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EXTENDED THEORY
OF THE
BÉNARD CONVECTION PROBLEM

Thesis submitted to the
University of Auckland
for the degree of
Doctor of Philosophy

by

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ACKNOWLEDGEMENTS

This research was performed under the supervision of Dr C.M. Segedin, whose help and encouragement were greatly appreciated.

The suggestion of the problem by Dr B.R. Morton, and the interest taken by Dr R.A. Wooding, are gratefully acknowledged.

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ABSTRACT

The onset of convection in a horizontal fluid layer, heated from below, is examined by means of perturbation analysis. The resulting eigenvalue system of equations is solved by means of a new extension of a Fourier series technique. Two sets of coupled effects are investigated:

(i) thermal buoyancy and surface-tension effects, and
(ii) thermal buoyancy and solute buoyancy effects.

For the first set of effects the magnetohydrodynamic problem is also studied.

For the surface-tension problem, attention is focussed on the case where the lower boundary is a rigid conductor and the upper free surface is subject to a general thermal condition. It is found that for this case the surface-tension and buoyancy forces reinforce each other and are tightly coupled. Cells formed by surface tension are approximately the same size as those formed by buoyancy. The streamline patterns produced by the two agencies acting separately are again similar.

When the fluid is electrically conducting and is in the presence of a vertical magnetic field, it is found that the field always has a stabilizing effect. When convection cells are formed in the presence of such a field, their horizontal dimensions are less than for cells formed in the absence of the field. The magnetic field accentuates
the difference between the cells induced by surface tension and those by buoyancy, and thus reduces the coupling between the destabilizing forces. Increase of magnetic field causes the buoyancy cell pattern to become more symmetrical, but causes the streamlines in surface-tension cells to become bunched near the surface. When the magnetic field is large, the transition from one type of cell to the other type is extremely sudden, at least when the upper surface is a good thermal conductor.

It has been found that, on the model considered, there can be no oscillatory instability for this problem. However, dimensional analysis reveals that, for a sufficiently flexible upper surface, oscillatory instability might in fact occur.

Finally the thermohaline problem, where the density varies with both temperature and the concentration of some solute, is studied. The eigenvalue equation is now found for general boundary conditions. The degree of coupling between the thermal and the solute effects again depends on the similarity between convection cells caused by the two agencies acting separately. (For one extreme case studied the coupling is zero for a certain range of parameters.) In this problem both monotonic and oscillatory instability can now occur.
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<td>$d/dz$ with respect to length $d/x$</td>
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\(L_u, L_v\)  
thermal boundary-condition parameters

\(M_u, M_v\)  
solute boundary-condition parameters

\(Q\)  
square of Hartmann number, \(\mu H^2 a^2/(4\pi \rho \nu \eta)\)

\(Q_a\)  
\(Q/\pi^2\)

\(R\)  
thermal Rayleigh number, \(g\alpha \beta \delta a^4/\kappa \nu\)

\(R_1\)  
\(R/\pi^4\)

\(S\)  
solute Rayleigh number, \(g\alpha \beta \delta a^4/\kappa \nu\)

\(S_1\)  
\(S/\pi^4\)

\(W\)  
function expressing \(z\)-dependence of \(w\)

\(W_1\)  
\(Wd/\pi \nu\)

\(X\)  
function expressing \(z\)-dependence of curl \(h_1\)

\(X_1\)  
\(XdHd/(4\pi^2 \rho \nu)\)

\(Z\)  
function expressing \(z\)-dependence of vorticity

\(\alpha\)  
thermal expansion coefficient

\(\alpha'\)  
solute expansion coefficient

\(\beta\)  
adverse vertical temperature gradient

\(\beta'\)  
adverse vertical solute concentration gradient

\(\gamma\)  
solute concentration perturbation

\(\zeta\)  
\(z\)-component of vorticity

\(\eta\)  
magnetic diffusivity

\(\theta\)  
temperature perturbation

\(\kappa\)  
thermal diffusivity

\(\kappa'\)  
solute diffusivity

\(\mu\)  
magnetic permeability
\( \nu \)  
kinematic viscosity

\( \xi \)  
z-component of curl \( h_1 \)

\( \rho \)  
density of fluid

\( \sigma_1 \)  
time-constant, \( \rho d^2/\pi^2 \nu \)

\( \sigma_0 \)  
rate of decrease of surface tension with increase of temperature at the surface

\( \omega \)  
time constant, \( \text{Im}(\sigma_1) \)

\( \Gamma \)  
function expressing \( z \)-dependence of \( \gamma \)

\( \Gamma_1 \)  
\( \Gamma \sigma_0 \beta d^2 \nu \)

\( \theta \)  
function expressing \( z \)-dependence of \( \theta \)

\( \theta_1 \)  
\( \theta \sigma_0 \beta d^2 \nu \)
The literature survey which forms Chapter 1 of this thesis covers a range considerably wider than the topics on which the author has made an original contribution. (The parts of the survey particularly pertinent here are sections 1(i), 1(ii), 2(i), 2(vii) and 2(viii).) It is thought that nearly all published papers on the onset of thermal convection in an initially-static stratified fluid are considered in the survey. On the borderlines of this subject with other types of hydrodynamic stability, with post-instability flows, and with geophysical and astrophysical applications, a selection has necessarily been made. Much of the content of this thesis has already been published by the author; such work has been integrated into the survey.

In our Chapter 2 the perturbation analysis presented in Chandrasekhar's treatise has been extended in two particular aspects. First the boundary condition, derived by Pearson (1958), applicable to the free surface of a liquid whose surface tension varies with temperature, is incorporated into the theory. Then the analysis has been extended to a two-component system such as a liquid containing a dissolved salt. Such a system has previously been considered, but in less detail, by several authors, the first of whom was Vertgeim (1955).

The differential equation system arising from our perturbation analysis forms an eigenvalue problem. A practical method of solving
this eigenvalue problem is described in Chapter 3. The following three chapters, which form the bulk of the original work in this thesis, contain the results of applying this method to some problems of practical importance.
1 The basic Rayleigh-Jeffreys problem
   (i)  Bénard's observations
   (ii) Theory
   (iii) Experiments

2 Additional effects
   (i)  Magnetic fields
   (ii) Rotation
   (iii) Radiation
   (iv) Electric fields
   (v)  Shear
   (vi) Viscoelasticity
   (vii) Surface tension
   (viii) Solvents
   (ix)  Suspensions
   (x)  Porous medium

3 Other configurations
   (i)  Vertical channels and cylinders
   (ii) Horizontal channels and cylinders
   (iii) Spheres and spherical shells
   (iv)  General confined configurations

4 Non-linear effects
   (i)  Steady finite-amplitude effects
   (ii) Heat transfer
   (iii) Induced convection
   (iv)  Time-dependent solutions
   (v)  Penetrative convection
   (vi) Property variations
   (vii) Direction of flow
   (viii) Type of cell pattern
   (ix)  Columnar convection
   (x)  Transition to turbulence
   (xi) Analogies

5 Geophysical and Astrophysical applications
   (i)  Earth physics
   (ii) Limnology and oceanography
   (iii) Meteorology
   (iv)  Astrophysics

6 Some other problems
   (i)  Rotating Couette flow
   (ii) Transition to turbulence in other flows