & -0.0002 & 0.0166 \\ 0.0074 & 0.0180 & -0.0013 & 0.3584 & 0.0802 & -0.0221 \\ 0.0013 & 0.2044 & -0.0002 & 0.0802 & 2.0204 & -0.0039 \\ -0.0010 & -0.0010 & 0.0166 & -0.0221 & -0.0039 & 0.5756 \end{bmatrix}$$

$$Y_1 = \begin{bmatrix} -89.9905 & 0.0074 & -0.0023 & 0.2109 & 0.0371 & -0.0282 \\ 0.2992 & -11.8997 & 1.2927 & 0.1252 & 0.0217 & -0.0167 \\ 14.9979 & -0.0017 & -165.8818 & -0.0474 & -0.0084 & 0.0059 \end{bmatrix} \tag{2.27}$$

$$Y_2 = \begin{bmatrix} -89.9793 & 0.0266 & -0.0051 & 0.4581 & 0.0832 & -0.0611 \\ 0.3208 & -13.6769 & 1.2980 & -0.3532 & -0.0646 & 0.0471 \\ 14.9952 & -0.0062 & -165.8812 & -0.1060 & -0.0193 & 0.0137 \end{bmatrix}$$

The integral action matrices K_{Ifi} and the state feedback control matrices K_{fi} are solved as

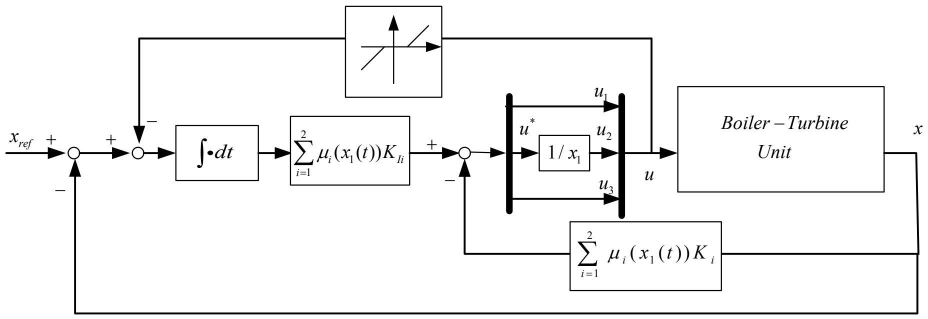


Figure 2.13: Integral state feedback control system with anti-windup for boiler-turbine unit

$$
\begin{aligned}
K_{If1} &= \begin{bmatrix} -0.0010 & 0.0002 & -0.0001 \\ -0.0019 & -0.0067 & 0.0000 \\ 0.0001 & -0.0001 & -0.0010 \end{bmatrix} \\
K_{f1} &= \begin{bmatrix} 0.0504 & -0.0022 & 0.0042 \\ -0.0005 & 0.0672 & -0.0003 \\ -0.0007 & 0.0007 & 0.0346 \end{bmatrix}, \\
K_{If2} &= \begin{bmatrix} -0.0016 & 0.0003 & -0.0001 \\ -0.0015 & -0.0065 & 0.0001 \\ 0.0004 & -0.0002 & -0.0052 \end{bmatrix} \\
K_{f2} &= \begin{bmatrix} 0.0504 & -0.0023 & 0.0042 \\ -0.0006 & 0.0771 & -0.0003 \\ -0.0007 & 0.0008 & 0.0346 \end{bmatrix}.
\end{aligned}
\tag{2.28}
$$

The guaranteed cost tracking control law for the boiler-turbine TS fuzzy models are

$$u(t) = [u_1(t)\ u_2(t)\ u_3(t)] = \sum_{j=1}^{2} \mu_j(x_1)[K_{Ifj}E(t) + K_{fj}x(t)]$$

with $u_2(t) = \frac{u_2^*(t)}{x_1(t)}$ and $E(t) = \int_0^t (x(t) - x_{ref}(t))dt$.

In order to prevent the windup caused by the saturations of the actuators, the tracking anti-windup strategy [80] is applied here, see Figure 2.13.

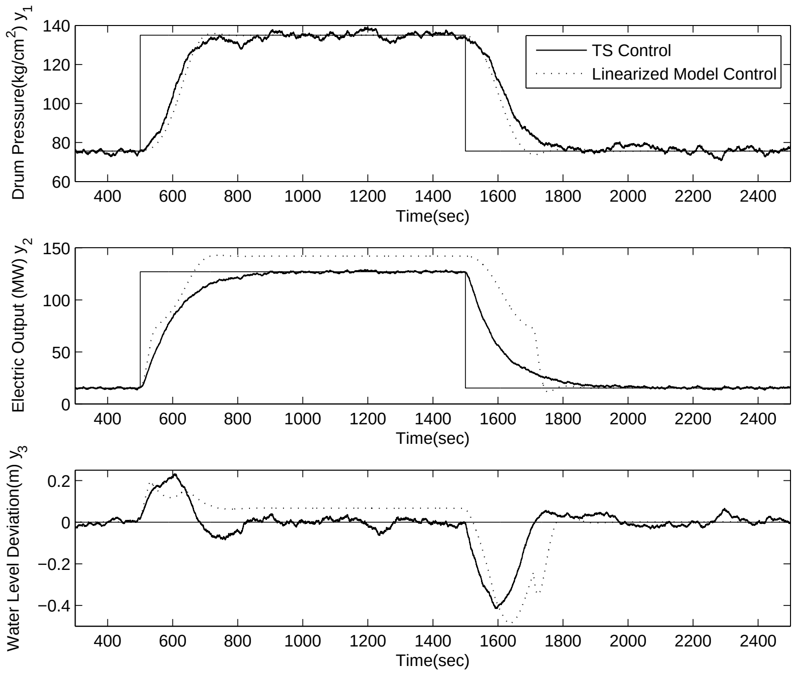


Figure 2.14: Boiler-turbine system outputs

The disturbances $w_1(t), w_2(t)$ and $w_3(t)$ are modeled as independent band-limited white noises with noise power spectrum density of 10. To show the benefits of the H_∞ fuzzy reference tracking approach presented in Theorem 2.3.2, the results are compared to a single H_∞ reference tracking controller that has been designed using a linearization of the system dynamics around the central operating point, but otherwise using the same design process outlined in Section 2.3.2. The simulation results for (1.3) with constraints (1.5) and H_∞ fuzzy reference tracking control (2.28) are shown in Figures 2.14 and 2.15 for the system output and input, respectively.

Both control approaches show similar tracking results for the output y_1, however the linear tracking controller fails to track the desired outputs y_2 and y_3.

Furthermore, the results for an implementation of the proposed H_∞ fuzzy tracking controller with and without an anti-windup strategy are compared. The implementation of an anti-windup concept is critical to the overall design process of an integral tracking con-

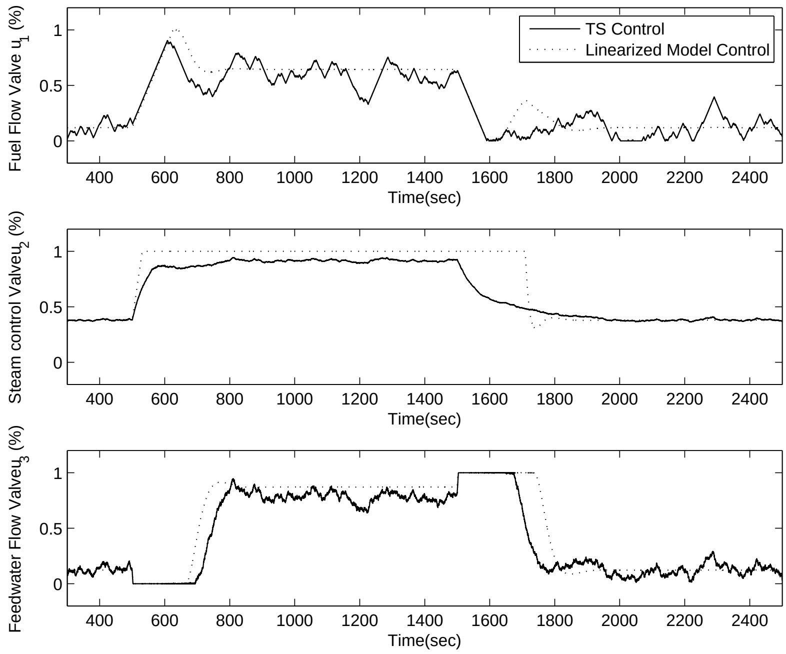


Figure 2.15: Boiler-turbine system inputs

troller for systems with input constraints. Otherwise, the control inputs can quickly saturate and the overall system may not be controllable, see Figure 2.16. The proposed anti-windup strategy avoids large overshoots as well as oscillatory behaviour in the output signals that could otherwise damage the system. Furthermore, the deviations in the water level y_3 may result in an overall system failure and an emergency shutdown.

2.4 Conclusion

A novel GA-based nonlinear model predictive control approach has been proposed for boiler-turbine systems. It has been shown that this control approach is capable of dealing well with the nonlinearities in the plant model and can be used for a wide operating range. Furthermore, a robust adaptive switching algorithm that uses GA-based nonlinear model predictive control as well as robust H_∞ fuzzy control was introduced to allow for a quick

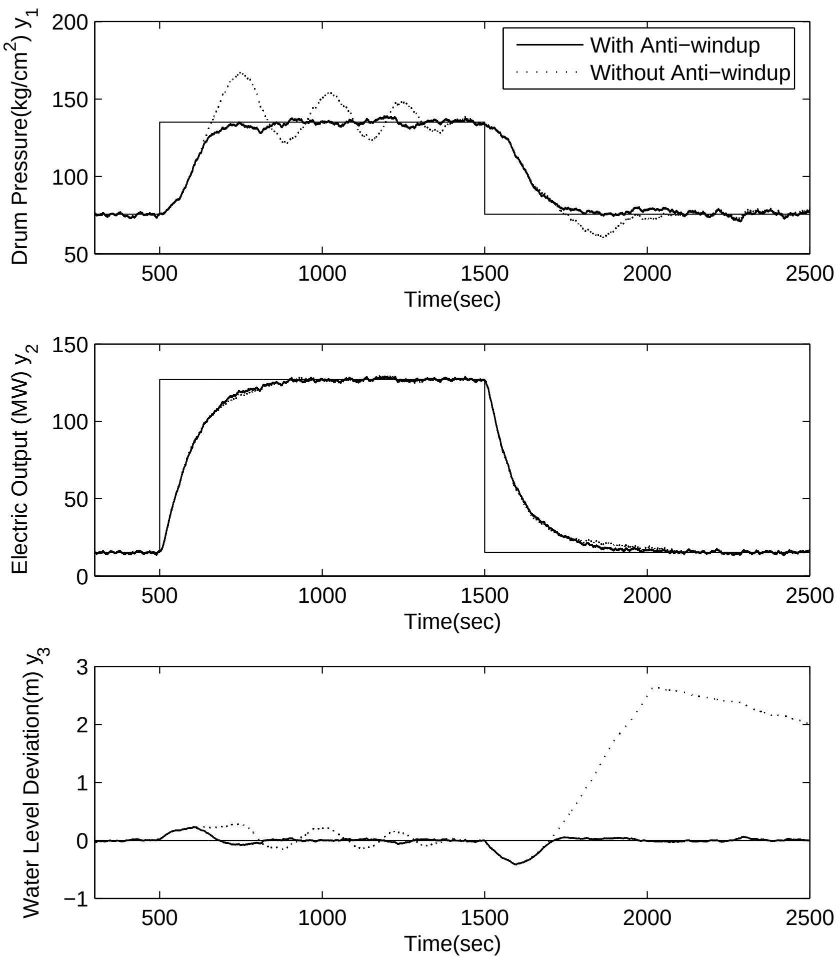


Figure 2.16: Impact anti-windup strategy

decay in the tracking error towards a steady state. The simulation results showcase the effectiveness of both approaches.

Even though the analysis was carried out for the specific boiler-turbine unit, this approach can easily be transferred to other nonlinear control tasks by careful adjustment of the configuration parameters [81].

The online solution presented in this work required the plant to be reasonably slow. An extension to faster plants and the handling of noise terms and uncertainties are still open topics for model predictive control incorporating GAs.

Part II

Control Synthesis for Polynomial Systems

Chapter 3

Stabilization of Nonlinear Polynomial System

3.1 Introduction

The control of polynomial systems is a nontrivial problem that stems from an inherently nonconvex relationship between the controller matrix and the Lyapunov function. To avoid this problem, it is customary to avoid the states that have nonzero entries in the system input matrix in the construction of a Lyapunov function, see for example [46, 45, 57, 30]. This is, however, not always practical. Furthermore, it introduces conservatism to the overall design approach.

The lack of a design approach that addresses this problem allowing a greater design freedom has been the motivation to investigate alternative modelling approaches that lead to overall less conservative control designs through a greater freedom in Lyapunov function candidates. The following approach addresses the nonconvexity of the problem by introducing an iterative algorithm. In general, this iterative procedure leads to an overall larger control problem, as more unknown coefficients in the Lyapunov function candidate have to be considered. Furthermore, the iterative procedure outlined below requires that several control problems have to be solved sequentially, which introduces a higher computational cost in the overall controller design. Results do, however, indicate that this higher cost tends to find superior solutions to the more commonly used restrictive Lyapunov function design and should more than make up the initial computational investment over the lifetime of the implemented controller.

The remainder of this chapter is organized as follows: Section 3.2 outlines the general control problem and presents the procedure for the iterative algorithm. The state feedback control problem is then extended to the common problem of polytropic uncertainties and the performance of the procedure is outlined with numerical examples in 3.3. This chapter closes with some concluding remarks in 3.4.

3.2 Main Results

In this section, the design of a state feedback controller for polynomial systems with polytropic uncertainties is presented. First, a derivation of the control laws without uncertainties is derived that is subsequently extended to the case of systems with polytropic uncertainties.

3.2.1 State Feedback Control for Polynomial Systems

Consider the dynamic system modelled by

$$\begin{aligned} \dot{x} =& A(x) + B(x)u, \\ y =& x, \end{aligned} \tag{3.1}$$

where $A(x)$ is a polynomial vector and $B(x)$ is a polynomial matrix of appropriate dimensions. The objective is to find a polynomial controller as

$$u = K(x) \tag{3.2}$$

such that the system (3.1) is asymptotic stable.

Theorem 3.2.1 *The polynomial system* (3.1) *is stabilizable via state feedback control if there exist a polynomial function* $V(x)$ *and a polynomial matrix* $K(x)$ *such that for* $\forall x \neq 0$

$$V(x) > 0 \tag{3.3}$$

and

$$\begin{aligned}&\frac{\partial V(x)}{\partial x}A(x) - \frac{1}{4}\frac{\partial V(x)}{\partial x}B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x}\\&+\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x)+K^T(x)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x)+K^T(x)\right)^T < 0.\end{aligned} \tag{3.4}$$

Proof: Note that for $\forall x \neq 0$

$$\begin{aligned}\dot{V}(x) =&\frac{\partial V(x)}{\partial x}\left[A(x)+B_u(x)K(x)\right] \leq \frac{\partial V(x)}{\partial x}\left[A(x)+B_u(x)K(x)\right]+K^T(x)K(x)\\=&\frac{\partial V(x)}{\partial x}A(x) - \frac{1}{4}\frac{\partial V(x)}{\partial x}B_u(x)B_u^T(x)\frac{\partial V(x)}{\partial x}\\&+\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x)+K^T(x)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x)+K^T(x)\right)^T.\end{aligned} \tag{3.5}$$

Thus, it follows from the Lyapunov stability theorem that the system (3.1) with (3.2) is asymptotic stable if (3.4) holds. ■

Even though we have separated the Lyapunov function and the controller matrix of the state feedback controller problem in (3.4), the problem cannot be directly transformed into a state-depended LMI using Schur Complement due to the negative term $-\frac{1}{4}\frac{\partial V(x)}{\partial x}B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x}$. Introducing a polynomial design vector $\varepsilon(x)$ of appropriate dimension, it is easy to verify that

$$\left(\varepsilon(x)-\frac{\partial V(x)}{\partial x}\right)B_u(x)B_u^T(x)\left(\varepsilon(x)-\frac{\partial V(x)}{\partial x}\right)^T \geq 0$$

for any $\varepsilon(x)$ and $\frac{\partial V(x)}{\partial x}$ of the same dimension, with equality for $\varepsilon(x) = \frac{\partial V(x)}{\partial x}$. An expansion yields

$$\begin{aligned}\frac{\partial V(x)}{\partial x}B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x} \geq & -\varepsilon(x)B_u(x)B_u^T(x)\varepsilon^T(x) + \varepsilon(x)B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x} \\ & + \frac{\partial V(x)}{\partial x}B_u(x)B_u^T(x)\varepsilon^T(x).\end{aligned} \tag{3.6}$$

Using (3.6) and (3.4), we can formulate the following theorem.

Theorem 3.2.2 *The polynomial system* (3.1) *is stabilizable via state feedback* (3.2)*, if there exist a polynomial function $V(x)$ satisfying* (3.3)*, a polynomial vector $\varepsilon(x)$ of appropriate dimensions, and a polynomial matrix $K(x)$ satisfying the following condition for* $\forall x \neq 0$

$$M(x) = \begin{bmatrix} M_{11}(x) & (*) \\ M_{21}(x) & -I \end{bmatrix} \prec 0, \tag{3.7}$$

with

$$\begin{aligned} M_{11}(x) =& \frac{\partial V(x)}{\partial x}A(x) + \frac{1}{4}\varepsilon(x)B_u(x)B_u^T(x)\varepsilon^T(x) - \frac{1}{2}\varepsilon(x)B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x}, \\ M_{21}(x) =& \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x) + K^T(x)\right)^T, \end{aligned} \tag{3.8}$$

Proof: It is obvious that using (3.6) in (3.4) yields

$$\begin{aligned} &\frac{\partial V(x)}{\partial x}A(x) + \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x) + K^T(x)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x) + K^T(x)\right)^T \\ &+ \frac{1}{4}\varepsilon(x)B_u(x)B_u^T(x)\varepsilon^T(x) - \frac{1}{2}\varepsilon(x)B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x} = \dot{\hat{V}}(x), \end{aligned} \tag{3.9}$$

thus if a $\hat{V}(x) > 0$ exists such that $\dot{\hat{V}}(x) < 0$, it is clear that $\dot{V}(x)$ is also negative and represents a sufficient condition for asymptotic stability that leads to (3.7) by applying Schur Complement. ■

Unfortunately, there are nonconvex expressions in (3.7) that cannot be solved directly. However, this nonconvexity can be overcome by applying an iterative SOS (ISOS) algorithm.

ISOS algorithm for state feedback control of polynomial systems.

Step 1: Linearize system (3.1). Use the state feedback approach described in [82] to find a solution to the linearized problem. Set $t = 1, \varepsilon_1(x) = x^T P, V_0 = x^T Px$.

Step 2: Solve the following SOS optimization problem in $V_t(x)$ and $K_t(x)$ with fixed auxiliary polynomial vector $\varepsilon_t(x)$ and some positive polynomials $\lambda_1(x)$ and $\lambda_2(x)$:

$$\begin{aligned} &\text{Minimize } \alpha_t \\ &\text{Subject to } V_t(x) - \lambda_1(x) && \text{is a SOS,} \\ &\qquad -v^T\left(M_t^\alpha(x) + \lambda_2(x)I\right)v && \text{is a SOS,} \end{aligned}$$

with

$$M_t^\alpha(x) \triangleq \begin{bmatrix} M_{11}(x) - \alpha_t V_{t-1}(x) & (*) \\ M_{21}(x) & -I \end{bmatrix}, \tag{3.10}$$

v of appropriate dimensions, and $M_{11}(x), M_{21}(x)$ are as in (3.8) with $V(x) \triangleq V_t(x), K(x) \triangleq K_t(x)$, and $\varepsilon(x) \triangleq \varepsilon_t(x)$.

If $\alpha_t < 0$, then $V_t(x)$ and $K_t(x)$ represent a feasible solution to the state feedback control problem of polynomial systems. Terminate the algorithm.

Step 3: Set $t = t + 1$ and solve the following SOS optimization problem in $V_t(x)$, $K_t(x)$, with $Z(x)$ as in Proposition 1.3.1 and the SOS decomposition of the Lyapunov function $V_t(x) = Z(x)^T Q_t Z(x), \varepsilon_t(x) = \varepsilon_{t-1}(x)$ as well as some positive polynomials $\lambda_1(x)$ and $\lambda_2(x)$:

$$\begin{aligned} &\text{Minimize trace}(Q_t) \\ &\text{Subject to } V_t(x) - \lambda_1(x) && \text{is a SOS,} \\ &\qquad -v^T\left(N_t^{\alpha}(x) + \lambda_2(x)I\right)v && \text{is a SOS,} \end{aligned}$$

with

$$N_t^{\alpha}(x) \triangleq \begin{bmatrix} M_{11}(x) - \alpha_{t-1}V_t(x) & (*) \\ M_{21}(x) & -I \end{bmatrix}, \tag{3.11}$$

v of appropriate dimensions, and $M_{11}(x), M_{21}(x)$ are as in (3.8) with $V(x) \triangleq V_t(x), K(x) \triangleq K_t(x)$, and $\varepsilon(x) \triangleq \varepsilon_t(x)$.

Step 4: Solve the following feasibility problem with $v_2 \in \mathbb{R}^{n+1}$ and some positive tolerance function $\delta(x) > 0, x \neq 0$:

$$v_2^T \begin{bmatrix} \delta(x) & (*) \\ \left(\varepsilon_t(x) - \frac{\partial V_t(x)}{\partial x}\right)^T & 1 \end{bmatrix} v_2 \qquad \text{is a SOS.}$$

If the problem is feasible go to Step 5. Else, set $t = t+1$ and $\varepsilon_t(x) = \frac{\partial V_{t-1}(x)}{\partial x}$ determined in Step 3 and go to Step 2.

Step 5: The system (3.1) may not be stabilizable with state feedback (3.2). Terminate the algorithm. ■

The term $-\frac{1}{2}\varepsilon(x)B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x}$ makes (3.5) nonconvex, hence the inequality cannot be solved directly by SOS decomposition. If, however, the auxiliary polynomial vector $\varepsilon(x)$ is fixed, (3.5) becomes convex and can be solved efficiently. Unfortunately, fixing $\varepsilon(x)$ generally does not yield a feasible solution. Therefore, we introduce $\alpha_t V_{t-1}(x)$ in (3.10) to relax the SOS decomposition in (3.7), where $V_{t-1}(x)$ is known from the previous step. This corresponds to the following Lyapunov inequalities:

$$V_t(x) > 0,$$
$$\dot{V}_t(x) \leq \alpha_t V_{t-1}(x).$$

Similar Lyapunov inequalities can be obtained for (3.11), where now α_{t-1} has a known value and thus the product $\alpha_{t-1}V(x)_t$ is convex. It is clear that any negative α in (3.10) or (3.11) yields a feasible solution of the SOS decomposition and the system (3.1) with (3.2) is asymptotic stable.

