Degenerate Elliptic Operators with Boundary Conditions via Form Methods

by

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Preface

The main subject in the first part of this thesis are *form methods*. Abstractly, form methods provide a means of both defining and studying unbounded operators in a Hilbert space. The probably most well-known instance of a form method is Kato's representation theorem for closed sectorial forms (1966). This result is commonly applied to obtain suitable realisations of elliptic differential operators in divergence form as unbounded operators in L^2-space.

Form methods tend to be quite robust, which is particularly useful for perturbation problems. Recently, Arendt and ter Elst (2008) have extended Kato's representation theorem to general sectorial forms, without the closedness condition and relaxing the former requirement that the form domain is embedded in the Hilbert space. This extension is well-suited for the *degenerate* elliptic setting and has also been applied to the Dirichlet-to-Neumann operator.

The main contributions of this thesis regarding form methods are the introduction of an abstract form method for *accretive* forms, the study of *compactly elliptic* forms including an application to the convergence of generalised Dirichlet-to-Neumann graphs and an investigation of the *regular part* of sectorial forms providing a formula for the important case of second-order differential sectorial forms. This part of the thesis includes joint work with Wolfgang Arendt, Tom ter Elst, James Kennedy and Hendrik Vogt.

In the final chapter of the thesis we consider a notion of a *weak trace* for elements of a Sobolev space, which is related to work of Maz'ya and arose in the study of the Laplacian with Robin boundary conditions and the Dirichlet-to-Neumann operator on arbitrary domains. Using tools from potential and lattice theory, we investigate the space of elements with weak trace zero. This is related to questions regarding the stability of the Dirichlet problem for varying domains.

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1 Introduction

This thesis is devoted to the study of extensions to the classical *form method* for linear elliptic operators in a Hilbert space. A particular focus is on applications to degenerate elliptic differential operators in $L^2(\Omega)$, where Ω is an open set in $\mathbb{R}^d$.

In this introduction, I first shall make precise what the term 'form method' means here, explain the usefulness of this approach and briefly provide some information regarding its history and development. This is followed by an overview of the thesis. I give a summary of every chapter, highlighting the respective main results. Then I provide details about the collaborative work and my contributions in this thesis.

1.1 General background for the form method

Anatoliĭ Mal'cev opens his *Foundations of Linear Algebra* as follows:

> "In linear algebra one studies three kinds of objects; matrices, linear spaces, and algebraic forms. The theories of these objects are so closely related that most problems of linear algebra have equivalent formulations in each of the three theories."

This classical and fundamental viewpoint remains fruitful also when applied in the topological and generally infinite-dimensional setting of functional analysis. In this thesis the term 'form method' denotes a procedure in Hilbert space that allows to generate operators (or extensions of operators) with certain properties by a correspondence principle between appropriate sesquilinear forms and the desired operators.

One particularly simple classical example of a form method for bounded linear operators is given by the Riesz–Fréchet representation theorem.

Example (Riesz–Fréchet)**.** Let V be a Hilbert space and let $\mathfrak{a}\colon V \times V \to \mathbb{C}$ be a continuous sesquilinear form. Suppose first that $u \in V$ is fixed. Then $\mathfrak{a}(u, \cdot)\colon V \to \mathbb{C}$ is an element of V^*, i.e., a bounded conjugate-linear functional in V. By the Riesz–Fréchet representation theorem there exists a unique $w \in V$ such that $\mathfrak{a}(u, v) = (w \,|\, v)_V$ for all $v \in V$. Hence, allowing u to vary in V, the map $T\colon u \mapsto w$ defines a bounded linear operator on V such that

$$\mathfrak{a}(u, v) = (Tu \,|\, v)_V \tag{1.1}$$

for all $u, v \in V$. ◇

Note that (1.1) gives an immediate one-to-one correspondence between continuous sesquilinear forms and bounded operators.

The second example is the well-known Lax–Milgram lemma.

Lemma (Lax–Milgram)**.** *Let V be a Hilbert space. Denote by V^* the conjugate-linear dual space of V. Let $\mathfrak{a}\colon V \times V \to \mathbb{C}$ be a continuous sesquilinear from. Suppose that $\mathfrak{a}$ is **coercive**, i.e., there exists a $\mu > 0$ such that*

$$\operatorname{Re} \mathfrak{a}(u, u) \geqslant \mu \|u\|_V^2$$

for all $u \in V$. Then there exists an invertible bounded operator $\mathcal{B}\colon V \to V^*$ such that

$$\mathfrak{a}(u, v) = \langle \mathcal{B}u, v \rangle_{V^* \times V} \tag{1.2}$$

for all $u, v \in V$.

So the Lax–Milgram lemma states that it is the invertible operators in $\mathcal{L}(V, V^*)$ which are associated via (1.2) with the coercive, continuous sesquilinear forms in V. Equivalently, it shows that the operator $T \in \mathcal{L}(V)$ in (1.1) is invertible for a coercive, continuous sesquilinear form $\mathfrak{a}$. We will make use of this in the following.

1.1.1 A form method for unbounded operators

The previous examples of form methods merely generate bounded linear operators. We are mainly concerned with generating unbounded operators, however, since differential operators generally are unbounded. We next discuss a construction that allows to generate unbounded operators using the Lax–Milgram lemma.

Let H and V be Hilbert spaces. Suppose that V is continuously and **densely embedded** into H, i.e., $V \subset H$, the closure of V in H is equal to H and there exists an $M > 0$ such that $\|u\|_H \leqslant M\|u\|_V$ for all $u \in V$. We denote the corresponding embedding by j. Let $\mathfrak{a}\colon V \times V \to \mathbb{C}$ be a coercive, continuous sesquilinear form. Then one can use (1.2) and the embedding of V in H to associate an unbounded operator A in H with $\mathfrak{a}$. More precisely, for all $x, f \in H$, we set $x \in D(A)$ and $Ax = f$ if and only if $x \in V$ and

$$\mathfrak{a}(x, v) = (f \,|\, v)_H$$

for all $v \in V$. The operator A has several remarkable properties. We shall single out the property that it has a bounded inverse and give an instructional proof of this property.

Proof. By the Lax–Milgram lemma there exists an invertible operator $T \in \mathcal{L}(V)$ such that

$$\mathfrak{a}(u, v) = (Tu \,|\, v)_V$$

for all $u, v \in V$. Let $x, f \in H$. Making use of the adjoint of the embedding j, it follows that $x \in D(A)$ and $Ax = f$ if and only if there exists a $u \in V$ such that $j(u) = x$ and

$$(Tu \,|\, v)_V = \mathfrak{a}(u, v) = (f \,|\, j(v))_H = (j^* f \,|\, v)_V$$

for all $v \in V$. Therefore $x \in D(A)$ and $Ax = f$ if and only if $x = jT^{-1}j^* f$. In particular, the bounded operator $jT^{-1}j^*$ is the inverse of A. □

Remarkably, this proof also shows that the inverse of A is compact if j is compact and that A is self-adjoint if $\mathfrak{a}$ is symmetric.

The notion of an *elliptic* form extends that of a coercive form by allowing for a shift with the inner product in H. Closely related is Kato's notion of *closed sectorial* forms, which requires that the form domain can be made into a Hilbert space in a certain way. The latter approach also allows to consider sectorial forms that are merely *closable*.

1.1.2 An application to the Dirichlet Laplacian

We give a simple example on how to apply the above form method. Let Ω be a bounded, open set in $\mathbb{R}^d$. Let $V = H^1_0(\Omega)$ and $H = L^2(\Omega)$. It is well-known that $H^1_0(\Omega)$ is compactly and densely embedded in $L^2(\Omega)$. Define $\mathfrak{a}\colon H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{C}$ by

$$\mathfrak{a}(u, v) = \int_\Omega \nabla u \cdot \nabla v.$$

Clearly the form $\mathfrak{a}$ is continuous. Moreover, by the Poincaré–Friedrichs inequality there exists a $\mu > 0$ such that

$$\mu \|u\|^2_{L^2(\Omega)} \leqslant \|\nabla u\|^2_{(L^2(\Omega))^d} \tag{1.3}$$

for all $u \in H^1_0(\Omega)$. Hence $\mathfrak{a}$ is coercive. By the above form method, the operator A associated with $\mathfrak{a}$ has a bounded inverse. Moreover, the inverse of A is compact since the embedding $j\colon H^1_0(\Omega) \to L^2(\Omega)$ is compact. Hence A has compact resolvent. Moreover, the operator A is self-adjoint as $\mathfrak{a}$ is symmetric.

Let $x \in D(A)$. Then it follows from the identity

$$\int_\Omega \nabla x \cdot \nabla v = \int_\Omega (Ax)\overline{v}$$

for all $v \in C_c^\infty(\Omega)$ that $Ax = -\Delta x$ in the sense of distributions. Note that x satisfies Dirichlet boundary conditions in the sense that $x \in H^1_0(\Omega)$. This shows that $-A$ is a realisation of the Dirichlet Laplacian in $L^2(\Omega)$.

1.1.3 Benefits of the form method

In this subsection we briefly address the question of why is it beneficial to use the form method. We shall focus on its application to formal elliptic differential operators. Then the form method is used to provide realisations of such operators in a suitable L^2-space. The sesquilinear form itself is usually directly obtained from the formal operator using integration by parts and the corresponding quadratic form often plays the role of a naturally associated energy functional.

Generally the classical function spaces, such as the continuously differentiable functions, tend to be too restrictive to allow good realisations. Alternatively, the approach based on distribution theory introduces additional difficulties due to the lack of structure of the corresponding spaces. While the form method is limited to the Hilbert space setting, it strikes a balance between the previous two extremes. The richer functional analytic setting allows an elegant and powerful theory based on Sobolev spaces and spectral theory.

For example, in Subsection 1.1.2 the form method directly provides a self-adjoint realisation of the Dirichlet Laplacian with compact resolvent. So by the spectral theorem there exists a corresponding orthonormal basis of eigenfunctions, which, among other things, immediately allows a description of the C_0-semigroup generated by the Dirichlet Laplacian. We stress that, in a sense, the only nontrivial ingredient in applying the form method in Subsection 1.1.2 is the Poincaré–Friedrichs inequality in (1.3). The latter is, however, a property of the form domain. This is remarkable since the form domain is far more 'stable' than the domain of the associated operator. It was observed by von Neumann that domains of unbounded operators are notoriously delicate [Neu29b, Satz 18]. So this gives a hint why the form method is particularly useful for perturbation problems. Moreover, as another application, forms sums frequently allow to give meaning to the sum of two unbounded operators.

1.1.4 Historical remarks

We give a brief and certainly incomplete historical overview of the development of the form method.

It is classical to study bounded operators in terms of sesquilinear forms. In the development of the theory of unbounded operators by von Neumann, however, the form methods where de-emphasised, see Alonso and Simon [AS80, Section 1]. For example, von Neumann uses Cayley transformations and the theory of unitary operators in his construction of extensions of symmetric operators in [Neu29a]. Friedrichs presented a very natural construction based on form methods of a unique maximal positive self-adjoint extension of a positive symmetric operator [Fri34], which answered a question of von Neumann [Neu29a, p. 103]. Still, the general usefulness of the form method employed in this construction was only recognised later. For example, Aronszajn opens in [Aro61] as follows:

> "Since the first quarter of this century the theory of quadratic forms has been somehow superseded by the theory of operators in the investigation of vector spaces, and with such rewarding success that the older theory has almost been forgotten – to an extent that younger mathematicians to-day may not be aware of some of the basic results in the theory."

In the early 1950's the ellipticity property of the sesquilinear form associated with an elliptic differential operator was established by Gårding [Går51; Går53]. In 1954 Aronszajn [Aro54] extended Gårding's results. Around the same time Lax and Milgram introduced their Lax–Milgram lemma while studying parabolic problems for elliptic differential operators [LM54]. In the following years Kato [Kat55] and J.-L. Lions [Lio57] established the abstract form method as a most convenient way to generate and study unbounded operators in a Hilbert space, having in mind the application to generate realisations of elliptic differential operators. In 1966 Kato presented his form method for closed sectorial forms [Kat80, Chapter VI], which nowadays is both well-known and commonly used. Kato's formulation provides an elegant one-to-one correspondence between m-sectorial operators and closed sectorial forms. McIntosh extended the theory of closed sectorial forms to accretive forms, see [McI66; McI68; McI70]. For positive symmetric forms Simon introduced the , which allowed to associate a positive self-adjoint operator even with nonclosable forms, see [Sim78a; Sim78b].

Of the more recent work, we single out two contributions. The first one is due to Vogt. He proved in [Vog09] that for a positive symmetric form associated with a real pure second-order differential expression, the abstract construction of the regular part respects the original form and yields a positive symmetric form that is again associated with a real pure second-order differential expression. The second one is by Arendt and ter Elst. In [AE12] they extended Kato's form method for closed sectorial forms to general sectorial forms. Moreover, their construction allows to use an arbitrary linear map from the from domain into the Hilbert space, instead of an (injective) embedding.

1.2 Outline of the thesis

In the appendix we gather general background material for the convenience of the reader. In particular, we recollect results about accretive operators and provide an introduction to graphs, i.e., multi-valued linear operators. Moreover, we gather facts about Kato's notion of the gap, introduce the Moore–Penrose generalised inverse along with two stability results by Izumino and recollect several basic properties of Sobolev spaces.

We give a summary of the following chapters. Chapter 2 provides basic prerequisites for the form method that are used in Chapters 3 and 4.

1.2.1 Chapter 3: The form method for accretive forms and operators

In Chapter 3 we study the prospects of a generalised form method for accretive forms to generate accretive operators. More precisely, the setting is as follows. Let V and H be Hilbert spaces, $\mathfrak{a}\colon V \times V \to \mathbb{C}$ a continuous, accretive sesquilinear form and $j \in \mathcal{L}(V, H)$ be such that $\operatorname{rg} j$ is dense in H. This should be compared with Subsection 1.1.1 where we required that $\mathfrak{a}$ is coercive and j is an (injective) embedding. In particular, in Chapter 3 we work with the same relaxed condition on the form domain as used by Arendt and ter Elst in [AE12, Section 2].

Throughout the chapter we are mostly concerned with accretive operators, as opposed to accretive graphs. We give a multitude of examples for many degenerate phenomena that can occur in the most general setting. In particular we showcase the pathological behaviour that can arise for accretive forms that are nonclosed in the sense of M^c^Intosh and for non-injective j. We give an abstract characterisation of when the graph associated with $(\mathfrak{a}, j)$ is an m-accretive operator. Furthermore, we investigate the class of operators that can be generated. In particular, we prove the following result in Theorem 3.29.

Theorem. *An accretive operator A in H can be generated by a continuous accretive form if and only if $\operatorname{rg}(I + A)$ is the range of a bounded operator on H.*

We obtain as a corollary that an accretive operator generated by a continuous accretive form is maximal accretive if and only if it is m-accretive. We also give several examples of accretive operators that cannot be generated. For the case that the associated graph is an m-accretive operator, we study form approximation and Ouhabaz-type invariance criteria.

1.2.2 Chapter 4: The form method for compactly elliptic forms

In this chapter we first introduce the notion of compactly elliptic forms. We then study the graphs associated with compactly elliptic forms.

Let V and H be Hilbert spaces, $j \in \mathcal{L}(V, H)$ and $\mathfrak{a}\colon V \times V \to \mathbb{C}$ a continuous sesquilinear form. Note that we assume neither that j is an (injective) embedding, nor that $\operatorname{rg} j$ is dense in H. The first main result is Theorem 4.9.

Theorem. *Suppose that* $\mathfrak{a}$ *is compactly elliptic. Let* A *be the graph associated with* $(\mathfrak{a}, j)$*. Then* A *is m-accretive if* $\mathfrak{a}$ *is accretive and* A *is self-adjoint if* $\mathfrak{a}$ *is symmetric.*

We then study form approximation in this setting. Let $(\mathfrak{a}_n)$ be a sequence of compactly elliptic forms on V. We suppose that the forms are all accretive or all symmetric and that $(\mathfrak{a}_n)$ suitably converges to $\mathfrak{a}$. We tackle the question of whether the associated graphs, which by the above theorem are m-accretive or self-adjoint, converge in the strong resolvent sense. There naturally arises a sufficient condition involving the dimensions of certain finite-dimensional subspaces $W_j(\mathfrak{a}_n)$ of V. In Theorem 4.19 we formulate the corresponding approximation result, which is the second main result of the chapter. We then investigate the strong convergence of the associated (degenerate) C_0-semigroups. Again there naturally arises a sufficient condition involving the dimensions of certain finite-dimensional subspaces of V.

Finally we apply the convergence result in Theorem 4.19 to generalised Dirichlet-to-Neumann graphs. Surprisingly, the condition on the dimensions of the spaces $W_j(\mathfrak{a}_n)$ turns out to be connected with the unique continuation property for elliptic operators.

In Section A.2 of the appendix we provide the required background on graphs and (degenerate) C_0-semigroups.

1.2.3 Chapter 5: The regular part of sectorial forms

For a general, possibly nonclosable, positive symmetric form, Simon [Sim78a] introduced a decomposition into a maximal closable *regular part* and a *singular part*. Using Kato's first representation theorem, a self-adjoint operator can be associated with the closure of the regular part. By a result of Arendt and ter Elst [AE12, Theorem 1.1] one can naturally associate an m-sectorial operator to any densely defined sectorial form. If the form is symmetric, then one reobtains the operator associated with Simon's regular part. This allows to generalise the notion of the regular and singular part to sectorial forms. In Chapter 5, we study this generalised regular and singular part in detail.

Throughout Chapter 5 we work in the setting of j-sectorial forms. We introduce a suitable definition of the regular part $\mathfrak{a}_{\mathrm{reg}}$ of a j-sectorial form $\mathfrak{a}$, and show that it is uniquely determined in a natural way. Moreover, we characterise when the singular part $\mathfrak{a}_s = \mathfrak{a} - \mathfrak{a}_{\mathrm{reg}}$ is sectorial. The main result of the chapter is Theorem 5.7

where we establish a formula of the regular part in terms of the real part of $\mathfrak{a}$, which is crucial in Chapter 6.

1.2.4 Chapter 6: The regular part of differential sectorial forms

In Chapter 6 we study the regular part of a differential sectorial form that represents a general linear second-order differential expression that may include lower-order terms. Loosely speaking, such a differential expression has the form

$$-\sum_{k,l=1}^{d} \partial_l c_{kl} \partial_k + \sum_{k=1}^{d} b_k \partial_k - \sum_{k=1}^{d} \partial_k d_k + c_0.$$

We impose mild conditions on the coefficients. Using the abstract formula for the regular part in Theorem 5.7 and the techniques introduced by Vogt in [Vog09], we obtain a formula for the regular part of the corresponding differential sectorial form from which it follows that the regular part is again a differential sectorial form. Furthermore, this formula allows to characterise when the singular part is sectorial and when the regular part of the real part is equal to the real part of the regular part. Remarkably, the presence of first-order terms introduces new phenomena compared to the pure second-order case. We give several interesting examples where we make use of our formula for the regular part.

1.2.5 Chapter 7: Elements of Sobolev space with weak trace zero

In Chapter 7 we consider a weak notion of the boundary trace for elements of the Sobolev space $W^{1,p}(\Omega)$ that is naturally defined via the approximation by functions that are continuous on $\overline{\Omega}$. This notion was introduced by Arendt and ter Elst in [AE11] and works for general open sets Ω.

Let $p \in (1, \infty)$. As usual, let $W_0^{1,p}(\Omega)$ be the closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$. Every $u \in W_0^{1,p}(\Omega)$ has weak trace zero and becomes, if extended by 0 outside of Ω, an element of $W^{1,p}(\mathbb{R}^d)$. The main result of the chapter is Theorem 7.38, which extends this extension property to all elements with weak trace zero. It, loosely speaking, states the following.

Theorem. *Let $u \in W^{1,p}(\Omega)$. Suppose that u has weak trace zero, i.e., suppose that there exists a sequence (u_n) in $W^{1,p}(\Omega) \cap C(\overline{\Omega})$ such that $u_n \to u$ in $W^{1,p}(\Omega)$ and the restriction of u_n to the boundary converges to zero in a suitable sense. Then the extension of u by 0 outside of Ω is an element of $W^{1,p}(\mathbb{R}^d)$.*

The proof relies crucially on an observation concerning the proof of [SZ99, Theorem 2.2] by Swanson and Ziemer.

Further, let $W_0^{1,p}(\overline{\Omega})$ be the space of restrictions to Ω of elements in $W^{1,p}(\mathbb{R}^d)$ that are 0 a.e. on $\mathbb{R}^d \setminus \overline{\Omega}$. We provide examples which show that, in general, elements with weak trace zero are not contained in $W_0^{1,p}(\Omega)$ and that not every element of $W_0^{1,p}(\overline{\Omega})$ has weak trace zero. In Section 7.5 we discuss consequences of the main result; this includes sufficient conditions on the domain Ω such that every element with weak trace zero is contained in $W_0^{1,p}(\Omega)$.

The space of elements with weak trace zero is a closed subspace of $W^{1,p}(\Omega)$. We will see that it is a closed lattice ideal of both $W^{1,p}(\Omega)$ and $W^{1,p}(\mathbb{R}^d)$ in a suitable sense. Using a result of Stollmann, this allows a description of the space of elements with weak trace zero based on the support of quasi continuous representatives. We compare this to very similar results for the space $W_0^{1,p}(\Omega)$.

The arguments throughout Chapter 7 require the notion of relative capacity and fine representatives of elements in Sobolev space. The necessary background and many references will be provided.

1.3 Contributions

The material in Chapter 3 is joint work in progress with Tom ter Elst and Hendrik Vogt [ESV13]. We introduce an abstract form method for accretive forms and operators which provides a common generalisation to both McIntosh's form method for closed accretive forms [McI68; McI70] and the recent results by Arendt and ter Elst [AE12, Section 2]. In particular, we neither assume that the form is closed nor that the form domain V is embedded in H. For this reason we consider general accretive operators and not only m-accretive operators. In the most general setting, various new phenomena occur, as well as new forms of degenerate behaviour. We provide a plethora of examples and give conditions which ensure a more regular behaviour.

Chapter 4 is based on joint work with Wolfgang Arendt, Tom ter Elst and James Kennedy in [AEKS13]. Our research was motivated by questions concerning generalised Dirichlet-to-Neumann graphs that were stimulated by [AE12, Subsection 4.4] and [AM12]. The newly introduced notion of compactly elliptic forms gives rise to a form method that is well-suited for the study of generalised Dirichlet-to-Neumann graphs and their stability, both in the accretive and symmetric setting. For this it is crucial that we do not assume that the form domain V is embedded in H.

The exposition in Chapter 4 is somewhat different from that in [AEKS13]. In this thesis I present the material from the general viewpoint of Chapter 3 and make use of results of Izumino about the stability of Moore–Penrose generalised inverses [Izu83]. Moreover, both the accretive and the symmetric case are considered at the same time. To relate the results of Chapter 4 to those in Chapter 3, it is important to note that the graphs associated with an accretive or symmetric compactly elliptic form are always m-sectorial graphs by Corollary 4.26.

The content in Chapter 5 is joint work with Tom ter Elst [ES11]. We study a generalised notion of the regular part for j-sectorial forms, which were introduced in [AE12, Section 3]. In Theorem 5.7 we prove a formula for the regular part that is useful to transfer results from the positive symmetric case. The work in this chapter extends some results about the regular part of positive symmetric forms by Simon [Sim78a].

The material in Chapter 6 is joint work with Tom ter Elst [ES13] that extends our previous results from [ES11, Section 4]. The formula for the regular part of a differential sectorial form in Theorem 6.5 extends results by Vogt [Vog09] and has immediate interesting consequences. The results in this chapter apply to a large class of forms associated with degenerate elliptic differential operators.

Chapter 7 contains my results about the space of elements with weak trace zero. Theorems 7.38 and 7.66 allow to relate this space to the spaces $W_0^{1,p}(\Omega)$ and $W_0^{1,p}(\overline{\Omega})$. In particular, the findings extend [AE11, Proposition 5.5] to domains with continuous boundary. Examples 7.56 and 7.55 show that the main result of the chapter, Theorem 7.38, is in a sense best possible. Applications of the results in Chapter 7 include domain approximation and the study of stability for problems with Dirichlet boundary conditions. Moreover, for $p = 2$ the space of elements with weak trace zero can be used as a form domain to obtain a realisation of the Dirichlet Laplacian in $L^2(\Omega)$, which is different from the usual realisation in Subsection 1.1.2 for irregular domains. This is analogous to using $W_0^{1,2}(\overline{\Omega})$ as the form domain, which gives rise to the pseudo Dirichlet Laplacian, see for example [AM95]. These remarks highlight that there is a considerable amount of freedom when considering problems with Dirichlet boundary conditions on very irregular domains.

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2 Preliminaries

In this chapter we collect various prerequisites for Chapters 3 and 4.

2.1 General remarks

In the part of this work where we study form methods, we are always in a Hilbert space setting. If not stated otherwise, we assume our Hilbert spaces to be complex and possibly nonseparable.

We will distinguish between *operators* and *graphs*. The operators and graphs considered here are always linear. We shall use the notion of *graph* to refer to what is commonly called a multi-valued operator or linear relation, whereas an *operator* will always be supposed to be a functional relation. We refer to Section A.2 for a succinct overview of the required theory.

2.2 Background of the form method

Let V, H be Hilbert spaces, and let $\mathfrak{a}\colon V \times V \to \mathbb{C}$ be a continuous sesquilinear form. Recall that $\mathfrak{a}$ is **continuous** if and only if there exists an $M \geqslant 0$ such that $|\mathfrak{a}(u,v)| \leqslant M\|u\|_V\|v\|_V$ for all $u, v \in V$. If V is continuously and densely embedded in H, then one defines the graph of an **operator A associated with the form $\mathfrak{a}$** in H as follows. Let $x, f \in H$. Then $x \in D(A)$ and $Ax = f$ if and only if $\mathfrak{a}(x,v) = (f \,|\, v)_H$ for all $v \in V$. Lions [Lio57, Theorem 3.6] proved the following theorem.

Theorem 2.1 (Lions). *Suppose that* V *is continuously and densely embedded in* H. *Moreover, suppose that* $\mathfrak{a}$ *is* ***elliptic****, i.e., there are* $\omega \in \mathbb{R}$ *and* $\mu > 0$ *such that*

$$\operatorname{Re} \mathfrak{a}(u, u) + \omega \|u\|_H^2 \geqslant \mu \|u\|_V^2$$

for all $u \in V$. *Then the operator* A *is m-sectorial.*

In [McI68] McIntosh improved Theorem 2.1 to the setting of accretive forms. Recall that $\mathfrak{a}$ is called **accretive** if

$$\operatorname{Re} \mathfrak{a}(u, u) \geqslant 0$$

for all $u \in V$.

Theorem 2.2 (McIntosh). *Suppose that* V *is continuously and densely embedded in* H. *Moreover, suppose that* $\mathfrak{a}$ *is accretive and that there exists a* $\mu > 0$ *such that*

$$\sup_{\|v\|_V \leqslant 1} |\mathfrak{a}(u, v) + (u \,|\, v)_H| \geqslant \mu \|u\|_V \tag{2.1}$$

for all $u \in V$. *Then the operator* A *is m-accretive.*

Clearly, if in Theorem 2.1 the ellipticity condition holds with $\omega = 1$, then (2.1) holds with the same value of μ (but $\mathfrak{a}$ does not need to be accretive). Note that if $\omega \in \mathbb{R}$ and $\mathfrak{a}' \colon V \times V \to \mathbb{C}$ is given by $\mathfrak{a}'(u, v) = \mathfrak{a}(u, v) + \omega (u \,|\, v)_H$, then the operator A' associated with $\mathfrak{a}'$ satisfies $A' = A + \omega I$, thus differs from A only by a shift. Traditionally the form $\mathfrak{a}$ in Theorem 2.1 does not have to be accretive, but in Theorem 2.2 the form $\mathfrak{a}$ is supposed to be accretive. One can relax the conditions in Theorem 2.2 by introducing a shift and replacing $\mathfrak{a}(u, v) + (u \,|\, v)_H$ by $\mathfrak{a}(u, v) + \omega' (u \,|\, v)_H$, but this essentially does not change the content. In order to avoid introducing such a shift, in the following sections we assume that $\mathfrak{a}$ is already accretive.

Finally, we formulate a recent generalisation of Theorem 2.1 where the Hilbert space V does not have to be embedded in the Hilbert space H.

Theorem 2.3 (Arendt and ter Elst [AE12, Theorem 2.1]). *Let* $j \colon V \to H$ *be a continuous linear map with dense range. Suppose* $\mathfrak{a}$ *is* ***j-elliptic****, i.e., there exist* $\omega \in \mathbb{R}$ *and* $\mu > 0$ *such that*

$$\operatorname{Re} \mathfrak{a}(u, u) + \omega \|j(u)\|_H^2 \geqslant \mu \|u\|_V^2$$

for all $u \in V$. *Define the graph of an operator* A *as follows. If* $x, f \in H$, *then* $x \in D(A)$ *and* $Ax = f$ *if and only if there exists a* $u \in V$ *such that* $j(u) = x$ *and* $\mathfrak{a}(u, v) = (f \,|\, j(v))_H$ *for all* $v \in V$. *Then* A *is well-defined and m-sectorial.*

In Section 3.1 we present a generation theorem which generalises both Theorem 2.2 and 2.3.

2.3 The abstract form method in the complete setting

In this section we introduce the notation that we will use in our investigation of the form method in the complete setting. In particular, the notation here will be used in Chapters 3 and 4. Moreover, we shall collect some basic properties of the introduced objects in the process.

Let V and H be Hilbert spaces. Let $\mathfrak{a}\colon V \times V \to \mathbb{C}$ be a continuous sesquilinear form and let $j \in \mathcal{L}(V, H)$. Suppose that $A \subset H \times H$ is such that $(x, f) \in A$ if and only if there exists a $u \in V$ such that $j(u) = x$ and $\mathfrak{a}(u, v) = (f \mid j(v))_H$ for all $v \in V$. Then A is a linear subspace of $H \times H$; so A is a graph in $H \times H$. We call A the **graph associated with** $(\mathfrak{a}, j)$ and say that $(\mathfrak{a}, j)$ **generates** A. If A is the graph of an operator, we also say that $(\mathfrak{a}, j)$ **generates an operator** and call A the **operator associated with** $(\mathfrak{a}, j)$.

The following is an easy observation.

Lemma 2.4. *Let $\rho \in \mathbb{C}$ and define $\mathfrak{b}\colon V \times V \to \mathbb{C}$ by*

$$\mathfrak{b}(u, v) = \mathfrak{a}(u, v) + \rho(j(u) \mid j(v))_H.$$

If A is the graph associated with $(\mathfrak{a}, j)$, then $\rho I + A$ is the graph associated with $(\mathfrak{b}, j)$.

On the one hand we are interested in conditions on $(\mathfrak{a}, j)$ that ensure that the graph A generated by $(\mathfrak{a}, j)$ satisfies a certain range condition. For example, if $\mathfrak{a}$ is accretive we want to know when A is m-accretive. On the other hand we are interested in conditions on $(\mathfrak{a}, j)$ which imply that A is an operator.

We define

$$D_j(\mathfrak{a}) \coloneqq \big\{u \in V : \text{there exists an } f \in H \text{ such that } \mathfrak{a}(u, v) = (f \mid j(v))_H \text{ for all } v \in V\big\}$$

and

$$V_j(\mathfrak{a}) \coloneqq \{u \in V : \mathfrak{a}(u, v) = 0 \text{ for all } v \in \ker j\}.$$

Then both $D_j(\mathfrak{a})$ and $V_j(\mathfrak{a})$ are subspaces of V, $D_j(\mathfrak{a}) \subset V_j(\mathfrak{a})$ and $V_j(\mathfrak{a})$ is closed in V. Suppose that A is the graph generated by $(\mathfrak{a}, j)$. Then $D_j(\mathfrak{a})$ is closely related to the domain of A; more precisely, $D(A) = j(D_j(\mathfrak{a}))$. This direct connection to the domain of the associated graph or operator highlights that $D_j(\mathfrak{a})$ is a delicate object which in applications frequently will be unknown. In comparison, the space $V_j(\mathfrak{a})$ is a simpler and more stable object. We will encounter conditions on $(\mathfrak{a}, j)$ which guarantee that $D_j(\mathfrak{a})$ is dense in $V_j(\mathfrak{a})$.

We define the operator $T_0 \in \mathcal{L}(V)$ by requiring that $(T_0 u \mid v)_V = \mathfrak{a}(u, v)$ for all $u, v \in V$. For all $\rho \in \mathbb{C}$ we define $T_\rho \coloneqq T_0 + \rho j^* j$. Then

$$(T_\rho u \mid v) = \mathfrak{a}(u, v) + \rho(j(u) \mid j(v))_H$$

for all $u, v \in V$. We will often use T_ρ instead of working directly with $\mathfrak{a}$ and j.

We shall see that sometimes restrictions of $\mathfrak{a}$ to suitable subspaces of V exhibit more regular behaviour. If V_1 is a closed subspace of V, we will call the pair $(\mathfrak{a}|_{V_1 \times V_1}, j|_{V_1})$ the **restriction** of $(\mathfrak{a}, j)$ to V_1.

Lemma 2.5. *Let V, H, $\mathfrak{a}$ and j be as above. Let A be the graph associated with $(\mathfrak{a}, j)$. Furthermore, let V_1 be a closed subspace of V such that $D_j(\mathfrak{a}) \subset V_1$, and let $(\mathfrak{a}_1, j_1)$ be the restriction of $(\mathfrak{a}, j)$ to V_1. Then the graph associated with $(\mathfrak{a}_1, j_1)$ is an extension of A.*

Proof. Let $(x, f) \in A$. Then there exists a $u \in D_j(\mathfrak{a})$ such that $j(u) = x$ and $\mathfrak{a}(u, v) = (f \,|\, j(v))_H$ for all $v \in V$. Since $u \in V_1$, it follows that $j_1(u) = x$ and $\mathfrak{a}_1(u, v) = (f \,|\, j_1(v))_H$ for all $v \in V_1$. Hence (x, f) is an element of the graph associated with $(\mathfrak{a}_1, j_1)$. □

The following gives an abstract characterisation of when the graph associated with $(\mathfrak{a}, j)$ is an operator.

Proposition 2.6. *Let V, H, $\mathfrak{a}$ and j be as above. Let A be the graph associated with $(\mathfrak{a}, j)$. Suppose that $\operatorname{rg} j$ is dense in H. Then the following are equivalent.*

(i) *A is an operator.*

(ii) *$D_j(\mathfrak{a}) \cap \ker j \subset \ker T_0$.*

(iii) *$D_j(\mathfrak{a}) \cap \ker j \subset \ker T_\rho$ for some (or all) $\rho \in \mathbb{C}$.*

Proof. '(i)⇒(ii)': Let $u \in D_j(\mathfrak{a}) \cap \ker j$. Then

$$(T_0 u \,|\, v)_V = \mathfrak{a}(u, v) = (Aj(u) \,|\, j(v))_H = 0$$

for all $v \in V$, whence $T_0 u = 0$.

'(ii)⇒(i)': Let $u \in D_j(\mathfrak{a})$, $f \in H$ and suppose that $j(u) = 0$ and $\mathfrak{a}(u, v) = (f \,|\, j(v))_H$ for all $v \in V$. Then $u \in D_j(\mathfrak{a}) \cap \ker j$ and hence $T_0 u = 0$ by Condition (ii). So $(f \,|\, j(v))_H = \mathfrak{a}(u, v) = (T_0 u \,|\, v)_V = 0$ for all $v \in V$. As $\operatorname{rg} j$ is dense in H, one deduces $f = 0$. By linearity this shows that A is an operator.

'(ii)⇔(iii)': This follows from $T_\rho = T_0 + \rho j^* j$. □

Note that it is an immediate consequence of Proposition 2.6 that $(\mathfrak{a}, j)$ generates an operator if $\operatorname{rg} j$ is dense and $V_j(\mathfrak{a}) \cap \ker j = \{0\}$.

Lemma 2.7. *Let A be the graph associated with $(\mathfrak{a}, j)$. Let $\rho \in \mathbb{C}$. Then $f \in \operatorname{rg}(\rho I + A)$ if and only if there exists a $u \in V$ such that $T_\rho u = j^* f$.*

Proof. Let $f \in H$. If $T_\rho u = j^* f$, then

$$\mathfrak{a}(u, v) + \rho(j(u) \,|\, j(v))_H = (T_\rho u \,|\, v)_V = (j^* f \,|\, v)_V = (f \,|\, j(v))_H$$

for all $v \in V$. So $(j(u), f - \rho j(u)) \in A$, hence $f \in \operatorname{rg}(\rho I + A)$. Conversely, suppose $f \in \operatorname{rg}(\rho I + A)$. Then there exists a $u \in D_j(\mathfrak{a})$ such that $\mathfrak{a}(u, v) = (f - \rho j(u) \mid j(v))_H$ for all $v \in V$. Hence $T_\rho u = j^* f$. $\square$

Lemma 2.8. *Let $\rho \in \mathbb{C}$ and $u \in V$. Then $u \in D_j(\mathfrak{a})$ if and only if $T_\rho u \in \operatorname{rg} j^*$.*

Proof. By definition, $u \in D_j(\mathfrak{a})$ if and only if there exists an $f \in H$ such that

$$(T_0 u \mid v)_V = \mathfrak{a}(u, v) = (f \mid j(v))_H = (j^* f \mid j(v))_V$$

for all $v \in V$. Now the claim follows from the inclusion $\operatorname{rg}(T_\rho - T_0) \subset \operatorname{rg} j^*$. $\square$

Lemma 2.9. *Let A be the graph associated with $(\mathfrak{a}, j)$. Suppose that $\rho(A) \neq \emptyset$ and j is compact. Then A has compact resolvent.*

Proof. Choose $\rho \in \mathbb{C}$ such that $(\rho I + A)^{-1} \in \mathcal{L}(H)$. Let $f \in H$. By Lemma 2.7 there exists a $u \in V$ such that $T_\rho u = j^* f$. Then $(\rho I + A)^{-1} f = j(u)$. Set $W \coloneqq (\ker T_\rho)^\perp$. We may assume that $u \in W$. Then u is unique. So by mapping $f \mapsto u$ we obtain a linear map $Z \colon H \to W$. Using the closed graph theorem, it is readily verified that Z is bounded. Since $(\rho I + A)^{-1} = jZ$, the graph A has compact resolvent. $\square$

Definition 2.10. If there exists an $\omega \in \mathbb{R}$ and a $\mu > 0$ such that

$$\operatorname{Re} \mathfrak{a}(u, u) + \omega \|j(u)\|_H^2 \geqslant \mu \|u\|_V^2$$

for all $u \in V$, we say that $\mathfrak{a}$ is **j-elliptic**.

The following is a straightforward generalisation of Theorem 2.3 that does not require $\operatorname{rg} j$ to be dense in H. The proof shows that the graph associated with a j-elliptic form $\mathfrak{a}$ fails to be an operator only in a trivial way.

Theorem 2.11. *Let V, H, $\mathfrak{a}$ and j be as above. If $\mathfrak{a}$ is j-elliptic, then $(\mathfrak{a}, j)$ is associated with an m-sectorial graph A such that $A[0] = (\operatorname{rg} j)^\perp$. In particular, if $\operatorname{rg} j$ is dense in H, then A is an operator.*

Proof. Set $H_1 \coloneqq \overline{\operatorname{rg} j}$. Define $j_1 \colon V \to H_1$ by $j_1(u) = u$. Let A_1 be the m-sectorial operator in H_1 associated with $(\mathfrak{a}, j_1)$ by Theorem 2.3. Set $H_2 \coloneqq H_1^\perp$. It is readily verified that $A = \operatorname{gr} A_1 \oplus (\{0\} \times H_2)$ and that A is m-sectorial. $\square$

Remark 2.12. To compare this with the general abstract setting, let $f \in H$ and $\rho \in \mathbb{C}$. Then $f \in A[0]$ if and only if there exists a $u \in \ker j$ such that $T_\rho u = j^* f$. So $\ker j$ plays a vital role here, not merely $\ker j^*$.

3

The form method for accretive forms and operators

In this chapter we introduce and study a form method for accretive forms and operators. Our motivation is to provide a common generalisation to both M^c^Intosh's generation theorem for accretive forms, Theorem 2.2, and the generation theorem for j-elliptic forms by Arendt and ter Elst, Theorem 2.3.

We start in Section 3.1 by presenting the new generation theorem. We study various sufficient conditions on the form which ensure that the generation theorem can be applied. Moreover, we investigate generation properties of suitable restrictions of the form. In Section 3.2 we give a necessary and sufficient condition in terms of operator ranges for an accretive operator to be associated with an accretive form. We give examples of accretive operators which can not be generated by an accretive form. In Section 3.3 we prove a basic form approximation result. We investigate generation properties of the dual form in Section 3.4. This is followed by Section 3.5, where we study a suitable sufficient condition for the range condition in the generation theorem that is adapted from M^c^Intosh [M^c^I68]. In Section 3.6 we transfer a characterisation by Ouhabaz for the invariance of closed convex sets under the associated semigroup to our setting. Finally, in Section 3.7 we briefly discuss how our results can be applied in a setting where the form domain is merely a pre-Hilbert space.

Throughout this chapter we provide various examples, give implications between many different conditions and highlight fundamental differences to the well-known elliptic theory. In Section A.1 of the appendix we collect various basic results about accretive operators. The material in this chapter is joint work with Tom ter Elst and Hendrik Vogt [ESV13].

3.1 The complete case

Let V and H be Hilbert spaces, $\mathfrak{a}\colon V \times V \to \mathbb{C}$ a sesquilinear form and $j \in \mathcal{L}(V, H)$. We assume that

(I) $\mathfrak{a}$ is continuous and accretive, and
(II) $j(V)$ is dense in H.

We point out that we do not assume j to be injective.

Since $\mathfrak{a}$ is continuous, there exists an operator $T_0 \in \mathcal{L}(V)$ such that

$$\mathfrak{a}(u, v) = (T_0 u \,|\, v)_V$$

for all $u, v \in V$. Clearly T_0 is accretive, hence m-accretive by Corollary A.6. Recall that

$$D_j(\mathfrak{a}) = \big\{u \in V : \text{there exists an } f \in H \text{ such that } \mathfrak{a}(u, v) = (f \,|\, j(v))_H \text{ for all } v \in V\big\}.$$

Note that for any $u \in D_j(\mathfrak{a})$, the element f on the right hand side is unique since $j(V)$ is dense in H. Here we use the notation $D_j(\mathfrak{a})$ instead of the notation $D_H(\mathfrak{a})$ that was introduced in [AE12] to emphasise that this space depends not only on H, but also on j. It will be convenient for the following to define the sesquilinear form $\mathfrak{b}\colon V \times V \to \mathbb{C}$ by

$$\mathfrak{b}(u, v) = \mathfrak{a}(u, v) + (j(u) \,|\, j(v))_H.$$

Since $\mathfrak{b}$ is continuous, there exists an m-accretive operator $T \in \mathcal{L}(V)$ such that

$$\mathfrak{b}(u, v) = (Tu \,|\, v)_V \tag{3.1}$$

for all $u, v \in V$. Clearly $T = T_0 + j^*j$. We assume throughout this chapter that $\mathfrak{a}$ and j satisfy Conditions (I) and (II), and we define T_0, T, $\mathfrak{b}$ and $D_j(\mathfrak{a})$ as above.

If $\mathfrak{b}(u, u) = 0$, then $\|j(u)\|_H^2 = 0$ since $\mathfrak{a}$ is accretive. Put differently,

$$(Tu \,|\, u)_V = 0 \quad \text{implies} \quad u \in \ker j. \tag{3.2}$$

In particular, $\ker T \subset \ker j$. As T is m-accretive, it follows from Proposition A.8 that $\ker T^* = \ker T \subset \ker j$. Hence

$$\overline{\operatorname{rg} j^*} \subset \overline{\operatorname{rg} T}. \tag{3.3}$$

Moreover, one has

$$\ker T \subset D_j(\mathfrak{a}) \cap \ker j. \tag{3.4}$$

If the statements in Proposition 2.6 are satisfied, we say that $(\mathfrak{a}, j)$ is **associated with an accretive operator** and call A the **operator associated with** $(\mathfrak{a}, j)$.

Now we state the generalised generation theorem, which is the main result of this section.

Theorem 3.1 (Generation theorem for m-accretive operators)**.** *Let* V, H, $\mathfrak{a}$ *and* j *be as above. Suppose that* $\mathfrak{a}$ *and* j *satisfy Conditions* (I) *and* (II). *Assume that the equivalent conditions of Proposition 2.6 are satisfied and let* A *be the operator associated with* $(\mathfrak{a}, j)$. *Then* A *is m-accretive if and only if*

$$\operatorname{rg} j^* \subset \operatorname{rg} T. \tag{3.5}$$

We first establish a simple general formula. By $T_0^{-1}[\cdot]$ we denote taking the preimage under T_0 (analogously for T).

Lemma 3.2. *Suppose* $\mathfrak{a}$ *and* j *satisfy Conditions* (I) *and* (II). *Then* $D_j(\mathfrak{a}) = T_0^{-1}[\operatorname{rg} j^*] = T^{-1}[\operatorname{rg} j^*]$. *In particular,* $T(D_j(\mathfrak{a})) \subset \operatorname{rg} j^*$.

Proof. Let $u \in V$. By definition, $u \in D_j(\mathfrak{a})$ if and only if there exists an $f \in H$ such that $\mathfrak{a}(u, v) = (f \,|\, j(v))_H$ for all $v \in V$. This is equivalent to the statement that there exists an $f \in H$ such that $(T_0 u \,|\, v)_V = \mathfrak{a}(u, v) = (j^* f \,|\, v)_V$ for all $v \in V$. Therefore $T_0 u \in \operatorname{rg} j^*$ if and only if $u \in D_j(\mathfrak{a})$. Now the second equality follows from $D_j(\mathfrak{a}) = D_j(\mathfrak{b})$. □

We will obtain Theorem 3.1 as a consequence of the following proposition.

Proposition 3.3. *Suppose* $\mathfrak{a}$ *and* j *satisfy Conditions* (I) *and* (II). *Assume that* $(\mathfrak{a}, j)$ *is associated with an accretive operator* A. *Let* $f \in H$. *Then* $f \in \operatorname{rg}(I + A)$ *if and only if there exists a* $u \in D_j(\mathfrak{a})$ *such that* $Tu = j^* f$. *In particular,* A *is m-accretive if and only if* $\operatorname{rg} j^* \subset T(D_j(\mathfrak{a}))$.

Proof. Let $f \in H$. Then $f \in \operatorname{rg}(I + A)$ if and only if there exists a $u \in D_j(\mathfrak{a})$ such that

$$(Tu \,|\, v)_V = \mathfrak{a}(u, v) + (j(u) \,|\, j(v))_H = ((I + A) j(u) \,|\, j(v))_H = (f \,|\, j(v))_H = (j^* f \,|\, v)_V$$

for all $v \in V$. Now the second statement follows from the above and the fact that the accretive operator A is m-accretive if and only if $\operatorname{rg}(I + A) = H$. □

Proof of Theorem 3.1. If A is m-accretive, then Proposition 3.3 implies that Condition (3.5) is satisfied. Conversely, suppose Condition (3.5) is satisfied. By Lemma 3.2, we obtain $T(D_j(\mathfrak{a})) = \operatorname{rg} j^*$. Therefore A is m-accretive by Proposition 3.3. This proves the theorem. □

Remark 3.4. 1. It follows from (3.4) that $(\mathfrak{a}, j)$ is associated with an accretive operator if and only if

$$D_j(\mathfrak{a}) \cap \ker j = \ker T.$$

Note that in general the latter equality does not hold with T_0 instead of T. Moreover, T may not be replaced by T_0 in Condition (3.5). Both can be observed in the example specified by $V = H$, $T_0 = 0$ and $j = I$, where H is a Hilbert space with $\dim H > 0$.

2. If V is finite-dimensional and $(\mathfrak{a}, j)$ is associated with an accretive operator A, then A is m-accretive. This follows from Theorem 3.1 since (3.3) implies $\operatorname{rg} j^* \subset \operatorname{rg} T$.

3. Suppose $(\mathfrak{a}, j)$ is associated with an accretive operator A. By Proposition 3.3 and Lemma 3.2 the operator A is m-accretive if and only if $T(D_j(\mathfrak{a})) = \operatorname{rg} j^*$.

4. If j is injective, then $(\mathfrak{a}, j)$ is associated with an accretive operator. If, in addition, T is bijective, then T^{-1} is bounded and the associated operator is m-accretive. Thus Theorem 2.2 of McIntosh [McI68, Theorem 3.1] is a special case of Theorem 3.1. We also point out that Condition (3.5) has already appeared in [McI66, Theorem 3.5] in the setting of injective j. Moreover, Theorem 2.3 is a special case of Theorem 3.1 by the following argument. Adopt the assumption of Theorem 2.3. We may shift $\mathfrak{a}$ such that $\omega = 0$. Then $\mathfrak{a}$ is accretive. By the ellipticity condition we have

$$\mu \|u\|_V^2 \leqslant \operatorname{Re} \mathfrak{a}(u, u) \leqslant |\mathfrak{b}(u, u)| \leqslant \|Tu\|_V \|u\|_V$$

for all $u \in V$. This implies that T is injective and has closed range. It follows from $\ker T^* = \ker T = \{0\}$ that $\operatorname{rg} T = V$. Hence T is invertible. Moreover, if $u \in V$ satisfies $\mathfrak{a}(u, u) = 0$, then $u = 0$. Hence $D_j(\mathfrak{a}) \cap \ker j = \{0\}$. Therefore $(\mathfrak{a}, j)$ is associated with an accretive operator A by Proposition 2.6. Moreover, A is m-accretive by Theorem 3.1. The same applies to the operator $e^{i\alpha} A$ for all $\alpha \in \mathbb{R}$ such that $|\alpha|$ is small. Hence A is m-sectorial.

The following finite-dimensional example shows that it is possible that $(\mathfrak{a}, j)$ is not associated with an accretive operator even though T is invertible.

Example 3.5. Let $V = \mathbb{C}^2$, $H = \mathbb{C}$ and $j(u_1, u_2) = u_2$. Define the form $\mathfrak{a} \colon V \times V \to \mathbb{C}$ by $\mathfrak{a}(u, v) = u_2 \overline{v_1} - u_1 \overline{v_2}$. Then clearly $T = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$, which is an invertible matrix. However, $(\mathfrak{a}, j)$ is not associated with an accretive operator. To prove this, let $u \in D_j(\mathfrak{a})$ and $f \in \mathbb{C}$ be such that

$$u_2 \overline{v_1} - u_1 \overline{v_2} = \mathfrak{a}(u, v) = (f \,|\, j(v))_{\mathbb{C}} = f\, \overline{v_2}$$

for all $v \in V$. Then $j(u) = u_2 = 0$ and $u_1 = -f$. This implies that $D_j(\mathfrak{a}) = \ker j = \mathbb{C} \times \{0\}$. Then the claim follows by Proposition 2.6. $\diamond$

Even if $(\mathfrak{a}, j)$ is associated with an accretive operator, the associated operator need not be m-accretive. To make matters worse, the restriction of $(\mathfrak{a}, j)$ to a closed subspace $W \subset V$ that contains $D_j(\mathfrak{a})$ and satisfies that $j(W)$ is dense in H need not be associated with an accretive operator, even if this is the case for $(\mathfrak{a}, j)$.

Example 3.6. Let $V = \ell_2$ and $H = \mathbb{C}$. By $(e_k)_{k \in \mathbb{N}}$ we denote the usual orthonormal basis in ℓ_2 such that e_k is 1 at the kth position and zero otherwise. Let $T_0 \in \mathcal{L}(V)$

be such that $T_0 e_k = \frac{1}{k} e_k$ for all $k \geqslant 3$, $T_0 e_1 = -e_2$ and $T_0 e_2 = e_1$. Observe that T_0 is m-accretive. Clearly $w := \big(0, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots\big) \in \ell_2$ is not in the range of T_0. Define $j \in \mathcal{L}(V, \mathbb{C})$ by $j(u) = (u \mid w)_{\ell_2}$. Set $T := T_0 + j^* j$ and note that $T \in \mathcal{L}(V)$ is m-accretive. As T_0 is injective and $j^*(\alpha) = \alpha w$ for all $\alpha \in \mathbb{C}$, the operator T is injective and $w \notin \operatorname{rg} T$. Hence T is not invertible. Define $\mathfrak{a} \colon V \times V \to \mathbb{C}$ by $\mathfrak{a}(u,v) = (T_0 u \mid v)_{\ell_2}$. Then T_0 and T are indeed the operators representing $\mathfrak{a}$ and $\mathfrak{b}$ in V.

Since $\operatorname{rg} j^* = (\ker j)^\perp = \operatorname{span}\{w\}$, it follows from Lemma 3.2 that $D_j(\mathfrak{a}) = \{0\}$. Therefore $(\mathfrak{a}, j)$ is associated with an accretive operator that is not m-accretive.

Let $W = \operatorname{span}\{e_1, e_2\} \subset V$ and define $\hat{\mathfrak{a}} := \mathfrak{a}|_{W \times W}$ and $\hat{j} := j|_W$. It is easily observed that we are now in the setting of Example 3.5. Therefore $(\hat{\mathfrak{a}}, \hat{j})$ is not associated with an accretive operator even though $D_j(\mathfrak{a}) \subset W$ and $(\mathfrak{a}, j)$ is associated with an accretive operator.

Furthermore, note that if we instead choose $W = \operatorname{span}\{e_2\}$, then $\mathfrak{a}|_{W \times W} = 0$ and $(\mathfrak{a}|_{W \times W}, j|_W)$ is associated with the m-accretive zero operator on $H = \mathbb{C}$. If one chooses $W = \operatorname{span}\{e_3\}$, then $(\mathfrak{a}|_{W \times W}, j|_W)$ is associated with an m-accretive operator that is different from the zero operator. In fact, a straightforward calculation shows that the associated operator in this case is $\frac{4}{3} I$. ◇

The previous example shows that taking seemingly suitable restrictions of $\mathfrak{a}$ and j does not need to give 'better' operators and can introduce surprising degrees of freedom. The next simple example illustrates that $(\mathfrak{a}, j)$ can be associated with a *nonclosed* accretive operator.

Example 3.7. Let V and H be Hilbert spaces and $j \in \mathcal{L}(V, H)$ with dense range. Choose the form $\mathfrak{a} = 0$ on $V \times V$. Then $(\mathfrak{a}, j)$ is associated with an accretive operator A. More precisely, $D(A) = \operatorname{rg} j$ and $A = 0$. Therefore A is m-accretive if and only if it is closed. The latter is equivalent to $\operatorname{rg} j = H$. Still, A is densely defined and closable. ◇

We now show that every densely defined, closed, accretive operator is, in the obvious way, associated with an accretive form. Note that the operator does not have to be m-accretive.

Example 3.8. Let R be a densely defined, closed, accretive operator in a Hilbert space H. Equip $V := D(R)$ with the inner product $(u \mid v)_V = (Ru \mid Rv)_H + (u \mid v)_H$. This makes V into a Hilbert space. Define the form $\mathfrak{a} \colon V \times V \to \mathbb{C}$ by $\mathfrak{a}(u,v) = (Ru \mid v)_H$. Then $\mathfrak{a}$ is accretive and continuous. Let $j \colon V \to H$ be the inclusion. Then j is continuous with dense range. Obviously $(\mathfrak{a}, j)$ is associated with an accretive operator. It is easy to verify that $D_j(\mathfrak{a}) = V$. So R is the associated operator. ◇

Next we give an example such that j is injective, whence $(\mathfrak{a}, j)$ is associated with an accretive operator, but such that $\operatorname{rg} T \cap \operatorname{rg} j^* = \{0\}$. In particular, the condition $\operatorname{rg} j^* \subset \operatorname{rg} T$ in Theorem 3.1 is clearly not fulfilled. The example is based on the following lemma.

Lemma 3.9. *Let* H *be an infinite-dimensional Hilbert space. Suppose* $R, S \in \mathcal{L}(H)$ *are self-adjoint, positive, injective operators such that* $\operatorname{rg} R \cap \operatorname{rg} S = \{0\}$*. Equip* $V = \operatorname{rg} R$ *with the inner product* $(u \mid v)_V = \big(R^{-1}u \mid R^{-1}v\big)_H$*. Let* $j\colon V \to H$ *be the inclusion. Then there exists an accretive form* $\mathfrak{a}\colon V \times V \to \mathbb{C}$ *such that* $\mathfrak{a}$ *and* j *satisfy Conditions* (I) *and* (II)*, and such that* $D_j(\mathfrak{a}) = \{0\}$*.*

Proof. Note that both $\operatorname{rg} R$ and $\operatorname{rg} S$ are dense, V is a Hilbert space and j is continuous with dense range. Moreover,

$$(j^*j(u) \mid v)_V = (j(u) \mid j(v))_H = \big(R^{-1}R^2u \mid R^{-1}v\big)_H = \big(R^2u \mid v\big)_V$$

for all $u, v \in V$. This shows that $j^*j = R^2|_V$ and $j^* = R^2$. Define the sesquilinear form $\mathfrak{a}\colon V \times V \to \mathbb{C}$ by

$$\mathfrak{a}(u,v) := \big(SR^{-1}u \mid R^{-1}v\big)_H = \big(RSR^{-1}u \mid v\big)_V.$$

So $T = R^2|_V + RSR^{-1}$. Let $u \in \operatorname{rg} T \cap \operatorname{rg} j^*$. Then there exist $v \in V$ and $f \in H$ such that $Tv = u = j^*f$. It follows that $RSR^{-1}v = R^2(f - v)$, whence $SR^{-1}v \in \operatorname{rg} R$. So $R^{-1}v = 0$ and hence $u = 0$. This proves that $D_j(\mathfrak{a}) = \{0\}$ by Lemma 3.2. $\square$

We point out that the form $\mathfrak{a}$ in Lemma 3.9 is symmetric and positive, but not elliptic. More precisely, by the above neither the conditions of Theorem 2.1 nor those of Theorem 2.3 are satisfied.

Example 3.10. Let $H = L^2(\mathbb{R})$. Define $R = \exp(-Q^4)$ and $S = \exp(-P^4)$, where Q is the multiplication operator with x in H (the so-called 'position operator') and P is the operator $i\frac{d}{dx}$ (the so-called 'momentum operator'). It is a consequence of Beurling's theorem (see [Hör91], for example) that $\operatorname{rg} R \cap \operatorname{rg} S = \{0\}$. So R and S are bounded linear operators that satisfy the conditions in Lemma 3.9. Alternatively, see the first step of the proof of [FW71, Theorem 3.6] for the construction of such operators on the Hilbert space ℓ_2.

By choosing V, $\mathfrak{a}$ and j as in Lemma 3.9, we obtain an example where j is injective and $(\mathfrak{a}, j)$ is associated with an accretive operator that has the domain $\{0\}$. $\diamond$

It is trivial to construct examples with $\ker T \neq \{0\}$ such that $(\mathfrak{a}, j)$ is associated with an m-accretive operator.

Example 3.11. Let $V = \mathbb{C}^2$, $H = \mathbb{C}$, $\mathfrak{a} = 0$ and $j(u) = u_1$. Then $T = j^*j$, and the equivalent conditions of Theorem 3.1 are satisfied. Moreover, $D_j(\mathfrak{a}) = \mathbb{C}^2$ and $\ker j = \{0\} \times \mathbb{C} = \ker T$. $\diamond$

A convenient sufficient condition for $(\mathfrak{a}, j)$ to be associated with an accretive operator is as follows.

Lemma 3.12. *Suppose* $\mathfrak{a}$ *and* j *satisfy Conditions* (I) *and* (II). *Suppose for all* $u \in V$ *with* $\mathfrak{b}(u,u) = 0$ *one has* $u = 0$. *Then* $D_j(\mathfrak{a}) \cap \ker j = \{0\}$ *and* $(\mathfrak{a}, j)$ *is associated with an accretive operator.*

Proof. Let $u \in D_j(\mathfrak{a}) \cap \ker j$. Let $f \in H$ be such that $\mathfrak{a}(u,v) = (f \,|\, j(v))_H$ for all $v \in H$. Then $\mathfrak{b}(u,u) = \mathfrak{a}(u,u) + \|j(u)\|_H^2 = (f \,|\, j(u))_H = 0$, whence $u = 0$. □

Another sufficient condition is as follows.

Lemma 3.13. *If* $j(D_j(\mathfrak{a}))$ *is dense in* H, *then* $(\mathfrak{a}, j)$ *is associated with an accretive operator.*

Proof. This follows from the fact that densely defined, accretive graphs are single-valued, see [HP97, Remark 3.1.42].

We provide a direct proof here to be self-contained. Let $u \in D_j(\mathfrak{a}) \cap \ker j$ and let $f \in H$ be such that $\mathfrak{a}(u,v) = (f \,|\, j(v))_H$ for all $v \in V$. Let $w \in D_j(\mathfrak{a})$ and $\lambda \in \mathbb{C}$. Then there exists a $g \in H$ such that

$$\mathfrak{a}(w,u) = (g \,|\, j(u))_H = 0.$$

Hence we obtain

$$0 \leqslant \operatorname{Re} \mathfrak{a}(w - \lambda u, w - \lambda u) = \operatorname{Re} \mathfrak{a}(w,w) - \operatorname{Re} (\lambda f \,|\, j(w))_H.$$

This shows that $(f \,|\, j(w))_H = 0$ for all $w \in D_j(\mathfrak{a})$. Therefore $f = 0$. Hence $T_0 u = j^* f = 0$. Now the statement follows from Proposition 2.6. □

We next give an example where $j(D_j(\mathfrak{a}))$ is dense in H, but such that the associated operator is not m-accretive.

