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Essays on Extreme Markets

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A thesis submitted in fulfilment of the requirements for the degree of

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Abstract

Various phenomena of interest are often most clearly manifested in the extremes. I consider a particular type of extreme market – a market ravaged by fire sales. In chapter 1 I present a theoretical model of fire sales that incorporates interaction between forced asset sales and declining price spirals in a heterogeneous multi-asset, multi-investor setting. I show that a unique equilibrium obtains in such a market and provide an analytical approximation of the resulting equilibrium fire-sale prices. Fire sales can drive cross-asset contagion through leverage and overlapping assets holdings. In addition, the leverage decisions of individual investors can impose externalities on other investors by exposing them to higher fire-sale risk. In chapter 2 I give empirical content to the fire-sale model by using market-estimated parameters to calculate model fire-sale prices for US stocks from 1982 to 2010. Model fire-sale prices predict the cross-section of stock returns in distressed markets (when the S&P 500 index declines by more than 10% over a quarter), thus lending support to the theoretical model. In chapter 3 I turn to identifying extreme stocks. Using a simple unit root specification I identify stocks that deviate from a pure random walk in log prices, measured by autocorrelation. Autocorrelation predicts US stock returns and this predictability is robust to a wide range of time-series risk factors and stock characteristics. Stocks with autocorrelation substantially below unity generate unusually persistent excess returns: a zero cost hedge portfolio based on such stocks generate statistically significant positive excess returns in every month post-formation up to horizons of 25 years. Abnormal returns around earning announcements indicate that this persistence is unlikely to be the result of biased investor expectations of future earnings. This suggests the possibility that downward deviations from the random walk norm might be priced.

Preface and Acknowledgements

Although a doctoral thesis is an individual project, it is not one that can be accomplished alone. I am fortunate to have had the support of many people throughout this journey. I would like at the outset to thank my supervisor, Professor Henk Berkman, for excellent guidance and practical support. Good ideas become good research only through the crucible of rigorous intellectual challenge. This thesis reflects arguments won and lost, and is the better for both. Thanks Henk.

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Paul Geertsema

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Name	Nature of Contribution
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Certification by Co-Authors

The undersigned hereby certify that:

- the above statement correctly reflects the nature and extent of the PhD candidate’s contribution to this work, and the nature of the contribution of each of the co-authors; and
- in cases where the PhD candidate was the lead author of the work that the candidate wrote the text.

Name	Signature	Date
Paul Koch	Paul Koch	25/09/2013

Part I

Introduction

Why should we care about extremes? Surely, one might argue, extreme markets are comparatively rare and not at all representative of the orderly, well-conducted and liquid markets that can be relied on to properly price financial assets. This argument is only partially correct. Even in orderly markets, the price an asset commands should bear some relation to its risk. Usually that risk remains latent; potential rather than actual. But it is still present, and should still be priced. And it is in extreme markets that assets often reveal the true extent of their otherwise hidden risk. Thus understanding how markets behave in extreme conditions has the potential to inform our understanding of financial risk and how it relates to expected returns, a relationship at the core of financial economics.

Normal markets are all normal in the same way, while extreme markets are each extreme in their own unique way[1]. So which of the many extremes should we consider? I choose to examine extreme markets from two different vantage points.

First, I consider fire sales. Fire sales are a particular type of extreme market, one that occurs when forced sellers encounter constrained buyers such that assets trade at prices substantially below any reasonable measure of their intrinsic value. I formulate a model of fire sales in chapter 1, and test the model empirically in chapter 2.

Second, I take a different approach and attempt to identify extreme stocks empirically. The intuition underlying my approach is to identify stocks with a price trajectory that strays from the beaten path. More formally, I estimate the time series autocorrelation of each stock; this can be interpreted as a measure of its deviation from a pure random walk. I proceed to investigate how this measure relates to realised returns.

Fire sales already has a well-established literature. Some more influential early papers include a theoretical model of fire sales within a given industry by Shleifer and Vishny (1992), an empirical demonstration of fire sales in the market for used aircraft by Pulvino (1998) and the cash-in-the-market fire-sale pricing model of Allen and Gale (1994). A recent survey of fire sales can be found in Shleifer and Vishny (2010a).

The fire-sale literature might be best understood as the intersection of several related literatures. Consider the necessary conditions for a fire sale: forced sellers and constrained buyers. Although investors might become forced sellers for any number of reasons, two mechanisms have received particular attention in the literature, namely, fund flows and leverage. Funds often experience investor withdrawals after experiencing sub-par investment performance (see for instance, Coval and Stafford (2007), de Souza (2010), Lou (2012) and Ben-David, Franzoni and Moussawi (2012)). Because funds have to raise cash to meet the redemptions required by investor withdrawals, this essentially makes them forced sellers. Leverage can have the same effect: as prices fall leveraged investors experience reductions in their equity and eventually become forced sellers (see for example Miller and Stiglitz (2010), Longstaff (2008) and Richardson, Saffi and Sigurdsson (2012)). Although the mechanisms underlying flow-induced selling and leverage-induced selling are different, the effect is the same. As the value of an investor's portfolio declines the investor becomes compelled to liquidate an increasing fraction of his portfolio.

---

[1] Tolstoy (1878) makes a similar observation in a different setting: “Happy families are all alike; every unhappy family is unhappy in its own way.” (Chapter 1, line 1)

Fire sales also require constrained buyers. This assertion can be best explained by considering a perfect market in which investors face no constraints or frictions. In such a market any forced selling will be absorbed by investors who would be willing to provide unlimited liquidity at any price fractionally below intrinsic value. However, markets are not perfect and truly unconstrained investors are a theoretical abstraction rather than a reality. The notion of constrained investors is closely related to the literature on limits to arbitrage. The ideal arbitrageur is unconstrained; in reality arbitrageurs face a variety of constraints, as discussed in Shleifer and Vishny (1997) and Gromb and Vayanos (2002), that hinders their ability to arbitrage deviations from intrinsic value[2].