Step 1 is the initialization of the iterative algorithm and necessary to find an initial value of $\varepsilon_1(x)$ to use in the following iterations. The optimization problem in Step 2 is a generalized eigenvalue minimization problem and guarantees the progressive reduction of α_t. Meanwhile, Step 3 ensures convergence of the algorithm. Step 4 updates $\varepsilon(x)$ and checks whether the iterative algorithm stalls, i.e. the gap between $\varepsilon(x)$ and $\frac{\partial V(x)}{\partial x}$ is smaller than some positive tolerance function $\delta(x)$.

Note that the iterative algorithm increases the iteration variable t twice per cycle (in Step 3 and Step 4). This is done to avoid confusion with the indexes.

3.2.2 Polytropic Stability Synthesis

The results presented in the previous section assume that all system parameters are known exactly. In this section, we extend the results to polynomial systems with polytropic uncertainties.

Consider the following system

$$\begin{aligned} \dot{x} =& A(x,\theta) + B_u(x,\theta)u, \\ y =& x, \end{aligned} \tag{3.12}$$

where the matrices $\cdot(x,\theta)$ are defined as follows

$$A(x,\theta) = \sum_{i=1}^{q} A_i(x)\theta_i, \quad B_u(x,\theta) = \sum_{i=1}^{q} B_{u_i}(x)\theta. \tag{3.13}$$

$\theta = \left[\theta_1, \ldots, \theta_q\right]^T \in \mathbb{R}^q$ is the vector of constant uncertainty and satisfies

$$\theta \in \Theta \triangleq \left\{\theta \in \mathbb{R}^q : \theta_i \geq 0, i = 1, \ldots, q, \sum_{i=1}^{q} \theta_i = 1\right\}. \tag{3.14}$$

We further define the following parameter dependent Lyapunov function

$$V(x) = \sum_{i=1}^{q} V_i(x)\theta_i. \tag{3.15}$$

With the results from the previous section, we can directly propose the main result the state feedback controller design for polynomial systems with polytropic uncertainties.

Theorem 3.2.3 *The polynomial system with polytropic uncertainties* (3.12) *is stabilizable with state feedback* (3.2) *if there exist a polynomial function* $V(x)$ *as in* (3.15)*, a polynomial vector* $\varepsilon(x) = \sum_{i=1}^{q} \varepsilon_i(x)\theta_i$ *of appropriate dimensions, a polynomial matrix* $K(x)$*, as well as some positive functions* $\lambda_1(x) > 0$ *and* $\lambda_2(x) > 0$ *satisfying the following conditions for* $x \neq 0, i = 1, \ldots, q$*:*

$$V_i(x) > 0 \tag{3.16}$$

and

$$M(x) = \sum_{i=1}^{q} M_i(x)\theta_i, \tag{3.17}$$

with

$$M_i(x) = \begin{bmatrix} M_{11}^i(x) & (*) \\ M_{21}^i(x) & -I \end{bmatrix} \prec 0, \tag{3.18}$$

with $M_{11}^i(x), M_{21}^i(x)$ *as in* (3.8) *for each subsystem of* (3.12), *respectively.*

Proof: This theorem follows directly from Theorem 3.2.2. ■

The same ISOS algorithm given in the previous section can be employed to solve for each subsystem of (3.13) with (3.16) and (3.18) with the same controller matrix (3.2) for all subsystems.

3.3 Numerical Examples

In this section, one example for the state feedback controller design for polynomial systems as well as one example for the state feedback controller for systems with polytropic uncertainty will be presented. Both examples are variations of the polynomial system control presented in [26].

3.3.1 State Feedback Control for Polynomial Systems

Consider the polynomial system

$$\dot{x} = \begin{bmatrix} -x_1 + x_1^2 - \frac{3}{2}x_1^3 - \frac{3}{4}x_1x_2^2 + \frac{1}{4}x_2 - x_1^2x_2 - \frac{1}{2}x_2^3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u. \tag{3.19}$$

The system is characterized by one pure integrator, thus the open-loop system is clearly not stable. We select $\lambda_1(x) = \lambda_2(x) = \delta(x) = 0.01\left(x_1^2 + x_2^2\right)$, set $K(x)$ to be of the form $K(x) = \mu_{10}x_1 + \mu_{01}x_2 + \mu_{11}x_1x_2 + \mu_{20}x_1^2 + \mu_{02}x_2^2$ and initially look for Lyapunov function candidates of degree 4. The ISOS algorithm for state feedback design for polynomial systems terminates with a feasible solution and $\|\mu_{11}\| \approx \|\mu_{20}\| \approx \|\mu_{02}\| < 0.01$. After setting $\mu_{11} = \mu_{20} = \mu_{02} = 0$ and initializing $\varepsilon_1(x)$ as the final value of $\frac{\partial V(x)}{\partial x}$, the algorithm terminates after 2 iterations and the following state feedback controller with:

$$K(x_1, x_2) = 3.12x_1 - 4.24x_2. \tag{3.20}$$

and Lyapunov matrix Q from $V(x) = Z(x)^T QZ(x)$ where $Z(x)$ is a vector of monomials up to a degree of 2

$$Z(x)^T = \begin{bmatrix} x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{bmatrix}, \tag{3.21}$$

$$Q = \begin{bmatrix} 1.5407 & -0.0053 & 0.1246 & -0.0122 & -0.0034 \\ -0.0053 & 0.7348 & -0.0355 & 0.0156 & 0.0225 \\ 0.1246 & -0.0355 & 0.8185 & 0.1798 & -0.2656 \\ -0.0122 & 0.0156 & 0.1798 & 0.8604 & 0.4648 \\ -0.0034 & 0.0225 & -0.2656 & 0.4648 & 1.3200 \end{bmatrix}. \tag{3.22}$$

It is noteworthy that it was possible to obtain a *linear* controller for the *polynomial* system (3.19). The closed loop response of the system for initial states $x_0 = [-3, 1]^T$ is shown in Figure 3.1, with the controller gains depicted in Figure 3.2.

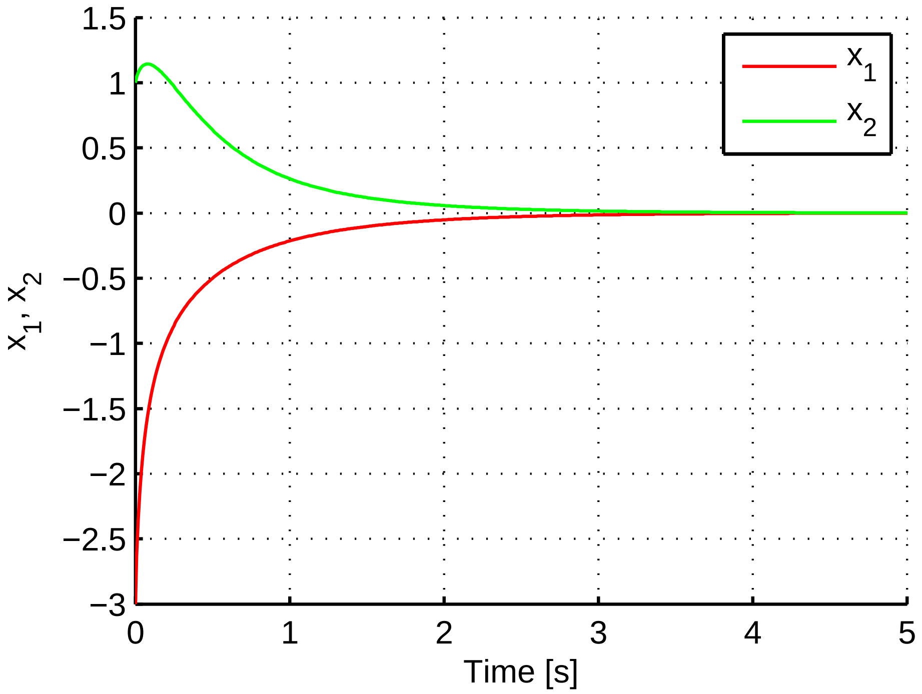


Figure 3.1: Polynomial state feedback control outputs

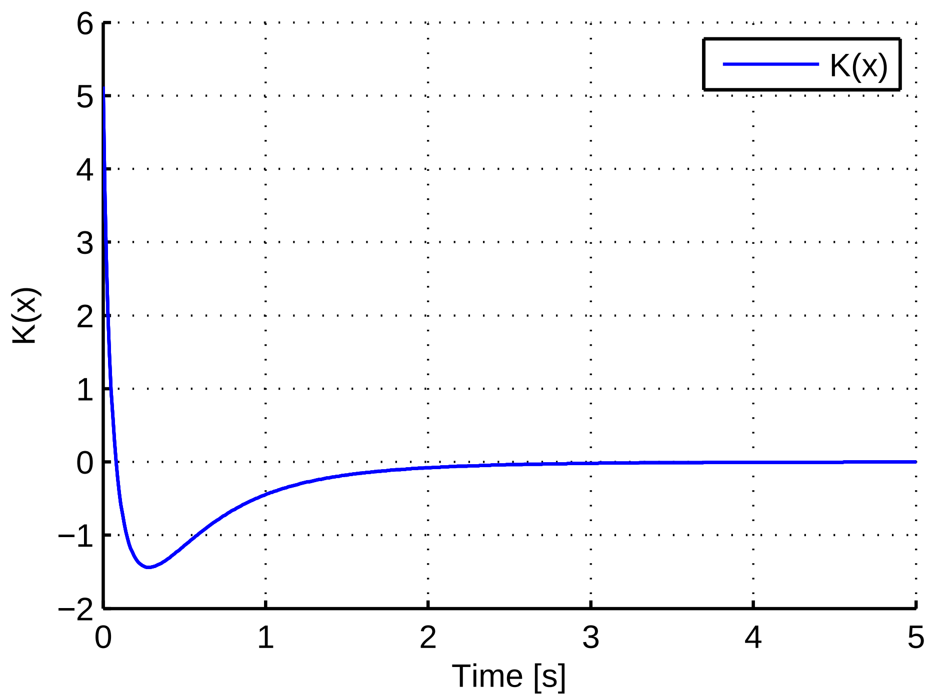


Figure 3.2: Polynomial state feedback control inputs

3.3.2 Polytropic Stability Synthesis

In this example, the previously discussed system will be extended to the case of polynomial system with polytropic uncertainties. Consider the polynomial system with $\beta \in [-1,1]$:

$$\begin{aligned}\dot{x} = &\begin{bmatrix} -x_1 + x_1^2 - \frac{3}{2}x_1^3 - \frac{3}{8}x_1x_2^2 + \frac{1}{4}x_2 - x_1^2x_2 - \frac{1}{4}x_2^3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1.1 \end{bmatrix} u, \\ &+\beta \left(\begin{bmatrix} \frac{3}{8}x_1x_2^2 - \frac{1}{4}x_2^3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u \right).\end{aligned} \tag{3.23}$$

First, the system (3.23) is transformed into form (3.13) with (3.14), i.e. $\theta_1 = 1, \theta_2 = 0$ for $\beta = -1$ and $\theta_1 = 0, \theta_2 = 1$ for $\beta = 1$. Next, we select $\lambda_1(x) = \lambda_2(x) = \delta(x) = 0.01\left(x_1^2 + x_2^2\right)$, set $K(x)$ to be of the form $K(x) = \mu_{10}x_1 + \mu_{01}x_2 + \mu_{11}x_1x_2 + \mu_{20}x_1^2 + \mu_{02}x_2^2$ and initially look for Lyapunov function candidates of degree 4. The ISOS algorithm for state feedback design for polynomial systems terminates with a feasible solution and $\|\mu_{11}\| \approx \|\mu_{20}\| \approx \|\mu_{02}\| < 0.01$. After setting $\mu_{11} = \mu_{20} = \mu_{02} = 0$, initializing $\varepsilon_1(x)$ as the final value of $\frac{\partial V(x)}{\partial x}$, and rerunning the algorithm terminates after 2 iterations and the following state feedback controller is obtained

$$K(x_1, x_2) = -1.64x_1 - 2.37x_2 \tag{3.24}$$

and Lyapunov matrices

$$Q_1 = \begin{bmatrix} 1.7083 & 0.0121 & 0.1648 & 0.1709 & -0.0085 \\ 0.0121 & 1.0235 & 0.0260 & 0.0277 & 0.0206 \\ 0.1648 & 0.0260 & 0.5886 & 0.4247 & -0.3003 \\ 0.1709 & 0.0277 & 0.4247 & 1.0610 & 0.3637 \\ -0.0085 & 0.0206 & -0.3003 & 0.3637 & 1.6723 \end{bmatrix}, \tag{3.25}$$

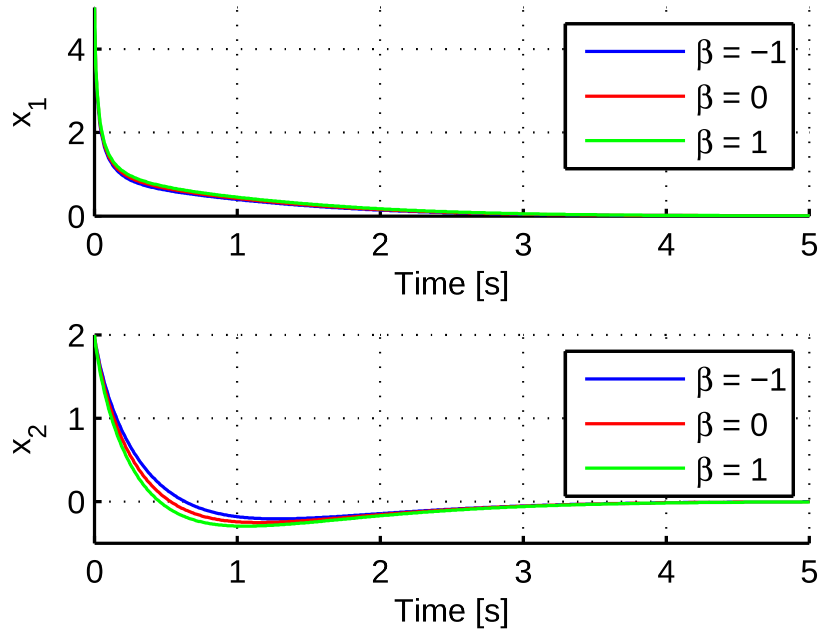


Figure 3.3: System response for polynomial state feedback control with polytropic uncertainties

and

$$Q_2 = \begin{bmatrix} 1.7783 & -0.0250 & 0.3120 & -0.0187 & 0.0450 \\ -0.0250 & 1.1901 & 0.0309 & -0.0127 & -0.0007 \\ 0.3120 & 0.0309 & 1.2522 & -0.2915 & 0.1870 \\ -0.0187 & -0.0127 & -0.2915 & 1.3923 & 0.0000 \\ 0.0450 & -0.0007 & 0.1870 & 0.0000 & 0.8953 \end{bmatrix}, \tag{3.26}$$

where $V_i = Z(x)^T Q_i Z(x)$, with $Z(x)$ as in (3.21).

It was once again possible to obtain a *linear* controller for the *polynomial* control problem. The system response for $\beta = -1, 0, 1$ are depicted in figure 3.3. It can be observed that the system responses for different values of β are quite similar. Further, it is noteworthy that the controller gains are similar in magnitude to the ones obtained for the single state feedback case. It has to be assumed that is has happened purely by accident, as the overall problem without a performance criterion allows for a wide range of feasible solutions.

For this particular example, it is also possible to restrict one of the controller gains to be positive and still find a *linear* controller.

It is also worth mentioning that keeping higher order controller terms do not significantly increase the overall system performance. Once again, this is most likely due to a lack of performance criterion.

3.4 Conclusion

In this chapter, the concept of an iterative design algorithm for the problem of polynomial system control with and without polytropic uncertainties has been presented. In detail, sufficient conditions for the existence of a controller that stabilizes the system and guarantees asymptotic stability have been formulated in terms of polynomial matrix inequalities. An iterative algorithm was introduced to deal with the nonconvex terms in the problem formulation, and the algorithm was able to obtain feasible solutions with very few iterations in numerical examples. Furthermore, it was possible to obtain *linear* controller gains for the *polynomial* system.

Chapter 4

Robust Nonlinear Control of Polynomial Systems with Norm-Bounded Uncertainties

4.1 Introduction

When dealing with real life control systems, it is important to ensure that the obtained control laws will work in the presence of uncertainties [83, 84]. Uncertainties can come from a lot of sources, for example simplification in the system model or parameter inaccuracies [85]. In general, the presence of uncertainties can degrade the system performance significantly, potentially even leading to instability of the overall control system. It is therefore necessary to carefully consider uncertainties in the system.

One way of looking at uncertainties is through the study of system with polytropic uncertainties, as has been presented in the previous chapter. Another way to model uncertainties is by looking at norm-bounded uncertainties. This topic is of particular interest for practical control applications as a lot of control systems can be approximated as (polynomial) system with some degree of norm-bounded uncertainty. Therefore, a vast amount of literature is available on system control with norm-bounded uncertainties, see [86, 87, 88, 89].

In this chapter, the process of designing a controller for polynomial system subject to norm-bounded uncertainties is investigated. Any controller for a system with norm-bounded uncertainties is said to be robust and the overall control system is considered robustly stable with respect to the system dynamics and within the assumed level of uncertainty. In the following, an extension of the previously introduced iterative design algorithm is presented, followed by some numerical examples to showcase the validity of the design approach. Lastly, some conclusions about the problem of polynomial systems with norm-bounded uncertainties are given in the last part of this chapter.

4.2 Main Results

In the following, we consider the uncertain polynomial system of the form

$$\dot{x} = A(x) + B(x)u + \Delta A(x) + \Delta B(x)u, \tag{4.1}$$

where x and u are the system's state and input, respectively. $A(x)$ and $B(x)$ are the polynomial system vector and matrix, respectively. Further, $\Delta A(x)$ and $\Delta B(x)$ are used to capture the uncertain parts of the system design. The following assumption will be used throughout the remainder of this thesis

Assumption 4.2.1 *The admissible parameter uncertainties considered here are assumed to be norm-bounded and can be described as*

$$\begin{bmatrix} \Delta A(x) & \Delta B(x) \end{bmatrix} = H(x)F(x)\begin{bmatrix} E_1(x) & E_2(x) \end{bmatrix}, \tag{4.2}$$

with known polynomial matrices $H(x), E_1(x), E_2(x)$ *of appropriate dimensions and* $F(x)$ *being an unknown state-depended matrix that satisfies*

$$\|F^T(x)F(x)\| \leq I. \tag{4.3}$$

Theorem 4.2.1 *The polynomial system* (4.1) *is controllable via polynomial feedback control of the form*

$$u = K(x) \tag{4.4}$$

if there exist a Lyapunov function $V(x)$*, a polynomial design vector* $\varepsilon(x)$ *of appropriate dimensions satisfying the following conditions for* $x \neq 0$

$$V(x) > 0, \tag{4.5}$$

and

$$M(x) = \begin{bmatrix} M_{11}(x) & (*) & (*) & (*) \\ M_{21}(x) & -I & (*) & (*) \\ M_{31}(x) & 0 & -2I & (*) \\ M_{41}(x) & 0 & 0 & -2I \end{bmatrix} \prec 0, \tag{4.6}$$

with

$$\begin{aligned} M_{11}(x) =& \frac{\partial V(x)}{\partial x}A(x) + \frac{1}{4}\varepsilon(x)B_u(x)B_u^T(x)\varepsilon^T(x) - \frac{1}{2}\varepsilon(x)B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x}, \\ M_{21}(x) =& \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x) + K^T(x)\right)^T, \\ M_{31}(x) =& \left(E_1(x) + E_2(x)K(x)\right), \\ M_{41}(x) =& H^T(x)\frac{\partial V^T(x)}{\partial x}. \end{aligned} \tag{4.7}$$

Proof: The case without uncertainty has been discussed in Theorem 3.2.1 in the previous chapter. Using triangular inequality on the uncertainty term yields

$$\begin{aligned}\Xi(x) =& \frac{\partial V(x)}{\partial x} H(x)F(x)\left(E_1(x)+E_2(x)K(x)\right)\\ \leq& \frac{1}{2}\frac{\partial V(x)}{\partial x} H(x)F(x)F(x)^T H(x)^T \frac{\partial V^T(x)}{\partial x}\\ &+\frac{1}{2}\left(E_1(x)+E_2(x)K(x)\right)^T\left(E_1(x)+E_2(x)K(x)\right)\end{aligned} \tag{4.8}$$

Therefore, using (3.9) and (4.8) combined suggests

$$\begin{aligned}\dot{V}(x) \leq& \frac{\partial V(x)}{\partial x}A(x) + \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x)+K^T(x)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x)+K^T(x)\right)^T\\ &+\frac{1}{4}\varepsilon(x)B_u(x)B_u^T(x)\varepsilon^T(x) - \frac{1}{2}\varepsilon(x)B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x} + \Xi(x)\\ \leq& \frac{\partial V(x)}{\partial x}A(x) + \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x)+K^T(x)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x)+K^T(x)\right)^T\\ &+\frac{1}{4}\varepsilon(x)B_u(x)B_u^T(x)\varepsilon^T(x) - \frac{1}{2}\varepsilon(x)B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x}\\ &+\frac{1}{2}\frac{\partial V(x)}{\partial x}H(x)F(x)F(x)^T H(x)^T\frac{\partial V^T(x)}{\partial x}\\ &+\frac{1}{2}\left(E_1(x)+E_2(x)K(x)\right)^T\left(E_1(x)+E_2(x)K(x)\right) < 0,\end{aligned} \tag{4.9}$$

Thus, if (4.6) and (4.5) hold, it is clear that this satisfies the Lyapunov stability criterion by applying Schur Complements and noting (4.3). ■

The following ISOS approach can be used to find a solution to the control problem of state feedback control for polynomial system with norm-bounded uncertainties

Step 1: Set $F(x) = 0$ and linearize the system (4.1). Use the state feedback approach described in [82] to find a solution of the linearized problems without uncertainty. Set $t = 1, \varepsilon_1(x) = x^T P, V_0 = x^T Px$.