Example 3.14. Let $H = L^2(0,\infty)$ and $V = H_0^1(0,\infty)$. Let j be the (injective) embedding of V into H. Define $\mathfrak{a} \colon V \times V \to \mathbb{C}$ by

$$\mathfrak{a}(u,v) = -\int_0^\infty u' \overline{v}.$$

Using the continuous representative of $u \in H_0^1(0,\infty)$, we obtain

$$2 \operatorname{Re} \mathfrak{a}(u,u) = -\int_0^\infty (u'\overline{u} + \overline{u'} u) = -\big[|u|^2\big]_0^\infty = |u(0)|^2 = 0$$

for all $u \in V$. Hence $\mathfrak{a}$ and j satisfy Conditions (I) and (II). It is easily observed that the operator A associated with $(\mathfrak{a}, j)$ is given by $Au = -u'$ and $D(A) = H_0^1(0,\infty)$.

Note that the operator B in H defined by $Bu = -u'$ and $D(B) = H^1(0,\infty)$ is accretive and strictly extends A. So $D(A)$ is dense, but A fails to be m-accretive.

We remark that the operator $-A$ is accretive and satisfies $(-A)^* = B$. Clearly $-A$ is closed and densely defined. Hence $-A$ is m-accretive by Proposition A.11. Note that the operator $-A$ is associated with $(-\mathfrak{a}, j)$, while $B = (-A)^*$ is associated with $(\tilde{\mathfrak{a}}, \tilde{j})$, where the form $\tilde{\mathfrak{a}}\colon H^1(0,\infty) \times H^1(0,\infty) \to \mathbb{C}$ is defined by $\tilde{\mathfrak{a}}(u,v) = -\int_0^\infty u'\overline{v}$ and $\tilde{j}$ is the embedding of $H^1(0,\infty)$ into H. $\diamond$

The next proposition is a direct consequence of Lemma 2.9.

Proposition 3.15. *Assume* $\mathfrak{a}$ *and* j *satisfy Conditions* (I) *and* (II). *Assume that* $(\mathfrak{a}, j)$ *is associated with an m-accretive operator* A. *Suppose that* $j\colon V \to H$ *is compact. Then* A *has compact resolvent.*

In the following example an m-accretive operator is associated with an accretive form corresponding to a second-order differential expression. Later in Section 3.5 after Proposition 3.55 we will briefly revisit this example.

Example 3.16. Let $a, b \in \mathbb{R}$ with $a < 0 < b$. Let $H = L^2(a,b)$, and let $V = H_0^1(a,b)$ with norm $\|u\|_V^2 = \int_a^b |u'|^2$. Let j be the embedding of V in H. Define $\mathfrak{a}\colon V \times V \to \mathbb{C}$ by

$$\mathfrak{a}(u,v) = i \int_a^b (\operatorname{sgn} x)\, u'(x)\overline{v'(x)}\, dx.$$

Then $\mathfrak{a}$ and j satisfy Conditions (I) and (II). Note that $\operatorname{Re}\mathfrak{a}(u,u) = 0$ for all $u \in V$. It is readily verified that the associated operator A is given by $Au = -i(\operatorname{sgn} \cdot u')'$ on the domain $D(A) = \{u \in H_0^1(a,b) : \operatorname{sgn} \cdot u' \in H^1(a,b)\}$. Since iA is a self-adjoint operator by [Naĭ68, Theorem 5 in §18.2], the operator A is m-accretive and Condition (3.5) is satisfied. It follows from Proposition 3.15 that A has compact resolvent.

In particular, we may choose $a = -1$ and $b = 1$. Then a straightforward calculation yields $(T_0 u)(s) = i(\operatorname{sgn} s)\big(u(s) + (|s|-1)u(0)\big)$ for all $s \in (-1,1)$ and $j^*j = (-\Delta^D_{(-1,1)})^{-1}|_{H_0^1(-1,1)}$, where $\Delta^D_{(-1,1)}$ denotes the Dirichlet Laplacian on $(-1,1)$. Note that T_0 is not injective since $s \mapsto 1 - |s|$ is an element of the kernel of T_0. In this example a direct verification of Condition (3.5) appears to be difficult. $\diamond$

As in Section 2.3 we define the subspace

$$V_j(\mathfrak{a}) := \{u \in V : \mathfrak{a}(u,v) = 0 \text{ for all } v \in \ker j\}.$$

It is immediate from the definitions that $D_j(\mathfrak{a}) \subset V_j(\mathfrak{a})$ and that $V_j(\mathfrak{a})$ is closed. Moreover, it is easily observed that

$$V_j(\mathfrak{a}) = V_j(\mathfrak{b}) = T^{-1}\big[(\ker j)^\perp\big] = (T^* \ker j)^\perp,$$

where $\perp$ denotes the orthogonal complement in V. Hence, if T is invertible, then it follows from Lemma 3.2 that $V_j(\mathfrak{a})$ is the closure of $D_j(\mathfrak{a})$.

The space $V_j(\mathfrak{a})$ plays an important role in the theory of j-elliptic forms as in Theorem 2.3. If $\mathfrak{a}$ is j-elliptic, then $D_j(\mathfrak{a})$ is dense in $V_j(\mathfrak{a})$ by [AE12, Proposition 2.3 (ii)], one has the (possibly nonorthogonal) decomposition $V = V_j(\mathfrak{a}) \oplus \ker j$ by [AE12, Theorem 2.5 (i)], and the associated operator is determined by the restriction $(\mathfrak{a}|_{V_j(\mathfrak{a})\times V_j(\mathfrak{a})}, j|_{V_j(\mathfrak{a})})$. If $\mathfrak{a}$ is merely accretive, then in general $D_j(\mathfrak{a})$ is not dense in $V_j(\mathfrak{a})$ even if j is injective and $(\mathfrak{a}, j)$ is associated with an m-accretive operator. An example for this is as follows.

Example 3.17. Let $V = H = \ell_2$. Let $S \in \mathcal{L}(V)$ be the right shift, so $Se_n = e_{n+1}$ for all $n \in \mathbb{N}$. Define $T' \in \mathcal{L}(V)$ by $T'e_n = 2^{-n}e_n$ and $j \in \mathcal{L}(V, H)$ by $j = (I - 2S^*)T' = T'(I - S^*)$. Then $j^* = T'(I - 2S)$ and hence $\operatorname{rg} j^* \subset \operatorname{rg} T'$. Since $T'(I - S^*)$ is the composition of two injective maps, it follows that j is injective and, in particular, $V_j(\mathfrak{a}) = V$. If $u \in V$, then

$$\|j(u)\|_H^2 = \|(I - 2S^*)T'u\|_H^2 \leqslant 9\|T'u\|_V^2 \leqslant \tfrac{9}{2}\big(T'u \,|\, u\big)_V.$$

Hence if one defines $\mathfrak{a}\colon V \times V \to \mathbb{C}$ by

$$\mathfrak{a}(u, v) = \tfrac{9}{2}\big(T'u \,|\, v\big)_V - (j(u) \,|\, j(v))_H,$$

then $\mathfrak{a}$ is continuous and accretive. As $(I - 2S)$ is injective, also $j^* = T'(I - 2S)$ is injective. Therefore j has dense range. Note that $\mathfrak{b}(u, v) = (Tu \,|\, v)_V$, where $T = \frac{9}{2}T'$. So $(\mathfrak{a}, j)$ is associated with an m-accretive operator by Theorem 3.1.

For all $n \in \mathbb{N}$ define $w_n := \sum_{k=1}^{n} e_k \in V$. Then $j(w_n) = T'e_n = 2^{-n}e_n$ for all $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} j(w_n) = 0$ in H. Moreover, define $w = \lim_{n\to\infty} T'w_n = \sum_{n=1}^{\infty} 2^{-n}e_n \in V$.

We show that $w \in (D_j(\mathfrak{a}))^\perp$. Let $u \in D_j(\mathfrak{a})$. By Lemma 3.2 there exists an $f \in H$ such that $T'u = j^*f$. Then

$$\big(u \,|\, T'w_n\big)_V = \big(T'u \,|\, w_n\big)_V = (j^*f \,|\, w_n)_V = (f \,|\, j(w_n))_H$$

for all $n \in \mathbb{N}$. For $n \to \infty$ we obtain $(u \,|\, w)_V = 0$. Since $w \neq 0$, it follows that $D_j(\mathfrak{a})$ is not dense in $V_j(\mathfrak{a})$.

We also point out that in this example $D_j(\mathfrak{a})$ is closed in V. To this end, observe that $D_j(\mathfrak{a}) = T^{-1}[\operatorname{rg} j^*] = \operatorname{rg}(\frac{1}{2}I - S)$. Therefore it suffices to show that $\frac{1}{2}I - S$ is Fredholm. To this end it is useful to consider the **Calkin algebra**, which is the quotient space of the bounded operators modulo the compact operators and becomes a C^*-algebra in the natural way; see [Dou72, Chapter 5]. Let π be the corresponding natural projection map. By [Dou72, Definition 5.14 and Theorem 5.17] the operator $\frac{1}{2}I - S$ is Fredholm if and only if $\pi(\frac{1}{2}I - S)$ is invertible in the Calkin algebra. As $SS^* - I$ is compact and $S^*S = I$, the operator $\pi(S)$ is unitary in the Calkin algebra. Hence $\pi(\frac{1}{2}I - S)$ is invertible in the Calkin algebra. ◇

The following example shows that $(\mathfrak{a}, j)$ can be associated with a *nonclosable* accretive operator. It is obtained by adapting Phillips' example for a nonclosed, maximal accretive (single-valued) operator in [Phi59, Footnote 6]. Note that the operator here is not maximal accretive. We will see in Proposition 3.37 that we cannot obtain a nonclosed, maximal accretive operator in our setting.

Example 3.18. Let $H = \ell_2(\mathbb{N})$, and let $(e_n)_{n\in\mathbb{N}}$ be the standard orthonormal basis. Equip the subset V of $\ell_2(\mathbb{N}_0)$ specified by

$$V \coloneqq \Big\{ u = (u_n)_{n\in\mathbb{N}_0} \in \ell_2(\mathbb{N}_0) : \sum_{n=0}^{\infty} |2^n u_n|^2 < \infty \Big\}$$

with the inner product

$$(u \,|\, v)_V \coloneqq \sum_{n=0}^{\infty} 4^n u_k \overline{v_k}$$

for all $u, v \in V$. Then V is a Hilbert space.

Define $y \coloneqq \sum_{n=2}^{\infty} 2^{-n} e_n \in H$. Let $j \in \mathcal{L}(V, H)$ be defined by

$$j(u) = u_0 e_1 + u_1 y + \sum_{n=2}^{\infty} u_n e_n.$$

Note that j has dense range in H. We show that j is injective. Let $u \in \ker j$. Then $u_0 = 0$. Moreover, $u_n = -u_1 2^{-n}$ for all $n \geqslant 2$. This implies that $u = 0$ because $(2^n u_n)_{n\in\mathbb{N}_0} \in \ell_2(\mathbb{N}_0)$ as $u \in V$.

Define $\mathfrak{a} \colon V \times V \to \mathbb{C}$ by

$$\mathfrak{a}(u, v) = u_1 \overline{v_0} - u_0 \overline{v_1}.$$

Then $\mathfrak{a}$ is accretive and continuous. So $\mathfrak{a}$ and j satisfy Conditions (I) and (II), and $(\mathfrak{a}, j)$ is associated with an accretive operator A since j is injective.

We first show that $D_j(\mathfrak{a}) = \{u \in V : u_0 = 0\}$.

'$\subset$' Let $u \in D_j(\mathfrak{a})$, and let $f \in H$ be such that

$$\mathfrak{a}(u, v) = (f \,|\, j(v))_H$$

for all $v \in V$. Hence

$$u_1 \overline{v_0} - u_0 \overline{v_1} = (f \,|\, j(v))_H = f_1 \overline{v_0} + (f \,|\, y)_H \overline{v_1} + \sum_{n=2}^{\infty} f_n \overline{v_n}$$

for all $v \in V$. This implies that $f_n = 0$ for all $n \geqslant 2$. Hence $(f \,|\, y)_H = 0$ and $u_0 = 0$.

'$\supset$' Let $u \in V$ be such that $u_0 = 0$. Set $f \coloneqq u_1 e_1$. Then

$$\mathfrak{a}(u, v) = u_1 \overline{v_0} = f_1 \overline{v_0} = (f \mid j(v))_H$$

for all $v \in V$, i.e., $u \in D_j(\mathfrak{a})$.

Now we show that the operator A associated with $(\mathfrak{a}, j)$ is not closable. Observe from the preceding calculations that

$$D(A) = \Big\{ u_1 y + \sum_{n=2}^{\infty} u_n e_n : u \in V \Big\}$$

and $Ay = e_1$, $Ae_n = 0$ for all $n \geqslant 2$. Note that $y_m \coloneqq \sum_{n=m}^{\infty} 2^{-n} e_n$ is in $D(A)$ for all $m \geqslant 2$. Then $\lim_{m\to\infty} y_m = 0$ in H, but $Ay_m = e_1$ for all $m \geqslant 2$. Hence A is not closable. Consequently, A cannot be densely defined by Lemma A.9.

For later use we note that

$$\operatorname{rg}(I + A) = \Big\{ u_1 y + \sum_{n=1}^{\infty} u_n e_n : u \in V \Big\}. \tag{3.6}$$

Finally, we point out that $D_j(\mathfrak{a})$ is closed in V, but $D_j(\mathfrak{a}) \neq V_j(\mathfrak{a}) = V$. $\diamond$

The next proposition explains why in the following we may restrict our attention to the case $\ker T = \{0\}$.

Proposition 3.19. *Assume* $\mathfrak{a}$ *and* j *satisfy Conditions* (I) *and* (II). *Let* W *be a closed subspace of* V *such that* $V = W \oplus \ker T$, *where the direct sum does not need to be orthogonal. Define* $\hat{\mathfrak{a}} \coloneqq \mathfrak{a}|_{W\times W}$ *and* $\hat{j} \coloneqq j|_W$. *Then* $\hat{\mathfrak{a}}$ *and* $\hat{j}$ *satisfy* (I) *and* (II). *Moreover, the following statements hold.*

(a) *Let* $\widehat{T}$ *be defined as in* (3.1) *with respect to* $\hat{\mathfrak{a}}$ *and* $\hat{j}$. *Then* $\ker \widehat{T} = \{0\}$.

(b) *The following three identities hold:*

$$\ker j = \ker \hat{j} \oplus \ker T,$$
$$D_j(\mathfrak{a}) = D_{\hat{j}}(\hat{\mathfrak{a}}) \oplus \ker T,$$
$$V_j(\mathfrak{a}) = V_{\hat{j}}(\hat{\mathfrak{a}}) \oplus \ker T.$$

(c) *One has* $D_j(\mathfrak{a}) \cap \ker j \subset \ker T$ *if and only if* $D_{\hat{j}}(\hat{\mathfrak{a}}) \cap \ker \hat{j} = \{0\}$, *and if this is the case, then* $A = \widehat{A}$, *where* A *and* $\widehat{A}$ *are the operators associated with* $(\mathfrak{a}, j)$ *and* $(\hat{\mathfrak{a}}, \hat{j})$, *respectively.*

(d) *Assume in addition that* $(\mathfrak{a}, j)$ *is associated with an m-accretive operator. Then there exists a unique operator* $Z\colon H \to W$ *such that* $TZ = j^*$. *Moreover,* Z *is bounded and* $(I + A)^{-1} = jZ$.

Proof. Obviously $\hat{\mathfrak{a}}$ is continuous and accretive. Since $\ker T \subset \ker j$ by (3.2), the map $\hat{j}$ has dense range. So $\hat{\mathfrak{a}}$ and $\hat{j}$ satisfy Conditions (I) and (II). Next we state some basic identities. If $u \in \ker T$, then $\mathfrak{b}(u,v) = (Tu \,|\, v)_V = 0$ for all $v \in V$. If $v \in \ker T$, then $v \in \ker T^*$ by Proposition A.8 (b) and hence $\mathfrak{b}(u,v) = 0$ for all $u \in V$. Then also $\mathfrak{a}(u,v) = 0$ for all $u,v \in V$ with $u \in \ker T$ or $v \in \ker T$.

(a) Let $u \in \ker \widehat{T}$. Then $\mathfrak{b}(u,v) = \big(\widehat{T}u \,|\, v\big)_V = 0$ for all $v \in W$. As also $\mathfrak{b}(u,v) = 0$ for all $v \in \ker T$, we obtain $\mathfrak{b}(u,v) = 0$ for all $v \in V$. Hence $Tu = 0$, i.e., $u \in \ker T$. Since $u \in W$, we deduce that $u = 0$. So $\ker \widehat{T} = \{0\}$.

(b) Since $\ker T \subset D_j(\mathfrak{a}) \subset V_j(\mathfrak{a})$ and $\ker T \subset \ker j$ by (3.4), it suffices to show the three identities

$$
\begin{aligned}
\ker j \cap W &= \ker \hat{j}, \\
D_j(\mathfrak{a}) \cap W &= D_{\hat{j}}(\hat{\mathfrak{a}}), \\
V_j(\mathfrak{a}) \cap W &= V_{\hat{j}}(\hat{\mathfrak{a}}).
\end{aligned}
$$

The first identity is clear. For the proof of the second identity let $u \in D_j(\mathfrak{a}) \cap W$. Then there exists an $f \in H$ such that $\hat{\mathfrak{a}}(u,v) = \mathfrak{a}(u,v) = (f \,|\, j(v))$ for all $v \in W$, so $u \in D_{\hat{j}}(\hat{\mathfrak{a}})$. Conversely, let $u \in D_{\hat{j}}(\hat{\mathfrak{a}})$. Then there exists an $f \in H$ such that $\mathfrak{a}(u,v) = (f \,|\, j(v))_H$ for all $v \in W$. But $\mathfrak{a}(u,v) = 0 = (f \,|\, j(v))_H$ for all $v \in \ker T$, by (3.2). Therefore $\mathfrak{a}(u,v) = (f \,|\, j(v))_H$ for all $v \in W + \ker T = V$. So $u \in D_j(\mathfrak{a})$ and hence $D_{\hat{j}}(\hat{\mathfrak{a}}) \subset D_j(\mathfrak{a}) \cap W$. The third identity is proved similarly.

(c) It follows from (b) that

$$
D_j(\mathfrak{a}) \cap \ker j = \big(D_{\hat{j}}(\hat{\mathfrak{a}}) \cap \ker \hat{j}\big) + \ker T.
$$

As $\ker \hat{j} \cap \ker T = \{0\}$, this shows that $D_j(\mathfrak{a}) \cap \ker j \subset \ker T$ if and only if $D_{\hat{j}}(\hat{\mathfrak{a}}) \cap \ker \hat{j} = \{0\}$.

For the proof of the second claim, let $u \in W$, $w \in \ker T$ and $f \in H$. Then by (3.2) one has $\hat{j}(u) = j(u + w)$, $\mathfrak{a}(u,v) = 0$ for all $v \in \ker T$ and $\mathfrak{a}(w,v) = 0$ for all $v \in V$. Therefore $\mathfrak{a}(u + w, v) = (f \,|\, j(v))_H$ for all $v \in V$ if and only if $\hat{\mathfrak{a}}(u,v) = (f \,|\, \hat{j}(v))_H$ for all $v \in W$. It follows that $A = \widehat{A}$.

(d) By Theorem 3.1 we have $\operatorname{rg} j^* \subset \operatorname{rg} T = \operatorname{rg} T|_W$. Note that $T|_W$ is injective. So the operator $Z \colon H \to W$ is given by $Z = (T|_W)^{-1} j^*$. It is closed as a composition of a bounded and a closed operator. Consequently, Z is bounded. To prove the final assertion, let $f \in H$. Then $TZf = j^* f$ and hence $(I + A)^{-1} f = jZf$. □

In Example 3.6 we saw that various degenerate behaviour can occur if we restrict $\mathfrak{a}$ and j to a closed subspace $W \subset V$ such that $D_j(\mathfrak{a}) \subset W$ and $j(W)$ is dense in H. The corollary to the following proposition shows that this does not happen if $(\mathfrak{a}, j)$ is associated with an m-accretive operator.

Proposition 3.20. *Assume* $\mathfrak{a}$ *and* j *satisfy Conditions* (I) *and* (II). *Assume that* $j\big(D_j(\mathfrak{a})\big)$ *is dense in* H. *Let* $W \subset V$ *be a closed subspace such that* $j\big(D_j(\mathfrak{a}) \cap W\big) = j\big(D_j(\mathfrak{a})\big)$. *Define* $\hat{\mathfrak{a}} := \mathfrak{a}|_{W\times W}$ *and* $\hat{j} := j|_W$. *Then* $(\mathfrak{a}, j)$ *and* $(\hat{\mathfrak{a}}, \hat{j})$ *are associated with accretive operators* A *and* $\widehat{A}$, *respectively. Moreover,* $\widehat{A}$ *is an extension of* A.

Proof. First note that $(\mathfrak{a}, j)$ is associated with an accretive operator A by Lemma 3.13. Also $\hat{j}$ satisfies Condition (II) since by assumption $j(D_j(\mathfrak{a})) \subset j(W) = \operatorname{rg} \hat{j}$.

It is immediately clear that $D_j(\mathfrak{a}) \cap W \subset D_{\hat{j}}(\hat{\mathfrak{a}})$. Therefore also $(\hat{\mathfrak{a}}, \hat{j})$ is associated with an accretive operator $\widehat{A}$ by Lemma 3.13. Let $x \in D(A)$ and $f = Ax$. Then there exists a $u \in D_j(\mathfrak{a}) \cap W$ such that $j(u) = x$. It follows that

$$\mathfrak{a}(u, v) = (f \,|\, j(v))_H$$

for all $v \in V$. This shows that $\widehat{A}$ is an extension of A. $\square$

Corollary 3.21. *Assume* $\mathfrak{a}$ *and* j *satisfy Conditions* (I) *and* (II). *Suppose* $(\mathfrak{a}, j)$ *is associated with an m-accretive operator* A. *Suppose* W *satisfies the assumptions in Proposition 3.20. Let* $\hat{\mathfrak{a}}$ *and* $\hat{j}$ *be defined as in Proposition 3.20. Then the operator* $\widehat{A}$ *associated with* $(\hat{\mathfrak{a}}, \hat{j})$ *is equal to* A.

Remark 3.22. 1. If W is a closed subspace of V such that $D_j(\mathfrak{a}) \subset W$, then the condition $j\big(W \cap D_j(\mathfrak{a})\big) = j\big(D_j(\mathfrak{a})\big)$ is clearly satisfied.

2. We note that in Proposition 3.20 one may choose $W := \overline{D_j(\mathfrak{a}) \cap (\ker T)^\perp}$, provided $j(D_j(\mathfrak{a}))$ is dense in H. This follows from Proposition 3.19 (b) using the decomposition $V = (\ker T)^\perp \oplus \ker T$.

3. Even if $(\mathfrak{a}, j)$ is associated with an m-accretive operator, in general it is not possible to choose $W := V_j(\mathfrak{a}) \cap (\ker j)^\perp$ in Proposition 3.20. For example, let $V = \mathbb{C}^2$, $H = \mathbb{C}$, define $j \in \mathcal{L}(V, H)$ by $j(u_1, u_2) = u_1$ and define $\mathfrak{a} \colon V \times V \to \mathbb{C}$ by $\mathfrak{a}(u, v) = u_2\overline{v_2} + i(u_1\overline{v_2} + u_2\overline{v_1})$. Then $\mathfrak{a}$ is j-elliptic (cf. Theorem 2.3) and $(\mathfrak{a}, j)$ is associated with an m-accretive operator. Moreover, $\ker j = \{0\} \times \mathbb{C}$ and $V_j(\mathfrak{a}) = \{(\alpha, -\alpha i) : \alpha \in \mathbb{C}\}$. Hence $V_j(\mathfrak{a}) \cap (\ker j)^\perp = \{0\}$.

The following example shows that the operator $\widehat{A}$ in Proposition 3.20 can indeed be a proper extension of A, even if j is injective and W is the closure of $D_j(\mathfrak{a})$ in V. In the construction of the example we rely on the following lemma.

Lemma 3.23. *Let* $A \geqslant I$ *be an unbounded self-adjoint operator in a Hilbert space* H. *Suppose* $f \in H \setminus D(A^{1/2})$. *Equip* $V := D(A^{1/2})$ *with the inner product* $(x \,|\, y)_V = (A^{1/2}x \,|\, A^{1/2}y)_H$. *Then the set*

$$\{x \in D(A) : (Ax \,|\, f)_H = 0\}$$

is dense in V.

Proof. Let $W = D(A)$ be equipped with the induced topology of V. Note that W is dense in V. It suffices to verify that $\varphi\colon W \to \mathbb{C}$ defined by $\varphi(x) = (Ax \,|\, f)_H$ is an unbounded linear functional. Then, by a well-known elementary fact of functional analysis, the kernel of φ is dense in W. Hence $\ker \varphi$ is dense in V, which proves the claim.

We complete the proof by showing that φ is unbounded. Assume φ is bounded. Then by the Riesz representation theorem there exists a $v \in V$ such that

$$(Ax \,|\, f)_H = \varphi(x) = (x \,|\, v)_V = \big(A^{1/2}x \,|\, A^{1/2}v\big)_H$$

for all $x \in D(A)$. This implies that $f \in D(A^{1/2})$, which is a contradiction. □

Example 3.24. Define an unbounded self-adjoint operator $\widehat{A} \geqslant I$ in a separable Hilbert space H by taking the countable disjoint sum of an operator as in Lemma 3.23. Equip $V_1 := D(\widehat{A}^{1/2})$ with the inner product $(u \,|\, v)_{V_1} = \big(\widehat{A}^{1/2}u \,|\, \widehat{A}^{1/2}v\big)$. Then there exists a closed infinite-dimensional subspace H_2 of H with $V_1 \cap H_2 = \{0\}$ such that the set

$$D := \{x \in D(\widehat{A}) : \big(\widehat{A}x \,|\, f\big)_H = 0 \text{ for all } f \in H_2\}$$

is dense in V_1.

Let V_2 and $\mathfrak{a}_2$ be given as in Example 3.10 for the Hilbert space $L^2(\mathbb{R})$. Since H_2 and $L^2(\mathbb{R})$ are isometrically isomorphic, we may identify H_2 and $L^2(\mathbb{R})$ and assume that $j_2\colon V_2 \to H_2$ is the inclusion. Then $D_{j_2}(\mathfrak{a}_2) = \{0\}$. Let $V = V_1 \oplus V_2$. Define $j\colon V \to H$ by $j(u_1, u_2) = u_1 + u_2$. Define $\mathfrak{a}\colon V \times V \to C$ by

$$\mathfrak{a}((u_1, u_2), (v_1, v_2)) = \big(\widehat{A}^{1/2}u_1 \,|\, \widehat{A}^{1/2}v_1\big)_H + \mathfrak{a}_2(u_2, v_2).$$

Then $\mathfrak{a}$ and j satisfy Conditions (I) and (II). Moreover, j is injective. Therefore $(\mathfrak{a}, j)$ is associated with an accretive operator A.

We determine $D_j(\mathfrak{a})$. Suppose $u \in D_j(\mathfrak{a})$ and let $f \in H$ be such that $\mathfrak{a}(u, v) = (f \,|\, j(v))_H$ for all $v \in V$. On the one hand, choosing $v_2 = 0$ yields $u_1 \in D(\widehat{A})$ and $\widehat{A}u_1 = f$. On the other hand, by choosing $v_1 = 0$, we obtain

$$\mathfrak{a}_2(u_2, v_2) = (f \,|\, v_2)_H = \big(\widehat{A}u_1 \,|\, Pv_2\big)_H = \big(P\widehat{A}u_1 \,|\, v_2\big)_{H_2}$$

for all $v_2 \in V_2$, where P is the orthogonal projection onto H_2 in H. Hence $u_2 \in D_{j_2}(\mathfrak{a}_2) = \{0\}$ and $P\widehat{A}u_1 = 0$. This shows that $u \in D \times \{0\}$. Conversely, assume that $u \in D \times \{0\}$. Then

$$\begin{aligned}\mathfrak{a}((u_1, 0), (v_1, v_2)) &= \big(\widehat{A}^{1/2}u_1 \,|\, \widehat{A}^{1/2}v_1\big)_H \\ &= \big(\widehat{A}u_1 \,|\, v_1\big)_H + \big(\widehat{A}u_1 \,|\, v_2\big)_H = \big(\widehat{A}u_1 \,|\, v_1 + v_2\big)_H\end{aligned}$$

for all $v = (v_1, v_2) \in V$. Thus $D_j(\mathfrak{a}) = D \times \{0\}$.

Let A be the operator associated with $(\mathfrak{a}, j)$. By construction, $D_j(\mathfrak{a})$ is dense in $W := V_1 \times \{0\}$. This implies that $D(A)$ is dense. Let $\hat{\mathfrak{a}} = \mathfrak{a}|_{W\times W}$ and $\hat{j} = j|_W$. Then $\widehat{A}$ is associated with $(\hat{\mathfrak{a}}, \hat{j})$. The operator $\widehat{A}$ is an extension of A by Proposition 3.20. Note, however, that $D(A)$ is a proper subset of $D(\widehat{A})$ since $\operatorname{rg} \widehat{A}$ is dense in H, whereas $\operatorname{rg} A \subset H_2^{\perp}$ fails to be dense in H. $\diamond$

We close this section with another example. It shows that in the setting of Proposition 3.20 one cannot expect to have any monotonicity of the domain of $\widehat{A}$ with respect to the choice of W.

Example 3.25. Let $\widehat{A} \geqslant I$ be an unbounded self-adjoint operator in a Hilbert space H. Equip $W := D(\widehat{A})$ with the inner product $(u \,|\, v)_W = \big(\widehat{A}u \,|\, \widehat{A}v\big)_H$. Let $w \in D(\widehat{A}) \setminus D(\widehat{A}^2)$ be such that $\|w\|_W = 1$. Then

$$W_1 := \{u \in D(\widehat{A}) : \big(\widehat{A}u \,|\, \widehat{A}w\big)_H = 0\}$$

is dense in H, which follows similarly as in the proof of Lemma 3.23. Moreover, we have the orthogonal decomposition $W = W_1 \oplus \operatorname{span}\{w\}$.

Set $V := \mathbb{C} \oplus W$ and define $j\colon V \to H$ by $j(\alpha, u) = u$. Define the form $\mathfrak{a}\colon V \times V \to \mathbb{C}$ by

$$\mathfrak{a}((\alpha,u),(\beta,v)) = \big(\widehat{A}u \,|\, v\big)_H + \alpha(w \,|\, v)_W - \overline{\beta}(u \,|\, w)_W.$$

Then $\mathfrak{a}$ and j satisfy Conditions (I) and (II).

We determine $D_j(\mathfrak{a})$. Let $(\alpha, u) \in D_j(\mathfrak{a})$ and $f \in H$ be such that $\mathfrak{a}((\alpha,u),(\beta,v)) = (f \,|\, j(v))_H$ for all $(\beta, v) \in V$. Choosing $\beta = 0$ and $v \in W_1$ shows $f = \widehat{A}u$. Moreover, if $\beta = 0$ and $v = w$, then

$$\big(\widehat{A}u \,|\, w\big)_H + \alpha\|w\|_W^2 = \mathfrak{a}((\alpha,u),(0,w)) = (f \,|\, w)_H = \big(\widehat{A}u \,|\, w\big)_H.$$

Therefore $\alpha = 0$. Furthermore, choosing $\beta = 1$ and $v = 0$ shows $(u \,|\, w)_W = 0$, i.e., $u \in W_1$. Conversely, if $u \in W_1$, then

$$\mathfrak{a}((0,u),(\beta,v)) = \big(\widehat{A}u \,|\, v\big)_H - \overline{\beta}(u \,|\, w)_W = \big(\widehat{A}u \,|\, j(\beta,v)\big)_H$$

for all $(\beta, v) \in V$. Therefore $D_j(\mathfrak{a}) = \{0\} \times W_1$. Note that $j(D_j(\mathfrak{a})) = W_1$ is dense in H. Therefore $(\mathfrak{a}, j)$ is associated with an accretive operator A. Moreover, A is the (proper) restriction of $\widehat{A}$ to W_1.

For simplicity, we now consider both W_1 and W directly as closed subspaces of V. Note that in this example $D_j(\mathfrak{a}) = W_1 \subset W$. So Proposition 3.20 applies to the restrictions of $(\mathfrak{a}, j)$ to both W_1 or W. Let $\mathfrak{a}_1 := \mathfrak{a}|_{W_1\times W_1}$, $\hat{\mathfrak{a}} := \mathfrak{a}|_{W\times W}$, $j_1 := j|_{W_1}$ and $\hat{j} := j|_W$. Then A is associated with $(\mathfrak{a}_1, j_1)$, the self-adjoint operator $\widehat{A}$ is associated with $(\hat{\mathfrak{a}}, \hat{j})$ and A is again associated with $(\mathfrak{a}, j)$, despite $W_1 \subset W \subset V$ and $D_j(\mathfrak{a}) = W_1$. In the j-elliptic case this cannot happen, see also Corollary 3.21.

Finally, note that $\hat{\jmath}$ is injective and $W = D_{\hat{\jmath}}(\hat{\mathfrak{a}})$ is the domain of $\hat{\mathfrak{a}}$. So Proposition 3.20 cannot be applied to any proper restriction of $(\hat{\mathfrak{a}}, \hat{\jmath})$. ◇

3.2 The class of accretive operators associated with an accretive form

Example 3.8 shows that all closed, accretive, densely defined operators on a Hilbert space H are associated with some accretive form. It is natural to ask if the same holds without the assumptions that the operator be densely defined or closed. Moreover, we are also interested in whether it can be arranged that the form domain is continuously embedded into H.

Definition 3.26. Let A be an operator in a Hilbert space H. We say that A **can be generated by an accretive form** if there exists a Hilbert space V, a linear map $j\colon V \to H$ and a form $\mathfrak{a}\colon V \times V \to \mathbb{C}$ such that $\mathfrak{a}$ and j satisfy Conditions (I) and (II), and such that A is associated with $(\mathfrak{a}, j)$. If j can be chosen to be injective, we say that A **can be generated by an embedded accretive form**.

In this section we characterise which accretive operators can be generated by an accretive form. Moreover, we provide examples of operators that cannot be generated. The following notion turns out to be essential.

Definition 3.27. A subspace $\mathcal{R}$ of H is an **operator range** in H if there exists an operator $R \in \mathcal{L}(H)$ such that $\mathcal{R} = \operatorname{rg} R$.

For an introduction to operator ranges we recommend [FW71]. We collect some properties of operator ranges that we will require later on.

Proposition 3.28. *Let H and K be Hilbert spaces.*

(a) *A subspace $\mathcal{R}$ of H is an operator range in H if and only if $\mathcal{R}$ can be given a Hilbert space structure such that it is continuously embedded into H.*

(b) *Let $R\colon H \supset D(R) \to K$ be a closed operator. Then $D(R)$ is an operator range in H and $\operatorname{rg} R$ is an operator range in K. Moreover, if $\mathcal{R}$ is an operator range in H, then $R(\mathcal{R})$ is an operator range in K.*

(c) *The operator ranges in H form a lattice with respect to taking sums and intersections of subspaces. The lattice of operator ranges in H contains the closed subspaces of H.*

(d) *Let $\mathcal{R}$ be an operator range in H. If $\mathcal{R}$ is nonclosed, then $\mathcal{R}$ has infinite codimension in H.*

(e) *If $\mathcal{R}$ and $\mathcal{S}$ are operator ranges in H such that $H = \mathcal{R} \oplus \mathcal{S}$, where the direct sum does not need to be orthogonal, then both $\mathcal{R}$ and $\mathcal{S}$ are closed in H.*

Proof. (a) This equivalence is part of [FW71, Theorem 1.1].

(b) The characterisations in [FW71, Theorem 1.1] can be easily extended to include closed operators between two possibly different Hilbert spaces. This shows that $D(R)$ is an operator range in H and that $\operatorname{rg} R$ is an operator range in K. Now the second statement readily follows from the fact that the composition of a closed and a bounded operator is closed.

(c) It is obvious that a closed subspace of H is an operator range in H. The first statement is proved in [FW71, Section 2].

(d) This is [FW71, Corollary of Theorem 2.3].

(e) See [FW71, Theorem 2.3].

$\square$

Our main result in this section is the following.

Theorem 3.29. *Let A be an accretive operator in a Hilbert space H. Then A can be generated by an accretive form if and only if $\operatorname{rg}(I + A)$ is an operator range. Moreover, if the orthogonal complement of $D(A)$ in H is zero or infinite dimensional, then A can be generated by an embedded accretive form.*

The necessity of the condition on $\operatorname{rg}(I + A)$ is shown in the following proposition.

Proposition 3.30. *Let A be an accretive operator in a Hilbert space H. If A can be generated by an accretive form, then $D(A)$, $\operatorname{rg}(I + A)$ and $\ker A$ are operator ranges in H.*

Proof. Suppose A is associated with $(\mathfrak{a}, j)$. By Proposition 3.19 (c) we may assume that $\ker T = \{0\}$. Then T^{-1} is a closed operator, so $D_j(\mathfrak{a}) = T^{-1}[\operatorname{rg} j^*]$ is an operator range in V by Proposition 3.28 (b). Composing this with j shows that $D(A) = j(D_j(\mathfrak{a}))$ is an operator range in H. It follows similarly that also $\operatorname{rg}(I + A) = j^{*-1}[TD_j(\mathfrak{a})]$ and $\ker A = j(\ker T_0 \cap D_j(\mathfrak{a}))$ are operator ranges in H. $\square$

For the proof of the other direction in Theorem 3.29 we need the following lemma.

Lemma 3.31. *Let H be a Hilbert space. Then the operator A in H with domain $D(A) = \{0\}$ can be generated by an accretive form $(\mathfrak{a}, j)$ such that $D_j(\mathfrak{a}) = \{0\}$. Moreover, if H is infinite dimensional, then A can be generated by an embedded accretive form.*

Proof. First suppose that H is finite dimensional. Let $(e_\alpha)_{\alpha \in I}$ be an orthonormal basis of H. Then similarly as in Example 3.6, we can choose $V_\alpha = \ell_2$, $j_\alpha \colon V \to \operatorname{span}\{e_\alpha\}$ and $\mathfrak{a}_\alpha \colon V_\alpha \times V_\alpha \to \mathbb{C}$ such that $D_{j_\alpha}(\mathfrak{a}_\alpha) = \{0\}$ for all $\alpha \in I$. Taking the direct sum over all $\alpha \in I$ gives a Hilbert space V, a linear map $j \colon V \to H$ and a form $\mathfrak{a}$ such that $\mathfrak{a}$ and j satisfy Conditions (I) and (II). Moreover, A is associated with $(\mathfrak{a}, j)$.

Now suppose that H is infinite dimensional. To show that A can be generated by an embedded accretive form, it suffices to obtain operators R and S on H as required in Lemma 3.9. If H is separable, we may assume that $H = L^2(\mathbb{R})$ and take the operators in Example 3.10. In the general case, we can take the direct sums of suitably many disjoint copies of the operators from the separable case. Note that only the Hilbert space dimension is of importance here. □

For the proof of Theorem 3.29 we also rely on Phillips' construction of extensions of dissipative operators as presented in [Phi59, Section I.1]. We recall the required results. Let A be an accretive operator in a Hilbert space H. The **Cayley transform** of A is the operator $J := (I - A)(I + A)^{-1}$ in H with domain $D(J) = \operatorname{rg}(I + A)$. The operator J is contractive as $\|(I - A)x\|^2 \leqslant \|(I + A)x\|^2$ for all $x \in D(A)$. Moreover, note that $I + J = 2(I + A)^{-1}$ is injective and that $\operatorname{rg}(I + J) = D(A)$. The operator A can be recovered from the equality

$$A(I + J)u = (I - J)u \tag{3.7}$$

for all $u \in D(J)$. The operator A is closed if and only if J is closed. Moreover, Phillips observed that every proper contractive extension J' of J such that $I + J'$ is injective corresponds to a proper accretive extension of A.

We can now prove the remaining direction of Theorem 3.29.

Proof of Theorem 3.29. Suppose that A is an accretive operator such that $\operatorname{rg}(I + A)$ is an operator range in H. Set $H_2 := D(A)^{\perp}$. By Lemma 3.31 there exist a Hilbert space V_2, a linear map $j_2 \colon V_2 \to H_2$ and a form $\mathfrak{a}_2 \colon V_2 \times V_2 \to \mathbb{C}$ such that $D_{j_2}(\mathfrak{a}_2) = \{0\}$. If H_2 is infinite dimensional, we may assume that j_2 is injective.

Let J be the contraction corresponding to A in the sense of Phillips, i.e., $J := (I - A)(I + A)^{-1}$ with domain $D(J) = \operatorname{rg}(I + A)$. By Proposition 3.28 (a) we can equip $X := D(J)$ with a Hilbert space structure such that X is continuously embedded into H. So there exists an $M > 0$ such that $\|Ju\|_H \leqslant \|u\|_H \leqslant M\|u\|_X$ for all $u \in X$.

Define $V := X \oplus V_2$, $j \colon V \to H$ by $j(u_1, u_2) = (I + J)u_1 + j_2(u_2)$ and $\mathfrak{a} \colon V \times V \to \mathbb{C}$ by

$$\begin{aligned}\mathfrak{a}((u_1, u_2), (v_1, v_2)) = ((I - J)u_1 \mid (I + J)v_1 + j_2(v_2))_H - (j_2(u_2) \mid (I - J)v_1)_H \\ + \mathfrak{a}_2(u_2, v_2).\end{aligned}$$

By the previous paragraph both $\mathfrak{a}$ and j are continuous. Observe that $\operatorname{rg} j = D(A) \oplus \operatorname{rg} j_2$ is dense in $H = \overline{D(A)} \oplus H_2$. It is readily verified that

$$\operatorname{Re} \mathfrak{a}((u_1, u_2), (u_1, u_2)) = \|u_1\|_H^2 - \|Ju_1\|_H^2 + \operatorname{Re} \mathfrak{a}_2(u_2, u_2) \geqslant 0.$$

So $\mathfrak{a}$ and j satisfy Conditions (I) and (II). Note that j is injective if H_2 is zero or infinite dimensional.

We show that $D_j(\mathfrak{a}) = X \times \{0\}$. On the one hand, for all $u_1 \in X$ we have

$$\mathfrak{a}((u_1, 0), (v_1, v_2)) = ((I - J)u_1 \mid j(v_1, v_2))_H$$

for all $(v_1, v_2) \in V$, and hence $(u_1, 0) \in D_j(\mathfrak{a})$. On the other hand, let $(u_1, u_2) \in D_j(\mathfrak{a})$ and $f \in H$ be such that $\mathfrak{a}((u_1, u_2), (v_1, v_2)) = (f \mid (I + J)v_1 + j_2(v_2))_H$ for all $(v_1, v_2) \in V$. By choosing $v_1 = 0$, we obtain

$$(f \mid j_2(v_2))_H = \mathfrak{a}((u_1, u_2), (0, v_2)) = ((I - J)u_1 \mid j_2(v_2))_H + \mathfrak{a}_2(u_2, v_2)$$

for all $v_2 \in V_2$. Therefore $u_2 \in D_{j_2}(\mathfrak{a}_2) = \{0\}$.

Clearly, $D_j(\mathfrak{a}) \cap \ker j = \{0\}$ since $I + J$ is injective. Therefore $(\mathfrak{a}, j)$ is associated with an accretive operator B. By the previous paragraph, $D(B) = j(D_j(\mathfrak{a})) = \operatorname{rg}(I + J) = D(A)$ and $B(I + J)u_1 = Bj(u_1, 0) = (I - J)u_1$ for all $u_1 \in X = D(J)$. By (3.7) it follows that $A = B$. □

The next two corollaries are special cases of Theorem 3.29. We will see in Example 3.36 below that the closability condition in the following corollary cannot be omitted. If A is densely defined, then A is automatically closable by Lemma A.9.

Corollary 3.32. *Let A be a closable, accretive operator in a Hilbert space H. Suppose that $D(A)$ is an operator range in H. Then A can be generated by an accretive form. Moreover, if the orthogonal complement of $D(A)$ in H is zero or infinite dimensional, then A can be generated by an embedded accretive form.*

Proof. By assumption, there exists a bounded operator $S \in \mathcal{L}(H)$ such that $\operatorname{rg}(S) = D(A)$. By Theorem 3.29 and Proposition 3.28 (b), it suffices to prove that $R = (I + A)S$ is a closed operator. This is clear since R is defined on the whole space H and closable as a composition of a bounded and a closable operator. □

Corollary 3.33. *Let A be a closed, accretive operator in a Hilbert space. Then A can be generated by an accretive form.*

Proof. The space $\operatorname{rg}(I + A)$ is an operator range since it is closed by Lemma A.4. □

It follows from (3.6) that Theorem 3.29 can be applied to the nonclosable operator from Example 3.18. We point out, however, that for this operator the orthogonal complement of its domain is merely one-dimensional. Hence the theorem does not state that this operator can be generated by an *embedded* accretive form, as was established in Example 3.18.

Next we give an example of an operator that can be generated by an accretive form, but not by an embedded accretive form. This shows that in general we cannot omit the condition on the dimension of the orthogonal complement of the operator domain in Corollary 3.32.

Example 3.34. Let H be a Hilbert space and let H_0 be a closed subspace of H. Suppose H_0 has finite, nonzero codimension in H. Define the accretive operator A in H by $A = 0$ on $D(A) = H_0$.

Let $V \subset H$ be a Hilbert space that is continuously embedded in H. Assume that V is dense in H and $H_0 \subset V$. Then V is a closed subspace of H since H_0 has finite codimension in H, so $V = H$ as vector spaces. By the bounded inverse theorem, the spaces V and H have equivalent inner products. Every continuous form on V is therefore associated with a bounded operator on H. This shows that the operator A cannot be generated by an embedded accretive form. $\diamond$

We next give examples of accretive operators that cannot be generated by an accretive form. The arguments are based on Proposition 3.30.

Example 3.35. Let $\widetilde{A}$ be a bounded accretive operator on an infinite-dimensional Hilbert space H. Suppose W is a nonclosed subspace of finite codimension in H. For example, W could be the kernel of an unbounded linear functional on H. Let $A = \widetilde{A}|_W$ with domain $D(A) = W$. Then $D(A)$ fails to be an operator range since a nonclosed operator range has infinite codimension in H by Proposition 3.28 (d). By Proposition 3.30 the operator A cannot be generated by an accretive form. $\diamond$

The next example shows that in general in Corollary 3.32 we cannot omit the closability assumption on A. In other words, not every accretive operator with a domain that is an operator range can be generated by an accretive form.

Example 3.36. Let $H = \ell_2(\mathbb{N}_0)$. Suppose φ is an unbounded linear functional on $\ell_2(\mathbb{N})$. Then the operator A given by $Ax = \varphi(x)e_0$ with domain $D(A) = \ell_2(\mathbb{N}) = \{e_0\}^\perp$ is accretive. Clearly, $D(A)$ is an operator range. But $\ker A$ fails to be an operator range since it is not closed and has codimension 2 in H. So A cannot be generated by an accretive form by Proposition 3.30. $\diamond$

Finally, we prove that Phillips' example of a maximal accretive operator that is not m-accretive cannot be generated by an accretive form.

Proposition 3.37. *Suppose A can be generated by an accretive form. Then A is maximal accretive if and only if A is m-accretive.*

Proof. Clearly, if A is m-accretive, then A is maximal accretive by Proposition A.5.

Now suppose that A is maximal accretive. Let J be the contraction corresponding to A by Phillips' extension theory. It follows that $D(J) = \operatorname{rg}(I + A)$ is dense since A is maximal accretive. So J has a unique bounded extension $\tilde{J}$ on H. Note that $\tilde{J}$ extends any contractive extension of J. Due to the maximality of A, the operator $I + \tilde{J}|_{\operatorname{span}\{z\}+D(J)}$ cannot be injective for any $z \in H \setminus D(J)$. In other words, $D(J) \oplus \ker(I + \tilde{J}) = H$, where the direct sum might not be orthogonal in H. Note that $\ker(I + \tilde{J})$ is an operator range in H since it is closed in H and that $D(J) = \operatorname{rg}(I + A)$

is an operator range in H by Proposition 3.30. It follows from Proposition 3.28 (e) that $\operatorname{rg}(I + A)$ is closed. Hence $\operatorname{rg}(I + A) = H$, so A is m-accretive. □

3.3 Form approximation

Assume that $\mathfrak{a}$ and j satisfy Conditions (I) and (II). Moreover, assume that $(\mathfrak{a}, j)$ is associated with an m-accretive operator A. It is natural to ask whether one can approximate $\mathfrak{a}$ by suitable j-elliptic forms $\mathfrak{a}_n \colon V \times V \to \mathbb{C}$ such that the operators A_n associated with $(\mathfrak{a}_n, j)$ converge to A in a suitable sense for $n \to \infty$.

The following result is a starting point for the study of resolvent convergence of m-accretive operators generated by accretive forms.

Theorem 3.38. *Assume $\mathfrak{a}$ and j satisfy Conditions* (I) *and* (II). *Suppose $(\mathfrak{a}, j)$ is associated with an m-accretive operator A.*

Let $\theta \in [0, \frac{\pi}{2})$. Let (B_n) be a sequence of bounded sectorial operators in V with vertex 0 and semi-angle θ. Moreover, suppose there exists an $M > 0$ and sequences of strictly positive numbers (δ_n) and (ε_n) such that $\lim \varepsilon_n^2/\delta_n = 0$ *and*

$$\delta_n \|u\|_V^2 \leqslant \operatorname{Re}(B_n u \mid u)_V \leqslant M\varepsilon_n \|u\|_V^2 \tag{3.8}$$

for all $u \in V$ and $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ define $\mathfrak{a}_n \colon V \times V \to C$ by

$$\mathfrak{a}_n(u, v) = \mathfrak{a}(u, v) + (B_n u \mid v)_V.$$

Then $\mathfrak{a}_n$ is j-elliptic and continuous, and $(\mathfrak{a}_n, j)$ is associated with an m-sectorial operator A_n for all $n \in \mathbb{N}$. Moreover,

$$\|(\lambda I + A_n)^{-1} - (\lambda I + A)^{-1}\| \to 0 \tag{3.9}$$

in the uniform operator norm for all $\lambda \in \rho(-A)$.

Proof. It suffices to prove (3.9) for $\lambda = 1$, cf. [Kat80, Subsection VIII.1.1]. For all $n \in \mathbb{N}$ let T_n be defined as in (3.1) with respect to $\mathfrak{a}_n$ and j. Then $T_n = T + B_n$. Let $f \in H$ and set $u := Zf$, where $Z \in \mathcal{L}(H, V)$ is as in Proposition 3.19 (d). Then u satisfies $Tu = j^* f$. Define $u_n := T_n^{-1} j^* f$ for all $n \in \mathbb{N}$. Observe that $j(u) = (I + A)^{-1} f$ and $j(u_n) = (I + A_n)^{-1} f$.

Let $n \in \mathbb{N}$ be fixed. Then we obtain

$$u_n - u = T_n^{-1} T u - T_n^{-1} T_n u = -T_n^{-1} B_n u$$

and $T_n(u_n - u) = -B_n u$. Hence

$$\|j(u_n) - j(u)\|_H^2 \leqslant \operatorname{Re}(T_n(u_n - u) \mid u_n - u)_V \leqslant \|B_n\|^2 \|T_n^{-1}\| \|u\|_V^2, \tag{3.10}$$

where the first inequality follows from the accretivity of $\mathfrak{a}_n$.

Due to (3.8) we obtain

$$\delta_n \|u\|_V^2 \leqslant \operatorname{Re}\,((T + B_n)u \,|\, u)_V \leqslant \|T_n u\|_V \|u\|_V$$

for all $u \in V$. Using (3.8) and [Kat80, (1.15) in Section VI.1], we deduce

$$|(B_n u \,|\, v)_V| \leqslant M(1 + \tan\theta)\varepsilon_n \|u\|_V \|v\|_V$$

for all $u, v \in V$. Hence

$$\|T_n^{-1}\| \leqslant \frac{1}{\delta_n} \quad \text{and} \quad \|B_n\| \leqslant M(1 + \tan\theta)\varepsilon_n \tag{3.11}$$

for all $n \in \mathbb{N}$. So by (3.10) we have

$$\|(I + A_n)^{-1} f - (I + A)^{-1} f\|_H^2 = \|j(u_n) - j(u)\|_H^2 \leqslant M^2(1 + \tan\theta)^2 \frac{\varepsilon_n^2}{\delta_n} \|Z\|^2 \|f\|_H^2.$$

This shows that the resolvents converge uniformly. □

The following is an interesting special case of Theorem 3.38.

Corollary 3.39. *Assume* $\mathfrak{a}$ *and* j *satisfy Conditions* (I) *and* (II). *Suppose* $(\mathfrak{a}, j)$ *is associated with an m-accretive operator* A. *For all* $n \in \mathbb{N}$ *define the form* $\mathfrak{a}_n \colon V \times V \to \mathbb{C}$ *by*

$$\mathfrak{a}_n(u, v) = \mathfrak{a}(u, v) + \frac{1}{n}(u \,|\, v)_V.$$

Then $\mathfrak{a}$ *is* j*-elliptic and continuous, and* $(\mathfrak{a}_n, j)$ *is associated with an m-sectorial operator* A_n *for all* $n \in \mathbb{N}$. *Moreover,*

$$\|(\lambda I + A_n)^{-1} - (\lambda I + A)^{-1}\| \to 0$$

in the uniform operator norm for all $\lambda \in \rho(-A)$.

Remark 3.40. It is not clear whether the upper bound $\operatorname{Re}\,(B_n u \,|\, u) \leqslant M\varepsilon_n \|u\|_V^2$ in (3.8) can be relaxed to $\lim_{n\to\infty} \|B_n\| = 0$. If T is invertible, however, the proof of Theorem 3.38 can be greatly simplified under relaxed assumptions.

3.4 The dual form

Assume $\mathfrak{a}$ and j satisfy Conditions (I) and (II). We define the **dual form** $\mathfrak{a}^* \colon V \times V \to \mathbb{C}$ by $\mathfrak{a}^*(u, v) = \overline{\mathfrak{a}(v, u)}$. Then obviously $\mathfrak{a}^*$ is continuous and accretive. So $\mathfrak{a}^*$ and j satisfy Conditions (I) and (II). While in the j-elliptic setting $(\mathfrak{a}^*, j)$ is always associated with the adjoint of the m-sectorial operator associated with $(\mathfrak{a}, j)$, the following reconsideration of Example 3.8 shows that this is no longer true in the accretive case, even if j is injective.

Example 3.41. Let R be a densely defined, closed, accretive operator in a Hilbert space H. Equip $V := D(R)$ with the inner product $(u \mid v)_V = (Ru \mid Rv)_H + (u \mid v)_H$ and let $j \colon V \to H$ be the inclusion. Define $\mathfrak{a} \colon V \times V \to \mathbb{C}$ by

$$\mathfrak{a}(u, v) = (Ru \mid v)_H.$$

We first prove that $D_j(\mathfrak{a}^*) = D(R) \cap D(R^*)$. Let $u \in D_j(\mathfrak{a}^*)$. There exists an $f \in H$ such that $\mathfrak{a}^*(u, v) = (f \mid j(v))_H$ for all $v \in V$, which means $(u \mid Rv)_H = (f \mid v)_H$ for all $v \in D(R)$. Hence $u \in D(R^*)$, and obviously we also have $u \in V = D(R)$. For the converse direction let $u \in D(R) \cap D(R^*)$. Set $f = R^*u$. Then

$$\mathfrak{a}^*(u, v) = (u \mid Rv)_H = (f \mid v)_H = (f \mid j(v))_H$$

for all $v \in D(R) = V$. Therefore $u \in D_j(\mathfrak{a}^*)$. Hence $D_j(\mathfrak{a}^*) = D(R) \cap D(R^*)$.

It is now clear that $R^*|_{D(R) \cap D(R^*)}$ is the operator associated with $(\mathfrak{a}^*, j)$.

The m-accretive operator $R := -A$ in $L^2(0, \infty)$ from Example 3.14 satisfies

$$D(R^*) = H^1(0, \infty) \not\subset H^1_0(0, \infty) = D(R).$$

In fact, it is well known that there exists an m-sectorial accretive operator R such that $D(R^*) \not\subset D(R)$. Moreover, one can even arrange that $D(R) \cap D(R^*) = \{0\}$. An operator with the latter property can be readily obtained by adapting the first part of the proof of [FW71, Theorem 3.6]. Choosing such an operator R, only a restriction of R^* is associated with $(\mathfrak{a}^*, j)$. In particular, the operator associated with $(\mathfrak{a}^*, j)$ does not even need to be densely defined. ◇

Still, if $(\mathfrak{a}, j)$ is associated with an accretive operator, then also $(\mathfrak{a}^*, j)$ is associated with an accretive operator, as the following proposition shows.

Proposition 3.42. *Assume* $\mathfrak{a}$ *and* j *satisfy Conditions* (I) *and* (II). *Then the following statements hold.*

(a) $D_j(\mathfrak{a}) \cap \ker j = D_j(\mathfrak{a}^*) \cap \ker j$.

(b) $(\mathfrak{a}, j)$ *is associated with an accretive operator if and only if* $(\mathfrak{a}^*, j)$ *is associated with an accretive operator.*

(c) $V_j(\mathfrak{a}) \cap \ker j = V_j(\mathfrak{a}^*) \cap \ker j$.

Proof. (a) It suffices to prove $D_j(\mathfrak{a}^*) \cap \ker j \subset D_j(\mathfrak{a})$. To this end, let $u \in D_j(\mathfrak{a}^*) \cap \ker j$. Then there exists an $f \in H$ such that $T_0^* u = j^* f$. Then

$$(T_0 u \mid u)_V = (u \mid T_0^* u)_V = (u \mid j^* f)_V = (j(u) \mid f)_H = 0.$$

So $\big((T_0 + T_0^*)u \,|\, u\big)_V = 0$. Since $T_0 + T_0^*$ is a positive (semi-definite) operator, it follows that $u \in \ker(T_0 + T_0^*)$ and $T_0 u = -T_0^* u = -j^* f \in \operatorname{rg} j^*$. Therefore $u \in D_j(\mathfrak{a})$.

(b) This is a consequence of (a), Proposition 2.6 and Proposition A.8.

(c) It suffices to prove $V_j(\mathfrak{a}^*) \cap \ker j \subset V_j(\mathfrak{a})$. To this end, let $u \in V_j(\mathfrak{a}^*) \cap \ker j$. Then $j(u) = 0$ and for all $v \in \ker j$ one has

$$\overline{\mathfrak{a}(v,u)} = \mathfrak{a}^*(u,v) = 0.$$

In particular, $\mathfrak{a}(u,u) = 0$ and $\operatorname{Re}(\mathfrak{a}(u,u)) = (\mathfrak{Re}\,\mathfrak{a})(u,u) = 0$. So by [Kat80, Equation VI.1.15] $(\mathfrak{Re}\,\mathfrak{a})(u,v) = 0$ for all $v \in V$. Therefore $\mathfrak{a}(u,v) + \mathfrak{a}^*(u,v) = 0$ for all $v \in V$. In particular, if $v \in \ker j$, then $\mathfrak{a}(u,v) = 0$ since $\mathfrak{a}^*(u,v) = 0$. So $u \in V_j(\mathfrak{a})$. □

Corollary 3.43. *Assume $j(D_j(\mathfrak{a}))$ is dense in H. Then $(\mathfrak{a}, j)$ is associated with an accretive operator A, and the operator associated with $(\mathfrak{a}^*, j)$ is a restriction of A^*.*

Proof. By Lemma 3.13 the operator A is well-defined. Hence, by Proposition 3.42 (b), also $(\mathfrak{a}^*, j)$ is associated with an accretive operator, which we denote by A_1. Let $u \in D_j(\mathfrak{a}^*)$ and $f \in H$ be such that $T_0^* u = j^* f$. Then for all $v \in D_j(\mathfrak{a})$ we obtain

$$(j(u) \,|\, Aj(v))_H = \overline{\mathfrak{a}(v,u)} = \mathfrak{a}^*(u,v) = (f \,|\, j(v))_H.$$

This shows that $j(u) \in D(A^*)$ and $A_1 j(u) = f = A^* j(u)$. □

Together with Example 3.41, the above corollary illustrates that even if $(\mathfrak{a}, j)$ is associated with an m-sectorial accretive operator A, the operator associated with $(\mathfrak{a}^*, j)$ in general is merely a proper restriction of A^* and thus not m-accretive. If $(\mathfrak{a}, j)$ is associated with an accretive operator A and the operator T is invertible, then the dichotomy of Example 3.41 does not occur. In fact, in this case also T^* is invertible and the operator A_1 associated with $(\mathfrak{a}^*, j)$ is m-accretive. Maximality of A_1 and Corollary 3.43 imply that $A_1 = A^*$.

