Hyperbolic Geometry and Reflection Groups

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Abstract

The n-dimensional pseudospheres are the surfaces in $\mathbf{R}^{n+1}$ given by the equations ${x_1}^2 + {x_2}^2 + \ldots + {x_k}^2 - {x_{k+1}}^2 - \cdots - {x_{n+1}}^2 = 1$ $(1 \leq k \leq n+1)$. The cases $k = 1, n+1$ give,respectively a pair of hyperboloids, and the ordinary n-sphere.

In the first chapter we consider the pseudospheres as surfaces in $E_{n+1,k}$, where $E_{m,k} = \mathbf{R}^k \times (i\mathbf{R})^{m-k}$, and investigate their geometry in terms of the linear algebra of these spaces.

The main objects of investigation are finite sequences of hyperplanes in a pseudosphere. To each such sequence we associate a square symmetric matrix, the Gram matrix, which gives information about angle and incidence properties of the hyperplanes. We find when a given matrix is the Gram matrix of some sequence of hyperplanes, and when a sequence is determined up to isometry by its Gram matrix.

We also consider subspaces of pseudospheres and projections onto them. This leads to an n-dimensional cosine rule for spherical and hyperbolic simplices.

In the second chapter we derive integral formulae for the volume of an n-dimensional spherical or hyperbolic simplex, both in terms of its dihedral angles and its edge lengths. For the regular simplex with common edge length γ we then derive power series for the volume, both in $u = \sin\gamma/2$, and in γ itself, and discuss some of the properties of the coefficients. In obtaining these series we encounter an interesting family of entire functions, $R_n(p)$ (n

a nonegative integer and $p \in \mathbf{C}$). We derive a functional equation relating $R_n(p)$ and $R_{n-1}(p)$.

Finally we classify, up to isometry, all tetrahedra with one or more vertices truncated, for which the dihedral angles along the edges formed by the truncations are all $\pi/2$, and the remaining dihedral angles are all submultiples of π. We show how to find the volumes of these polyhedra, and find presentations and small generating sets for the orientation-preserving subgroups of their reflection groups.

For particular families of these groups, we find low index torsion free subgroups, and construct associated manifolds and manifolds with boundary. In particular, we find a sequence of manifolds with totally geodesic boundary of genus $g \geq 2$, which we conjecture to be of least volume among such manifolds.

Contents