Step 2: Solve the following SOS optimization problem in $V_t(x)$ and $K_t(x)$ with fixed auxiliary polynomial vector $\varepsilon_t(x)$ and some positive polynomials $\lambda_1(x)$ and $\lambda_2(x)$:

$$\begin{aligned}
&\text{Minimize } \alpha_t \\
&\text{Subject to } V_t(x) - \lambda_1(x) && \text{is a SOS,} \\
&\qquad -v^T\left(M_t^{\alpha}(x) + \lambda_2(x)I\right)v && \text{is a SOS,}
\end{aligned}$$

with

$$M_t^{\alpha}(x) \triangleq \begin{bmatrix} M_{11}(x) - \alpha_t V_{t-1}(x) & (*) & (*) & (*) \\ M_{21}(x) & -I & (*) & (*) \\ M_{31}(x) & 0 & -2I & (*) \\ M_{41}(x) & 0 & 0 & -2I \end{bmatrix}, \tag{4.10}$$

v of appropriate dimensions, and $M_{11}(x), M_{21}(x), M_{31}(x), M_{41}(x)$ are as in (4.7) with $V(x) \triangleq V_t(x), K(x) \triangleq K_t(x)$, and $\varepsilon(x) \triangleq \varepsilon_t(x)$.

If $\alpha_t < 0$, then $V_t(x)$ and $K_t(x)$ represent a feasible solution to the H_∞ state feedback control problem of polynomial systems. Terminate the algorithm.

Step 3: Set $t = t+1$ and solve the following SOS optimization problem in $V_t(x)$, $K_t(x)$, with $Z(x)$ as in Proposition 2.2 and the SOS decomposition of the Lyapunov function $V_t(x) = Z(x)^T Q_t Z(x), \varepsilon_t(x) = \varepsilon_{t-1}(x)$ as well as some positive polynomials $\lambda_1(x)$ and $\lambda_2(x)$:

$$\begin{aligned}
&\text{Minimize } \operatorname{trace}(Q_t) \\
&\text{Subject to } V_t(x) - \lambda_1(x) && \text{is a SOS,} \\
&\qquad -v^T\left(N_t^{\alpha}(x) + \lambda_2(x)I\right)v && \text{is a SOS,}
\end{aligned}$$

with

$$N_t^{\alpha}(x) \triangleq \begin{bmatrix} M_{11}(x) - \alpha_{t-1}V_t(x) & (*) & (*) & (*) \\ M_{21}(x) & -I & (*) & (*) \\ M_{31}(x) & 0 & -2I & (*) \\ M_{41}(x) & 0 & 0 & -2I \end{bmatrix}, \qquad (4.11)$$

v of appropriate dimensions, and $M_{11}(x), M_{21}(x), M_{31}(x), M_{41}(x)$ are as in (4.7) with $V(x) \triangleq V_t(x), K(x) \triangleq K_t(x),$ and $\varepsilon(x) \triangleq \varepsilon_t(x)$.

Step 4: Solve the following feasibility problem with $v_2 \in \mathbb{R}^{n+1}$ and some positive tolerance function $\delta(x) > 0, x \neq 0$:

$$v_2^T \begin{bmatrix} \delta(x) & (*) \\ \left(\varepsilon_t(x) - \frac{\partial V_t(x)}{\partial x}\right)^T & 1 \end{bmatrix} v_2 \qquad \text{is a SOS.}$$

If the problem is feasible go to Step 5. Else, set $t = t+1$ and $\varepsilon_t(x) = \frac{\partial V_{t-1}(x)}{\partial x}$ determined in Step 3 and go to Step 2.

Step 5: The system (4.1) may not be stabilizable with state feedback control (4.4). Terminate the algorithm. ■

The term $-\frac{1}{2}\varepsilon(x)B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x}$ makes (4.6) nonconvex, hence the inequality cannot be solved directly by SOS decomposition. If, however, the auxiliary polynomial vector $\varepsilon(x)$ is fixed, (4.6) becomes convex and can be solved efficiently. Unfortunately, fixing $\varepsilon(x)$ generally does not yield a feasible solution. Therefore, we introduce $\alpha_t V_{t-1}(x)$ in (4.10) to relax the SOS decomposition in (4.6), where $V_{t-1}(x)$ is known from the previous step. This corresponds to the following Lyapunov inequalities:

$$V_t(x) > 0,$$
$$\dot{V}_t(x) \leq \alpha_t V_{t-1}(x).$$

Similar Lyapunov inequalities can be obtained for (4.11), where now α_{t-1} has a known value and thus the product $\alpha_{t-1}V(x)_t$ is convex. It is clear that any negative α in (4.10) or (4.11) yields a feasible solution of the SOS decomposition and the system (4.1) with uncertainties as in (4.2)(4.3) is asymptotic stable with state feedback control (4.4).

Step 1 is the initialization of the iterative algorithm and necessary to find an initial value of $\varepsilon_1(x)$ to use in the following iterations. The optimization problem in Step 2 is a generalized eigenvalue minimization problem and guarantees the progressive reduction of α_t. Meanwhile, Step 3 ensures convergence of the algorithm. Step 4 updates $\varepsilon(x)$ and checks whether the iterative algorithm stalls, i.e. the gap between $\varepsilon(x)$ and $\frac{\partial V(x)}{\partial x}$ is smaller than some positive tolerance function $\delta(x)$.

Note that the iterative algorithm increases the iteration variable t twice per cycle (in Step 3 and Step 4). This is done to avoid confusion with the indexes.

4.3 Numerical Example

In the following, a numerical example is used to demonstrate the validity of the iterative design approach.

Consider the following polynomial system

$$\begin{aligned}
\dot{x} =& A(x) + B(x)u + H(x)F(x)\left(E_1(x) + E_2(x)u\right), \\
&A(x) = \begin{bmatrix} -x_1 + x_1^2 - \frac{3}{2}x_1^3 - \frac{3}{8}x_1x_2^2 + \frac{1}{4}x_2 - x_1^2x_2 - \frac{1}{4}x_2^3 \\ 0 \end{bmatrix}, \quad B(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&H(x) = 1, \quad E_1(x) = \begin{bmatrix} \frac{3}{8}x_1x_2^2 - \frac{1}{4}x_2^3 \\ 0 \end{bmatrix}, \quad E_2(x) = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}, \quad F(x) = I\sin(x)
\end{aligned} \tag{4.12}$$

The system has one pure integrator, thus it is clearly open-loop unstable. We select $\lambda_1(x) = \lambda_2(x) = \delta(x) = 0.01\left(x_1^2 + x_2^2\right)$, set the controller to be a function of x up to a degree of 3 and choose to look for Lyapunov function candidates of degree 4. By using

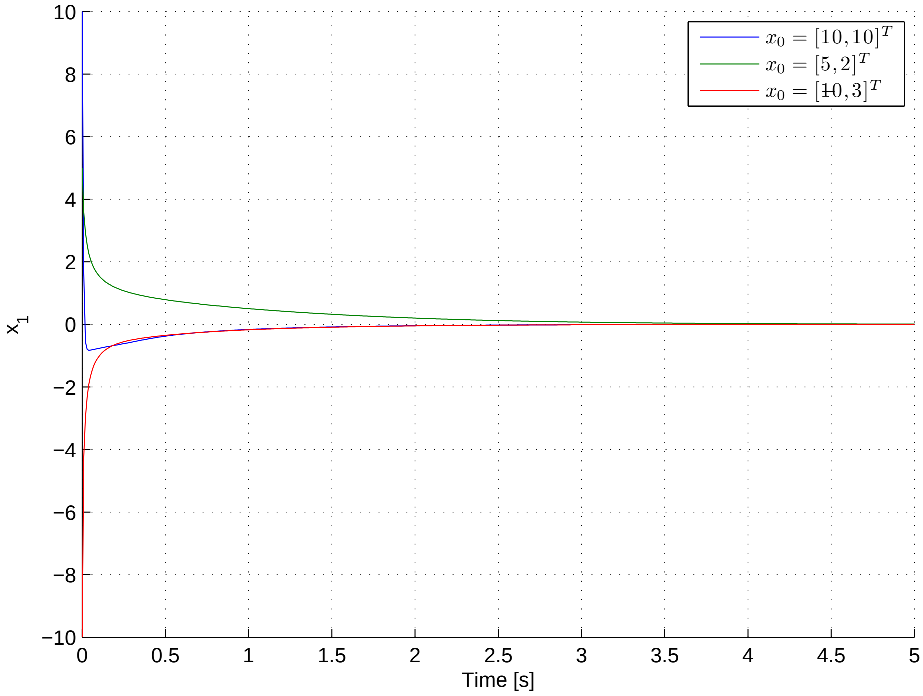


Figure 4.1: History of x_1 for different initial conditions

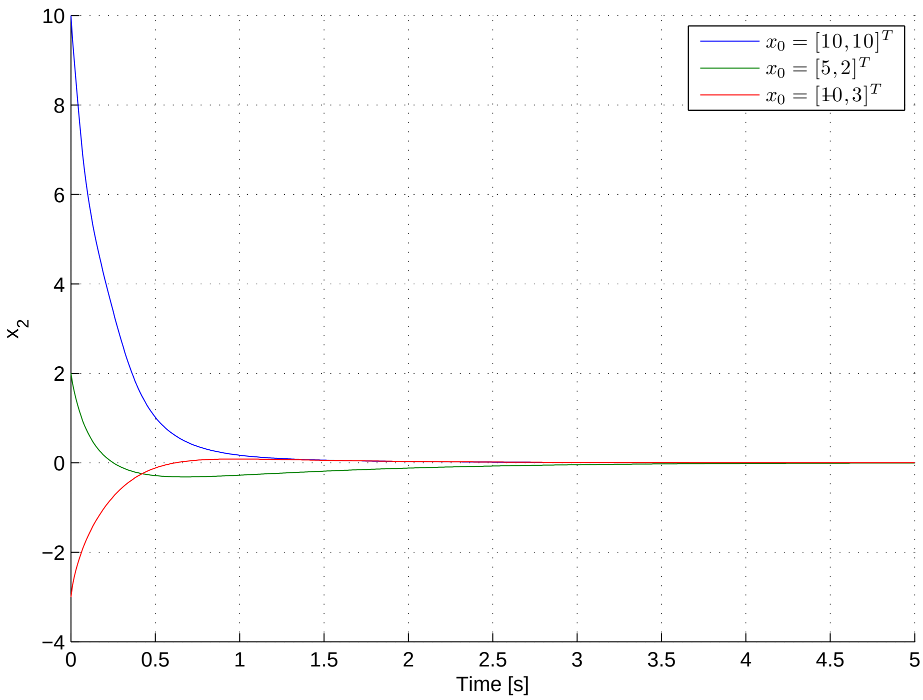


Figure 4.2: History of x_2 for different initial conditions

the ISOS algorithm presented above, we were able to obtain the following controller after 3 iterations

$$K(x) = -2.2740x_1 - 4.812x_2 \tag{4.13}$$

with the following Lyapunov matrix

$$Q = \begin{bmatrix} 1.8654 & 0.0676 & 0.1469 & 0.1406 & 0.0260 \\ 0.0676 & 0.9160 & 0.0247 & 0.0360 & -0.0160 \\ 0.1469 & 0.0247 & 0.6280 & 0.5129 & -0.2520 \\ 0.1406 & 0.0360 & 0.5129 & 1.0884 & 0.3016 \\ 0.0260 & -0.0160 & -0.2520 & 0.3016 & 1.2188 \end{bmatrix}, \tag{4.14}$$

with $V(x) = Z(x)^T QZ(x), Z(x) = \begin{bmatrix} x_1 & x_2 & x_2^2 & x_1x_2 & x_2^2 \end{bmatrix}^T$ and noting that the coefficients for the higher order controller terms were almost zero, thus allowing to find a controller to be linear by initializing $\varepsilon(x)_1 = \frac{\partial V(x)}{\partial x}$ with the results from the first ISOS run and readjusting for a linear $K(x)$. The simulation results for different initial conditions are depicted in figure 4.1 and 4.2 for x_1 and x_2, respectively. The proposed controller efficiently stabilizes the system and the system states converge towards a steady state.

4.4 Conclusion

In this chapter, sufficient conditions for the existence of a state feedback controller for systems with norm-bounded uncertainties has been presented. An iterative algorithm has been used to address the nonconvexity in the problem formulation and the effectiveness of the approach has been outlined with a numerical example.

Chapter 5

Nonlinear H_∞ State Feedback Control for Polynomial Systems with Polytropic Uncertainties

5.1 Introduction

The problem of designing a nonlinear H_∞ controller has attracted considerable attention for more than three decades, see for example [90, 91, 92, 93] and references therein. This interest stems from the nature of the H_∞ control problem. Generally speaking, the aim is to design a controller such that the resulting closed-loop system is stable and a prescribed level of attenuation from the exogenous disturbance input to the controlled output in L_2/l_2-norm is fulfilled. There are two common approaches available to address nonlinear H_∞ control problems: One is based on the theory of dissipative energy [94] and theory of differential games [90]; The other is based on the nonlinear version of the bounded real lemma as developed in [95, 39]. The underlying idea behind both approaches is the conversion of the nonlinear H_∞ control problem into the solvability form of the Hamilton-Jacobi equation

(HJE). Unfortunately, this representation is NP-hard and it is generally very difficult to find a global solution.

In recent years, several approaches utilizing SOS decompositions to achieve nonlinear H_∞ control for polynomial system have been presented, e.g. [45, 46, 57, 30] and references therein. The systems discussed are represented in a state dependent linear-like form and the authors assumed that the control input matrix has some zero rows and that the Lyapunov function only depends on the states whose rows are zero. These assumptions, however, lead to conservatism in the controller design.

The remainder of this chapter is organized as follows: The main results for a single polynomial system, as well as the subsequent extension to the case of systems with polytropic uncertainties is discussed in section 5.2. Some numerical examples are provided in 5.3 to showcase the efficiency of the proposed algorithm before the chapter closes with some final remarks in 5.4.

5.2 Main Results

5.2.1 Nonlinear H_∞ Control for Polynomial Systems

Consider the following dynamic model of a polynomial system

$$\begin{aligned} \dot{x} &= A(x) + B_u(x)u + B_\omega(x)\omega, \\ z &= C_z(x) + D_z(x)u, \end{aligned} \tag{5.1}$$

where $\omega \in \mathbb{R}^p$ is the disturbance input and z is the regulated output. $A(x), C_z(x)$ are polynomial vectors and $B_u(x), B_\omega(x), D_z(x)$ are polynomial matrices of appropriate dimensions. The objective of state feedback H_∞ control is to find a controller $K(x)$ such that the closed-loop system with

$$u = K(x) \tag{5.2}$$

is asymptotically stable and the L_2 gain from the disturbance input to the controlled output is less than a prescribed value $\gamma > 0$. In detail, the following condition must hold:

$$\int_0^\infty z^T z dt \leq \gamma^2 \int_0^\infty \omega^T \omega dt. \tag{5.3}$$

Theorem 5.2.1 *The polynomial system* (5.1) *is stabilizable with a prescribed H_∞ performance $\gamma > 0$ via state feedback* (5.2) *if there exist a polynomial function $V(x)$ and a polynomial matrix $K(x)$ such that for $\forall x \neq 0$ such that*

$$V(x) > 0, \tag{5.4}$$

and

$$\begin{aligned} &\frac{\partial V(x)}{\partial x} A(x) - \frac{1}{4} \frac{\partial V(x)}{\partial x} B_u(x) B_u^T(x) \frac{\partial V^T(x)}{\partial x} \\ &+ \left(\frac{1}{2}\frac{\partial V(x)}{\partial x} B_\omega(x)\right) \frac{1}{\gamma^2} \left(\frac{1}{2}\frac{\partial V(x)}{\partial x} B_\omega(x)\right)^T \\ &+ \left(\frac{1}{2}\frac{\partial V(x)}{\partial x} B_u(x) + K^T(y)\right) \left(\frac{1}{2}\frac{\partial V(x)}{\partial x} B_u(x) + K^T(y)\right)^T \\ &+ \left(C_z(x) + D_z(x) K(y)\right)^T \left(C_z(x) + D_z(x) K(y)\right) < 0. \end{aligned} \tag{5.5}$$

Proof: The case without disturbance has been discussed in Theorem 3.2.1, thus the closed loop system is asymptotically stable with $\omega = 0$. This leave the contribution of

$B_\omega(x)\omega$ and z to the H_∞ control problem, i.e.

$$\begin{aligned}
\Xi(x) =& \frac{\partial V(x)}{\partial x} B_\omega(x)\omega = \frac{\partial V(x)}{\partial x} B_\omega(x)\omega + \left(\gamma^2 \omega^T \omega - z^T z\right) - \left(\gamma^2 \omega^T \omega - z^T z\right) \\
=& -\left(\frac{1}{2\gamma}\frac{\partial V(x)}{\partial x} B_\omega(x) - \gamma\omega^T\right)\left(\frac{1}{2\gamma}\frac{\partial V(x)}{\partial x} B_\omega(x) - \gamma\omega^T\right)^T \\
&+ \left(\frac{1}{2}\frac{\partial V(x)}{\partial x} B_\omega(x)\right)\frac{1}{\gamma^2}\left(\frac{1}{2}\frac{\partial V(x)}{\partial x} B_\omega(x)\right)^T + \left(\gamma^2 \omega^T \omega - z^T z\right) \\
&+ \left(C_z(x) + D_z(x)K(x)\right)^T \left(C_z(x) + D_z(x)K(x)\right) \\
\leq& \left(\frac{1}{2}\frac{\partial V(x)}{\partial x} B_\omega(x)\right)\frac{1}{\gamma^2}\left(\frac{1}{2}\frac{\partial V(x)}{\partial x} B_\omega(x)\right)^T + \left(\gamma^2 \omega^T \omega - z^T z\right) \\
&+ \left(C_z(x) + D_z(x)K(x)\right)^T \left(C_z(x) + D_z(x)K(x)\right) = \tilde{\Xi}(x).
\end{aligned} \tag{5.6}$$

Using (3.9) and adding $\tilde{\Xi}(x)$ from (5.6), we have

$$\begin{aligned}
\dot{V}(x) \leq& \frac{\partial V(x)}{\partial x} A(x) - \frac{1}{4}\frac{\partial V(x)}{\partial x} B_u(x) B_u^T(x) \frac{\partial V^T(x)}{\partial x} \\
&+ \left(\frac{1}{2}\frac{\partial V(x)}{\partial x} B_\omega(x)\right)\frac{1}{\gamma^2}\left(\frac{1}{2}\frac{\partial V(x)}{\partial x} B_\omega(x)\right)^T \\
&+ \left(\frac{1}{2}\frac{\partial V(x)}{\partial x} B_u(x) + K^T(y)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x} B_u(x) + K^T(y)\right)^T \\
&+ \left(C_z(x) + D_z(x)K(y)\right)^T \left(C_z(x) + D_z(x)K(y)\right) \\
&+ \left(\gamma^2 \omega^T \omega - z^T z\right).
\end{aligned} \tag{5.7}$$

Thus, if (5.5) holds, we have

$$\dot{V}(x) < -z^T z + \gamma^2 \omega^T \omega.$$

Integrating both sides of the inequality yields

$$\begin{aligned}
\int_0^\infty \dot{V}(x) dt &\leq \int_0^\infty \left(-z^T z + \gamma^2 \omega^T \omega\right) dt, \\
V(x(\infty)) - V(x(0)) &\leq \int_0^\infty \left(-z^T z + \gamma^2 \omega^T \omega\right) dt.
\end{aligned}$$

Noting that with initial conditions $x(0) = 0$ and $V(x(\infty)) \geq 0$, we obtain

$$\int_0^\infty z^T z dt \leq \gamma^2 \int_0^\infty \omega^T \omega dt.$$

Hence (5.3) holds and H_∞ performance is fulfilled. ■

Theorem 5.2.2 *The polynomial system* (5.1) *is stabilizable with prescribed H_∞ performance $\gamma > 0$ via state feedback* (5.2)*, if there exist a polynomial function $V(x)$ satisfying* (5.4)*, a polynomial vector $\varepsilon(x)$ of appropriate dimensions, and a polynomial matrix $K(x)$ satisfying the following condition for $\forall x \neq 0$*

$$M(x) = \begin{bmatrix} M_{11}(x) & (*) & (*) & (*) \\ M_{21}(x) & -I & (*) & (*) \\ M_{31}(x) & 0 & -I & (*) \\ M_{41}(x) & 0 & 0 & -\gamma^2 I \end{bmatrix} \prec 0, \tag{5.8}$$

with

$$\begin{aligned} M_{11}(x) =& \frac{\partial V(x)}{\partial x} A(x) + \frac{1}{4}\varepsilon(x) B_u(x) B_u^T(x) \varepsilon^T(x) - \frac{1}{2}\varepsilon(x) B_u(x) B_u^T(x) \frac{\partial V^T(x)}{\partial x}, \\ M_{21}(x) =& \left(\frac{1}{2}\frac{\partial V(x)}{\partial x} B_u(x) + K^T(x)\right)^T, \\ M_{31}(x) =& C_z(x) + D_z(x) K(x), \\ M_{41}(x) =& \left(\frac{1}{2}\frac{\partial V(x)}{\partial x} B_\omega(x)\right)^T. \end{aligned} \tag{5.9}$$

Proof: Using (3.6) in (5.5) yields

$$\begin{aligned}
&\frac{\partial V(x)}{\partial x}A(x)+\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x)+K^T(x)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x)+K^T(x)\right)^T \\
&+\frac{1}{4}\varepsilon(x)B_u(x)B_u^T(x)\varepsilon^T(x)-\frac{1}{2}\varepsilon(x)B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x} \\
&+\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_\omega(x)\right)\frac{1}{\gamma^2}\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_\omega(x)\right)^T \\
&+\left(C_z(x)+D_z(x)K(x)\right)^T\left(C_z(x)+D_z(x)K(x)\right)<0,
\end{aligned} \tag{5.10}$$

which is a sufficient condition for H_∞ stability. Applying Schur Complement results in (5.8). ■

With this, the following iterative SOS algorithm for H_∞ control polynomial systems can be proposed.

Step 1: Linearize system (5.1) and set $\omega = 0$. Use the state feedback approach described in [82] to find a solution to the linearized problem without disturbance. Set $t = 1, \varepsilon_1(x) = x^T P, V_0 = x^T P x$.