We close this section with an easy observation connecting the radicals of $\mathfrak{a}$ and $\mathfrak{a}^*$. The (left) **radical** of $\mathfrak{a}$ is defined by

$$R(\mathfrak{a}) := \{u \in V : \mathfrak{a}(u,v) = 0 \text{ for all } v \in V\}.$$

Clearly $R(\mathfrak{a})$ is closed and $R(\mathfrak{a}) = \ker T_0$. Since $\ker T_0 = \ker T_0^*$, we obtain $R(\mathfrak{a}) = R(\mathfrak{a}^*)$. This in particular shows that the left radical of $\mathfrak{a}$ agrees with the right radical.

3.5 The McIntosh condition

Recall that we always assume that $\mathfrak{a}$ and j satisfy Conditions (I) and (II). Consider the following condition that, in the setting of injective j, was introduced by McIntosh in [McI68].

(III) There exists a $\mu > 0$ such that

$$\sup_{\substack{v \in V \\ \|v\|_V \leqslant 1}} |\mathfrak{a}(u,v) + (j(u) \,|\, j(v))_H| \geqslant \mu \|u\|_V$$

for all $u \in V$.

It is easy to verify that (III) implies that the operator T is injective and has closed range. Since T is m-accretive, it follows that $(\operatorname{rg} T)^\perp = \ker T^* = \ker T = \{0\}$. Hence T is invertible. Thus (III) is valid if and only if T is invertible. Note that if T_0 is invertible, then T is invertible by Proposition A.12.

If Conditions (I), (II) and (III) are satisfied and $(\mathfrak{a}, j)$ is associated with an accretive operator, then the associated operator is m-accretive. This follows immediately from Theorem 3.1.

Note that if there exists a $\rho > 0$ such that $|\mathfrak{b}(u,u)| \geqslant \rho \|u\|_V^2$ for all $u \in V$, then Condition (III) is valid with $\mu = \rho$.

The following example, in which Condition (III) is satisfied, shows that using a non-injective map j allows a variety of new phenomena that do not occur in the j-elliptic or embedded accretive case. It is particularly remarkable that in this example $V_j(\mathfrak{a}) = D_j(\mathfrak{a})$ while the associated operator can be unbounded and m-accretive. In both the j-elliptic setting of Theorem 2.3 and the embedded accretive setting of Theorem 2.2 the property $V_j(\mathfrak{a}) = D_j(\mathfrak{a})$ implies that the associated operator is bounded. This follows from Lemma 3.45 below or, for the j-elliptic case, from an inspection of the proof of [Kat80, Theorem VI.2.1 (ii)] together with [AE12, Theorem 2.5 (ii)].

Example 3.44. Let H be a Hilbert space and $B \in \mathcal{L}(H)$ an accretive operator. Let $V = H \times H$ and define $j \in \mathcal{L}(V, H)$ by $j(u) = u_2$. Define the sesquilinear form $\mathfrak{a} \colon V \times V \to \mathbb{C}$ by

$$\mathfrak{a}(u,v) = \left(\begin{pmatrix} B & -I \\ I & 0 \end{pmatrix} u \,\middle|\, v \right)_V = (Bu_1 \,|\, v_1)_H - (u_2 \,|\, v_1)_H + (u_1 \,|\, v_2)_H.$$

Then $\mathfrak{a}$ and j satisfy Conditions (I) and (II). Moreover, $T = \left(\begin{smallmatrix} B & -I \\ I & I \end{smallmatrix}\right)$, whence T is invertible with $T^{-1} = (I+B)^{-1} \left(\begin{smallmatrix} I & I \\ -I & B \end{smallmatrix}\right)$. Thus also Condition (III) is satisfied.

Since $j^*(g) = (0, g)$ for all $g \in H$, it follows that $\operatorname{rg} j^*$ is closed and $\operatorname{rg} j^* = (\ker j)^\perp$. Therefore $V_j(\mathfrak{a}) = D_j(\mathfrak{a})$. We have $V_j(\mathfrak{a}) = \{u \in V : (Bu_1 - u_2 \,|\, v_1)_H = 0 \text{ for all } v_1 \in H\} = \operatorname{gr} B$, the graph of B. Hence by Condition (iii) in Proposition 2.6 $(\mathfrak{a}, j)$ is associated with an accretive operator if and only if B is injective. Moreover $V_j(\mathfrak{a}) + \ker j = H \times \operatorname{rg} B$. This shows that $V_j(\mathfrak{a}) + \ker j = V$ if and only if B is surjective.

Assume that B is injective. Then the associated operator A is m-accretive, $D(A) = j(V_j(\mathfrak{a})) = \operatorname{rg} B$ and $(I + A)B = (I + B)$. Therefore $A = B^{-1}$.

We want to consider a concrete example in more detail. To this end, choose an injective positive operator $B \in \mathcal{L}(H)$ that is not invertible. Then B has dense range, but $\operatorname{rg} B$ is not closed. The latter implies that there does not exist a $\rho > 0$ such that $(Bx \mid x)_H \geqslant \rho \|x\|_H^2$ for all $x \in H$. Note that $\mathfrak{a}$ and j satisfy Condition (III) and that $\mathfrak{b}(u,u) = 0$ implies $u = 0$, but there does not exist a $\rho > 0$ such that $|\mathfrak{b}(u,u)| \geqslant \rho \|u\|_V^2$ for all $u \in V$. So $(\mathfrak{a}, j)$ is associated with an unbounded m-accretive operator. Moreover, $V_j(\mathfrak{a}) + \ker j \neq V$ and $H = j(V) \neq j(V_j(\mathfrak{a})) = \operatorname{rg} B$. $\diamond$

We are interested in when $D_j(\mathfrak{a}) = V_j(\mathfrak{a})$, which occurs in Examples 3.5, 3.44 or 3.60, for example. This is equivalent to $(\ker j)^\perp \cap \operatorname{rg} T \subset \operatorname{rg} j^* \cap \operatorname{rg} T$. So if Condition (III) holds, then this is equivalent to $\operatorname{rg} j^*$ being closed. By Banach's closed range theorem, see [Kat80, Theorem IV.5.13], $\operatorname{rg} j^*$ is closed if and only if $\operatorname{rg} j$ is closed. Since the range of j is dense in H by Condition (II), we have proved the following lemma.

Lemma 3.45. *Suppose Condition* (III) *is satisfied. Then* $D_j(\mathfrak{a}) = V_j(\mathfrak{a})$ *if and only if* j *is surjective.*

Lemma 3.46. *Suppose Condition* (III) *is satisfied. Then one has the following.*

(a) $D_j(\mathfrak{a})$ *is dense in* $V_j(\mathfrak{a})$.

(b) $T(V_j(\mathfrak{a}) \cap \ker j) = T^*(V_j(\mathfrak{a}^*) \cap \ker j)$.

(c) $T(V_j(\mathfrak{a}) \cap \ker j) = \big(V_j(\mathfrak{a}) + \ker j\big)^\perp$.

(d) $V_j(\mathfrak{a}) + \ker j$ *is dense in* V *if and only if* $V_j(\mathfrak{a}) \cap \ker j = \{0\}$.

(e) $\overline{V_j(\mathfrak{a}) + \ker j} = \overline{V_j(\mathfrak{a}^*) + \ker j}$.

Proof. (a) Since $D_j(\mathfrak{a}) = T^{-1} \operatorname{rg} j^*$ and $V_j(\mathfrak{a}) = T^{-1}\big((\ker j)^\perp\big)$, the statement follows from the continuity of T^{-1} and the density of $\operatorname{rg} j^*$ in $(\ker j)^\perp$.

(b) Note that the identity in (b) is equivalent to

$$(\ker j)^\perp \cap T \ker j = (\ker j)^\perp \cap T^* \ker j.$$

Thus it suffices to show that $(\ker j)^\perp \cap T \ker j \subset T^* \ker j$. Let $u \in (\ker j)^\perp \cap T \ker j$. Define $v := (T^*)^{-1} u$. Then

$$(Tv \mid v)_V = (v \mid T^* v)_V = \big((T^*)^{-1} u \mid u\big)_V = \big(u \mid T^{-1} u\big)_V = 0$$

since $T^{-1} u \in \ker j$ and $u \in (\ker j)^\perp$. Hence $v \in \ker j$ by (3.2).

(c) We show the inclusion '$\subset$'. Clearly $T\big(V_j(\mathfrak{a}) \cap \ker j\big) \subset T V_j(\mathfrak{a}) = (\ker j)^\perp$. Let $u \in V_j(\mathfrak{a}) \cap \ker j$ and set $w = (T^{-1})^* T u$. Then

$$(Tw \mid w)_V = (w \mid T^* w)_V = \big((T^{-1})^* T u \mid T u\big)_V = (Tu \mid u)_V = 0$$

since $Tu \in (\ker j)^{\perp}$ and $u \in \ker j$. Therefore $w \in \ker j$ by (3.2). Hence $Tu = T^*w \in T^* \ker j = V_j(\mathfrak{a})^{\perp}$.
We show the inclusion '$\supset$'. Clearly

$$T^{-1}\big((V_j(\mathfrak{a}) + \ker j)^{\perp}\big) \subset T^{-1}\big((\ker j)^{\perp}\big) = V_j(\mathfrak{a}).$$

Let $u \in (V_j(\mathfrak{a}) + \ker j)^{\perp}$. Then $u \in V_j(\mathfrak{a})^{\perp} = \overline{T^* \ker j} = T^* \ker j$. Moreover,

$$\big(T(T^{-1}u) \,|\, T^{-1}u\big)_V = \big(u \,|\, T^{-1}u\big)_V = \big((T^*)^{-1}u \,|\, u\big)_V = 0$$

since $(T^*)^{-1}u \in \ker j$ and $u \in (\ker j)^{\perp}$. Therefore $T^{-1}u \in \ker j$ by (3.2).

(d) This follows immediately from (c).

(e) This statement follows from applying (c) on both sides of (b) and taking the orthogonal complement. □

If Condition (III) holds, then $D_j(\mathfrak{a})$ is dense in $V_j(\mathfrak{a})$. It is natural to investigate when Condition (III) is still satisfied if one restricts to $V_j(\mathfrak{a})$.

Proposition 3.47. *Suppose* $\mathfrak{a}$ *and* j *satisfy Conditions* (I), (II) *and* (III). *Let* $\hat{\mathfrak{a}} := \mathfrak{a}|_{V_j(\mathfrak{a}) \times V_j(\mathfrak{a})}$ *and* $\hat{j} := j|_{V_j(\mathfrak{a})}$. *Then* $\hat{\mathfrak{a}}$ *and* $\hat{j}$ *satisfy Condition* (III) *if and only if* $V_j(\mathfrak{a}) + \ker j = V$.

Proof. We note that $\widehat{T} \in \mathcal{L}(V_j(\mathfrak{a}))$ can be defined for $(\hat{\mathfrak{a}}, \hat{j})$ as in (3.1) since $\hat{\mathfrak{a}}$ satisfies Condition (I).

'$\Rightarrow$': Assume that $\hat{\mathfrak{a}}$ and $\hat{j}$ satisfy Condition (III), i.e., assume that $\widehat{T}$ is invertible. Let $J\colon V_j(\mathfrak{a}) \hookrightarrow V$ be the natural embedding. Define $P := (TJ\widehat{T}^{-1})^*\colon V \to V_j(\mathfrak{a})$. Let $u \in V_j(\mathfrak{a})$ and $v \in \ker j$. Then $\widehat{T}^{-1}u \in V_j(\mathfrak{a}) = V_j(\mathfrak{b})$ and

$$(u \,|\, Pv)_{V_j(\mathfrak{a})} = \big(TJ\widehat{T}^{-1}u \,|\, v\big)_V = \mathfrak{b}(\widehat{T}^{-1}u, v) = 0.$$

Therefore $P|_{\ker j} = 0$. For all $u, v \in V_j(\mathfrak{a})$ we have

$$(u \,|\, Pv)_{V_j(\mathfrak{a})} = \mathfrak{b}(\widehat{T}^{-1}u, v) = \hat{\mathfrak{b}}(\widehat{T}^{-1}u, v) = (u \,|\, v)_{V_j(\mathfrak{a})},$$

whence $P|_{V_j(\mathfrak{a})} = I$. This shows that $V_j(\mathfrak{a}) \cap \ker j = \{0\}$. We next show that $V_j(\mathfrak{a}) + \ker j$ is closed. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $V_j(\mathfrak{a}) + \ker j$ and $u \in V$. Suppose $\lim u_k = u$ in V. Since P is continuous and both $V_j(\mathfrak{a})$ and $\ker j$ are closed, it follows that $Pu = \lim Pu_k \in V_j(\mathfrak{a})$ and $(I - JP)u = \lim(I - JP)u_k \in \ker j$. This shows that $u \in V_j(\mathfrak{a}) + \ker j$. Therefore $V_j(\mathfrak{a}) + \ker j$ is closed. Now we apply Lemma 3.46 (d) to obtain $V_j(\mathfrak{a}) + \ker j = V$.

'$\Leftarrow$': Assume that $V_j(\mathfrak{a}) + \ker j = V$. Lemma 3.46 (d) yields that $V_j(\mathfrak{a}) \cap \ker j = \{0\}$. Let $P\colon V \to V_j(\mathfrak{a})$ be the projection along $\ker j$. It follows from the closed graph theorem that P is bounded. Let $\varepsilon = \mu/2 > 0$, where μ is the constant from Condition (III) for $\mathfrak{a}$. Let $u \in V_j(\mathfrak{a}) \setminus \{0\}$. Then there exists a $v \in V \setminus \{0\}$

such that $|\mathfrak{b}(u,v)| \geqslant \varepsilon\|u\|_V\|v\|_V$. Since $v - Pv \in \ker j$ and $u \in V_j(\mathfrak{a})$, we obtain $\mathfrak{b}(u,v) = \mathfrak{b}(u,Pv)$. Since $\mathfrak{b}(u,v) \neq 0$, this implies that $Pv \neq 0$. Moreover,

$$|\mathfrak{b}(u,Pv)| \geqslant \varepsilon\|u\|_V\|v\|_V \geqslant \varepsilon\delta\|u\|_{V_j(\mathfrak{a})}\|Pv\|_{V_j(\mathfrak{a})},$$

where $\delta = \|P\|^{-1}$. This shows that $\hat{\mathfrak{a}}$ and $\hat{j}$ satisfy Condition (III). □

Remark 3.48. The assumption that $\mathfrak{a}$ and j satisfy Conditions (I), (II) and (III) in Proposition 3.47 does not imply that $j(V_j(\mathfrak{a}))$ is dense in H, i.e., $\hat{j}$ does not need to satisfy Condition (II), cf. Example 3.5. However, if also $\hat{\mathfrak{a}}$ and $\hat{j}$ satisfy Condition (III), then $\operatorname{rg}\hat{j} = j(V_j(\mathfrak{a})) = j(V_j(\mathfrak{a}) + \ker j) = j(V)$ and $\hat{j}$ does satisfy Condition (II).

It follows from Proposition 2.6 that $D_j(\mathfrak{a}) \cap \ker j = \{0\}$ implies that the associated operator is well defined. We next give an example where T is invertible and $D_j(\mathfrak{a}) \cap \ker j = \{0\}$, but $V_j(\mathfrak{a}) \cap \ker j \neq \{0\}$. This shows that $V_j(\mathfrak{a}) \cap \ker j = \{0\}$ is not a necessary condition for $(\mathfrak{a}, j)$ to be associated with an accretive operator, even if $(\mathfrak{a}, j)$ is associated with an m-accretive operator.

Example 3.49. Let H and H_1 be Hilbert spaces such that $H_1 \subsetneq H$ is dense and $\|u\|_H \leqslant \|u\|_{H_1}$ for all $u \in H_1$. Denote the embedding of H_1 into H by j_1. Let $V = H_1 \times \mathbb{C}$ and $j\colon V \to H$ be defined by $j(u,\alpha) = j_1(u)$. Then $\operatorname{rg} j$ is dense in H and $\ker j = \{(0,\alpha) : \alpha \in \mathbb{C}\}$. Since

$$((u,\alpha) \,|\, j^* f)_V = (j(u,\alpha) \,|\, f)_H = (j_1(u) \,|\, f)_H = (u \,|\, j_1^* f)_{H_1} + (\alpha \,|\, 0)_{\mathbb{C}} = ((u,\alpha) \,|\, (j_1^* f, 0))_V,$$

for all $(u,\alpha) \in V$ and $f \in H$, it follows that $\operatorname{rg} j^* = \operatorname{rg} j_1^* \times \{0\}$.

There exists an $x \in H_1$ such that $x \notin \operatorname{rg} j_1^*$ and $\|x\|_{H_1} = 1$. Define the form $\mathfrak{a}\colon V \times V \to \mathbb{C}$ by

$$\mathfrak{a}((u,\alpha),(v,\beta)) \coloneqq (u \,|\, v)_{H_1} + (\alpha x \,|\, v)_{H_1} - (u \,|\, \beta x)_{H_1} - (j_1(u) \,|\, j_1(v))_H.$$

Clearly $\mathfrak{a}$ is accretive. It is easily observed that T is given by

$$T(u,\alpha) = (u + \alpha x, -(u \,|\, x)_{H_1}).$$

A straightforward calculation shows that T is invertible with

$$T^{-1}(v,\beta) = \big(v - ((v \,|\, x)_{H_1} + \beta)x, (v \,|\, x)_{H_1} + \beta\big).$$

So Condition (III) is valid. Moreover,

$$\begin{aligned} V_j(\mathfrak{a}) &= \{(u,\alpha) \in V : -(u \,|\, \beta x)_{H_1} = \mathfrak{b}((u,\alpha),(0,\beta)) = 0 \text{ for all } \beta \in \mathbb{C}\} \\ &= \{(u,\alpha) \in V : (u \,|\, x)_{H_1} = 0\} \end{aligned}$$

and

$$D_j(\mathfrak{a}) = T^{-1}\big(\operatorname{rg} j_1^* \times \{0\}\big) = \Big\{\big(w - (w\,|\,x)_{H_1}x, (w\,|\,x)_{H_1}\big) : w \in \operatorname{rg} j_1^*\Big\}.$$

Thus $V_j(\mathfrak{a}) \cap \ker j = \ker j \neq \{0\}$, but

$$D_j(\mathfrak{a}) \cap \ker j = \Big\{(0, (w\,|\,x)_{H_1}) : w \in \operatorname{rg} j_1^*,\ w - (w\,|\,x)_{H_1}x = 0\Big\} = \{0\}.$$

It follows from Theorem 3.1 that $(\mathfrak{a}, j)$ is associated with an m-accretive operator A.

Lemma 3.46 implies that $D_j(\mathfrak{a})$ is dense in $V_j(\mathfrak{a})$ and that $V_j(\mathfrak{a}) + \ker j$ is not dense in V.

We next determine the behaviour of the restriction of $\mathfrak{a}$ to $V_j(\mathfrak{a})$. Let $\hat{\mathfrak{a}} := \mathfrak{a}|_{V_j(\mathfrak{a})\times V_j(\mathfrak{a})}$ and $\hat{j} := j|_{V_j(\mathfrak{a})}$ as in Proposition 3.47. Then $\hat{j}$ has dense range since $D(A) = j(D_j(\mathfrak{a})) = \hat{j}(V_j(\mathfrak{a}))$ is dense in H. Moreover, for all $(u, \alpha), (v, \beta) \in V_j(\mathfrak{a})$ we have

$$\hat{\mathfrak{a}}((u, \alpha), (v, \beta)) = (u\,|\,v)_{H_1} - (j_1(u)\,|\,j_1(v))_H$$

because $(u\,|\,x)_{H_1} = (v\,|\,x)_{H_1} = 0$. This shows that $\widehat{T}(u, \alpha) = (u, 0)$ and that $\widehat{T}$ is not invertible. Therefore $\hat{\mathfrak{a}}$ and $\hat{j}$ do not satisfy Condition (III) whilst $\mathfrak{a}$ and j do. It is easily observed that $V_{\hat{j}}(\hat{\mathfrak{a}}) = V_j(\mathfrak{a})$ and $V_{\hat{j}}(\hat{\mathfrak{a}}) \cap \ker \hat{j} = \{0\} \times \mathbb{C} \neq \{0\}$. ◇

At the end of Section 3.4 we observed the following.

Proposition 3.50. *Suppose* $\mathfrak{a}$ *and* j *satisfy Conditions* (I), (II) *and* (III). *Suppose that* $(\mathfrak{a}, j)$ *is associated with an accretive operator* A. *Then* A^* *is m-accretive and associated with* $(\mathfrak{a}^*, j)$.

Curiously, we shall see that as a consequence of Proposition 3.50 not every m-accretive operator can be generated by an embedded accretive form that satisfies Condition (III). This was observed in [McI70, Introduction and Theorem 4.2]. We extend this result to a slightly more general setting.

Corollary 3.51. *Suppose* $\mathfrak{a}$ *and* j *satisfy Conditions* (I), (II) *and* (III). *Suppose that* $(\mathfrak{a}, j)$ *is associated with an accretive operator* A *such that* iA *is maximal symmetric. If* $V_j(\mathfrak{a}) + \ker j = V$, *then* iA *is self-adjoint. In particular, if* j *is injective, then* iA *is self-adjoint.*

Proof. Suppose that $V_j(\mathfrak{a}) + \ker j = V$. By Proposition 3.47 the restrictions $\hat{\mathfrak{a}} := \mathfrak{a}|_{V_j(\mathfrak{a})\times V_j(\mathfrak{a})}$ and $\hat{j} := j|_{V_j(\mathfrak{a})}$ still satisfy Condition (III). Moreover, $\hat{j}$ is injective by Lemma 3.46 (d). By Corollary 3.21 the operator A is associated with $(\hat{\mathfrak{a}}, \hat{j})$. So without loss of generality we can assume that j is injective.

Note that for all $u, v \in D_j(\mathfrak{a})$ we have

$$\mathfrak{a}(u, v) = (Aj(u)\,|\,j(v))_H = -i(iAj(u)\,|\,j(v))_H = -(j(u)\,|\,Aj(v))_H = -\mathfrak{a}^*(u, v).$$

As $D_j(\mathfrak{a})$ is dense in $V_j(\mathfrak{a}) = V$ by Lemma 3.46 (a), we obtain $\mathfrak{a}^* = -\mathfrak{a}$. In particular, $-A = A^*$ and $-A$ is m-accretive by Proposition 3.50. Therefore both $i, -i \in \rho(iA)$. This shows that iA is self-adjoint. □

Remark 3.52. Suppose that A is an m-accretive operator such that iA is maximal symmetric but not self-adjoint. Note that the operator $-A$ in Example 3.14 is an example of such an operator. Then by Corollary 3.51 the operator A cannot be generated by an embedded accretive form that satisfies Condition (III). It is not clear whether A can be generated by a general non-embedded form that satisfies Condition (III).

For completeness we mention the following positive result due to McIntosh, see [McI70, Section 3, Example (c)].

Proposition 3.53. *Let A be m-accretive such that $D(A^{1/2}) = D(A^{*1/2})$. Then A can be generated by an embedded accretive form that satisfies Condition* (III)*.*

We close this section with some references to other sufficient or equivalent conditions for Condition (III). McIntosh introduced Condition (III) in [McI68] as a closedness condition for densely defined, accretive forms. In that way he generalised Kato's theory for closed sectorial forms. In [McI70] he formulated a more general abstract closedness condition for general densely defined, separated sesquilinear forms. The connection between the latter abstract condition and Condition (III) is explained by [McI70, Proposition 3.3 and Theorem 3.4]. The following reformulation extends this to our case where j need not be injective. For notation and underlying theory we refer to [McI70] and [Sch71].

Proposition 3.54. *Suppose $\mathfrak{a}$ and j satisfy Conditions* (I) *and* (II)*. Suppose* $\ker T = \{0\}$*. Let X and Y denote the vector space V without topology. Then $(X, Y, \mathfrak{b})$ is a separated dual pair. Denote by X_τ the space X equipped with the locally convex Mackey topology induced by this dual pair. Then the operator T is invertible if and only if the topology of X_τ is that of V.*

Proof. Since $\ker T = \ker T^* = \{0\}$, the dual pair $(X, Y, \mathfrak{b})$ is clearly separated.

Suppose T is invertible. Therefore every continuous linear functional on V is of the form $\mathfrak{b}(\cdot, y)$ for some $y \in Y$. Since V is a Mackey space, this implies that X_τ carries the same topology as V.

Conversely, suppose X_τ carries the same topology as V. Let $z \in V$. Then $(\cdot \,|\, z)_V$ is a continuous linear functional in X_τ. Hence there exists a $y \in Y$ such that $(x \,|\, z)_V = \mathfrak{b}(x, y) = (x \,|\, T^* y)_V$ for all $x \in X$. Therefore T^* is surjective. By the bounded inverse theorem T^* and T are invertible. □

For general densely defined, symmetric sesquilinear forms, McIntosh's closedness condition in [McI70] can be recast in terms of Krein spaces as done in [Fle99] and [FHS00]. This can be utilised in our setting for embedded accretive forms $\mathfrak{a}$ such that $i\mathfrak{a}$ is symmetric. An accretive form $\mathfrak{a}$ is called **conservative** if $\operatorname{Re} \mathfrak{a}(u,u) = 0$ for all $u \in V$. Moreover, an operator is called **conservative** if $\operatorname{Re}(Ax \mid x) = 0$ for all $x \in D(A)$. An m-accretive operator A is conservative if and only if iA is self-adjoint, see [Phi59, Lemma 1.1.4 and 1.1.5]. The following is now a consequence of [Bog74, Theorem V.1.3].

Proposition 3.55. *Suppose* $\mathfrak{a}$ *and* j *satisfy Conditions* (I) *and* (II). *Moreover, suppose that* $\mathfrak{a}$ *is conservative. Let* X *be the vector space* V *without topology. Then* T_0 *is invertible if and only if* $(X, i\mathfrak{a})$ *is a Krein space. If* T_0 *is invertible, then* $\mathfrak{a}$ *and* j *satisfy Condition* (III).

The above proposition can be used to establish that the form in Example 3.16 satisfies Condition (III) provided $b \neq -a$. For details see [Fle99, Lemma 6]. It is not clear whether Condition (III) is also satisfied for the case $b = -a$.

3.6 Ouhabaz type invariance criteria

The following well-known result relates the invariance of closed convex sets under a C_0-semigroup of contraction operators to properties of the generator. For a proof, see for example [Ouh96, Theorem 2.2 and Proposition 2.3].

Proposition 3.56. *Let* A *be an m-accretive operator in* H. *Denote by* S *the* C_0*-semigroup generated by* $-A$. *Let* C *be a closed, convex subset of* H *and let* P *be the associated orthogonal projection onto* C. *Then the following statements are equivalent.*

(i) $S_t C \subset C$ *for all* $t > 0$.

(ii) $\lambda(\lambda I + A)^{-1} C \subset C$ *for all* $\lambda > 0$.

(iii) $\operatorname{Re}(Ax \mid x - Px) \geqslant 0$ *for all* $x \in D(A)$.

If the negative generator of the C_0-semigroup is associated with a j-elliptic accretive form, then one can conveniently characterise invariance by the form itself [Ouh96]. In this section we generalise the following result to our setting.

Proposition 3.57 (Arendt, ter Elst [AE12, Proposition 2.9]). *Suppose* $\mathfrak{a}$ *is* j*-elliptic and accretive. Let* A *be the m-accretive operator associated with* $(\mathfrak{a}, j)$. *Denote by* S *the* C_0*-semigroup generated by* $-A$. *Let* C *be a closed, convex subset of* H *and* P *be the associated orthogonal projection onto* C. *Then the following statements are equivalent.*

(i) $S_t C \subset C$ *for all* $t > 0$.

(ii) *For all* $u \in V$ *there exists a* $w \in V$ *such that*

$$Pj(u) = j(w) \quad \textit{and} \quad \operatorname{Re} \mathfrak{a}(u, u - w) \geqslant 0.$$

Note that Statement (ii) in Proposition 3.57 in particular states that $Pj(V) \subset j(V)$.

Now assume that an m-accretive operator A is associated with $(\mathfrak{a}, j)$. By Theorem A.13 the operator $-A$ generates a C_0-semigroup S on H. The following example shows that even if C is a closed subspace of H that is invariant under S, in general we do not have $Pj(V) \subset j(V)$, where P is the orthogonal projection onto C in H.

Example 3.58. Let H_1 be a Hilbert space and $R \geqslant I$ a self-adjoint operator in H_1 such that $D(R) \neq H_1$. Let $V := H_1 \times D(R)$, $H := H_1 \times H_1$ and define $j \in \mathcal{L}(V, H)$ by $j(u_1, u_2) = (u_1, u_1 + u_2)$. It is clear that j is injective and has dense range. We first determine $j^* \in \mathcal{L}(H, V)$. Let $(u_1, u_2) \in V$ and $(x, y) \in H$. Then

$$\begin{aligned}((u_1, u_2) \,|\, j^*(x, y))_V = (j(u_1, u_2) \,|\, (x, y))_H &= (u_1 \,|\, x + y)_{H_1} + (u_2 \,|\, y)_{H_1} \\ &= (u_1 \,|\, x + y)_{H_1} + \big(Ru_2 \,|\, RR^{-2}y\big)_{H_1}.\end{aligned}$$

This shows that $j^*(x, y) = (x + y, R^{-2}y)$ and $j^*j(u_1, u_2) = (2u_1 + u_2, R^{-2}(u_1 + u_2))$.

Define the sesquilinear form $\mathfrak{a}\colon V \times V \to \mathbb{C}$ by

$$\mathfrak{a}(u, v) = \left(\begin{pmatrix} 0 & -R \\ R^{-1} & R^{-1} \end{pmatrix} u \,\middle|\, v\right)_V.$$

Clearly, $\mathfrak{a}$ is continuous and accretive because

$$\mathfrak{a}(u, u) = -(Ru_2 \,|\, u_1)_{H_1} + (u_1 + u_2 \,|\, Ru_2)_{H_1} = 2i \operatorname{Im} (u_1 \,|\, Ru_2)_{H_1} + (u_2 \,|\, Ru_2)_{H_1}.$$

Let $T_0 \in \mathcal{L}(V)$ be such that $\mathfrak{a}(u, v) = (T_0 u \,|\, v)_V$ for all $u, v \in V$. Clearly T_0 is invertible and $T = T_0 + j^*j$. Therefore T is invertible by Proposition A.12 and Condition (III) is satisfied. Hence $(\mathfrak{a}, j)$ is associated with an m-accretive operator.

Let $u \in V$ and $f \in H$ be such that $\mathfrak{a}(u, v) = (f \,|\, j(v))_H$ for all $v \in V$. That means

$$(-Ru_2 \,|\, v_1)_{H_1} + (u_1 + u_2 \,|\, Rv_2)_{H_1} = (f_1 + f_2 \,|\, v_1)_{H_1} + (f_2 \,|\, v_2)_{H_1}$$

for all $v \in V$. This implies that $-Ru_2 = f_1 + f_2$ and $u_1 + u_2 = R^{-1}f_2$. Hence $u_1 \in D(R)$ and $D_j(\mathfrak{a}) = D(R) \times D(R)$. We obtain $D(A) = D(R) \times D(R)$ and $A(w_1, w_2) = (Rw_1 - 2Rw_2, Rw_2)$. A straightforward calculation shows that for all $\lambda > 0$ we have

$$(\lambda I + A)^{-1} = (\lambda I + R)^{-1} \begin{pmatrix} I & 2R(\lambda I + R)^{-1} \\ 0 & I \end{pmatrix}.$$

Obviously $C := H_1 \times \{0\}$ is an invariant subspace of $(\lambda I + A)^{-1}$ for all $\lambda > 0$. By Proposition 3.56 this shows that C is an invariant subspace of the C_0-semigroup generated by $-A$.

Denote by P the orthogonal projection onto C in H. Let $u \in H \setminus D(R)$. Then

$$Pj(u,0) = P(u,u) = (u,0),$$

but $(u,0) \notin \operatorname{rg} j$. To prove this, assume that there exists a $(u_1,u_2) \in V$ such that $j(u_1,u_2) = (u,0)$. Then $u_1 = u$ and $u_2 = -u$, which is a contradiction since $u \notin D(R)$ and $u_2 \in D(R)$.

Note that in this example one has $Pj(D_j(\mathfrak{a})) \subset j(D_j(\mathfrak{a}))$, so $PD(A) \subset D(A)$. $\diamond$

It is easy to give an example such that the associated C_0-semigroup leaves a closed, convex set invariant, but such that the operator domain of the generator is not left invariant by the corresponding projection. For example, the Laplacian in $L_2(\mathbb{R})$ with domain $H^2(\mathbb{R})$ is m-dissipative and it generates a positive C_0-semigroup. The projection onto the positive real-valued cone, however, does not leave $H^2(\mathbb{R})$ invariant.

Alternatively, in the following example a closed subspace is invariant under the semigroup, but the projection onto this closed subspace does not leave the operator domain invariant.

Example 3.59. Let H_1 be a Hilbert space and $R \geqslant I$ be a self-adjoint unbounded operator in H_1. Let $V := D(R) \times H_1$ and $H := H_1 \times H_1$. Then the sesquilinear form $\mathfrak{a}\colon V \times V \to \mathbb{C}$ defined by

$$\mathfrak{a}(u,v) = (Ru_1 - u_2 \,|\, Rv_1)_{H_1} + (u_2 \,|\, v_2)_{H_1}$$

is continuous and elliptic. Therefore by Theorem 2.1 the associated operator A is m-sectorial. A straightforward calculation shows that $x = (u_1,u_2) \in D(A)$ if and only if $u_1 \in D(R)$ and $Ru_1 - u_2 \in D(R)$.

The closed subspace $C := H_1 \times \{0\}$ is left invariant by the C_0-semigroup generated by $-A$, which is easily deduced from [Ouh96, Theorem 2.1]. Let $u_1 \in D(R) \setminus D(R^2)$ and set $u_2 := Ru_1$. Then $u = (u_1,u_2) \in D(A)$, but $Pu \notin D(A)$, where P denotes the orthogonal projection onto C in H. $\diamond$

Still, one might hope that $PD(A) \subset j(V)$ if P is the orthogonal projection onto a closed subspace that is invariant under the C_0-semigroup generated by an m-accretive operator A. The following basic example shows that this is not true in general.

Example 3.60. Let $H = L^2(\mathbb{R})$, $V = H^1(\mathbb{R})$ and let $j\colon V \to H$ be the inclusion. Define $\mathfrak{a}\colon V \times V \to \mathbb{C}$ by

$$\mathfrak{a}(u,v) = \int_{\mathbb{R}} u' \overline{v}$$

for all $u,v \in V$. Then $\operatorname{Re} \mathfrak{a}(u,u) = 0$ for all $u \in V$, whence $\mathfrak{a}$ is accretive. Clearly $\mathfrak{a}$

and j satisfy Condition (I) and (II), $D_j(\mathfrak{a}) = H^1(\mathbb{R}) = V$ and the associated operator A is the derivative on $H^1(\mathbb{R})$.

Let S be the C_0-semigroup generated by $-A$. Then $(S_t u)(x) = u(x-t)$ for all $t > 0$, $u \in L^2(\mathbb{R})$ and a.e. $x \in \mathbb{R}$. Let

$$C \coloneqq L^2(0,\infty) = \Big\{u \in L^2(\mathbb{R}) : u(x) = 0 \text{ for a.e. } x \in (-\infty, 0)\Big\}.$$

Then C is a closed subspace that is invariant under S. The orthogonal projection P from H onto C is given by $Pu = \mathbb{1}_{(0,\infty)}u$. Obviously $PD(A) \not\subset j(V)$.

Note that Condition (III) is not satisfied in this example. $\diamond$

The preceding examples show that if the associated C_0-semigroup leaves some closed subspace invariant, neither the form domain nor the domain of the operator needs to be left invariant by the corresponding projection. Therefore it is not surprising that some approximation arises if one transfers Proposition 3.57 to our setting.

Proposition 3.61. *Assume $\mathfrak{a}$ and j satisfy Conditions* (I) *and* (II). *Assume that $(\mathfrak{a}, j)$ is associated with an m-accretive operator A. Denote by S the C_0-semigroup generated by $-A$. Let C be a closed, convex subset of H and P be the associated orthogonal projection onto C. Then the following statements are equivalent.*

(i) $S_t C \subset C$ *for all* $t > 0$.

(ii) *For all $u \in D_j(\mathfrak{a})$ and for all sequences $(w_k)_{k\in\mathbb{N}}$ in V such that $\lim j(w_k) = Pj(u)$ one has*
$$\lim_{k\to\infty} \operatorname{Re} \mathfrak{a}(u, u - w_k) \geqslant 0.$$

(iii) *For all $u \in D_j(\mathfrak{a})$ there exists a sequence $(w_k)_{k\in\mathbb{N}}$ in $D_j(\mathfrak{a})$ (or equivalently, in V) such that $\lim j(w_k) = Pj(u)$ and*
$$\limsup_{k\to\infty} \operatorname{Re} \mathfrak{a}(u, u - w_k) \geqslant 0.$$

Proof. '(i)⇒(ii)': Let $u \in D_j(\mathfrak{a})$. Then $x \coloneqq j(u) \in D(A)$. Let $(w_k)_{k\in\mathbb{N}}$ be a sequence in V such that $\lim j(w_k) = Px$. Since $\mathfrak{a}(u, u - w_k) = (Ax \,|\, x - j(w_k))_H$ for all $k \in \mathbb{N}$, the limit exists if $k \to \infty$ and

$$\operatorname{Re} \mathfrak{a}(u, u - w_k) = \operatorname{Re}\,(Ax \,|\, x - j(w_k))_H \to \operatorname{Re}\,(Ax \,|\, x - Px)_H \geqslant 0,$$

where we used Proposition 3.56 '(i)⇒(iii)' in the last step.

'(ii)⇒(iii)': This follows from the fact that $D(A) = j(D_j(\mathfrak{a}))$ is dense in H.

'(iii)⇒(i)': Let $x \in D(A)$. Then there exists a $u \in D_j(\mathfrak{a})$ such that $j(u) = x$. Let

$(w_k)_{k\in\mathbb{N}}$ be a sequence in $D_j(\mathfrak{a})$ such that $\lim j(w_k) = Px$ and

$$\limsup_{k\to\infty} \operatorname{Re} \mathfrak{a}(u, u - w_k) \geqslant 0.$$

Since also

$$\mathfrak{a}(u, u - w_k) = (Ax \,|\, x - j(w_k))_H \to (Ax \,|\, x - Px)_H$$

for $k \to \infty$, we obtain $\operatorname{Re} (Ax \,|\, x - Px)_H \geqslant 0$. Then (i) follows from Proposition 3.56. □

3.7 Remarks on the incomplete case

In the following, let H be a Hilbert space and V_0 be a semi-definite inner product space, i.e., V_0 is a vector space equipped with a nonnegative symmetric sesquilinear form that makes it into a semi-normed space. Let $j_0 \colon V_0 \to H$ be continuous with dense range and $\mathfrak{a}_0 \colon V_0 \times V_0 \to \mathbb{C}$ be a continuous, accretive sesquilinear form. We denote the Hausdorff completion of V_0 by V, see [Bou66, Chapter II, §3, Theorem 3] for technical details. Then there exist unique extensions of $\mathfrak{a}_0$ and j_0 to the Hausdorff completion V which we denote by $\mathfrak{a}$ and j, respectively. Note that V is a Hilbert space and $\mathfrak{a}$ and j satisfy Conditions (I) and (II).

We are interested in conditions on $\mathfrak{a}_0$ and j_0 which imply that $(\mathfrak{a}, j)$ is associated with an m-accretive operator. Clearly, one can express the conditions of Proposition 2.6 and Theorem 3.1 in terms of Cauchy sequences introduced by the completion.

A more easily verified sufficient condition is as follows. Assume that there exists a $\rho > 0$ such that

$$|\mathfrak{b}_0(u)| \geqslant \rho \|u\|_{V_0}^2 \tag{3.12}$$

for all $u \in V_0$. This implies that the extensions $\mathfrak{a}$ and j satisfy Condition (III) and that $\mathfrak{b}(u, u) = 0$ implies $u = 0$. Therefore $(\mathfrak{a}, j)$ is associated with an m-accretive operator.

We note that Condition (3.12) is not necessary to ensure that $(\mathfrak{a}, j)$ is associated with an m-accretive operator, cf. Example 3.44. Still, this condition suffices to cover the incomplete j-sectorial case as considered in [AE12, Section 3].

4

The form method for compactly elliptic forms

In this chapter we introduce a rather accessible condition on a continuous form $\mathfrak{a}$ which ensures that $(\mathfrak{a}, j)$ is associated with a graph that satisfies an appropriate range condition. More precisely, in both the accretive and symmetric case this condition yields that the resolvent set of the associated graph is not empty. It allows us to obtain generation theorems for both self-adjoint and m-accretive operators and graphs. The condition itself is of a very standard nature and known to hold in many cases.

A most interesting application is to generate and study generalised Dirichlet-to-Neumann graphs for symmetric second-order elliptic differential operators on a Lipschitz domain. In particular, we investigate the strong resolvent convergence of generalised Dirichlet-to-Neumann graphs. The results in this chapter are joint work with Wolfgang Arendt, Tom ter Elst and James Kennedy [AEKS13]. We point out that the exposition here is somewhat different to that in [AEKS13] and that we only present a selection of the results there. In particular, we will here use the theory of the *Moore–Penrose generalised inverse* and properties of the *gap* between two closed subspaces of a Hilbert space. For this background material, we refer to Sections A.4 and A.3 in the appendix, respectively.

4.1 Compactly elliptic forms

Let V be a Hilbert space and $\mathfrak{a}\colon V \times V \to \mathbb{C}$ a continuous sesquilinear form. We say that $\mathfrak{a}$ is **compactly elliptic** if there exists a Hilbert space $\widetilde{H}$ and a *compact* linear operator $\tilde{j} \in \mathcal{L}(V, \widetilde{H})$ such that $\mathfrak{a}$ is $\tilde{j}$-elliptic, i.e., such that there exists an $\omega \in \mathbb{R}$ and a $\mu > 0$ such that

$$\operatorname{Re} \mathfrak{a}(u,u) + \omega \|\tilde{j}(u)\|_{\widetilde{H}}^2 \geqslant \mu \|u\|_V^2 \tag{4.1}$$

for all $u \in V$. In this section we study the graphs associated with forms that are compactly elliptic. We are particularly interested in the case when $\mathfrak{a}$ is accretive or symmetric.

The following lemma shows that a form is compactly elliptic if it is coercive on the complement of a finite-dimensional subspace.

Lemma 4.1. *Let V be a Hilbert space and $\mathfrak{a}\colon V \times V \to \mathbb{C}$ a continuous sesquilinear form. Let $\mu > 0$. Suppose that $V = V_+ \oplus V_-$, where V_+ is a closed subspace of V, $\dim V_- < \infty$ and*

$$\operatorname{Re} \mathfrak{a}(u,u) \geqslant \mu \|u\|_V^2$$

for all $u \in V_+$. Then $\mathfrak{a}$ is compactly elliptic.

Proof. Let $M > 0$ be such that $|\mathfrak{a}(u,v)| \leqslant M \|u\|_V \|v\|_V$ for all $u, v \in V$. Let $\widetilde{H} = V_-$ and $\tilde{j} = P_-$, where P_- is the projection of V onto V_- along the decomposition $V = V_+ \oplus V_-$. Observe that $\tilde{j}$ is compact. Set $\omega := (\mu/2 + M + 2M^2/\mu) > 0$. Let $u \in V$. Then $u = u_1 + u_2$ with $u_1 \in V_+$ and $u_2 \in V_-$. Using the Peter–Paul inequality, we obtain

$$\begin{aligned}
\operatorname{Re} \mathfrak{a}(u,u) + \omega \|\tilde{j}(u)\|_{\widetilde{H}}^2 &\geqslant \mu \|u_1\|_V^2 + (\omega - M)\|u_2\|_V^2 - 2M \|u_1\|_V \|u_2\|_V \\
&\geqslant \mu \|u_1\|_V^2 + \Big(\frac{\mu}{2} + \frac{2M^2}{\mu}\Big)\|u_2\|_V^2 - \frac{\mu}{2}\|u_1\|_V^2 - \frac{2M^2}{\mu}\|u_2\|_V^2 \\
&= \frac{\mu}{2}\big(\|u_1\|_V^2 + \|u_2\|_V^2\big) \geqslant \frac{\mu}{4}\|u\|_V^2.
\end{aligned}$$

This shows that $\mathfrak{a}$ is $\tilde{j}$-elliptic. □

Remark 4.2. Adopt the notation and assumptions of Lemma 4.1. If $\mathfrak{a}$ is symmetric and the decomposition $V = V_+ \oplus V_-$ is orthogonal with respect to $\mathfrak{a}$, then $(V, \mathfrak{a})$ is a so-called Pontryagin space [Bog74], i.e., a Krein space with a finite rank of indefiniteness.

Proposition 4.3. *Let $\mathfrak{a}$ be a compactly elliptic form and $B \in \mathcal{L}(V)$. Define $\mathfrak{b}\colon V \times V \to \mathbb{C}$ by*

$$\mathfrak{b}(u,v) = \mathfrak{a}(u,v) + (Bu \,|\, v)_V.$$

Then one has the following.

(a) *If B is compact, then $\mathfrak{b}$ is compactly elliptic.*

(b) *There exists a $\delta > 0$ such that if $\|B\| < \delta$ then $\mathfrak{b}$ is compactly elliptic.*

Proof. Let $\omega \in \mathbb{R}$, $\mu > 0$ and $\tilde{j}$ be as in the compact ellipticity condition of $\mathfrak{a}$.
(a) By the Peter–Paul inequality

$$|(Bu \,|\, u)_V| \leqslant \|Bu\|_V \|u\|_V \leqslant \frac{\mu}{2}\|u\|_V^2 + \frac{1}{2\mu}\|Bu\|_V^2.$$

Let $H' := \widetilde{H} \oplus V$ and $j' \in \mathcal{L}(V, H')$ be given by $j'(u) = (\tilde{j}(u), Bu)$. Then j' is compact and

$$\operatorname{Re} \mathfrak{b}(u,u) + \big(\omega + \frac{1}{2\mu}\big)\|j'(u)\|_{H'}^2 \geqslant \frac{\mu}{2}\|u\|_V^2$$

for all $u \in V$. So $\mathfrak{b}$ is compactly elliptic.

(b) Set $\delta = \mu/2$. If $\|B\| < \delta$, then $|(Bu \,|\, u)_V| < \frac{\mu}{2}\|u\|_V^2$. Now it is easily checked that $\mathfrak{b}$ is compactly elliptic. □

In the following, we suppose that $\mathfrak{a}$ is compactly elliptic. Let $\omega \in \mathbb{R}$, $\mu > 0$ and $\tilde{j}$ be as in (4.1). Let H be a Hilbert space and $j \in \mathcal{L}(V, H)$. The next proposition highlights why we are mainly interested in the case when j is not injective.

Proposition 4.4. *Let V and H be Hilbert spaces. Let $\mathfrak{a}\colon V \times V \to \mathbb{C}$ be compactly elliptic. Suppose $j \in \mathcal{L}(V, H)$ is injective. Then $\mathfrak{a}$ is j-elliptic.*

Proof. By Lemma A.38 there exists a $\rho > 0$ such that

$$\omega\|\tilde{j}(u)\|_{\widetilde{H}}^2 \leqslant \tfrac{\mu}{2}\|u\|_V^2 + \rho\|j(u)\|_H^2$$

for all $u \in V$. Then, using that $\mathfrak{a}$ is compactly elliptic, we obtain

$$\operatorname{Re} \mathfrak{a}(u,u) + \rho\|j(u)\|_H^2 \geqslant \tfrac{\mu}{2}\|u\|_V^2$$

for all $u \in V$. Hence $\mathfrak{a}$ is j-elliptic. □

Next we introduce some technical notation that will be useful in the following. Set $K := \omega\tilde{j}^*\tilde{j}$. Then $K \in \mathcal{L}(V)$ is a compact operator. Recall that $T_\rho = T_0 + \rho j^* j$ for all $\rho \in \mathbb{C}$.

Lemma 4.5. *Let $\rho \in \mathbb{C}$ be such that $\operatorname{Re}\rho \geqslant 0$. Then the operator T_ρ is Fredholm. In particular, $\operatorname{rg} T_\rho$ is closed and $\ker T_\rho$ is finite-dimensional. Moreover, $\operatorname{ind} T_\rho = 0$.*

Proof. Since $\mathfrak{a}$ is compactly elliptic, we have

$$\operatorname{Re}\big((T_\rho + K)u \,|\, u\big)_V = \operatorname{Re}\big(\mathfrak{a}(u,u) + \rho\|j(u)\|_H^2\big) + \omega\|\tilde{j}(u)\|_{\widetilde{H}}^2 \geqslant \mu\|u\|_V^2 \tag{4.2}$$

for all $u \in V$. It follows that the operator $R := T_\rho + K$ is injective and has closed range. As R is m-accretive, $(\operatorname{rg} R)^\perp = \ker R^* = \ker R = \{0\}$. This proves that R is invertible. Because K is compact, T_ρ is invertible in the Calkin algebra. By [Dou72, Theorem 5.17], T_ρ is a Fredholm operator. Finally, $\operatorname{ind}(T_\rho) = \operatorname{ind}(R) = 0$ by [Dou72, Theorem 5.36]. □

We define

$$W_j(\mathfrak{a}) := \{u \in \ker j : \mathfrak{a}(u,v) = 0 \text{ for all } v \in V\}.$$

Then $W_j(\mathfrak{a})$ is a closed subspace of V such that $W_j(\mathfrak{a}) \subset D_j(\mathfrak{a}) \subset V_j(\mathfrak{a})$. The next two lemmas show that $W_j(\mathfrak{a})$ has special properties if $\mathfrak{a}$ is accretive or symmetric.

Lemma 4.6. *Suppose $\mathfrak{a}$ is accretive. Let $\rho \in \mathbb{C}$ be such that* $\operatorname{Re}\rho > 0$. *Then* $\ker T_\rho = W_j(\mathfrak{a})$ *and* $\operatorname{rg} T_\rho = W_j(\mathfrak{a})^\perp$. *Moreover, $W_j(\mathfrak{a}) = W_j(\mathfrak{a}^*)$.*

Proof. Obviously, $W_j(\mathfrak{a}) \subset \ker T_\rho$. Now let $u \in \ker T_\rho$. Note that both T_ρ and T_0 are accretive. Therefore $0 = \operatorname{Re}(T_\rho u \mid u)_V \geqslant (\operatorname{Re}\rho)\|j(u)\|_H^2 \geqslant 0$. Hence $u \in \ker j$ and $T_0 u = T_\rho u = 0$. So $\mathfrak{a}(u,v) = (T_0 u \mid v)_V = 0$ for all $v \in V$. This shows that $u \in W_j(\mathfrak{a})$. By Lemma 4.5 and since T_ρ is m-accretive, it follows from Proposition A.8 (b) that $\operatorname{rg} T_\rho = (\ker T_\rho^*)^\perp = (\ker T_\rho)^\perp = W_j(\mathfrak{a})^\perp$.

For the last statement note that $W_j(\mathfrak{a}) = \ker T_1 = \ker T_1^* = W_j(\mathfrak{a}^*)$. ☐

Lemma 4.7. *Suppose $\mathfrak{a}$ is symmetric. Let $\rho \in \mathbb{C}$ be such that* $\operatorname{Re}\rho \geqslant 0$ *and* $\operatorname{Im}\rho \neq 0$. *Then* $\ker T_\rho = W_j(\mathfrak{a})$ *and* $\operatorname{rg} T_\rho = W_j(\mathfrak{a})^\perp$. *Moreover, $W_j(\mathfrak{a}) = W_j(\mathfrak{a}^*)$.*

Proof. As we can put the real part of ρ into $\mathfrak{a}$, we may assume without loss of generality that $\operatorname{Re}\rho = 0$. Obviously, $W_j(\mathfrak{a}) \subset \ker T_\rho$. Now let $u \in \ker T_\rho$. Then $0 = \operatorname{Im}(T_\rho u \mid u)_V = \operatorname{Im}\rho\|j(u)\|_H^2$. Hence $u \in \ker j$ and $T_0 u = T_\rho u = 0$. So $\mathfrak{a}(u,v) = (T_0 u \mid v)_V = 0$ for all $v \in V$. This shows that $u \in W_j(\mathfrak{a})$. By Lemma 4.5 it follows that $\operatorname{rg} T_\rho = (\ker T_\rho^*)^\perp = (\ker T_{\overline{\rho}})^\perp = W_j(\mathfrak{a})^\perp$. The last claim is trivial. ☐

The proof of Theorem 4.9, which is the main result of this section, will be based on the next proposition. The notion $T_\rho^\dagger$ refers to the Moore–Penrose generalised inverse of T_ρ, see Section A.4.

Proposition 4.8. *Let H be a Hilbert space and $j \in \mathcal{L}(V,H)$. Suppose that $\mathfrak{a}$ is compactly elliptic. Let A be the graph associated with $(\mathfrak{a}, j)$. Suppose that either $\mathfrak{a}$ is accretive and $\rho = 1$ or that $\mathfrak{a}$ is symmetric and $\rho \in \{-i, i\}$. Then $\rho \in \rho(-A)$ and*

$$(\rho I + A)^{-1} = j T_\rho^\dagger j^*. \tag{4.3}$$

Proof. Let $f \in H$. We abbreviate $W := W_j(\mathfrak{a})$. Clearly, $W \subset \ker j$ and hence $\operatorname{rg} j^* \subset (\ker j)^\perp \subset W^\perp$. Observe that by either Lemma 4.6 or Lemma 4.7 we have $\operatorname{rg} T_\rho = W^\perp$ and $\ker T_\rho = W$. So there exists a $u \in W^\perp$ such that $T_\rho u = j^* f$. By Lemma 2.7 this implies $f \in \operatorname{rg}(\rho I + A)$.

Finally we establish (4.3). By Remark A.33.4 it follows that $T_\rho^\dagger T_\rho = P_{W^\perp}$, where $P_{W^\perp}$ denotes the orthogonal projection in V onto $W^\perp$. So $u = T_\rho^\dagger T_\rho u = T_\rho^\dagger j^* f$. Hence

$$(\rho I + A)^{-1} f = j(u) = j T_\rho^\dagger j^* f.$$

This concludes the proof. ☐

Theorem 4.9. *Let* H *be a Hilbert space and* $j \in \mathcal{L}(V, H)$*. Suppose that* $\mathfrak{a}$ *is compactly elliptic. Let* A *be the graph associated with* $(\mathfrak{a}, j)$*. Then* A *is m-accretive if* $\mathfrak{a}$ *is accretive and* A *is self-adjoint if* $\mathfrak{a}$ *is symmetric.*

Proof. Let $A \subset H \times H$ be the graph associated with $(\mathfrak{a}, j)$. It is clear that A is symmetric if $\mathfrak{a}$ is symmetric and accretive if $\mathfrak{a}$ is accretive. To verify that A is self-adjoint or m-accretive, it suffices to check the range condition $\operatorname{rg}(\rho I + A) = H$, where ρ is as in Proposition 4.8. The range condition follows from (4.3). □

Remark 4.10. We point out that if $\mathfrak{a}$ is compactly elliptic, then also $\mathfrak{a}^*$ is compactly elliptic. So suppose that $\mathfrak{a}$ is accretive and compactly elliptic and let $j \in \mathcal{L}(V, H)$. Let A and B be the graphs associated with $(\mathfrak{a}, j)$ and $(\mathfrak{a}^*, j)$, respectively. Then A and B are m-accretive by Theorem 4.9. It is easily verified that $B \subset A^*$, where A^* is the adjoint graph of A. Then $B = A^*$ by Proposition A.18 and Proposition A.16.

4.2 Form approximation

In this section we study form approximation in the setting of compactly elliptic forms. We suppose that the forms uniformly satisfy the compact ellipticity condition of Section 4.1. Both for accretive and symmetric forms we obtain sufficient conditions such that the associated graphs converge in the strong resolvent sense.

Throughout this section we shall use the following notation. Let V be a Hilbert space. Let $\mathfrak{a}\colon V \times V \to \mathbb{C}$ be a continuous sesquilinear form. Let H be a Hilbert space and $j \in \mathcal{L}(V, H)$. Moreover, suppose that $\mathfrak{a}_n\colon V \times V \to \mathbb{C}$ is a continuous sesquilinear form for all $n \in \mathbb{N}$. Suppose that $(\mathfrak{a}_n)$ is **uniformly compactly elliptic**, i.e., there exists a Hilbert space $\widetilde{H}$, a compact operator $\tilde{j} \in \mathcal{L}(V, \widetilde{H})$, an $\omega \in \mathbb{R}$ and a $\mu > 0$ such that

$$\operatorname{Re} \mathfrak{a}_n(u, u) + \omega \|\tilde{j}(u)\|_{\widetilde{H}}^2 \geqslant \mu \|u\|_V^2$$

for all $u \in V$ and $n \in \mathbb{N}$.

Definition 4.11. We say that $(\mathfrak{a}_n)$ **converges weakly** to $\mathfrak{a}$ as $n \to \infty$ if for every sequence (w_n) in V such that $w_n \rightharpoonup w$ weakly in V it follows that

$$\lim_{n\to\infty} \mathfrak{a}_n(u, w_n) = \mathfrak{a}(u, w)$$

for all $u \in V$ and

$$\lim_{n\to\infty} \mathfrak{a}_n(w_n, v) = \mathfrak{a}(w, v)$$

for all $v \in V$.

We say that $(\mathfrak{a}_n)$ **converges uniformly** to $\mathfrak{a}$ as $n \to \infty$ if there exists a sequence (α_n) in $[0, \infty)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and

$$|\mathfrak{a}_n(u, v) - \mathfrak{a}(u, v)| \leqslant \alpha_n \|u\|_V \|v\|_V \tag{4.4}$$

for all $u, v \in V$ and $n \in \mathbb{N}$.

We require a number of auxiliary results before we can prove the results about resolvent convergence in Theorem 4.19, which is our main result in this section. Let $\rho \in \mathbb{C}$ be such that $\operatorname{Re} \rho \geqslant 0$. We define the operator $T \coloneqq T_\rho \in \mathcal{L}(V)$ as in Section 4.1. Analogously, for all $n \in \mathbb{N}$ define the operator $T_n \coloneqq T_\rho^{(n)} \in \mathcal{L}(V)$ by requiring that

$$(T_n u \mid v)_V = \big(T_\rho^{(n)} u \mid v\big)_V = \mathfrak{a}_n(u, v) + \rho(j(u) \mid j(v))_H$$

for all $u, v \in V$.

If $(\mathfrak{a}_n)$ converges weakly (or uniformly) to $\mathfrak{a}$, then also $\mathfrak{a}$ is compactly elliptic. Moreover, if $(\mathfrak{a}_n)$ converges uniformly to $\mathfrak{a}$, then by (4.4) it also converges weakly to $\mathfrak{a}$ as every weakly convergent sequence is bounded. It is easy to see that $(\mathfrak{a}_n)$ converges uniformly to $\mathfrak{a}$ if and only if $T_n \to T$ uniformly. The next lemma gives a similar characterisation for the weak convergence.

Lemma 4.12. *The sequence $(\mathfrak{a}_n)$ converges weakly to $\mathfrak{a}$ as $n \to \infty$ if and only if both $T_n \to T$ strongly and $T_n^* \to T^*$ strongly.*

Proof. Observe that $(\mathfrak{a}_n)$ converges weakly to $\mathfrak{a}$ if and only if for every sequence (w_n) in V such that $w_n \rightharpoonup w$ weakly in V one has both $T_n^* w_n \rightharpoonup T^* w$ weakly in V and $T_n w_n \rightharpoonup T w$ weakly in V. Now the claim follows from Lemma A.36. □

Remark 4.13. Suppose that $(\mathfrak{a}_n)$ converges weakly to $\mathfrak{a}$. Then $T_n \to T$ strongly by Lemma 4.12. Therefore $\sup_{n \in \mathbb{N}} \|T_n\| < \infty$.

