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**An Investigation of Students' Understanding and  
Representation of Derivative in a Graphic Calculator-  
Mediated Teaching and Learning Environment**

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**A thesis submitted in fulfilment of the requirements for the degree of  
Doctor of Philosophy in Mathematics Education  
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*For my Mama and sisters*

## ABSTRACT

This research is a collective case study that investigates the constitutive relationship between students' representational competences and mathematical understanding of derivative. Its goal was to describe the representational abilities characterising different ways of knowing, and these were categorised as procedure-oriented, process-oriented, object-oriented, concept-oriented and versatile.

The study was conducted in four Form 7 classrooms, in their Mathematics with Calculus classes, where graphic calculators were used in the teaching and learning of derivative. The choice of the context was based on the belief that the use of graphic calculators might encourage a multi-representational approach to teaching, and support the development of students' multi-representational way of thinking. The research data were collected both from teachers and their students. These data comprise teacher and student interviews, with the first interviews conducted before their lessons on derivative and the second after the lesson. The students were also given pre-lesson and post-lesson tests in order to triangulate student data.

A Representational Framework of Knowing Derivative was constructed as an analysis tool, and used to explore students' representational abilities and ways of knowing. From the analysis, the students' cognitive processes were construed, together with the nature of their representational, cognitive and conceptual schemas. The representational framework of knowing was later refined to present an empirically-based theoretical framework that bridges the gap between what was theorised and what was observed.

The results of the study suggest that the relationship between students' ways of knowing and their representational abilities is mediated by the following factors: (i) the students' interpretation of the mathematical notion; (ii) the representational nature of their interpretations of derivative, and the representational aspects in their problem solving activities; and (iii) the nature of the representational links that they have formed between procedures, processes, objects and sub-concepts that were construed to constitute their conceptual and cognitive schemas of derivative. With regard to the use of the graphic calculator, this research has noted a possible contribution of the graphic calculator in the development of students' multi-representational ways of thinking and learning.

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## PUBLICATIONS ARISING FROM THIS THESIS

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