I have outlined some necessary conditions for a fire sale, but what happens if a fire sale occurs? If a fire sale occurs in a single asset, then there is no direct contagion effect. However, a fire sale might depress prices sufficiently to bring about the forced liquidation of whole portfolios. This in turn might depress the prices of other, apparently unrelated, assets. This intuition provides a link to the vast literature on financial contagion. (See for instance, Kaminsky and Reinhart (2000), Allen and Gale (2000), Longstaff (2009) and Iyer and Peydro (2010). Moser (2003) and Hasman (2012) are useful surveys of the literature on financial contagion.) The model of Kiyotaki and Moore (2002) is particularly relevant since it explicitly considers the negative feedback loop between falling asset prices and binding leverage constraints.

If fire sales do potentially give rise to cross-asset contagion, that raises a further point to consider. When financial contagion poses a systemic risk, asset prices should reflect that risk. Therefore, any actions by an investor that increases the risk of an asset being subjected to a fire sale (for instance, by taking on more leverage) also increases the risk and decreases the value of that asset *ex-ante*. This imposes externalities on other investors that own the asset – they have lost money (today) solely because of the actions of a third party. This is a point forcefully raised by the theoretical model proposed in Wagner (2011). This model endogenously models the market participation choices of three types of investors – leveraged investors, credit providers and liquidity providers. Wagner (2011) shows that the market clearing prices of assets reflect the fire-sale risk associated with assets owned by liquidation-prone investors. Basically, an asset held by liquidation-prone investors (such as highly leveraged hedge funds) is risky because such an asset is both more likely to be involved in a fire sales and to sell at a lower price if a fire sale occurs.

---

[2] A recent survey of the limits of arbitrage literature can be found in Gromb and Vayanos (2010)

Understanding fire sales are important, not only because it can shed light on asset pricing more generally, but also because fire sales impose significant costs on the real economy. Dislocated asset prices distort price signals and disrupt investment in the real economy (Shleifer and Vishny, 2010b). The excessive price volatility associated with fire sales can lead to the collapse of otherwise sound asset managers and firms. Furthermore, the anticipation of fire sales might prompt investors to hold more of their portfolios in unproductive liquid assets such as cash, rather than in productive assets. Fire sales thus have real welfare implications; as such there is a case to be made for regulatory intervention aimed at preventing the frequency and severity of fire sales. But to successfully contain and prevent fire sales, it is first necessary to understand the mechanisms underlying fire sales. The model I present in chapter 1 and test in chapter 2 aims to contribute in this direction.

In chapter 3 I take a different approach. Rather than focus on markets, I consider individual assets. How can extreme behaviour in assets be quantified? And how does such extreme behaviour impact realised returns? Motivated by the work of Phillips and Yu (2011) and Phillips, Wu and Yu (2011) on identifying bubbles in aggregate price indices, I use a simple variant of a unit root test to quantify how far stocks deviate from a pure random walk in log prices. The underlying intuition is that deviations from a pure random walk mark out stocks as “extreme”. To my knowledge, there is no theory to suggest that past deviations from a random walk should be priced. In this sense chapter 3 has a more empirical orientation than the previous chapters.

The rest of this thesis is set out as follows. Part II collects three essays on extreme markets. Chapter 1 presents a model of fire sales. Chapter 2 uses the model presented in chapter 1 to calculate model fire-sale prices as a proxy for fire-sale risk. I show that model fire-sale prices predict realised US stock returns in distressed markets, thus empirically validating the model. Chapter 3 introduces autocorrelation as a measure of the deviation of a stock from a pure random walk. I show that extreme realisations of autocorrelation predict realised returns in US stocks and is robust to a range of time series risk factors and stock characteristics. Part III concludes.

Part II

Essays on Extreme Markets

Chapter 1

Fire Sales – Theory

Abstract

I formulate a general model of fire sales in which multiple heterogeneous investors, each investing in multiple assets, become forced sellers because of exogenous price shocks. Simultaneously prices endogenously adjust based on the volume of forced sales. This induces strategic interaction between investors mediated by price changes. I show that equilibrium fire-sale prices exist, are unique and can be calculated to arbitrary accuracy using the method of successive approximations. I derive an analytical price approximation shown (numerically) to explain 98% of the equilibrium price adjustment. Analytical statics derived from this approximation allows me to quantify fire-sale contagion effects. Further, I show that a change in leverage by a single investor has spillover effects such that risk externalities are imposed on other investors.

1.1 Introduction

I consider a model of fire sales and contagion in a multi-investor, multi-asset setting. Fire sales matter because they impose significant welfare losses. When assets are sold at a significant discount to fundamentals it implies a high discount rate, unrelated to the fundamental risk inherent in the asset. Such distortions in discount rates can lead to the inefficient allocation of capital. Some examples of misallocation in bank lending include Diamond and Rajan (2011) and Shleifer and Vishny (2010c), in which the expectation of windfall profits from holding fire-sale assets displaces productive lending. Another consequence of fire sales is that assets often end with a buyer of last resort who might lack the expertise to put the asset to best use as in Shleifer and Vishny (1992). Fire sales in one asset or asset class can also lead to contagion to other assets or asset classes. Wagner (2011) presents a model in which investors are concerned about the possibility of being involved in a (costly) fire sale alongside other investors. He shows that in such a setting investors might rationally deviate from a diversified portfolio to reduce their exposure to assets held by liquidation-prone investors and that as a consequence equilibrium asset prices reflect fire-sale risk.

My paper is in the spirit of Wagner (2011) in that I consider fire sales in a multi-investor, multi-asset setting[1]. The approach I take differs from Wagner (2011) in two respects. First, the Wagner (2011) model is a general asset pricing model in which joint liquidation (fire-sale) risk is priced endogenously. By contrast, mine is a purely mechanical model of fire sales, rather than a fully fledged asset pricing model. Wagner (2011) shows that fire-sale risk should be reflected in expected returns; my contribution, which I view as complementary, is to *quantify* that fire-sale risk. Second, Wagner (2011) models investors as a unit mass of infinitesimally small investors. This "small investor" assumption implies that no individual investor can affect asset prices unilaterally and, therefore, there is no strategic interaction in such a setting[2]. On the other hand, I explicitly model the strategic interdependence between investor portfolio choices that result from endogenous asset price changes – to the best of my knowledge, this is a novel contribution to the literature on fire sales.