Lemma 4.14. *Suppose that $(\mathfrak{a}_n)$ converges weakly to $\mathfrak{a}$. Let $u \in V$ and (u_n) be a sequence in V such that $u_n \rightharpoonup u$ weakly in V. If $\lim_{n\to\infty} \mathfrak{a}_n(u_n, u_n) = \mathfrak{a}(u, u)$, then $\lim_{n\to\infty} \|u_n - u\|_V = 0$.*

Proof. The assumptions imply

$$\lim_{n\to\infty} \mathfrak{a}_n(u_n - u, u_n - u) = 2\mathfrak{a}(u, u) - \lim_{n\to\infty} \mathfrak{a}_n(u_n, u) - \lim_{n\to\infty} \mathfrak{a}_n(u, u_n) = 0.$$

Moreover, $\lim_{n\to\infty} \|\tilde{j}(u_n - u)\|_{\tilde{H}}^2 = 0$ as $\tilde{j}$ is compact. By the uniform $\tilde{j}$-ellipticity we have

$$\mu \|u_n - u\|_V^2 \leqslant \operatorname{Re} \mathfrak{a}(u_n - u, u_n - u) + \omega \|\tilde{j}(u_n - u)\|_{\tilde{H}}^2$$

for all $n \in \mathbb{N}$. So the lemma follows from taking the limit. □

Lemma 4.15. *Suppose that $(\mathfrak{a}_n)$ converges weakly to $\mathfrak{a}$. Let $u \in V$ and $w_n \in W_j(\mathfrak{a}_n)$ for all $n \in \mathbb{N}$. Suppose that $w_n \rightharpoonup u$ weakly in V. Then $u \in W_j(\mathfrak{a})$ and $\lim_{n\to\infty} \|w_n - u\|_V = 0$.*

Proof. Clearly $j(u) = \lim_{n\to\infty} j(w_n) = 0$. Moreover, $\mathfrak{a}(u, v) = \lim_{n\to\infty} \mathfrak{a}_n(w_n, v) = 0$ for all $v \in V$. Therefore $u \in W_j(\mathfrak{a})$. So $\mathfrak{a}_n(w_n, w_n) = 0 = \mathfrak{a}(u, u)$ for all $n \in \mathbb{N}$. Now the norm convergence in V follows from Lemma 4.14. □

Lemma 4.16. *Suppose that* $(\mathfrak{a}_n)$ *converges weakly to* $\mathfrak{a}$. *Suppose* $\dim W_j(\mathfrak{a}_n) = W_j(\mathfrak{a})$ *for all* $n \in \mathbb{N}$. *Then, after going to a subsequence, for all* $w \in W_j(\mathfrak{a})$ *there exists a sequence* (w_n) *such that* $w_n \in W_j(\mathfrak{a}_n)$ *for all* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty}\|w_n - w\| = 0$.

Proof. Let $d := \dim W_j(\mathfrak{a})$. Since $W_j(\mathfrak{a}) \subset \ker T_0$, it follows from Lemma 4.5 that $d < \infty$. For all $n \in \mathbb{N}$, let $(f_1^{(n)}, \dots, f_d^{(n)})$ be an orthonormal basis of $W_j(\mathfrak{a}_n)$. After going to a subsequence, there exist $f_1, \dots, f_d \in V$ such that $f_k^{(n)} \rightharpoonup f_k$ for all $k \in \{1, \dots, d\}$. By Lemma 4.15 we obtain $f_k \in W_j(\mathfrak{a})$ and $\lim_{n\to\infty}\|f_k^{(n)} - f_k\| = 0$ for all $k \in \{1, \dots, d\}$. This implies that $(f_1, \dots, f_d)$ is an orthonormal basis of $W_j(\mathfrak{a})$. Now the claim follows by linearity. □

Proposition 4.17. *Suppose that* $(\mathfrak{a}_n)$ *converges weakly to* $\mathfrak{a}$. *Denote by* P *and* P_n *the orthogonal projections of* V *onto* $W_j(\mathfrak{a})$ *and* $W_j(\mathfrak{a}_n)$, *respectively. Then one has* $\lim_{n\to\infty}\|(I - P)P_n\| = 0$. *In particular,* $\limsup_{n\to\infty} \dim W_j(\mathfrak{a}_n) \leqslant \dim W_j(\mathfrak{a})$.

Proof. We first prove that $\lim_{n\to\infty}\|(I - P)P_n\| = 0$. Suppose not. Then, after going to a subsequence, there exists an $\varepsilon > 0$ such that $\|(I - P)P_n\| \geqslant 2\varepsilon$ for all $n \in \mathbb{N}$. So there exist $u_n \in W_j(\mathfrak{a}_n)$ such that $\|u_n\|_V \leqslant 1$ and $\|(I - P)u_n\| \geqslant \varepsilon$ for all $n \in \mathbb{N}$. After going to a subsequence, there exists a $u \in W_j(\mathfrak{a})$ such that $\lim_{n\to\infty} u_n = u$ by Lemma 4.15. Hence $\varepsilon \leqslant \|(I - P)u_n\| \to 0$ as $n \to \infty$, a contradiction. The second statement is a consequence of Proposition A.30. □

Proposition 4.18. *Adopt the notation and conditions of Proposition 4.17. Suppose*

$$\lim_{n\to\infty} \dim W_j(\mathfrak{a}_n) = \dim W_j(\mathfrak{a}). \tag{4.5}$$

Then $P_n \to P$ *uniformly.*

Proof. In view of Proposition A.29, it suffices to prove that $\lim_{n\to\infty}\|(I - P)P_n\| = 0$ and $\lim_{n\to\infty}\|(I - P_n)P\| = 0$. In Proposition 4.17 we established the former.

It remains to prove that $\lim_{n\to\infty}\|(I - P_n)P\| = 0$. Suppose not. Then, after going to a subsequence, there exists an $\varepsilon > 0$ such that $\|(I - P_n)P\| \geqslant 2\varepsilon$ for all $n \in \mathbb{N}$. So there exist $u_n \in W_j(\mathfrak{a})$ such that $\|u_n\| \leqslant 1$ and $\|(I - P_n)u_n\| \geqslant \varepsilon$ for all $n \in \mathbb{N}$. After going to a subsequence, we may suppose that there exists a $w \in W_j(\mathfrak{a})$ such that $\lim_{n\to\infty} u_n = w$. By Lemma 4.16 we find elements $w_n \in W_j(\mathfrak{a}_n)$ such that $\lim_{n\to\infty} w_n = w$. Then $\varepsilon \leqslant \|(I - P_n)u_n\| = \|(I - P_n)(u_n - w_n)\| \leqslant \|u_n - w_n\| \to 0$ as $n \to \infty$ gives a contradiction. □

The next theorem is the main result of this section. Its proof is based on convergence results for the Moore–Penrose generalised inverse by Izumino, see Section A.4.

Theorem 4.19. *Suppose that* $(\mathfrak{a}_n)$ *is a uniformly compactly elliptic sequence of accretive or symmetric forms. Suppose that* $(\mathfrak{a}_n)$ *converges weakly to* $\mathfrak{a}$. *Moreover, suppose* (4.5) *holds.*

Let A *be the graph associated with* $(\mathfrak{a}, j)$ *and for all* $n \in \mathbb{N}$ *let* A_n *be the graph associated with* $(\mathfrak{a}_n, j)$. *Then* $A_n \to A$ *in the strong resolvent sense. Moreover, if* j *is compact, then the resolvents converge uniformly.*

Proof. Let $\rho = 1$ in the accretive case and let $\rho = i$ in the symmetric case. Note that in both cases $W_j(\mathfrak{a}_n) = \ker T_n$ and $\operatorname{rg} T_n = (\ker T_n)^\perp$ for all $n \in \mathbb{N}$ by Lemma 4.6 and Lemma 4.7. By Proposition 4.18 we know that $\lim_{n\to\infty} P_n = P$. Due to (A.4) it follows that $T_n T_n^\dagger = T_n^\dagger T_n = (I - P_n) \to (I - P) = TT^\dagger = T^\dagger T$ uniformly as $n \to \infty$. Moreover, $T_n \to T$ strongly by Lemma 4.12.

We next show that $\sup_{n\in\mathbb{N}} \|T_n^\dagger\| < \infty$. To this end, let $v \in V$ and set $u_n := T_n^\dagger v$ for all $n \in \mathbb{N}$. By the uniform boundedness principle it suffices to show that (u_n) is bounded in V. Due to (A.5) we have $u_n \in W_j(\mathfrak{a}_n)^\perp$ for all $n \in \mathbb{N}$. Moreover, $T_n u_n = T_n T_n^\dagger v = (I - P_n)v$. Let $K = \omega \tilde{j}^* \tilde{j}$. It follows from (4.2) that

$$\mu \|u_n\|_V \leqslant \|(T_n + K)u_n\|_V \leqslant \|v\|_V + \|Ku_n\|_V \tag{4.6}$$

for all $n \in \mathbb{N}$. So (u_n) is bounded in V if and only if (Ku_n) is bounded in V.

Assume that (u_n), and hence (Ku_n), is not bounded in V. We shall derive a contradiction. After going to a subsequence, we may suppose that $\alpha_n := \|Ku_n\| \to \infty$ as $n \to \infty$. Define $\hat{u}_n := \alpha_n^{-1} u_n$ for all $n \in \mathbb{N}$. Then by (4.6), after going to a subsequence, there exists a $\hat{u} \in V$ such that $\hat{u}_n \rightharpoonup \hat{u}$ weakly in V. As K is compact, it follows that $\|K\hat{u}\|_V = 1$. In particular, $\hat{u} \neq 0$.

On the one hand, $T_n \hat{u}_n = \alpha_n^{-1} T_n u_n = \alpha_n^{-1}(I - P_n)v \to 0$ as $n \to \infty$. Moreover, $T_n \hat{u}_n \rightharpoonup T\hat{u}$ weakly in V since $(\mathfrak{a}_n)$ converges weakly to $\mathfrak{a}$. So $\hat{u} \in \ker T = W_j(\mathfrak{a})$. On the other hand, $0 = P_n \hat{u}_n \rightharpoonup P\hat{u}$ weakly in V by Lemma A.36. So $\hat{u} \in W_j(\mathfrak{a})^\perp$. Together this implies $\hat{u} = 0$, a contradiction.

Using Proposition A.35 we establish that $T_n^\dagger \to T^\dagger$ strongly. Now it follows from (4.3) that

$$(\rho I + A_n)^{-1} = jT_n^\dagger j^* \to jT^\dagger j^* = (\rho I + A)^{-1} \tag{4.7}$$

strongly. So $A_n \to A$ in the strong resolvent sense.

Finally, suppose that j is compact. By Lemma 4.12 and Proposition A.35, we also obtain that $(T_n^\dagger)^* = (T_n^*)^\dagger \to (T^*)^\dagger = (T^\dagger)^*$ strongly. Then $jT_n^\dagger \to jT^\dagger$ uniformly by Lemma A.37. So by (4.7) the resolvents converge uniformly. □

Remark 4.20. Adopt the notation and conditions of Theorem 4.19. Suppose in addition that $(\mathfrak{a}_n)$ converges uniformly to $\mathfrak{a}$. Then, arguing as in the first paragraph of the proof of Theorem 4.19, it follows from Proposition A.34 that $T_n^\dagger \to T^\dagger$ uniformly. So by (4.7) the resolvents converge uniformly. Note that in this case we do not need to first prove $\sup_{n\in\mathbb{N}} \|T_n^\dagger\| < \infty$.

We close this section with a simple example which shows that (4.5) is not necessary for strong resolvent convergence.

Example 4.21. Let $V = \mathbb{C}^2$ and $H = \mathbb{C}$. Let $\mathfrak{a}\colon V \times V \to \mathbb{C}$ be given by $\mathfrak{a}(u, v) = 0$. For all $n \in \mathbb{N}$, let $\mathfrak{a}_n\colon V \times V \to \mathbb{C}$ be defined by $\mathfrak{a}_n(u, v) = \frac{1}{n} u_2 \overline{v}_2$. Then $(\mathfrak{a}_n)$ is a sequence of symmetric forms that converges uniformly to $\mathfrak{a}$. Clearly, the sequence $(\mathfrak{a}_n)$ is uniformly compactly elliptic. Define $j\colon V \to H$ by $j(u) = u_1$.

On the one hand, it is readily checked that $W_j(\mathfrak{a}) = \ker j$ and $W_j(\mathfrak{a}_n) = \{0\}$ for all $n \in \mathbb{N}$. So (4.5) does not hold. On the other hand, an easy calculation shows $A = A_n = 0$ for all $n \in \mathbb{N}$. So obviously $A_n \to A$ in the strong resolvent sense. $\diamond$

4.3 Lower boundedness, sectoriality and convergence of the associated semigroups

In this section we will study the lower boundedness and sectoriality of graphs associated with compactly elliptic forms. Let V and H be Hilbert spaces. Let $\mathfrak{a}\colon V \times V \to \mathbb{C}$ be a continuous sesquilinear form. Suppose that $\mathfrak{a}$ is compactly elliptic. Let $j \in \mathcal{L}(V, H)$. We first derive a criterion that ensures that the graph associated with $(\mathfrak{a}, j)$ is lower bounded. Next we consider a sequence of accretive or symmetric forms converging weakly to $\mathfrak{a}$. We investigate when the associated graphs are uniformly sectorial. The results are useful to establish the convergence of the associated (degenerate) C_0-semigroups.

Proposition 4.22. *Let* $V_1 := V_j(\mathfrak{a}) \cap (V_j(\mathfrak{a}) \cap \ker j)^{\perp}$ *and* $V_2 := V_j(\mathfrak{a}) \cap \ker j$*. Then the graph associated with* $(\mathfrak{a}, j)$ *is lower bounded if* $\mathfrak{a}(w, v) = 0$ *for all* $w \in V_2$ *and* $v \in V_1$*. If* $D_j(\mathfrak{a})$ *is dense in* $V_j(\mathfrak{a})$*, then the latter condition is necessary.*

Proof. Note that $V_j(\mathfrak{a}) = V_1 \oplus V_2$ and that $j_1 := j|_{V_1}$ is injective. Let A be the graph associated with $(\mathfrak{a}, j)$. Let $(x, f) \in A$. Then there exists a $u \in D_j(\mathfrak{a})$ such that $j(u) = x$ and $\mathfrak{a}(u, u) = (f \,|\, x)_H$. Then $u = u_1 + u_2$ with $u_1 \in V_1$ and $u_2 \in V_2$. Observe that $\operatorname{Re} \mathfrak{a}(u, u) = \operatorname{Re} \mathfrak{a}(u_1, u_1)$ and $j(u) = j_1(u_1)$. This proves that A is lower bounded since the restriction of $(\mathfrak{a}, j)$ to V_1 is j_1-elliptic by Proposition 4.4.

Next, suppose that A is lower bounded and that $D_j(\mathfrak{a})$ is dense in $V_j(\mathfrak{a})$. Then there exists a $\rho > 0$ such that

$$\operatorname{Re} \mathfrak{a}(u, u) + \rho \|j(u)\|_H^2 \geqslant 0$$

for all $u \in V_j(\mathfrak{a})$. Let $u_1 \in V_1$. Then

$$\operatorname{Re} \mathfrak{a}(u_1, u_1) + \operatorname{Re} \mathfrak{a}(u_2, u_1) + \rho \|j(u_1)\|_H^2 \geqslant 0$$

for all $u_2 \in V_2$. This implies $\mathfrak{a}(u_2, u_1) = 0$ for all $u_2 \in V_2$. $\square$

In the setting of Theorem 4.9 we obtain the following descriptions of $D_j(\mathfrak{a})$ and $V_j(\mathfrak{a})$.

Lemma 4.23. *Adopt the notation and assumptions of Theorem 4.9. Then* $D_j(\mathfrak{a}) = T_\rho^\dagger \operatorname{rg} j^* \oplus W_j(\mathfrak{a})$ *and* $V_j(\mathfrak{a}) = T_\rho^\dagger (\ker j)^\perp \oplus W_j(\mathfrak{a})$. *In particular,* $D_j(\mathfrak{a})$ *is dense in* $V_j(\mathfrak{a})$.

Proof. Let $u \in V$. Then $u \in D_j(\mathfrak{a})$ if and only if $T_\rho u \in \operatorname{rg} j^*$ by Lemma 2.8. Moreover, $u \in V_j(\mathfrak{a})$ if and only if $T_\rho u \in (\ker j)^\perp = \overline{\operatorname{rg} j^*}$. It is obvious that $W_j(\mathfrak{a}) \subset D_j(\mathfrak{a}) \subset V_j(\mathfrak{a})$. Recall that $T_\rho^\dagger T_\rho = T_\rho T_\rho^\dagger$ is the orthogonal projection in V onto $W_j(\mathfrak{a})^\perp$ (see the proof of Proposition 4.8). Now both decompositions are readily verified. Finally, the last statement follows from the fact that $T_\rho^\dagger \operatorname{rg} j^*$ is dense in $T_\rho^\dagger (\ker j)^\perp$. □

Remark 4.24. Suppose $\mathfrak{a}$ is accretive. Then $D_j(\mathfrak{a})$ is dense in $V_j(\mathfrak{a})$ by Lemma 4.23. Moreover, the graph associated with $(\mathfrak{a}, j)$ is trivially lower bounded. So let $V_1 \coloneqq V_j(\mathfrak{a}) \cap (V_j(\mathfrak{a}) \cap \ker j)^\perp$ and $V_2 \coloneqq V_j(\mathfrak{a}) \cap \ker j$. Now it follows from Proposition 4.22 that $\mathfrak{a}(w, v) = 0$ for all $w \in V_2$ and $v \in V_1$.

The next proposition helps to characterise the graphs that we can obtain in the setting of Theorem 4.9.

Proposition 4.25. *Adopt the notation and assumptions of Theorem 4.9. Set* $V_1 \coloneqq V_j(\mathfrak{a}) \cap (V_j(\mathfrak{a}) \cap \ker j)^\perp$. *Let* $(\mathfrak{a}_1, j_1)$ *be the restriction of* $(\mathfrak{a}, j)$ *to* V_1. *Then* $\mathfrak{a}_1$ *is* j_1*-elliptic and* A *is associated with* $(\mathfrak{a}_1, j_1)$.

Proof. Let $(\hat{\mathfrak{a}}, \hat{j})$ be the restriction of $(\mathfrak{a}, j)$ to $V_j(\mathfrak{a})$. Clearly, $\hat{\mathfrak{a}}$ is still compactly elliptic and accretive or symmetric. So $(\hat{\mathfrak{a}}, \hat{j})$ generates an m-accretive or self-adjoint graph $\widehat{A}$. Let $(x, f) \in A$. Then there exists a $u \in V_j(\mathfrak{a})$ such that $j(u) = x$ and $\mathfrak{a}(u, v) = (f \mid j(v))_H$ for all $v \in V$. This implies that $A \subset \widehat{A}$. It follows that $\widehat{A} = A$ by Proposition A.16.

Next, note that $\mathfrak{a}_1$ is compactly elliptic and that j_1 is injective. Consequently, $\mathfrak{a}_1$ is j_1-elliptic by Proposition 4.4. In particular, $(\mathfrak{a}_1, j_1)$ generates an m-sectorial graph A_1. Let $(x, f) \in A_1$. Then there exists a $u \in V_1$ such that $j_1(u) = x$ and $\mathfrak{a}_1(u, v) = (f \mid j(v))_H$ for all $v \in V_1$. As $u \in V_j(\mathfrak{a})$, we have $\mathfrak{a}(u, v) = 0 = (f \mid j(v))_H$ for all $v \in V_j(\mathfrak{a}) \cap \ker j$. By linearity it follows that $(x, f) \in \widehat{A} = A$. Hence $A_1 = A$ by Proposition A.16. □

So in the accretive or symmetric case, compactly elliptic forms only generate graphs that are accretive and sectorial or self-adjoint and lower bounded. We formulate this as a corollary.

Corollary 4.26. *Adopt the notation and assumptions of Theorem 4.9. Then the graph* A *associated with* $(\mathfrak{a}, j)$ *is m-sectorial. In particular,* A *is lower bounded.*

The following simple example shows that in the general case (i.e., if $\mathfrak{a}$ is only continuous but neither accretive nor symmetric) the graph associated with $(\mathfrak{a}, j)$ need not be lower bounded. Moreover, the graph associated with $(\mathfrak{a}, j)$ can have empty resolvent set.

Example 4.27. Let $V = \mathbb{C}^2$ and $H = \mathbb{C}$. Let $\mathfrak{a}\colon V \times V \to \mathbb{C}$ be given by $\mathfrak{a}(u, v) = u_2\overline{v}_1$. Clearly, the form $\mathfrak{a}$ is compactly elliptic. Define $j\colon V \to H$ by $j(u) = u_1$. It is easily verified that the graph $A = \mathbb{C} \times \mathbb{C}$ is associated with $(\mathfrak{a}, j)$. This graph is not lower bounded. ◇

We require the following additional notation and assumptions. Let $\mathfrak{a}_n\colon V \times V \to \mathbb{C}$ be a continuous sesquilinear form for all $n \in \mathbb{N}$. We suppose that $(\mathfrak{a}_n)$ is uniformly compactly elliptic and that $(\mathfrak{a}_n)$ converges weakly to $\mathfrak{a}$. By Corollary 4.26 the graph associated with $(\mathfrak{a}_n, j)$ is lower bounded for all $n \in \mathbb{N}$. In the remaining part of this section we study, in particular, when these graphs are uniformly lower bounded. We then use the results in combination with Theorem 4.19 to obtain a sufficient criterion for the associated (degenerate) C_0-semigroups to converge strongly.

Lemma 4.28. *Let $u \in V$ and $w_n \in V_j(\mathfrak{a}_n) \cap \ker j$ for all $n \in \mathbb{N}$. Suppose that $w_n \rightharpoonup u$ weakly in V. Then $u \in V_j(\mathfrak{a}) \cap \ker j$ and $\lim_{n\to\infty} \|w_n - u\|_V = 0$.*

Proof. Clearly $j(u) = \lim_{n\to\infty} j(w_n) = 0$. Moreover, $\mathfrak{a}(u, v) = \lim_{n\to\infty} \mathfrak{a}_n(w_n, v) = 0$ for all $v \in \ker j$. Therefore $u \in V_j(\mathfrak{a}) \cap \ker j$. So $\mathfrak{a}_n(w_n, w_n) = 0 = \mathfrak{a}(u, u)$ for all $n \in \mathbb{N}$. Now the norm convergence in V follows from Lemma 4.14. □

Lemma 4.29. *One has $\dim(V_j(\mathfrak{a}) \cap \ker j) < \infty$. Furthermore, suppose that*

$$\dim(V_j(\mathfrak{a}_n) \cap \ker j) = \dim(V_j(\mathfrak{a}) \cap \ker j)$$

for all $n \in \mathbb{N}$. Then, after going to a subsequence, for all $u \in V_j(\mathfrak{a}) \cap \ker j$ there exists a sequence (u_n) such that $u_n \in V_j(\mathfrak{a}_n) \cap \ker j$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|u_n - u\|_V = 0$.

Proof. Note that $\mathfrak{a}(u, u) = 0$ for all $u \in V_j(\mathfrak{a}) \cap \ker j$. By (4.2) it follows that $\|Ku\|_V \geqslant \mu\|u\|_V$ for all $u \in V_j(\mathfrak{a}) \cap \ker j$, where $K := \omega\tilde{j}^*\tilde{j}$ comes from the compact ellipticity condition, as before. Consequently, as K is compact, $d := \dim V_j(\mathfrak{a}) \cap \ker j < \infty$. Now the claim follows by the same argument as in the proof of Lemma 4.16, using Lemma 4.28 instead of Lemma 4.15. □

Remark 4.30. It is possible to obtain a version of Proposition 4.17 or Proposition 4.18 for the sequence of the spaces $V_j(\mathfrak{a}_n) \cap \ker j$ instead of $W_j(\mathfrak{a}_n)$. For details we refer to [AEKS13]. Furthermore, we point out that analogous results could be also obtained for $\ker T_\rho^{(n)}$ or $(\operatorname{rg} T_\rho^{(n)})^\perp$ for all $\rho \in \mathbb{C}$ with $\operatorname{Re}\rho \geqslant 0$.

Proposition 4.31. *Suppose that*

$$\lim_{n\to\infty} \dim(V_j(\mathfrak{a}_n) \cap \ker j) = \dim(V_j(\mathfrak{a}) \cap \ker j). \tag{4.8}$$

Then there exist $\omega' \in \mathbb{R}$ and $\mu' > 0$ such that

$$\operatorname{Re} \mathfrak{a}_n(u, u) + \omega'\|j(u)\|_H^2 \geqslant \mu'\|u\|_V^2$$

for all $n \in \mathbb{N}$ *and* $u \in V_j(\mathfrak{a}_n) \cap (V_j(\mathfrak{a}_n) \cap \ker j)^\perp$.

Proof. Suppose not. Let $V^{(n)} := V_j(\mathfrak{a}_n) \cap (V_j(\mathfrak{a}_n) \cap \ker j)^\perp$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$ be such that $\varepsilon < \mu$. Then, after going to a subsequence, there exists a $u_n \in V^{(n)}$ such that

$$\operatorname{Re} \mathfrak{a}_n(u_n, u_n) + n\|j(u_n)\|_H^2 < \varepsilon \|u_n\|_V^2$$

for all $n \in \mathbb{N}$. After renormalisation, we may suppose that $\|\tilde{j}(u_n)\|_{\tilde{H}} = 1$. By the uniform $\tilde{j}$-ellipticity, we obtain

$$\begin{aligned} \omega\|\tilde{j}(u_n)\|_{\tilde{H}}^2 + \varepsilon\|u_n\|_V^2 &> \operatorname{Re} \mathfrak{a}_n(u_n, u_n) + \omega\|\tilde{j}(u_n)\|_{\tilde{H}}^2 + n\|j(u_n)\|_H^2 \\ &\geqslant \mu\|u_n\|_V^2 + n\|j(u_n)\|_H^2 \end{aligned}$$

for all $n \in \mathbb{N}$. Hence

$$\omega > (\mu - \varepsilon)\|u_n\|_V^2 + n\|j(u_n)\|_H^2 \tag{4.9}$$

for all $n \in \mathbb{N}$. So, after going to a subsequence, there exists a $u \in V$ such that $u_n \rightharpoonup u$ weakly in V and $u \neq 0$ because $\|\tilde{j}(u)\|_{\tilde{H}} = 1$. Note that $j(u) = 0$ by (4.9). Let $v \in \ker j$. Then $\mathfrak{a}(u, v) = \lim_{n\to\infty} \mathfrak{a}_n(u_n, v) = 0$. So on the one hand $u \in V_j(\mathfrak{a}) \cap \ker j$. On the other hand, let $w \in V_j(\mathfrak{a}) \cap \ker j$. By Lemma 4.29 there exists a sequence (w_n) in V with $w_n \in V^{(n)}$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} w_n = w$. Then $(u \,|\, w)_V = \lim_{n\to\infty} (u_n \,|\, w_n)_V = 0$. So $u \in (V_j(\mathfrak{a}) \cap \ker j)^\perp$. This is a contradiction. □

Theorem 4.32. *Suppose that* $(\mathfrak{a}_n)$ *is a uniformly compactly elliptic sequence of accretive or symmetric forms. Suppose that* $(\mathfrak{a}_n)$ *converges weakly to* $\mathfrak{a}$. *Finally, suppose that* (4.8) *holds. For all* $n \in \mathbb{N}$, *let* A_n *be the graph associated with* $(\mathfrak{a}_n, j)$. *Then the graphs* A_n *are uniformly sectorial. In particular, the graphs are uniformly lower bounded.*

Proof. By Remark 4.13 there exists an $M > 0$ such that

$$|\mathfrak{a}_n(u, u)| \leqslant M\|u\|_V^2$$

for all $u \in V$ and $n \in \mathbb{N}$.

Choose $\omega' \in \mathbb{R}$ and $\mu' > 0$ as in Proposition 4.31. Let $n \in \mathbb{N}$ and $(x, f) \in A_n$. Then there exists a $u \in V_j(\mathfrak{a}_n) \cap (V_j(\mathfrak{a}_n) \cap \ker j)^\perp$ such that $\mathfrak{a}_n(u, u) = (f \,|\, x)_H$. By Proposition 4.31 it follows that

$$\operatorname{Re} (f \,|\, x)_H + \omega'\|x\|_H^2 \geqslant \mu'\|u\|_V^2 \geqslant \frac{\mu'}{M}|\operatorname{Im} (f \,|\, x)_H|.$$

This completes the proof. □

By Example 4.21 the condition (4.8) is not necessary for uniform lower boundedness. The following example shows that (4.8) is not sufficient for strong resolvent convergence.

Example 4.33. Let $V = \mathbb{C}^2$ and $H = \mathbb{C}$. Let $\mathfrak{a}\colon V \times V \to \mathbb{C}$ be given by $\mathfrak{a}(u, v) = 0$. For all $n \in \mathbb{N}$, let $\mathfrak{a}_n\colon V \times V \to \mathbb{C}$ be defined by $\mathfrak{a}_n(u, v) = \frac{1}{n} u_2 \bar{v}_1 + \frac{1}{n} u_1 \bar{v}_2$. Then $(\mathfrak{a}_n)$ is a sequence of symmetric forms that converges uniformly to $\mathfrak{a}$. Clearly, the sequence $(\mathfrak{a}_n)$ is uniformly compactly elliptic. Define $j\colon V \to H$ by $j(u) = u_1$.

On the one hand, it is readily checked that $V_j(\mathfrak{a}) = V$ and $V_j(\mathfrak{a}_n) = \ker j$ for all $n \in \mathbb{N}$. So $\dim(V_j(\mathfrak{a}) \cap \ker j) = 1 = \dim(V_j(\mathfrak{a}_n) \cap \ker j)$ for all $n \in \mathbb{N}$. On the other hand, $W_j(\mathfrak{a}) = \ker j$ and $W_j(\mathfrak{a}_n) = \{0\}$ for all $n \in \mathbb{N}$. So (4.5) does not hold. Let A be the graph generated by $(\mathfrak{a}, j)$, and let A_n be the graph generated by $(\mathfrak{a}_n, j)$ for all $n \in \mathbb{N}$. An easy calculation shows that $A = \mathbb{C} \times \{0\}$ and $A_n = \{0\} \times \mathbb{C}$ for all $n \in \mathbb{N}$. So $(iI + A_n)^{-1} = 0$ and $(iI + A)^{-1} = -iI$. In particular, the graphs are uniformly lower bounded, but one does not have strong resolvent convergence. ◇

Conversely, the next example shows that (4.5), and in particular strong resolvent convergence, is not sufficient for uniform lower boundedness.

Example 4.34. Let $V = \mathbb{C}^2$ and $H = \mathbb{C}$. Let $\mathfrak{a}\colon V \times V \to \mathbb{C}$ be given by $\mathfrak{a}(u, v) = u_2 \bar{v}_1 + u_1 \bar{v}_2$. For all $n \in \mathbb{N}$, let $\mathfrak{a}_n\colon V \times V \to \mathbb{C}$ be defined by $\mathfrak{a}_n(u, v) = u_2 \bar{v}_1 + \frac{1}{n} u_2 \bar{v}_2 + u_1 \bar{v}_2$. Then $(\mathfrak{a}_n)$ is a sequence of symmetric forms that converges uniformly to $\mathfrak{a}$. Clearly, the sequence $(\mathfrak{a}_n)$ is uniformly compactly elliptic. Define $j\colon V \to H$ by $j(u) = u_1$.

On the one hand, it is readily checked that $V_j(\mathfrak{a}) = \ker j$ and

$$V_j(\mathfrak{a}_n) = \{(u_1, -n u_1) : u_1 \in \mathbb{C}\}$$

for all $n \in \mathbb{N}$. So (4.8) does not hold. On the other hand, $W_j(\mathfrak{a}) = W_j(\mathfrak{a}_n) = \{0\}$ for all $n \in \mathbb{N}$. So (4.5) holds and one has strong resolvent convergence. Let A be the graph generated by $(\mathfrak{a}, j)$, and let A_n be the graph generated by $(\mathfrak{a}_n, j)$ for all $n \in \mathbb{N}$. An easy calculation shows that $A = \{0\} \times \mathbb{C}$ and $A_n = -nI$ for all $n \in \mathbb{N}$. In particular, the graphs A_n are not uniformly lower bounded. ◇

Finally, we discuss the convergence of the associated (degenerate) C_0-semigroups.

Theorem 4.35. *Adopt the notation and assumptions of Theorem 4.32. For all $n \in \mathbb{N}$, denote by S_n the (degenerate) C_0-semigroup generated by $-A_n$. Let S be the (degenerate) C_0-semigroup generated by $-A$. If $A_n \to A$ in the strong resolvent sense, then $S_n(t) \to S(t)$ strongly as $n \to \infty$ for all $t > 0$. In particular, this is the case if (4.5) holds.*

Proof. By Theorem 4.32 and Corollary 4.26 the graphs A_n are uniformly m-sectorial. So we can apply Theorem A.28. □

We close with an example where the associated graphs are uniformly lower bounded and the corresponding (degenerate) C_0-semigroups converge strongly for all $t > 0$, but where (4.8) is not satisfied. Moreover, in this example the corresponding (degenerate) C_0-semigroups do not converge strongly at $t = 0$. It is easy to see that in the uniformly j-elliptic case strong resolvent convergence of the generators is sufficient for the strong convergence of the (trivially degenerate) C_0-semigroups for all $t \geqslant 0$, see for example [Kat80, Theorem IX.2.16]. This shows that in the compactly elliptic setting form approximation is more delicate.

Example 4.36. We shall modify Example 4.34. We use the notation from there, with the exception that $\mathfrak{a}_n \colon V \times V \to \mathbb{C}$ be instead defined by $\mathfrak{a}_n(u, v) = u_2\overline{v}_1 - \frac{1}{n}u_2\overline{v}_2 + u_1\overline{v}_2$ for all $n \in \mathbb{N}$. As before, condition (4.8) does not hold. An easy calculation shows that $A_n = nI$ for all $n \in \mathbb{N}$. In particular, the operators A_n are uniformly lower bounded. The (degenerate) C_0-semigroup generated by $-A = \{0\} \times \mathbb{C}$ is given by $S(t) = 0$ for all $t \geqslant 0$, while the C_0-semigroup generated by $-A_n$ is given by $S_n(t) = \exp(-nt)$ for all $t \geqslant 0$ and $n \in \mathbb{N}$. So $S_n(t) \to S(t)$ strongly for all $t > 0$, but not at $t = 0$. ◇

4.4 An application to Dirichlet-to-Neumann graphs

In this section we give an interesting application of the results of the previous sections to generalised Dirichlet-to-Neumann graphs. First, we shall use the generation result Theorem 4.9 to define these graphs. Then we study the strong resolvent convergence of such graphs. Finally we address the question of whether the corresponding (degenerate) C_0-semigroups converge strongly using the results of Section 4.3. In the following, let $\Omega \subset \mathbb{R}^d$ be Lipschitz and $V = H^1(\Omega)$.

Definition 4.37. Let $\mu > 0$. Let $C \colon \Omega \to \mathbb{R}^{d\times d}$ and $m \colon \Omega \to \mathbb{R}$ be bounded measurable maps. Suppose that $C(x)$ is a symmetric matrix for all $x \in \Omega$ such that $C(x)\xi \cdot \xi \geqslant \mu|\xi|^2$ for all $\xi \in \mathbb{R}^d$ and $x \in \Omega$, where by $\zeta \cdot \eta$ we denote the Euclidean inner product of $\zeta, \eta \in \mathbb{C}^d$. Then we say that (C, m) are **μ-elliptic symmetric coefficients**.

Define the continuous sesquilinear form $\mathfrak{a} \colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ by

$$\mathfrak{a}(u, v) = \int_\Omega C\nabla u \cdot \nabla v + \int_\Omega m u \overline{v}.$$

We call $\mathfrak{a}$ the **form in $H^1(\Omega)$ associated with (C, m)**.

In the following, let $\mu > 0$ and let (C, m) be μ-elliptic symmetric coefficients. Let $\mathfrak{a}$ be the form in $H^1(\Omega)$ associated with (C, m). Then $\mathfrak{a}$ is symmetric and compactly elliptic. In fact, let $\widetilde{H} = L^2(\Omega)$ and $\tilde{\jmath}$ be the embedding of $H^1(\Omega)$ into $L^2(\Omega)$. Then $\tilde{\jmath}$ is compact by [EE87, Theorem V.4.17] and $\mathfrak{a}$ is $\tilde{\jmath}$-elliptic. Let B^D be the m-sectorial operator in $L^2(\Omega)$ associated with the restriction of $(\mathfrak{a}, \tilde{\jmath})$ to $H^1_0(\Omega)$. Note that B^D has compact resolvent by Lemma 2.9.

Now let Γ be the boundary of Ω. Let $H = L^2(\Gamma)$ and $j \colon H^1(\Omega) \to L^2(\Gamma)$ be the trace map $j(u) = \operatorname{Tr} u$, see Theorem A.48. By Theorem 4.9 and Corollary 4.26 the graph N associated with $(\mathfrak{a}, j)$ is self-adjoint and lower-bounded. We call N the **generalised Dirichlet-to-Neumann graph associated with** $(\mathbf{C}, \mathbf{m})$. Note that N is a graph in $L^2(\Gamma)$.

Example 4.38. Let $\lambda \in \mathbb{R}$. Suppose $C(x) = I$ and $m(x) = -\lambda$ for all $x \in \Omega$. Then we shall use the notation $D_\lambda := N$. By the above, the graph D_λ is self-adjoint and lower bounded. Note that B^D is the realisation of $-\Delta - \lambda I$ in $L^2(\Omega)$ with Dirichlet boundary conditions. Denote the Dirichlet Laplacian on Ω by Δ^D. If $\lambda = 0$, then D_λ is the classical Dirichlet-to-Neumann operator for Ω. We refer to [AE12, Subsection 4.4] for details.

First suppose that $\lambda \in \mathbb{R} \setminus \sigma(-\Delta^D)$. Then D_λ is equal to the Dirichlet-to-Neumann operator associated with $-\Delta - \lambda I$. So if $(\phi, \psi) \in L^2(\Gamma) \times L^2(\Gamma)$, then $(\phi, \psi) \in D_\lambda$ if and only if there exists a $u \in H^1(\Omega)$ such that

$$\begin{cases} -\Delta u - \lambda u = 0, \\ \operatorname{Tr} u = \phi, \\ \psi = \partial_\nu u, \end{cases}$$

where both $-\Delta u - \lambda u = 0$ and $\partial_\nu u$ are understood in a suitable weak sense, see [AE11] or [AM12]. In particular, D_λ is an operator.

Now suppose $\lambda \in \sigma(-\Delta^D)$. Then the situation mostly stays the same, with the exception that the graph D_λ need not be an operator any more. To see this, assume that Ω is C^2. Let $u \in H^1_0(\Omega) \setminus \{0\}$ be an eigenfunction for the eigenvalue λ, i.e., a weak solution of $-\Delta x = \lambda x$ in $H^1_0(\Omega)$. Then it is known that $u \in H^2(\Omega)$. In particular, $\partial_\nu u$ exists. Moreover, $\partial_\nu u \neq 0$ since otherwise the extension of u by 0 would be a weak solution of $-\Delta x = \lambda x$ in $H^1(\mathbb{R}^d)$, which is impossible; see for example Proposition 4.39. So D_λ is not an operator as $(0, \partial_\nu u) \in D_\lambda$. $\diamond$

The following result is based on a connection between the *unique continuation property* and the space $W_j(\mathfrak{a})$.

Proposition 4.39. *Let $\mathfrak{a}$ and j be as above. If the second-order coefficients C are Lipschitz continuous, then $W_j(\mathfrak{a}) = \{0\}$.*

Proof. Let $u \in W_j(\mathfrak{a})$. As Ω is Lipschitz, $\ker j = H^1_0(\Omega)$. So $u \in H^1_0(\Omega)$ satisfies $\mathfrak{a}(u, v) = 0$ for all $v \in H^1(\Omega)$. Denote by $\hat{u}\colon \mathbb{R}^d \to \mathbb{C}$ and $\hat{m}\colon \mathbb{R}^d \to \mathbb{R}$ the extensions by 0 of u and m. Then $\hat{u} \in H^1(\mathbb{R}^d)$. We can construct a bounded, Lipschitz continuous extension $\hat{C}\colon \mathbb{R}^d \to \mathbb{R}^{d \times d}$ of C such that $\hat{C}(x)$ is symmetric and $\hat{C}(x)\xi \cdot \xi \geqslant \mu' |\xi|^2$ for all $\xi \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$, where $0 < \mu' < \mu$. Then we have

$$\int_{\mathbb{R}^d} \hat{C} \nabla \hat{u} \cdot \nabla v + \int_{\mathbb{R}^d} \hat{m} \hat{u} \bar{v} = \mathfrak{a}(u, v|_\Omega) = 0$$

for all $v \in C_c^\infty(\mathbb{R}^d)$. So $\hat{u}$ is a weak solution of a strongly elliptic equation in $\mathbb{R}^d$ with Lipschitz continuous second-order coefficients. Moreover, $\hat{u}$ vanishes on an open set. Now it follows from [GL87, Theorem 1.1] or [AKS62, Section 5, Remark 3] that $u = 0$. □

Remark 4.40. In two dimensions, i.e., if Ω is a bounded open set in $\mathbb{R}^2$ with Lipschitz boundary, then $W_j(\mathfrak{a}) = \{0\}$ for general μ-elliptic symmetric coefficients (C, m) by [Ale12].

In what follows we additionally assume that the coefficients C are Lipschitz continuous. Let (C_n, m_n) be μ-elliptic symmetric coefficients for all $n \in \mathbb{N}$. Suppose that $\lim_{n\to\infty} C_n = C$ in $L^\infty(\Omega; \mathbb{R}^{d\times d})$ and that $m_n \to m$ weak* in $L^\infty(\Omega)$. Note that we do not require the coefficients C_n to be Lipschitz continuous. For all $n \in \mathbb{N}$, let $\mathfrak{a}_n$ be the form in $H^1(\Omega)$ associated with (C_n, m_n).

Lemma 4.41. *The sequence $(\mathfrak{a}_n)$ converges weakly to $\mathfrak{a}$.*

Proof. Let $u \in H^1(\Omega)$ and (u_n) be a sequence in $H^1(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$. As $\tilde{j}$ is compact, it follows that $\lim_{n\to\infty} u_n = u$ in $L^2(\Omega)$. Moreover, (u_n) is bounded in $H^1(\Omega)$ and (m_n) is bounded in $L^\infty(\Omega)$. Hence

$$\mathfrak{a}_n(u_n, v) - \mathfrak{a}(u, v) = \int_\Omega (C_n - C)\nabla u_n \cdot \nabla v + \int_\Omega C\nabla(u_n - u) \cdot \nabla v$$
$$+ \int_\Omega m_n(u_n - u)\bar{v} + \int_\Omega (m_n - m)u\bar{v} \to 0$$

for all $v \in H^1(\Omega)$. □

The uniform $\tilde{j}$-ellipticity of $(\mathfrak{a}_n)$ follows from the boundedness of (m_n) in $L^\infty(\Omega)$. As $W_j(\mathfrak{a}) = \{0\}$ by Proposition 4.39, condition (4.5) is automatically satisfied by Proposition 4.17. So we can apply Theorem 4.19 to obtain the uniform resolvent convergence of the associated graphs. We consider the following concrete example, which also shows that the corresponding (degenerate) C_0-semigroups need not converge strongly.

Example 4.42. Suppose $C(x) = C_n(x) = I$ for all $x \in \Omega$. Let $\lambda \in \sigma(-\Delta^D)$ be the smallest positive eigenvalue of $-\Delta^D$. Let (λ_n) be a sequence in $\mathbb{R}$ such that $\lambda_n < \lambda$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \lambda_n = \lambda$. Suppose $m(x) = -\lambda$ and $m_n(x) = -\lambda_n$ for all $n \in \mathbb{N}$. This falls into the above setting. In particular, $(\mathfrak{a}_n)$ is uniformly $\tilde{j}$-elliptic and converges weakly to $\mathfrak{a}$. Observe that the graph D_λ is associated with $(\mathfrak{a}, j)$, whereas the operators D_{λ_n} are associated with $(\mathfrak{a}_n, j)$ for all $n \in \mathbb{N}$. Note that D_λ does not need to be an operator since $\lambda \in \sigma(-\Delta^D)$, see Example 4.38. Still, by Proposition 4.39 we have $W_j(\mathfrak{a}) = \{0\}$. So $D_{\lambda_n} \to D_\lambda$ in the strong resolvent sense.

Next we discuss whether the operators D_{λ_n} are uniformly lower bounded in $n \in \mathbb{N}$ and whether one has strong convergence of the corresponding (degenerate) C_0-semigroups. For all $n \in \mathbb{N}$ denote by S_n the C_0-semigroup generated by D_{λ_n}. By [AM12, Proposition 5 and Proposition 3], there exists a sequence (μ_n) in $\mathbb{R}$ such that μ_n is an eigenvalue of the operator D_{λ_n} for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mu_n = -\infty$. Hence the operators D_{λ_n} are not uniformly lower bounded in $n \in \mathbb{N}$. Moreover, the sequence $(S_n(t))$ is unbounded in $\mathcal{L}(H)$ for all $t > 0$ and therefore does not converge strongly. $\diamond$

Finally we show that the condition $0 \in \rho(B^D)$ implies that the corresponding (degenerate) C_0-semigroups converge strongly. Suppose that $0 \in \rho(B^D)$. Let $u \in V_j(\mathfrak{a}) \cap \ker j$. Then $u \in H_0^1(\Omega)$ and $B^D u = 0$. Since $0 \in \rho(B^D)$, it follows that $u = 0$ and $V_j(\mathfrak{a}) \cap \ker j = \{0\}$. By Theorem 4.35 and Remark 4.30, the corresponding (degenerate) C_0-semigroups converge strongly. Note, however, that in the setting of Example 4.42 the condition $0 \in \rho(B^D)$ amounts to choosing $\lambda \in \mathbb{R} \setminus \sigma(-\Delta^D)$. In particular, in this case the graphs D_{λ_n} are eventually operators and the corresponding C_0-semigroups are not degenerate.

4.5 Notes and remarks

We point out that there is a connection between compactly elliptic forms and *Fredholm closed forms* which were introduced by McIntosh in [McI70, Section 6]. The forms considered there are continuous forms on the direct product of two possibly different Hilbert spaces V_1 and V_2, where both V_1 and V_2 are continuously and densely embedded into H. Such a form $\mathfrak{a}\colon V_1 \times V_2 \to \mathbb{C}$ is called **Fredholm closed** if for some $\rho \in \mathbb{C}$ the operator $T_\rho \in \mathcal{L}(V_1, V_2)$ is Fredholm, where T_ρ satisfies

$$(T_\rho u \,|\, v)_{V_2} = \mathfrak{a}(u, v) + \rho (u \,|\, v)_H$$

for all $u \in V_1$ and $v \in V_2$. The corresponding generation theorem states that the operator A associated with $\mathfrak{a}$ has a nonempty **Fredholm set**; in fact, $(\rho I + A)$ is a densely defined, closed Fredholm operator with the same index as T_ρ. We shall not go into details here and refer for definitions and properties of unbounded Fredholm operators and the Fredholm set to the literature; see for example [Gol66, Chapter IV], [KS63], or [GK60].

In Lemma 4.5 we established that for a compactly elliptic form the operator T_ρ is Fredholm for all $\rho \in \mathbb{C}$ such that $\operatorname{Re} \rho \geqslant 0$. So if j is a dense embedding, a compactly elliptic form is Fredholm closed. In this case, however, the notion of compact ellipticity is not very interesting as we observed in Proposition 4.4. It is really the non-injectivity of the map j which gives rise to new behaviour and makes the results in the previous sections interesting.

Still, we expect that mutatis mutandis the theory of Fredholm closed forms can be extended to the case in which one merely has $j_1 \in \mathcal{L}(V_1, H)$ and $j_2 \in \mathcal{L}(V_2, H)$ instead of two dense embeddings. This would yield a generation theorem for closed graphs with nonempty Fredholm set. Of course, in this more general setting the associated graphs are less regular. In particular, if A is the associated graph and $\rho \in \mathbb{C}$ is such that T_ρ is Fredholm, then in general one cannot expect that the index of T_ρ is equal to that of $\rho I + A$. As a simple example, observe that the associated operator in Example 4.27 has Fredholm set $\mathbb{C}$, but the index differs from that of T_ρ. While the approach presented here is focused on the accretive and symmetric case, it features the more accessible condition of compact ellipticity, allows a general linear operator $j \in \mathcal{L}(V, H)$ instead of a dense (injective) embedding and ensures that the associated graph has a nonempty resolvent set. Still, the graphs associated with accretive or symmetric compactly elliptic forms are always m-sectorial according to Corollary 4.26.

Some of the results in the previous sections slightly extend those in [AEKS13]. Specifically this is the case for the statement in Remark 4.20 and for the accretive case in Theorem 4.19. It is remarkable that under the assumptions of Theorem 4.19 one actually obtains $T_n^\dagger \to T^\dagger$ strongly. In particular, in this case it is not the operators j^* and j in (4.7) that are essential for the convergence. This should be compared with Theorem 3.38, where the situation is quite different. Consequently, on the one hand one might take this as an indication that the condition (4.5) in Theorem 4.19 is too strong. On the other hand, in the application to the Dirichlet-to-Neumann graphs one has automatically $W_j(\mathfrak{a}) = \{0\}$ as soon as the second-order coefficients are Lipschitz or the dimension is 2, see Proposition 4.39. This follows from the connection between the unique continuation property and $W_j(\mathfrak{a}) = \{0\}$. It would be interesting to further the investigation of this connection. The remarkable example of Filonov [Fil01b] of a pure second-order elliptic divergence form operator with a nonzero compactly supported smooth solution shows that (4.5) is not always satisfied in this setting. It is natural to ask whether one can have resolvent convergence of Dirichlet-to-Neumann graphs if $W_j(\mathfrak{a}_n) = \{0\}$ for all $n \in \mathbb{N}$ and $W_j(\mathfrak{a}) \neq \{0\}$.

We note that it is possible to relax the regularity assumptions on the set Ω in Section 4.4. We assumed that Ω is Lipschitz, but the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ remains compact for bounded open sets Ω with continuous boundary [EE87, Theorem V.4.17], for example. Similarly, for the existence of a continuous trace map from $H^1(\Omega)$ into $L^2(\Gamma)$, it is not necessary that Ω is Lipschitz. In [AE11] the classical Dirichlet-to-Neumann operator is considered on (basically arbitrarily) rough domains via the form method in the incomplete sectorial case. There the notion of the *weak trace*, which we consider in Chapter 7, is essential.

5

The regular part of sectorial forms

In [Sim78a] Simon introduced a decomposition of a general, possibly nonclosable, positive symmetric form into the sum of two positive symmetric forms: a maximal closable *regular part* and a *singular part*. In this chapter we study a generalisation of this decomposition for j-sectorial forms, which is made possible by Kato's first representation theorem [Kat80, Section VI.2.1] and the generation theorem for j-sectorial forms in [AE12, Section 3].

In Section 5.1 we first recall the necessary notation and results from [AE12]. Then we define the regular and singular part of a j-sectorial form $\mathfrak{a}$ and prove that they are unique in a natural sense. Moreover, we deduce a formula for the regular part in terms of the real part of the form. This is the main result of this chapter, which will be used in Section 5.2 and in Chapter 6.

In Section 5.2 we study the real part of the regular part and properties of the singular part. Since a sectorial form is closable if and only if its real part is closable, one might expect that taking the real part of a form commutes with taking the regular part of the form. We present a counterexample which shows that this is not always the case. Furthermore, we characterise when the singular part is j-sectorial.

The results in this chapter are joint work with Tom ter Elst [ES11]. We wish to thank Wolfgang Arendt for stimulating discussions and Brian Davies for raising the question of whether $\mathfrak{Re}(\mathfrak{a}_{\mathrm{reg}}) = (\mathfrak{Re}\,\mathfrak{a})_{\mathrm{reg}}$.

5.1 The regular part expressed in terms of the real part

Let $\mathfrak{a}\colon D(\mathfrak{a}) \times D(\mathfrak{a}) \to \mathbb{C}$ be a sesquilinear form. The form domain $D(\mathfrak{a})$ is merely supposed to be a vector space. Let H be a Hilbert space and $j\colon D(\mathfrak{a}) \to H$ be a linear map with dense range. We allow that j is not injective. Throughout this chapter we assume that the form $\mathfrak{a}$ is **j-sectorial**, i.e., there exist a **vertex** $\gamma \in \mathbb{R}$ and a **semi-angle** $\theta \in [0, \frac{\pi}{2})$ such that

$$\mathfrak{a}(u) - \gamma \|j(u)\|_H^2 \in \Sigma_\theta \tag{5.1}$$

for all $u \in D(\mathfrak{a})$, where $\Sigma_\theta := \{z \in \mathbb{C} : z = 1 \text{ or } |\arg(z)| \leqslant \theta\}$.

One of the main results in [AE12] is the following generation theorem.

Theorem 5.1. *Let $\mathfrak{a}$ be a sesquilinear form, H a Hilbert space and $j\colon D(\mathfrak{a}) \to H$ a linear map. Suppose $\mathfrak{a}$ is j-sectorial and $j(D(\mathfrak{a}))$ is dense in H. Then there exists an m-sectorial operator A in H such that for all $x, f \in H$ one has $x \in D(A)$ and $Ax = f$ if and only if there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $D(\mathfrak{a})$ with the following three properties:*

(I) $\lim_{n\to\infty} j(u_n) = x$ *in* H,
(II) $\lim_{n,m\to\infty} \operatorname{Re} \mathfrak{a}(u_n - u_m, u_n - u_m) = 0$, *and,*
(III) $\lim_{n\to\infty} \mathfrak{a}(u_n, v) = (f \mid j(v))_H$ *for all* $v \in D(\mathfrak{a})$.

The operator A in Theorem 5.1 is called the **operator associated** with $(\mathfrak{a}, j)$.

If A is the operator as in Theorem 5.1, then due to a result by Kato [Kat80, Theorem VI.2.7] there exists a unique densely defined, closed sectorial form $\mathfrak{a}_c$ in H such that the operator A is 'classically' associated with $\mathfrak{a}_c$. In [AE12, Proposition 3.10] it is shown that $j(D(\mathfrak{a}))$ is a form core of $\mathfrak{a}_c$. In [AE12] this has been used to generalise the notion of the regular part of a semi-bounded symmetric form to j-sectorial forms. Since this regular part is a form living in H, we call it in the following the H-regular part. More precisely, the **H-regular part** of the j-sectorial form $\mathfrak{a}$ is the form $\mathfrak{a}_r$ with form domain $D(\mathfrak{a}_r) = j(D(\mathfrak{a}))$ defined by $\mathfrak{a}_r = \mathfrak{a}_c|_{j(D(\mathfrak{a}))\times j(D(\mathfrak{a}))}$.

Note that $\mathfrak{a}_r$ is a densely defined, closable sectorial form in H, and A is classically associated with its closure. If $\mathfrak{a}$ is symmetric and j is the inclusion map, then $\mathfrak{a}_r$ coincides with the regular part defined by Simon [Sim78a]. However, one would like to have a decomposition of the original j-sectorial form $\mathfrak{a}$ into a sum of a regular and a singular part. Since $\mathfrak{a}_r$ is defined on $j(D(\mathfrak{a}))$, this would make only sense provided that $D(\mathfrak{a}) \subset H$ and j is the corresponding inclusion map.

We prefer a slightly modified notion of the regular part. We define the **j-regular part** $\mathfrak{a}_{reg}$ of the j-sectorial form $\mathfrak{a}$ to be the form with form domain $D(\mathfrak{a}_{reg}) = D(\mathfrak{a})$ and $\mathfrak{a}_{reg}(u, v) = \mathfrak{a}_c(j(u), j(v))$ for all $u, v \in D(\mathfrak{a})$. It is clear that also this definition coincides with Simon's in the symmetric case when j is the inclusion map $D(\mathfrak{a}) \hookrightarrow H$. Moreover, A is associated with $(\mathfrak{a}_{reg}, j)$ and $(\mathfrak{a}_{reg})_{reg} = \mathfrak{a}_{reg}$. Note that since $\mathfrak{a}_{reg}$ is defined on $D(\mathfrak{a})$, we immediately can write $\mathfrak{a}$ as the sum of $\mathfrak{a}_{reg}$ and the **j-singular part** $\mathfrak{a}_s := \mathfrak{a} - \mathfrak{a}_{reg}$ with the form domain $D(\mathfrak{a}_s) = D(\mathfrak{a})$.

If γ is as in (5.1), then

$$(u \mid v)_{\mathfrak{a}} = (\mathfrak{Re}\, \mathfrak{a})(u, v) + (1 - \gamma)(j(u) \mid j(v))_H \tag{5.2}$$

is a semi-inner product on $D(\mathfrak{a})$, where we denote by $\mathfrak{Re}$ and $\mathfrak{Im}$ the real and imaginary parts of a form. We always equip $D(\mathfrak{a})$ with this semi-inner product. There exists a Hilbert space V and an isometric map $q \colon D(\mathfrak{a}) \to V$ such that $q(D(\mathfrak{a}))$ is dense in V. Note that the pair (V, q) is unique, up to unitary equivalence, since it is the Hausdorff completion of the semi-inner product space $D(\mathfrak{a})$. There exist unique continuous extensions of $\mathfrak{a}$ and j to V, which we denote by $\tilde{\mathfrak{a}}$ and $\tilde{j}$.

We set

$$V(\tilde{\mathfrak{a}}) := V_{\tilde{j}}(\tilde{\mathfrak{a}}) = \{u \in V : \tilde{\mathfrak{a}}(u, v) = 0 \text{ for all } v \in \ker \tilde{j}\}.$$

It follows from [AE12, Theorem 2.5] that $V = \ker \tilde{j} \oplus V(\tilde{\mathfrak{a}})$. This is to be understood as a direct (but not necessarily orthogonal) sum of closed vector subspaces of V. Denote by $P_{V(\tilde{\mathfrak{a}})}$ the projection of V onto $V(\tilde{\mathfrak{a}})$ along this decomposition. Note that if $\mathfrak{a}$, and therefore also $\tilde{\mathfrak{a}}$, is symmetric, then the decomposition is orthogonal.

It is proved in Proposition 3.10 and Theorem 2.5 of [AE12] that

$$\mathfrak{a}_c(\tilde{j}(u), \tilde{j}(v)) = \tilde{\mathfrak{a}}\big(P_{V(\tilde{\mathfrak{a}})} u, P_{V(\tilde{\mathfrak{a}})} v\big)$$

for all $u, v \in V$. Hence

$$\mathfrak{a}_{\mathrm{reg}}(u, v) = \mathfrak{a}_r\big(\tilde{j}(q(u)), \tilde{j}(q(v))\big) = \mathfrak{a}_c\big(\tilde{j}(q(u)), \tilde{j}(q(v))\big) = \tilde{\mathfrak{a}}\big(P_{V(\tilde{\mathfrak{a}})} q(u), P_{V(\tilde{\mathfrak{a}})} q(v)\big) \tag{5.3}$$

for all $u, v \in D(\mathfrak{a})$.

Definition 5.2. Let H be a Hilbert space, W a vector space and $j \colon W \to H$ a linear map with dense range. Let $\mathfrak{b} \colon W \times W \to \mathbb{C}$ be a j-sectorial form. The form $\mathfrak{b}$ is called **j-closable** if every Cauchy sequence $(u_n)_{n \in \mathbb{N}}$ in $D(\mathfrak{b})$ with $\lim_{n\to\infty} j(u_n) = 0$ in H also satisfies that $\lim_{n\to\infty} \mathfrak{b}(u_n, u_n) = 0$.

It is easily seen that the j-sectorial form $\mathfrak{a}$ is j-closable if and only if $\tilde{j}$ is injective.

Proposition 5.3. *Assume the notation and conditions of Theorem 5.1. Then the* j*-regular part* $\mathfrak{a}_{\mathrm{reg}}$ *is the unique* j*-closable form with form domain* $D(\mathfrak{a}_{\mathrm{reg}}) = D(\mathfrak{a})$ *such that* A *is associated with* $(D(\mathfrak{a}_{\mathrm{reg}}), j)$.

Proof. The map $u \mapsto P_{V(\tilde{\mathfrak{a}})} q(u)$ from $D(\mathfrak{a})$ into $V(\tilde{\mathfrak{a}})$ is isometric and has dense range. Hence one can use $V(\tilde{\mathfrak{a}})$ as the completion of $(D(\mathfrak{a}), \|\cdot\|_{\mathfrak{a}_{\mathrm{reg}}})$. The corresponding continuous extension of j is $\tilde{j}|_{V(\tilde{\mathfrak{a}})}$. But $\tilde{j}|_{V(\tilde{\mathfrak{a}})}$ is injective. Therefore the form $\mathfrak{a}_{\mathrm{reg}}$ is j-closable.