Step 2: Solve the following SOS optimization problem in $V_t(x)$ and $K_t(x)$ with fixed auxiliary polynomial vector $\varepsilon_t(x)$ and some positive polynomials $\lambda_1(x)$ and $\lambda_2(x)$:

$$\begin{aligned}
&\text{Minimize } \alpha_t \\
&\text{Subject to } V_t(x)-\lambda_1(x) && \text{is a SOS,} \\
&\qquad -v^T\left(M_t^\alpha(x)+\lambda_2(x)I\right)v && \text{is a SOS,}
\end{aligned}$$

with

$$M_t^\alpha(x) \triangleq \begin{bmatrix} M_{11}(x)-\alpha_t V_{t-1}(x) & (*) & (*) & (*) \\ M_{21}(x) & -I & (*) & (*) \\ M_{31}(x) & 0 & -I & (*) \\ M_{41}(x) & 0 & 0 & -\gamma^2 I \end{bmatrix}, \tag{5.11}$$

v of appropriate dimensions, and $M_{11}(x), M_{21}(x), M_{31}(x), M_{41}(x)$ are as in (8.1) with $V(x) \triangleq V_t(x), K(x) \triangleq K_t(x),$ and $\varepsilon(x) \triangleq \varepsilon_t(x)$.

If $\alpha_t < 0$, then $V_t(x)$ and $K_t(x)$ represent a feasible solution to the H_∞ state feedback control problem of polynomial systems. Terminate the algorithm.

Step 3: Set $t = t+1$ and solve the following SOS optimization problem in $V_t(x)$, $K_t(x)$, with $Z(x)$ as in Proposition 1.3.2 and the SOS decomposition of the Lyapunov function $V_t(x) = Z(x)^T Q_t Z(x), \varepsilon_t(x) = \varepsilon_{t-1}(x)$ as well as some positive polynomials $\lambda_1(x)$ and $\lambda_2(x)$:

$$\begin{aligned} &\text{Minimize trace}(Q_t) \\ &\text{Subject to } V_t(x) - \lambda_1(x) && \text{is a SOS,} \\ &\qquad -v^T\left(N_t^\alpha(x) + \lambda_2(x)I\right)v && \text{is a SOS,} \end{aligned}$$

with

$$N_t^\alpha(x) \triangleq \begin{bmatrix} M_{11}(x) - \alpha_{t-1}V_t(x) & (*) & (*) & (*) \\ M_{21}(x) & -I & (*) & (*) \\ M_{31}(x) & 0 & -I & (*) \\ M_{41}(x) & 0 & 0 & -\gamma^2 I \end{bmatrix}, \tag{5.12}$$

v of appropriate dimensions, and $M_{11}(x), M_{21}(x), M_{31}(x), M_{41}(x)$ are as in (8.1) with $V(x) \triangleq V_t(x), K(x) \triangleq K_t(x),$ and $\varepsilon(x) \triangleq \varepsilon_t(x)$.

Step 4: Solve the following feasibility problem with $v_2 \in \mathbb{R}^{n+1}$ and some positive tolerance function $\delta(x) > 0, x \neq 0$:

$$v_2^T \begin{bmatrix} \delta(x) & (*) \\ \left(\varepsilon_t(x) - \frac{\partial V_t(x)}{\partial x}\right)^T & 1 \end{bmatrix} v_2 \qquad \text{is a SOS.}$$

If the problem is feasible go to Step 5. Else, set $t = t+1$ and $\varepsilon_t(x) = \frac{\partial V_{t-1}(x)}{\partial x}$ determined in Step 3 and go to Step 2.

Step 5: The system (5.1) may not be stabilizable with H_∞ performance γ by state feedback (5.2). Terminate the algorithm.

The term $-\frac{1}{2}\varepsilon(x)B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x}$ makes (5.8) non-convex, hence the inequality cannot be solved directly by SOS decomposition. If, however, the auxiliary polynomial vector $\varepsilon(x)$ is fixed, (5.8) becomes convex and can be solved efficiently. Unfortunately, fixing $\varepsilon(x)$ generally does not yield a feasible solution. Therefore, we introduce $\alpha_t V_{t-1}(x)$ in (5.11) to relax the SOS decomposition in (5.8). This corresponds to the following Lyapunov inequalities:

$$\begin{aligned} V_t(x) &> 0, \\ \dot{V}_t(x) &\leq \alpha_t V_{t-1}(x). \end{aligned}$$

Similar Lyapunov inequalities can be obtained for (5.12). It is clear that any negative α in (5.11) or (5.12) yields a feasible solution of the SOS decomposition and the system (5.1) with (5.2) can be stabilized with H_∞ performance γ with state feedback control.

Step 1 is the initialization of the iterative algorithm and necessary to find an initial value of $\varepsilon_1(x)$ to use in the following iterations. The optimization problem in Step 2 is a generalized eigenvalue minimization problem and guarantees the progressive reduction of α_t. Meanwhile, Step 3 ensures convergence of the algorithm. Step 4 updates $\varepsilon(x)$ and checks whether the iterative algorithm stalls, i.e. the gap between $\varepsilon(x)$ and $\frac{\partial V(x)}{\partial x}$ is smaller than some positive tolerance function $\delta(x)$.

Note that the iterative algorithm increases the iteration variable t twice per cycle (in Step 3 and Step 4). This is done to avoid confusion with the indexes.

5.2.2 Polytropic H_∞ Control Synthesis

The results from the previous section assume that all system parameters are known exactly. In this section, the results are extended to polynomial system with polytropic uncertainties. Consider the system

$$\begin{aligned}\dot{x} =& A(x,\theta)+B_u(x,\theta)u+B_\omega(x,\theta)w,\\ z =& C_z(x,\theta)+D_z(x,\theta)u,\end{aligned} \tag{5.13}$$

where the matrices $\cdot(x,\theta)$ are defined as follows

$$\begin{aligned}&A(x,\theta)=\sum_{i=1}^{q}A_i(x)\theta_i,\quad B_u(x,\theta)=\sum_{i=1}^{q}B_{u_i}(x)\theta,\quad B_\omega(x,\theta)=\sum_{i=1}^{q}B_{\omega_i}(x)\theta,\\ &C_z(x,\theta)=\sum_{i=1}^{q}C_{z_i}(x)\theta,\quad D_z(x,\theta)=\sum_{i=1}^{q}D_{z_i}(x)\theta.\end{aligned} \tag{5.14}$$

$\theta=\begin{bmatrix}\theta_1,\ldots,\theta_q\end{bmatrix}^T\in\mathbb{R}^q$ is the vector of constant uncertainty and satisfies

$$\theta\in\Theta\triangleq\left\{\theta\in\mathbb{R}^q:\theta_i\geq 0, i=1,\ldots,q,\sum_{i=1}^{q}\theta_i=1\right\}. \tag{5.15}$$

We further define the following parameter dependent Lyapunov function

$$V(x)=\sum_{i=1}^{q}V_i(x)\theta_i. \tag{5.16}$$

With the results from the previous section and the discussions in Chapter 3.2.2, we can directly propose the theorem for robust H_∞ state feedback controller design for polynomial systems with polytropic uncertainties.

Theorem 5.2.3 *The polynomial system with parametric uncertainties* (5.13) *is stabilizable with prescribed H_∞ performance $\gamma>0$ via state feedback control* (5.2) *if there exist a polynomial function $V(x)$ as in* (5.16), *a polynomial vector $\varepsilon(x)=\sum_{i=1}^{q}\varepsilon_i(x)\theta_i$ of appropriate dimensions, a polynomial matrix $K(x)$, as well as some positive functions $\lambda_1(x)>0$ and*

$\lambda_2(x) > 0$ *satisfying the following conditions for* $x \neq 0, i = 1, \ldots, q$*:*

$$V_i(x) > 0 \tag{5.17}$$

and

$$M(x) = \sum_{i=1}^{q} M_i(x)\theta_i, \tag{5.18}$$

with

$$M_i(x) = \begin{bmatrix} M_{11}^i(x) & (*) & (*) & (*) \\ M_{21}^i(x) & -I & (*) & (*) \\ M_{31}^i(x) & 0 & -I & (*) \\ M_{41}^i(x) & 0 & 0 & -\gamma^2 I \end{bmatrix} \prec 0, \tag{5.19}$$

with $M_{11}^i(x), M_{21}^i(x), M_{31}^i(x), M_{41}^i(x)$ *as in* (5.9) *for each subsystem of* (5.13)*, respectively.*

Proof: This follows directly from Theorem 5.2.2. ■

The iterative algorithm from the previous section can be adjusted to reflect the changes from Theorem 5.2.2 to Theorem 5.2.3 as follows.

Step 1: Linearize each system from (5.13) and set $\omega = 0$. Use the state feedback approach described in [82] to find a solution to each of the linearized problems without disturbance. For $i = 1, \ldots, q$, set $t = 1$ and $[\varepsilon_i(x)]_1 = x^T P_i, [V_i(x)]_0 = x^T P_i x$.

Step 2: Solve the following SOS optimization problem in $[V_i(x)]_t$ and $K_t(x)$ with fixed auxiliary polynomial vectors $[\varepsilon_i(x)]_t$ and some positive polynomials λ_1 and λ_2 for

$i = 1, \ldots, q$:

$$\begin{aligned} &\text{Minimize } \alpha_t \\ &\text{Subject to } [V_i(x)]_t - \lambda_1(x), && \text{is a SOS,} \\ &\qquad - v^T \left([M_i^{\alpha}(x)]_t + \lambda_2(x) I\right) v && \text{is a SOS,} \end{aligned}$$

with

$$[M_i^{\alpha}(x)]_t \triangleq \begin{bmatrix} M_{11}^i(x) - \alpha_t [V_i(x)]_{t-1} & (*) & (*) & (*) \\ M_{21}^i(x) & -I & (*) & (*) \\ M_{31}^i(x) & 0 & -I & (*) \\ M_{41}^i(x) & 0 & 0 & -\gamma^2 I \end{bmatrix}, \tag{5.20}$$

v of appropriate dimensions, and $M_{11}^i(x), M_{21}^i(x), M_{31}^i(x), M_{41}^i(x)$ are as in (5.9) with $V(x) \triangleq [V_i(x)]_t, K(y) \triangleq K_t(y)$, and $\varepsilon(x) \triangleq [\varepsilon_i(x)]_t$ for each subsystem of (5.14), respectively.

If $\alpha_t < 0$, then $V_t(x) = \sum_{i=1}^q [V_i(x)]_t \theta_i$ and $K_t(x)$ represent a feasible solution. Terminate the algorithm.

Step 3: Set $t = t + 1$ and solve the following SOS optimization problem in $[V_i(x)]_t, K_t(x)$, with $Z(x)$ as in Proposition 1.3.2 and the SOS decomposition of $[V_i(x)]_t = Z(x)^T [Q_i]_t Z(x)$, and $[\varepsilon_i(x)]_t = [\varepsilon_i(x)]_{t-1}$ as well as some positive polynomials $\lambda_1(x)$ and $\lambda_2(x)$ for $i = 1, \ldots, q$:

$$\begin{aligned} &\text{Minimize } \sum_{i=1}^{q} \text{trace}([Q_i]_t) \\ &\text{Subject to } [V_i(x)]_t - \lambda_1(x) && \text{is a SOS,} \\ &\qquad - v^T \left([N_i^{\alpha}(x)]_t + \lambda_2(x) I\right) v && \text{is a SOS,} \end{aligned}$$

with

$$[N_i^{\alpha}(x)]_t \triangleq \begin{bmatrix} M_{11}^i(x) - \alpha_{t-1}[V_i(x)]_t & (*) & (*) & (*) \\ M_{21}^i(x) & -I & (*) & (*) \\ M_{31}^i(x) & 0 & -I & (*) \\ M_{41}^i(x) & 0 & 0 & -\gamma^2 I \end{bmatrix}, \qquad (5.21)$$

v of appropriate dimensions, and $M_{11}^i(x), M_{21}^i(x), M_{31}^i(x), M_{41}^i(x)$ as in (5.9) with $V(x) \triangleq [V_i(x)]_t, K(x) \triangleq K_t(x)$, and $\varepsilon(x) \triangleq [\varepsilon_i(x)]_t$ for each subsystem of (5.13), respectively.

Step 4: Solve the following feasibility problem with $v_2 \in \mathbb{R}^{n+1}$ and some positive tolerance function $\delta(x) > 0, x \neq 0$ for $i = 1, \ldots, q$:

$$v_2^T \begin{bmatrix} \delta(x) & (*) \\ \left(\varepsilon_t^i(x) - \frac{\partial V_t^i(x)}{\partial x}\right)^T & 1 \end{bmatrix} v_2 \qquad \text{is a SOS.}$$

If the problem is feasible go to Step 5. Else, set $t = t+1$ and $[\varepsilon_i(x)]_t = \left[\frac{\partial V_i(x)}{\partial x}\right]_{t-1}$, for $i = 1, \ldots, q$ determined in Step 3 and go to Step 2.

Step 5: The system (5.13) may not be stabilizable with H_∞ performance γ by state feedback control (5.2). Terminate the algorithm. ■

5.3 Numerical Example

Consider the following polynomial system with norm-bounded uncertainties with $\beta \in [-1, 1]$:

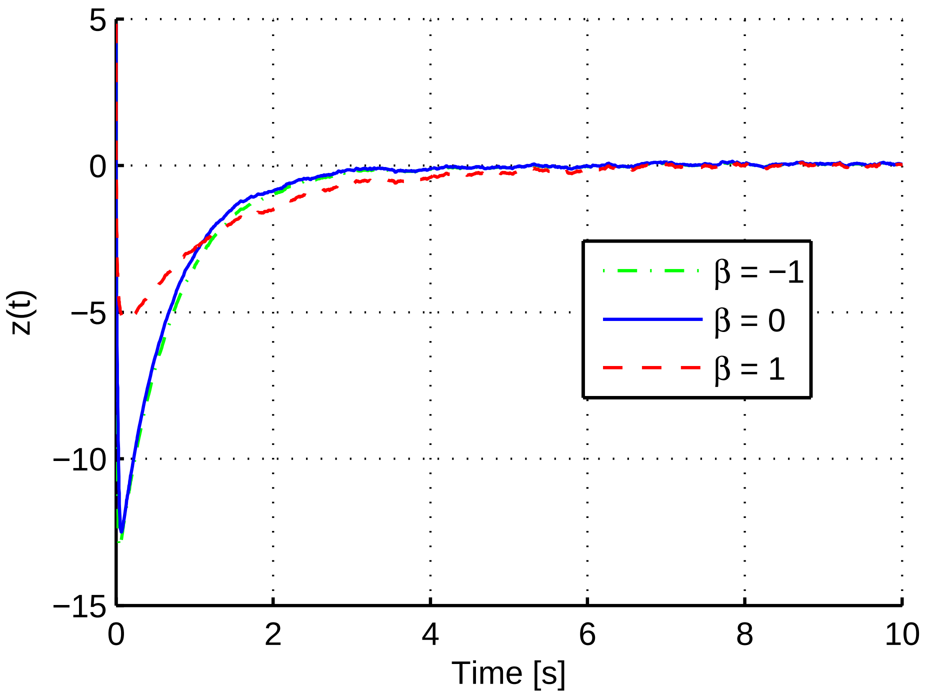


Figure 5.1: Regulated output

$$\begin{aligned}\dot{x} =& \begin{bmatrix} -x_1 + x_1^2 - \frac{3}{2}x_1^3 - \frac{3}{8}x_1x_2^2 + \frac{1}{4}x_2 - x_1^2x_2 - \frac{1}{4}x_2^3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1.1 \end{bmatrix} u + \begin{bmatrix} 1.25 \\ 0 \end{bmatrix} \omega, \\ &+\beta\left(\begin{bmatrix} \frac{3}{8}x_1x_2^2 - \frac{1}{4}x_2^3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u + \begin{bmatrix} 0.25 \\ 0 \end{bmatrix} \omega\right), \\ z =& u. \end{aligned} \tag{5.22}$$

We select $\lambda_1(x) = \lambda_2(x) = \delta(x) = 0.01\left(x_1^2 + x_2^2\right)$, set the controller to be a function of x up to a degree of 3 and choose to look for Lyapunov function candidates of degree 4. The ISOS algorithm terminates with a feasible solution for $\gamma^2 = 1.423$ after 3 iterations with very small coefficients for the higher order terms in $K(x)$. Thus, we initialize $\varepsilon_t^i(x) = \frac{\partial V_i(x)}{\partial x}$, adjust $K(x)$ to be linear and rerun the algorithm for a linear controller. After 3 iterations, the following H_∞ controller for the polynomial system (6.14) with polytropic uncertainties

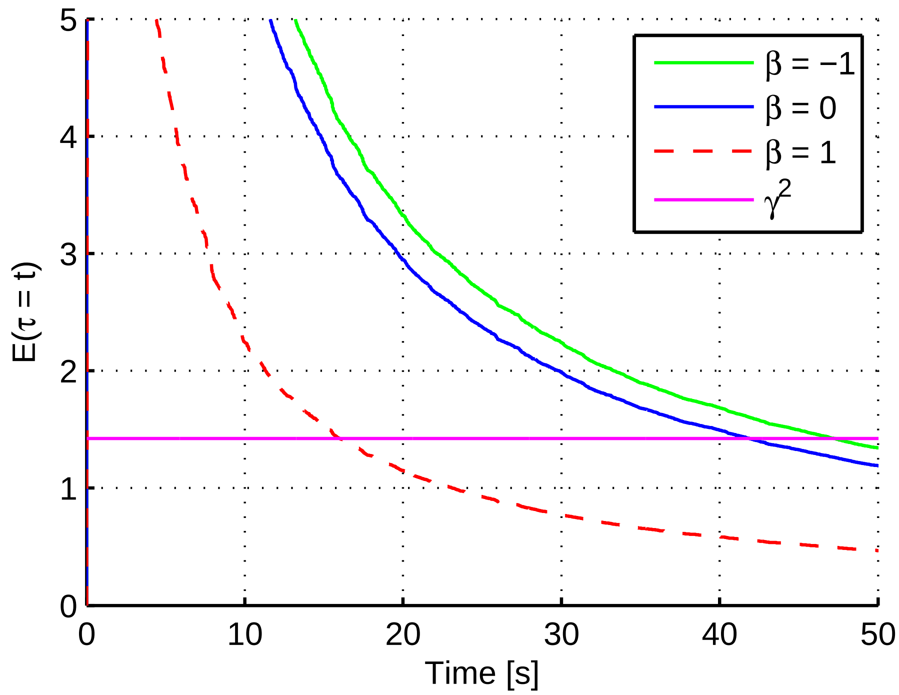


Figure 5.2: Energy ratio $E(\tau) = \frac{\int_0^\tau z^T z dt}{\int_0^\tau \omega^T \omega dt}$

has been obtained

$$K(x) = 1.137x_1 - 0.653x_2, \tag{5.23}$$

with Lyapunov functions

$$Q_0 = \begin{bmatrix} 1.2940 & -0.0322 & 0.1408 & 0.0515 & -0.0128 \\ -0.0322 & 0.6587 & -0.0028 & -0.0367 & -0.1138 \\ 0.1408 & -0.0028 & 0.3003 & 0.1737 & -0.1311 \\ 0.0515 & -0.0367 & 0.1737 & 0.5404 & 0.3576 \\ -0.0128 & -0.1138 & -0.1311 & 0.3576 & 1.0156 \end{bmatrix} \tag{5.24}$$

and

$$Q_1 = \begin{bmatrix} 0.2912 & -0.0485 & -0.0169 & -0.0093 & 0.0068 \\ -0.0485 & 0.4639 & 0.0015 & -0.0065 & -0.0033 \\ -0.0169 & 0.0015 & 0.1692 & 0.0942 & -0.1621 \\ -0.0093 & -0.0065 & 0.0942 & 0.2807 & 0.0054 \\ 0.0068 & -0.0033 & -0.1621 & 0.0054 & 0.6608 \end{bmatrix} \tag{5.25}$$

for $V_i(x) = Z(x)^T Q_i Z(x), i = 1,2$ and $Z(x) = \begin{bmatrix} x_1 & x_2 & x_2^2 & x_1x_2 & x_2^2 \end{bmatrix}^T$.

Once again, it was possible to obtain a *linear* controller for the *polynomial* system. For initial states $x_0^T = \begin{bmatrix} 10 & 10 \end{bmatrix}^T$ and a disturbance modelled with Gaussian white noise with power density spectrum of 0.01, the trajectories of the regulated output are depicted in Figure 5.1. Figure 5.2 depicts the overall energy in the system. It can be observed that $E(\tau) = \frac{\int_0^\tau z^T z}{\int_0^\tau \omega^T \omega}$ falls below the prescribed performance value after 20 seconds for $\beta = 1$, and after around 50 seconds for any admissible value of β.

5.4 Conclusion

An iterative procedure to obtain a H_∞ state feedback controller for polynomials with polytropic uncertainties has been presented in this section. Sufficient conditions for the existence of a H_∞ controller have been derived in terms of bilinear matrix inequalities. An iterative algorithm has been proposed that results in a polynomial controller that avoids rational components encountered when inverting the Lyapunov function in traditional control approaches. Further, the Lyapunov function has been shown to be true function of all system states and is not restricted to only incorporates states which corresponding rows in the control matrix are zeros. A numerical example has been provided to show the effectiveness of the proposed approach.

Chapter 6

Robust Nonlinear H_∞ State Feedback Control for Polynomial Systems with Norm-Bounded Uncertainties

6.1 Introduction

The motivation for this chapter stems from the concepts provided in the previous chapters. In dealing with real life applications, we would like to ensure that our controller is robust enough to stabilize the system in the presence of disturbances. Furthermore, it is desirable to ensure that the controller is optimized in a way that the overall system response ensures that the effect of disturbances on the system output is minimized. This chapter is organized as follows. In section 6.2, the main results for the robust nonlinear H_∞ state feedback control problem for polynomial systems with norm-bounded uncertainties are presented. A numerical example is provided in section 6.3 to showcase the validity of the design approach before the chapter is concluded with final remarks in section 6.4.