The fundamental characteristic of a fire sale is that the affected asset trades at a price significantly below its true or fundamental value. In general, this requires that two conditions hold. First, it requires one or more forced sellers. Second, some constraints should exist that stop buyers from taking advantage of fire-sale prices. (For a recent survey of the literature on fire sales, the reader is referred to Shleifer and Vishny (2010a)).

Forced sellers imply the existence of duress or binding constraints of some sort, otherwise sellers would not sell at a price so far below fundamental value. A non-exhaustive list of constraints that might plausibly lead to a forced sale could include: bankruptcy, financial distress (as documented in the market for commercial aircraft by Pulvino (1998)), sale of collateral by lenders, margin calls, borrower covenants, adverse changes in funding conditions (see for instance, Brunnermeier and Pedersen (2008)), defence against a hostile takeover (for corporates), rating agency models, regulatory capital requirements (for regulated entities such as banks and insurers), self-enforced risk management, stop loss limits (trading desks), fund withdrawals (mutual funds and hedge funds, see Brunnermeier and Pedersen (2005) and Coval and Stafford (2007)), counter-party credit monitoring and legal constraints on assets held (regulated entities again).

---

[1] Much of the theoretical literature on fire sales, starting with Shleifer and Vishny (1992) and Allen and Gale (1994), considers a single risky asset over time. Although a single asset setting is instructive in showing the various ways in which fire sales can occur and how they might evolve, I generalise to a setting with many assets. This allows me to to link the fire-sale literature to that of financial contagion.

[2] A similar point is made by Feeney and King (2001) in the context of parimutuel betting: "Much of the theoretical work analysing parimutuel systems, however, makes a critical 'small player' assumption that one individual cannot influence the actions of others. In other words, there are always enough players in the system so that the effect of one player's actions on the information and returns of other players can be ignored. ... The small player assumption greatly simplifies the modelling of parimutuel systems, but it is very strong. For example, if there are few players or if some players wager relatively large sums of money then the interdependency of returns in a parimutuel system means that the 'small player' assumption is likely to be violated"

However, the presence of a forced seller, in itself, does not lead to a fire sale. In a competitive market with no frictions or information asymmetries, some buyer should be willing to trade with a forced seller at a small discount to fundamental asset value. Hence, the second requirement, that there should be some constraints on potential buyers. This is really equivalent to a constraint on perfect arbitrage in the spirit of the literature on limits to arbitrage (see Gromb and Vayanos (2010) for a recent survey). A non-exhaustive list of constraints that might prevent buyers from taking advantage of fire sales could include: limited market participation (Allen and Gale, 1994), specialist knowledge or skills concerning the asset (as in Shleifer and Vishny (1992) in a corporate setting), asymmetric information, funding constraints, industry wide distress, agency problems facing outside investors[3], opacity of assets (consider for example sub-prime CDO's) and liquidity hoarding (as in Acharya, Shin and Yorulmazer (2009)).

I abstract from the exact mechanism causing a forced sale; instead, I assume that investors are subjected to some binding constraint expressed as an asset ratio. For expository reasons I label this ratio the debt-to-asset or leverage ratio; however, a similar analysis would apply to any constraint that can be expressed as an equity ratio. Likewise, I abstract from the exact mechanism constraining potential buyers. Instead, I model buyer constraints with an exogenous liquidity parameter such that prices are linearly decreasing in the volume of forced sales. This reflects the economic intuition that price decreases are necessary to induce buyers to overcome any constraints or costs they might face, and that the marginal buyer will require an ever lower price as the volume of assets put up for sale increases. These assumptions allow me to keep the model general, with a pure focus on fire sales and related contagion effects.

This means that my model is *not* an asset pricing model – instead, it is an attempt to model the mechanics of a fire sale in a strategic multi-investor, multi-asset setting, assuming the necessary impetus for a fire sale is in place. Thus it is a conditional model in the sense that it presumes a fire-sale environment. In basic terms, I seek to answer the question "How much will the price of this asset fall in a fire sale?" rather than the more general question "How much is this asset worth given the presence of fire-sale risk?". This allows me to *quantify* fire-sale risk at the asset level rigorously, which in turn opens the way to direct empirical tests of the hypothesis that expected returns should reflect fire-sale risk.

Within my model I show that equilibrium fire-sale prices exist. This equilibrium is unique, and it can be calculated to arbitrary accuracy using the method of successive approximations. Moreover, I derive an analytical approximation to equilibrium prices, which I validate numerically. From this approximation I derive (approximate) model statics and policy implications. A change in leverage by one investor imposes externalities, in the form of higher fire-sale risk and lower expected fire-sale prices, on other investors. In addition, a shock to one asset – leading to a fire sale – translates directly into contagion to other asset prices. Such contagion has implications for risk management; in particular, fire-sale risk is dependent not only on asset characteristics, but also on the overall pattern of asset ownership and investor leverage in the market.

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[3] Outside investors will need to contract specialist managers for the assets they acquire, which could give rise to costly agency problems.

1.2 Related literature

There has been renewed academic interest in fire sales recently, perhaps motivated by the global financial crisis of 2007-2008. In this subsection I discuss several recent working papers most closely related to the work presented in this chapter and the next[4].

Among the recent papers dealing with fire sales Cont and Wagalath (2012) is perhaps the closest to this chapter in their theoretical approach. The major similarities are that, as in this chapter, Cont and Wagalath (2012) considers a multi-asset multi-investor setting in which changes in prices give rise to selling pressure on funds through a specified "deleveraging schedule" – this can be viewed as a generalisation of the binding leverage requirement of my model. They also employ a "price impact" function, which plays a similar role to the price response function in my model. However, the objective in Cont and Wagalath (2012) is to characterise the covariance structure of returns generated by fire sales, while I focus on characterising equilibrium fire-sale prices and asset sales. In an empirical application of their model to two distinct market shocks (the hedge fund unwind of 2007 and the collapse of Lehman Brothers in 2008) Cont and Wagalath (2012) show that their model can detect the presence of fire-sale events.