To prove the uniqueness, let $\mathfrak{b}$ be a j-closable form with form domain $D(\mathfrak{b}) = D(\mathfrak{a})$ such that A is associated with $(\mathfrak{b}, j)$. We show that $\mathfrak{b} = \mathfrak{a}_{\mathrm{reg}}$. Denote the completion of $(D(\mathfrak{b}), \|\cdot\|_{\mathfrak{b}})$ by $(V_{\mathfrak{b}}, q_{\mathfrak{b}})$ and the continuous extensions of j and $\mathfrak{b}$ to $V_{\mathfrak{b}}$ by $\tilde{j}_{\mathfrak{b}}$ and $\tilde{\mathfrak{b}}$. We know that $\tilde{j}_{\mathfrak{b}}$ is injective, whence $V_{\mathfrak{b}}(\tilde{\mathfrak{b}}) = V_{\mathfrak{b}}$. Applying [AE12, Theorem 2.5], we obtain that

$$\mathfrak{b}(u,v) = \tilde{\mathfrak{b}}\big(q_{\mathfrak{b}}(u), q_{\mathfrak{b}}(v)\big) = \mathfrak{a}_c\big(\tilde{j}_{\mathfrak{b}}(q_{\mathfrak{b}}(u)), \tilde{j}_{\mathfrak{b}}(q_{\mathfrak{b}}(v))\big) = \mathfrak{a}_c\big(j(u), j(v)\big) = \mathfrak{a}_{\mathrm{reg}}(u,v)$$

for all $u, v \in D(\mathfrak{a})$, where $\mathfrak{a}_c$ denotes the unique closed sectorial form in H associated with the m-sectorial operator A. This finishes the proof. □

Remark 5.4. It is not hard to see that the j-regular part in general differs from the original form even in the complete case, i.e. if $D(\mathfrak{a})$ is a Hilbert space; see also the example below. For a detailed discussion of the complete case, see [AE12, Section 2]. If $D(\mathfrak{a})$ is a Hilbert space, then one can choose the sequence $(u_k)_{k\in\mathbb{N}}$ in Theorem 5.1 to be constant and the associated operator is much more easily defined. Thus one is tempted to require a sensible definition of the regular part to preserve a form in the complete case. In fact, this is possible by considering the decomposition

$$V = V(\tilde{\mathfrak{a}}) \oplus \overline{q(\ker j)} \oplus \big\{u \in \ker \tilde{j} : (\mathfrak{Re}\,\tilde{\mathfrak{a}})(u,v) = 0 \text{ for all } v \in q(\ker j)\big\}.$$

Here both sums are direct, but the they are not orthogonal in general. If $U(\tilde{\mathfrak{a}}) \coloneqq V(\tilde{\mathfrak{a}}) \oplus \overline{q(\ker j)}$ and $P_{U(\tilde{\mathfrak{a}})}$ denotes the projection onto $U(\tilde{\mathfrak{a}})$ along the above decomposition, a candidate for such an alternative notion of the regular part is the form $(u,v) \mapsto \tilde{\mathfrak{a}}\big(P_{U(\tilde{\mathfrak{a}})}q(u), P_{U(\tilde{\mathfrak{a}})}q(v)\big)$ with form domain $D(\mathfrak{a})$. This would also be consistent with Simon's definition if j is the inclusion map. In some sense, however, the complete case is not as nice as it seems, since a corresponding uniqueness statement to the one in Proposition 5.3 does not hold in this setting. This is substantiated by the following example. Therefore we decided to define the j-regular part as introduced before.

Example 5.5. Let H be a nontrivial Hilbert space and set $X \coloneqq H \times H$ with the natural inner product. Define $j \in \mathcal{L}(X, H)$ by $j(u_1, u_2) = u_1 + u_2$. Define the forms $\mathfrak{a}$ and $\mathfrak{b}$ with form domains $D(\mathfrak{a}) = D(\mathfrak{b}) = X$ by $\mathfrak{a}(u,v) = (u_1 \,|\, v_1)_H$ and $\mathfrak{b}(u,v) = (u_2 \,|\, v_2)_H$, where $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Then $\mathfrak{a}$ and $\mathfrak{b}$ are j-sectorial. Moreover, both $D(\mathfrak{a})$ and $D(\mathfrak{b})$ are Hilbert spaces such that $j(D(\mathfrak{a}))$ and $j(D(\mathfrak{b}))$ are dense in H. It is easy to verify that the zero operator is the operator associated with both $(\mathfrak{a}, j)$ and $(\mathfrak{b}, j)$. Moreover, $\ker j = \{(x, -x) : x \in H\}$ and $\mathfrak{a}(u,v) = \mathfrak{b}(u,v)$ for all $u, v \in \ker j$. But $\mathfrak{a} \neq \mathfrak{b}$. Thus there is no version of Proposition 5.3 for a regular part that preserves the complete case. Note that actually $V(\mathfrak{a}) = \{0\} \times H$, whereas $V(\mathfrak{b}) = H \times \{0\}$. Also $\mathfrak{a}_{\mathrm{reg}} = \mathfrak{b}_{\mathrm{reg}} = 0$. ◇

Motivated by (5.3), we are going to describe $P_{V(\tilde{\mathfrak{a}})}$ more explicitly. Set $\mathfrak{h} \coloneqq \mathfrak{Re}\,\mathfrak{a}$. Note that (V, q) is also the completion of the inner product space $D(\mathfrak{h})$. Let $\tilde{\mathfrak{h}}$ be the continuous extension of $\mathfrak{h}$ to V. Then $\mathfrak{Re}\,\tilde{\mathfrak{a}} = \tilde{\mathfrak{h}}$. Therefore the inner product on

V is given by

$$(u \,|\, v)_V = \tilde{\mathfrak{h}}(u, v) + (1-\gamma)\big(\tilde{j}(u) \,|\, \tilde{j}(v)\big)_H. \tag{5.4}$$

If we apply the above to $\mathfrak{h}$ instead of $\mathfrak{a}$, we obtain the *orthogonal* decomposition

$$V = \ker \tilde{j} \oplus V(\tilde{\mathfrak{h}}) =: V_1 \oplus V_2,$$

where $V_1 = \ker \tilde{j}$ and $V_2 = V(\tilde{\mathfrak{h}}) = (\ker \tilde{j})^{\perp}$. Denote the projections of V along this decomposition onto V_1 and V_2 by π_1 and π_2.

Lemma 5.6. *There exists a unique* $T \in \mathcal{L}(V, V_1)$ *such that*

$$(\mathfrak{Im}\,\tilde{\mathfrak{a}})(u, v) = \tilde{\mathfrak{h}}(Tu, v)$$

for all $u \in V$ *and* $v \in V_1$.

Proof. By the Riesz–Fréchet theorem, there exists a unique self-adjoint operator $R \in \mathcal{L}(V)$ such that

$$(\mathfrak{Im}\,\tilde{\mathfrak{a}})(u, v) = (Ru \,|\, v)_V$$

for all $u, v \in V$. Define $T := \pi_1 R \in \mathcal{L}(V, V_1)$. Then it follows from (5.4) that

$$(\mathfrak{Im}\,\tilde{\mathfrak{a}})(u, v) = (Tu \,|\, v)_V = \tilde{\mathfrak{h}}(Tu, v)$$

for all $u \in V$ and $v \in V_1$. This proves existence. Next, let $T' \in \mathcal{L}(V, V_1)$ and suppose that $(\mathfrak{Im}\,\tilde{\mathfrak{a}})(u, v) = \tilde{\mathfrak{h}}(T'u, v)$ for all $u \in V$ and $v \in V_1$. Then

$$0 = \tilde{\mathfrak{h}}((T - T')u, v) = \big((T-T')u \,|\, v\big)_V = \big((T-T')u \,|\, v\big)_{V_1}$$

for all $u \in V$ and $v \in V_1$, where the second equality again follows from (5.4). Thus $T = T'$. □

Let $T \in \mathcal{L}(V, V_1)$ be as in Lemma 5.6. Then $T_{11} := T|_{V_1}$ is self-adjoint since both $\tilde{\mathfrak{h}}$ and $\mathfrak{Im}\,\tilde{\mathfrak{a}}$ are symmetric. Therefore $(I_{V_1} + iT_{11}) \in \mathcal{L}(V_1)$ is invertible.

If $u \in V$, then obviously

$$u = \Big(i(I_{V_1} + iT_{11})^{-1} T\pi_2 u + \pi_1 u\Big) + \Big(\pi_2 u - i(I_{V_1} + iT_{11})^{-1} T\pi_2 u\Big).$$

Moreover, the first parenthesised summand clearly lies in $V_1 = \ker \tilde{j}$. We check that the second summand, which we abbreviate in the following calculation by w, lies in $V(\tilde{\mathfrak{a}})$. Let $v \in V_1$. Then

$$\begin{aligned}
\tilde{\mathfrak{a}}(w, v) &= \tilde{\mathfrak{h}}((\pi_1 I_V + iT)w, v) \\
&= \tilde{\mathfrak{h}}((\pi_1 I_V + iT)(\pi_1 w + \pi_2 w), v) \\
&= -\tilde{\mathfrak{h}}(iT\pi_2 u, v) + \tilde{\mathfrak{h}}(iT\pi_2 u, v) = 0.
\end{aligned}$$

So $w \in V(\tilde{\mathfrak{a}})$. Therefore

$$P_{V(\tilde{\mathfrak{a}})}u = w = \pi_2 u - i(I_{V_1} + iT_{11})^{-1}T\pi_2 u$$

for all $u \in V$. We substitute this into (5.3) and obtain

$$\begin{aligned}
\mathfrak{a}_{\mathrm{reg}}(u,v) &= \tilde{\mathfrak{a}}\big(P_{V(\tilde{\mathfrak{a}})}q(u), \pi_2 q(v) - i(I_{V_1} + iT_{11})^{-1}T\pi_2 q(v)\big) \\
&= \tilde{\mathfrak{a}}\big(\pi_2 q(u), \pi_2 q(v)\big) - i\tilde{\mathfrak{a}}\big((I_{V_1} + iT_{11})^{-1}T\pi_2 q(u), \pi_2 q(v)\big) \\
&= \tilde{\mathfrak{a}}\big(\pi_2 q(u), \pi_2 q(v)\big) - i\tilde{\mathfrak{h}}\big((I_{V_1} + iT_{11})^{-1}T\pi_2 q(u), \pi_2 q(v)\big) \\
&\quad + \tilde{\mathfrak{h}}\big((I_{V_1} + iT_{11})^{-1}T\pi_2 q(u), T\pi_2 q(v)\big) \\
&= \tilde{\mathfrak{a}}\big(\pi_2 q(u), \pi_2 q(v)\big) + \tilde{\mathfrak{h}}\big((I_{V_1} + iT_{11})^{-1}T\pi_2 q(u), T\pi_2 q(v)\big)
\end{aligned}$$

for all $u, v \in D(\mathfrak{a})$, where we used in the second step the definition of $V(\tilde{\mathfrak{a}})$ and that $P_{V(\tilde{\mathfrak{a}})}q(u) \in V(\tilde{\mathfrak{a}})$, in the third step the definition of T, and in the fourth step the orthogonality of the arguments in the second term.

We have proved the following.

Theorem 5.7. *Let $\mathfrak{a}$ be a sesquilinear form, H a Hilbert space and $j\colon D(\mathfrak{a}) \to H$ a linear map. Assume that $\mathfrak{a}$ is j-sectorial and $j(D(\mathfrak{a}))$ is dense in H. Let (V, q) be the completion of $D(\mathfrak{a})$. Let $\tilde{\mathfrak{a}}\colon V \times V \to \mathbb{C}$ and $\tilde{j}\colon V \to H$ be the continuous extensions of $\mathfrak{a}$ and j, respectively. Let $\tilde{\mathfrak{h}}$ be the real part of $\tilde{\mathfrak{a}}$. Let π_1 be the orthogonal projection of V onto $V_1 := \ker \tilde{j}$ and let $\pi_2 = I_V - \pi_1$. Then π_2 is the orthogonal projection of V onto $V(\tilde{\mathfrak{h}}) = \{u \in V : \tilde{\mathfrak{h}}(u,v) = 0 \text{ for all } v \in \ker \tilde{j}\}$. Moreover, there exists a unique operator $T \in \mathcal{L}(V, V_1)$ such that*

$$(\mathfrak{Im}\,\tilde{\mathfrak{a}})(u,v) = \tilde{\mathfrak{h}}(Tu, v)$$

for all $u \in V$ and $v \in V_1$. Then the operator $T_{11} := T|_{V_1} \in \mathcal{L}(V_1)$ is self-adjoint. Define the operator $\Pi \in \mathcal{L}(V)$ by

$$\Pi u = \pi_2 u - i(I_{V_1} + iT_{11})^{-1}T\pi_2 u. \tag{5.5}$$

Then the regular part of $\mathfrak{a}$ is given by

$$\begin{aligned}
\mathfrak{a}_{\mathrm{reg}}(u,v) &= \tilde{\mathfrak{a}}(\Pi q(u), \Pi q(v)) && (5.6) \\
&= \tilde{\mathfrak{a}}\big(\pi_2 q(u), \pi_2 q(v)\big) + \tilde{\mathfrak{h}}\big((I_{V_1} + iT_{11})^{-1}T\pi_2 q(u), T\pi_2 q(v)\big) && (5.7)
\end{aligned}$$

for all $u, v \in D(\mathfrak{a})$.

Remark 5.8. Note that the vertex γ of the form $\mathfrak{a}$ is not unique. For different admissible values of γ this in general leads to different Hilbert spaces V. These Hilbert spaces are, however, isomorphic as normed spaces. Therefore the continuous extensions $\tilde{\mathfrak{a}}$ and $\tilde{j}$ are independent of γ. Consequently also the operator T is independent of γ.

5.2 About the singular part and the real part of the regular part

Theorem 5.7 enables us to deduce a formula for the real part of the j-regular part of a j-sectorial form. Therefore we can characterise when it agrees with the regular part of the real part in terms of the uniquely determined operator T.

Proposition 5.9. *Assume the notation and conditions of Theorem 5.7. Let* $\mathfrak{h}$ *be the real part of the form* $\mathfrak{a}$. *Then*

$$\begin{aligned}\big(\mathfrak{Re}(\mathfrak{a}_{\mathrm{reg}})\big)(u,v) - \mathfrak{h}_{\mathrm{reg}}(u,v) &= \tilde{\mathfrak{h}}\big((I_{V_1} + T_{11}^2)^{-1} T\pi_2 q(u), T\pi_2 q(v)\big) \\ &= \big((I_{V_1} + T_{11}^2)^{-1/2} T\pi_2 q(u) \,|\, (I_{V_1} + T_{11}^2)^{-1/2} T\pi_2 q(v)\big)_V\end{aligned} \tag{5.8}$$

for all $u, v \in D(\mathfrak{a})$. *Moreover,* $\mathfrak{Re}(\mathfrak{a}_{\mathrm{reg}}) = \mathfrak{h}_{\mathrm{reg}}$ *if and only if* $T\pi_2 = 0$.

Proof. It follows from (5.3) applied to $\mathfrak{h}$ that

$$\mathfrak{h}_{\mathrm{reg}}(u,v) = \tilde{\mathfrak{h}}(\pi_2 q(u), \pi_2 q(v)) = (\mathfrak{Re}\,\tilde{\mathfrak{a}})(\pi_2 q(u), \pi_2 q(v)) \tag{5.9}$$

for all $u, v \in D(\mathfrak{a})$. Combining this with (5.7), one deduces that

$$\begin{aligned}\big(\mathfrak{Re}(\mathfrak{a}_{\mathrm{reg}})\big)(u,v) - \mathfrak{h}_{\mathrm{reg}}(u,v) &= \tfrac{1}{2}\tilde{\mathfrak{h}}\big((I_{V_1} + iT_{11})^{-1} T\pi_2 q(u), T\pi_2 q(v)\big) \\ &\quad + \tfrac{1}{2}\overline{\tilde{\mathfrak{h}}\big((I_{V_1} + iT_{11})^{-1} T\pi_2 q(v), T\pi_2 q(u)\big)}\end{aligned}$$

for all $u, v \in D(\mathfrak{a})$. Since $\tilde{\mathfrak{h}}$ is symmetric and

$$(I_{V_1} + iT_{11})^{-1} + (I_{V_1} - iT_{11})^{-1} = 2(I_{V_1} + T_{11}^2)^{-1},$$

one establishes that (5.8) is valid.

Finally, suppose $\mathfrak{Re}(\mathfrak{a}_{\mathrm{reg}}) = \mathfrak{h}_{\mathrm{reg}}$. Then (5.8) gives $\|(I_{V_1} + T_{11}^2)^{-1/2} T\pi_2 q(u)\|_V^2 = 0$ for all $u \in D(\mathfrak{a})$. This implies that $T\pi_2 q(u) = 0$ for all $u \in D(\mathfrak{a})$. The density of $q(D(\mathfrak{a}))$ in V yields $T\pi_2 = 0$. The converse direction is trivial. □

Remark 5.10. By (5.3) it is obvious that $\mathfrak{Re}(\mathfrak{a}_{\mathrm{reg}}) = \mathfrak{h}_{\mathrm{reg}}$ if and only if $\mathfrak{Re}(\mathfrak{a}_r) = \mathfrak{h}_r$. Also note that $\mathfrak{Re}(\mathfrak{a}_{\mathrm{reg}})(u,u) - \mathfrak{h}_{\mathrm{reg}}(u,u) \geqslant 0$ for all $u \in D(\mathfrak{a})$ by (5.8). Moreover, since T is bounded on V it follows from (5.8) and (5.9) that there exists a $C > 0$ such that

$$\mathfrak{h}_{\mathrm{reg}}(u,u) \leqslant (\mathfrak{Re}(\mathfrak{a}_{\mathrm{reg}}))(u,u) \leqslant C\big(\mathfrak{h}_{\mathrm{reg}}(u,u) + \|j(u)\|_H^2\big)$$

for all $u \in D(\mathfrak{a})$. Hence $D(\overline{\mathfrak{a}_r}) = D(\overline{\mathfrak{Re}(\mathfrak{a}_r)}) = D(\overline{\mathfrak{h}_r})$, which is Proposition 3.10 (iv) in [AE12]. In fact, $D(\overline{\mathfrak{a}_r}) = D(\overline{\mathfrak{h}_r}) = \tilde{j}(V)$.

Proposition 5.9 has inter alia an interesting consequence for the singular part of sectorial forms.

Corollary 5.11. *Assume the notation and conditions of Theorem 5.7. If* $\mathfrak{Re}(\mathfrak{a}_{\mathrm{reg}}) = \mathfrak{h}_{\mathrm{reg}}$, *then* $\mathfrak{a}_s$ *is* j*-sectorial with vertex* 0.

Proof. It follows from Proposition 5.9 that $T\pi_2 = 0$. Let $u, v \in D(\mathfrak{a})$. Then

$$\mathfrak{a}_{\mathrm{reg}}(u,v) = \tilde{\mathfrak{a}}\big(\pi_2 q(u), \pi_2 q(v)\big)$$

by Theorem 5.7. Moreover,

$$\tilde{\mathfrak{a}}\big(\pi_2 q(u), \pi_1 q(v)\big) = (\tilde{\mathfrak{h}} + i\,\mathfrak{Im}\,\tilde{\mathfrak{a}})\big(\pi_2 q(u), \pi_1 q(v)\big) = i\tilde{\mathfrak{h}}\big(T\pi_2 q(u), \pi_1 q(v)\big) = 0.$$

Similarly it follows that $\tilde{\mathfrak{a}}(\pi_1 q(u), \pi_2 q(v)) = 0$. Hence

$$\mathfrak{a}_s(u,v) = \mathfrak{a}(u,v) - \tilde{\mathfrak{a}}\big(\pi_2 q(u), \pi_2 q(v)\big) = \tilde{\mathfrak{a}}\big(\pi_1 q(u), \pi_1 q(v)\big).$$

Therefore $\mathfrak{a}_s$ is j-sectorial with vertex 0. □

We continue by studying further properties of the singular part. In the following lemma we provide formulas for the real and imaginary part of the singular part.

Lemma 5.12. *Assume the notation and conditions of Theorem 5.7. Then*

$$\big(\mathfrak{Re}\,(\mathfrak{a}_s)\big)(u,v) = \tilde{\mathfrak{h}}(\pi_1 q(u), \pi_1 q(v)) - \tilde{\mathfrak{h}}\big((I_{V_1} + T_{11}^2)^{-1} T\pi_2 q(u), T\pi_2 q(v)\big) \tag{5.10}$$

and

$$\begin{aligned}\big(\mathfrak{Im}\,(\mathfrak{a}_s)\big)(u,v) = {}& \tilde{\mathfrak{h}}(T\pi_1 q(u), \pi_1 q(v)) + \tilde{\mathfrak{h}}(T\pi_2 q(u), \pi_1 q(v)) \\ &+ \tilde{\mathfrak{h}}(\pi_1 q(u), T\pi_2 q(v)) \\ &+ \tilde{\mathfrak{h}}\big(T_{11}(I_{V_1} + T_{11}^2)^{-1} T\pi_2 q(u), T\pi_2 q(v)\big)\end{aligned} \tag{5.11}$$

for all $u, v \in D(\mathfrak{b})$.

Proof. The formulas follow from a straightforward calculation using (5.7). □

Next, we characterise when the singular part is j-sectorial.

Proposition 5.13. *Assume the notation and conditions of Theorem 5.7. Then* $\mathfrak{a}_s$ *is* j*-sectorial if and only if there exists an* $M > 0$ *such that*

$$\|T\pi_2 q(u)\|_V^2 \leqslant M \|j(u)\|_H^2 \tag{5.12}$$

for all $u \in D(\mathfrak{a})$.

Proof. First, suppose that $\mathfrak{b} := \mathfrak{a}_s$ is j-sectorial with vertex $-\omega_s$. Without loss of generality, we may assume that $\tilde{\mathfrak{h}}$ is the inner product on V and that $\omega_s > 0$. Then it follows from (5.10) that

$$\|u\|_{\mathfrak{b}}^2 = \|\pi_1 q(u)\|_V^2 + (1 + \omega_s)\|\tilde{j}(q(u))\|_H^2 - \|(I_{V_1} + T_{11}^2)^{-1/2} T\pi_2 q(u)\|_V^2 \tag{5.13}$$

for all $u \in D(\mathfrak{b})$. Let $v \in V_2$. There are $u_1, u_2, \ldots \in D(\mathfrak{b})$ such that $\lim q(u_n) = v$. Since $\|u_n\|_{\mathfrak{b}}^2 \geqslant 0$ for all $n \in \mathbb{N}$, it follows from (5.13) that

$$(1+\omega_s)\|\tilde{j}(v)\|_H^2 - \|(I_{V_1} + T_{11}^2)^{-1/2} T\pi_2 v\|_V^2 \geqslant 0.$$

Hence

$$\|Tv\|_V^2 \leqslant (1+\omega_s)\|I_{V_1} + T_{11}^2\| \|\tilde{j}(v)\|_H^2. \tag{5.14}$$

Note that $\tilde{j}(\pi_2 q(u)) = j(u)$ for all $u \in D(\mathfrak{b})$. Hence (5.12) follows from (5.14).

For the converse, note that we may assume without loss of generality that $\|u\|_V^2 = \tilde{\mathfrak{h}}(u,u)$ for all $u \in V$. By (5.10) and (5.12), there exists an $\omega_s > 0$ such that

$$\operatorname{Re} \mathfrak{b}(u,u) \geqslant \|\pi_1 q(u)\|_V^2 - \omega_s \|j(u)\|_H^2 \tag{5.15}$$

for all $u \in D(\mathfrak{a})$. By (5.11), we have

$$\begin{aligned} |\operatorname{Im} \mathfrak{b}(u,u)| \leqslant \|T\pi_1 q(u)\|_V \|\pi_1 q(u)\|_V + 2\|T\pi_2 q(u)\|_V \|\pi_1 q(u)\|_V \\ + \|T_{11}(I_{V_1} + T_{11}^2)^{-1}\| \|T\pi_2 q(u)\|_V^2 \end{aligned}$$

for all $u \in D(\mathfrak{a})$. Using (5.12) and taking $C > 0$ sufficiently large, we obtain

$$|\operatorname{Im} \mathfrak{b}(u,u)| \leqslant C\big(\|\pi_1 q(u)\|_V^2 + \|j(u)\|_H^2\big).$$

for all $u \in D(\mathfrak{a})$. By (5.15) this shows that $\mathfrak{a}_s = \mathfrak{b}$ is j-sectorial. □

If the singular part is j-sectorial, then by Theorem 5.1 it is also associated with some m-sectorial operator. Since the regular part should capture in some sense as much as possible of the original form, one expects that operator to be trivial. This is indeed the case.

Proposition 5.14. *Assume the notation and conditions of Theorem 5.7. Moreover, assume that $\mathfrak{a}_s$ is j-sectorial. Then $(\mathfrak{a}_s)_{\mathrm{reg}} = 0$.*

Proof. Assume that $\mathfrak{b} := \mathfrak{a}_s$ is sectorial with vertex $-\omega_s$. We prove the proposition by showing that the operator B associated with $(\mathfrak{b}, j)$ is zero. Without loss of generality, we may assume that $\tilde{\mathfrak{h}}$ is the inner product on V and that $\omega_s > 0$. It follows from (5.13) that

$$\|\pi_1 q(u)\|_V^2 \leqslant \|u\|_{\mathfrak{b}}^2 \tag{5.16}$$

for all $u \in D(\mathfrak{b})$.

Next, let $x \in D(B)$ and $f = Bx$. Then there exists a Cauchy sequence $(u_n)_{n\in\mathbb{N}}$ in $D(\mathfrak{b})$ such that $\lim j(u_n) = x$ in H and $\lim \mathfrak{b}(u_n, v) = (f \,|\, j(v))_H$ for all $v \in D(\mathfrak{b})$. Then also $\lim \tilde{j}(\pi_2 q(u_n)) = x$ in H. Hence $(T\pi_2 q(u_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in V_1 by (5.12) and $w_2 = \lim T\pi_2 q(u_n)$ exists in V_1. Similarly $w_1 = \lim \pi_1 q(u_n)$ exists in V_1 by (5.16). Using (5.10) and (5.11), we therefore obtain

$$\lim_{n\to\infty} (\mathfrak{Re}\, \mathfrak{b})(u_n, v) = \tilde{\mathfrak{h}}(w_1, \pi_1 q(v)) - \tilde{\mathfrak{h}}((I_{V_1} + T_{11}^2)^{-1} w_2, T\pi_2 q(v))$$

and

$$\lim_{n\to\infty} (\mathfrak{Im}\, \mathfrak{b})(u_n, v) = \tilde{\mathfrak{h}}(Tw_1, \pi_1 q(v)) + \tilde{\mathfrak{h}}(w_2, \pi_1 q(v)) + \tilde{\mathfrak{h}}(w_1, T\pi_2 q(v)) \\ + \tilde{\mathfrak{h}}(T(I_{V_1} + T_{11}^2)^{-1} w_2, T\pi_2 q(v))$$

for all $v \in D(\mathfrak{b}) = D(\mathfrak{a})$. This yields

$$\tilde{\mathfrak{h}}(w_1 + iTw_1 + iw_2, v) + \tilde{\mathfrak{h}}\big(iw_1 - (I_{V_1} + T_{11}^2)^{-1}(I - iT)w_2, T\pi_2 v\big) = \big(f \,|\, \tilde{j}(v)\big)_H \quad (5.17)$$

first for all $v \in q(D(\mathfrak{a}))$ and then by density for all $v \in V$. Choosing $v = w_1 + iTw_1 + iw_2$ gives $w_1 + iTw_1 + iw_2 = 0$. Therefore $w_2 = i(I + iT)w_1$. Substituting this into (5.17) shows $f = 0$. So $B = 0$. □

We next give an example of a sectorial form where the real part of the regular part differs from the regular part of the real part and, in addition, the singular part is still sectorial. We also provide an example such that in addition the form domain is a dense subset of H. We need a lemma.

Lemma 5.15. *Let X be an infinite dimensional, separable normed space. Then there exists a dense linear subspace W of $X \times X$ such that $\pi_1|_W$ is injective, where π_1 denotes the natural projection onto the first component in $X \times X$.*

Proof. Set $Z := X \times X$. Since X is separable, there is a sequence $(a_n)_{n\in\mathbb{N}}$ in Z such that $\{a_n : n \in \mathbb{N}\}$ is dense in Z. Choose $(x_1, y_1) \in Z \setminus (\{0\} \times X)$ such that $\|(x_1, y_1) - a_1\| < 1$. Let $n \in \mathbb{N}$ and suppose $(x_1, y_1), \dots, (x_n, y_n)$ are chosen. Then there exists an element $(x_{n+1}, y_{n+1}) \in Z \setminus (\operatorname{span}\{x_1, \dots, x_n\} \times X)$ such that $\|(x_{n+1}, y_{n+1}) - a_{n+1}\| < \frac{1}{n+1}$. Then $\{x_n : n \in \mathbb{N}\}$ is linearly independent and $\{(x_n, y_n) : n \in \mathbb{N}\}$ is dense in Z. Now choose $W := \operatorname{span}\{(x_n, y_n) : n \in \mathbb{N}\}$. □

Example 5.16. Let H be an infinite dimensional, separable Hilbert space. Let S be a densely defined, closed operator in H such that $\|Sx\| \geqslant \|x\|$ for all $x \in D(S)$. Set $V := D(S) \times D(S)$ and define $\tilde{\mathfrak{h}} \colon V \times V \to \mathbb{C}$ by

$$\tilde{\mathfrak{h}}\big((u_1, u_2), (v_1, v_2)\big) = (Su_1 \,|\, Sv_1)_H + (Su_2 \,|\, Sv_2)_H.$$

Note that $\tilde{\mathfrak{h}}$ is an inner product on V. We equip V with this inner product, which turns V into a Hilbert space. Define the linear map $\tilde{j} \colon V \to H$ by $\tilde{j}(u_1, u_2) = u_2$. Then $V_1 := \ker \tilde{j} = D(S) \times \{0\}$ and $V_2 := (\ker \tilde{j})^\perp = \{0\} \times D(S)$. Therefore the orthogonal projections from V onto V_1 and V_2, denoted by π_1 and π_2, are simply the respective coordinate projections. Define the form $\tilde{\mathfrak{a}}\colon V \times V \to \mathbb{C}$ by

$$\tilde{\mathfrak{a}}\big((u_1,u_2),(v_1,v_2)\big) = (S(u_1+iu_2) \,|\, Sv_1)_H + (S(u_2+iu_1) \,|\, Sv_2)_H.$$

Note that $\tilde{\mathfrak{a}}$ is $\tilde{j}$-sectorial and $D(\tilde{\mathfrak{a}})$ is a Hilbert space. Let $R = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right) \in \mathcal{L}(V)$. Then $\mathfrak{Re}\,\tilde{\mathfrak{a}} = \tilde{\mathfrak{h}}$ and

$$\tilde{\mathfrak{a}}(u,v) = \tilde{\mathfrak{h}}(u,v) + i\tilde{\mathfrak{h}}(Ru,v)$$

for all $u, v \in V$. By Lemma 5.15 there exists a dense linear subspace V_0 of V such that $\tilde{j}(V_0)$ is dense in H and $j := \tilde{j}|_{V_0}$ is injective. Consider the form $\mathfrak{a}\colon V_0 \times V_0 \to \mathbb{C}$ given by $\mathfrak{a} = \tilde{\mathfrak{a}}|_{V_0\times V_0}$. Then $\mathfrak{a}$ is a j-sectorial form with $j(D(\mathfrak{a}))$ dense in H. Moreover, if T is the operator from Theorem 5.7, then $T = \pi_1 R$. Since $T\pi_2 = \pi_1 R \pi_2 \neq 0$, we obtain that $\mathfrak{Re}(\mathfrak{a}_{\mathrm{reg}}) \neq (\mathfrak{Re}\,\mathfrak{a})_{\mathrm{reg}}$ by Proposition 5.9.

It follows from the construction, Theorem 5.7 and Proposition 5.9 that

$$(\mathfrak{Re}\,\mathfrak{a})_{\mathrm{reg}}\big((u_1,u_2),(v_1,v_2)\big) = (Su_2 \,|\, Sv_2)_H$$

and

$$\mathfrak{a}_{\mathrm{reg}}\big((u_1,u_2),(v_1,v_2)\big) = (\mathfrak{Re}(\mathfrak{a}_{\mathrm{reg}}))\big((u_1,u_2),(v_1,v_2)\big) = 2(Su_2 \,|\, Sv_2)_H$$

for all $(u_1,u_2),(v_1,v_2) \in V_0$. Hence the operator associated with, for example, $((\mathfrak{Re}\,\mathfrak{a})_{\mathrm{reg}}, j)$ is equal to S^*S.

It is now easy to give an example where the form domain is a subset of H. Define the form $\mathfrak{b}$ in H by setting $D(\mathfrak{b}) = j(D(\mathfrak{a}))$ and $\mathfrak{b}(j(u),j(v)) = \mathfrak{a}(u,v)$. Then $\mathfrak{b}$ is a densely defined sectorial form in H with $\mathfrak{Re}(\mathfrak{b}_{\mathrm{reg}}) \neq (\mathfrak{Re}\,\mathfrak{b})_{\mathrm{reg}}$.

For example, one may take $S = I_H$ in the above. Then $\mathfrak{b}_{\mathrm{reg}}(y_1,y_2) = 2(y_1 \,|\, y_2)_H$ and $\mathfrak{b}_{\mathrm{reg}}$ is continuous. Therefore $\mathfrak{b}_s = \mathfrak{b} - \mathfrak{b}_{\mathrm{reg}}$ is sectorial. Note that 0 is not a vertex for $\mathfrak{b}_s$. ◇

5.3 Notes and remarks

A different approach to the regular part of a positive symmetric form based on parallel sums is given in [HSS09]. Furthermore, we point out that, in the context of nonlinear phenomena and discontinuous media, the relaxation of a functional is a notion which in the linear setting corresponds to the regular part of positive symmetric forms [Mos94; Bra02; Dal93].

6

The regular part of second-order differential sectorial forms

In this chapter we present a formula for the regular part of a sectorial form which represents a general linear second-order differential expression, possibly including lower-order terms. Loosely speaking, such a differential expression has the form

$$-\sum_{k,l=1}^{d} \partial_l c_{kl} \partial_k + \sum_{k=1}^{d} b_k \partial_k - \sum_{k=1}^{d} \partial_k d_k + c_0.$$

The formula is given in terms of the original coefficients. It shows that the regular part is again a differential sectorial form.

In Section 6.1 we introduce notation and assumptions. In particular, we make precise what we mean by a 'differential sectorial form'. In Section 6.2 we derive the formula for the regular part of a differential sectorial form. This formula is the main result of this chapter. It is established using the results of Chapter 5 and the techniques introduced by Vogt in [Vog09]. In Section 6.3 we study when the singular part of a differential sectorial form is sectorial. We provide examples which show that the singular part is not always sectorial and that lower-order terms can introduce new behaviour that does not occur in the pure second-order case.

The material in this chapter is joint work with Tom ter Elst, see [ES11, Section 4] and [ES13]. We wish to thank El Maati Ouhabaz for raising the question of whether the formula for the regular part can be extended to allow lower-order terms.

6.1 The definition of differential sectorial forms

We introduce the form $\mathfrak{a}$ that is considered throughout this chapter. Let $\Omega \subset \mathbb{R}^d$ be open. Let $H = L^2(\Omega)$ and suppose $D(\mathfrak{a})$ is a vector subspace of H that contains $C_c^\infty(\Omega)$. Suppose that $\partial_k u \in L^1_{\mathrm{loc}}(\Omega)$ for all $k \in \{1, \dots, d\}$ and $u \in D(\mathfrak{a})$. For all $k, l \in \{1, \dots, d\}$, let c_{kl}, b_k, d_k and c_0 be measurable functions from Ω into $\mathbb{C}$. Suppose that $c_0 \in L^\infty(\Omega)$. Define $C \colon \Omega \to \mathbb{C}^{d\times d}$ by $C(x) = (c_{kl}(x))_{k,l=1}^d$ and $b, d \colon \Omega \to \mathbb{C}^d$ by $b(x) = (b_k(x))_{k=1}^d$ and $d(x) = (d_k(x))_{k=1}^d$. Suppose that $C\nabla u \cdot \nabla u \in L^1(\Omega)$ for all $u \in D(\mathfrak{a})$, where $\zeta \cdot \eta$ denotes the Euclidean inner product of $\zeta, \eta \in \mathbb{C}^d$. Moreover, suppose there exists a $\theta \in [0, \pi/2)$ such that $C(x)\xi \cdot \xi \in \Sigma_\theta$ for all $\xi \in \mathbb{C}^d$ and $x \in \Omega$. Define $a_{kl} := \frac{1}{2}(c_{kl} + \overline{c_{lk}})$ and $b_{kl} := \frac{1}{2i}(c_{kl} - \overline{c_{lk}})$ for all $k, l \in \{1, \dots, d\}$. Then $C = A + iB$, where $A = (a_{kl})$ and $B = (b_{kl})$. Both A and B can be considered as measurable maps from Ω into $\mathbb{C}^{d\times d}$ that have values in the Hermitian matrices. Note that $A(x)$ is a positive semi-definite Hermitian matrix for all $x \in \Omega$. Hence $A(x)$ admits a unique positive semi-definite square root $A^{1/2}(x)$ for all $x \in \Omega$. Furthermore, suppose there exists a $K > 0$ such that

$$|\overline{b(x)} \cdot \xi| \leqslant K\|A^{1/2}(x)\xi\|_{\mathbb{C}^d} \qquad \text{and} \qquad |d(x) \cdot \xi| \leqslant K\|A^{1/2}(x)\xi\|_{\mathbb{C}^d} \tag{6.1}$$

for all $\xi \in \mathbb{C}^d$ and $x \in \Omega$.

In the next lemma it is shown that the measurable map B can be suitably factorised 'modulo' $A^{1/2}$. This is based on the sectoriality of C and will allow us to effectively replace B by a bounded, measurable map. We also need that the map $x \mapsto A^{1/2}(x)$ is measurable from Ω into $\mathbb{C}^{d\times d}$, which will be proved first.

Lemma 6.1. *The maps $A^{1/2} \colon \Omega \to \mathbb{C}^{d\times d}$ and $P_{\ker} \colon \Omega \to \mathbb{C}^{d\times d}$ are measurable, where the matrix $P_{\ker}(x)$ is the orthogonal projection from $\mathbb{C}^d$ onto $\ker A(x)$ for all $x \in \Omega$. Moreover, there exists a unique bounded, measurable map $Z \colon \Omega \to \mathbb{C}^{d\times d}$ such that $Z(x)$ is Hermitian, $Z(x)P_{\ker}(x) = 0$ and $A^{1/2}(x)Z(x)A^{1/2}(x) = B(x)$ for all $x \in \Omega$. Finally, $B(x)P_{\ker}(x) = 0$ for all $x \in \Omega$.*

Proof. It is not hard to see that one can approximate the function $f \colon [0, \infty) \to [0, \infty)$ given by $f(t) = t^{1/2}$ pointwise by a sequence of real polynomials $(p_n)_{n\in\mathbb{N}}$. Since $A(x)$ is diagonalisable, it follows from [Gan59, Theorem V.4.1] that the sequence $(p_n(A(x)))_{n\in\mathbb{N}}$ converges for all $x \in \Omega$ to the positive semi-definite square root of $A(x)$. Therefore $A^{1/2}$ is measurable. Similarly the map $P_{\ker}$ is measurable since $P_{\ker}(x) = \mathbb{1}_{\{0\}}(A(x))$ for all $x \in \Omega$.

Define $g \colon [0, \infty) \to [0, \infty)$ by $g(t) = t^{-1/2}$ if $t > 0$ and $g(0) = 0$. Since one can approximate g pointwise by real polynomials and $A(x)$ is diagonalisable for all $x \in \Omega$, one deduces similarly that $x \mapsto g(A(x))$ is measurable from Ω into $\mathbb{C}^{d\times d}$.

Define $Z\colon \Omega \to \mathbb{C}^{d\times d}$ by

$$Z(x) \coloneqq g(A(x))B(x)g(A(x)).$$

Then Z is measurable from Ω into $\mathbb{C}^{d\times d}$.

By the sectoriality condition one has

$$|B\xi \cdot \xi| \leqslant (\tan\theta)\, A\xi \cdot \xi \tag{6.2}$$

for all $\xi \in \mathbb{C}^d$. Therefore for all $\xi, \eta \in \mathbb{C}^d$ one deduces that

$$|B\xi \cdot \eta| \leqslant (\tan\theta)\big(A\xi \cdot \xi\big)^{1/2}\big(A\eta \cdot \eta\big)^{1/2} \tag{6.3}$$

by [Kat80, Inequalities (1.15) in Section VI.1.2]. It is a straightforward consequence of (6.3) that $\ker A(x) \subset \ker B(x)$ for all $x \in \Omega$. Hence it follows that $B(x)P_{\ker}(x) = 0$ and $Z(x)P_{\ker}(x) = 0$ for all $x \in \Omega$.

As $fg = gf = \mathbb{1}_{(0,\infty)}$, $\mathbb{1}_{(0,\infty)}(A) + P_{\ker} = I$ and $BP_{\ker} = 0$ we obtain

$$A^{1/2}ZA^{1/2} = \mathbb{1}_{(0,\infty)}(A)B\mathbb{1}_{(0,\infty)}(A) = B \tag{6.4}$$

pointwise on Ω. If $\xi \in \mathbb{C}^d$, then it follows from (6.2) that

$$\begin{aligned}|Z\xi \cdot \xi| &= |g(A)Bg(A)\xi \cdot \xi| \\ &\leqslant (\tan\theta)\big(A^{1/2}g(A)\xi \cdot A^{1/2}g(A)\xi\big) = (\tan\theta)\, |\mathbb{1}_{(0,\infty)}(A)\xi|^2 \leqslant (\tan\theta)\, |\xi|^2\end{aligned}$$

pointwise on Ω. Hence Z is bounded by $\tan\theta$. Finally, the uniqueness of the map Z is easily deduced. □

Next we suitably factorise the maps b and d.

Lemma 6.2. *Let $A^{1/2}$ and $P_{\ker}$ be as in Lemma 6.1. There exist unique bounded, measurable maps $X, Y\colon \Omega \to \mathbb{C}^d$ such that $A^{1/2}(x)X(x) = \overline{b(x)}$, $P_{\ker}(x)X(x) = 0$, $A^{1/2}(x)Y(x) = d(x)$ and $P_{\ker}(x)Y(x) = 0$ for all $x \in \Omega$.*

Proof. Let g be as in the proof of Lemma 6.1. Then, as before, the map $x \mapsto g(A(x))$ is measurable from Ω into $\mathbb{C}^{d\times d}$. Therefore also the map $X\colon \Omega \to \mathbb{C}^d$ defined by $X(x) = g(A(x))\overline{b(x)}$ is measurable. Moreover,

$$|X(x) \cdot \xi| = |\overline{b(x)} \cdot g(A(x))\xi| \leqslant K\|A^{1/2}(x)g(A(x))\xi\|_{\mathbb{C}^d} \leqslant K\|\xi\|_{\mathbb{C}^d}$$

for all $\xi \in \mathbb{C}^d$ and $x \in \Omega$. This proves that X is bounded. Then arguing as in (6.4), one deduces $A^{1/2}(x)X(x) = \overline{b(x)}$. Existence and boundedness of Y are proved similarly. The uniqueness of X and Y is easily deduced. □

In the following, let $A^{1/2}$ and Z be as in Lemma 6.1, and let X and Y be as in Lemma 6.2. We define the form $\mathfrak{a}\colon D(\mathfrak{a}) \times D(\mathfrak{a}) \to \mathbb{C}$ by

$$\mathfrak{a}(u,v) = \int_\Omega \sum_{k,l=1}^{d} c_{kl}(\partial_k u)\overline{\partial_l v} + \int_\Omega \sum_{k=1}^{d} b_k(\partial_k u)\overline{v} + \int_\Omega \sum_{k=1}^{d} d_k u\,\overline{\partial_k v} + \int_\Omega c_0 u\overline{v}. \quad (6.5)$$

Using the measurable maps $A^{1/2}$, Z, X and Y, we obtain

$$\mathfrak{a}(u,v) = \big((I+iZ)A^{1/2}\nabla u \,|\, A^{1/2}\nabla v\big) + \big(A^{1/2}\nabla u \,|\, vX\big) + \big(uY \,|\, A^{1/2}\nabla v\big) + (c_0 u \,|\, v) \quad (6.6)$$

for all $u, v \in D(\mathfrak{a})$. In particular, it follows that the first-order terms in (6.5) are indeed integrable. Since X, Y and Z are bounded, the next lemma follows from (6.6).

Lemma 6.3. *The form $\mathfrak{a}$ is a sectorial form in $L^2(\Omega)$.*

Let $\mathfrak{b}$ be a sesquilinear form in $L^2(\Omega)$. If $\mathfrak{b}$ is equal to the form $\mathfrak{a}$ for an appropriate choice of the coefficient functions c_{kl}, b_k, d_k and c_0 with $k, l \in \{1, \dots, d\}$, then we shall call $\mathfrak{b}$ a **differential sectorial form**. The main result of this chapter establishes that, under suitable mild conditions which we next introduce, the regular part of $\mathfrak{a}$ is also a differential sectorial form.

We introduce two conditions. We say that $\mathfrak{a}$ satisfies **Condition (L)** if

(i) $D(\mathfrak{a}) \cap L^\infty(\Omega)$ is invariant under multiplication with $C_c^\infty(\Omega;\mathbb{R})$ functions,

(ii) there exists a $\psi \in C_b^1(\mathbb{R};\mathbb{R})$ such that $\psi(0) = 0$, $\psi'(0) = 1$, and $\psi \circ (\operatorname{Re} u) + i\psi \circ (\operatorname{Im} u) \in D(\mathfrak{a})$ for all $u \in D(\mathfrak{a})$, and,

(iii) $c_{kl} + \overline{c_{lk}}$ is real-valued for all $k, l \in \{1, \dots, d\}$.

This is the condition introduced in [Vog09], adapted to complex vector spaces. For real second-order coefficients and if $D(\mathfrak{a}) = H^1(\Omega)$ or $D(\mathfrak{a}) = H_0^1(\Omega)$, the form $\mathfrak{a}$ satisfies Condition (L) by [GT01, Sections 7.4 and 7.5]. Furthermore, we say that $\mathfrak{a}$ satisfies **Condition (B)** if

(i) $D(\mathfrak{a})$ is invariant under multiplication with $C_c^\infty(\Omega;\mathbb{R})$ functions, and,

(ii) $c_{kl} \in L_{loc}^\infty(\Omega)$ for all $k, l \in \{1, \dots, d\}$.

6.2 The formula for the regular part

We use the notation as introduced in Section 6.1. In particular, we assume that the form $\mathfrak{a}$ is as in (6.5).

In this section we derive a formula for the regular part of the form $\mathfrak{a}$. To this end, we assume that $\mathfrak{a}$ satisfies Condition (L) or (B). It will be immediate from the obtained formula that both the regular and singular part of $\mathfrak{a}$ continue to be *differential* sectorial forms. Note that this is not at all clear as the definition of the regular part is rather abstract.

To be able to make use of Theorem 5.7, we must first construct a suitable completion of the pre-Hilbert space $(D(\mathfrak{a}), \|\cdot\|_{\mathfrak{a}})$ that allows us to get hold of the corresponding continuous extensions of $\mathfrak{a}$ and of the embedding of $D(\mathfrak{a})$ into H. We note that the embedding of $D(\mathfrak{a})$ into H corresponds to the map j in Theorem 5.7.

Let $\mathfrak{h}$ be the real part of $\mathfrak{a}$. Then

$$\begin{aligned}\mathfrak{h}(u,v) = \big(A^{1/2}\nabla u \,|\, A^{1/2}\nabla v\big) + \tfrac{1}{2}\big(A^{1/2}\nabla u \,|\, v(X+Y)\big) \\ + \tfrac{1}{2}\big(u(X+Y) \,|\, A^{1/2}\nabla v\big) + ((\operatorname{Re} c_0)u \,|\, v)\end{aligned}$$

for all $u, v \in D(\mathfrak{a})$. Let $\mathcal{H}$ be the Hilbert space $L^2(\Omega) \times (L^2(\Omega))^d$ with the usual inner product. Let $\gamma_0 \in \mathbb{R}$ be a vertex of the sectorial form $\mathfrak{a}$. Since X and Y are bounded, there exists a $\gamma \leqslant \gamma_0$ such that the sesquilinear form $\langle\cdot,\cdot\rangle \colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ defined by

$$\begin{aligned}\big\langle (u_1, w_1), (u_2, w_2) \big\rangle = (w_1 \,|\, w_2) + \tfrac{1}{2}(w_1 \,|\, u_2(X+Y)) \\ + \tfrac{1}{2}(u_1(X+Y) \,|\, w_2) + ((1 - \gamma + \operatorname{Re} c_0)u_1 \,|\, u_2)\end{aligned}$$

defines an equivalent inner product on $\mathcal{H}$. Note that γ is also a vertex of $\mathfrak{a}$. We shall fix this value of γ and use it in $\|\cdot\|_{\mathfrak{a}}$, see (5.2).

Let $\mathcal{H}'$ denote the space $L^2(\Omega) \times (L^2(\Omega))^d$ equipped with the inner product $\langle\cdot,\cdot\rangle$. Define the map $\Phi \colon (D(\mathfrak{a}), \|\cdot\|_{\mathfrak{a}}) \to \mathcal{H}'$ by

$$\Phi(u) = (u, A^{1/2}\nabla u).$$

Then Φ is an isometry. Hence the completion V of $D(\mathfrak{a})$ can be realised as the closure of $\Phi(D(\mathfrak{a}))$ in $\mathcal{H}'$ equipped with the inner product of $\mathcal{H}'$. In particular, the map Φ corresponds to the map q in Theorem 5.7. Note that V is also equal to the closure of $\Phi(D(\mathfrak{a}))$ in $\mathcal{H}$. The map $\tilde{j} \colon V \to H$ defined by $\tilde{j}(u, w) = u$ is the continuous extension of the embedding of $D(\mathfrak{a})$ into H. Furthermore, due to (6.6) the continuous extension of the form $\mathfrak{a}$ is the form $\tilde{\mathfrak{a}} \colon V \times V \to \mathbb{C}$ given by

$$\tilde{\mathfrak{a}}((u_1, w_1), (u_2, w_2)) = ((I + iZ)w_1 \,|\, w_2) + (w_1 \,|\, u_2 X) + (u_1 Y \,|\, w_2) + (c_0 u_1 \,|\, u_2). \tag{6.7}$$

Then the real part $\tilde{\mathfrak{h}}$ of $\tilde{\mathfrak{a}}$ is given by

$$\begin{aligned}\tilde{\mathfrak{h}}((u_1, w_1), (u_2, w_2)) = (w_1 \,|\, w_2) + \tfrac{1}{2}(w_1 \,|\, u_2(X+Y)) \\ + \tfrac{1}{2}(u_1(X+Y) \,|\, w_2) + ((\operatorname{Re} c_0)u_1 \,|\, u_2).\end{aligned} \tag{6.8}$$

Note that

$$\big\langle (u_1, w_1), (u_2, w_2) \big\rangle = ((u_1, w_1) \,|\, (u_2, w_2))_{\tilde{\mathfrak{h}}}$$

for all $(u_1, w_1), (u_2, w_2) \in V$.

Next, set $V_1 \coloneqq \ker \tilde{j}$ and $V_2 \coloneqq V_1^\perp$. Let π_1 and π_2 be the orthogonal projections in V onto V_1 and V_2, respectively. We shall show that π_1 and π_2 can be represented by multiplication operators in $\mathcal{H}$. To this end, let V_s be such that $V_s \subset (L^2(\Omega))^d$ and $V_1 = \{0\} \times V_s$. Clearly V_s is a closed subspace of $(L^2(\Omega))^d$. Consider the orthogonal projection from $(L^2(\Omega))^d$ onto V_s. If Condition (L) is valid, then it follows as in [Vog09, Proof of Theorem 1] that this projection is the multiplication operator associated with a measurable function $Q \colon \Omega \to \mathbb{C}^{d \times d}$ that has values in the orthogonal projection matrices. Alternatively, also Condition (B) suffices for this conclusion by an inspection of the proof in [Vog09]. To be more self-contained, we provide the proof for this case.

Lemma 6.4. *Suppose that Condition* (B) *is satisfied. Let* P *be the orthogonal projection from* $(L^2(\Omega))^d$ *onto* V_s. *Then there exists a measurable function* $Q \colon \Omega \to \mathbb{C}^{d \times d}$ *that has values in the orthogonal projection matrices such that* P *is the multiplication operator in* $(L^2(\Omega))^d$ *associated with* Q.

Proof. Let $w \in V_s$ and $\varphi \in C_c^\infty(\Omega; \mathbb{R})$. Then there exists a sequence (u_n) in $D(\mathfrak{a})$ such that $\Phi(u_n) \to (0, w)$ in $\mathcal{H}$ for $n \to \infty$. Hence $u_n \to 0$ in $L^2(\Omega)$ and $A^{1/2} \nabla u_n \to w$ in $(L^2(\Omega))^d$. After going to a subsequence, we may assume that $u_n \to 0$ pointwise a.e. By assumption, $\varphi u_n \in D(\mathfrak{a})$ for all $n \in \mathbb{N}$. Moreover, it is clear that $\varphi u_n \to 0$ in $L^2(\Omega)$. Next, observe that

$$A^{1/2} \nabla(\varphi u_n) = u_n A^{1/2} \nabla \varphi + \varphi A^{1/2} \nabla u_n$$

for all $n \in \mathbb{N}$. By Condition (B) the function $A \nabla \varphi \cdot \nabla \varphi$ is bounded. So $u_n A^{1/2} \nabla \varphi \to 0$ in $(L^2(\Omega))^d$. Hence $A^{1/2} \nabla(\varphi u_n) \to \varphi w$ in $(L^2(\Omega))^d$. This proves that $\varphi w \in V_s$.

Let $f \in (L^2(\Omega))^d$ and define $M \coloneqq [f \neq 0]$. There exists a sequence (φ_n) in $C_c^\infty(\Omega; \mathbb{R})$ such that $\varphi_n \to \mathbb{1}_M$ a.e. and $\|\varphi_n\|_\infty \leqslant 1$ for all $n \in \mathbb{N}$. It follows that $\varphi_n P f \to \mathbb{1}_M P f$ in $(L^2(\Omega))^d$. Since V_s is closed, this implies that $\mathbb{1}_M P f \in V_s$ by the first part of the proof. Then

$$\|f - \mathbb{1}_M P f\|_{(L^2(\Omega))^d} = \|\mathbb{1}_M (f - P f)\|_{(L^2(\Omega))^d} \leqslant \|f - P f\|_{(L^2(\Omega))^d}.$$

Hence $\mathbb{1}_M P f = P f$. This shows that P is a local operator in the sense of [AT05]. It follows from the vector-valued version of Zaanen's theorem [AT05, Theorem 2.3] that there exists a measurable function $Q \colon \Omega \to \mathbb{C}^{d \times d}$ such that $Pf(x) = Q(x) f(x)$ for all $f \in (L^2(\Omega))^d$ and a.e. $x \in \Omega$. By [AT05, Corollary 2.4], one has $|Q(x)| \leqslant 1$ and $Q(x) = Q^2(x)$ for a.e. $x \in \Omega$. Hence one may assume that Q has values in the orthogonal projection matrices. □

The map Q is determined up to a set of measure zero. We point out that Q only depends on the second-order coefficients $(c_{kl})_{k,l=1}^d$ and the form domain of $\mathfrak{a}$. Now we are able to state the main result of this chapter.

Theorem 6.5. *Let* $\mathfrak{a}$ *be defined as in Section 6.1. Assume that* $\mathfrak{a}$ *satisfies Condition* (L) *or* (B). *Let* $Q\colon \Omega \to \mathbb{C}^{d\times d}$ *be as before and set* $W := Q(I + iQZQ)^{-1}Q$ *and* $P := I - Q$. *Then the regular part of* $\mathfrak{a}$ *is given by*

$$\begin{aligned}\mathfrak{a}_{\mathrm{reg}}(u,v) &= \big((I + iZ + ZWZ)PA^{1/2}\nabla u \,|\, PA^{1/2}\nabla v\big)\\ &\quad + \big(\overline{X}^t(I - iWZ)PA^{1/2}\nabla u \,|\, v\big) + \big(u \,|\, \overline{Y}^t(I + iW^*Z)PA^{1/2}\nabla v\big)\\ &\quad - \big((\overline{X}^t WY)u \,|\, v\big) + (c_0 u \,|\, v).\end{aligned} \tag{6.9}$$

Proof. We will use and adopt the notation of Theorem 5.7. First we represent the operators π_2, $T\pi_2$ and $(I_{V_1} + iT_{11})^{-1}$ by multiplication operators.

Observe that $(u,w) \mapsto (0, w + \frac{1}{2}u(X+Y))$ is the orthogonal projection of $\mathcal{H}'$ onto $\{0\} \times (L^2(\Omega))^d$. Moreover, considering $\{0\} \times (L^2(\Omega))^d$ as a (closed) subspace of $\mathcal{H}'$, the map $(0,w) \mapsto (0, Qw)$ is the orthogonal projection of $\{0\} \times (L^2(\Omega))^d$ onto V_1. Since $V_1 \subset V \subset \mathcal{H}'$, the map π_1 is given by

$$\pi_1(u,w) = \big(0, Qw + \tfrac{1}{2}uQ(X+Y)\big)$$

for all $(u,w) \in V$. Therefore

$$\pi_2(u,w) = \big(u, (I-Q)w - \tfrac{1}{2}uQ(X+Y)\big)$$

for all $(u,w) \in V$.

Let $(u_1,w_1),(u_2,w_2) \in V$. It follows from (6.7) that

$$\begin{aligned}(\mathfrak{Im}\,\tilde{\mathfrak{a}})\big((u_1,w_1),(u_2,w_2)\big) &= (Zw_1 \,|\, w_2) - \tfrac{i}{2}(w_1 \,|\, u_2(X-Y))\\ &\qquad + \tfrac{i}{2}(u_1(X-Y) \,|\, w_2) + ((\operatorname{Im} c_0)u_1 \,|\, u_2).\end{aligned}$$

So if $(u_2,w_2) \in V_1$, then $u_2 = 0$ and

$$\begin{aligned}(\mathfrak{Im}\,\tilde{\mathfrak{a}})\big((u_1,w_1),(0,w_2)\big) &= (Zw_1 \,|\, w_2) + \tfrac{i}{2}(u_1(X-Y) \,|\, w_2)\\ &= \big(QZw_1 + \tfrac{i}{2}u_1 Q(X-Y) \,|\, w_2\big)\\ &= \tilde{\mathfrak{h}}\big((0, QZw_1 + \tfrac{i}{2}u_1 Q(X-Y)),(0,w_2)\big).\end{aligned}$$

Hence the operator $T \in \mathcal{L}(V,V_1)$ in Theorem 5.7 is given by

$$T(u,w) = \big(0, QZw + \tfrac{i}{2}uQ(X-Y)\big)$$

for all $(u,w) \in V$. Then $T_{11} = T|_{V_1}$ is given by $T_{11}(0,w) = (0, QZw)$ for all $(0,w) \in V_1$. If $w \in V_s$, then $(I_{V_1} + iT_{11})(0,w) = (0,(I + iQZ)w) = (0,(I+iQZQ)w)$. Observe that the map $w \mapsto (I + iQZQ)w$ is invertible both as a map from V_s into V_s and as a map from $(L^2(\Omega))^d$ into $(L^2(\Omega))^d$ since Z has values in the Hermitian matrices.

Therefore $(I + iQZQ)^{-1}w \in V_s$ and $(I_{V_1} + iT_{11})(0, (I + iQZQ)^{-1}w) = (0, w)$ for all $w \in V_s$. So we have

$$(I_{V_1} + iT_{11})^{-1}(0, w) = \big(0, (I + iQZQ)^{-1}w\big) = (0, Ww)$$

for all $(0, w) \in V_1$. Next note that

$$T\pi_2(u, w) = \big(0, QZ(I - Q)w - \tfrac{1}{2}uQZQ(X + Y) + \tfrac{i}{2}uQ(X - Y)\big) \tag{6.10}$$

for all $(u, w) \in V$.

Now we plug the representations for π_2, $T\pi_2$ and $(I_{V_1} + iT_{11})^{-1}$ into (5.5). First note that

$$i(W - Q) = iQ\big((I + iQZQ)^{-1} - I\big)Q = Q(I + iQZQ)^{-1}QZQ = WZQ. \tag{6.11}$$

Similarly

$$i(W - Q) = QZW. \tag{6.12}$$

Then, by a straightforward computation, it follows from (6.11) and (5.5) that

$$\Pi(u, w) = \big(u, (I - iWZ)Pw - uWY\big)$$

for all $(u, w) \in V$, where Π is as in Theorem 5.7. Let $(u_1, w_1), (u_2, w_2) \in V$. Then

$$\begin{aligned}
&\tilde{a}\big(\Pi(u_1, w_1), \Pi(u_2, w_2)\big) \\
&\quad = ((I + iZW^*)(I + iZ)(I - iWZ)Pw_1 \,|\, Pw_2) \\
&\qquad - (Q(I + iZ)(I - iWZ)Pw_1 \,|\, u_2WY) + ((I - iWZ)Pw_1 \,|\, u_2X) \\
&\qquad - (u_1Y \,|\, W^*(I - iZ)(I - iWZ)Pw_2) + (u_1Y \,|\, (I - iWZ)Pw_2) \\
&\qquad + (u_1Q(I + iZ)WY \,|\, u_2WY) - (u_1WY \,|\, u_2X) - (u_1QY \,|\, u_2WY) + (c_0u_1 \,|\, u_2).
\end{aligned} \tag{6.13}$$

We simplify (6.13). It follows from (6.12) that

$$Q(I + iZ)(I - iWZ)P = 0. \tag{6.14}$$

This simplifies the first-order terms in (6.13) that involve w_1 and u_2. Using $W^* = W^*Q$, (6.14) and $PW = 0$, one establishes that

$$P(I + iZW^*)(I + iZ)(I - iWZ)P = P(I + iZ)(I - iWZ)P = P(I + iZ + ZWZ)P.$$

This simplifies the second-order terms in (6.13). Moreover, one readily verifies that $(I + iQZQ)^{-1}$ maps the range of Q into itself. So

$$(I + iQZQ)^{-1}Q = Q(I + iQZQ)^{-1}Q.$$

Therefore

$$W^* + W = 2Q(I - iQZQ)^{-1}(I + iQZQ)^{-1}Q = 2W^*W. \tag{6.15}$$

Since $W^*P = 0$, one deduces that

$$-W^*(I - iZ)(I - iWZ)P = iW^*WZP + iW^*ZP + W^*ZWZP.$$

But

$$W^*ZWZP = W^*(QZW)ZP = iW^*(W - Q)ZP = iW^*WZP - iW^*ZP,$$

where we used (6.12) in the second step. Together with (6.15) one establishes that

$$-W^*(I - iZ)(I - iWZ)P + (I - iWZ)P = \big(I + i(2W^*W - W)Z\big)P = (I + iW^*Z)P.$$

This simplifies the first-order terms that involve u_1 and w_2. Finally, for the terms involving u_1 and u_2, observe that

$$Q(I + iZ)W - Q = W + iQZW - Q = 0$$

by (6.12). Now the theorem follows from (6.13) and Theorem 5.7. □

Theorem 6.5 shows that the regular part $\mathfrak{a}_{\mathrm{reg}}$ is indeed a differential sectorial form. The most remarkable aspect of (6.9) is the appearance of the zeroth-order term involving X and Y, even if $c_0 = 0$, i.e., if the original form $\mathfrak{a}$ did not have a zeroth-order term. This new term can affect the vertex of the singular part. A simple concrete example where this happens is given in Example 6.11.