6.2 Main Results

Consider the following polynomial system with norm-bounded uncertainties

$$\begin{aligned} \dot{x} &= A(x) + B_u(x) + B_\omega(x) + \Delta A(x) + \Delta B_u(x), \\ z &= C_z(x) + D_z(x)u, \end{aligned} \tag{6.1}$$

where $x \in \mathbb{R}^n$ are the system states, $u \in \mathbb{R}^m$ is the input and z is the controlled output. $A(x)$ and $C_z(x)$ are polynomial vectors and B_u, B_ω, D_z are polynomial matrices of appropriate dimensions. The disturbance signal is ω, whereas the norm-bounded uncertainties of the system are captured in $\Delta A(x)$ and $\Delta B_u(x)$. The objective of a state feedback H_∞ control is to find a controller $K(x)$ such that the system (6.1) with

$$u = K(x) \tag{6.2}$$

is asymptotically stable and the L_2 gain from the disturbance input to the controlled output is less than a prescribed value $\gamma > 0$, that is

$$\int_0^\infty z^T z dt \le \gamma^2 \int_0^\infty \omega^T \omega dt. \tag{6.3}$$

The following assumption is used for the norm-bounded uncertainty

Assumption 6.2.1 *The admissible parameter uncertainties considered here are assumed to be norm-bounded and can be described as*

$$\begin{bmatrix} \Delta A(x) & \Delta B(x) \end{bmatrix} = H(x)F(x) \begin{bmatrix} E_1(x) & E_2(x) \end{bmatrix}, \tag{6.4}$$

with known polynomial matrices $H(x), E_1(x), E_2(x)$ of appropriate dimensions and $F(x)$ being an unknown state-depended matrix that satisfies

$$\|F^T(x)F(x)\| \leq I. \tag{6.5}$$

Theorem 6.2.1 *The polynomial system* (6.1) *is stabilizable with a prescribed H_∞ performance $\gamma > 0$ via state feedback controller* (6.2) *if there exist a polynomial function $V(x)$ and a polynomial matrix $K(x)$ such that $\forall x \neq 0$*

$$V(x) > 0 \tag{6.6}$$

and

$$\begin{aligned}
&\frac{\partial V(x)}{\partial x}A(x) - \frac{1}{4}\frac{\partial V(x)}{\partial x}B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x} \\
&+\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_\omega(x)\right)\frac{1}{\gamma^2}\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_\omega(x)\right)^T \\
&+\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x)+K^T(y)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x)+K^T(y)\right)^T \\
&+\left(C_z(x)+D_z(x)K(y)\right)^T\left(C_z(x)+D_z(x)K(y)\right) \\
&+\frac{1}{2}\frac{\partial V(x)}{\partial x}H(x)F(x)F(x)^T H(x)^T\frac{\partial V^T(x)}{\partial x} \\
&+\frac{1}{2}\left(E_1(x)+E_2(x)K(x)\right)^T\left(E_1(x)+E_2(x)K(x)\right) < 0.
\end{aligned} \tag{6.7}$$

Proof: The proof follows directly from Theorem 4.2.1 and Theorem 5.2.1. ■

Theorem 6.2.2 *The polynomial system* (6.1) *is stabilizable with H_∞ norm $\gamma > 0$ via polynomial state feedback control and*

$$u = K(x) \tag{6.8}$$

if there exist a Lyapunov function $V(x)$, a polynomial design vector $\varepsilon(x)$ of appropriate dimensions satisfying the following conditions for $x \neq 0$

$$V(x) > 0, \tag{6.9}$$

and

$$M(x) = \begin{bmatrix} M_{11}(x) & (*) & (*) & (*) & (*) & (*) \\ M_{21}(x) & -I & (*) & (*) & (*) & (*) \\ M_{31}(x) & 0 & -2I & (*) & (*) & (*) \\ M_{41}(x) & 0 & 0 & -2I & (*) & (*) \\ M_{51}(x) & 0 & 0 & 0 & -I & (*) \\ M_{61}(x) & 0 & 0 & 0 & 0 & -\gamma^2 \end{bmatrix} \prec 0, \tag{6.10}$$

where

$$\begin{aligned} M_{11}(x) =& \frac{\partial V(x)}{\partial x} A(x) + \frac{1}{4}\varepsilon(x)B_u(x)B_u^T(x)\varepsilon^T(x) - \frac{1}{2}\varepsilon(x)B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x}, \\ M_{21}(x) =& \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x) + K^T(x)\right)^T, \\ M_{31}(x) =& \left(E_1(x) + E_2(x)K(x)\right), \\ M_{41}(x) =& H^T(x)\frac{\partial V^T(x)}{\partial x}, \\ M_{51}(x) =& C_z(x) + D_z(x)K(x), \\ M_{61}(x) =& \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_\omega(x)\right)^T. \end{aligned} \tag{6.11}$$

Proof: This follows directly from applying Schur Complement to Theorem 6.2.1. ■

The term $-\frac{1}{2}\varepsilon(x)B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x}$ makes (6.10) nonconvex, hence the inequality cannot be solved directly by SOS decomposition. Therefore, the following iterative SOS algorithm is proposed

Step 1: Linearize the system (6.1) and set $\omega = 0, F(x) = 0$. Use the state feedback approach described in [82] to find a solution to the linearized problem without disturbance. Set $t = 1$ and $\varepsilon_1(x) = x^T P, V_0 = x^T P x$.

Step 2: Solve the following SOS optimization problem in $V_t(x)$ and $K_t(x)$ with fixed auxiliary polynomial vector $\varepsilon_t(x)$ and some positive polynomials $\lambda_1(x)$ and $\lambda_2(x)$:

$$\begin{aligned} &\text{Minimize } \alpha_t \\ &\text{Subject to } V_t(x) - \lambda_1(x) && \text{is a SOS,} \\ &\qquad -v^T\left(M_t^\alpha(x) + \lambda_2(x) I\right) v && \text{is a SOS,} \end{aligned}$$

with

$$M_t^\alpha(x) \triangleq \begin{bmatrix} M_{11}(x) - \alpha_t V(x)_{t-1} & (*) & (*) & (*) & (*) & (*) \\ M_{21}(x) & -I & (*) & (*) & (*) & (*) \\ M_{31}(x) & 0 & -2I & (*) & (*) & (*) \\ M_{41}(x) & 0 & 0 & -2I & (*) & (*) \\ M_{51}(x) & 0 & 0 & 0 & -I & (*) \\ M_{61}(x) & 0 & 0 & 0 & 0 & -\gamma^2 \end{bmatrix}, \qquad (6.12)$$

v of appropriate dimensions, and $M_{11}(x), M_{21}(x), M_{31}(x), M_{41}(x), M_{51}(x), M_{61}(x)$ are as in (6.11) with $V(x) \triangleq V_i(x)_t, K(x) \triangleq K_t(x)$, and $\varepsilon(x) \triangleq \varepsilon_t(x)$.

If $\alpha_t < 0$, then $V_t(x) = V(x)_t$ and $K_t(x)$ represent a feasible solution. Terminate the algorithm.

Step 3: Set $t = t+1$ and solve the following SOS optimization problem in $V(x)_t, K_t(x)$, with $Z(x)$ of appropriate dimensions as in Proposition 1.3.1 and the SOS decomposition of $V(x)_t = Z(x)^T Q_t Z(x)$, and $\varepsilon(x)_t = \varepsilon(x)_{t-1}$ as well as some positive polynomials $\lambda_1(x)$ and $\lambda_2(x)$:

$$\begin{aligned} &\text{Minimize trace}(Q_t) \\ &\text{Subject to } V_t(x) - \lambda_1(x) && \text{is a SOS,} \\ &\qquad -v^T\,(N_t^\alpha(x) + \lambda_2(x)I)\,v && \text{is a SOS,} \end{aligned}$$

with

$$N_t^\alpha(x) \triangleq \begin{bmatrix} M_{11}(x) - \alpha_{t-1}V(x)_t & (*) & (*) & (*) & (*) & (*) \\ M_{21}(x) & -I & (*) & (*) & (*) & (*) \\ M_{31}(x) & 0 & -2I & (*) & (*) & (*) \\ M_{41}(x) & 0 & 0 & -2I & (*) & (*) \\ M_{51}(x) & 0 & 0 & 0 & -I & (*) \\ M_{61}(x) & 0 & 0 & 0 & 0 & -\gamma^2 \end{bmatrix}, \tag{6.13}$$

v of appropriate dimensions, and $M_{11}(x), M_{21}(x), M_{31}(x), M_{41}(x), M_{51}(x), M_{61}(x)$ are as in (6.11) with $V(x) \triangleq V_i(x)_t, K(x) \triangleq K_t(x),$ and $\varepsilon(x) \triangleq \varepsilon_t(x)$.

Step 4: Solve the following feasibility problem with $v_2 \in \mathbb{R}^{n+1}$ and some positive tolerance function $\delta(x) > 0, x \neq 0$ for $i = 1, \dots, q$:

$$v_2^T \begin{bmatrix} \delta(x) & (*) \\ \left(\varepsilon_t(x) - \frac{\partial V_t(x)}{\partial x}\right)^T & 1 \end{bmatrix} v_2 \qquad \text{is a SOS.}$$

If the problem is feasible go to Step 5. Else, set $t = t+1$ and $\varepsilon_t(x) = \frac{\partial V_{t-1}(x)}{\partial x}$ determined in Step 3 and go to Step 2.

Step 5: The system (6.1) may not be stabilizable with H_∞ performance γ by state feedback control (6.2). Terminate the algorithm.

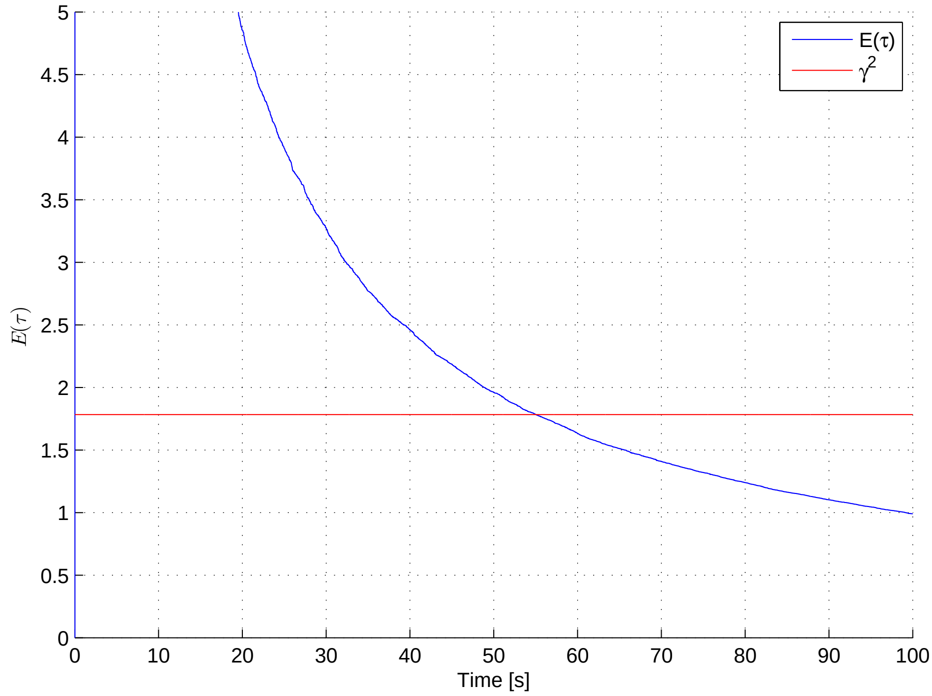


Figure 6.1: Energy ratio $E(\tau) = \frac{\int_0^\tau z^T z dt}{\int_0^\tau \omega^T \omega dt}$

6.3 Numerical Example

Consider the following polynomial system with polytropic uncertainties

$$
\begin{aligned}
\dot{x} =& A(x) + B(x)u + H(x)F(x)\left(E_1(x) + E_2(x)u\right) + B_\omega \omega, \\
& A(x) = \begin{bmatrix} -x_1 + x_1^2 - \frac{3}{2}x_1^3 - \frac{3}{8}x_1x_2^2 + \frac{1}{4}x_2 - x_1^2x_2 - \frac{1}{4}x_2^3 \\ 0 \end{bmatrix}, \\
& B(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_\omega = \begin{bmatrix} 1.25 \\ 0 \end{bmatrix}, \quad H(x) = 1, \\
& E_1(x) = \begin{bmatrix} \frac{3}{8}x_1x_2^2 - \frac{1}{4}x_2^3 \\ 0 \end{bmatrix}, \quad E_2(x) = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}, \quad F(x) = I\sin(x) \\
z =& u.
\end{aligned}
\tag{6.14}
$$

We select $\lambda_1(x) = \lambda_2(x) = \delta(x) = 0.01\left(x_1^2 + x_2^2\right)$, set the controller to be a function of x up to a degree of 3 and choose to look for Lyapunov function candidates of degree 4.

The ISOS algorithm terminates without finding a feasible solution. Therefore, the degree of the Lyapunov function candidate is increased to 6 and the algorithm is restarted. The ISOS algorithm terminates with a feasible solution for $\gamma^2 = 1.783$ after 3 iterations with very small coefficients for the higher order terms in $K(x)$. Thus, we set these to zero, initialize $\varepsilon_1(x) = \frac{\partial V(x)}{\partial x}$ from the previous solution and rerun the algorithm for a linear controller. After 3 iterations, the following H_∞ controller for the polynomial system (6.14) with polytropic uncertainties has been obtained

$$K(x) = -1.893x_1 - 2.642x_2, \tag{6.15}$$

Note that the Lyapunov function is omitted here due to its size. The smallest eigenvalue of Q has been found as 1.376×10^{-3}.

Once again, it was possible to obtain a *linear* controller for the *polynomial* system. The disturbance has been modelled as Gaussian white noise with power density spectrum of 0.01, and Figure 6.1 shows the ratio of the regulated output energy to the noise energy over time. The ratio clearly falls below the design threshold value of γ^2 after less than 60 seconds.

6.4 Conclusion

An iterative design algorithm for the problem of designing a robust H_∞ controller for polynomial systems with norm-bounded uncertainties has been presented in this chapter. In detail, sufficient conditions for the existence of a controller that stabilizes the system with H_∞ performance γ in the presence of norm-bounded uncertainties has been derived in the form of polynomial matrix inequalities. The nonconvex components of these conditions have been addressed using an iterative design algorithm, and a numerical example has been provided to show the effectiveness of the proposed procedure. Furthermore, it was possible to obtain *linear* controller gains for the *polynomial* system.

Chapter 7

Nonlinear H_∞ Output Feedback Control for Polynomial Systems with Polytropic Uncertainties

7.1 Introduction

The results in Chapters 3 to 6 were derived under the assumption that all system states are available for the controller design. This is, however, only rarely the case in real life control problems. Therefore, a lot of research has been undertaken in the field of static output control, see [96] and references therein for a comprehensive survey. Among other things, the authors prove that any dynamic output feedback problem can be transformed into a static output feedback problem. Therefore, it is possible to design a full order dynamic output feedback control law within the framework of static output feedback control. The converse, however, is not true.

Compared to the linear case, the study of polynomial static output feedback is a rather new field, see for example [33, 26]. In relation to the design of state feedback control for polynomial systems, the design of a static output feedback controller represents a more complex problem and several approaches to deal with the resulting problems have been proposed. In particular, [33] suggests to use the Hermite Stability Criterion and use a SOS/moment primal/dual approach to generate a monotone sequence that converges to the global optimum. In [26], an upper bound is introduced to limit the effect of the nonconvex terms. To determine a suitable upper bound is, however, hard and the overall closed loop stability can only be guarantee in a neighborhood of the origin.

In this chapter, the design problem of nonlinear H_∞ output feedback for polynomial systems is discussed. In 7.2 the results for a nominal polynomial system are presented in form of solvability conditions of polynomial matrix inequalities that are subsequently addressed by a relaxation of the nonconvex terms and solved with an interactive SOS algorithm. The presented framework is successively extended to the case of polynomial systems with polytropic uncertainties. In 7.3 a numerical example is provided to show the effectiveness of the proposed design. Some closing remarks are made in 7.4.

7.2 Main Results

The first part of this section investigates the problem of designing a H_∞ output feedback controller for a polynomial system. In the second part, the results are extended to the case of polynomial system with polytropic uncertainties.

7.2.1 Nonlinear H_∞ Output Feedback Control for Polynomial Systems

Consider the following dynamic model of a polynomial system

$$\begin{aligned} \dot{x} =& A(x) + B_u(x)u + B_\omega(x)\omega, \\ y =& C_y(x) + D_y(x)u, \\ z =& C_z(x) + D_z(x)u, \end{aligned} \tag{7.1}$$

where $\omega \in \mathbb{R}^p$ is the disturbance input, y and z are the measured and regulated output, respectively. $A(x), C_y(x), C_z(x)$ are polynomial vectors and $B_u(x), B_\omega(x), C_z(x), D_z(x)$ are polynomial matrices of appropriate dimensions. The objective of static output feedback H_∞ control is to find a controller $K(y)$ such that the closed-loop system with

$$u = K(y) \tag{7.2}$$

is asymptotically stable and the L_2 gain from the disturbance input to the controlled output is less than a prescribed value $\gamma > 0$. In detail, the following condition must hold:

$$\int_0^\infty z^T z dt \leq \gamma^2 \int_0^\infty \omega^T \omega dt. \tag{7.3}$$

Theorem 7.2.1 *The polynomial system* (7.1) *is stabilizable with a prescribed H_∞ performance $\gamma > 0$ via static output feedback* (7.2) *if there exist a polynomial function $V(x)$ and a polynomial matrix $K(y)$ such that for $\forall x \neq 0$ such that*

$$V(x) > 0, \tag{7.4}$$

and

$$\begin{aligned}&\frac{\partial V(x)}{\partial x}A(x) - \frac{1}{4}\frac{\partial V(x)}{\partial x}B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x}\\&+\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_\omega(x)\right)\frac{1}{\gamma^2}\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_\omega(x)\right)^T\\&+\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x)+K^T(y)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x)+K^T(y)\right)^T\\&+\left(C_z(x)+D_z(x)K(y)\right)^T\left(C_z(x)+D_z(x)K(y)\right) < 0.\end{aligned} \tag{7.5}$$

Proof: Note that for $\forall x \neq 0$

$$
\begin{aligned}
\dot{V}(x) =& \frac{\partial V(x)}{\partial x}\left[A(x) + B_u(x)K(y) + B_\omega(x)\omega\right] \\
\leq& \frac{\partial V(x)}{\partial x}\left[A(x) + B_u(x)K(y) + B_\omega(x)\omega\right] + \left(\gamma\omega^T\omega - z^Tz\right) \\
& - \left(\gamma\omega^T\omega - z^Tz\right) + K^T(y)K(y) \\
=& \frac{\partial V(x)}{\partial x}A(x) - \frac{1}{4}\frac{\partial V(x)}{\partial x}B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x} + \Theta(x,y)\Theta(x,y)^T \\
& + \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_\omega(x)\right)\frac{1}{\gamma^2}\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_\omega(x)\right)^T + z^Tz \\
& - \Theta_\omega(x,\omega)\Theta_\omega(x,\omega)^T + \left(\gamma^2\omega^T\omega - z^Tz\right) \\
\leq& \frac{\partial V(x)}{\partial x}A(x) - \frac{1}{4}\frac{\partial V(x)}{\partial x}B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x} + \Theta(x,y)\Theta(x,y)^T \\
& + \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_\omega(x)\right)\frac{1}{\gamma^2}\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_\omega(x)\right)^T + z^Tz \\
& + \left(\gamma^2\omega^T\omega - z^Tz\right),
\end{aligned}
\tag{7.6}
$$

with

$$
\begin{aligned}
\Theta(x,y) &= \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x) + K^T(y)\right) \\
\Theta_\omega(x,\omega) &= \left(\frac{1}{2\gamma}\frac{\partial V(x)}{\partial x}B_\omega(x) - \gamma\omega^T\right).
\end{aligned}
$$

Thus, if there exist a $V(x) > 0$ such that (7.5) holds, it follows that

$$
\dot{V}(x) < -z^Tz + \gamma^2\omega^T\omega.
$$

Integrating both sides of the inequality yields

$$
\begin{aligned}
\int_0^\infty \dot{V}(x)dt &\leq \int_0^\infty \left(-z^Tz + \gamma^2\omega^T\omega\right)dt, \\
V(x(\infty)) - V(x(0)) &\leq \int_0^\infty \left(-z^Tz + \gamma^2\omega^T\omega\right)dt.
\end{aligned}
$$

Noting that with initial conditions $x(0) = 0$ and $V(x(\infty)) \geq 0$, we obtain

$$\int_0^\infty z^T z dt \leq \gamma^2 \int_0^\infty \omega^T \omega dt. \tag{7.7}$$

Hence (7.5) holds and H_∞ performance is fulfilled.

To proof asymptotic stability for the closed-loop system (7.1) with (7.2), the disturbance is set $\omega(t) = 0$. From (7.7) it is obvious that $\dot{V}(x) < 0$, hence the Lyapunov stability theorem is fulfilled and the closed-loop system (7.1) with (7.2) is asymptotically stable. ■

Theorem 7.2.2 *The polynomial system* (7.1) *is stabilizable with prescribed H_∞ performance $\gamma > 0$ via static output feedback* (7.2)*, if there exist a polynomial function $V(x)$ satisfying* (7.4) *and* (7.5)*, a polynomial vector $\varepsilon(x)$ of appropriate dimensions, and a polynomial matrix $K(y)$ satisfying the following condition for $\forall x \neq 0$*

$$M(x,y) = \begin{bmatrix} M_{11}(x) & (*) & (*) & (*) \\ M_{21}(x,y) & -I & (*) & (*) \\ M_{31}(x,y) & 0 & -I & (*) \\ M_{41}(x) & 0 & 0 & -\gamma^2 I \end{bmatrix} \prec 0, \tag{7.8}$$

with

$$\begin{aligned} M_{11}(x) =& \frac{\partial V(x)}{\partial x} A(x) + \frac{1}{4}\varepsilon(x) B_u(x) B_u^T(x) \varepsilon^T(x) - \frac{1}{2}\varepsilon(x) B_u(x) B_u^T(x) \frac{\partial V^T(x)}{\partial x}, \\ M_{21}(x,y) =& \left(\frac{1}{2} \frac{\partial V(x)}{\partial x} B_u(x) + K^T(y) \right)^T, \\ M_{31}(x,y) =& C_z(x) + D_z(x) K(y), \\ M_{41}(x) =& \left(\frac{1}{2} \frac{\partial V(x)}{\partial x} B_\omega(x) \right)^T. \end{aligned} \tag{7.9}$$

Proof: Using (3.6) in (7.5) yields

$$\begin{aligned}
&\frac{\partial V(x)}{\partial x}A(x)+\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x)+K^T(y)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x)+K^T(y)\right)^T\\
&+\frac{1}{4}\varepsilon(x)B_u(x)B_u^T(x)\varepsilon^T(x)-\frac{1}{2}\varepsilon(x)B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x}\\
&+\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_\omega(x)\right)\frac{1}{\gamma^2}\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_\omega(x)\right)^T\\
&+\left(C_z(x)+D_z(x)K(y)\right)^T\left(C_z(x)+D_z(x)K(y)\right)<0,
\end{aligned}\tag{7.10}$$

which is a sufficient condition for H_∞ stability. Applying Schur Complement results in (7.8). ■

With this, the following iterative SOS algorithm for H_∞ control polynomial systems can be proposed.