Blocher (2013) proposes a model of overlapping holdings that generates spillover effects (externalities) as a result of "peer flows"[5]. As he puts it: "*All of these studies focus on the effect of a mutual fund's own flows predicting its own future returns, whereas my contribution is to consider a mutual fund's peers in the prediction of returns and fund flows through the channel of common stock holdings.*" (Blocher (2013), p6). In the model presented by Blocher (2013), the similarity in asset holdings between two funds is measured as the sum of the asset-by-asset product of their portfolio allocations, divided by the product of the (Euclidean) distance of each funds asset allocations from the origin in n-dimensional asset space[6]. Using this similarity measure, Blocher (2013) goes on to estimate the volume of flow a fund experiences because of its similarity to other funds that receive inflows: for a given fund this is calculated as the fund flows of all other funds, weighted by their relative similarity[7]. Using mutual fund data from Morningstar covering 1998 to 2009 Blocher (2013) shows that this measure of peer flow explains fund returns over and above the impact of the fund's own fund flows and after controlling for the common time-series risk factors.

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[4] Except for Greenwood and Thesmar (2011), all the papers discussed here are unpublished working papers as of September 2013. To the best of my knowledge these papers became publicly available only after this chapter was publicly disseminated in December 2011 (see Geertsema (2011))

[5] See also Antón and Polk (2010) for an alternative model of stocks connected by overlapping holdings.

[6] More precisely, let $h_{i,j}$ be the weight invested by fund $i \in [1...M]$ in asset $j \in [1..N]$, assuming each fund's total holdings are normalised to unity so the holding can be interpreted as percentages.

The model I present is similar to that of Blocher (2013) in that both models explicitly consider overlapping assets. However, the motivation of the model in Blocher (2013) is to explain fund flows while the primary motivation of my model is to derive fire-sale prices[8]. In addition, the model of Blocher (2013) does not consider investor heterogeneity – the definition of peer flow makes no distinction between highly leveraged investors and cash only investors. Because Blocher (2013) applies his model only to mutual funds this does not matter as mutual funds are typically barred from taking on leverage. However, my model is intended to apply more generally to both leveraged and non-leveraged investors. Finally, the model presented by Blocher (2013) is essentially static; it does not consider the second order effects that arise as selling promotes further selling and so on. In contrast my model explicitly models the cascading interaction between asset sales and declining prices that gives rise to higher order effects.

A model with more of a focus on systematic risk is due to Caccioli et al. (2012). In their model, institutions (referred to as "banks") are linked by overlapping asset holdings. It is assumed that at least some banks take on leverage. If the value of assets held by a bank falls sufficiently to completely erode its equity, the bank liquidates its entire portfolio of assets. Prices adjust based on the volume of assets liquidated – this is similar to the price response mechanism in my model. However, in their model, banks only sell assets at the point of bankruptcy – no prices move until at least one bank fails. This is in contrast to my model where asset sales of investors are regulated by the requirement to maintain a target leverage ratio. Caccioli et al. (2012) do not attempt to provide an equilibrium concept for their model and thus resort to an algorithm for calculating the impact of shocks in their model. Shocks take the form of either exogenous bank failure or an exogenous drop in the price of an asset. Calculation of the impact of a shock terminates when no further banks become insolvent. Using their model Caccioli et al. (2012) show that there are critical levels of leverage. Below this critical of level of leverage contagion does not occur, while above this level the network is subjected to rare but catastrophic cascades of bank failures.

Greenwood and Thesmar (2011) introduce the notion of stock price "fragility". Intu-

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Then the similarity between funds 1 and 2 is defined as $s_{A,B} = \frac{\sum_{j=1}^{N} h_{1,i} h_{2,j}}{\sqrt{\sum_{j=1}^{N} h_{1,j}^2}\sqrt{\sum_{j=1}^{N} h_{2,j}^2}}$. Note, I have simplified and recast the formulas of Blocher (2013) to be more consistent with the notation used in this chapter. For the original definitions see Blocher (2013), Appendix B, page 45.

[7] Again, to be more precise, for fund i "peer flow" is calculated as $p_i = \sum_{j=1, j \neq i}^{N} \frac{s_{i,j}}{\sum_{i=i}^{M} s_{i,j}} flow_i$

[8] Because the equilibrium in my model determines both asset holdings and prices endogenously, my model could in principle be used to explain fund flows. However, in this chapter the primary focus is understanding fire-sale prices.

itively fragility measures the price pressure an asset might experience because of the correlated (flow driven) liquidity needs of investors in that asset[9]. To calculate fragility Greenwood and Thesmar (2011) use ownership data from US mutual funds. They show that their estimates of fragility have predictive power for future realised volatility.

1.3 Model

1.3.1 Motivation

The purpose of the model presented here is to describe what happens in a fire sale scenario – when forced sellers are met in the market by constrained buyers.

I make a distinction between forced sellers, whom I model explicitly as individual agents (hereafter "investors"), and the remainder of the investor universe (i.e. the constrained buyers). Instead of directly modelling the constrained buyers, I model the effect of buyer constraints on the price available to forced sellers; the price is linearly decreasing in the volume of forced sales as explained in 1.3.5.

The leverage ratio (measured as the debt-to-asset ratio) is assumed to be a binding constraint on investors – this is what makes them forced sellers when prices decrease. The underlying assumption is that the starting point (period $t = 0$) is one of "equilibrium". Each investor has decided on his own allocation of assets and degree of leverage, and the assumption is that this is optimal for that investor. Note, I do not specify how this initial equilibrium is arrived at or how it should be calculated. I simply assume that, before the exogenous shock, each investor has taken on as much leverage as is optimal for him or her. I then require that, post the exogenous shock, the investor has to re-establish this initial leverage ratio. In reality, not all leveraged investors will necessarily be bound to maintain their leverage ratio's – those investors should not be included amongst the forced sellers in this model. On the other hand, this requirement is a reasonable approximation of the situation faced by many investors that are trading on margin, borrowing using collateral or otherwise compelled to liquidate assets when prices decline substantially.