We make a brief remark about the function Q. Clearly, the form $\mathfrak{a}$ is closable if and only if $Q = 0$ a.e. Moreover, it is not hard to see that if U is an open subset of Ω and $\mathfrak{a}$ is strongly elliptic on U, then $Q(x) = 0$ for a.e. $x \in U$.

6.3 About the sectoriality of the singular part

We suppose that $\mathfrak{a}$ is a second-order differential sectorial form as defined in Section 6.1. Moreover, we assume that $\mathfrak{a}$ satisfies the conditions of Theorem 6.5, i.e., we assume that $\mathfrak{a}$ satisfies Condition (L) or (B). In this section we characterise in various ways when the singular part $\mathfrak{a}_s = \mathfrak{a} - \mathfrak{a}_{\mathrm{reg}}$ of the form $\mathfrak{a}$ is sectorial. We will see that the presence of lower-order terms can lead to a more diverse behaviour than possible in the pure second-order case.

In the following, we shall use the notation of Theorem 6.5. In particular, $P = I - Q$. Let $\mathfrak{a}^p$ be the differential sectorial form that belongs to the pure second-order differential expression

$$-\sum_{k,l=1}^{d} \partial_l c_{kl} \partial_k.$$

We denote the regular part and the singular part of $\mathfrak{a}^p$ by $\mathfrak{a}^p_{\mathrm{reg}}$ and $\mathfrak{a}^p_s$, respectively. By applying Theorem 6.5 to both $\mathfrak{a}$ and $\mathfrak{a}^p$, we obtain

$$\begin{aligned}\mathfrak{a}_{\mathrm{reg}}(u,v) = \mathfrak{a}^p_{\mathrm{reg}}(u,v) &+ \big((I - iWZ)PA^{1/2}\nabla u \,|\, vX\big) + \big(uY \,|\, (I + iW^*Z)PA^{1/2}\nabla v\big) \\ &- \big(uWY \,|\, vX\big) + (c_0 u \,|\, v)\end{aligned} \tag{6.16}$$

for all $u, v \in D(\mathfrak{a})$. Therefore with (6.6) one deduces that

$$\begin{aligned}\mathfrak{a}_s(u,v) = \mathfrak{a}^p_s(u,v) &+ \big((Q + iWZP)A^{1/2}\nabla u \,|\, vX\big) + \big(uY \,|\, (Q - iW^*ZP)A^{1/2}\nabla v\big) \\ &+ (uWY \,|\, vX)\end{aligned} \tag{6.17}$$

for all $u, v \in D(\mathfrak{a})$.

We need the following lemma.

Lemma 6.6. *If* $QZ(I-Q)A^{1/2} = 0$ *a.e., then* $QZ = ZQ$ *a.e.*

Proof. Let P_{ker} be as in Lemma 6.1. If $u \in D(\mathfrak{a})$, then $P_{\mathrm{ker}}A^{1/2}\nabla u = 0$ by definition of P_{ker}. So by density $P_{\mathrm{ker}}w = 0$ for all $(u, w) \in V$. In particular, $P_{\mathrm{ker}}w = 0$ for all $w \in V_s$ and $P_{\mathrm{ker}}Qw = 0$ for all $w \in (L^2(\Omega))^d$. This implies that $P_{\mathrm{ker}}(x)Q(x) = 0$ for a.e. $x \in \Omega$. It follows by duality that $Q(x)P_{\mathrm{ker}}(x) = 0$ for a.e. $x \in \Omega$.

By assumption there exists a null set $N \subset \Omega$ such that $Q(x)Z(x)(I - Q(x))A(x)\xi = 0$ for all $x \in \Omega \setminus N$ and $\xi \in \mathbb{C}^d$. Then $Q(x)Z(x)(I - Q(x))\xi = 0$ for all $x \in \Omega \setminus N$ and $\xi \in \operatorname{rg} A(x)$. But $Q(x)P_{\mathrm{ker}}(x) = 0 = Z(x)P_{\mathrm{ker}}(x)$ for a.e. $x \in \Omega$. So $Q(x)Z(x)(I - Q(x))\xi = 0$ for a.e. $x \in \Omega$ and $\xi \in \mathbb{C}^d$. Therefore $QZ(I - Q) = 0$ a.e. By taking the adjoint we obtain that $(I - Q)ZQ = 0$ a.e. This yields $QZ = ZQ$ a.e. □

Now we are ready to characterise when the singular part of $\mathfrak{a}$ is sectorial.

Proposition 6.7. *The following statements are equivalent.*

(i) $\mathfrak{a}_s$ *is sectorial.*

(ii) $QZ = ZQ$ *a.e.*

(iii) *For all* $u \in D(\mathfrak{a})$ *one has*

$$T\pi_2\Phi(u) = \big(0, -\tfrac{1}{2}uQZQ(X+Y) + \tfrac{i}{2}uQ(X-Y)\big).$$

(iv) *For all* $u, v \in D(\mathfrak{a})$ *one has*

$$\mathfrak{a}^p_{\mathrm{reg}}(u, v) = \big((I + iZ)(I - Q)A^{1/2}\nabla u \,|\, (I - Q)A^{1/2}\nabla v\big). \tag{6.18}$$

(v) $\mathfrak{a}^p_s$ *is sectorial.*

Proof. '(i)⇒(ii)': Let $\tau \in C_c^\infty(\Omega; \mathbb{R})$ and $\xi \in \mathbb{R}^d$. For all $\lambda > 0$ define $u_\lambda \in C_c^\infty(\Omega)$ by $u_\lambda(x) = e^{i\lambda x\cdot\xi}\tau(x)$. Since $\mathfrak{a}_s$ is sectorial, it follows from Proposition 5.13 that there exists an $M > 0$ such that

$$\tilde{\mathfrak{h}}(T\pi_2\Phi(u_\lambda), T\pi_2\Phi(u_\lambda)) \leqslant M\|u_\lambda\|^2_{L^2(\Omega)}.$$

Expanding the terms using (6.8) and (6.10) gives

$$\int_\Omega |QZ(I - Q)A^{1/2}(i\lambda\tau\xi + \nabla\tau) + \tfrac{1}{2}\tau QZQ(X + Y) + \tfrac{i}{2}\tau Q(X - Y)|^2 \leqslant M\|\tau\|^2_{L^2(\Omega)}.$$

Dividing both sides by λ^2 and letting $\lambda \to \infty$ shows that $\tau QZ(I - Q)A^{1/2}\xi = 0$ a.e. By linearity this implies that $QZ(I - Q)A^{1/2} = 0$ a.e. Now it follows from Lemma 6.6 that $QZ = ZQ$ a.e.

'(ii)⇒(iii)': This is immediate, using (6.10).

'(iii)⇒(i)': This is a consequence of Proposition 5.13.

'(ii)⇒(iv)': By assumption $QZ = ZQ$ and therefore $WZP = WQZP = 0$ a.e. Then (6.18) follows by applying Theorem 6.5 to $\mathfrak{a}^p$.

'(iv)⇒(v)': By applying Theorem 6.5 to $\mathfrak{Re}\,(\mathfrak{a}^p)$ and using (6.18), we obtain

$$\mathfrak{Re}\,(\mathfrak{a}^p_{\mathrm{reg}}) = (\mathfrak{Re}\,\mathfrak{a}^p)_{\mathrm{reg}}.$$

Then it follows from Corollary 5.11 that $\mathfrak{a}^p_s$ is sectorial.

'(v)⇒(ii)': Suppose $\mathfrak{a}^p_s$ is sectorial and let $\gamma_s < 0$ be a corresponding vertex. Using (6.10) with $X = Y = 0$, it follows from Proposition 5.13 that there exists an $M > 0$ such that

$$\|QZ(I - Q)A^{1/2}\nabla u\|_{(L^2(\Omega))^d} \leqslant M\|u\|^2_{L^2(\Omega)} \tag{6.19}$$

for all $u \in D(\mathfrak{a})$. Now (ii) follows as in the proof of '(i)⇒(ii)'. □

Remark 6.8. Suppose that $\mathfrak{a}_s$ is sectorial. Then by Proposition 6.7 '(i)⇒(ii)', (6.16) and (6.17) it follows that

$$\begin{aligned}\mathfrak{a}_{\mathrm{reg}}(u, v) = {} & \big((I + iZ)(I - Q)A^{1/2}\nabla u \,|\, (I - Q)A^{1/2}\nabla v\big) \\ & + \big((I - Q)A^{1/2}\nabla u \,|\, vX\big) + \big(uY \,|\, (I - Q)A^{1/2}\nabla v\big) \\ & - \big(Q(I + iZ)^{-1}QY)u \,|\, vX\big) + (c_0 u \,|\, v).\end{aligned} \tag{6.20}$$

and

$$\mathfrak{a}_s(u,v) = \mathfrak{a}_s^p(u,v) + \big(QA^{1/2}\nabla u \,|\, vX\big) + \big(uY \,|\, QA^{1/2}\nabla v\big) + (uWY \,|\, vX)$$

for all $u, v \in D(\mathfrak{a})$.

Next we characterise when the regular part of the real part equals the real part of the regular part.

Lemma 6.9. *One has* $(\mathfrak{Re}\,\mathfrak{a})_{\mathrm{reg}} = \mathfrak{Re}(\mathfrak{a}_{\mathrm{reg}})$ *if and only if both* $(I + iZ)QX = (I - iZ)QY$ *and* $QZ = ZQ$ *a.e.*

Proof. By Proposition 5.9, we know that $(\mathfrak{Re}\,\mathfrak{a})_{\mathrm{reg}} = \mathfrak{Re}(\mathfrak{a}_{\mathrm{reg}})$ if and only if $T\pi_2 = 0$.
'$\Rightarrow$': Suppose $T\pi_2 = 0$. By Proposition 5.13 the form $\mathfrak{a}_s$ is sectorial. Therefore it follows from Proposition 6.7 '(i)$\Rightarrow$(ii)' and '(i)$\Rightarrow$(iii)' that $QZ = ZQ$ and $iQ(X - Y) = QZQ(X + Y)$ a.e. Now the claim follows by rearranging the terms.
'$\Leftarrow$': After rearranging terms, we obtain $iQ(X - Y) = QZQ(X + Y)$ a.e. and $QZ = ZQ$ a.e. Hence it follows directly from (6.10) that $T\pi_2 = 0$. □

In the remainder of this section we present two examples which showcase Theorem 6.5 and Proposition 6.7. We first require some prerequisites. Let $K \subset [0, 1]$ be compact such that its interior is empty and its Lebesgue measure $|K|$ is strictly positive. We consider the positive symmetric form $\mathfrak{t}$ with $D(\mathfrak{t}) = H^1(\mathbb{R})$ defined by

$$\mathfrak{t}(u,v) = \int_{\mathbb{R}} \mathbb{1}_K u' \overline{v'}.$$

Define $\Phi^{(\mathfrak{t})} \colon D(\mathfrak{t}) \to L^2(\mathbb{R}) \times L^2(\mathbb{R})$ by $\Phi^{(\mathfrak{t})}(u) = (u, \mathbb{1}_K u')$.

Lemma 6.10. *Let* $B \subset K$ *be measurable. Then there exists a sequence* $(\psi_n)_{n\in\mathbb{N}}$ *in* $D(\mathfrak{t})$ *such that* $\lim_{n\to\infty} \Phi^{(\mathfrak{t})}(\psi_n) = (0, \mathbb{1}_B)$ *in* $L^2(\mathbb{R}) \times L^2(\mathbb{R})$.

Proof. The lemma can be proved by using [Vog09, Corollary 2], which itself is based on the deep result [RW85, Theorem 1.1]. This immediately yields that $\mathfrak{t}_{\mathrm{reg}} = 0$. Hence the equivalent statements of Proposition 6.7 hold for the form $\mathfrak{t}$. Inspection of the proof of Proposition 6.7 shows that $Q^{(\mathfrak{t})}(x) = \mathbb{1}_K(x)$. This implies that $V_s^{(\mathfrak{t})} = L^2(K)$, which is a reformulation of the lemma's statement.

The following direct proof is obtained by adapting [FŌT94, Proof of Theorem 3.1.6]. Let $B \subset K$ be measurable. For all $n \in \mathbb{N}$ and $k \in \{1, \ldots, n\}$ define

$$I_{n,k} \coloneqq \big[\tfrac{k-1}{n}, \tfrac{k}{n}\big], \qquad \alpha_{n,k} \coloneqq \frac{|I_{n,k} \cap B|}{1 + |(I_{n,k} \cap K) \setminus B|} \quad \text{and} \quad \beta_{n,k} \coloneqq \frac{\alpha_{n,k}}{|I_{n,k} \setminus K|}.$$

Note that $\beta_{n,k}$ is well-defined since K is nowhere dense. Fix $n \in \mathbb{N}$ and consider the function $\varphi_n \colon \mathbb{R} \to \mathbb{R}$ given by

$$\varphi_n = \sum_{k=1}^{n} \big(\mathbb{1}_{I_{n,k} \cap B} - \alpha_{n,k} \mathbb{1}_{(I_{n,k} \cap K) \setminus B} - \beta_{n,k} \mathbb{1}_{I_{n,k} \setminus K} \big).$$

Then $\varphi_n \in L^\infty(\mathbb{R})$, $\operatorname{supp} \varphi_n \subset [0,1]$, $[\varphi_n > 0] = B$ and

$$\int_{I_{n,k}} \varphi_n = |I_{n,k} \cap B| - \alpha_{n,k} |(I_{n,k} \cap K) \setminus B| - \beta_{n,k} |I_{n,k} \setminus K| = 0$$

for all $k \in \{1, \ldots, n\}$. Define the function $\psi_n \colon \mathbb{R} \to \mathbb{R}$ by

$$\psi_n(x) = \int_0^x \varphi_n(t) \, dt.$$

It is a consequence of [Bre83, Lemma VIII.2] that $\psi_n \in H^1(\mathbb{R}) = D(\mathfrak{t})$. Let $x \in [0,1]$. If $k \in \{1, \ldots, n\}$ is such that $x \in I_{n,k}$, then it follows from the above properties of φ_n that

$$|\psi_n(x)| \leqslant \int_{I_{n,k}} |\varphi_n| = \int_{I_{n,k} \cap B} \varphi_n - \int_{I_{n,k} \setminus B} \varphi_n = 2 \int_{I_{n,k} \cap B} \varphi_n \leqslant 2 |I_{n,k}| = \frac{2}{n}.$$

This implies that $\lim_{n \to \infty} \psi_n = 0$ in $L^2(\mathbb{R})$.

On the other hand, one has

$$\begin{aligned} \int_{\mathbb{R}} |\mathbb{1}_K \psi_n' - \mathbb{1}_B|^2 &= \int_K |\varphi_n - \mathbb{1}_B|^2 = \sum_{k=1}^{n} \alpha_{n,k}^2 |(I_{n,k} \cap K) \setminus B| \\ &\leqslant \sum_{k=1}^{n} |I_{n,k} \cap B|^2 \leqslant \sum_{k=1}^{n} \frac{1}{n^2} = \frac{1}{n} \end{aligned}$$

for all $n \in \mathbb{N}$. Thus $\lim_{n \to \infty} \Phi^{(\mathfrak{t})}(\psi_n) = (0, \mathbb{1}_B)$ in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$, as required. $\square$

Note that if $\mathfrak{a}$ is a pure second-order differential sectorial form, then $\mathfrak{a}_s$ is sectorial if and only if $(\mathfrak{Re}\, \mathfrak{a})_{\mathrm{reg}} = \mathfrak{Re}(\mathfrak{a}_{\mathrm{reg}})$ by Lemma 6.9 and Proposition 6.7. We now present an example of a form $\mathfrak{a}$ with lower-order terms such that $\mathfrak{a}_s$ is sectorial while at the same time $(\mathfrak{Re}\, \mathfrak{a})_{\mathrm{reg}} \neq \mathfrak{Re}(\mathfrak{a}_{\mathrm{reg}})$. Moreover, the example shows that if $\mathfrak{a}_s$ is sectorial, then 0 need not be a vertex for $\mathfrak{a}_s$. By the previous comments and Corollary 5.11, this is again a phenomenon that does not occur for differential sectorial forms that are purely of second order.

Example 6.11. Let $K \subset [0,1]$ be a compact set with empty interior and strictly positive Lebesgue measure $|K|$. Consider the form $\mathfrak{a}\colon H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to \mathbb{C}$ given by

$$\mathfrak{a}(u,v) = \int_{\mathbb{R}} \mathbb{1}_K u'\overline{v'} + \int_{\mathbb{R}} \mathbb{1}_K u'\overline{v} - \int_{\mathbb{R}} \mathbb{1}_K u\overline{v'} + \int_{\mathbb{R}} \mathbb{1}_K u\overline{v}.$$

Then $\mathfrak{a}$ is sectorial in $L^2(\mathbb{R})$. More precisely,

$$(\mathfrak{Re}\,\mathfrak{a})(u,v) = \int_{\mathbb{R}} \mathbb{1}_K u'\overline{v'} + \int_{\mathbb{R}} \mathbb{1}_K u\overline{v}$$

and

$$|\mathrm{Im}\,\mathfrak{a}(u,u)| \leqslant \mathrm{Re}\,\mathfrak{a}(u,u)$$

for all $u, v \in H^1(\mathbb{R})$. So $\mathfrak{a}$ has vertex 0. It follows from Lemma 6.10 that we may take $Q = \mathbb{1}_K$. Clearly $Z = 0$, so $\mathfrak{a}_s$ is sectorial by Proposition 6.7. Using (6.20), we obtain

$$\mathfrak{a}_{\mathrm{reg}}(u,v) = 2\int_{\mathbb{R}} \mathbb{1}_K u\overline{v}$$

and hence

$$\mathfrak{a}_s(u,v) = \int_{\mathbb{R}} \mathbb{1}_K u'\overline{v'} + \int_{\mathbb{R}} \mathbb{1}_K u'\overline{v} - \int_{\mathbb{R}} \mathbb{1}_K u\overline{v'} - \int_{\mathbb{R}} \mathbb{1}_K u\overline{v}$$

for all $u, v \in H^1(\mathbb{R})$. It is easily seen that

$$\mathfrak{Re}(\mathfrak{a}_{\mathrm{reg}}) = \mathfrak{a}_{\mathrm{reg}} \neq \tfrac{1}{2}\mathfrak{a}_{\mathrm{reg}} = (\mathfrak{Re}\,\mathfrak{a})_{\mathrm{reg}}.$$

Now let $u \in C_c^\infty(\mathbb{R})$ be such that $u|_{[0,1]} = 1$. Then $\mathrm{Re}\,\mathfrak{a}_s(u,u) = -|K| < 0$. This shows that 0 is not a vertex of $\mathfrak{a}_s$.

Finally, if $\mathfrak{b}\colon H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to \mathbb{C}$ is the form without zeroth-order term given by

$$\mathfrak{b}(u,v) = \int_{\mathbb{R}} \mathbb{1}_K u'\overline{v'} + \int_{\mathbb{R}} \mathbb{1}_K u'\overline{v} - \int_{\mathbb{R}} \mathbb{1}_K u\overline{v'},$$

then

$$\mathfrak{b}_{\mathrm{reg}}(u,v) = \int_{\mathbb{R}} \mathbb{1}_K u\overline{v}$$

for all $u, v \in H^1(\mathbb{R})$. So $\mathfrak{b}_{\mathrm{reg}}$ contains a nontrivial zeroth-order term. $\diamond$

Finally we provide an example of a differential sectorial form that is purely of second-order and satisfies both of the Conditions (L) and (B), but such that Z and Q do not commute. Therefore the singular part is not sectorial and the regular part is not of the form (6.18) by Proposition 6.7.

Example 6.12. Let $d = 2$, $\Omega = \mathbb{R}^2$, $H = L^2(\mathbb{R}^2)$ and let K be a compact, nowhere dense subset of $[0,1]$. Define the form $\mathfrak{h}$ on $D(\mathfrak{h}) = H^1(\mathbb{R}^2)$ by

$$\mathfrak{h}(u,v) = \int_{\mathbb{R}^2} \Big(\mathbb{1}_{K\times\mathbb{R}} (\partial_x u)\overline{\partial_x v} + (\partial_y u)\overline{\partial_y v} \Big).$$

We determine the measurable map Q. To this end, define the isometry $\Phi\colon D(\mathfrak{h}) \to (L^2(\mathbb{R}^2))^3$ by

$$\Phi(u) = \big(u, \mathbb{1}_{K\times\mathbb{R}} \partial_x u, \partial_y u\big)$$

and let V be the closure of $\Phi(D(\mathfrak{h}))$ in $(L^2(\mathbb{R}^2))^3$. Set

$$V_s \coloneqq \Big\{ (v,w) \in (L^2(\mathbb{R}^2))^2 : (0,v,w) \in V \Big\}.$$

We next show that $V_s = L^2(K\times\mathbb{R}) \times \{0\}$.

Let $(v,w) \in V_s$ be given. Then there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $D(\mathfrak{h})$ such that $\lim_{n\to\infty} \Phi(u_n) = (0,v,w)$. Then

$$(w \,|\, \chi)_H = \lim_{n\to\infty} (\partial_y u_n \,|\, \chi)_H = -\lim_{n\to\infty} (u_n \,|\, \partial_y \chi)_H = 0$$

for all $\chi \in C_c^\infty(\mathbb{R}^2)$ by integrating by parts. Hence $w = 0$ and $V_s \subset L^2(K\times\mathbb{R}) \times \{0\}$. Conversely, let $B \subset K$ be measurable and $\tau \in C_c^\infty(\mathbb{R})$. Let $(\psi_n)_{n\in\mathbb{N}}$ be as in Lemma 6.10. Then $\psi_n \otimes \tau \in D(\mathfrak{h})$ and

$$\Phi(\psi_n \otimes \tau) = \big(\psi_n \otimes \tau, (\mathbb{1}_K \psi_n') \otimes \tau, \psi_n \otimes \tau'\big)$$

for all $n \in \mathbb{N}$. Therefore

$$\lim_{n\to\infty} \Phi(\psi_n \otimes \tau) = (0, \mathbb{1}_B \otimes \tau, 0).$$

Due to the density of elementary tensors of the form $\mathbb{1}_B \otimes \tau$ in $L^2(K\times\mathbb{R})$, it follows that $V_s = L^2(K\times\mathbb{R}) \times \{0\}$.

Define $Q\colon \mathbb{R}^2 \to \mathbb{C}^{2\times 2}$ by

$$Q(x,y) = \mathbb{1}_{K\times\mathbb{R}}(x,y) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then the orthogonal projection from $(L^2(\mathbb{R}^2))^2$ onto V_s is the multiplication operator associated with Q. Define $A, B\colon \mathbb{R}^2 \to \mathbb{C}^{2\times 2}$ by $A = \mathbb{1}_{K\times\mathbb{R}} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $B =$

$\mathbb{1}_{K\times\mathbb{R}}\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Then the form $\mathfrak{a}$ with $D(\mathfrak{a}) = D(\mathfrak{h}) = H^1(\mathbb{R}^2)$ given by

$$\begin{aligned}\mathfrak{a}(u,v) &= \int_{\mathbb{R}^2} (A+iB)\nabla u \cdot \nabla v \\ &= \int_{\mathbb{R}^2} \Big(\mathbb{1}_{K\times\mathbb{R}}(\partial_x u + i\partial_y u)\overline{\partial_x v} + (\partial_y u + i\mathbb{1}_{K\times\mathbb{R}}\partial_x u)\overline{\partial_y v}\Big)\end{aligned}$$

is sectorial and has real part $\mathfrak{h}$. Moreover, $Z = \mathbb{1}_{K\times\mathbb{R}}\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and $QZ = ZQ$ does not hold a.e. Therefore all of the equivalent statements in Proposition 6.7 are false. In particular, $\mathfrak{Re}(\mathfrak{a}_{\mathrm{reg}}) \neq (\mathfrak{Re}\,\mathfrak{a})_{\mathrm{reg}}$. Explicitly, it follows from Theorem 6.5 that

$$\mathfrak{a}_{\mathrm{reg}}(u,v) = \int_{\mathbb{R}^2} (1+\mathbb{1}_{K\times\mathbb{R}})(\partial_y u)\overline{\partial_y v}$$

and

$$(\mathfrak{Re}\,\mathfrak{a})_{\mathrm{reg}}(u,v) = \int_{\mathbb{R}^2} (\partial_y u)\overline{\partial_y v}$$

for all $u, v \in D(\mathfrak{a})$. Moreover,

$$\big(\mathfrak{Re}(\mathfrak{a}_s)\big)(u,v) = \int_{K\times\mathbb{R}} \Big((\partial_x u)\overline{\partial_x v} - (\partial_y u)\overline{\partial_y v}\Big)$$

for all $u, v \in D(\mathfrak{a})$. So clearly the singular part $\mathfrak{a}_s$ of $\mathfrak{a}$ is not sectorial. $\diamond$

7

Elements of Sobolev space with weak trace zero

Let Ω be a nonempty open subset of $\mathbb{R}^d$ and let $p \in (1, \infty)$. In this chapter we study the notion of a *weak trace* for elements in the Sobolev space $W^{1,p}(\Omega)$. The notion was quite recently introduced in [AE11] for $p = 2$.

We start by providing general prerequisites in Section 7.1. This is followed by a treatment of the relative capacity and related notions in Section 7.2. In Section 7.3 we will prove that elements in $W^{1,p}(\Omega)$ with weak trace zero can be extended by 0 to obtain elements in $W^{1,p}(\mathbb{R}^d)$, which is the main result in this chapter. We proceed to collect related results pertaining the space $W_0^{1,p}(\Omega)$ in Section 7.4. In Section 7.5 we give corollaries and applications of the main extension result. This is followed by a potential theoretic description of the space of elements with weak trace zero in Section 7.6. In Section 7.7 we briefly discuss two other approaches to define traces for elements of Sobolev space on general domains.

Throughout we compare the space of elements in $W^{1,p}(\Omega)$ with weak trace zero with other related spaces such as $W_0^{1,p}(\Omega)$. In particular, we provide examples which show that an element in $W^{1,p}(\Omega)$ with weak trace zero does not need to be an element of $W_0^{1,p}(\Omega)$. Moreover, we present sufficient conditions on the set Ω which ensure that elements in $W^{1,p}(\Omega)$ with weak trace zero are contained in $W_0^{1,p}(\Omega)$. The results presented here extend the corresponding results from [AE11], specifically [AE11, Proposition 5.5]. The required standard results about Sobolev spaces are collected in Section A.6 of the Appendix.

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7.1 Preliminaries

In the following let Ω be a nonempty open subset of $\mathbb{R}^d$, where $d \in \mathbb{N}$. We usually work with the Sobolev space $W^{1,p}(\Omega)$ for a $p \in [1, \infty)$ and refer to Ω as the **domain** of the Sobolev space. In general we do not assume that Ω is bounded, connected or of finite Lebesgue measure. We denote the boundary of Ω by Γ. By $\mathcal{H}^{d-1}$ we denote the $(d-1)$-dimensional *Hausdorff measure* in $\mathbb{R}^d$. The Lebesgue spaces on subsets of Γ are always considered with respect to $\mathcal{H}^{d-1}$. In this chapter we shall assume that all function spaces are real. We make this assumption since we will use truncation and lattice theoretic arguments. However, in most cases it is obvious that the obtained results also hold in the complex case simply by considering the real and imaginary part separately. Elements of $W^{1,p}(\Omega)$ are by definition only elements of $L^p(\Omega)$ and hence equivalence classes. We shall refer to a specific representative of an element in $W^{1,p}(\Omega)$ as a Sobolev *function* and identify it with its equivalence class.

We start by defining the weak trace for elements of the Sobolev space $W^{1,p}(\Omega)$. This is a straightforward generalisation of the notion introduced in [AE11].

Definition 7.1. Let $p \in [1, \infty)$. Let $u \in W^{1,p}(\Omega)$. Let $r \in [1, \infty)$. We call $\varphi \in L^r(\Gamma)$ a **weak r-trace** of u if there exists a sequence (u_n) in $W^{1,p}(\Omega) \cap C(\overline{\Omega})$ such that $u_n|_\Gamma \in L^r(\Gamma)$ for all $n \in \mathbb{N}$ and

$$u_n \to u \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad u_n|_\Gamma \to \varphi \text{ in } L^r(\Gamma).$$

Moreover, we define the set

$$V_r^p(\Omega) = \Big\{ u \in W^{1,p}(\Omega) : \text{the zero function in } L^r(\Gamma) \text{ is a weak r-trace of } u \Big\}.$$

Remark 7.2. The sequence (u_n) in the above definition may be assumed to be in $W^{1,p}(\Omega) \cap C_c(\overline{\Omega})$, where we denote by $C_c(\overline{\Omega})$ the space of continuous functions on $\overline{\Omega}$ with compact support in $\overline{\Omega}$. In fact, let $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$. Let $r \in [1, \infty)$ and suppose $u|_\Gamma \in L^r(\Gamma)$. Let $\zeta \in C_c^\infty(\mathbb{R}^d)$ be such that $0 \leqslant \zeta \leqslant 1$ and $\zeta(x) = 1$ for all $|x| \leqslant 1$. Define $\zeta_n(x) = \zeta(\frac{x}{n})$ for all $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$. Using dominated convergence, it is readily verified that $\zeta_n u \to u$ in $W^{1,p}(\Omega)$ and $(\zeta_n u)|_\Gamma \to u|_\Gamma$ in $L^r(\Gamma)$.

In [AE11] the exposition was confined to the Hilbert space case $p = r = 2$ and some assumptions on Ω were made to reduce technicalities. We shall work in the p-dependent setting and in addition not make latter assumptions.

Next we introduce several other spaces that are related to $V_r^p(\Omega)$. As usual, let $W_0^{1,p}(\Omega)$ be the closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$. Proposition A.40 implies that $W^{1,p}(\mathbb{R}^d) = W_0^{1,p}(\mathbb{R}^d)$. It follows immediately from the definitions that if $u \in$

$W_0^{1,p}(\Omega)$ then $u \in V_r^p(\Omega)$ for all $r \in [1, \infty)$. If $u \colon \Omega \to \mathbb{R}$ is a measurable function, we denote by $u^* \colon \mathbb{R}^d \to \mathbb{R}$ the **extension of u by 0**. We shall use the same notation also for $u \in L^p(\Omega)$ and understand u^* as an element of $L^p(\mathbb{R}^d)$ in the obvious way. It is not hard to see that $u^* \in W^{1,p}(\mathbb{R}^d)$ for all $u \in W_0^{1,p}(\Omega)$. One can consider $W_0^{1,p}(\Omega)$ as a closed subspace of $W^{1,p}(\mathbb{R}^d)$ by identifying $u \in W_0^{1,p}(\Omega)$ and u^*.

Remark 7.3. The space $W_0^{1,p}(\Omega)$ is of great importance since its elements can be considered to 'vanish at the boundary' in a sense compatible with the Sobolev space structure. So the space $W_0^{1,p}(\Omega)$ arises naturally when dealing with Dirichlet boundary conditions or if one wants to express in a weak way that two elements of $W^{1,p}(\Omega)$ have the same boundary values, for example. Note however, that functions in $W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ need not vanish pointwise on Γ in general. For $p = 2$ it has been shown in [BW06] that functions in $W_0^{1,2}(\Omega) \cap C(\overline{\Omega})$ vanish pointwise on Γ if and only if Ω is regular in 2-capacity, which is an extremely weak regularity condition on a domain. So if $\Omega = B(0,1) \setminus \{0\}$ is the punctured unit ball in $\mathbb{R}^2$, which is not regular in 2-capacity, then there are functions in $W_0^{1,2}(\Omega) \cap C(\overline{\Omega})$ that do not vanish in 0.

Another reasonable choice for a subspace of elements of $W^{1,p}(\Omega)$ that 'vanish at the boundary' is given by

$$W_0^{1,p}(\overline{\Omega}) = \Big\{ u|_\Omega : u \in W^{1,p}(\mathbb{R}^d) \text{ such that } u = 0 \text{ a.e. on } \mathbb{R}^d \setminus \overline{\Omega} \Big\}.$$

This space was used for example in [AD08]. For us the next very similar space will be more important. Regarding the tilde in the notation, we follow [Gri85, Definition 1.3.2.5]. Let $\widetilde{W}_0^{1,p}(\Omega)$ be the space defined by

$$\begin{aligned} \widetilde{W}_0^{1,p}(\Omega) &= \Big\{ u|_\Omega : u \in W^{1,p}(\mathbb{R}^d) \text{ such that } u = 0 \text{ a.e. on } \mathbb{R}^d \setminus \Omega \Big\} \\ &= \Big\{ u \in W^{1,p}(\Omega) : u^* \in W^{1,p}(\mathbb{R}^d) \Big\}. \end{aligned}$$

Clearly, $W_0^{1,p}(\Omega) \subset \widetilde{W}_0^{1,p}(\Omega) \subset W_0^{1,p}(\overline{\Omega})$. In fact, if Γ is a Lebesgue nullset then $\widetilde{W}_0^{1,p}(\Omega) = W_0^{1,p}(\overline{\Omega})$. The space $\big\{ u^* : u \in \widetilde{W}_0^{1,p}(\Omega) \big\}$ is closed in $W^{1,p}(\mathbb{R}^d)$ since a convergent sequence in $W^{1,p}(\mathbb{R}^d)$ has a subsequence that converges pointwise almost everywhere on $\mathbb{R}^d$. Moreover, if $u \in \widetilde{W}_0^{1,p}(\Omega)$ then $\nabla u = 0$ a.e. on $\mathbb{R}^d \setminus \Omega$ by Proposition A.41. Consequently one has $\|u\|_{W^{1,p}(\Omega)} = \|u^*\|_{W^{1,p}(\mathbb{R}^d)}$ for all $u \in \widetilde{W}_0^{1,p}(\Omega)$. Hence $\widetilde{W}_0^{1,p}(\Omega)$ is a closed subspace of $W^{1,p}(\Omega)$.

We define the space $\widetilde{W}^{1,p}(\Omega)$ by

$$\widetilde{W}^{1,p}(\Omega) = \overline{W^{1,p}(\Omega) \cap C_c(\overline{\Omega})},$$

where the closure is taken in $W^{1,p}(\Omega)$. Then it follows from Proposition A.40 that

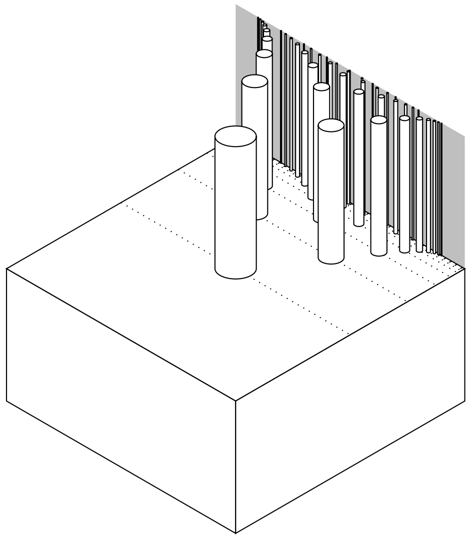

Figure 7.1. An example of a domain $\Omega \subset \mathbb{R}^3$ where the weak trace is not unique, see [AE11, Example 4.4].

$u|_\Omega \in \widetilde{W}^{1,p}(\Omega)$ for all $u \in W^{1,p}(\mathbb{R}^d)$. Hence $\widetilde{W}_0^{1,p}(\Omega)$ is a subspace of $\widetilde{W}^{1,p}(\Omega)$.

Remark 7.4. In general $\widetilde{W}^{1,p}(\Omega)$ is not equal to $W^{1,p}(\Omega)$, although this is true if Ω has continuous boundary by Theorem A.45. It is easy to obtain counterexamples where Ω is not **topologically regular**, i.e., where the interior of $\overline{\Omega}$ is different from Ω. However, it is neither sufficient nor necessary for the equality $\widetilde{W}^{1,p}(\Omega) = W^{1,p}(\Omega)$ that Ω is topologically regular. If $\Omega = \mathbb{R}^d \setminus \{0\}$ where $d \geqslant 2$ and $p \leqslant d$, then Ω is not topologically regular, but $\widetilde{W}^{1,p}(\Omega) = W^{1,p}(\Omega)$ as $W_0^{1,p}(\Omega) = W^{1,p}(\mathbb{R}^d)$ by [EE87, Corollary VIII.6.4]. Conversely, in [Kol81] a counterexample with a topologically regular bounded open set Ω is constructed.

For a discussion of positive and negative results concerning the density of $W^{1,p}(\Omega) \cap C_c(\overline{\Omega})$ in $W^{1,p}(\Omega)$ see [O'Fa97, Section 1].

Example 7.5. Let $\Omega = (-1,0) \cup (0,1)$. Then it is easily checked that $\widetilde{W}^{1,p}(\Omega) = W^{1,p}((0,1)) \neq W^{1,p}(\Omega)$ and $\widetilde{W}_0^{1,p}(\Omega) = W_0^{1,p}(\overline{\Omega}) = W_0^{1,p}((0,1)) \neq W_0^{1,p}(\Omega)$. Still, $V_r^p(\Omega) = W_0^{1,p}(\Omega)$ for all $r \in [1,\infty)$. An example of a topologically regular domain Ω such that $W_0^{1,p}(\Omega) \neq \widetilde{W}_0^{1,p}(\Omega) = W_0^{1,p}(\overline{\Omega})$ is given in [Hedoo, Example on p. 94]; see also Examples 7.54 and 7.56. ◇

Since only elements of $\widetilde{W}^{1,p}(\Omega)$ can have a weak trace, it follows that in general not every element of $W^{1,p}(\Omega)$ has a weak trace. In fact, if Ω has a sufficiently sharp outward pointing cusp, then the restriction of a function in $W^{1,p}(\Omega) \cap C(\overline{\Omega})$ to the boundary does not need to be locally integrable at the tip of the cusp. Such an example is given in [AE11, Example 9.1].

Moreover, depending on the geometry of Ω it is possible that an element of

$W^{1,p}(\Omega)$ has more than one weak r-trace, or put differently, it is possible that the zero function has nontrivial weak r-traces.

Example 7.6. In [AE11, Example 4.4] the bounded and connected domain $\Omega \subset \mathbb{R}^3$ depicted in Figure 7.1 was considered. We denote the part of the boundary represented by the grey rectangle by Γ_s. The cylinders on top of the box have radii that become small rapidly towards Γ_s. In particular, one has $\mathcal{H}^{d-1}(\Gamma) < \infty$. It was established that $\mathbb{1}_{\Gamma_s}$ is a weak 2-trace of the zero function in $H^1(\Omega)$. As $\mathcal{H}^{d-1}(\Gamma_s) > 0$ this shows that the weak 2-trace is not unique in $H^1(\Omega)$.

We note that in this example Ω has very low density at the part of the boundary where the weak r-trace is not unique. In fact, this is the case for all known examples where the weak trace is not unique. A connection between Ω having Lebesgue density 0 and the nonuniqueness phenomenon was suggested in [BG10, p. 941]. $\diamond$

Lemma 7.7. *Let* $p \in [1, \infty)$ *and* $r \in [1, \infty)$. *Then* $V_r^p(\Omega)$ *is a closed subspace of* $W^{1,p}(\Omega)$. *Next, let* $u \in V_r^p(\Omega)$. *Then* $u \vee 0 \in V_r^p(\Omega)$, $u \wedge 1 \in V_r^p(\Omega)$ *and* $\zeta u \in V_r^p(\Omega)$ *for all* $\zeta \in C_c^\infty(\mathbb{R}^d)$.

Proof. We first show that $V_r^p(\Omega)$ is a closed subspace of $W^{1,p}(\Omega)$. It is obvious that $V_r^p(\Omega)$ is a linear subspace. Let $w \in W^{1,p}(\Omega)$ and suppose (w_n) is a sequence in $V_r^p(\Omega)$ such that $w_n \to w$ in $W^{1,p}(\Omega)$. Then for all $n \in \mathbb{N}$ there exists a $w_n' \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ such that $\|w_n - w_n'\|_{1,p} < \frac{1}{n}$ and $\|w_n'|_\Gamma\|_r < \frac{1}{n}$. Hence $w_n' \to w$ in $W^{1,p}(\Omega)$ and $w_n'|_\Gamma \to 0$ in $L^r(\Gamma)$. So $w \in V_r^p(\Omega)$.

The remaining claims follow similarly using Proposition A.41. $\square$

Definition 7.8. We denote the **locally finite part** of Γ by

$$\Gamma_{\text{loc}} = \Big\{ z \in \Gamma : \exists r > 0 \text{ such that } \mathcal{H}^{d-1}\big(\Gamma \cap B(z, r)\big) < \infty \Big\}.$$

Remark 7.9. As usual, the set $B(z, r)$ in the definition of the locally finite part Γ_{loc}, denotes the open ball in $\mathbb{R}^d$ centred in z with radius r. It is clear that Γ_{loc} is relatively open in Γ and σ-compact, i.e., a countable union of compact sets.

We note that $(\Gamma_{\text{loc}}, \mathscr{B}(\Gamma_{\text{loc}}), \mathcal{H}^{d-1})$ is a locally finite Borel regular measure space by [EG92, Theorem 2.1.1]. In particular, if $K \subset \Gamma_{\text{loc}}$ is compact, then $\mathcal{H}^{d-1}(K) < \infty$. Moreover, if $d > 1$ then this measure space is atomless by [Fre03, Exercise 264Yg].

Example 7.10. Let $\Omega \subset \mathbb{R}^2$ be the interior of the Koch snowflake as depicted in Figure 7.2. Then $\Gamma_{\text{loc}} = \emptyset$. Let $p \in [1, \infty)$. If $r \in [1, \infty)$ and $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ is such that $u|_\Gamma \in L^r(\Gamma)$, then $u|_\Gamma = 0$. It follows from Proposition A.43 that $u \in W_0^{1,p}(\Omega)$. Consequently $V_r^p(\Omega) = W_0^{1,p}(\Omega)$ for all $r \in [1, \infty)$. Of course, in this case it is more natural to use the s-dimensional Hausdorff measure $\mathcal{H}^s$ on Γ, where $s = \frac{\log 4}{\log 3}$. We refer to [Wal91] for a study of the trace operator for Sobolev spaces on sufficiently regular fractal domains. $\diamond$

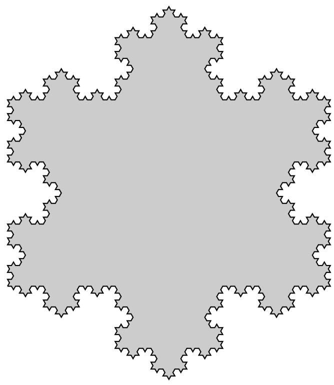

Figure 7.2. The interior of the Koch snowflake

The following proposition shows that if Γ has finite $(d-1)$-dimensional Hausdorff measure then, at least for bounded weak traces, the r-dependence is not really relevant.

Proposition 7.11. *Let* $p \in [1, \infty)$. *Let* $u \in W^{1,p}(\Omega)$ *and* $\varphi \in L^\infty(\Gamma)$. *Let* $r \in (1, \infty)$. *If* φ *is a weak* 1*-trace of* u*, then* φ *is a weak* r*-trace of* u*. Moreover, if* $\mathcal{H}^{d-1}(\Gamma_{\mathrm{loc}}) < \infty$*, then* φ *is a weak* r*-trace of* u *if and only if it is a weak* 1*-trace of* u*.*

Proof. Suppose that φ is a weak 1-trace of u. It follows from Proposition A.41 that then $\varphi \vee 0$ is a weak 1-trace of $u \vee 0$. So we may assume that $u \geqslant 0$ a.e. on Ω and $\varphi \geqslant 0$ a.e. on Γ. Let $M > 0$ be such that $\|\varphi\|_\infty \leqslant M$. Let (u_n) be a sequence in $W^{1,p}(\Omega) \cap C(\overline{\Omega})$ such that $u_n \geqslant 0$ for all $n \in \mathbb{N}$, $u_n \to u$ in $W^{1,p}(\Omega)$ and $u_n|_\Gamma \to \varphi$ in $L^1(\Gamma)$. Set $\theta \coloneqq 1 - \frac{1}{r}$. Then by the interpolation inequality for Lebesgue spaces one obtains

$$\begin{aligned}\|\varphi - (u_n \wedge M)|_\Gamma\|_r &\leqslant \|\varphi - (u_n \wedge M)|_\Gamma\|_1^{1-\theta} \|\varphi - (u_n \wedge M)|_\Gamma\|_\infty^\theta \\ &\leqslant (2M)^\theta \|\varphi - (u_n \wedge M)|_\Gamma\|_1^{1-\theta}.\end{aligned}$$

So, again using Proposition A.41, it follows that $\varphi = \varphi \wedge M$ is a weak r-trace of $u \wedge M$.

Define $v \coloneqq u - u \wedge M$. Then $v \geqslant 0$ a.e. on Ω and $v \in V_1^p(\Omega)$. Let (v_n) be a sequence in $W^{1,p}(\Omega) \cap C(\overline{\Omega})$ such that $v_n \geqslant 0$ for all $n \in \mathbb{N}$, $v_n \to v$ in $W^{1,p}(\Omega)$ and $v_n|_\Gamma \to 0$ in $L^1(\Gamma)$. Let $N > 0$. Then by the interpolation inequality for Lebesgue spaces one obtains

$$\|(v_n \wedge N)|_\Gamma\|_r \leqslant \|(v_n \wedge N)|_\Gamma\|_1^{1-\theta} \|(v_n \wedge N)|_\Gamma\|_\infty^\theta \leqslant N^\theta \|(v_n \wedge N)|_\Gamma\|_1^{1-\theta}.$$

So $v \wedge N \in V_r^p(\Omega)$. As $v \wedge N \to v$ in $W^{1,p}(\Omega)$ as $N \to \infty$, it follows from Lemma 7.7 that $v \in V_r^p(\Omega)$. Hence φ is a weak r-trace of $u = v + u \wedge M$.

Suppose now that $\mathcal{H}^{d-1}(\Gamma_{\mathrm{loc}}) < \infty$ and that φ is a weak r-trace of u. It is clear that $\varphi = 0$ a.e. on $\Gamma \setminus \Gamma_{\mathrm{loc}}$. Moreover, $L^r(\Gamma_{\mathrm{loc}})$ is continuously embedded into $L^1(\Gamma_{\mathrm{loc}})$. This implies the claim. $\square$

Corollary 7.12. *Let* $p \in [1,\infty)$*. If* $r \in (1,\infty)$*, then* $V_1^p(\Omega) \subset V_r^p(\Omega)$*. Moreover, if* $\mathcal{H}^{d-1}(\Gamma_{loc}) < \infty$*, then* $V_r^p(\Omega) = V_1^p(\Omega)$ *for all* $r \in (1,\infty)$*.*

We shall make use of the class of locally integrable functions on Γ_{loc} to obtain a larger space of elements that have weak trace zero. We work with the Borel regular and locally finite measure space $(\Gamma_{loc}, \mathscr{B}(\Gamma_{loc}), \mathcal{H}^{d-1})$. We say that a measurable function $\varphi\colon \Gamma_{loc} \to \mathbb{R}$ is **locally integrable** on Γ_{loc} if $\varphi \mathbb{1}_K \in L^1(\Gamma_{loc})$ for all $K \subset \Gamma_{loc}$ compact. Moreover, let $L^1_{loc}(\Gamma_{loc})$ be the vector space of all locally integrable functions on Γ_{loc}, where we identify functions that agree $\mathcal{H}^{d-1}$-a.e. We equip $L^1_{loc}(\Gamma_{loc})$ with the locally convex topology induced by the seminorms $\|\cdot\|_K\colon \varphi \mapsto \|\varphi \mathbb{1}_K\|_1$, where $K \subset \Gamma_{loc}$ is compact. Since Γ_{loc} is σ-compact, it follows that $L^1_{loc}(\Gamma_{loc})$ is metrizable. Clearly $L^1_{loc}(\Gamma_{loc})$ is complete and hence a Fréchet space. Moreover, a sequence which converges in $L^1_{loc}(\Gamma_{loc})$ has a subsequence such that the corresponding representatives converge pointwise almost everywhere.

Definition 7.13. Let $p \in [1,\infty)$ and $u \in W^{1,p}(\Omega)$. We say that u has **weak trace zero** if there exists a sequence (u_n) in $W^{1,p}(\Omega) \cap C_c(\overline{\Omega})$ such that $\lim_{n\to\infty} u_n = u$ in $W^{1,p}(\Omega)$, $u_n(z) = 0$ for all $z \in \Gamma \setminus \Gamma_{loc}$ and $n \in \mathbb{N}$, and $u_n|_{\Gamma_{loc}} \to 0$ in $L^1_{loc}(\Gamma_{loc})$. Moreover, we define the space

$$V^p(\Omega) = \Big\{ u \in W^{1,p}(\Omega) : u \text{ has weak trace zero} \Big\}.$$

It is immediately clear that $V_r^p(\Omega) \subset V^p(\Omega)$ for all $r \in [1,\infty)$. Moreover, the statements in Lemma 7.7 also hold for $V^p(\Omega)$.

Remark 7.14. Suppose Ω is bounded and Lipschitz. Clearly $\Gamma_{loc} = \Gamma$. Let $p \in [1,\infty)$. Then it follows from Theorem A.46 that $W_0^{1,p}(\Omega) = \widetilde{W}_0^{1,p}(\Omega) = W_0^{1,p}(\overline{\Omega})$. Moreover, it is clear that if $u \in W_0^{1,p}(\Omega)$ then $u \in V_r^p(\Omega)$ for all $r \in [1,\infty)$. Conversely, suppose that $u \in V^p(\Omega)$. Then there exists a sequence (u_n) in $W^{1,p}(\Omega) \cap C(\overline{\Omega})$ such that $\lim_{n\to\infty} u_n = u$ and $u_n|_\Gamma \to 0$ $\mathcal{H}^{d-1}$-a.e. on Γ. By Theorem A.48 one has $\operatorname{Tr} u = \lim_{n\to\infty} \operatorname{Tr} u_n$ in $L^p(\Gamma)$. It follows that $\operatorname{Tr} u = 0$ in $L^p(\Gamma)$. Now Theorem A.49 implies that $u \in W_0^{1,p}(\Omega)$. Therefore $W_0^{1,p}(\Omega) = V^p(\Omega)$.

7.2 The relative capacity

In this section we collect some more specialised prerequisites. We start by recalling notation and results for the relative capacity that was introduced for $p = 2$ in [AW03, Section 1]. There the relative capacity was realised as a capacity associated with a certain Dirichlet form, see [BH91, Section I.8]. A thorough direct study of the relative capacity for $p \in (1,\infty)$ can be found in [Bie09b; Bie09a]. We point out that the case $p = 1$ is not covered.

Definition 7.15 (Relative p-capacity)**.** Let $p \in (1, \infty)$. The **relative p-capacity** of a subset $A \subset \overline{\Omega}$ is defined as

$$\begin{aligned}\operatorname{cap}_{p,\Omega}(A) = \inf\{\|u\|_{1,p}^p : u \in \widetilde{W}^{1,p}(\Omega) &\text{ and there exists a relatively open} \\ &\text{neighbourhood } O \text{ of } A \text{ in } \overline{\Omega} \text{ such that } u \geqslant 1 \text{ a.e. on } O \cap \Omega\}.\end{aligned}$$

We call $P \subset \overline{\Omega}$ **relatively p-polar** if $\operatorname{cap}_{p,\Omega}(P) = 0$. We say that a pointwise property holds **relatively p-quasi everywhere** (relatively p-q.e.) on A if there exists a relatively p-polar set $P \subset \overline{\Omega}$ such that it holds for all points in $A \setminus P$.

Remark 7.16. If $\Omega = \mathbb{R}^d$, then the relative p-capacity is equal to the usual p-capacity. So $\operatorname{cap}_p(A) = \operatorname{cap}_{p,\mathbb{R}^d}(A)$ for all $A \subset \mathbb{R}^d$. We shall use the above notation without the qualifier 'relative' if we refer to the p-capacity. In particular, a set $A \subset \mathbb{R}^d$ is called p-polar if $\operatorname{cap}_p(A) = 0$. Note that $u|_\Omega \in \widetilde{W}^{1,p}(\Omega)$ for all $u \in W^{1,p}(\mathbb{R}^d)$. This shows that

$$\operatorname{cap}_{p,\Omega}(A) \leqslant \operatorname{cap}_p(A) \tag{7.1}$$

for all $A \subset \overline{\Omega}$. In particular, if $A \subset \overline{\Omega}$ is p-polar then A is relatively p-polar. Moreover, if $A \subset \Omega$, then A is p-polar if and only if A is relatively p-polar by [Bie09a, Corollary 3.15]; see also Lemma 7.18.

It is clear by the above definition that $\operatorname{cap}_{p,\Omega}(A) \geqslant |A|$ for all $A \subset \Omega$, where $|A|$ denotes the Lebesgue measure of A. We consider the capacity as a means of measuring subsets of $\mathbb{R}^d$ which are not negligible with respect to the structure of the Sobolev space, despite being Lebesgue nullsets. For example, the Lebesgue–Besicovitch differentiation theorem [EG92, Theorem 1.7.1] states that a locally Lebesgue integrable function has a Lebesgue point almost everywhere. So in measure theoretic terminology such functions are approximately continuous. Taking into consideration that a Sobolev space has considerably more structure, one naturally expects that elements of a Sobolev space have 'more' Lebesgue points than merely almost everywhere. It turns out that an element in $W^{1,p}(\mathbb{R}^d)$ has Lebesgue points everywhere in $\mathbb{R}^d \setminus P$, where P is a p-polar set [EG92, Theorem 4.8.1] or [MZ97, Theorem 2.55]. To emphasise how fine the notion of the p-capacity actually is, we note that a p-polar set has $(d-1)$-dimensional Hausdorff measure 0, see [EG92, Theorem 4.7.4] or [MZ97, Theorem 2.53]. Curiously the latter property is not true for relatively p-polar sets, see [AW03, Example 4.3] and Example 7.44.

We next provide some basic properties of the relative p-capacity and then study relatively p-quasi continuous representatives of elements in $\widetilde{W}^{1,p}(\Omega)$. The following proposition collects several basic results that can be found in [Bie09a, Subsection 3.1] or [Bie09b, Section 3].

Proposition 7.17. *Let* $p \in (1, \infty)$. *The relative* p*-capacity has the following properties.*

(i) $\operatorname{cap}_{p,\Omega}(\emptyset) = 0$.

(ii) *If* $A \subset B \subset \overline{\Omega}$, *then* $\operatorname{cap}_{p,\Omega}(A) \leqslant \operatorname{cap}_{p,\Omega}(B)$.

(iii) *If* (A_n) *is a sequence of increasing subsets of* $\overline{\Omega}$, *then*

$$\lim_{n\to\infty} \operatorname{cap}_{p,\Omega}(A_n) = \operatorname{cap}_{p,\Omega}\left(\bigcup_{n=1}^{\infty} A_n\right).$$

(iv) *If* (K_n) *is a sequence of decreasing compact subsets of* $\overline{\Omega}$, *then*

$$\lim_{n\to\infty} \operatorname{cap}_{p,\Omega}(K_n) = \operatorname{cap}_{p,\Omega}\left(\bigcap_{n=1}^{\infty} K_n\right).$$

(v) *If* $A \subset \overline{\Omega}$, *then*

$$\operatorname{cap}_{p,\Omega}(A) \;=\; \inf\bigl\{\operatorname{cap}_{p,\Omega}(O) \;:\; O \subset \overline{\Omega} \text{ is relatively open in } \overline{\Omega} \text{ and } A \subset O\bigr\}.$$

(vi) *If* (A_n) *is a sequence of subsets of* $\overline{\Omega}$, *then*

$$\operatorname{cap}_{p,\Omega}\left(\bigcup_{n=1}^{\infty} A_n\right) \leqslant \sum_{n=1}^{\infty} \operatorname{cap}_{p,\Omega}(A_n).$$

(vii) *If* $K \subset \overline{\Omega}$ *is compact, then*

$$\operatorname{cap}_{p,\Omega}(K) = \inf\bigl\{\|u\|_{1,p}^{p} : u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \text{ such that } u(x) \geqslant 1 \text{ for all } x \in K\bigr\}.$$

In particular, the relative p*-capacity is a normed* Choquet capacity *and an outer measure.*

The following result shows that the capacity and the relative capacity are comparable as long as one stays inside of a compact subset of Ω.

Lemma 7.18 (cf. [Bie09a, Example 3.12])**.** *Let* $K \subset \Omega$ *be compact. Then there exists a* $C > 0$ *such that*

$$\operatorname{cap}_p(A) \leqslant C \operatorname{cap}_{p,\Omega}(A)$$

for all $A \subset K$.

The following basic proposition, which is mostly a consequence of the Sobolev embedding theorem A.50, shows why the p-capacity is usually only considered for $p \leqslant d$. We have adapted the arguments given in [EE87, Section VIII.6].

Proposition 7.19. *Let* $p > d$. *Let* $A \subset \mathbb{R}^d$. *Then* $\operatorname{cap}_p(A) = 0$ *if and only if* $A = \emptyset$.

Proof. Suppose that $\operatorname{cap}_p(A) = 0$. We assume that $A \neq \emptyset$ and deduce a contradiction. We may suppose that $A = \{x_0\}$ for an $x_0 \in \mathbb{R}^d$. By Proposition 7.17 (vii) there exists a sequence (u_n) in $W^{1,p}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ such that $u_n \to 0$ in $W^{1,p}(\mathbb{R}^d)$ and $u_n(x_0) \geqslant 1$ for all $n \in \mathbb{N}$. Due to Proposition A.41 we may assume that $0 \leqslant u_n(x) \leqslant 1$ for all $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$. Let $\zeta \in C_c^\infty(\mathbb{R}^d)$ be such that $\zeta(x_0) \neq 0$. Clearly, $(1 - u_n)\zeta \to \zeta$ in $W^{1,p}(\Omega)$. So it follows from Theorem A.50 that $0 = (1 - u_n(x_0))\zeta(x_0) \to \zeta(x_0) \neq 0$ for $n \to \infty$. This is a contradiction. □

Definition 7.20. A measurable function $\tilde{u}\colon \overline{\Omega} \to \mathbb{R}$ is called **relatively p-quasi continuous** if for every $\varepsilon > 0$ there exists a set $V \subset \overline{\Omega}$ relatively open in $\overline{\Omega}$ such that $\operatorname{cap}_{p,\Omega}(V) < \varepsilon$ and the restriction of $\tilde{u}$ to $\overline{\Omega} \setminus V$ is continuous.

A measurable function $\tilde{u}\colon \Omega \to \mathbb{R}$ is called **p-quasi continuous** if for every $\varepsilon > 0$ there exists an open set $V \subset \Omega$ such that $\operatorname{cap}_p(V) < \varepsilon$ and the restriction of $\tilde{u}$ to $\Omega \setminus V$ is continuous.

The following standard construction of a quasi-continuous representative is taken from [Bie09b, Lemma 3.17]; see also [EE87, Theorem VIII.5.2].