Step 1: Linearize system (7.1) and set $\omega=0$. Use the static output feedback approach described in [82] to find a solution to the linearized problem without disturbance. Set $t=1,\varepsilon_1(x)=x^TP,V_0=x^TPx$.

Step 2: Solve the following SOS optimization problem in $V_t(x)$ and $K_t(y)$ with fixed auxiliary polynomial vector $\varepsilon_t(x)$ and some positive polynomials $\lambda_1(x)$ and $\lambda_2(x)$:

Minimize α_t

Subject to $V_t(x)-\lambda_1(x)$ is a SOS,

$-v^T\left(M_t^\alpha(x,y)+\lambda_2(x)I\right)v$ is a SOS,

with

$$M_t^\alpha(x,y)\triangleq\begin{bmatrix} M_{11}(x)-\alpha_tV_{t-1}(x) & (*) & (*) & (*)\\ M_{21}(x,y) & -I & (*) & (*)\\ M_{31}(x,y) & 0 & -I & (*)\\ M_{41}(x) & 0 & 0 & -\gamma^2I\end{bmatrix},\tag{7.11}$$

v of appropriate dimensions, and $M_{11}(x), M_{21}(x,y), M_{31}(x,y), M_{41}(x)$ are as in (7.9) with $V(x) \triangleq V_t(x), K(y) \triangleq K_t(y)$, and $\varepsilon(x) \triangleq \varepsilon_t(x)$.

If $\alpha_t < 0$, then $V_t(x)$ and $K_t(y)$ represent a feasible solution to the H_∞ static output feedback control problem of polynomial systems. Terminate the algorithm.

Step 3: Set $t = t+1$ and solve the following SOS optimization problem in $V_t(x)$, $K_t(y)$, with $Z(x)$ as in Proposition 1.3.2 and the SOS decomposition of the Lyapunov function $V_t(x) = Z(x)^T Q_t Z(x), \varepsilon_t(x) = \varepsilon_{t-1}(x)$ as well as some positive polynomials $\lambda_1(x)$ and $\lambda_2(x)$:

$$\begin{aligned} &\text{Minimize trace}(Q_t) \\ &\text{Subject to } V_t(x) - \lambda_1(x) \qquad \text{is a SOS}, \\ &\qquad -v^T\left(N_t^\alpha(x,y) + \lambda_2(x) I\right) v \qquad \text{is a SOS}, \end{aligned}$$

with

$$N_t^\alpha(x,y) \triangleq \begin{bmatrix} M_{11}(x) - \alpha_{t-1} V_t(x) & (*) & (*) & (*) \\ M_{21}(x,y) & -I & (*) & (*) \\ M_{31}(x,y) & 0 & -I & (*) \\ M_{41}(x) & 0 & 0 & -\gamma^2 I \end{bmatrix}, \tag{7.12}$$

v of appropriate dimensions, and $M_{11}(x), M_{21}(x,y), M_{31}(x,y), M_{41}(x)$ are as in (7.9) with $V(x) \triangleq V_t(x), K(y) \triangleq K_t(y)$, and $\varepsilon(x) \triangleq \varepsilon_t(x)$.

Step 4: Solve the following feasibility problem with $v_2 \in \mathbb{R}^{n+1}$ and some positive tolerance function $\delta(x) > 0, x \neq 0$:

$$v_2^T \begin{bmatrix} \delta(x) & (*) \\ \left(\varepsilon_t(x) - \frac{\partial V_t(x)}{\partial x}\right)^T & 1 \end{bmatrix} v_2 \qquad \text{is a SOS.}$$

If the problem is feasible go to Step 5. Else, set $t = t + 1$ and $\varepsilon_t(x) = \frac{\partial V_{t-1}(x)}{\partial x}$ determined in Step 3 and go to Step 2.

Step 5: The system (7.1) may not be stabilizable with H_∞ performance γ by static output feedback (7.2). Terminate the algorithm.

The term $-\frac{1}{2}\varepsilon(x)B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x}$ makes (7.5) non-convex, hence the inequality cannot be solved directly by SOS decomposition. If, however, the auxiliary polynomial vector $\varepsilon(x)$ is fixed, (7.5) becomes convex and can be solved efficiently. Unfortunately, fixing $\varepsilon(x)$ generally does not yield a feasible solution. Therefore, we introduce $\alpha_t V_{t-1}(x)$ in (7.11) to relax the SOS decomposition in (7.5). This corresponds to the following Lyapunov inequalities:

$$
\begin{aligned}
V_t(x) &> 0, \\
\dot{V}_t(x) &\leq \alpha_t V_{t-1}(x).
\end{aligned}
$$

Similar Lyapunov inequalities can be obtained for (7.12). It is clear that any negative α in (7.11) or (7.12) yields a feasible solution of the SOS decomposition and the system (7.1) with (7.2) can be stabilized with H_∞ performance γ with static output feedback control.

Step 1 is the initialization of the iterative algorithm and necessary to find an initial value of $\varepsilon_1(x)$ to use in the following iterations. The optimization problem in Step 2 is a generalized eigenvalue minimization problem and guarantees the progressive reduction of α_t. Meanwhile, Step 3 ensures convergence of the algorithm. Step 4 updates $\varepsilon(x)$ and checks whether the iterative algorithm stalls, i.e. the gap between $\varepsilon(x)$ and $\frac{\partial V(x)}{\partial x}$ is smaller than some positive tolerance function $\delta(x)$.

Note that the iterative algorithm increases the iteration variable t twice per cycle (in Step 3 and Step 4). This is done to avoid confusion with the indexes.

7.2.2 Polytropic H_∞ Output Feedback Synthesis

The results from the previous section assume that all system parameters are known exactly. In this section, the results are extended to polynomial system with polytropic uncertainties. Consider the system

$$\begin{aligned} \dot{x} =& A(x,\theta) + B_u(x,\theta)u + B_\omega(x,\theta)w, \\ y =& C_y(x,\theta), \\ z =& D_z(x,\theta) + D_z(x,\theta)u, \end{aligned} \tag{7.13}$$

where the matrices $\cdot(x,\theta)$ are defined as follows

$$\begin{aligned} A(x,\theta) &= \sum_{i=1}^{q} A_i(x)\theta_i, \quad B_u(x,\theta) = \sum_{i=1}^{q} B_{u_i}(x)\theta, \quad B_\omega(x,\theta) = \sum_{i=1}^{q} B_{\omega_i}(x)\theta, \\ C_y(x,\theta) &= \sum_{i=1}^{q} C_{y_i}(x)\theta, \quad C_z(x,\theta) = \sum_{i=1}^{q} C_{z_i}(x)\theta, \quad D_z(x,\theta) = \sum_{i=1}^{q} D_{z_i}(x)\theta. \end{aligned} \tag{7.14}$$

$\theta = \begin{bmatrix} \theta_1, \ldots, \theta_q \end{bmatrix}^T \in \mathbb{R}^q$ is the vector of constant uncertainty and satisfies

$$\theta \in \Theta \triangleq \left\{ \theta \in \mathbb{R}^q : \theta_i \geq 0, i = 1, \ldots, q, \sum_{i=1}^{q} \theta_i = 1 \right\}. \tag{7.15}$$

We further define the following parameter dependent Lyapunov function

$$V(x) = \sum_{i=1}^{q} V_i(x)\theta_i. \tag{7.16}$$

With the results from the previous section and the discussions in Chapter 7.2.1, we can directly propose the theorem for robust H_∞ static output feedback controller design for polynomial systems with polytropic uncertainties.

Theorem 7.2.3 *The polynomial system with parametric uncertainties* (7.13) *is stabilizable with prescribed H_∞ performance $\gamma > 0$ via static output feedback control* (7.2) *if there exist a polynomial function $V(x)$ as in* (7.16)*, a polynomial vector $\varepsilon(x) = \sum_{i=1}^{q} \varepsilon_i(x)\theta_i$ of appro-*

priate dimensions, a polynomial matrix $K(y)$, as well as some positive functions $\lambda_1(x) > 0$ and $\lambda_2(x) > 0$ satisfying the following conditions for $x \neq 0, i = 1, \ldots, q$:

$$V_1(x) > 0 \tag{7.17}$$

and

$$M(x,y) = \sum_{i=1}^{q} M_i(x,y)\theta_i, \tag{7.18}$$

with

$$M_i(x,y) = \begin{bmatrix} M_{11}^i(x) & (*) & (*) & (*) \\ M_{21}^i(x,y) & -I & (*) & (*) \\ M_{31}^i(x,y) & 0 & -I & (*) \\ M_{41}^i(x) & 0 & 0 & -\gamma^2 I \end{bmatrix} \prec 0, \tag{7.19}$$

with $M_{11}^i(x), M_{21}^i(x,y), y, M_{31}^i(x,y), M_{41}^i(x)$ as in (7.9) *for each subsystem of* (7.13)*, respectively.*

Proof: This follows directly from Theorem 7.2.2. ■

The iterative algorithm from the previous section can be adjusted to reflect the changes from Theorem 7.2.2 to Theorem 7.2.3 as follows.

Step 1: Linearize each system from (7.13) and set $\omega = 0$. Use the static output feedback approach described in [82] to find a solution to each of the linearized problems without disturbance. For $i = 1, \ldots, q$, set $t = 1$ and $[\varepsilon_i(x)]_1 = x^T P_i$, $[V_i(x)]_0 = x^T P_i x$.

Step 2: Solve the following SOS optimization problem in $[V_i(x)]_t$ and $K_t(y)$ with fixed auxiliary polynomial vectors $[\varepsilon_i(x)]_t$ and some positive polynomials λ_1 and λ_2 for

$i = 1, \ldots, q$:

$$\begin{aligned} &\text{Minimize } \alpha_t \\ &\text{Subject to } [V_i(x)]_t - \lambda_1(x), && \text{is a SOS,} \\ &\qquad -v^T\left([M_i^{\alpha}(x)]_t + \lambda_2(x)I\right)v && \text{is a SOS,} \end{aligned}$$

with

$$[M_i^{\alpha}(x,y)]_t \triangleq \begin{bmatrix} M_{11}^i(x) - \alpha_t\,[V_i(x)]_{t-1} & (*) & (*) & (*) \\ M_{21}^i(x,y) & -I & (*) & (*) \\ M_{31}^i(x,y) & 0 & -I & (*) \\ M_{41}^i(x) & 0 & 0 & -\gamma^2 I \end{bmatrix}, \tag{7.20}$$

v of appropriate dimensions, and $M_{11}^i(x), M_{21}^i(x,y), M_{31}^i(x,y), M_{41}^i(x)$ are as in (7.9) with $V(x) \triangleq [V_i(x)]_t, K(y) \triangleq K_t(y)$, and $\varepsilon(x) \triangleq [\varepsilon_i(x)]_t$ for each subsystem of (7.13), respectively.

If $\alpha_t < 0$, then $V_t(x) = \sum_{i=1}^{q} [V_i(x)]_t\,\theta_i$ and $K_t(y)$ represent a feasible solution. Terminate the algorithm.

Step 3: Set $t = t+1$ and solve the following SOS optimization problem in $[V_i(x)]_t, K_t(y)$, with $Z(x)$ as in Proposition 1.3.2. Further, the SOS decomposition of $[V_i(x)]_t = Z(x)^T [Q_i]_t Z(x)$, and $[\varepsilon_i(x)]_t = [\varepsilon_i(x)]_{t-1}$ as well as some positive polynomials $\lambda_1(x)$ and $\lambda_2(x)$ for $i = 1, \ldots, q$:

$$\begin{aligned} &\text{Minimize } \sum_{i=1}^{q} \text{trace}([Q_i]_t) \\ &\text{Subject to } [V_i(x)]_t - \lambda_1(x) && \text{is a SOS,} \\ &\qquad -v^T\left([N_i^{\alpha}(x,y)]_t + \lambda_2(x)I\right)v && \text{is a SOS,} \end{aligned}$$

with

$$[N_i^{\alpha}(x,y)]_t \triangleq \begin{bmatrix} M_{11}^i(x) - \alpha_{t-1}[V_i(x)]_t & (*) & (*) & (*) \\ M_{21}^i(x,y) & -I & (*) & (*) \\ M_{31}^i(x,y) & 0 & -I & (*) \\ M_{41}^i(x) & 0 & 0 & -\gamma^2 I \end{bmatrix}, \tag{7.21}$$

v of appropriate dimensions, and $M_{11}^i(x), M_{21}^i(x,y), M_{31}^i(x,y), M_{41}^i(x)$ as in (7.9) with $V(x) \triangleq [V_i(x)]_t, K(y) \triangleq K_t(y)$, and $\varepsilon(x) \triangleq [\varepsilon_i(x)]_t$ for each subsystem of (7.13), respectively.

Step 4: Solve the following feasibility problem with $v_2 \in \mathbb{R}^{n+1}$ and some positive tolerance function $\delta(x) > 0, x \neq 0$ for $i = 1, \ldots, q$:

$$v_2^T \begin{bmatrix} \delta(x) & (*) \\ \left(\varepsilon_t^i(x) - \frac{\partial V_t^i(x)}{\partial x}\right)^T & 1 \end{bmatrix} v_2 \qquad \text{is a SOS.}$$

If the problem is feasible go to Step 5. Else, set $t = t+1$ and $[\varepsilon_i(x)]_t = \left[\frac{\partial V_i(x)}{\partial x}\right]_{t-1}$, for $i = 1, \ldots, q$ determined in Step 3 and go to Step 2.

Step 5: The system (7.13) may not be stabilizable with H_∞ performance γ by static output feedback control (7.2). Terminate the algorithm. ■

7.3 Numerical Example

Consider the following polynomial system with polytropic uncertainties with $\beta \in [-1, 1]$:

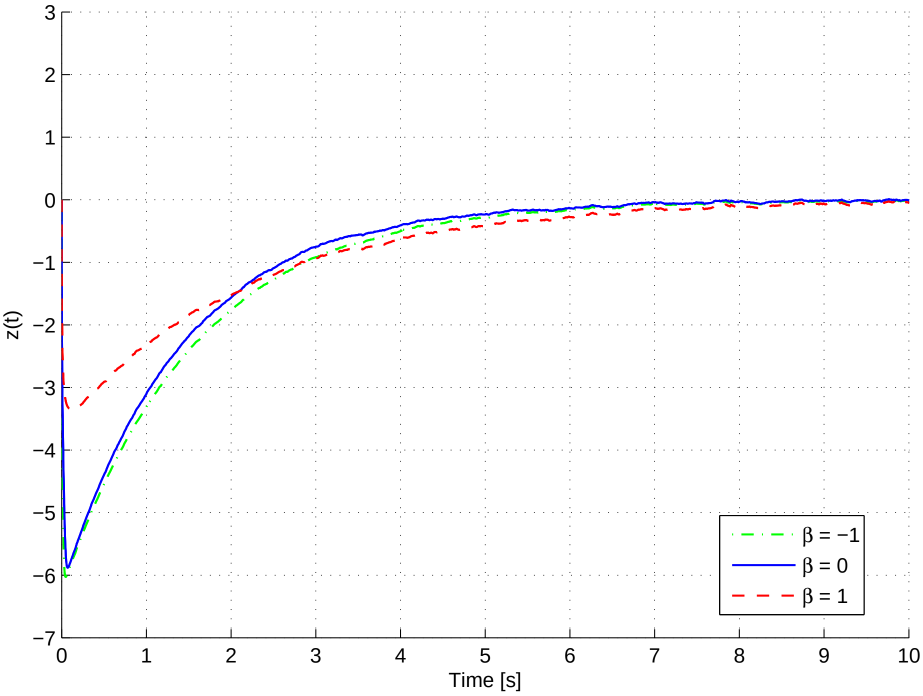


Figure 7.1: Regulated output

$$
\begin{aligned}
\dot{x} = & \begin{bmatrix} -x_1 + x_1^2 - \frac{3}{2}x_1^3 - \frac{3}{8}x_1x_2^2 + \frac{1}{4}x_2 - x_1^2x_2 - \frac{1}{4}x_2^3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1.1 \end{bmatrix} u + \begin{bmatrix} 1.25 \\ 0 \end{bmatrix} \omega, \\
& + \beta \left(\begin{bmatrix} \frac{3}{8}x_1x_2^2 - \frac{1}{4}x_2^3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u + \begin{bmatrix} 0.25 \\ 0 \end{bmatrix} \omega \right), \qquad (7.22) \\
y = & x_1 - x_2, \\
z = & u.
\end{aligned}
$$

First, we bring (7.22) in form of (7.13) by setting $\theta_1 = 1, \theta_2 = 0$ for $\beta = -1$ and $\theta_1 = 0, \theta_2 = 1$ for $\beta = 1$. Next, we select $\lambda_1(x) = \lambda_2(x) = \delta(x) = 0.01\left(x_1^2 + x_2^2\right)$, set the controller to be a function of y up to a degree of 3 and choose to look for Lyapunov function candidates of degree 6. The ISOS algorithm terminates with a feasible solution for $\gamma^2 = 1.423$ after 3 iterations with very small coefficients for the higher order terms in

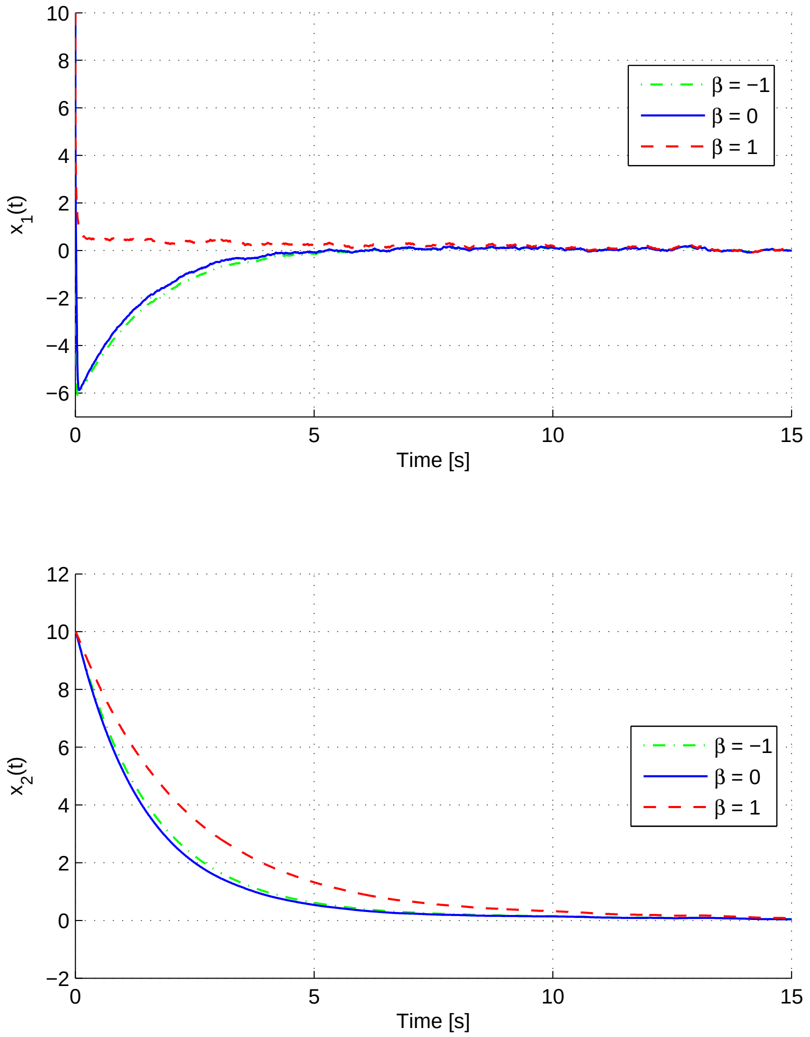


Figure 7.2: System response

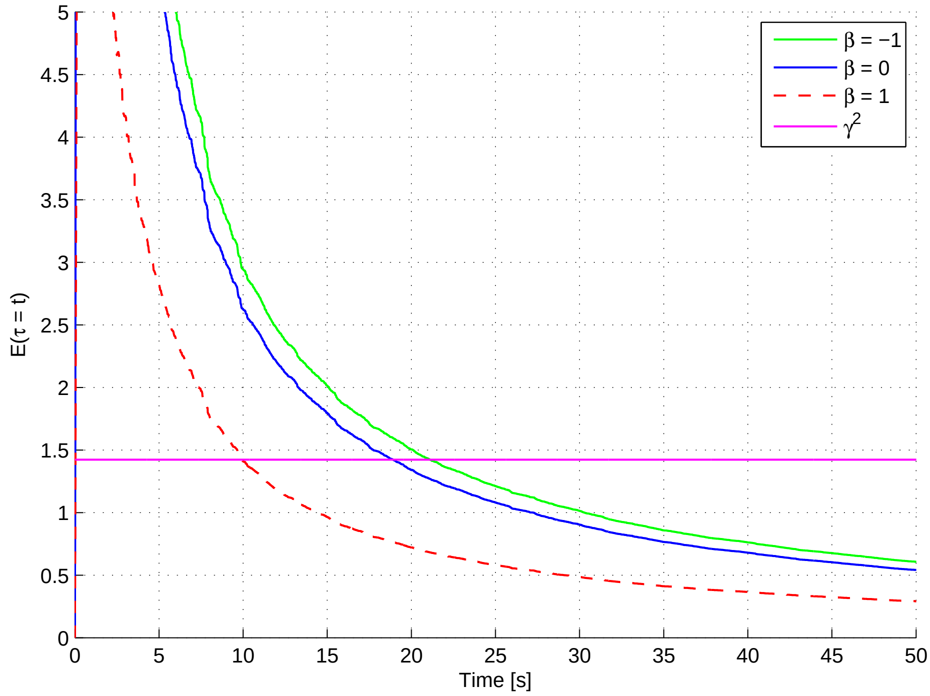


Figure 7.3: Energy ratio $E(\tau) = \frac{\int_0^\tau z^T z dt}{\int_0^\tau \omega^T \omega dt}$

$K(y)$. Thus, we set these to zero, initialize $\varepsilon_1^i(x) = \frac{\partial V^i(x)}{\partial x}$ from the previous results and rerun the algorithm for a linear controller. After 3 iterations, the following H_∞ controller for the polynomial system (6.14) with polytropic uncertainties has been obtained:

$$K(y) = 0.389y. \tag{7.23}$$

The Lyapunov function matrices have been omitted here due to their size. The smallest eigenvalue of Q_1, Q_2 were obtained as 1.781×10^{-4} and 3.564×10^{-3}, respectively. Once again, it was possible to obtain a *linear* controller for the *polynomial* system. For initial states $x_0^T = \begin{bmatrix} 10 & 10 \end{bmatrix}^T$ and a disturbance modelled with Gaussian white noise with power density spectrum of 0.01. The regulated output for different values of β are shown in Figure 7.1, with the state trajectories depicted in Figure 7.2. Figure 7.3 shows the progression of the Energy ratio over time $E(\tau) = \frac{\int_0^\tau z^T z dt}{\int_0^\tau \omega^T \omega dt}$. It can be observed that the system is falls below the prescribed performance value after around 22 seconds for for all admissible values of β.