The initial pattern of ownership is exogenous and allows for multiple investors to be invested in multiple assets. I rule out short positions for the modelled investors. (This requirement aids tractability.) In reality investors can and do take on short positions. I argue that disallowing short sales may not be as restrictive as it might appear at first glance. If short positions are only a small fraction of the total investor portfolio then the overall effect of the short positions are not likely to be economically significant. On the other hand, if an investor has large short positions (i.e. is short overall), such an investor will benefit from a large negative price shock and will not become a forced seller – instead, we can imagine such an investor buying stock from forced sellers to cover his own short positions and take profit. This provision of liquidity is modelled via the price response function described later in this section.

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[9] More precisely, the fragility $G_{i,t}$ of a stock i at time t is calculated as $G_{i,t} = \frac{1}{\theta^2_{t,t}} W'_{i,t}\Omega_t W_{i,t}$ where $W_{i,t}$ is the vector of the ownership interests in asset i , Ω_t is the variance-covariance matrix of investor fund flows and $\theta_{i,t}$ is the the market capitalisation of stock i.

Additional borrowing or equity-raising by investors is ruled out. In the long run and under benign market conditions, investors may of course have access to capital markets. But fire sales are neither benign nor long-run, and hence I believe it is reasonable to rule out access to capital markets for those investors that are forced to liquidate assets.

I make no assumptions about investor preferences. I don't need to. The modelled investors corresponds to those investors that become *forced* sellers in a fire sale. Hence, their preferences are irrelevant - they *have* to sell.

1.3.2 Description

I consider a three period model of N investors and M assets. In the first period ($t = 0$), each investor owns a portfolio consisting of some mix of the available assets. In addition, each investor's portfolio is partly funded by a given amount of debt. Both the composition of the initial portfolios and the levels of debt are exogenously specified.

In the second period ($t = 1$) each asset is subjected to an exogenous negative shock. The reduction in the value of investor portfolios implies that investor leverage ratios will now be higher than before. Therefore, at $t = 2$ each investor sells a portion of their portfolio to reduce their debt to the point where their original leverage ratio is re-established. If an investor's equity is reduced below zero he is bankrupted and his entire portfolio is liquidated. A crucial assumption is the price at which the investors can sell at $t = 2$. Initially, I take the liquidation price in period two as given and determine the fraction of each portfolio sold. I then extend the model by endogenising the liquidation price, linking it to the volume of portfolio sales in the same period.

1.3.3 Setup and notation

I start with N investors indexed by $i \in [1..N]$ with access to M assets in fixed supply indexed by $j \in [1..M]$. The units of asset j held by investor i is denoted by $h_{i,j}$. Each investor i has debt of d_i (dollar face value). For convenience, let

$$A_i \equiv \sum_{j=1}^{M} h_{i,j} \tag{1.3.1}$$

be the value of each investor's portfolio at $t = 0$ prices (we assume $p_j = 1$ at $t = 0$). I also define

$$H_i \equiv \sum_{j=1}^{M} h_{i,j} p_j \tag{1.3.2}$$

This might be thought of as the valuation of the *initial* portfolio holdings $\{h_{i,1}, ..., h_{i,M}\}$ of investor i valued at the *new* $t = 2$ prices $\{p_1, ..., p_M\}$. I enforce a positive equity constraint initially; assets exceed debt for all investors; $0 \leq d_i < A_i$. The initial period $t = 0$ assets holdings and debt are thus as described in Table 1.1 below.

Table 1.1: Asset holdings and debt at $t = 0$

	Asset 1 holdings	$\cdots$	Asset M holdings	Investor debt
Investor 1	$h_{1,1}$	$\cdots$	$h_{1,M}$	d_1
$\vdots$	$\vdots$		$\vdots$	$\vdots$
Investor N	$h_{N,1}$	$\cdots$	$h_{N,M}$	d_N

Asset holdings are infinitely divisible and non-negative – short positions are not allowed. Therefore, $h_{i,j} \geq 0$ for all $i \in [1..N]$ and for all $j \in [1..M]$.[10] Without loss of generality I normalise the $t = 0$ prices of assets to unity. The initial leverage ratio of investor i is then given by

$$L_i(0) \equiv \frac{Debt_i(0)}{Assets_i(0)} = \frac{d_i}{A_i} \tag{1.3.3}$$

It is sometimes convenient to use the debt-equity ratio instead of the leverage ratio, therefore, I define the debt-equity ratio as

$$D_i \equiv \frac{Debt_i}{Equity_i} = \frac{d_i}{A_i - d_i}. \tag{1.3.4}$$

At $t = 1$ there occurs an exogenous negative shock to assets, such that prices reduce to s_j with $0 < s_j \leq 1$.[11] These negative shocks have the effect of increasing investor leverage ratios:

$$L_i(1) = \frac{Debt_i(1)}{Assets_i(1)} = \frac{d_i}{\sum_{j=1}^{M} h_{i,j} s_j} > L_i(0) \tag{1.3.5}$$

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[10] Hereafter a subscript i will be taken to mean $i \in [1..N]$ (ie applicable to all investors) and similarly a subscript j will be taken to mean $j \in [1..M]$ (ie applicable to all assets). Where relevant, the time is indicated by the period in parenthesis. Therefore, $K_{i,j}(t)$ means variable K for any investor i and any asset j at time t.

[11] Provided that $s_j < 1$ for at least one asset. If $s_j = 1$ for all assets then the model is trivially in equilibrium without the need for any portfolio sales – by requiring $s_j < 1$ for at least one asset $j \in [1..M]$ this trivial equilibrium is ruled out.

At $t = 2$ investors re-establish their original leverage ratios by each selling a fraction of their portfolio at prices p_j (exogenously specified, for now). If debt exceeds assets at the new prices for any investor (before considering any sales), that investor is bankrupted and their entire portfolio is liquidated. Otherwise the investor re-establishes his original leverage ratio by selling a portion of his assets pro-rata.