Lemma 7.21. *Let* $u \in \widetilde{W}^{1,p}(\Omega)$. *Suppose* (u_n) *is a sequence in* $W^{1,p}(\Omega) \cap C_c(\overline{\Omega})$ *such that* $u_n \to u$ *in* $W^{1,p}(\Omega)$. *Then, possibly after going to a subsequence, there exists a relatively* p*-polar set* $P \subset \overline{\Omega}$ *and a measurable function* $\tilde{u}\colon \overline{\Omega} \to \mathbb{R}$ *such that* $\tilde{u}(x) := \lim_{n\to\infty} u_n(x)$ *for all* $x \in \overline{\Omega} \setminus P$, $\tilde{u} = u$ *a.e. on* Ω *and* $\tilde{u}$ *is relatively* p*-quasi continuous.*

Proof. After going to a subsequence, we may assume that

$$\sum_{n=1}^{\infty} 2^{np} \|u_{n+1} - u_n\|_{1,p}^p < \infty$$

and that (u_n) converges to u pointwise a.e. on Ω. We define

$$G_n := \left\{x \in \overline{\Omega} : |u_{n+1} - u_n| > 2^{-n}\right\}$$

for all $n \in \mathbb{N}$. Then G_n is relatively open in $\overline{\Omega}$ and $2^n|u_{n+1} - u_n| \geqslant 1$ on G_n for all $n \in \mathbb{N}$. Observe that $|u_{n+1} - u_n| \in \widetilde{W}^{1,p}(\Omega)$ by Proposition A.41. This implies that

$$\operatorname{cap}_{p,\Omega}(G_n) \leqslant 2^{np} \|u_{n+1} - u_n\|_{1,p}^p$$

for all $n \in \mathbb{N}$. Consequently,

$$\sum_{n=1}^{\infty} \operatorname{cap}_{p,\Omega}(G_n) < \infty.$$

If $m \in \mathbb{N}$, then by Proposition 7.17 (vi) there exists an $n_m \in \mathbb{N}$ such that

$$\operatorname{cap}_{p,\Omega}\Big(\bigcup_{n=n_m}^{\infty} G_n\Big) < \frac{1}{m}.$$

Set $V_m := \bigcup_{n \geqslant n_m} G_n$ for all $m \in \mathbb{N}$ and $P := \bigcap_{m \in \mathbb{N}} V_m$. It follows from Proposition 7.17 (v) that $\operatorname{cap}_{p,\Omega}(P) = 0$.

Note that

$$u_n = u_1 + \sum_{k=1}^{n-1} (u_{k+1} - u_k)$$

for all $n \in \mathbb{N}$. Moreover, the sequence of continuous functions (u_n) converges uniformly on $\overline{\Omega} \setminus V_m$ for all $m \in \mathbb{N}$. Therefore

$$\tilde{u}(x) := \lim_{n \to \infty} u_n(x) = u_1(x) + \sum_{k=1}^{\infty} \big(u_{k+1}(x) - u_k(x)\big)$$

exists for all $x \in \overline{\Omega} \setminus P$. We set $\tilde{u}(x) := 0$ for all $x \in P$. Note that P is a Lebesgue nullset. So (u_n) converges to $\tilde{u}$ pointwise a.e. on Ω. In particular, $\tilde{u} = u$ a.e. on Ω. Hence the restriction of $\tilde{u}$ to $\overline{\Omega} \setminus V_m$ is continuous for all $m \in \mathbb{N}$. This finishes the proof. □

The following is now an immediate consequence.

Corollary 7.22. *Every $u \in \widetilde{W}^{1,p}(\Omega)$ has a relatively p-quasi continuous representative $\tilde{u}\colon \overline{\Omega} \to \mathbb{R}$.*

Theorem 7.23 (cf. [Bie09a, Theorem 3.26]). *Let $u, v \in \widetilde{W}^{1,p}(\Omega)$ and let $U \subset \overline{\Omega}$ be relatively open in $\overline{\Omega}$. Suppose $u \leqslant v$ a.e. on $U \cap \Omega$. Let $\tilde{u}$ and $\tilde{v}$ be relatively p-quasi continuous representatives of u and v, respectively. Then there exists a relatively p-polar set $P \subset U$ such that $\tilde{u}(x) \leqslant \tilde{v}(x)$ for all $x \in U \setminus P$.*

Corollary 7.24. *Let $u \in \widetilde{W}^{1,p}(\Omega)$. Then the relatively p-quasi continuous representative $\tilde{u}$ is unique up to a relatively p-polar set.*

Remark 7.25. For the special case $\Omega = \mathbb{R}^d$, one obtains by the previous results a p-quasi continuous representative for every $u \in W^{1,p}(\mathbb{R}^d)$, which is unique up to a p-polar set. It is not difficult to see that p-quasi continuity is a local property, i.e., a function $\tilde{u}\colon \Omega \to \mathbb{R}$ is p-quasi continuous on Ω if and only if $\tilde{u}$ is p-quasi continuous on a neighbourhood of every point in Ω. Locally inside of Ω, however, the p-capacity and the relative p-capacity are comparable by Lemma 7.18 and (7.1).

This helps to obtain the following consistency property. Let $u \in \widetilde{W}^{1,p}(\Omega)$ and let $\tilde{u}\colon \overline{\Omega} \to \mathbb{R}$ be a relatively p-quasi continuous representative of u. Then $\tilde{u}|_\Omega$ is a p-quasi continuous representative of u, which is unique up to a p-polar set by Corollary 7.24. For the full details we refer to [Bie09a, Theorem 3.31 and Example 3.32].

The next result is a special case of [Bie09a, Theorem 3.29]. It shows that relatively p-quasi continuous representatives behave well with respect to the convergence in $W^{1,p}(\Omega)$.

Proposition 7.26. *Let $u \in \widetilde{W}^{1,p}(\Omega)$ and suppose (u_n) is a sequence in $\widetilde{W}^{1,p}(\Omega)$ such that $u_n \to u$ in $W^{1,p}(\Omega)$. Let $\tilde{u}$ and $\tilde{u}_n$ be relatively p-quasi continuous representatives of u and u_n for all $n \in \mathbb{N}$, respectively. Then, after going to a subsequence, there exists a relatively p-polar set $P \subset \overline{\Omega}$ such that $\lim_{n\to\infty} \tilde{u}_n(x) = \tilde{u}(x)$ for all $x \in \overline{\Omega} \setminus P$.*

Next we show that a (relatively) p-quasi continuous representative exhibits other desirable fine properties. This allows to obtain a representative for an element in a Sobolev space that simultaneously exhibits several additional regularity properties. In the literature the existence of a corresponding representative for each single one of these regularity properties is well established. While the existence of a 'simultaneous' representative is not immediately obvious, it can be directly verified by an inspection of the respective proofs that the pointwise limit of an appropriate smooth approximation sequence exhibits all required properties. In [EG92, Chapter 4] the first part of the following theorem can be found.

Theorem 7.27 (A representative for an element of Sobolev space)**.** *Let $p \in (1, \infty)$. Let $u \in W^{1,p}(\Omega)$. We set*

$$\Omega_{k,y} = \{x \in \Omega : x = (y_1, \dots, y_{k-1}, t, y_k, \dots, y_{d-1}) \text{ for a } t \in \mathbb{R}\} \tag{7.2}$$

for every $k \in \{1, \dots, d\}$ and $y \in \mathbb{R}^{d-1}$. We may consider $\Omega_{k,y}$ as an open subset of $\mathbb{R}$.

Then there exists a measurable function $\tilde{u} \colon \Omega \to \mathbb{R}$ and a p-polar set $P \subset \Omega$ such that the following properties hold:

(i) *$\tilde{u}$ is a p-quasi continuous representative of u;*

(ii) *there exists a sequence (u_n) in $W^{1,p}(\Omega) \cap C^\infty(\Omega)$ converging to u in $W^{1,p}(\Omega)$ and to $\tilde{u}$ pointwise everywhere on $\Omega \setminus P$;*

(iii) *every point in $\Omega \setminus P$ is a Lebesgue point of $\tilde{u}$;*

(iv) *for $\mathcal{L}^{d-1}$-a.e. $y \in \mathbb{R}^{d-1}$ and all $k \in \{1, \dots, d\}$ the one-dimensional function $\tilde{u}|_{\Omega_{k,y}}$ is absolutely continuous on each compact interval of $\Omega_{k,y}$.*

Moreover, if $u \in \widetilde{W}^{1,p}(\Omega)$, then there exists a measurable function $\tilde{u} \colon \overline{\Omega} \to \mathbb{R}$ and a relatively p-polar set $P \subset \overline{\Omega}$ such that $\tilde{u}|_\Omega$ satisfies all of the above properties and such that the following properties hold:

(i)′ *$\tilde{u}$ is a relatively p-quasi continuous representative of u;*

(ii)′ *there exists a sequence* (u_n) *in* $W^{1,p}(\Omega) \cap C_c(\overline{\Omega})$ *converging to* u *in* $W^{1,p}(\Omega)$ *and to* $\tilde{u}$ *pointwise everywhere on* $\overline{\Omega} \setminus P$.

Proof. Let $u \in W^{1,p}(\Omega)$ and let $\tilde{u}$ be a p-quasi continuous representative of u. It follows from Lemma 7.21, Theorem A.39 and Remark 7.25 that $\tilde{u}$ satisfies (ii). If $p > d$, then it follows from Proposition 7.19 that $\tilde{u} \in C(\Omega)$. So (iii) is clearly satisfied if $p > d$. For $p \in (1, d]$ property (iii) follows from [MZ97, Theorem 2.55]. Property (iv) follows from [EG92, Theorem 4.9.2 (i)] since $\tilde{u}$ satisfies (iii).

Suppose now that $u \in \widetilde{W}^{1,p}(\Omega)$ and let $\tilde{u} \colon \overline{\Omega} \to \mathbb{R}$ be a relatively p-quasi continuous representative of u. Then (i)′ and (ii)′ follow from Lemma 7.21. The claims about $\tilde{u}|_\Omega$ follow by Remark 7.25 from the first part of the theorem. □

Remark 7.28. The first part of the theorem does extend to $p = 1$ for a suitable version of the 1-capacity, see [EG92, Section 4]. It is not clear whether the notion of the relative p-capacity can be extended to obtain a Choquet capacity also for the case $p = 1$. In particular, the existence of a relatively 1-quasi continuous representative is unclear.

We finally collect a few results about the p-fine topology and the Lebesgue density topology. Both are topologies in $\mathbb{R}^d$ that are finer than the Euclidean topology.

Definition 7.29. Suppose $p \in (1, d]$ and let $E \subset \mathbb{R}^d$. Then E is called **p-thin** at a point $x \in \mathbb{R}^d$ if

$$\int_0^1 \left(\frac{\operatorname{cap}_p(E \cap B(x, r))}{r^{d-p}} \right)^{1/(p-1)} \frac{1}{r} \, dr < \infty. \tag{7.3}$$

One defines

$$b_p(E) = \Big\{ x \in \mathbb{R}^d : E \text{ is not p-thin at } x \Big\}.$$

If $b_p(E) \subset E$, then E is called **p-finely closed**.

Suppose that $p \in (1, d]$. It is obvious that if $E \subset F \subset \mathbb{R}^d$ then $b_p(E) \subset b_p(F)$.

The next proposition collects a few important properties of the p-fine topology. The corresponding statements can be found in [MZ97, Remark 2.135, Theorem 2.136 and Corollary 2.143], for example.

Proposition 7.30. *Suppose* $p \in (1, d]$. *The family of complements of* p*-finely closed sets defines a topology in* $\mathbb{R}^d$ *that is finer than the Euclidean topology. We call this topology the* **p*-fine topology***. *Moreover, let* $E \subset \mathbb{R}^d$. *Then the closure of* E *in the* p*-fine topology is given by* $E \cup b_p(E)$. *Furthermore,* E *is* p*-polar if and only if* $b_p(E) = \emptyset$.

Remark 7.31. Suppose that $d \geqslant 2$ and $p \in (1, d]$. As usual, let $\Omega \subset \mathbb{R}^d$ be open and $\Gamma = \partial\Omega$. It follows from Proposition 7.30 that the boundary of Ω in the p-fine topology, which we call the **p-fine boundary** of Ω, is equal to $b_p(\Omega) \setminus \Omega$. Clearly $b_p(\Omega) \subset \overline{\Omega}$. Consequently the p-fine boundary of Ω is equal to $b_p(\Omega) \cap \Gamma$.

The following lemma is readily obtained using the description of the p-fine closure in Proposition 7.30. We give its simple proof to make a first use of the introduced notation.

Lemma 7.32. *Suppose* $p \in (1, d]$. *Let* $u\colon \mathbb{R}^d \to \mathbb{R}$ *and* $x_0 \in \mathbb{R}^d$. *Then* u *is continuous at* x_0 *in the* p*-fine topology if and only if*

$$\left\{x \in \mathbb{R}^d : |u(x) - u(x_0)| \geqslant \varepsilon\right\} \tag{7.4}$$

is p*-thin at* x_0 *for all* $\varepsilon > 0$.

Proof. '$\Leftarrow$': Let $\varepsilon > 0$ and E be the set in (7.4). By assumption E is p-thin at x_0. Then $x_0 \notin E \cup b_p(E)$. Hence $U := \mathbb{R}^d \setminus (E \cup b_p(E))$ is an p-finely open neighbourhood of x_0 such that $|u(x) - u(x_0)| < \varepsilon$ for all $x \in U$. Therefore u is continuous at x_0 in the p-fine topology.

'$\Rightarrow$': Let $\varepsilon > 0$ and $U \subset \mathbb{R}^d$ be a p-finely open neighbourhood of x_0 such that $|u(x) - u(x_0)| < \varepsilon$ for all $x \in U$. Let E be the set in (7.4). Then $E \subset \mathbb{R}^d \setminus U$ and $\mathbb{R}^d \setminus U$ is p-finely closed. So $x_0 \notin \mathbb{R}^d \setminus U$ and $b_p(E) \subset b_p(\mathbb{R}^d \setminus U) \subset \mathbb{R}^d \setminus U$. Hence E is p-thin at x_0.

□

The next result connects p-quasi continuity to continuity with respect to the p-fine topology. It is a special case of [MZ97, Theorem 2.145].

Proposition 7.33. *Suppose* $p \in (1, d]$. *Let* $u\colon \mathbb{R}^d \to \mathbb{R}$ *be a function. Then* u *is* p*-quasi continuous if and only if there exists a* p*-polar set* $P \subset \mathbb{R}^d$ *such that* u *is* p*-finely continuous at every point in* $\mathbb{R}^d \setminus P$.

Next we consider the Lebesgue density topology.

Definition 7.34. Let $E \subset \mathbb{R}^d$. We say that E is **open in the Lebesgue density topology** if E is Lebesgue measurable and every point of E is a Lebesgue density point of E, i.e.,

$$\liminf_{r \to 0+} \frac{|E \cap B(x, r)|}{|B(x, r)|} = 1 \tag{7.5}$$

for all $x \in E$. The family of these sets define a topology, which we call the **Lebesgue density topology**.

Remark 7.35. It is obvious that the Lebesgue density topology is finer than the Euclidean topology. Moreover, in (7.5) one can equivalently replace the open balls $B(x, r)$ by open d-dimensional cubes centred in x with side-length $2r$, for example. We point out, however, that if $d > 1$ then one does not obtain the same topology if one measures the density with respect to more general 'd-dimensional rectangles' centred at x. Details regarding different density topologies can be found in [Fug71, Subsection 4.9] and in [LMZ86, Chapter 6].

Fuglede observed that (7.3) allows to compare the 2-fine topology with the Lebesgue density topology, see [LMZ86, Remarks and Comments, p. 326–327] and [Fug71, Subsection 5.6]. In fact, the following holds for all $p \in (1, d]$.

Proposition 7.36 (cf. [MZ97, Corollary 2.51 and Remark 2.135]). *Suppose* $p \in (1, d]$. *Then the* p*-fine topology is coarser than the Lebesgue density topology.*

7.3 Extendability of elements with weak trace zero

In this section we show that if $u \in W^{1,p}(\Omega)$ has weak trace zero then $u \in \widetilde{W}_0^{1,p}(\Omega)$. The next proposition is an essential technical ingredient for the proof. It is an intermediate result in the proof of [SZ99, Theorem 2.2]. The latter theorem gives a necessary and sufficient condition for an element of $W^{1,p}(\Omega)$ to be in $W_0^{1,p}(\Omega)$; see Theorem 7.46 in the following section.

Proposition 7.37 (see the proof of [SZ99, Theorem 2.2]). *Suppose* $p \in (1, \infty)$. *Let* $u \in W^{1,p}(\Omega)$. *Suppose that*

$$\lim_{r \to 0+} \frac{1}{r^d} \int_{B(z,r) \cap \Omega} |u| = 0 \tag{7.6}$$

for $\mathcal{H}^{d-1}$*-a.e.* $z \in \Gamma$. *Then* $u \in \widetilde{W}_0^{1,p}(\Omega)$.

The proof of Proposition 7.37 is based on several elaborate results on functions of bounded variation and fine properties of Sobolev functions. Therefore I shall not attempt to provide a self-contained exposition here and refer to [SZ99] for the full details. We will, however, continue with a sketch of the proof of Proposition 7.37.

Sketch of the proof. Let $\tilde{u} \colon \Omega \to \mathbb{R}$ be the p-quasi continuous representative of u. Since a p-polar set has $(d-1)$-dimensional Hausdorff measure 0, it follows from Theorem 7.27 that $\tilde{u}$ has a Lebesgue point $\mathcal{H}^{d-1}$-a.e. in Ω. Due to (7.6) the extension of $\tilde{u}$ by 0, which we denote by $\tilde{u}^*$, has a Lebesgue point $\mathcal{H}^{d-1}$-a.e. in $\mathbb{R}^d$. Let $x_0 \in \mathbb{R}^d \setminus \Omega$ be a Lebesgue point of $\tilde{u}^*$. Then

$$\lim_{r \to 0+} \frac{|[\tilde{u}^* > t] \cap B(x_0, r)|}{|B(x_0, r)|} = \begin{cases} 0 & \text{for all } t > 0, \text{ and} \\ 1 & \text{for all } t < 0. \end{cases}$$

So for all $t \in \mathbb{R} \setminus \{0\}$ the point x_0 is not contained in the measure theoretic boundary of the superlevel set $[\tilde{u}^* > t]$. Consequently, in a more technical notation, $\mathcal{H}^{d-1}(\partial_m[\tilde{u}^* > t] \setminus \Omega) = 0$ for all $t \in \mathbb{R} \setminus \{0\}$. Using this with a suitable version of the coarea formula implies that $\tilde{u}^*$ is locally of bounded variation in $\mathbb{R}^d$.

Applying the notation (7.2) in Theorem 7.27 for the domain $\mathbb{R}^d$, one has

$$(\mathbb{R}^d)_{k,y} = \Big\{ x \in \mathbb{R}^d : x = (y_1, \ldots, y_{k-1}, t, y_k, \ldots, y_{d-1}) \text{ for a } t \in \mathbb{R} \Big\}$$

for all $k \in \{1, \ldots, d\}$ and $y \in \mathbb{R}^{d-1}$. Clearly $(\mathbb{R}^d)_{k,y}$ is isomorphic to $\mathbb{R}$ for all $k \in \{1, \ldots, d\}$ and $y \in \mathbb{R}^{d-1}$. We shall say that $\tilde{u}^*$ satisfies a property along almost all lines parallel to the coordinate axes (along a.a.l.), if it holds for the one-dimensional function $\tilde{u}^*|_{(\mathbb{R}^d)_{k,y}} : \mathbb{R} \to \mathbb{R}$ for $\mathcal{L}^{d-1}$-a.e. $y \in \mathbb{R}^{d-1}$ and all $k \in \{1, \ldots, d\}$. By [Fed69, Theorem 4.5.9 (29)] the function $\tilde{u}^*$ is continuous along a.a.l. since $\tilde{u}^*$ has a Lebesgue point $\mathcal{H}^{d-1}$-a.e. and is locally of bounded variation in $\mathbb{R}^d$. Moreover, $\tilde{u}^*$ is of bounded variation on every compact interval along a.a.l. It is not hard to check that $\tilde{u}^*$ maps one-dimensional Lebesgue nullsets into nullsets along a.a.l. Using an appropriate characterisation for absolute continuity, the last three statements imply that $\tilde{u}^*$ is absolutely continuous on every compact interval along a.a.l. It follows that $\tilde{u}^*$ has classical partial derivatives in $L^p(\mathbb{R}^d)$ along a.a.l. By [EG92, Theorem 4.9.2 (ii)], which is a kind of converse of Theorem 7.27 (iv), the function $\tilde{u}^*$ is a representative of an element of $W^{1,p}(\mathbb{R}^d)$. □

We point out that the proof of Proposition 7.37 does extend to the case $p = 1$. For this it suffices to note that also for elements of $W^{1,1}(\Omega)$ it is true that $\mathcal{H}^{d-1}$-a.e. point in Ω is a Lebesgue point, see for example [EG92, Theorem 4.8.1 and Theorem 5.6.3]. Moreover, for $p = 1$ the above proposition is a special case of [Swa07, Theorem 5.2]; see Theorem 7.47.

The following is the main result of this section. It allows to relate the property that an element has weak trace zero to another notion for elements of $W^{1,p}(\Omega)$ to vanish at the boundary. Moreover, it shows that the notion of the weak trace is both sensible and useful.

Theorem 7.38. *Let* $p \in (1, \infty)$ *and* $u \in V^p(\Omega)$. *Then* $u \in \widetilde{W}_0^{1,p}(\Omega)$.

Proof. First suppose in addition that $u \geqslant 0$ a.e. on Ω and $u \in L^\infty(\Omega)$. Let $M > 0$ be such that $u \leqslant M$ a.e. on Ω. We shall prove that u can be approximated in $W^{1,p}(\Omega)$ by elements of $\widetilde{W}_0^{1,p}(\Omega)$.

Let (u_n) be a sequence in $W^{1,p}(\Omega) \cap C_c(\overline{\Omega})$ as in Definition 7.13. In particular, $u_n(z) = 0$ for all $z \in \Gamma \setminus \Gamma_{loc}$ and $n \in \mathbb{N}$. After going to a subsequence, we may assume that there is a $\mathcal{H}^{d-1}$-nullset $N \subset \Gamma_{loc}$ such that $u_n(z) \to 0$ for all $z \in \Gamma \setminus N$.

Let $\tilde{u}\colon \overline{\Omega} \to \mathbb{R}$ be a relatively p-quasi continuous representative of u as in Lemma 7.21 defined with respect to (u_n), possibly after going to a subsequence. So there exists a relatively p-polar set $P \subset \overline{\Omega}$ such that $\tilde{u}(x) = \lim_{n\to\infty} u_n(x)$ for all $x \in \overline{\Omega} \setminus P$. Since $\tilde{u}$ is relatively p-quasi continuous one deduces from Proposition 7.17 (v) and (vi) that for all $m \in \mathbb{N}$ there exists a $V_m \subset \overline{\Omega}$ relatively open in $\overline{\Omega}$ such that $\operatorname{cap}_{p,\Omega}(V_m) < \frac{1}{m}$, $P \subset V_m$ and the restriction of $\tilde{u}$ to $\overline{\Omega} \setminus V_m$ is continuous. So there exists a $\zeta_m \in \widetilde{W}^{1,p}(\Omega)$ such that $\|\zeta_m\|_{1,p}^p < \frac{1}{m}$ and $\zeta_m \geqslant 1$ a.e. on $V_m \cap \Omega$ for all $m \in \mathbb{N}$. Moreover, due to Proposition A.41 we may additionally assume that $0 \leqslant \zeta_m \leqslant 1$ a.e. on Ω for all $m \in \mathbb{N}$. Set $v_m := u - \zeta_m u$ for all $m \in \mathbb{N}$. Then $v_m \in W^{1,p}(\Omega)$ by Lemma A.42. Moreover,

$$\begin{aligned}\|u - v_m\|_{1,p}^p &= \|\zeta_m u\|_{1,p}^p \\ &\leqslant \int_\Omega |\zeta_m|^p |u|^p + 2^p \int_\Omega |\nabla \zeta_m|^p |u|^p + 2^p \int_\Omega |\zeta_m|^p |\nabla u|^p \\ &\leqslant (2M)^p \|\zeta_m\|_{1,p}^p + 2^p \int_\Omega |\zeta_m|^p |\nabla u|^p\end{aligned}$$

for all $m \in \mathbb{N}$. Note that $\zeta_m \to 0$ in $W^{1,p}(\Omega)$ for $m \to \infty$. So after going to a subsequence we may assume that $\zeta_m \to 0$ a.e. on Ω. Then it follows that $\int |\zeta_m|^p |\nabla u|^p \to 0$ by dominated convergence. Hence we have established that $v_m \to u$ in $W^{1,p}(\Omega)$.

Let $m \in \mathbb{N}$ be fixed. We will show that $v_m \in \widetilde{W}_0^{1,p}(\Omega)$. Let $z \in \Gamma \setminus N$. We consider the two disjoint cases that $z \in V_m$ or that $z \in \overline{\Omega} \setminus V_m$. Suppose first that $z \in V_m$. Then there exists an $R > 0$ such that $(B(z,R) \cap \overline{\Omega}) \subset V_m$. So

$$\frac{1}{r^d} \int_{B(z,r)\cap\Omega} |v_m| \leqslant \frac{1}{r^d} \int_{B(z,r)\cap\Omega} M|1 - \zeta_m| = 0$$

for all $r \in (0, R)$ since $\zeta_m = 1$ a.e. on $V_m \cap \Omega$. Next suppose that $z \in \overline{\Omega} \setminus V_m$. Then $\tilde{u}(z) = \lim_{n\to\infty} u_n(z) = 0$ since $z \notin P$ and $z \notin N$. Moreover, $\tilde{u}|_{\overline{\Omega}\setminus V_m}$ is continuous. Let $\varepsilon > 0$. Then there exists an $R > 0$ such that $|\tilde{u}(x)| < \varepsilon$ for all $x \in B(z,R) \cap (\overline{\Omega} \setminus V_m)$. Hence

$$\frac{1}{r^d} \int_{B(z,r)\cap\Omega} |v_m| \leqslant \frac{1}{r^d} \int_{B(z,r)\cap\Omega\cap V_m} M|1 - \zeta_m| + \frac{1}{r^d} \int_{B(z,r)\cap(\Omega\setminus V_m)} |\tilde{u}| \leqslant \varepsilon |B(0,1)|$$

for all $r \in (0, R)$. So we have verified that v_m satisfies (7.6) for all $z \in \Gamma \setminus N$. Therefore $v_m \in \widetilde{W}_0^{1,p}(\Omega)$ by Proposition 7.37. As $v_m \to u$ in $W^{1,p}(\Omega)$ and $\widetilde{W}_0^{1,p}(\Omega)$ is closed in $W^{1,p}(\Omega)$, it follows that $u \in \widetilde{W}_0^{1,p}(\Omega)$.

We now reduce the general case to the case considered above. First suppose that $u \geqslant 0$ a.e. on Ω. Let $M > 0$. Using Proposition A.41, we obtain $u \wedge M \in V^p(\Omega)$. By the above we know that $u \wedge M \in \widetilde{W}_0^{1,p}(\Omega)$. Moreover, $u \wedge M \to u$ in $W^{1,p}(\Omega)$ for $M \to \infty$. As $\widetilde{W}_0^{1,p}(\Omega)$ is a closed subspace of $W^{1,p}(\Omega)$, this implies $u \in \widetilde{W}_0^{1,p}(\Omega)$. Finally, let us consider the general case. It follows from Proposition A.41 that both $u \vee 0$ and $(-u) \vee 0$ are elements of $V^p(\Omega)$. By linearity and the above we obtain $u \in \widetilde{W}_0^{1,p}(\Omega)$. □

In Section 7.1 we immediately established several inclusions between the various introduced spaces. In combination with Theorem 7.38 we can summarise the relationships in the following way.

Corollary 7.39. *One has the following inclusion properties.*

$$
\begin{tikzcd}
W_0^{1,p}(\Omega) \arrow[r, hook] & V^p(\Omega) \arrow[r, hook, "T.\ 7.38"] & \widetilde{W}_0^{1,p}(\Omega) \arrow[r, hook, "*"] \arrow[ddr, hook] & W^{1,p}(\mathbb{R}^d) & \\
& & \cap & & \\
& & W_0^{1,p}(\overline{\Omega}) \arrow[r, phantom, "\subset"] & \widetilde{W}^{1,p}(\Omega) \arrow[r, hook] & W^{1,p}(\Omega)
\end{tikzcd}
$$

All the spaces along the 'middle' path are equipped with the norm of $W^{1,p}(\Omega)$*. The arrow* $\hookrightarrow$ *expresses that the first space is a closed subspace of the second, and correspondingly for the arrow* $\overset{*}{\hookrightarrow}$ *after an extension by* 0*.*

Remark 7.40. Clearly the space $W_0^{1,p}(\overline{\Omega})$ is a linear subspace of $W^{1,p}(\Omega)$. If Γ has Lebesgue measure zero, then $W_0^{1,p}(\overline{\Omega})$ is closed in $W^{1,p}(\Omega)$ as $W_0^{1,p}(\overline{\Omega}) = \widetilde{W}_0^{1,p}(\Omega)$. In general $W_0^{1,p}(\overline{\Omega})$ does not need to be closed in $W^{1,p}(\Omega)$. Still, one can make $W_0^{1,p}(\overline{\Omega})$ into a Banach space that is continuously embedded into $W^{1,p}(\Omega)$ by equipping it with a suitable quotient topology of $W^{1,p}(\mathbb{R}^d)$. Since we do not require these results in this chapter, we will not provide further details.

In Section 7.5 we will give more corollaries and examples to Theorem 7.38. To this end we need some additional material that we introduce in the next section.

7.4 Related results

In this section we collect several related results for the space $W_0^{1,p}(\Omega)$. The following theorem is a special case of [AH96, Theorem 9.1.3], where the more delicate case of higher-order derivatives is considered. For this special case a simplified proof based on the fact that $W^{1,p}(\mathbb{R}^d)$ is closed under truncation is given in [AH96, Section 9.2].

Theorem 7.41 (Havin and Bagby)**.** *Let* $p \in (1, \infty)$ *and let* $u \in W^{1,p}(\mathbb{R}^d)$*. Then* $u|_\Omega \in W_0^{1,p}(\Omega)$ *if and only if*

$$
\lim_{r \to 0+} \frac{1}{r^d} \int_{B(z,r)} |u| = 0
$$

for p*-quasi every* $z \in \mathbb{R}^d \setminus \Omega$*. So if* $\tilde{u}$ *is a* p*-quasi continuous representative of* u*, then* $u|_\Omega \in W_0^{1,p}(\Omega)$ *if and only if* $\tilde{u}(z) = 0$ *for* p*-quasi every* $z \in \mathbb{R}^d \setminus \Omega$*.*

Remark 7.42. The result as stated above extends to the case $p = 1$ for a suitable notion of the 1-capacity that was studied in [FZ72]. This can be readily obtained from the proof in [AH96, Section 9.2]. We also refer to [AH96, Section 10.3] for a discussion in a more general setting that includes the higher-order case.

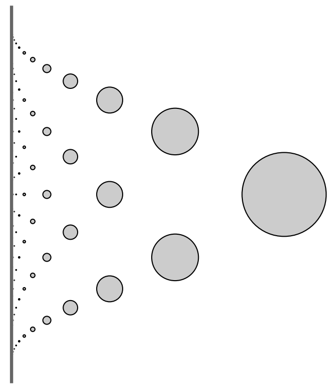

Figure 7.3. An example of an open set $\Omega \subset \mathbb{R}^2$ where the weak trace is not unique and the 2-fine boundary is not all of Γ.

The next result is a refinement of Theorem 7.41 in that it requires the p-quasi continuous representative to vanish only on a part of Γ. The formulation relies on the p-fine topology. Recall that the p-fine topology is finer than the Euclidean topology by Proposition 7.30. Consequently, the p-fine boundary of Ω is contained in Γ; see also Remark 7.31.

Theorem 7.43 (cf. [MZ97, Theorem 2.147]). *Let* $p \in (1, d]$. *Let* $u \in W^{1,p}(\mathbb{R}^d)$ *and let* $\tilde{u}$ *be a* p*-quasi continuous representative of* u. *Then* $u|_\Omega \in W_0^{1,p}(\Omega)$ *if and only if* $\tilde{u}(z) = 0$ *for* p*-quasi every* z *in the* p*-fine boundary of* Ω.

The following example shows that the p-fine boundary of Ω can be a proper subset of Γ. We point out that the arguments can be adapted to work in any dimension $d \geqslant 2$ and for all $p \in (1, d]$.

Example 7.44. We shall consider a simplified nonconnected 2-dimensional version of Example 7.6. Let (r_n) be a decreasing sequence of positive numbers such that $r_n \leqslant 4^{-n}$ for all $n \in \mathbb{N}$. Let $\Omega \subset \mathbb{R}^2$ be given by

$$\Omega = \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{m-1} B\big((2^{-m}, \tfrac{k}{m}), r_m\big);$$

see also Figure 7.3.

Clearly $\Gamma_s := \{0\} \times [0, 1]$ is contained in Γ. For all $n \in \mathbb{N}$ consider the function $u_n \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ defined by $u_n = 0$ on

$$\bigcup_{m=1}^{n} \bigcup_{k=1}^{m-1} \overline{B\big((2^{-m}, \tfrac{k}{m}), r_m\big)}$$

and $u_n = 1$ elsewhere in $\overline{\Omega}$. It follows that $u_n \to 0$ in $W^{1,p}(\Omega)$ and $u_n \geqslant 1$ in a relative neighbourhood of Γ_s in $\overline{\Omega}$. Hence, $\operatorname{cap}_{p,\Omega}(\Gamma_s) = 0$, but $\mathcal{H}^{d-1}(\Gamma_s) > 0$.

Moreover, $\mathbb{1}_{\Gamma_s}$ is a weak r-trace of zero in $W^{1,p}(\Omega)$ for all $r \in [1, \infty)$. We note that $\operatorname{cap}_p(\Gamma_s) > 0$.

Suppose now that $p = 2$. We shall show that Γ_s is disjoint from the 2-fine boundary of Ω if one chooses the sequence (r_n) appropriately. Take (r_n) such that for all $n \in \mathbb{N}$ one has

$$\sum_{m=n}^{\infty} (m-1) \operatorname{cap}_2(B(0, r_m)) < 2^{-n}. \tag{7.7}$$

Let $z \in \Gamma_s$. Let $n \in \mathbb{N}$ and $s \in (2^{-(n+1)}, 2^{-n}]$. Then it follows from (7.7) that

$$\operatorname{cap}_2(\Omega \cap B(z,s))\frac{1}{s} \leqslant \operatorname{cap}_2\big(\Omega \cap \big([0, 2^{-n}] \times [0,1]\big)\big)2^{n+1} \leqslant 2^{-n}2^{n+1} = 2.$$

Therefore

$$\int_0^1 \operatorname{cap}_2(\Omega \cap B(z,s))\frac{1}{s}\,ds < \infty.$$

This means that Ω is 2-thin at z. Hence z is not in the 2-fine boundary of Ω by Remark 7.31. ◇

The following extends the corresponding result for $p = 2$ in [AW03, Theorem 2.3].

Theorem 7.45 (cf. [Bie09a, Corollary 4.4]). *Let $p \in (1, \infty)$ and let $u \in \widetilde{W}^{1,p}(\Omega)$. Let $\tilde{u}\colon \overline{\Omega} \to \mathbb{R}$ be a relatively p-quasi continuous representative of u. Then $u \in W_0^{1,p}(\Omega)$ if and only if $\tilde{u}(z) = 0$ for relatively p-quasi every $z \in \Gamma$.*

For $p = 2$, the notion of the relative capacity has recently been extended to Sobolev spaces on σ-compact Riemannian manifolds, see [ABE12, Section 5]. In particular, a corresponding version of Theorem 7.45 holds in the manifold setting, see [ABE12, Theorem 5.2].

The next result by Swanson and Ziemer characterises $W_0^{1,p}(\Omega)$ in some sense intrinsically in $W^{1,p}(\Omega)$. In its proof Proposition 7.37 was established as an intermediate result. We recall that the latter proposition was a central ingredient for Theorem 7.38,

Theorem 7.46 (cf. [SZ99, Theorem 2.2]). *Let $p \in (1, \infty)$ and let $u \in W^{1,p}(\Omega)$. Then $u \in W_0^{1,p}(\Omega)$ if and only if*

$$\lim_{r \to 0+} \frac{1}{r^d} \int_{\Omega \cap B(z,r)} |u| = 0$$

for p-quasi every $z \in \Gamma$.

The following result extends Theorem 7.46 to the case $p = 1$. It is the lack of reflexivity that complicates matters in this case. See for example [FZ72, Section 4], where a version of the 1-capacity is studied with tools from geometric measure theory.

Theorem 7.47 (cf. [Swa07, Theorem 5.2]). *Let* $u \in W^{1,1}(\Omega)$. *Then* $u \in W_0^{1,1}(\Omega)$ *if and only if*

$$\lim_{r\to 0+} \frac{1}{r^d} \int_{\Omega\cap B(z,r)} |u| = 0$$

for $\mathcal{H}^{d-1}$*-a.e.* $z \in \Gamma$.

7.5 Consequences of the extension result

In this section we give corollaries to Theorem 7.38 and provide examples where $V^p(\Omega) \neq W_0^{1,p}(\Omega)$ or $V^p(\Omega) \neq W_0^{1,p}(\overline{\Omega})$. The following is an immediate consequence of Theorem 7.38 and Theorem A.46.

Corollary 7.48. *Suppose that* Ω *has continuous boundary. Then* $V^p(\Omega) = W_0^{1,p}(\Omega)$.

Corollary 7.49. *Suppose that* Ω *is topologically regular and* $p > d$. *Then* $V^p(\Omega) = W_0^{1,p}(\Omega)$.

Proof. Let $u \in V^p(\Omega)$. We shall show that $u \in W_0^{1,p}(\Omega)$. By Theorem 7.38 we have $u^* \in W^{1,p}(\mathbb{R}^d)$. As $p > d$, we may assume that $u^* \in W^{1,p}(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$ by the Sobolev embedding theorem, Theorem A.50. Clearly $u^*(z) = 0$ for all $z \in \mathbb{R}^d \setminus \overline{\Omega}$. Let $z \in \Gamma$. As Ω is topologically regular, there exists a sequence $z_n \in \mathbb{R}^d \setminus \overline{\Omega}$ such that $\lim_{n\to\infty} z_n = z$. It follows from the continuity of u^* that $u^*(z) = 0$. So $u^*|_{\overline{\Omega}} \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ vanishes pointwise on Γ. Now Proposition A.43 implies that $u \in W_0^{1,p}(\Omega)$. □

Remark 7.50. Corollary 7.49 does not hold without the assumption that Ω is topologically regular. For example, let $\Omega = \mathbb{R}^d \setminus \{0\}$ for a $d \geqslant 2$ and suppose that $p > d$. We show that $W_0^{1,p}(\Omega) \neq V^p(\Omega)$. Let $\zeta \in C_c^\infty(\mathbb{R}^d)$ be such that $\zeta(0) = 1$. Clearly $\zeta \in V^p(\Omega)$ as $\mathcal{H}^{d-1}(\{0\}) = 0$. Assume, for contradiction, that $\zeta \in W_0^{1,p}(\Omega)$. Then there exists a sequence (ζ_n) in $C_c^\infty(\mathbb{R}^d)$ such that $\operatorname{supp} \zeta_n \subset \Omega$ and $\zeta_n \to \zeta$ in $W^{1,p}(\Omega)$. Then $\zeta_n^* \to \zeta^*$ in $W^{1,p}(\mathbb{R}^d)$. By the Sobolev embedding theorem, Theorem A.50, it follows that $0 = \zeta_m(0) \to \zeta(0) = 1$, which is a contradiction. Hence $\zeta \notin W_0^{1,p}(\Omega)$.

We point out that for $p \leqslant d$ and $\Omega = \mathbb{R}^d \setminus \{0\}$ one has $W_0^{1,p}(\Omega) = W^{1,p}(\mathbb{R}^d)$, see [EE87, Corollary VIII.6.4].

Due to connections to the stability of the Dirichlet problem, it is well studied when $W_0^{1,p}(\Omega)$ and $W_0^{1,p}(\overline{\Omega})$ are equal. In the case that Ω is topologically regular, several potential theoretic conditions have been presented in [AH96, Theorem 11.4.1] that are equivalent to this equality. The following theorem lists two of these conditions. In combination with Theorem 7.38 this immediately yields sufficient conditions on Ω which ensure that $V^p(\Omega) = W_0^{1,p}(\Omega)$.

Theorem 7.51 (cf. [AH96, Theorem 11.4.1]). *Suppose that Ω is topologically regular and $p \in (1, d]$. The following statements are equivalent.*

(i) $W_0^{1,p}(\Omega) = W_0^{1,p}(\overline{\Omega})$.

(ii) $\mathrm{cap}_p(G \setminus \Omega) = \mathrm{cap}_p(G \setminus \overline{\Omega})$ *for all* $G \subset \mathbb{R}^d$ *open.*

(iii) $\mathrm{cap}_p(\Gamma \setminus b_p(\mathbb{R}^d \setminus \overline{\Omega})) = 0$.

The following is a basic example for a measure geometric condition that can be obtained using properties of the p-fine topology. Its proof is inspired by the techniques in [Hed00, p. 95].

Theorem 7.52. *Let* $p \in (1, d]$. *Suppose that there exists a* p*-polar set* $P \subset \mathbb{R}^d$ *such that no point in* $\Gamma \setminus P$ *is a Lebesgue density point of* $\overline{\Omega}$. *Then* $V^p(\Omega) = W_0^{1,p}(\Omega) = W_0^{1,p}(\overline{\Omega})$.

Proof. First observe that $(\mathbb{R}^d \setminus \overline{\Omega}) \cup b_p(\mathbb{R}^d \setminus \overline{\Omega})$ is p-finely closed by Proposition 7.30. Hence the set $\overline{\Omega} \setminus b_p(\mathbb{R}^d \setminus \overline{\Omega})$, which is its complement in $\mathbb{R}^d$, is p-finely open. By Proposition 7.36 the set $\overline{\Omega} \setminus b_p(\mathbb{R}^d \setminus \overline{\Omega})$ is open in the Lebesgue density topology. Consequently, every point of $\overline{\Omega} \setminus b_p(\mathbb{R}^d \setminus \overline{\Omega})$ is a Lebesgue density point of $\overline{\Omega}$. Let $z \in \Gamma \setminus P$. It follows from the assumption that $z \notin \overline{\Omega} \setminus b_p(\mathbb{R}^d \setminus \overline{\Omega})$. Hence $z \in b_p(\mathbb{R}^d \setminus \overline{\Omega})$. This shows that $\Gamma \setminus P \subset b_p(\mathbb{R}^d \setminus \overline{\Omega})$.

Now let $u \in W^{1,p}(\mathbb{R}^d)$ be such that $u = 0$ a.e. on $\mathbb{R}^d \setminus \overline{\Omega}$. Let $\tilde{u}$ be a p-quasi continuous representative of u. By Theorem 7.23 it follows that $\tilde{u}(x) = 0$ for p-quasi every $x \in \mathbb{R}^d \setminus \overline{\Omega}$. We may suppose that $\tilde{u}(x) = 0$ for all $x \in \mathbb{R}^d \setminus \overline{\Omega}$. Furthermore, the function $\tilde{u}$ is p-finely continuous p-quasi everywhere in $\mathbb{R}^d$ by Proposition 7.33. Enlarging P if necessary, we may suppose that $\tilde{u}$ is p-finely continuous at every point in $\mathbb{R}^d \setminus P$. Let $x_0 \in \mathbb{R}^d \setminus P$. Suppose that $\tilde{u}(x_0) \neq 0$ and let $\varepsilon = |\tilde{u}(x_0)|/2$. Then

$$E := \left\{ x \in \mathbb{R}^d : |\tilde{u}(x) - \tilde{u}(x_0)| \geqslant \varepsilon \right\}$$

is p-thin at x_0 by Lemma 7.32. Clearly $\mathbb{R}^d \setminus \overline{\Omega} \subset E$. Hence $\mathbb{R}^d \setminus \overline{\Omega}$ is p-thin at x_0. So $x_0 \notin b_p(\mathbb{R}^d \setminus \overline{\Omega})$ and therefore in particular $x_0 \notin \Gamma \setminus P$. This shows that $\tilde{u}(x) = 0$ for p-quasi every $x \in \mathbb{R}^d \setminus \Omega$. Hence $u|_\Omega \in W_0^{1,p}(\Omega)$ by Theorem 7.41. We have proved $W_0^{1,p}(\overline{\Omega}) = W_0^{1,p}(\Omega)$. Now it follows from Theorem 7.38 that $V^p(\Omega) = W_0^{1,p}(\Omega)$. □

Remark 7.53. Suppose $p \in (1, d]$. If Ω is Lipschitz, then clearly the conditions of Theorem 7.52 are satisfied for $P = \emptyset$. Next suppose that $\Omega \subset \mathbb{R}^2$ is a domain as depicted in Figure 7.4. We assume that there are holes arbitrarily close to all boundary points in the grey line segment on the left. Then Ω does not have continuous boundary. Moreover, we can arrange that the Lebesgue density of the holes is negligible at all points in the boundary. Then this domain satisfies $W_0^{1,p}(\Omega) = W_0^{1,p}(\overline{\Omega})$ by Theorem 7.52.

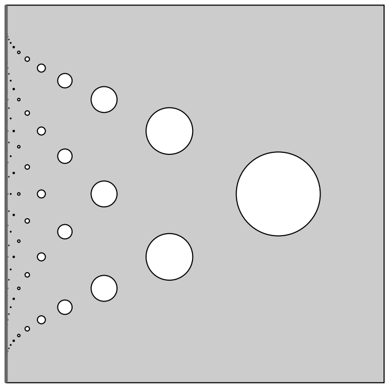

Figure 7.4. An open set $\Omega \subset \mathbb{R}^2$ that does not have continuous boundary, but which satisfies $W_0^{1,p}(\Omega) = W_0^{1,p}(\overline{\Omega})$ by Theorem 7.52.

We finally present several examples.

Example 7.54. Suppose $d > 1$ and $p \in (1, d)$. Choosing $h(r) = r^{d-1}$, $\alpha = 1$ and $c(r) = r^{d-p}$ in [AH96, Theorem 5.4.1] yields a compact set $E \subset \mathbb{R}^d$ such that $\mathcal{H}^{d-1}(E) = 0$, but $\operatorname{cap}_p(E) > 0$. We may suppose $E \subset B(0, 1)$. Let $\Omega = B(0, 2) \setminus E$, see Figure 7.5. Let $\zeta \in C_c^\infty(\mathbb{R}^d)$ be such that $\operatorname{supp} \zeta \subset B(0, 2)$ and $\zeta = 1$ in $B(0, 1)$. Note that $\zeta \in V^p(\Omega)$. But $\zeta \notin W_0^{1,p}(\Omega)$ by Theorem 7.41. $\diamond$

Note that the domain Ω in Example 7.54 is not topologically regular. In the next example we shall use the construction from [Hed00, Example, p. 94] to obtain a topologically regular domain Ω such that $W_0^{1,p}(\Omega) \neq W_0^{1,p}(\overline{\Omega})$ and $V^p(\Omega) = W_0^{1,p}(\overline{\Omega})$. Moreover, we point out that the construction can be adapted to allow $p = d$; see [AH96, Theorem 5.4.1].

Example 7.55. Suppose $d > 1$ and $p \in (1, d)$. Let $E \subset B(0, 1)$ be as in Example 7.54. Let (x_n) be a sequence of pairwise distinct elements in $B(0, 1) \setminus E$ such that the set of limit points of (x_n) is equal to E. We may suppose that $\operatorname{dist}(x_n, E) < \frac{1}{n}$ for all $n \in \mathbb{N}$. Let (r_n) be a sequence of strictly positive numbers such that $r_n \leqslant 2^{-n}$, $B(x_n, 2r_n) \subset (B(0, 1) \setminus E)$ and $B(x_k, 2r_k) \cap B(x_n, 2r_n) = \emptyset$ for all $k, n \in \mathbb{N}$ such that

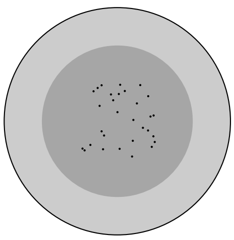

Figure 7.5. A ball where in the interior a compact set of positive p-capacity and $\mathcal{H}^{d-1}$-measure zero has been removed.

$k \neq n$. By further decreasing the elements of (r_n), if necessary, we may suppose that

$$\operatorname{cap}_p\Big(\bigcup_{n\in\mathbb{N}} B(x_n, r_n)\Big) < \operatorname{cap}_p(E).$$

We define

$$\Omega := B(0,2) \setminus \Big(E \cup \bigcup_{n\in\mathbb{N}} \overline{B(x_n, r_n)}\Big).$$

We first prove a few topological properties of Ω.

Claim 1: Ω is open. It suffices to show that

$$E \cup \bigcup_{n\in\mathbb{N}} \overline{B(x_n, r_n)} \tag{7.8}$$

is closed. This is clear as the set in (7.8) contains its limit points.

Claim 2: $\overline{\Omega}$ is equal to the set

$$A := \overline{B(0,2)} \setminus \bigcup_{n\in\mathbb{N}} B(x_n, r_n).$$

Since A is closed and $\Omega \subset A$, clearly $\overline{\Omega} \subset A$. Conversely, it is readily verified that every point in A is a limit point of Ω. This shows that $\overline{\Omega} = A$.

Claim 3: Ω is topologically regular. First observe that

$$\Gamma = \overline{\Omega} \setminus \Omega = \partial B(0,2) \cup E \cup \bigcup_{n\in\mathbb{N}} \partial B(x_n, r_n).$$

Clearly $\partial\overline{\Omega} \subset \Gamma$. The converse inclusion follows by observing that every point in Γ is a limit point of $\mathbb{R}^d \setminus \overline{\Omega}$.

By choosing $G = B(0,2)$ in Theorem 7.51 (ii), it follows that $W_0^{1,p}(\Omega) \neq W_0^{1,p}(\overline{\Omega})$. We shall prove that $V^p(\Omega) = W_0^{1,p}(\overline{\Omega})$. The inclusion $V^p(\Omega) \subset W_0^{1,p}(\overline{\Omega})$ follows from Theorem 7.38. For the converse, let $u \in W_0^{1,p}(\overline{\Omega})$. We first assume in addition that $u \in L^\infty(\Omega)$ and $u \geqslant 0$ a.e. on Ω. Let

$$\Omega_n := B(0,2) \setminus \bigcup_{k=1}^{n} \overline{B(x_k, r_k)}$$

for all $n \in \mathbb{N}$. Then Ω_n is Lipschitz and $\Omega \subset \Omega_n$ for all $n \in \mathbb{N}$. So it follows from Remark 7.14 that $u^*|_{\Omega_n} \in W_0^{1,p}(\Omega_n)$ for all $n \in \mathbb{N}$. For all $n \in \mathbb{N}$ let $u_n \in C_c^\infty(\Omega_n)$ be such that $0 \leqslant u_n \leqslant 2\|u\|_\infty$ and $\|u^*|_{\Omega_n} - u_n\|_{W^{1,p}(\Omega_n)} \leqslant \frac{1}{n}$. Then $u_n|_\Omega \to u$ in $W^{1,p}(\Omega)$. Since $\mathcal{H}^{d-1}(E) = 0$, it follows that $\lim_{n\to\infty} \mathcal{H}^{d-1}(\Gamma \setminus \partial\Omega_n) = 0$. This implies that $u_n|_\Gamma \to 0$ in $L^1(\Gamma)$. Hence $u \in V_1^p(\Omega)$. For the general case, we note that $W_0^{1,p}(\overline{\Omega}) = \widetilde{W}_0^{1,p}(\Omega)$ since Γ is a Lebesgue nullset as $\mathcal{H}^{d-1}(\Gamma) < \infty$.

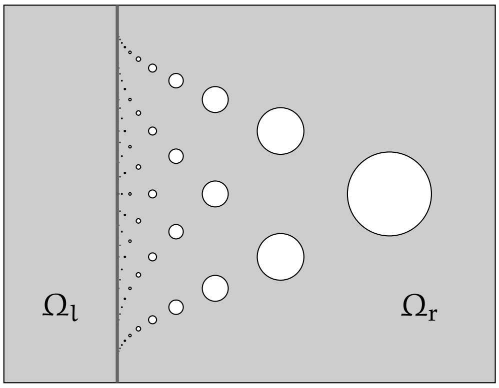

Figure 7.6. This modified version of Figure 7.4 shows an open set $\Omega = \Omega_l \cup \Omega_r \subset \mathbb{R}^2$ where $W_0^{1,p}(\Omega) = V^p(\Omega) \neq W_0^{1,p}(\overline{\Omega})$ by Example 7.56.

Therefore $W_0^{1,p}(\overline{\Omega})$ is a closed subspace of $W^{1,p}(\Omega)$ and by Proposition A.41 we can approximate u in $W^{1,p}(\Omega)$ by elements in $W_0^{1,p}(\overline{\Omega}) \cap L^\infty(\Omega)$. Then $u \in V_1^p(\Omega)$ by Lemma 7.7. So $V_1^p(\Omega) = W_0^{1,p}(\overline{\Omega})$. This shows that $W_0^{1,p}(\Omega)$ is properly contained in $V^p(\Omega) = W_0^{1,p}(\overline{\Omega})$. $\diamond$

For simplicity we assume in the following example that $d = 2$. It is clear that one can similarly obtain examples for every $d \geqslant 2$.

Example 7.56. Suppose that $d = 2$ and $p \in (1, d]$. We consider a domain $\Omega \subset \mathbb{R}^2$ as depicted in Figure 7.6. More precisely, suppose that $\Omega = \big((-1,1) \times (0,1)\big) \setminus \overline{U}$, where $U \subset (0,1)^2$ is the open set considered in Example 7.44. Clearly Ω is topologically regular. Using the same argument as in Example 7.55, we obtain that $W_0^{1,p}(\Omega) \neq W_0^{1,p}(\overline{\Omega})$, provided the radii of the holes decrease sufficiently quickly. Next we show that $V^p(\Omega) = W_0^{1,p}(\Omega)$. Let $u \in V^p(\Omega)$. Set $\Omega_l \coloneqq \Omega \cap \big((-1,0) \times (0,1)\big)$ and $\Omega_r \coloneqq \Omega \cap \big((0,1) \times (0,1)\big)$ as indicated in Figure 7.6. Then $\Omega = \Omega_l \cup \Omega_r$, $u|_{\Omega_l} \in V^p(\Omega_l)$ and $u|_{\Omega_r} \in V^p(\Omega_r)$. So it follows from Theorem 7.52 that $u|_{\Omega_l} \in W_0^{1,p}(\Omega_l)$ and $u|_{\Omega_r} \in W_0^{1,p}(\Omega_r)$. Consequently $u \in W_0^{1,p}(\Omega)$. This shows that $V^p(\Omega) = W_0^{1,p}(\Omega)$ is properly contained in $W_0^{1,p}(\overline{\Omega})$. $\diamond$

Remark 7.57. Suppose $d > 1$ and $p \in (1, d]$. By suitably combining Example 7.56 and Example 7.55, one can obtain a topologically regular bounded open set Ω such that $\mathcal{H}^{d-1}(\Gamma) < \infty$ and $V^p(\Omega)$ is different from both $W_0^{1,p}(\Omega)$ and $W_0^{1,p}(\overline{\Omega})$. We point out that it can be arranged that Ω is connected. For example, the domain in Example 7.54 is connected by [LMZ86, Exercise 6.C.22]. This can be used to show that the domain in Example 7.55 is connected.

7.6 Order ideals and the space of elements with weak trace zero

We repeat that in this chapter we always deal with real spaces in order to simplify the exposition.

Definition 7.58. An **ordered vector space** is a vector space V equipped with a partial order $\leqslant$ such that the following two properties are satisfied.

(i) If $x, y \in V$ and $x \leqslant y$, then $x + z \leqslant y + z$ for all $z \in V$.

(ii) If $x, y \in V$ and $x \leqslant y$, then $\alpha x \leqslant \alpha y$ for all $\alpha \geqslant 0$.

A **Riesz space** is an ordered vector space $(V, \leqslant)$ such that for all $x, y \in V$ there exists the supremum $x \vee y := \sup\{x, y\}$ and the infimum $x \wedge y := \inf\{x, y\}$ with respect to $\leqslant$. Let $(V, \leqslant)$ be a Riesz space. Then $|x| := x \vee -x$ for all $x \in V$. A vector subspace W of V is called an **ideal** of V if for all $x \in W$ and $y \in V$ the relation $|y| \leqslant |x|$ implies $y \in W$. If V is in addition a Banach space such that $|u| \leqslant |v|$ implies $\|u\| \leqslant \|v\|$ for all $u, v \in V$, then it is called a **Banach lattice**.

Example 7.59. Let (X, Σ, μ) be a measure space. Let $p \in [1, \infty)$ and let $L^p(X)$ be the usual Lebesgue space of p-integrable measurable functions on X that are identified if they are equal μ-a.e. We define a partial order $\leqslant$ in $L^p(X)$ by letting $u \leqslant v$ if and only if $u \leqslant v$ μ-a.e. on X. Then $(L^p(X), \leqslant)$ is a Riesz space. Furthermore, $L^p(X)$ is a Banach lattice.

Let $A \in \Sigma$ be a measurable subset of X. Then the set

$$\{u \in L^p(X) : u = 0 \ \mu\text{-a.e. on } A\} \tag{7.9}$$

is easily seen to be a closed ideal of $L^p(X)$. ◇

The following description of closed ideals of $L^p(X)$ is a special case of [Sch74, Example 2 on p. 157].

Proposition 7.60 (Schaefer). *Let (X, Σ, μ) be a σ-finite measure space. Let $p \in [1, \infty)$. Then every closed ideal of $L^p(X)$ is of the form (7.9) for some measurable $A \in \Sigma$.*

Remark 7.61. It was also shown in [Sch74, Example 2 on p. 157] that the statement of the proposition does in general not extend to $p = \infty$.

It follows from Proposition A.41 that $W^{1,p}(\Omega)$ and $\widetilde{W}^{1,p}(\Omega)$ are Riesz subspaces of $L^p(\Omega)$. Moreover, by Theorem 7.23 the induced order on $\widetilde{W}^{1,p}(\Omega)$ is compatible with the finer structure of the Sobolev space in the following sense.

Corollary 7.62. *Let $u, v \in \widetilde{W}^{1,p}(\Omega)$. Then $u \leqslant v$ a.e. on Ω if and only if $\tilde{u} \leqslant \tilde{v}$ relatively p-q.e. on $\overline{\Omega}$.*

Remark 7.63. It follows from Remark 7.25 that a corresponding statement holds for the p-quasi continuous representatives in $W^{1,p}(\Omega)$ for a general open $\Omega \subset \mathbb{R}^d$.

We are interested in the closed ideals of Sobolev spaces. Taking the finer structure of Sobolev spaces into consideration, it is not surprising that the Lebesgue measure in general does not suffice any more to distinguish closed ideals.

Example 7.64. Let $A \subset \overline{\Omega}$ be Borel measurable. Define

$$D_0(A) = \Big\{u \in \widetilde{W}^{1,p}(\Omega) : \tilde{u} = 0 \text{ relatively p-q.e. on } A\Big\}. \tag{7.10}$$

Then $D_0(A)$ is closed ideal in $\widetilde{W}^{1,p}(\Omega)$. The two claims follow from Proposition 7.26 and Corollary 7.62. $\diamond$

The following result shows that closed ideals in $\widetilde{W}^{1,p}(\Omega)$ or $W^{1,p}(\mathbb{R}^d)$ can be described similarly as those of $L^p(X)$ in Proposition 7.60, by using quasi continuous representatives and capacity instead of the Lebesgue measure. We note that the proof of [Sto93, Theorem 1] also works for $p \neq 2$ by the remark on [Sto93, p. 267].

Theorem 7.65 (Stollmann, [Sto93, Theorem 1])**.** *Suppose* $p \in (1, \infty)$. *Let* W *be a closed ideal of* $\widetilde{W}^{1,p}(\Omega)$. *Then there exists a Borel measurable set* $A \subset \overline{\Omega}$ *such that* $W = D_0(A)$, *where* $D_0(A)$ *is defined as in* (7.10).

Using Proposition A.41 it is readily checked that $W_0^{1,p}(\Omega)$ is a closed ideal of both $\widetilde{W}^{1,p}(\Omega)$ and $W^{1,p}(\mathbb{R}^d)$. Then Theorem 7.65 allows to obtain an easy proof for both Theorem 7.45 and Theorem 7.41. For details, see [AW03, Theorem 2.3] and [AM95, Theorem 1.1]. We next show that Theorem 7.65 can be used to obtain similar descriptions for $V^p(\Omega)$.

Theorem 7.66. *The space* $V^p(\Omega)$ *is a closed ideal of* $\widetilde{W}^{1,p}(\Omega)$ *and, after extension by* 0*, of* $W^{1,p}(\mathbb{R}^d)$. *Moreover, there exist Borel sets* $A \subset \Gamma$ *and* $M \subset \mathbb{R}^d \setminus \Omega$ *such that*

$$\begin{aligned} V^p(\Omega) &= \Big\{u \in \widetilde{W}^{1,p}(\Omega) : \tilde{u} = 0 \textit{ relatively } p\textit{-q.e. on } A\Big\} \\ &= \Big\{u|_\Omega : u \in W^{1,p}(\mathbb{R}^d) \textit{ such that } \tilde{u} = 0 \; p\textit{-q.e. on } M\Big\}. \end{aligned} \tag{7.11}$$

Proof. It follows as in the proof of Lemma 7.7 that the space $V^p(\Omega)$ is closed in $\widetilde{W}^{1,p}(\Omega)$. Let $u \in \widetilde{W}^{1,p}(\Omega)$, $w \in V^p(\Omega)$ and suppose $|u| \leqslant |w|$. Due to Proposition A.41 we may suppose $0 \leqslant u \leqslant w$ a.e. on Ω. Let (u_n) and (w_n) be sequences in $W^{1,p}(\Omega) \cap C_c(\overline{\Omega})$ such that $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} w_n = w$ in $W^{1,p}(\Omega)$, $w_n|_{\Gamma_{\text{loc}}} \to 0$ in $L^1_{\text{loc}}(\Gamma_{\text{loc}})$ and $w_n(z) = 0$ for all $z \in \Gamma \setminus \Gamma_{\text{loc}}$ and $n \in \mathbb{N}$. We may assume that $u_n \geqslant 0$ and $w_n \geqslant 0$ for all $n \in \mathbb{N}$. Define $v_n := u_n \wedge w_n$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} v_n = u$ in $W^{1,p}(\Omega)$ by Proposition A.41. Moreover, $0 \leqslant v_n|_\Gamma \leqslant w_n|_\Gamma$ for all $n \in \mathbb{N}$. It follows that $u \in V^p(\Omega)$. Hence $V^p(\Omega)$ is a closed ideal in $\widetilde{W}^{1,p}(\Omega)$.