7.4 Conclusion

An iterative procedure to obtain a H_∞ static output feedback controller for polynomials with polytropic uncertainties has been presented in this section. Sufficient conditions for the existence of a H_∞ controller have been derived in terms of bilinear matrix inequalities. An iterative algorithm has been proposed that results in a polynomial controller that avoids rational components encountered when inverting the Lyapunov function in traditional control approaches. Further, the Lyapunov function has been shown to be true function of all system states and is not restricted to only incorporates states which corresponding rows in the control matrix are zeros. A numerical example has been provided to show the effectiveness of the proposed approach.

Chapter 8

Robust Nonlinear H_∞ Output Feedback Control for Polynomial Systems with Norm-Bounded Uncertainties

8.1 Introduction

In this chapter, the problem of designing a robust H_∞ static output feedback controller for polynomial systems with norm-bounded uncertainties is investigated. In detail, section 8.2 will outline how the state feedback results from chapter 6 can be extended to the static output case. A numerical example will be presented in section 8.3 before this chapter concludes with some final remarks in section 8.4.

8.2 Main Results

Consider the following polynomial system with norm-bounded uncertainties

$$\begin{aligned} \dot{x} &= A(x) + B_u(x) + B_\omega(x) + \Delta A(x) + \Delta B_u(x), \\ y &= C_y(x) + D_y(x)u, \\ z &= C_z(x) + D_z(x)u, \end{aligned} \tag{8.1}$$

where $x \in \mathbb{R}^n$ are the system states, $u \in \mathbb{R}^m$ is the input, y and z are the measured output and the controlled output, respectively. $A(x), C_y(x), C_z(x)$ are polynomial vectors and B_u, B_ω, D_z are polynomial matrices of appropriate dimensions. The disturbance signal is ω, whereas the norm-bounded uncertainties of the system are captured in $\Delta A(x)$ and $\Delta B_u(x)$. The objective of a state feedback H_∞ control is to find a controller $K(y)$ such that the system (8.1) with

$$u = K(y) \tag{8.2}$$

is asymptotically stable and the L_2 gain from the disturbance input to the controlled output is less than a prescribed value $\gamma > -$, that is

$$\int_0^\infty z^T z dt \leq \gamma^2 \int_0^\infty \omega^T \omega dt. \tag{8.3}$$

The following assumption is used for the norm-bounded uncertainty

Assumption 8.2.1 *The admissible parameter uncertainties considered here are assumed to be norm-bounded and can be described as*

$$\begin{bmatrix} \Delta A(x) & \Delta B_u(x) \end{bmatrix} = H(x)F(x) \begin{bmatrix} E_1(x) & E_2(x) \end{bmatrix}, \tag{8.4}$$

with known polynomial matrices $H(x), E_1(x), E_2(x)$ *of appropriate dimensions and* $F(x)$ *being an unknown state-depended matrix that satisfies*

$$\|F^T(x)F(x)\| \leq I. \tag{8.5}$$

Theorem 8.2.1 *The polynomial system* (8.1) *is stabilizable with a prescribed* H_∞ *performance* $\gamma > 0$ *via static output feedback controller* (8.2) *if there exist a polynomial function* $V(x)$ *and a polynomial matrix* $K(y)$ *such that* $\forall x \neq 0$

$$V(x) > 0 \tag{8.6}$$

and

$$\begin{aligned}
&\frac{\partial V(x)}{\partial x}A(x) - \frac{1}{4}\frac{\partial V(x)}{\partial x}B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x} \\
&+ \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_\omega(x)\right)\frac{1}{\gamma^2}\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_\omega(x)\right)^T \\
&+ \left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x) + K^T(y)\right)\left(\frac{1}{2}\frac{\partial V(x)}{\partial x}B_u(x) + K^T(y)\right)^T \\
&+ \left(C_z(x) + D_z(x)K(y)\right)^T\left(C_z(x) + D_z(x)K(y)\right) \\
&+ \frac{1}{2}\frac{\partial V(x)}{\partial x}H(x)F(x)F(x)^T H(x)^T\frac{\partial V^T(x)}{\partial x} \\
&+ \frac{1}{2}\left(E_1(x) + E_2(x)K(y)\right)^T\left(E_1(x) + E_2(x)K(y)\right) < 0.
\end{aligned} \tag{8.7}$$

Proof: The proof can be obtained in a similar manner as has been done for the state feedback case in Chapter 6 for Theorem 6.2.1. ■

Theorem 8.2.2 *The polynomial system* (8.1) *is stabilizable with* H_∞ *norm* $\gamma > 0$ *via polynomial static output feedback control* (8.2) *if there exist a Lyapunov function* $V(x)$*, a polynomial design vector* $\varepsilon(x)$ *of appropriate dimensions and a controller matrix* $K(y)$ *as in* (8.2) *satisfying the following conditions for* $x \neq 0$

$$V(x) > 0, \tag{8.8}$$

and

$$M(x,y) = \begin{bmatrix} M_{11}(x) & (*) & (*) & (*) & (*) & (*) \\ M_{21}(x,y) & -I & (*) & (*) & (*) & (*) \\ M_{31}(x,y) & 0 & -2I & (*) & (*) & (*) \\ M_{41}(x) & 0 & 0 & -2I & (*) & (*) \\ M_{51}(x,y) & 0 & 0 & 0 & -I & (*) \\ M_{61}(x) & 0 & 0 & 0 & 0 & -\gamma^2 \end{bmatrix} \prec 0, \tag{8.9}$$

where

$$\begin{aligned} M_{11}(x) =& \frac{\partial V(x)}{\partial x} A(x) + \frac{1}{4}\varepsilon(x) B_u(x) B_u^T(x) \varepsilon^T(x) - \frac{1}{2}\varepsilon(x) B_u(x) B_u^T(x) \frac{\partial V^T(x)}{\partial x}, \\ M_{21}(x,y) =& \left(\frac{1}{2}\frac{\partial V(x)}{\partial x} B_u(x) + K^T(y)\right)^T, \\ M_{31}(x,y) =& \left(E_1(x) + E_2(x) K(y)\right), \\ M_{41}(x) =& H^T(x)\frac{\partial V^T(x)}{\partial x}, \\ M_{51}(x,y) =& C_z(x) + D_z(x) K(y), \\ M_{61}(x) =& \left(\frac{1}{2}\frac{\partial V(x)}{\partial x} B_\omega(x)\right)^T. \end{aligned} \tag{8.10}$$

Proof: This follows directly from applying Schur Complement to Theorem 8.2.2. ■

The term $-\frac{1}{2}\varepsilon(x)B_u(x)B_u^T(x)\frac{\partial V^T(x)}{\partial x}$ makes (8.9) nonconvex, hence the inequality cannot be solved directly by SOS decomposition. The following iterative SOS algorithm is proposed

Step 1: Linearize the system (8.1) and set $\omega = 0, F(x) = 0$. Use the static output feedback approach described in [82] to find a solution to each of the linearized problems without disturbance. For $i = 1, \ldots, q$, set $t = 1$ and $\varepsilon_1(x) = x^T P, V_0(x) = x^T Px$.

Step 2: Solve the following SOS optimization problem in $V_t(x)$ and $K_t(y)$ with fixed auxiliary polynomial vector $\varepsilon_t(x)$ and some positive polynomials $\lambda_1(x)$ and $\lambda_2(x)$:

$$\begin{aligned} &\text{Minimize } \alpha_t \\ &\text{Subject to } V_t(x) - \lambda_1(x) && \text{is a SOS,} \\ &\qquad - v^T\left(M_t^\alpha(x,y) + \lambda_2(x) I\right) v && \text{is a SOS,} \end{aligned}$$

with

$$M_t^\alpha(x,y) \triangleq \begin{bmatrix} M_{11}(x) - \alpha_t V_{t-1}(x) & (*) & (*) & (*) & (*) & (*) \\ M_{21}(x,y) & -I & (*) & (*) & (*) & (*) \\ M_{31}(x,y) & 0 & -2I & (*) & (*) & (*) \\ M_{41}(x) & 0 & 0 & -2I & (*) & (*) \\ M_{51}(x,y) & 0 & 0 & 0 & -I & (*) \\ M_{61}(x) & 0 & 0 & 0 & 0 & -\gamma^2 \end{bmatrix}, \qquad (8.11)$$

v of appropriate dimensions, and $M_{11}(x), M_{21}(x,y), M_{31}(x,y), M_{41}(x), M_{51}(x,y), M_{61}(x)$ are as in (8.10) with $V(x) \triangleq V_t(x), K(y) \triangleq K_t(y),$ and $\varepsilon(x) \triangleq \varepsilon_t(x)$.

If $\alpha_t < 0$, then $V_t(x) = V_t(x)$ and $K_t(y)$ represent a feasible solution. Terminate the algorithm.

Step 3: Set $t = t+1$ and solve the following SOS optimization problem in $V_t(x), K_t(y)$, with $Z(x)$ as in Proposition 1.3.1 and the SOS decomposition of $V_t(x) = Z(x)^T Q_t Z(x)$, and $\varepsilon_t(x) = \varepsilon_{t-1}(x)$ as well as some positive polynomials $\lambda_1(x)$ and

$\lambda_2(x)$:

$$\begin{aligned} &\text{Minimize } \sum_{i=1}^{q} \text{trace}(Q_t) \\ &\text{Subject to } V(x)_t - \lambda_1(x) && \text{is a SOS,} \\ &\qquad -v^T \left(N_t^{\alpha}(x,y) + \lambda_2(x) I\right) v && \text{is a SOS,} \end{aligned}$$

with

$$N^{\alpha}(x,y)_t \triangleq \begin{bmatrix} M_{11}(x) - \alpha_{t-1} V_t(x) & (*) & (*) & (*) & (*) & (*) \\ M_{21}(x,y) & -I & (*) & (*) & (*) & (*) \\ M_{31}(x,y) & 0 & -2I & (*) & (*) & (*) \\ M_{41}(x) & 0 & 0 & -2I & (*) & (*) \\ M_{51}(x,y) & 0 & 0 & 0 & -I & (*) \\ M_{61}(x) & 0 & 0 & 0 & 0 & -\gamma^2 \end{bmatrix}, \qquad (8.12)$$

v of appropriate dimensions, and $M_{11}(x), M_{21}(x,y), M_{31}(x,y), M_{41}(x), M_{51}(x,y), M_{61}(x)$ are as in (8.10) with $V(x) \triangleq V_t(x), K(y) \triangleq K_t(y),$ and $\varepsilon(x) \triangleq \varepsilon_t(x)$.

Step 4: Solve the following feasibility problem with $v_2 \in \mathbb{R}^{n+1}$ and some positive tolerance function $\delta(x) > 0, x \neq 0$ for $i = 1, \ldots, q$:

$$v_2^T \begin{bmatrix} \delta(x) & (*) \\ \left(\varepsilon_t(x) - \frac{\partial V_t(x)}{\partial x}\right)^T & 1 \end{bmatrix} v_2 \qquad \text{is a SOS.}$$

If the problem is feasible go to Step 5. Else, set $t = t+1$ and $\varepsilon_t(x) = \frac{\partial V_{t-1}(x)}{\partial x}$ determined in Step 3 and go to Step 2.

Step 5: The system (8.1) may not be stabilizable with H_∞ performance γ by static output feedback control (8.2). Terminate the algorithm.

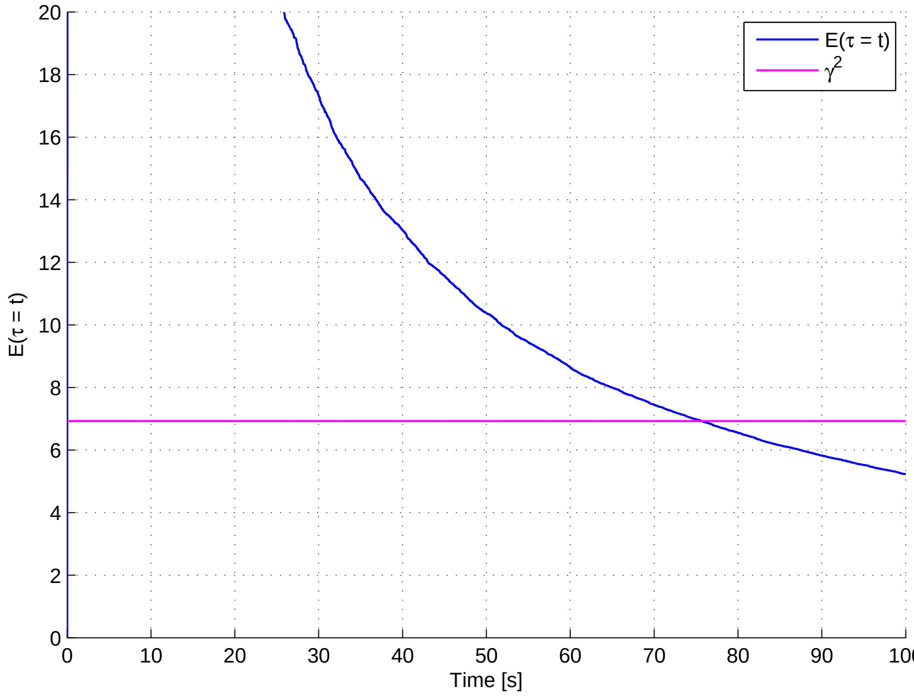


Figure 8.1: Energy ratio $E(\tau) = \frac{\int_0^\tau z^T z dt}{\int_0^\tau \omega^T \omega dt}$

8.3 Numerical Example

Consider the following polynomial system with polytropic uncertainties

$$
\begin{aligned}
\dot{x} =& A(x) + B(x)u + H(x)F(x)\left(E_1(x) + E_2(x)u\right) + B_\omega(x)\omega, \\
A(x) =& \begin{bmatrix} -x_1 + x_1^2 - \frac{3}{2}x_1^3 - \frac{3}{8}x_1x_2^2 + \frac{1}{4}x_2 - x_1^2x_2 - \frac{1}{4}x_2^3 \\ 0 \end{bmatrix}, \\
B(x) =& \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_\omega = \begin{bmatrix} 1.25 \\ 0 \end{bmatrix}, \quad H(x) = 1, \\
E_1(x) =& \begin{bmatrix} \frac{3}{8}x_1x_2^2 - \frac{1}{4}x_2^3 \\ 0 \end{bmatrix}, \quad E_2(x) = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}, \quad F(x) = I\sin(x), \\
y =& x_1 - x_2, \\
z =& x_1 - x_2 + u.
\end{aligned}
\tag{8.13}
$$

We select $\lambda_1(x) = \lambda_2(x) = \delta(x) = 0.01\left(x_1^2 + x_2^2\right)$, set the controller to be a function of y up to a degree of 3 and choose to look for Lyapunov function candidates of degree 4. The ISOS algorithm terminates without finding a feasible solution. Therefore, the degree of the Lyapunov function candidate is increased to 6 and the algorithm is restarted. The ISOS algorithm terminates with a feasible solution for $\gamma^2 = 6.921$ after 5 iterations with very small coefficients for the higher order terms in $K(y)$. Thus, we set these to zero, initialize $\varepsilon_1(x) = \frac{\partial V(x)}{\partial x}$ from the previous solution and rerun the algorithm for a linear controller. After 3 iterations, the following H_∞ controller for the polynomial system (8.13) with norm-bounded uncertainties has been obtained

$$K(y) = 0.189y. \tag{8.14}$$

Note that the Lyapunov function is omitted here due to its size. The smallest eigenvalue of Q has been found as 4.511×10^{-5}.

Once again, it was possible to obtain a *linear* controller for the *polynomial* system. The disturbance has been modelled as Gaussian white noise with power density spectrum of 0.01, and Figure 6.1 shows the ratio of the regulated output energy to the noise energy over time. The ratio clearly falls below the design threshold value of γ^2 after less than 80 seconds.

8.4 Conclusion

An iterative design algorithm for the problem of designing a robust H_∞ static output controller for polynomial systems with norm-bounded uncertainties has been presented in this chapter. In detail, sufficient conditions for the existence of a controller that stabilizes the system with H_∞ performance γ in the presence of norm-bounded uncertainties has been derived in the form of polynomial matrix inequalities. The nonconvex components of these conditions have been addressed using an iterative design algorithm, and numerical examples have been provided to show the effectiveness of the proposed procedure. Furthermore, it was possible to obtain *linear* controller gains for the *polynomial* system.

Chapter 9

Conclusion

9.1 Summary of Thesis

This thesis consist of two parts. In Part I, the controller design for a highly nonlinear, highly coupled boiler-turbine systems has been discussed. In Chapter 2, a novel approach to online model predictive control with genetic algorithms has been presented. In particular, the careful design considerations necessary to use this stochastic artificial intelligence approach to obtain a suitable system response have been discussed. Furthermore, a switching control scheme that combines the benefits of the H_∞ fuzzy reference tracking control design with the advantages of the model predictive control based algorithm has been presented.

The great versatility and design freedom of that come with the implementation of genetic algorithms in model predictive control problems allowed a well rounded customized controller design. In general, an extension of the results is possible to other control problems, as long as the system dynamics are sufficiently slow. This restriction is also the greatest drawback of the proposed control regime, and thus must be seen as a specialist solution for a niche group of control problems.

In Part II, the more general case of polynomial system control was investigated. An iterative sum of squares decomposition algorithm has been presented and applied to a variety of state feedback as well as output feedback control problems. The outlined representation of the problem is less conservative than other available control approaches and allows for more design freedom in the choice of the form and structure of the (higher order) Lyapunov functions and control matrices, that can be both formed without assumptions on the system structure.

In particular, Chapter 3 introduces the basic state feedback control for polynomial system as well as the case for polynomial systems with polytropic uncertainties. The control procedure for system with norm-bounded uncertainties was outlined in Chapter 4. In Chapter 5 and Chapter 6, the effective H_∞ control of systems with polytropic as well as norm-bounded uncertainties has been derived, respectively. The discussion on polynomial system control has been concluded with an investigation how the iterative algorithm can be extended to the output feedback case for systems with polytropic and norm-bounded uncertainties in Chapter 7 and Chapter 8, respectively.

Sufficient conditions for stability and performance of state feedback and static output feedback controllers have been presented in the form of polynomial matrix inequalities. To avoid the nonconvex expressions in the problem formulation, a novel iterative design algorithm for polynomial system design has been presented. This approach avoids several of the most common problems found in other approaches. The controller does not directly depend on the inverse of a polynomial Lyapunov matrix, thus rational controllers can be avoided. Further, there is no restriction on the sparsity of the input or control matrix to be able to form a suitable Lyapunov function, thus the presented iterative procedure can be readily implemented for non-sparse systems. Moreover, the Lyapunov function is not restricted to be a function of only the system states which corresponding rows in the control input matrix are zero.

Generally speaking, the biggest problem of the proposed controller synthesis is the computational complexity arising from higher order multivariate polynomials and their SOS decompositions. To the best of the author's knowledge, this is a problem common to all design approaches for polynomial systems, and thus the controller design is generally limited to systems with only a few system states. The resulting SDP realizations of SOS decompositions requirements quickly grow to a point that the SDP becomes too large to be handled efficiently and numerical errors grow rapidly, making an efficient design impossible. This has also lead to the choice of presented numerical examples.

The main contributions of this thesis are:

- A novel approach to control highly nonlinear systems using online model predictive control utilizing genetic algorithms to obtain the optimal input sequence.
- A novel take on the robust controller synthesis problem for polynomial systems with or without polytropic or norm-bounded uncertainties.
- A less restrictive design algorithm for the control synthesis that avoids rational feedback gains and was often able to obtain *linear* controllers for the *polynomial* control problems.

As a result, this thesis provides an integrated approach for the controller synthesis for polynomial systems and represents a valuable and meaningful contribution to the development to the framework of polynomial system control. Furthermore, Part I of this thesis suggests novel solutions to highly nonlinear system control of importance to the power generating community.

9.2 Future Work

In general, nonlinear systems control is still an open area that requires a lot more research work. In particular, further research could be directed to the following areas:

1. Application of online model predictive control theories with GAs to other (groups of) systems. With the advance of modern computation hardware, an application of the

proposed methodologies to faster systems or other hard nonlinear control problems is desirable in the future.

2. Time-delayed polynomial systems. Time-delayed systems have become some of the most studied areas of control engineering and to the best of the author's knowledge no general framework has been developed for polynomial systems yet. A successful extension of the iterative procedures to time-delayed systems could provide a novel methodology for the controller synthesis of networked control systems.

3. Polynomial Filtering. Further research addressing the problem of polynomial system filtering would be desirable, with extension to robust performance H_∞ methods for polytropic and norm-bounded uncertainties. Also, a reduced order filter design would be beneficially to limit the strain on numerical methods.

4. Rational systems. Existing methods are already able to address rational control systems as long as the denominator of the system is always positive or negative for all system states. However, this requirement is very restrictive and it would be beneficial to find a less restrictive extension of the presented polynomial control synthesis to rational systems.

5. Extension of polynomial methods to other highly nonlinear systems with system states having fractional exponents. It is currently not possible to apply polynomial control problems to systems where system states have fractional exponents. An extension in this direction, together with a discussion on rational system could potentially lead to an extension to highly nonlinear systems of interest such as the boiler-turbine model from Part I.

Appendix

Appendix A

Schur Complement

The Schur Complement is a standard tool in the LMI context, see for example [42]. Consider a LMI

$$\begin{bmatrix} A(x) & B(x) \\ B(x)^T & D(x) \end{bmatrix} \succeq 0, \tag{A.1}$$

where $A(x) = A(x)^T, D(x) = D(x)^T$ and $B(x)$ is affine dependend on x, (A.1) is equivalent to

$$D(x) > 0, \quad A(x) - B(x)D(x)^{-1}B(x)^T \geq 0. \tag{A.2}$$

This relation holds vice versa.

Bibliography

[1] R. Dorf and R. Bishop, *Mondern control systems*. Upper Saddle River: Prentice Hall, 2005.

[2] P. Melin and O. Castillo, *Modelling, simulation and control of non-linear dynamical systems: an intelligent approach using soft computing and fractal theory*. London, New York: Taylor & Francis, 2002.