1.3.4 Leverage and portfolio sales

At period $t = 2$ investor i sells an identical fraction $x_i = f_i(p_1, ..., p_M)$ of each asset held. Note that the sale fraction x_i is the same for *each* of the assets held by that investor – each investor liquidates his portfolio strictly pro-rata. This assumption is a somewhat restrictive one because, in reality, investors might well choose to sell different fractions of each asset to maximise their final net worth. In particular, if there is an idiosyncratic shock that affects only some assets but not others, an investor might rationally decide that selling the unaffected assets at fundamental value is preferable to selling those assets that have just experienced a negative price shock and are now trading below fundamental value. In other words, an investor would liquidate assets sequentially in descending price order to minimize the discount to fundamental value. On the other hand the pro-rata portfolio sales assumption is not that unrealistic when considering a systematic shock in which all assets decrease proportionately in price. Because all assets are trading at the same discount to fundamental value, there is no benefit in selling more of one asset than another, so the assumption that investors sell the same fraction of each asset is more realistic[12].

I enforce $p_j \leq s_j$ for now, namely, the prices cannot exceed the period $t = 1$ price. In addition, prices are strictly positive, such that $p_j > 0$.[13] Post asset-sale holdings and sales are summarised in Table 1.2 below.

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[12] As a robustness check I perform a numerical simulation (see subsection 1.8.4 in the Appendix for details) in which I compare equilibrium prices derived using pro-rata asset sales against equilibrium prices obtained using sequential asset sales in descending price order. It turns out that whether assets are liquidated pro-rata or sequentially in descending price does not materially change the equilibrium price outcome. Equilibrium prices obtained using the two assumptions are highly correlated ($\rho > 0.97$) and the pro-rata equilibrium price adjustment explains 95% or more of the variation in sequential sales equilibrium price adjustment. Based on this analysis I opt for the analytically simpler assumption of pro-rata portfolio sales.

[13] These assumptions are required while prices are exogenous to our model. In the next subsection I will endogenise prices, so I will no longer need to make these assumptions. However, the same constraints emerge endogenously.

Table 1.2: Post-sale asset holdings and asset sales at $t = 1$

	Post-sale Holdings			Fraction Sold		
	Asset 1	$\cdots$	Asset M	Asset 1	$\cdots$	Asset M
Investor 1	$(1 - x_1)h_{1,1}$	$\cdots$	$(1 - x_1)h_{1,M}$	$x_1 h_{1,1}$	$\cdots$	$x_1 h_{1,M}$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
Investor N	$(1 - x_N)h_{N,1}$	$\cdots$	$(1 - x_N)h_{N,M}$	$x_N h_{N,1}$	$\cdots$	$x_N h_{N,M}$
Price	p_1	$\cdots$	p_M			

I proceed to derive the fraction sold by each investor, assuming the investor remains solvent at the new prices. (If not solvent, the investor is bankrupted and is liquidated such that $x_i = 1$). Because I assumed investor solvency in this case, we have $H_i > d_i$. I denote the portfolio sale fraction of investor i in the solvent case by f_i^s to distinguish it from the general case, which admits investor bankruptcy. The new level of leverage, after portfolio sales, is then given by

$$L_i(2) = \frac{Debt_i(2)}{Assets_i(2)} = \frac{Debt_i(1) - Sales_i(2)}{Assets_i(2)} = \frac{d_i - f_i^s H_i}{(1 - f_i^s)H_i} \tag{1.3.6}$$

To re-establish initial leverage, I require that the new leverage ratio $L_i(2)$ equal the initial leverage ratio $L_i(0)$

$$L_i(2) = L_i(0)$$

or, recalling that $L_i(0) = \frac{d_i}{A_i}$ from equation (1.3.3)

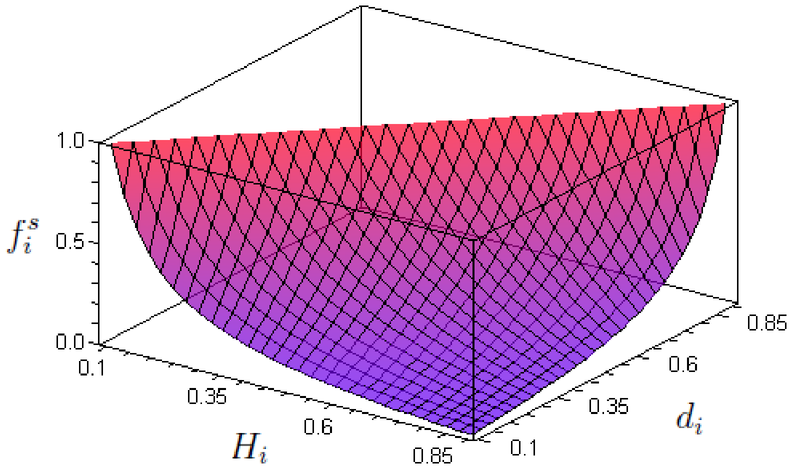
$$\frac{d_i - f_i^s H_i}{(1 - f_i^s)H_i} = \frac{d_i}{A_i}$$

Rewriting for f_i^s and rearranging slightly, I obtain

$$f_i^s = \frac{A_i - H_i}{H_i} \times \frac{d_i}{A_i - d_i} = \frac{A_i - H_i}{H_i} D_i \tag{1.3.7}$$

This provides us with a convenient economic interpretation of f_i^s (the fraction of the portfolio investor i needs to sell to re-establish the original leverage ratio). The first product term, $(A_i - H_i)/H_i$, corresponds to the percentage change in the value of the initial portfolio (expressed in terms of the new portfolio value), while the second product term, $d_i/(A_i - d_i)$ or D_i, corresponds to the initial debt-equity ratio. In other words, the portfolio sale fraction f_i^s is the product of the percentage change in the value of the portfolio and the debt-equity ratio. Note that this relationship must hold for *any* new prices p_j if leverage ratios are to be re-established under those prices. The portfolio sale fraction f_i^s can be described graphically, as in Figure 1.3.1 below.