It follows from Corollary 7.39 that $V^p(\Omega)$ is, when extending elements by 0 to $\mathbb{R}^d$, a closed subspace of $W^{1,p}(\mathbb{R}^d)$. If $u \in W^{1,p}(\mathbb{R}^d)$ and $w \in V^p(\Omega)$ are such that $0 \leqslant u \leqslant w^*$ a.e. on $\mathbb{R}^d$, then $u|_\Omega \in \widetilde{W}^{1,p}(\Omega)$ and by the above $u|_\Omega \in V^p(\Omega)$.

Now it follows from Theorem 7.65 that there exist Borel sets $A \subset \overline{\Omega}$ and $M \subset \mathbb{R}^d$ such that (7.11) holds. It remains to show that we can arrange that $A \cap \Omega = \emptyset$ and $M \cap \Omega = \emptyset$. Let $K \subset \Omega$ be compact. There exists a $\zeta \in C_c^\infty(\Omega)$ such that $\zeta = 1$ on K. Clearly $\zeta \in V^p(\Omega)$. Hence $\operatorname{cap}_{p,\Omega}(A \cap K) = 0$ and $\operatorname{cap}_p(M \cap K) = 0$. Exhausting Ω with compact sets finishes the proof. □

Remark 7.67. Let $M \subset \mathbb{R}^d \setminus \Omega$ be as in Theorem 7.66.

1. It is clear that M may be changed by a p-polar set without affecting (7.11). It follows from Theorem 7.23 that

$$W_0^{1,p}(\overline{\Omega}) = \Big\{u \in W^{1,p}(\mathbb{R}^d) : \tilde{u} = 0 \text{ p-q.e. on } \mathbb{R} \setminus \overline{\Omega}\Big\}. \tag{7.12}$$

Hence we may assume $\mathbb{R} \setminus \overline{\Omega} \subset M$ by Theorem 7.38.

2. It can happen that both $\operatorname{cap}_p(\Gamma \setminus M) > 0$ and $\operatorname{cap}_p(\Gamma \cap M) > 0$. In fact, suppose $d > 1$ and $p \in (1, d]$. Then there exists a bounded open set Ω such that $W_0^{1,p}(\Omega) \subsetneq V^p(\Omega) \subsetneq W_0^{1,p}(\overline{\Omega})$ by Remark 7.57. By the previous remark we may assume that $\mathbb{R} \setminus \overline{\Omega} \subset M$. Then it follows from Theorem 7.41 that $\operatorname{cap}_p(\Gamma \setminus M) > 0$. The property $\operatorname{cap}_p(\Gamma \cap M) > 0$ is clear by (7.12).

3. In general, the set M is not uniquely determined up to a p-polar set. This is in contrast to Proposition 7.60, where the measurable set $A \subset X$ is determined up to a μ-nullset. For example, suppose that Ω is a bounded Lipschitz domain. Then $W_0^{1,p}(\Omega) = V^p(\Omega) = W_0^{1,p}(\overline{\Omega})$ by Remark 7.14. So one can choose for example $M = \mathbb{R}^d \setminus \Omega$ or $M = \mathbb{R}^d \setminus \overline{\Omega}$ in Theorem 7.66. We note, however, that $\mathbb{R}^d \setminus \Omega$ is the p-fine closure of $\mathbb{R}^d \setminus \overline{\Omega}$ by the argument in the proof of Theorem 7.52.

7.7 Other notions of traces

To complement our study of weak traces, in this section we shall briefly discuss two other approaches to introduce traces in Sobolev spaces on general domains. First we shall make some remarks about the 'classical' trace theory for Sobolev spaces, which is based on regularity assumptions on the domain. We refer to [JW84] for an overview of the classical theory and a general treatment that includes other classes of function spaces and fractional-order Sobolev spaces. The results in [JW84] characterise the trace space as a suitable Besov space and assert the existence of continuous extension and restriction operators, provided the domain satisfies certain regularity conditions. The essential notion there is that of an *s-set*.

Definition 7.68. Let $A \subset \mathbb{R}^d$ and $s \in (0, d]$. Then A is called an **s-set** if there exist constants $c_1, c_2 > 0$ such that

$$c_1 r^s \leqslant \mathcal{H}^s(A \cap B(x, r)) \leqslant c_2 r^s$$

for all $x \in A$ and $r \in (0, 1]$.

It follows directly from the definition that an s-set has Hausdorff dimension s. The following is a special case of [JW84, Theorem VIII.1]. Note that the assumptions are satisfied if Ω is a bounded Lipschitz domain.

Theorem 7.69. *Let $p \in (1, \infty)$. Suppose that Ω is a d-set and Γ is a $(d-1)$-set. Moreover, suppose that there exists a bounded linear extension operator from $W^{1,p}(\Omega)$ into $W^{1,p}(\mathbb{R}^d)$. Then there exists a bounded linear trace operator from $W^{1,p}(\Omega)$ onto the Besov space $B^{p,p}_{\beta}(\Gamma)$ where $\beta = 1 - \frac{1}{p}$.*

Under weak additional assumptions, the kernel of the trace operator in Theorem 7.69 has been described in [Mar87] and [Wal91]. These descriptions extend Theorem A.49 to more general domains and fractional-order Sobolev spaces.

7.7.1 Maz'ya's approach

The first approach that we consider in this section is due to Maz'ya. It is closely related to the notion of the weak trace. In [Maz85, Section 3.6 and Section 4.11] Maz'ya considers the Sobolev-type space $W^1_{p,r}(\Omega, \Gamma)$ defined as the completion of the space

$$Y^{p,r}(\Omega) := \left\{ u \in W^{1,1}(\Omega) \cap C^\infty(\Omega) \cap C(\overline{\Omega}) : u|_\Gamma \in L^r(\Gamma) \right\} \tag{7.13}$$

with respect to the norm

$$\|\nabla u\|_{L^p(\Omega)} + \|u|_\Gamma\|_{L^r(\Gamma)}.$$

It is clear that elements of this completion have traces in $L^r(\Gamma)$. Based on the isoperimetric inequality, Maz'ya in particular shows that the following remarkable Friedrichs-type inequality holds for general open sets Ω. There exists a $C > 0$ such that

$$\|u\|_{L^{d/(d-1)}(\Omega)} \leqslant C\big(\|\nabla u\|_{L^1(\Omega)} + \|u|_\Gamma\|_{L^{d/(d-1)}(\Gamma)}\big)$$

for all $u \in Y^{1,d/(d-1)}(\Omega)$, see [Maz85, Theorem 3.6.3]. As a more concrete application of these results, suppose that Ω has finite Lebesgue measure and let $p = r = 2$. Then there exists a $C > 0$ such that

$$\|u\|_{L^{2d/(d-1)}(\Omega)} \leqslant C\big(\|\nabla u\|_{L^2(\Omega)} + \|u|_\Gamma\|_{L^2(\Gamma)}\big) \tag{7.14}$$

for all $u \in Y^{2,2}(\Omega)$, see [Maz85, Subsection 4.11.1]. Maz'ya's inequality (7.14) plays a central role in [Dan00; AW03; AE11]. It implies that the completion $W^1_{2,2}(\Omega, \Gamma)$

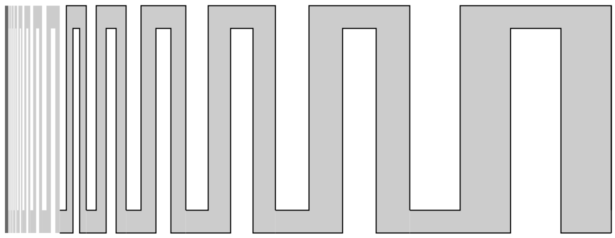

Figure 7.7. A domain where the points in the dark grey line segment on the left are 'inaccessible' from the inside with respect to the geodesic metric.

can be understood as a closed subspace of $H^1(\Omega) \oplus L^2(\Gamma)$. We shall adopt this viewpoint. In [Dan00; AW03] it was observed that the intersection of $\{0\} \oplus L^2(\Gamma)$ and $W^1_{2,2}(\Omega, \Gamma)$ can be nontrivial. So while every element of $W^1_{2,2}(\Omega, \Gamma)$ has a unique trace, this does not need to be true after a projection on the $H^1(\Omega)$ component. In particular, this shows why a weak trace is not unique in general.

7.7.2 Shvartsman's approach

The second approach that we discuss in this section is due to Shvartsman [Shv10] and very recent. It allows to define a trace for all functions in $W^{1,p}(\Omega)$ on a general connected domain $\Omega \subset \mathbb{R}^d$ provided that $p > d$. The latter condition implies that functions in $W^{1,p}(\Omega)$ are locally Hölder continuous. Fix $\alpha \in (0, 1]$ and define a metric ρ on Ω by

$$\rho(x, y) := \|x - y\|^\alpha + \inf_\gamma \int_\gamma \operatorname{dist}(z, \Gamma)^{\alpha-1} \, ds(z),$$

where the infimum is taken over all rectifiable curves γ in Ω that join x to y. If $\alpha = 1$, then the second term in the definition of ρ is simply the geodesic metric on Ω. Now suppose that $\alpha = (p - d)/(p - 1)$. Then $\alpha \in (0, 1)$. In [BS01, Theorem 3.2] it was proved that functions in $W^{1,p}(\Omega)$ are uniformly continuous on Ω with respect to ρ. So every function admits a unique continuous extension to the Cauchy completion of (Ω, ρ), which we denote by (Ω^*, ρ^*). It is not hard to see that ρ induces the Euclidean topology on Ω. It follows that one can consider Ω as an open subset of Ω^*. We set $\Gamma^* := \Omega^* \setminus \Omega$. Note that while every element of Γ^* corresponds to a point in Γ, in general this is not a one-to-one correspondence. Consider for example the domain $\Omega = (-2, 2)^2 \setminus ([-1, 1] \times \{0\})$, a box with a slit. Then this procedure will introduce 'upper' and 'lower' boundary points in Γ^* for the interior of the slit, which corresponds to cutting the domain open at the slit. Note that this is basically opposite to what the weak trace does. Moreover, not every point in Γ corresponds to an element of Γ^* as we shall see in Example 7.70. Shvartsman's trace for an element $u \in W^{1,p}(\Omega)$ is a continuous function defined on Γ^*, namely the restriction to Γ^* of the continuous extension of u to (Ω^*, ρ^*). In particular, this trace splits where the domain is on 'different sides' of the boundary and it is not defined on the 'inaccessible' part of Γ. Furthermore, Shvartsman gives a description of the trace space, i.e., the space of continuous functions on Γ^* that are traces of elements in $W^{1,p}(\Omega)$. Finally, we provide two examples of connected domains where 'large parts' of their boundary are 'inaccessible'. The first example is similar to the comb-like domain in [Shv10, Figure 3].

Example 7.70. In Figure 7.7 a connected domain in $\mathbb{R}^2$ is depicted where some points in Γ are 'inaccessible' from Ω in the metric ρ. More precisely, let $x \in \Omega$ and z be a boundary point on the grey line segment on the left. Suppose (x_n) is a sequence of points in Ω such that $x_n \to z$ in the Euclidean metric. Then the geodesic distance from x to x_n is not bounded for $n \to \infty$. In particular, (x_n) cannot be a Cauchy sequence in (Ω, ρ). ◇

Example 7.71. We consider the domain in Figure 7.1. Let $x \in \Omega$ and $z \in \Gamma_s$ be a boundary point in the inside of the grey rectangle. Then there exists a sequence (x_n) of points in Ω such that $x_n \to z$ in the Euclidean metric. While the geodesic distance from x to x_n is uniformly bounded in n, this is certainly not true for the metric ρ as $\alpha \in (0, 1)$ and due to the constant proximity to the boundary inside of the thinner and thinner cylinders. It follows that all boundary points in the inside of the grey rectangle are 'inaccessible' from Ω in the metric ρ. ◇

7.8 Notes and remarks

The notion of what we here call the *weak trace* has been introduced and studied in [AE11], although it had previously been used in a more general context in [AE12, Subsection 4.3]. It was Daners' study of the Maz'ya inequality (7.14) and Robin boundary value problems [Dan00] that stimulated some of the developments in [AW03] and [AE12] and which led to the introduction of this notion.

Considering the practical importance of traces for elements in Sobolev spaces it is not surprising that there exists a vast array of different approaches that extend the classical trace results for bounded Lipschitz domains. For $p = r = 2$, the notion under investigation here meshes well with the form method in the general sectorial setting. It allows to obtain meaningful traces for elements in Sobolev spaces on very rough domains. This allows to study objects like the Dirichlet-to-Neumann operator [AE11] or elliptic operators with Robin boundary conditions [AW03] on general domains. The notion is very natural in that it uses an approximation by Sobolev functions that are continuous on the closure of the domain. Furthermore, it is accompanied by the corresponding notion of the relative capacity, which has been encountered before in the study of Dirichlet forms [BH91, Section I.8].

However, a single element of $W^{1,p}(\Omega)$ can have multiple weak traces or none at all. If one is willing to work in a different Sobolev-type space, one can overcome this by using Maz'ya's spaces $W^1_{p,r}(\Omega, \Gamma)$, which we mentioned in Section 7.7. In the case $p = r = 2$, and specifically when studying Robin problems, the problem of nonuniqueness can be resolved by going to the regular part of the corresponding sectorial form, see [AW03, Section 3]. This approach has been employed in a very recent study of the principal eigenvalue of generalised Robin problems on arbitrary bounded domains [Dan13, Section 2].

A

Appendix

A.1 Basic properties of accretive operators

In this section we collect basic results about linear accretive operators for the reader's convenience. Most of the results are standard and can be found in a more general setting in the literature, see for example [Phi69] and [HP97, Chapter 3].

Definition A.1. Let A be an operator in a Hilbert space H with domain $D(A)$. We say that A is **accretive** if $\operatorname{Re}(Ax\,|\,x) \geqslant 0$ for all $x \in D(A)$. If A is accretive and $(I+A)$ is surjective, we say that A is **m-accretive**. The operator A is called **maximal accretive** if for every accretive operator B with $A \subset B$ it follows that $A = B$.

We first show that every m-accretive operator A is densely defined and satisfies $(0, \infty) \subset \rho(-A)$.

Lemma A.2. *Let A be an m-accretive operator. Then $D(A)$ is dense in H, i.e., A is densely defined.*

Proof. Let $f \in D(A)^\perp$. Then there exists an $x \in D(A)$ such that $(I+A)x = f$. Hence

$$((I+A)x\,|\,y) = 0$$

for all $y \in D(A)$. Choosing $y = x$ yields $\|x\|^2 = -\operatorname{Re}(Ax\,|\,x) \leqslant 0$. Thus $x = 0$ and $f = (I+A)x = 0$. ☐

Lemma A.3. *Let A be an operator. Then A is m-accretive if and only if $(0, \infty) \subset \rho(-A)$ and $(\lambda I + A)^{-1}$ is accretive for all $\lambda > 0$.*

Proof. '⇒': Let A be m-accretive. Since

$$\|x\|^2 \leqslant \operatorname{Re}\left((I+A)x \,|\, x\right) \leqslant \|(I+A)x\|\|x\|$$

for all $x \in D(A)$, it follows that $(I+A)\colon D(A) \to H$ is bijective with a bounded inverse. This means $1 \in \rho(-A)$. Let $R := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ be the open half-plane and $S := R \cap \rho(-A)$. Clearly S is open in R. We show that S is closed in R as well. To this end, assume $\lambda \in S$. Let $x \in D(A)$ and set $f := (\lambda I + A)x$. Then

$$\operatorname{Re}\lambda\|(\lambda I+A)^{-1}f\|^2 = \operatorname{Re}\lambda\|x\|^2 \leqslant \operatorname{Re}\left((\lambda I+A)x \,|\, x\right) = \operatorname{Re}\left(f \,|\, x\right) \leqslant \|f\|\|x\|.$$

Consequently we have the estimate

$$\|(\lambda I+A)^{-1}\| \leqslant \frac{1}{\operatorname{Re}\lambda}. \tag{A.1}$$

Let now $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in S such that $\lambda_n \to \mu \in R$ for $n \to \infty$. Due to the analyticity of the resolvent map, the norm of $(\lambda_n I + A)^{-1}$ would blow up for $n \to \infty$ if $\mu \in \sigma(-A)$. As this cannot happen due to (A.1), we infer $\mu \in S$. Therefore S is both open and closed in R, hence $S = R$ since $1 \in S$.
Let $\lambda > 0$. Let $f \in H$, set $x := (\lambda I + A)^{-1} f \in D(A)$ and observe

$$\operatorname{Re}\left((\lambda I+A)^{-1}f \,|\, f\right) = \operatorname{Re}\left(x \,|\, (\lambda I+A)x\right) \geqslant 0.$$

Hence $(\lambda I + A)^{-1}$ is accretive. This completes the proof of the 'only if' part.
'⇐': For the converse, let $x \in D(A)$, $\lambda > 0$ and $f := (\lambda I + A)x$. Then

$$\operatorname{Re}\left((\lambda I+A)x \,|\, x\right) = \operatorname{Re}\left(f \,|\, (\lambda I+A)^{-1}f\right) \geqslant 0.$$

Taking the limit $\lambda \searrow 0$ shows that A is accretive. Surjectivity of $(I + A)$ is obvious. □

Lemma A.4. *Let A be an accretive operator. Then $\operatorname{rg}(I+A)$ is closed if and only if A is closed.*

Proof. Define the map $F\colon \operatorname{gr} A \to \operatorname{rg}(I+A)$ by $F(x, Ax) = (I+A)x$. Give $\operatorname{gr} A$ the norm $\|(x, Ax)\|_{\operatorname{gr} A} = \|x\| + \|Ax\|$ and $\operatorname{rg}(I+A)$ the induced norm of H. Clearly F is continuous and surjective. The inequality $\|x\| + \|Ax\| \leqslant 2\|x\| + \|(I+A)x\| \leqslant 3\|(I+A)x\|$ for all $x \in D(A)$ implies that F is bijective and bicontinuous. This proves the claim. □

Proposition A.5. *Let A be an operator in H. Then A is m-accretive if and only if A is closed and maximal accretive.*

Proof. '⇒': Assume that A is m-accretive and $A \subset B$ for an accretive operator B. Then $(I+A) \subset (I+B)$, where both operators are bijective. Let $x \in D(B)$. Then there exists an $x' \in D(A)$ such that $(I+A)x' = (I+B)x$. Using the injectivity of the operator $I+B$, we deduce that $x = x' \in D(A)$. So $A = B$. Hence A is maximal accretive. Since $1 \in \rho(-A)$, it is clear that A is closed.

'$\Leftarrow$': This direction follows from [Phi59, Lemma 1.1.3 and the Corollary of Theorem 1.1.1 on p. 201]. We give a proof to be self-contained.
Assume that $f \in (\operatorname{rg}(I+A))^{\perp}$. If $f \in D(A)$, then

$$0 = \operatorname{Re}((I+A)f \mid f) = \|f\|^2 + \operatorname{Re}(Af \mid f) \geqslant \|f\|^2$$

and $f = 0$. Therefore we can define an extension A_1 of A by $D(A_1) = \operatorname{span}\{D(A), f\}$ and

$$A_1(x + \alpha f) = Ax + \alpha f$$

for all $x \in D(A)$ and $\alpha \in \mathbb{C}$. Observe that

$$\operatorname{Re}(A_1(x+\alpha f) \mid x + \alpha f) = \operatorname{Re}(Ax \mid x) + \|\alpha f\|^2 \geqslant 0$$

for all $x \in D(A)$ and $\alpha \in \mathbb{C}$. This shows that A_1 is accretive. Maximality of A implies that $A = A_1$, i.e., $f = 0$. Since A is closed, the range of $I + A$ is closed by Lemma A.4. Therefore $\operatorname{rg}(I+A) = H$ and A is m-accretive. □

Corollary A.6. *If $A \in \mathcal{L}(H)$ is accretive, then A is m-accretive.*

Lemma A.7. *Let A be a densely defined, accretive operator. Then $\ker A \subset \ker A^*$.*

Proof. Let $y \in \ker A$. Then for all $x \in D(A)$ and $\lambda \in \mathbb{C}$ we have

$$\operatorname{Re}(Ax \mid \lambda y) + \operatorname{Re}(Ax \mid x) = \operatorname{Re}(A(x+\lambda y) \mid x + \lambda y) \geqslant 0.$$

So $(Ax \mid y) = 0$ for all $x \in D(A)$. Therefore $y \in D(A^*)$ and $A^* y = 0$. □

If A is an m-accretive operator, then it is densely defined by Lemma A.2. Therefore its adjoint operator exists.

Proposition A.8. *Let A be an m-accretive operator. Then we have the following.*

(a) A^* *is m-accretive.*

(b) $\ker A = \ker A^*$.

Proof. Lemma A.3 yields that also A^* is m-accretive. Then (b) follows from Lemma A.7. □

Lemma A.9. *Let A be a densely defined, accretive operator. Then A is closable and its closure $\overline{A}$ is accretive.*

Proof. Let (x_n) be a sequence in $D(A)$ and $y \in H$. Suppose that $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} Ax_n = y$. We need to prove that $y = 0$. To this end, let $\alpha > 0$ and $z \in D(A)$. By accretivity of A we have

$$\|\tfrac{1}{\alpha}x_n + z\| \leqslant \|(I + \alpha A)(\tfrac{1}{\alpha}x_n + z)\|.$$

Letting $n \to \infty$ and using the triangle inequality yields

$$\|z\| \leqslant \|y + z + \alpha Az\| \leqslant \|y + z\| + \alpha\|Az\|.$$

Since this holds for any $\alpha > 0$, it follows that

$$\|z\| \leqslant \|y + z\|$$

for all $z \in D(A)$. This implies that $y = 0$ because $D(A)$ is dense in H.

As the graph of $\overline{A}$ is simply the closure of $\operatorname{gr} A$ in $H \times H$, the accretivity of $\overline{A}$ is obvious. □

Note that the following is a consequence of Proposition A.5 and Lemma A.9.

Corollary A.10. *Let A be a densely defined, accretive operator. Then there exists an m-accretive extension of A.*

Proposition A.11. *Let A be a densely defined, closed, accretive operator. Then A is m-accretive if and only if A^* is accretive.*

Proof. If A is m-accretive, then A^* is m-accretive by Proposition A.8.

To prove the converse, assume that A^* is accretive. Let B be a maximal accretive extension of A, which exists thanks to Zorn's lemma. Then B is closed by Lemma A.9. Hence B is m-accretive by Proposition A.5 and B^* is m-accretive by Proposition A.8. Let $y \in D(B^*)$. Then

$$(Ax \,|\, y) = (Bx \,|\, y) = (x \,|\, B^* y)$$

for all $x \in D(A) \subset D(B)$. Hence A^* is an extension of B^*. So $A^* = B^*$. Since A is closed and densely defined, one obtains $A = A^{**} = B^{**} = B$. Thus A is m-accretive. □

We will need the following perturbation result for an invertible m-accretive operator. The part about the invertibility of the operator $A + S$ appears to be new.

Proposition A.12. *Let A be an m-accretive operator in H. Let S be a bounded sectorial operator on H with vertex 0 and semi-angle θ. Suppose A is invertible. Then the operator $A + S$ is m-accretive and invertible. Moreover,*

$$\|(A + S)^{-1}\| \leqslant 2\|A^{-1}\| + (1 + \tan\theta)^2 \|S\| \, \|A^{-1}\|^2.$$

Proof. Clearly, the operator $A+S$ is densely defined, accretive and closed. Since also its adjoint operator $(A+S)^* = A^* + S^*$ is accretive, the operator $A+S$ is m-accretive by Proposition A.11.

First suppose that there exists an $\varepsilon > 0$ such that $\operatorname{Re}(Ax\,|\,x) \geqslant \varepsilon\,\|x\|^2$ for all $x \in D(A)$. Then $\varepsilon\,\|x\|^2 \leqslant \operatorname{Re}((A+S)x\,|\,x) \leqslant \|(A+S)x\|\,\|x\|$ and hence $\varepsilon\,\|x\| \leqslant \|(A+S)x\|$ for all $x \in D(A)$. This implies that $A+S$ is injective and has closed range. By Proposition A.8 we obtain

$$\big(\operatorname{rg}(A+S)\big)^{\perp} = \ker(A+S)^* = \ker(A+S) = \{0\}.$$

Hence $A+S$ is invertible. By the second resolvent identity we have

$$(A+S)^{-1} - A^{-1} = -A^{-1}S(A+S)^{-1}.$$

Let $P = \operatorname{Re} S = \frac{1}{2}(S+S^*)$. Then by [Kat80, Theorem VI.3.2] there exists a symmetric operator $B \in \mathcal{L}(H)$ such that $\|B\| \leqslant \tan\theta$ and $S = P^{1/2}(I+iB)P^{1/2}$. By plugging the latter into the above equation, we obtain

$$(A+S)^{-1} - A^{-1} = -\big(A^{-1}P^{1/2}(I+iB)\big)\big(P^{1/2}(A+S)^{-1}\big).$$

If $x \in D(A)$, then

$$\|P^{1/2}x\|^2 = (Px\,|\,x) \leqslant \operatorname{Re}((A+S)x\,|\,x) \leqslant \|(A+S)x\|\,\|x\|.$$

So $\|P^{1/2}(A+S)^{-1}x\|^2 \leqslant \|x\|\,\|(A+S)^{-1}x\|$ for all $x \in H$. Let $x \in H$. Then

$$\begin{aligned}
\|(A+S)^{-1}x\| &\leqslant \|A^{-1}x\| + \|A^{-1}P^{1/2}(I+iB)\|\,\|P^{1/2}(A+S)^{-1}x\| \\
&\leqslant \|A^{-1}x\| + \|A^{-1}P^{1/2}(I+iB)\|\,\|x\|^{1/2}\|(A+S)^{-1}x\|^{1/2} \\
&\leqslant \|A^{-1}x\| + \tfrac{1}{2}\|A^{-1}P^{1/2}(I+iB)\|^2\,\|x\| + \tfrac{1}{2}\|(A+S)^{-1}x\|.
\end{aligned}$$

Hence

$$\begin{aligned}
\|(A+S)^{-1}x\| &\leqslant 2\|A^{-1}x\| + \|A^{-1}P^{1/2}(I+iB)\|^2\,\|x\| \\
&\leqslant 2\|A^{-1}\|\,\|x\| + (1+\tan\theta)^2\|S\|\,\|A^{-1}\|^2\,\|x\|.
\end{aligned}$$

This proves the norm estimate.

Now we prove the general case. Let $\varepsilon > 0$. Replacing A by $\varepsilon I + A$ gives

$$\|(\varepsilon I + A + S)^{-1}\| \leqslant 2\|(\varepsilon I + A)^{-1}\| + (1+\tan\theta)^2\|S\|\,\|(\varepsilon I + A)^{-1}\|^2.$$

Since A is invertible, it follows that

$$\sup_{\varepsilon \in (0,1]} 2\|(\varepsilon I + A)^{-1}\| + (1 + \tan\theta)^2 \|S\| \, \|(\varepsilon I + A)^{-1}\|^2 < \infty.$$

Hence $A + S$ is invertible as the operator norm of the resolvent does not blow up for $\varepsilon \searrow 0$. □

Our main motivation to study m-accretive operators is the following theorem. It highlights the usefulness of m-accretive operators for the study of evolution equations.

Theorem A.13 (Phillips, cf. [Phi59, Theorem 1.1.3]). *Let A be a linear operator in a Hilbert space H. Then A is m-accretive if and only if $-A$ is the generator of a C_0-semigroup of contraction operators on H.*

A.2 General theory of graphs

An efficient introduction to multi-valued operators, which we call *graphs* here, can be found in [Haa06, Appendix A]. Alternatively, the monograph [Cro98] deals extensively with this subject. For the corresponding semigroup theory we refer to [FY99]. Here we only present the theory that is required in the main text. We reserve the term *operator* for the single-valued setting and assume that the reader is familiar with bounded and unbounded linear operators.

Let H be a Hilbert space. A **graph** in H is a linear subspace of the Cartesian product $H \times H$. In the following, let A be a graph in H. If A is the graph of an operator, we say that A *is* an operator. For all $x \in H$, we define the set $A[x] = \{f \in H : (x, f) \in A\}$. Then A is an operator if and only if $A[0] = \{0\}$. If A happens to be the graph of a bounded operator with domain H, we say that A is a bounded operator and write $A \in \mathcal{L}(H)$. So, effectively, we identify an operator with its graph.

We define the **range** of A by

$$\operatorname{rg} A := \{f \in H : \text{there exists an } x \in H \text{ such that } (x, f) \in A\},$$

the **domain** of A by

$$D(A) := \{x \in H : \text{there exists an } f \in H \text{ such that } (x, f) \in A\}$$

and the **kernel** of A by

$$\ker A := \{x \in H : (x, 0) \in A\}.$$

The graph A is called **closed** if it is closed as a subset of $H \times H$. Taking the closure of A in $H \times H$, we obtain a closed graph $\overline{A}$ which is called the **closure** of A. The **inverse** of A is the graph given by

$$A^{-1} := \{(f, x) \in H \times H : (x, f) \in A\}.$$

If $B \in \mathcal{L}(H)$ and $\alpha \in \mathbb{C}$, then we define the graph $B + \alpha A$ by

$$B + \alpha A := \{(x, Bx + \alpha f) \in H \times H : (x, f) \in A\}.$$

For example, if $\lambda \in \mathbb{C}$, then

$$(\lambda I - A)^{-1} = \{(\lambda x - f, x) \in H \times H : (x, f) \in A\}.$$

The set

$$\rho(A) := \left\{\lambda \in \mathbb{C} : (\lambda I - A)^{-1} \in \mathcal{L}(H)\right\}$$

is called the **resolvent set** of A. Note that if $\lambda \in \rho(A)$, then for all $f \in H$ there exists a unique 'solution' $x_f \in H$ such that $f \in (\lambda I - A)[x_f]$ and the map $f \mapsto x_f$ is continuous. It is easily checked that if $\rho(A)$ is not empty, then A is a closed graph and $A[0]$ is closed. Much more is true.

Proposition A.14 (cf. [FY99, Theorem 1.6 and Theorem 1.8]). *Let A be a graph. Then $\rho(A)$ is an open subset of $\mathbb{C}$. Moreover, the map $R(\cdot, A) \colon \rho(A) \to \mathcal{L}(H)$ given by*

$$\lambda \mapsto (\lambda I - A)^{-1}$$

is holomorphic and satisfies the resolvent identity

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda) R(\mu, A) R(\lambda, A) \tag{A.2}$$

for all $\lambda, \mu \in \rho(A)$.

If there exists a $\lambda_0 \in \rho(A)$ such that $R(\lambda_0, A)$ is compact, then we say that A has **compact resolvent**. Note that then $R(\lambda, A)$ is compact for all $\lambda \in \rho(A)$ by (A.2).

The **adjoint** of A is the graph A^* in H that is defined as follows. If $y, g \in H$ then $(y, g) \in A^*$ if and only if $(f \mid y)_H = (x \mid g)_H$ for all $(x, f) \in A$. It is readily verified that $A^* = (JA)^\perp$, where the orthogonal complement is taken with respect to $H \times H$ and $J \colon H \times H \to H \times H$ is defined by $J(x, f) = (-f, x)$. This shows that the adjoint is always a closed graph.

The graph A is called **accretive** if $\operatorname{Re}(f \mid x) \geqslant 0$ for all $(x, f) \in A$. It is called **symmetric** if $(f \mid x) \in \mathbb{R}$ for all $(x, f) \in A$. Moreover, A is called **sectorial** if there exist $\gamma \in \mathbb{R}$ and $\theta \in [0, \frac{\pi}{2})$ such that $(f \mid x) - \gamma \|x\|^2 \in \Sigma_\theta$ for all $(x, f) \in A$, where

$$\Sigma_\theta = \{z \in \mathbb{C} : z = 0 \text{ or } |\arg z| \leqslant \theta\}.$$

In this case one calls γ a **vertex** and θ a **semi-angle** of A. If there exists an $\omega \in \mathbb{R}$ such that $\omega I + A$ is accretive, then we say that A is **lower bounded**. So accretive and sectorial graphs are trivially lower bounded. The graph A is called **self-adjoint** if A is symmetric and $\operatorname{rg}(iI + A) = \operatorname{rg}(-iI + A) = H$. If A is accretive and $\operatorname{rg}(I + A) = H$, then A is said to be **m-accretive**. Furthermore, if A is sectorial such that $\omega I + A$ is m-accretive for an $\omega \in \mathbb{R}$, then we say that A is **m-sectorial**.

The next lemma is an elementary observation.

Lemma A.15. *Let A and B be graphs in H such that $A \subset B$. Let $\rho \in \mathbb{C}$ and suppose that $\operatorname{rg}(\rho I + A) = H$ and $\ker(\rho I + B) = \{0\}$. Then $A = B$.*

It follows from Lemma A.15 that an m-accretive, m-sectorial or self-adjoint graph is maximal in the following sense.

Proposition A.16. *Let A and B be graphs in H such that $A \subset B$. Suppose that A is m-accretive (or m-sectorial or self-adjoint) and that B is accretive (or sectorial or symmetric, respectively). Then $A = B$.*

The next result shows that the type of graphs that we are mainly interested in are a direct sum of a densely defined operator and a closed subspace. This was studied more generally in [Rof85].

Proposition A.17. *Let A be a graph in H. Suppose that A is m-accretive (or m-sectorial or self-adjoint). Let $H_1 = A[0]^{\perp}$. Then the **single-valued part** of A defined by $A^{\circ} := A \cap (H_1 \times H_1)$ is an m-accretive (or m-sectorial or self-adjoint, respectively) operator in H_1. Moreover, $D(A)$ is dense in H_1 and $H = \overline{D(A)} \oplus A[0]$ is an orthogonal decomposition.*

Proof. Suppose that A is m-accretive. As $-1 \in \rho(A)$, the space $A[0]$ is closed. If $(x, f) \in A$, then $\operatorname{Re}(f + h \,|\, x) \geqslant 0$ for all $h \in A[0]$. This shows that $D(A) \subset A[0]^{\perp}$. Clearly, A° is an accretive operator. Now let $f \in H_1$. As A is m-accretive, there exists an $x \in D(A) \subset H_1$ such that $(x, f - x) \in A$. Then $(x, f - x) \in A^{\circ}$. So A° is an m-accretive operator in H_1. In particular, $D(A^{\circ}) = D(A)$ is dense in H_1 by Lemma A.2. This concludes the proof in the m-accretive case.

If A is m-sectorial or self-adjoint, the proof is similar. $\square$

The following is a consequence of Proposition A.17 and the corresponding well-known properties for operators, see Proposition A.8.

Proposition A.18. *Let A be a graph in H. If A is m-accretive (or m-sectorial or self-adjoint), then also A^* is m-accretive (or m-sectorial or self-adjoint, respectively).*

Definition A.19. Let H be a Hilbert space. A map $S\colon [0, \infty) \to \mathcal{L}(H)$ is called a **(degenerate) C_0-semigroup** in H if it has the following two properties.

(i) The semigroup law $S(t + s) = S(t)S(s)$ holds for all $t, s \geqslant 0$.

(ii) The map $S(\cdot)x\colon [0,\infty) \to H$ is continuous for all $x \in H$.

It follows from the above definition that a (degenerate) C_0-semigroup S is **exponentially bounded**, i.e., there exist $M > 0$ and $\omega \in \mathbb{R}$ such that $\|S(t)\| \leqslant M\exp(\omega t)$ for all $t \geqslant 0$. If the latter holds with $M = 1$, then S is called **quasi-contractive**. If $\|S(t)\| \leqslant 1$ for all $t \geqslant 0$ then S is called **contractive**.

Proposition A.20 (cf. [Haa06, Proposition A.8.1]). *Let H be a Hilbert space. Let S be a (degenerate) C_0-semigroup. Then there exists a unique graph A, called the **generator** of S, such that there exists an $\omega \in \mathbb{R}$ for which one has $\lambda \in \rho(A)$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ and*

$$R(\lambda, A)x = \int_0^\infty \exp(-\lambda t)S(t)x\,dt$$

for all $x \in H$.

Remark A.21. Let A be the generator of the semigroup S and let $\alpha \in \mathbb{R}$. Then it is easily checked that $\alpha I + A$ is the generator of the semigroup T given by $T(t) = \exp(\alpha t)S(t)$ for all $t \geqslant 0$.

Theorem A.22 (graph version of Lumer–Phillips, cf. [FY99, Theorem 2.7 and Theorem 2.4]). *Let H be a Hilbert space and A be an m-accretive graph in H. Then $-A$ is the generator of a (degenerate) C_0-semigroup S in H. Moreover, $\|S(t)\| \leqslant 1$ for all $t \geqslant 0$. Furthermore, along the orthogonal decomposition $H = \overline{D(A)} \oplus A[0]$, one has $S = S_0 \oplus 0$, where S_0 is the C_0-semigroup in $\overline{D(A)}$ associated with A°.*

Remark A.23. Due to Remark A.21, it follows from Theorem A.22 that the negative of an m-sectorial graph is the generator of a (degenerate) C_0-semigroup S that is quasi-contractive.

Definition A.24. Let H be a Hilbert space and let A, A_n be graphs in H for all $n \in \mathbb{N}$. We say that $A_n \to A$ in the **strong resolvent sense** if there exists a $\lambda_0 \in \rho(A)$ such that $\lambda_0 \in \rho(A_n)$ for all large $n \in \mathbb{N}$ and $R(\lambda_0, A_n) \to R(\lambda_0, A)$ in the strong operator topology. If $R(\lambda_0, A_n) \to R(\lambda_0, A)$ in the uniform operator topology, then we say that $A \to A_n$ in the **uniform resolvent sense**.

The following two theorems show that the convergence of a sequence of graphs in the strong or uniform resolvent sense is a property that is, in a suitable sense, independent of the point in the resolvent set.

Theorem A.25 (cf. [Haa06, Corollary A.5.2]). *Let H be a Hilbert space and let A, A_n be graphs in H for all $n \in \mathbb{N}$. Define the set*

$$\Omega \coloneqq \rho(A) \cap \Big\{\lambda \in \mathbb{C} : \textit{there exists a } N_0 \in \mathbb{N} \textit{ such that } \lambda \in \rho(A_n) \textit{ for all } n \geqslant N_0 \textit{ and } \sup_{n \geqslant N_0} \|R(\lambda, A_n)\| < \infty\Big\}.$$

Suppose $A_n \to A$ *in the strong resolvent sense. Then* $R(\lambda, A_n) \to R(\lambda, A)$ *in the strong operator topology for all* $\lambda \in \Omega$.

Theorem A.26 (cf. [Haa06, Proposition A.5.3]). *Let* H *be a Hilbert space and let* A, A_n *be graphs in* H *for all* $n \in \mathbb{N}$. *Let* Ω *be as in Theorem A.25. Suppose that* $A_n \to A$ *in the uniform resolvent sense. Then* $\Omega = \rho(A)$ *and* $R(\lambda, A_n) \to R(\lambda, A)$ *in the uniform operator topology for all* $\lambda \in \rho(A)$.

Next we are interested when the (degenerate) C_0-semigroups converge strongly provided their generators converge in the strong resolvent sense.

Proposition A.27 (cf. [FY99, Section 3.6]). *Let* H *be a Hilbert space and* A *an m-sectorial graph in* H. *Suppose that* A *is sectorial with vertex* $-\omega$ *and semi-angle* $\theta \in [0, \frac{\pi}{2})$. *Let* γ *be the contour in* $\mathbb{C}$ *formed by combining*

$$\gamma_{\pm} = \{\omega + \rho \exp(\pm i\theta') : \rho \geqslant 1\}$$

and

$$\gamma_0 = \{\omega + \exp(i\alpha) : -\theta' \leqslant \alpha \leqslant \theta'\},$$

where $\theta' = \frac{3}{4}\pi - \frac{\theta}{2}$. *Then* $\gamma \subset \rho(-A)$ *and the (degenerate)* C_0*-semigroup* S *generated by* $-A$ *is given by*

$$S(t)x = \frac{1}{2\pi i} \int_{\gamma} \exp(\lambda t) R(\lambda, -A) x \, d\lambda$$

for all $x \in H$.

Using Proposition A.27, the following result is easily obtained. We point out that in the classical setting of nondegenerate C_0-semigroups stronger results hold, see for example [Kat80, Theorem IX.2.16].

Theorem A.28 (Trotter–Kato for graphs, cf. [FY99, Section 3.6]). *Let* H *be a Hilbert space. Let* A, A_n *be m-sectorial graphs in* H. *Suppose that the graphs* A_n *are uniformly sectorial and that* $A_n \to A$ *in the strong resolvent sense. Let* S *and* S_n *be the (degenerate)* C_0*-semigroups generated by* $-A$ *and* $-A_n$ *for all* $n \in \mathbb{N}$, *respectively. Then* $S_n(t) \to S(t)$ *in the strong operator topology for all* $t > 0$.

In Theorem A.28 convergence at $t = 0$ cannot be expected, see also Example 4.36.

A.3 The gap in Hilbert space

Let H be a Hilbert space. Suppose M and N are closed subspaces of H. The **gap** (or **opening**) $\hat{\delta}(M, N)$ between M and N, denoted by $\hat{\delta}(M, N)$, is defined as follows. Set

$$\delta(M, N) := \sup_{\substack{u \in M \\ \|u\| \leqslant 1}} \operatorname{dist}(u, N),$$

and $\hat{\delta}(M,N) := \max\{\delta(M,N), \delta(N,M)\}$. Clearly $\hat{\delta}(M,N) \leqslant 1$. For a more general discussion of the gap in the setting of Banach spaces we refer to [Kat80, Section IV.2] or [GK60, Section 1].

Let P_M and P_N be the orthogonal projections of H onto M and N, respectively. Then it is easily checked that

$$\delta(M,N) = \|(I - P_N)P_M\| = \|(P_M - P_N)P_M\| \leqslant \|P_M - P_N\|.$$

As pointed out by Kato in [Kat80, Footnote 1, p. 198], it follows from the next proposition that $\hat{\delta}(M,N) = \|P_M - P_N\|$.

Proposition A.29 (special case of [Kat58, Lemma 221]). *Let H be a Hilbert space. Let P and Q be orthogonal projections in H. If $\|(I-P)Q\| < 1$ and $\|(I-Q)P\| < 1$, then*

$$\|P - Q\| = \|(I-P)Q\| = \|(I-Q)P\|.$$

The following result can be useful to compare the Hilbert space dimension of two finite-dimensional subspaces.

Proposition A.30 (cf. [Kat80, Lemma IV.2.3]). *Let H be a Hilbert space. Let P and Q be orthogonal projections in H. Suppose that $\dim \operatorname{rg} P < \infty$. If $\|(I-P)Q\| < 1$, then $\dim \operatorname{rg} Q \leqslant \dim \operatorname{rg} P$.*

Proof. We give a proof by contrapositive. Suppose $\dim \operatorname{rg} P < \dim \operatorname{rg} Q$. Then $P|_{\operatorname{rg} Q} \colon \operatorname{rg} Q \to H$ cannot be injective. Hence there exists an $x \in \operatorname{rg} Q$ such that $\|x\| = 1$ and $Px = 0$. So $\|(I-P)Q\| \geqslant \|(I-P)x\| = \|x\| = 1$. □

Finally, we recollect a criterion for two closed subspaces to have the same Hilbert space dimension.

Proposition A.31 (cf. [GK60, Theorem 1.2]). *Let H be a Hilbert space. Let P and Q be orthogonal projections in H. If $\|P - Q\| < 1$, then $\dim \operatorname{rg} P = \dim \operatorname{rg} Q$.*

A.4 The Moore–Penrose generalised inverse

We require a few basic results about the *Moore–Penrose generalised inverse* (also called *pseudoinverse*) for a bounded linear operator in Hilbert space. A thorough discussion of generalised inverses for both matrices and operators with many pointers to the literature is given in [BG03]. Apart from the existence of the Moore–Penrose generalised inverse, we content ourselves here with results concerning the approximation of the Moore–Penrose generalised inverse due to Izumino [Izu83].

Definition A.32 (existence and uniqueness; see for example [BG03, Theorem 9.3]). Let H_1 and H_2 be Hilbert spaces. Let $T \in \mathcal{L}(H_1, H_2)$. If T has closed range in

H_2, then there exists a unique operator $T^\dagger \in \mathcal{L}(H_2, H_1)$, called the **Moore–Penrose generalised inverse**, such that the following four identities hold:

$$TT^\dagger T = T, \quad T^\dagger TT^\dagger = T^\dagger, \quad (TT^\dagger)^* = TT^\dagger, \quad (T^\dagger T)^* = T^\dagger T. \tag{A.3}$$

Remark A.33. 1. If T is invertible, then clearly $T^\dagger = T^{-1}$ satisfies (A.3).

2. One has $(T^*)^\dagger = (T^\dagger)^*$. This is readily verified with (A.3).

3. It follows from the first two identities in (A.3) that $TT^\dagger TT^\dagger = TT^\dagger$ and $T^\dagger TT^\dagger T = T^\dagger T$. The remaining two identities in (A.3) imply that $TT^\dagger$ and $T^\dagger T$ are orthogonal projections in H_2 and H_1, respectively. Clearly, $\operatorname{rg} T^\dagger T = (\ker T^\dagger T)^\perp$. Hence $\operatorname{rg} T = \operatorname{rg} TT^\dagger T \subset \operatorname{rg} TT^\dagger \subset \operatorname{rg} T$ and $\ker T \subset \ker T^\dagger T \subset \ker TT^\dagger T = \ker T$. This proves that $TT^\dagger$ is the orthogonal projection of H_2 onto $\operatorname{rg} T$ and that $T^\dagger T$ is the orthogonal projection of H_1 onto $(\ker T)^\perp$. In particular, this shows that closedness of $\operatorname{rg} T$ is a necessary condition for the existence of the Moore–Penrose generalised inverse.

4. Let H be a Hilbert space and $T \in \mathcal{L}(H)$. Suppose that $\operatorname{rg} T = (\ker T)^\perp$. Hence T has closed range. Denote by P the orthogonal projection in H onto $\ker T$. Then it follows by the previous remark that

$$TT^\dagger = T^\dagger T = I - P. \tag{A.4}$$

Moreover, as $\operatorname{rg} T^\dagger = \operatorname{rg} T^\dagger TT^\dagger \subset \operatorname{rg} T^\dagger T \subset \operatorname{rg} T^\dagger$, we obtain

$$\operatorname{rg} T^\dagger = \operatorname{rg}(T^\dagger T) = \operatorname{rg}(TT^\dagger) = (\ker T)^\perp = \operatorname{rg} T. \tag{A.5}$$

In fact, $T|_{\operatorname{rg} T}$ is invertible as an operator on $\operatorname{rg} T$. So $T^\dagger = (T|_{\operatorname{rg} T})^{-1}(I - P)$ by (A.4).

While the Moore–Penrose generalised inverse has a multitude of applications, its behaviour under perturbation is quite delicate. More precisely, suppose that T and T_n are bounded operators on H such that T and T_n have closed range for all $n \in \mathbb{N}$ and such that $T_n \to T$ uniformly. Then in general the Moore–Penrose generalised inverses $T_n^\dagger$ need not converge to $T^\dagger$ as $n \to \infty$. If T is invertible, however, then $T_n^\dagger \to T^\dagger = T^{-1}$ uniformly.

The following two propositions from [Izu83] give convenient equivalent conditions for the uniform and strong convergence of the Moore–Penrose generalised inverses.

Proposition A.34 (cf. [Izu83, Proposition 2.4])**.** *Let T and T_n be operators in $\mathcal{L}(H_1, H_2)$ for all $n \in \mathbb{N}$. Suppose that T and T_n have closed range for all $n \in \mathbb{N}$ and that $T_n \to T$ uniformly. Then the following statements are equivalent.*

(i) $T_n^\dagger \to T^\dagger$ *uniformly.*

(ii) $T_n T_n^\dagger \to TT^\dagger$ *uniformly.*

(iii) $T_n^\dagger T_n \to T^\dagger T$ *uniformly.*

Proposition A.35 (cf. [Izu83, Proposition 2.3]). *Let* T *and* T_n *be operators in* $\mathcal{L}(H_1, H_2)$ *for all* $n \in \mathbb{N}$. *Suppose that* T *and* T_n *have closed range for all* $n \in \mathbb{N}$ *and that* $T_n \to T$ *strongly. The following statements are equivalent.*

(i) $T_n^\dagger \to T^\dagger$ *strongly.*

(ii) $\sup_{n\in\mathbb{N}}\|T_n^\dagger\| < \infty$, $T_n T_n^\dagger \to TT^\dagger$ *strongly and* $T_n^\dagger T_n \to T^\dagger T$ *strongly.*

A.5 Miscellaneous auxiliary results from functional analysis

Lemma A.36. *Let* H *be a Hilbert space and* $T, T_n \in \mathcal{L}(H)$ *for all* $n \in \mathbb{N}$. *The following statements are equivalent.*

(i) *If* $x_n \rightharpoonup x$ *weakly in* H, *then* $T_n x_n \rightharpoonup Tx$ *weakly in* H.

(ii) $T_n^* \to T^*$ *strongly.*

Proof. '(i)⇒(ii)': Suppose not. Then there exist $y \in H$ and $\varepsilon > 0$ such that, after going to a subsequence, $\|(T_n^* - T^*)y\| \geqslant \varepsilon$ for all $n \in \mathbb{N}$. So for all $n \in \mathbb{N}$ there exists an $x_n \in H$ such that $\|x_n\| = 1$ and

$$\varepsilon \leqslant |(x_n \,|\, (T_n^* - T^*)y)|. \tag{A.6}$$

After going to a subsequence, there exists an $x \in H$ such that $x_n \rightharpoonup x$ weakly in H. Then $T_n x_n \rightharpoonup Tx$ weakly in H by assumption and $Tx_n \rightharpoonup Tx$ weakly in H by continuity. So

$$(x_n \,|\, (T_n^* - T^*)y) = ((T_n - T)x_n \,|\, y) \to 0$$

as $n \to \infty$. This contradicts (A.6).

'(ii)⇒(i)': Let $x_n \rightharpoonup x$ weakly in H. Note that $\sup_{n\in\mathbb{N}}\|x_n\| < \infty$. So

$$(T_n x_n - Tx \,|\, y) = (x_n \,|\, (T_n^* - T^*)y) + (x_n - x \,|\, T^* y) \to 0$$

as $n \to \infty$ for all $y \in H$. □

Lemma A.37. *Let* H *be a Hilbert space. Let* $K, T, T_n \in \mathcal{L}(H)$ *for all* $n \in \mathbb{N}$. *Suppose that* K *is compact and that* $T_n^* \to T^*$ *strongly. Then* $KT_n \to KT$ *uniformly.*

Proof. Suppose not. Then, after going to a subsequence, there exist $\varepsilon > 0$ and a sequence (x_n) in H such that $\|x_n\| = 1$ and $\|KT_n x_n - KTx_n\| \geqslant \varepsilon$ for all $n \in \mathbb{N}$. We may assume that there exists an $x \in H$ such that $x_n \rightharpoonup x$ weakly in H. Then $T_n x_n \rightharpoonup Tx$ weakly in H by Lemma A.36. Moreover, $Tx_n \rightharpoonup Tx$ weakly in H. So $\varepsilon \leqslant \lim_{n\to\infty}\|KT_n x_n - KTx_n\| = 0$, a contradiction. □

The next lemma gives a folklore estimate for the norm of a compact operator in terms of an injective operator.

Lemma A.38 (for example, cf. [Bre11, Example 6.13]). *Let* V, H_1 *and* H_2 *be Hilbert spaces. Suppose* $K \in \mathcal{L}(V, H_1)$ *is compact and* $S \in \mathcal{L}(V, H_2)$ *is injective. Then for all* $\varepsilon > 0$ *there exists a* $C \geqslant 0$ *such that*

$$\|Kx\|_{H_1}^2 \leqslant \varepsilon \|x\|_V^2 + C\|Sx\|_{H_2}^2$$

for all $x \in V$.

Proof. Suppose not. Then there exists an $\varepsilon > 0$ and a sequence (x_n) in V such that

$$\|Kx_n\|_{H_1}^2 > \varepsilon \|x_n\|_V^2 + n\|Sx_n\|_{H_2}^2$$

for all $n \in \mathbb{N}$. Since we may assume that $\|x_n\|_V = 1$ for all $n \in \mathbb{N}$, after going to a subsequence there exists an $x \in V$ such that $x_n \rightharpoonup x$ weakly in V. On the one hand, as K is compact, $\lim_{n\to\infty} Kx_n = Kx$ and $\|Kx\|_{H_1}^2 \geqslant \varepsilon$. In particular, $x \neq 0$. On the other hand, it follows from the weak convergence of (Sx_n) in H_2 that $\liminf_{n\to\infty} \|Sx_n\|_{H_2}^2 \geqslant \|Sx\|_{H_2}^2$. This implies that

$$\|Kx\|_{H_1}^2 \geqslant \varepsilon + M\|Sx\|_{H_2}^2$$

for all $M > 0$. Therefore $\|Sx\| = 0$ and hence $x = 0$. This is a contradiction. □

A.6 Standard results on Sobolev spaces

In this section Ω is a nonempty open subset of $\mathbb{R}^d$. To simplify the exposition we shall consider only real spaces. Let $p \in [1, \infty)$. By $W^{1,p}(\Omega)$ we denote the **Sobolev space** of elements $u \in L^p(\Omega)$ that have distributional derivatives $D_j u \in L^p(\Omega)$ for all $j \in \{1, \ldots, d\}$. We equip $W^{1,p}(\Omega)$ with the norm

$$\|u\|_{1,p} = \big(\|u\|_p^p + \sum_{j=1}^{d} \|D_j u\|_p^p\big)^{1/p},$$

which makes it into a Banach space that is reflexive if $p \in (1, \infty)$. As usual, $W_0^{1,p}(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$. The following result shows that elements of $W^{1,p}(\Omega)$ can be approximated by smooth functions.

Theorem A.39 (Meyers and Serrin, cf. [EE87, Theorem V.3.2]). *The linear subspace* $W^{1,p}(\Omega) \cap C^\infty(\Omega)$ *is dense in* $W^{1,p}(\Omega)$.

If the domain is equal to $\mathbb{R}^d$, one has the following density property.

Proposition A.40 (cf. [Neč12, Proposition 2.2.6]). *The linear subspace $C_c^\infty(\mathbb{R}^d)$ is dense in $W^{1,p}(\mathbb{R}^d)$.*

The next proposition is a direct consequence of [GT01, Theorem 7.8 and Section 7.5]. It shows in particular that the space $W^{1,p}(\Omega)$ is closed under truncation. Consequently, $W^{1,p}(\Omega)$ is a Riesz subspace of $L^p(\Omega)$.

Proposition A.41. *The maps $u \mapsto u \vee 0$, $u \mapsto u \wedge 1$ and $u \mapsto |u|$ are continuous maps from $W^{1,p}(\Omega)$ into itself. Moreover, let $u \in W^{1,p}(\Omega)$. Then $\nabla u = 0$ a.e. on the set $[u = 0]$.*

We will also need the following basic product rule, see [EG92, Theorem 4.2.4].

Lemma A.42. *Let $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then $uv \in W^{1,p}(\Omega)$ and*

$$\|uv\|_{1,p} \leqslant \|u\|_{1,p}\|v\|_\infty + \|u\|_\infty\|v\|_{1,p}.$$

The next result gives a sufficient criterion for a continuous Sobolev function to be contained in $W_0^{1,p}(\Omega)$. It follows from the proof of [Bre83, Theorem IX.17]. Note that the converse fails without further assumptions on Ω.

Proposition A.43. *Let $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$. If $u|_\Gamma = 0$, then $u \in W_0^{1,p}(\Omega)$.*

For some of the following domain regularity conditions we shall only give a verbal definition. For details see for example [Gri85, Section 1.2].

Definition A.44. Let $\Omega \subset \mathbb{R}^d$ be open. If the interior of $\overline{\Omega}$ is equal to Ω, then Ω is called **topologically regular**. One says that Ω has **continuous boundary** if for all $z \in \Gamma$ there exists a neighbourhood of z in which Ω lies, in local orthogonal coordinates, above the graph of a continuous function from $\mathbb{R}^{d-1} \to \mathbb{R}$. If the previous statement holds with a Lipschitz continuous function, then Ω is called **Lipschitz**.

Clearly, a Lipschitz domain has continuous boundary. Moreover, note that a domain with continuous boundary is topologically regular.

The following result shows that, for quite general domains, elements in $W^{1,p}(\Omega)$ can be approximated by smooth functions that are continuous on $\overline{\Omega}$.

Theorem A.45 (cf. [EE87, Theorem V.4.7]). *Let Ω have continuous boundary and let $p \in [1, \infty)$. Then the set of restrictions to Ω of all functions in $C_c^\infty(\mathbb{R}^d)$ is dense in $W^{1,p}(\Omega)$.*

Moreover, an inspection of the proof of [EE87, Theorem V.4.7] yields the following result. Alternatively, see [Gri72, p. 77, 78] in combination with [EE87, Theorem V.4.4] for a proof in the bounded setting.

Theorem A.46. *Let Ω have continuous boundary and let $p \in [1, \infty)$. Let $u \in W^{1,p}(\mathbb{R}^d)$ be such that $u = 0$ a.e. on $\mathbb{R}^d \setminus \overline{\Omega}$. Then $u|_\Omega \in W_0^{1,p}(\Omega)$.*

It is well-known that Sobolev spaces on bounded Lipschitz domains are particularly well-behaved. The next result establishes the existence of a continuous extension operator in this setting and can be found in [EG92, Theorem 4.4.1] or [Gri85, Theorem 1.4.3.1], for example.

Theorem A.47. *Let Ω be bounded and Lipschitz. Let $p \in [1, \infty)$. Then there exists a bounded linear operator $E\colon W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$ such that $(Eu)|_\Omega = u$ a.e. for all $u \in W^{1,p}(\Omega)$.*

On Lipschitz domains one also has the existence of a well-behaved **trace operator Tr**. Moreover, the elements $u \in W^{1,p}(\Omega)$ such that $\operatorname{Tr} u = 0$ are contained in $W_0^{1,p}(\Omega)$.

Theorem A.48 (cf. [EG92, Theorem 4.3.1])**.** *Let Ω be bounded and Lipschitz. Let $p \in [1, \infty)$. Then there exists a unique bounded linear operator $\operatorname{Tr}\colon W^{1,p}(\Omega) \to L^p(\Gamma)$ such that $\operatorname{Tr} u = u|_\Gamma$ for all $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$.*

Theorem A.49 (cf. [Neč12, Theorem 2.4.10])**.** *Let Ω be bounded and Lipschitz. Let $p \in [1, \infty)$ and let Tr be the operator from Theorem A.48. Then*

$$W_0^{1,p}(\Omega) = \Big\{ u \in W^{1,p}(\Omega) : \operatorname{Tr} u = 0 \textit{ in } L^p(\Gamma) \Big\}.$$

A generalisation of Theorem A.49 to higher- and fractional-order Sobolev spaces on Lipschitz domains can be found in [Mar87].

We shall make use of the following simplified version of the Sobolev embedding theorem. For a proof in a more general setting see [Ada75, Theorem 5.4]. The space $C_b(\Omega)$ is the space of bounded continuous functions on Ω equipped with the sup-norm.

Theorem A.50. *Let Ω be bounded Lipschitz or equal to $\mathbb{R}^d$. Then the following statements hold.*

(i) *If $p < d$, then define $p^* \coloneqq dp/(d-p)$. It follows that $W^{1,p}(\Omega)$ is continuously embedded into $L^q(\Omega)$ for all $q \in [p, p^*]$.*

(ii) *If $p = d$, then $W^{1,p}(\Omega)$ is continuously embedded into $L^q(\Omega)$ for all $q \in [p, \infty)$.*

(iii) *If $p > d$, then there exists a $C > 0$ such that $\|u\|_\infty \leqslant C\|u\|_{1,p}$ for all $u \in W^{1,p}(\Omega)$. Consequently, $W^{1,p}(\Omega)$ is continuously embedded into $C_b(\Omega)$.*

Remark A.51. Note that Theorem A.50 immediately implies the corresponding local results for general $\Omega \subset \mathbb{R}^d$. In particular, if $p > d$, then every element in $W^{1,p}(\Omega)$ has a continuous representative.

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