[3] C. Sparrow, *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors*. New York: Springer, 1982.

[4] C. E. Thomas, *Process technology equipment and systems*. Clifton Park, NY: Thomson Delmar Learning, 2007.

[5] T. F. Lammers, *Steam plant operation*, 9th ed. New York, NY: McGraw-Hill, 2011.

[6] J. Wu, J. Shen, M. Krug, S. Nguang, and Y. Li, "GA-based nonlinear predictive switching control for a boiler-turbine system," *Journal of Control Theory and Applications*, vol. 10, no. 1, pp. 100 – 106, 2012.

[7] W. Tan, H. J. Marquez, T. Chen, and J. Liu, "Analysis and control of a nonlinear boiler-turbine unit," *Journal of Process Control*, vol. 15, no. 8, pp. 883 – 891, 2005.

[8] A. Ben-Abdennour and K. Lee, "An autonomous control system for boiler-turbine units," *Energy Conversion, IEEE Transactions on*, vol. 11, no. 2, pp. 401 – 406, 1996.

[9] R. Bell and K. Åström, "Dynamic models for boiler-turbine-alternator units: data logs and parameter estimation for a 160 MW unit," Lund Institute of Technology, Tech. Rep. LUTFD2/TFTR-3192)/1-127, 1987.

[10] W. Tan, F. Fang, L. Tian, C. Fu, and J. Liu, "Linear control of a boilerturbine unit: Analysis and design," *ISA Transactions*, vol. 47, no. 2, pp. 189 – 197, 2008.

[11] P.-C. Chen and J. S. Shamma, "Gain-scheduled ℓ_1-optimal control for boiler-turbine dynamics with actuator saturation," *Journal of Process Control*, vol. 14, no. 3, pp. 263 – 277, 2004.

[12] F. Fang, W. Tan, and J.-Z. Liu, "Nonlinear output tracking control for the coordianted system of boiler-turbine unit," in *Proceedings of the Chinese Society of Electrical Engineering*, 2005, pp. 147 – 151.

[13] U.-C. Moon and K. Lee, "A boiler-turbine system control using a fuzzy auto-regressive moving average (FARMA) model," *IEEE Transactions on Energy Conversion*, vol. 18, no. 1, pp. 142 – 148, 2003.

[14] R. Dimeo and K. Lee, "Boiler-turbine control system design using a genetic algorithm," *IEEE Transactions on Energy Conversion*, vol. 10, no. 4, pp. 752 – 759, 1995.

[15] W. Kwon, S. Kim, and P. Park, "On the multivariable robust control of a boiler-turbine system," in *IFAC Symposium on power systems and power plant control*, 1989, pp. 219 – 223.

[16] W. Tan, Y. Niu, and J. Liu, "Robust control for a nonlinear boiler-turbine system," *Control Theory and Applications*, vol. 16, no. 6, pp. 863 – 867, 1999.

[17] U.-C. Moon, Y.-M. Park, and K. Y. Lee, "A development of a power system stabilizer with a fuzzy auto-regressive moving average model," *Fuzzy Sets and Systems*, vol. 102, no. 1, pp. 95 – 101, 1999.

[18] L. Sun, Y. Zhang, and L. Wang, “The application of neural network α-th order inverse control method in boiler-turbine coordinated control system,” in *Proceedings of the International Conference on Design and Applications*, vol. 2, 2010, pp. 500 – 504.

[19] T. Zhang, J. Lu, and Z. Hua, “Fuzzy gain scheduled model predictive control for boiler-turbine coordinated system,” in *Proceedings of the Chinese Society of Electrical Engineering*, 25, Ed., vol. 4, 2005, pp. 158 – 165.

[20] G. Chesi, “LMI techniques for optimization over polynomials in control: A survey,” *IEEE Transactions on Automatic Control*, vol. 55, no. 11, pp. 2500 – 2510, 2010.

[21] M. Krug, S. Saat, and S. K. Nguang, “Robust H_∞ static output feedback controller design for parameter dependent polynomial systems: An iterative sums of squares approach,” *Journal of the Franklin Institute*, vol. 350, no. 2, pp. 318 – 330, 2013.

[22] H. Lam and L. Seneviratne, “Stability analysis of polynomial fuzzy-model-based control systems under perfect/imperfect premise matching,” *IET Control Theory and Applications*, vol. 5, no. 15, pp. 1689 – 1697, 2011.

[23] L.-W. Huang, C.-Y. Cheng, and G. Yu, “Design of polynomial controllers for a two-link robot arm using sum-of-squares approachs,” in *Proceedings of the Asian Control Conference*, 2011, pp. 436 – 440.

[24] B. Sanjaya, B. Trilaksono, and A. Syaichu-Rohman, “Static output feedback control synthesis for nonlinear polynomial fuzzy systems using a sum of squares approach,” in *Proceedings of International Conference on Instrumentation, Communication, Information Technology and Biomedical Engineering*, 2011, pp. 214 – 217.

[25] T. Jennawasin, T. Narikiyo, and M. Kawanishi, “An improved SOS-based stabilization condition for uncertain polynomial systems,” in *Proceedings of the SICE Annual Conference*, 2010, pp. 3030 – 3034.

[26] D. Zhao and J.-L. Wang, “Robust static output feedback design for polynomial nonlinear systems,” *International Journal of Robust and Nonlinear Control*, vol. 20, no. 14, pp. 1637 – 1654, 2010.

[27] M. Belhaouane, R. Mtar, H. Belkhiria Ayadi, and N. Benhadj Braiek, “An LMI technique for the global stabilization of nonlinear polynomial systems,” *International Journal of Computers, Communications and Control*, vol. 4, no. 4, pp. 335 – 348, 2009.

[28] K. Tanaka, H. Yoshida, H. Ohtake, and H. Wang, “A sum-of-squares approach to modeling and control of nonlinear dynamical systems with polynomial fuzzy systems,” *IEEE Transactions on Fuzzy Systems*, vol. 17, no. 4, pp. 911–922, 2009.

[29] Q. Zheng and F. Wu, “Regional stabilisation of polynomial non-linear systems using rational Lyapunov functions,” *International Journal of Control*, vol. 82, no. 9, pp. 1605 – 1615, 2009.

[30] D. Zhao and J. Wang, “An improved H_∞ synthesis for parameter-dependent polynomial nonlinear systems using SOS programming,” in *Proceedings of the American Control Conference*, 2009, pp. 796 – 801.

[31] D. Cheng and H. Qi, “Global stability and stabilization of polynomial systems,” in *Proceedings of the IEEE Conference on Decision and Control*, 2007, pp. 1746 – 1751.

[32] C. Ebenbauer and F. Allgöwer, “Analysis and design of polynomial control systems using dissipation inequalities and sum of squares,” *Computers and Chemical Engineering*, vol. 30, no. 10-12, pp. 1590 – 1602, 2006.

[33] D. Henrion and J.-B. Lasserre, “Convergent relaxations of polynomial matrix inequalities and static output feedback,” *IEEE Transactions on Automatic Control*, vol. 51, no. 2, pp. 192 – 202, 2006.

[34] P. A. Parrilo and B. Sturmfels, “Minimizing polynomial functions,” *Algorithmic and quantitative real algebraic geometry, DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, vol. 60, pp. 83 – 99, 2003.

[35] W. Assawinchaichote, S. K. Nguang, P. Shi, and E.-K. Boukas, “Fuzzy state-feedback control design for nonlinear systems with -stability constraints: An LMI approach,” *Mathematics and Computers in Simulation*, vol. 78, no. 4, pp. 514 – 531, 2008.

[36] M. Belhaouane, M. Ghariani, H. Belkhiria Ayadi, and N. Benhadj Braiek, “Improved results on robust stability analysis and stabilization for a class of uncertain nonlinear systems,” *Mathematical Problems in Engineering*, 2010.

[37] J. Lasserre, “Global optimization with polynomials and the problem of moments,” *SIAM Journal on Optimization*, vol. 11, no. 3, pp. 796 – 817, 2001.

[38] L. Vandenberghe and S. P. Boyd, “Semidefinite programming,” *SIAM Review*, vol. 38, no. 1, pp. 49 – 95, 1996.

[39] J. C. Willems, “Dissipative dynamical systems part I: General theory,” *Archive for Rational Mechanics and Analysis*, vol. 45, pp. 321 – 351, 1972.

[40] A. Liapunov and A. Fuller, *The general problem of the stability of motion*. London; Washington, DC: Taylor & Francis, 1992, edited and Translated by A.T. Fuller; includes Biography of Lyapunov by V.I. Smirnov and a bibliography of his work by J.F. Barrett.

[41] P. Kokotovi and M. Arcak, “Constructive nonlinear control: a historical perspective,” *Automatica*, vol. 37, no. 5, pp. 637 – 662, 2001.

[42] S. P. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear matrix inequalities in system and control theory*. Philadelphia: SIAM, 1994.

[43] P. A. Parrilo, "Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization," Ph.D. Thesis, California Institute of Technology, May 2000.

[44] K. Murty and S. Kabadi, "Some NP-complete problems in quadratic and nonlinear programming," *Mathematically Programming*, vol. 39, no. 2, pp. 117 – 129, 1987.

[45] S. Prajna, A. Papachristodoulou, and F. Wu, "Nonlinear control synthesis by sum of squares optimization: a Lyapunov-based approach," in *Proceedings of the Asian Control Conference*, vol. 1, 2004, pp. 157 – 165.

[46] A. Papachristodoulou and S. Prajna, "On the construction of Lyapunov functions using the sum of squares decomposition," in *Proceedings of the IEEE Conference on Decision and Control*, vol. 3, 2002, pp. 3482 – 3487.

[47] H. Hindi, "A tutorial on convex optimization," in *Proceedings of the American Control Conference*, vol. 4, 2004, pp. 3252 – 3265.

[48] A. Papachristodoulou and S. Prajna, "A tutorial on sum of squares techniques for systems analysis," in *Proceedings of the American Control Conference*, vol. 4, 2005, pp. 2686 – 2700.

[49] B. Reznick, "Some concrete aspects of Hilbert's 17th problem," *Contemporary Mathematics*, vol. 253, pp. 251 – 272, 2000.

[50] S. Prajna, A. Papachristodoulou, and P. A. Parrilo, "Introducing SOSTOOLS: a general purpose sum of squares programming solver," in *Proceedings of the IEEE Conference on Decision and Control*, vol. 1, 2002, pp. 741 – 746.

[51] J. Lofberg, "YALMIP: a toolbox for modeling and optimization in Matlab," in *IEEE International Symposium on Computer Aided Control Systems Design*, 2004, pp. 284 – 289.

[52] M. Grant and S. Boyd, “CVX: Matlab software for disciplined convex programming, version 2.0 beta,” http://cvxr.com/cvx, Sep. 2012.

[53] ——, “Graph implementations for nonsmooth convex programs,” in *Recent Advances in Learning and Control*, ser. Lecture Notes in Control and Information Sciences, V. Blondel, S. Boyd, and H. Kimura, Eds. Springer Verlag, 2008, pp. 95 – 110.

[54] D. Henrion, J.-B. Lasserre, and J. Löfberg, “GloptiPoly 3: moments, optimization and semidefinite programming,” *Optimization Methods and Software*, vol. 24, no. 4, pp. 761 – 779, 2009.

[55] C.-T. Hsu and S.-L. Chen, “Exact linearization of a voltage-controlled 3-pole active magnetic bearing system,” *IEEE Transactions on Control Systems Technology*, vol. 10, no. 4, pp. 618 – 625, 2002.

[56] A. Pisarchik and B. Kuntsevich, “Control of multistability in a directly modulated diode laser,” *IEEE Journal of Quantum Electronics*, vol. 38, no. 12, pp. 1594 – 1598, 2002.

[57] H.-J. Ma and G.-H. Yang, “Fault-tolerant control synthesis for a class of nonlinear systems: Sum of squares optimization approach,” *International Journal of Robust and Nonlinear Control*, vol. 19, no. 5, pp. 591 – 610, 2009.

[58] L. Z. Chen, S. K. Nguang, and X. D. Chen, *Modelling and optimization of biotechnological processes - artificial intelligence approaches*, ser. Studies in Computational Intelligence, J. Kacprzyk, Ed. Springer, 2006, vol. 15.

[59] Y. Wang, “Fuzzy clustering analysis by using genetic algorithm,” *ICIC Express Letters*, vol. 2, no. 4, pp. 331 – 337, 2008.

[60] G. Carrillo-Ureta, P. Roberts, and V. Becerra, "Genetic algorithms for optimal control of beer fermentation," in *Proceedings of the IEEE International Symposium on Intelligent Control*, 2001, pp. 391 – 396.

[61] M. Samadi and K. Salahshoor, "Online fault disgnostic monitoring of a NMPC controlled hybrid CSTR plant in start-up operation using a modified particle filtering algorithm," in *Proceedings of the IEEE International Conference on Control Applications*, vol. 17, San Antonio, 2008, pp. 733 – 738.

[62] B. Bouzouita, F. Bouani, V. Wertz, and M. Ksouri, "MIMO robust predictive control applied to a three tank system," in *Proceedings of the Mediterranean Conference on Control and Automation*, vol. 16, Ajaccio, 2008, pp. 1880 – 1885.

[63] C. Venkateswarlu and A. Reddy, "Nonlinear model predictive control of reactive distillation based on stochastic optimization," *Industrial & Engineering Chemistry Research*, vol. 47, no. 18, pp. 6649 – 6690, 2008.

[64] J. H. Holland, *Adaptation in natural and artificial systems*, 1st ed. Cambridge, Mass: MIT Press, 1992.

[65] T. Bäck, *Evolutionary algorithms in theory and practice : evolution strategies, evolutionary programming, genetic algorithms*. New York: Oxford Univeristy Press, 1996.

[66] D. E. Goldberg, *Genetic algorithms in search, optimization, and machine learning*. Reading, Mass.: Addison-Wesley, 1989.

[67] L. Chambers, *Practical handbook of genetic algorithms*. Boca Raton: CRC Press, 1999.

[68] S. N. Sivanandam and S. N. Deepa, *Introduction to genetic algorithms*. Berlin: Springer, 2007.

[69] M. Mitchell, *An Introduction to Genetic Algorithms*. Bradford Book, 1998.

[70] R. E. King, *Computational intelligence in control engineering*, ser. Control engineering series. New York: Marcel Dekker, Inc., 1999.

[71] A. Zilouchian and M. Jamshidi, *Intelligent control systems using soft computing methodologies*. Boca Raton, FL: CRC Press, 2001.

[72] M. Jamshidi, *Robust control systems with genetic algorithms*. Boca Raton, FL: CRC Press, 2003.

[73] I. Dumitrache and C. Buiu, "Genetic learning of fuzzy controllers," *Mathematics and Computers in Simulation*, vol. 49, no. 12, pp. 13 – 26, 1999.

[74] B.-S. Chen and Y.-M. Cheng, "A structure-specified H_∞ optimal control design for practical applications: a genetic approach," *IEEE Transactions on Control Systems Technology*, vol. 6, no. 6, pp. 707 – 718, 1998.

[75] B.-S. Chen, B.-K. Lee, and S.-C. Peng, "Maximum likelihood parameter estimation of F-ARIMA processes using the genetic algorithm in the frequency domain," *Signal Processing, IEEE Transactions on*, vol. 50, no. 9, pp. 2208 – 2220, 2002.

[76] N. Jiang, Z. Zhao, and L. Ren, "Design of structural modular neural networks with genetic algorithm," *Advances in Engineering Software*, vol. 34, no. 1, pp. 17 – 24, 2003.

[77] W. Tan, H. Marquez, and T. Chen, "Multivariable robost controller design for a boiler system," *IEEE Trans. on Control System Technology*, vol. 10, no. 5, pp. 735 – 742, 2002.

[78] A. Abdennour and K. Lee, "An autonomous system for boiler-turbine units," *IEEE Trans. on Energy Conversion*, vol. 11, no. 2, pp. 401 – 406, 2004.

[79] F. Alturki and A. Ben-Abdennour, “Neuro-fuzzy control of a steam boiler-turbine unit,” in *Proceedings of the IEEE International Conference on Control Applications*, vol. 2, Hawaii, 1999, pp. 1050 – 1055.

[80] M. Tharayil and A. Alleyne, “A generalized PID error governing scheme for SMART/SBLI control,” in *Proceedings of the American Control Conference*, vol. 1, 2002, pp. 346 – 351.

[81] M. Srinivas and L. Patnaik, “Genetic algorithms: a survey,” *Computer*, vol. 27, no. 6, pp. 17 – 26, Jun 1994.

[82] Y.-Y. Cao, J. Lam, and Y.-X. Sun, “Static output feedback stabilization: An ILMI approach,” *Automatica*, vol. 34, no. 12, pp. 1641 – 1645, 1998.

[83] J. C. Doyle, B. A. Francis, and A. R. Tannenbaum, *Feedback control theory*. New York: Maxwell Macmillian International, 1992.

[84] G. E. Dullerud and F. G. Paganini, *A course in robust control theory: a convex approach*. New York: Springer, 2000.

[85] P. Dorato, *Robust control*. New York: Institute of Electrical and Electronics Engineers, 1987.

[86] D. Huang and S. K. Nguang, “Robust H_∞ static output feedback control of fuzzy systems: an ILMI approach,” *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 36, no. 1, pp. 216 – 222, 2006.

[87] I. Petersen and D. McFarlane, “Optimal guaranteed cost control and filtering for uncertain linear systems,” *IEEE Transactions on Automatic Control*, vol. 39, no. 9, pp. 1971 – 1977, 1994.

[88] G. Wei, Z. Wang, and H. Shu, “Robust filtering with stochastic nonlinearities and multiple missing measurements,” *Automatica*, vol. 45, no. 3, pp. 836 – 841, 2009.

[89] Q.-L. Han and L. Yu, "Robust stability of linear neutral systems with nonlinear parameter perturbations," *IEE Proceedings on Control Theory and Applications*, vol. 151, no. 5, pp. 539 – 546, 2004.

[90] J. Ball and J. Helton, "H_∞ control for nonlinear plants: connections with differential games," in *Proceedings of the Conference on Decision and Control*, 1989, pp. 956 – 962.

[91] T. Başar and G. J. Olsder, *Dynamic noncooperative game theory*. London; New York: Academic Press, 1995.

[92] A. van der Schaft, "L_2-gain analysis of nonlinear systems and nonlinear state-feedback H_∞ control," *IEEE Transactions on Automatic Control*, vol. 37, no. 6, pp. 770 – 784, 1992.

[93] A. Isidori and A. Astolfi, "Disturbance attenuation and H_∞-control via measurement feedback in nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 37, no. 9, pp. 1283 – 1293, 1992.

[94] T. Başar, "Optimum performance levels for minimax filters, predictors and smoothers," *Systems & Control Letters*, vol. 16, no. 5, pp. 309 – 317, 1991.

[95] D. J. Hill and P. J. Moylan, "Dissipative dynamical systems: Basic input-output and state properties," *Journal of the Franklin Institute*, vol. 309, no. 5, pp. 327 – 357, 1980.

[96] V. Syrmos, C. Abdallah, P. Dorato, and K. Grigoriadis, "Static output feedback–a survey," *Automatica*, vol. 33, no. 2, pp. 125 – 137, 1997.

Publications

- Peer-reviewed journal papers

    1. Matthias Krug, Shakir Saat and Sing Kiong Nguang, Robust H_∞ static output feedback controller design for parameter dependent polynomial systems: An iterative sums of squares approach, *Journal of the Franklin Institute*, vol. 350, no. 2, 318 - 330, 2013
    2. Jie Wu, Jiong Shen, Matthias Krug, Sing Kiong Nguang and Yiguo Li, GA-based nonlinear predictive switching control for a boiler-turbine system, *Journal of Control Theory and Applications*, vol. 10, no. 1, pp. 100 - 106, 2012.
    3. Sing Kiong Nguang, Shakir Saat, Matthias Krug, Static output feedback controller design for uncertain polynomial systems: an iterative sum of squares approach, *IET Control Theory and Applications*, vol. 5, no. 9, pp. 1079 - 1084, 2011.
    4. Matthias Krug, Sing Kiong Nguang, Jie Wu and Jiong Shen, GA-based model predictive control of boiler-turbine systems, *International Journal of Innovative Computing, Information and Control*, vol. 6, no. 11, pp. 5237 - 5248, 2010.

- Conference papers

    1. Matthias Krug, Shakir Saat and Sing Kiong Nguang, Nonlinear robust H_∞ static output feedback controller design for parameter dependent polynomial systems:

An iterative sum of squares approach, *Proceedings of the IEEE Conference on Decision and Control*, 2011, pp. 3502 - 3507.

2. Faiz Rasool, Matthias Krug and Sing Kiong Nguang, Robust H_∞ output feedback control of networked control systems with multiple quantizers, *Proceedings of IEEE Conference on Industrial Electronics and Applications,* 2011, pp. 1541 - 1546.

3. Faiz Rasool, Matthias Krug and Sing Kiong Nguang, Robust H_∞ output feedback control of discrete-time networked systems with adaptive quantizers, *Proceedings of IEEE Conference on Industrial Electronics and Applications*, 2011, pp. 1535 - 1540.

4. Sing Kiong Nguang, Matthias Krug and Shakir Saat, Nonlinear static output feedback controller design for uncertain polynomial systems: an iterative sum of squares approach, *Proceedings of the IEEE Conference on Industrial Electronics and Applications*, 2011, pp. 979 - 984.

5. Shakir Saat, Matthias Krug and Sing Kiong Nguang, Nonlinear H_∞ static output feedback controller design for polynomial systems: An iterative sums of squares approach, *Proceedings of the IEEE Conference on Industrial Electronics and Applications*, 2011, pp. 985 - 990.

6. Shakir Saat, Matthias Krug and Sing Kiong Nguang, A nonlinear static output controller design for polynomial systems: An iterative sums of squares approach, *Proceedings of the International Conference on Mechatronics,* 2011, pp. 1 - 6.

7. Jie Wu, Matthias Krug, Sing Kiong Nguang, Jiong Shen and Yiguo Li, H_∞ fuzzy tracking control for boiler-turbine systems, *Proceedings of the IEEE Conference on Control and Automation,* 1980 - 1985. 2009.

Vita

Matthias Krug was born in Erfurt, Germany, on February 1, 1983. He received his B.Sc. form Hamburg University of Technology, Germany, in 2007. He is currently working at the University of Auckland, New Zealand as a professional teaching fellow.

His research interests include polynomial systems, robust control, evolutionary algorithms and optimization.

He is a student member of IEEE.