Figure 1.3.1: Optimal portfolio liquidation fraction f_i^s as a function of pre-sale portfolio value H_i and investor debt d_i



The limit of $f_i^s = 1$ (selling the entire portfolio) is approached at the plane defined by $H_i = d_i$, that is

$$\lim_{H_i \to d_i} \frac{A_i - H_i}{H_i} \times \frac{d_i}{A_i - d_i} = 1 \tag{1.3.8}$$

Economically, this corresponds to the notion that an investor targeting a lower leverage ratio will have to sell an ever larger fraction of the portfolio the closer the remaining equity $(H_i - d_i)$ approaches zero.

It is instructive to examine the statics of f_i^s in terms of the pre-sale portfolio value H_i

$$\frac{\partial f_i^s}{\partial H_i} = -\frac{A_i d_i}{(A_i - d_i)\, H_i^2} \leq 0 \quad \text{and} \quad \frac{\partial^2 f_i^s}{\partial H_i^2} = \frac{2 A_i d_i}{(A_i - d_i)\, H_i^3} \geq 0$$

Thus the portfolio sale fraction f_i^s is decreasing in the pre-sale portfolio value H_i at an increasing rate. I now consider how initial debt influences the portfolio sale fraction f_i^s.

$$\frac{\partial f_i^s}{\partial d_i} = \frac{A_i - H_i}{H_i} \times \frac{A_i}{(A_i - d_i)^2} \geq 0 \quad \text{and} \quad \frac{\partial^2 f_i^s}{\partial d_i^2} = \frac{A_i - H_i}{H_i} \times \frac{2A_i}{(A_i - d_i)^3} \geq 0$$

The portfolio sale fraction f_i^s is increasing in the level of initial debt at an increasing rate. Taken together, this implies that the portfolio sale fraction f_i^s is highly non-linear in investor equity $(H_i - d_i)$. As equity declines and approaches zero the optimal portfolio sale fraction rapidly increases towards 100% (see Figure 1.3.1 above).

After substituting in the definition of H_i and A_i, f_i^s (as per equation (1.3.7)) can be rewritten as

$$\begin{aligned} f_i^s &= \frac{A_i - H_i}{H_i} \times \frac{d_i}{A_i - d_i} \\ &= \left(\frac{A_i}{H_i} - 1 \right) \times \frac{d_i}{A_i - d_i} \\ &= \left(\frac{\sum_{j=1}^{M} h_{i,j}}{\sum_{j=1}^{M} h_{i,j} p_j} - 1 \right) \times \frac{d_i}{\left(\sum_{j=1}^{M} h_{i,j}\right) - d_i} \end{aligned} \tag{1.3.9}$$

Note that the above expression assumes the investor is solvent. In the event of bankruptcy (where $H_i < d_i$) the portfolio sale fraction for that investor is 1. Therefore, the more general expression for the portfolio sale fraction, in which I explicitly consider the possibility of investor bankruptcy, is given by

$$x_i = f_i(p_1, p_2, ..., p_M) = \begin{cases} f_i^s(p_1, p_2, ..., p_M) & H_i > d_i \quad \text{(Solvent)} \\ 1 & H_i \leq d_i \quad \text{(Bankrupt)} \end{cases}$$

Because the limit of $f_i^s = 1$ is approached as $H_i \rightarrow d_i$, this can be written more compactly as

$$f_i = \min(f_i^s, 1) \tag{1.3.10}$$

1.3.5 Price response

I now extend the model by determining portfolio sales prices endogenously. Broadly, I would like prices to be decreasing in the volume of forced sales. This allows me to abstract from the exact mechanism which might be constraining potential buyers (of which there are potentially many). I focus instead on the *impact* of constraints on buyers[14]. Buyers require a price discount to overcome their constraints, whatever those constraints might be. Additionally, as the volume of forced sales increases the marginal buyer faces ever tighter constraints, such that an ever larger price discount is needed to surmount those constraints. This might be thought of as a liquidity effect – as the volume of forced sales increases the price decreases. To model the impact of constraints on buyers, I assume a constant liquidity parameter δ_j for each asset j[15]. The liquidity parameter δ_j equals the percentage change in quantity sold divided by the percentage change in price (this is equivalent to the price elasticity concept in economics).

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[14] Thus, instead of directly modelling the buyers, I instead model the price discount they require to provide liquidity to forced sellers.

[15] At first glance, *constant* liquidity appears a strong assumption – there is no reason to expect liquidity to be constant, irrespective of the size of the shock. However, both the shock and the liquidity are exogenously specified in this setup. Therefore, nothing prevents us from choosing, for each asset, a liquidity parameter that we consider appropriate given the size of the shock applied to that asset.

$$\delta_j = \frac{\%\Delta Q_j}{\%\Delta p_j} \tag{1.3.11}$$

I define $\%\Delta p_j$ as the percentage change in price for asset j between $t = 1$ and $t = 2$

$$\%\Delta p_j = \frac{\Delta p_j}{p_j(1)} = \frac{p_j(2) - p_j(1)}{p_j(1)} = \frac{p_j - s_j}{s_j} \tag{1.3.12}$$

and I define $\%\Delta Q_j$ as the percentage of asset j sold (by all investors) at time $t = 2$

$$\%\Delta Q_j = \frac{\Delta Q_j}{Q_j(1)} = \frac{Q_j(2) - Q_j(1)}{Q_j(1)} = -\frac{\sum_{i=1}^{N} h_{i,j} x_i}{\sum_{i=1}^{N} h_{i,j}} \tag{1.3.13}$$

Substituting equation (1.3.12) and (1.3.13) into equation (1.3.11) and solving for p_j yields

$$p_j = s_j - \frac{s_j}{\delta_j}\left(\frac{\sum_{i=1}^{N} h_{i,j} x_i}{\sum_{i=1}^{N} h_{i,j}}\right) \tag{1.3.14}$$

Some constraints must be imposed on δ_j; if $\delta_j = 1$, then prices would go to zero if all investors sold all their assets. Likewise, $\delta_j < 1$ implies the existence of negative prices. Because I would like to ensure strictly positive prices, I assume the supply of assets is strictly elastic for each asset, $\delta_j > 1$. It is then easy to show that if *all* investors sold *all* their assets, the resulting price would be at its minimum (because price is strictly decreasing in portfolio sales) and the price of asset j is then given by

$$p_j^{min} = s_j - \frac{s_j}{\delta_j} > 0 \tag{1.3.15}$$

I assume that the price response (equation (1.3.14)) is binding in period $t = 2$. Prices and portfolio sale fractions are thus resolved simultaneously in period $t = 2$.

Price p_j is linearly decreasing in the portfolio sale fractions x_i, as should be expected given our assumption of constant liquidity. Analytically, the statics of equation (1.3.14) with regards to the portfolio sale fraction x_i of investor i is given by

$$\begin{aligned}\frac{\partial p_j}{\partial x_i} &= -\frac{h_{i,j}}{\sum_{i=1}^{N} h_{i,j}} \times \frac{s_j}{\delta_j} \\ &= -\left(\%\text{ of asset } j \text{ held by investor } i \times \frac{\text{Asset } j \text{ shock}}{\text{Asset } j \text{ liquidity}}\right)\end{aligned}$$

The strength of the negative price response of asset j to portfolio sales by investor i depends on the shock s_j relative to liquidity δ_j, and the fraction of asset j held by investor i, that is $h_{i,j}/(\sum_{i=1}^{N} h_{i,j})$.

Crucially, the new price p_j in equation (1.3.14) depends on the initial portfolios and subsequent portfolio sales of *all* investors. This induces strategic dependence between the investors – the portfolio sales of one investor moves the prices of all assets owned by him and thereby influences the amount other investors need to sell, and vice versa.

Another way of looking at fire-sale risk is to introduce a fire-sale multiplier – how much an initial price shock is aggravated by the effect of fire sales in that (and other) assets. This yields the fire-sale multiplier m_j.

$$\begin{aligned} m_j &\equiv \frac{\text{Total price change}}{\text{Price change due to shock}} \\ &= \frac{1 - p_j}{1 - s_j} \end{aligned}$$

Rewriting the above suggests another way of presenting the equilibrium price p_j

$$p_j = 1 - m_j(1 - s_j)$$

The new price is equal to the initial price (1) less the fire-sale multiplier (m_j) times the exogenous shock $(1 - s_j)$.

1.3.6 Model assumptions

For completeness, these are the assumptions placed on model parameters.

1. $h_{i,j} \geq 0$. Initial investor holdings of each asset are weakly positive (no short positions).
2. $G_j = h_{1,j} + h_{2,j} + \cdots + h_{N,j} > 0$. The total stock of each asset (across the entire set of investors) is strictly positive for each asset. This is necessary for the price response to be well defined for each asset.
3. $\delta_j > 1$. The supply of assets at $t = 2$ is strictly elastic. This rule out zero or negative prices for any level of portfolio sales.
4. $0 < s_j \leq 1$ with $s_j < 1$ for at least one $j \in [1..M]$. The period $t = 1$ (post-shock) price is strictly positive and less than or equal to unity. At least one asset must experience a non-zero price shock, otherwise the model is trivially in equilibrium without requiring any portfolio sales.
5. $0 \leq d_i < A_i = h_{i,1} + h_{i,2} + \cdots + h_{i,M}$. For each investor initial debt is weakly positive and less than the value of the investor's initial portfolio at $t = 0$ prices.

1.4 Equilibrium

1.4.1 Introduction

Equilibrium is reached when all investors have either become bankrupt or have re-established their leverage ratios at asset prices simultaneously determined through the price response mechanism outlined earlier. It is this interdependence between portfolio sales and asset prices that makes it particularly challenging to establish equilibrium. Consider the mechanics of our model: asset prices are subjected to a negative exogenous shock in period one. This results in higher leverage ratios for all investors holding those assets; however, investors are compelled to keep their leverage ratios at their pre-shock levels (this requirement creates the necessary set of forced sellers in our model). To do so, investors need to sell some fraction of their portfolio. This would be straightforward if they were all price takers; however, in our model this is not the case. When investors sell assets this drives down the price of those assets in line with our assumption of a constant liquidity parameter. The constant liquidity assumption is the mechanism by which I model the other necessary component of fire sales – constrained buyers. As the volume of forced asset sales increase, outside buyers need increasingly higher price discounts to overcome any constraints they face and enter the market. So, when investors sell assets they realise that doing so will drive down prices. Moreover, an investor will need to consider not only the price-impact of his own assets sales, but also the price-impact of sales by other investors. Those other investors, in turn, will need to take into account asset sales by investors other than themselves. In other words, the actions of investors are strategically interdependent. This suggests a game-theoretic interpretation: investors sell a fraction their portfolio in order to re-establish their leverage ratios, rationally taking into account the optimal strategies of other investors to achieve the same result.

More formally, *equilibrium* is a set of portfolio sale fractions and a set of final asset prices such that each investor either re-establishes his original leverage ratio at those prices or goes bankrupt with his entire portfolio liquidated; at the same time prices are determined by the volume of each asset sold by investors, according to the price response mechanism. If in equilibrium no investors are bankrupted, I term this a *solvent equilibrium* to distinguish it from the more general equilibrium, which admits bankrupt investors.

The formal proof for the existence and uniqueness of equilibrium is set out in subsection 1.8.1 in the Appendix.

1.4.2 Existence and uniqueness

It is not obvious *a priori* that every combination of allowed model parameters should result in equilibrium. The standard approach to establish equilibrium would be to derive the equilibrium portfolio sale fractions and asset prices analytically. I did so (for the special case of two investors and two assets) by substituting prices into the two investor's best response functions, then solving those best response functions simultaneously[16]. This yields a polynomial of degree 4, for which there are known to exist general algebraic solutions. Even so, the resulting analytical expressions are unwieldy (several pages long) and are also hard to interpret economically because the equilibrium quantities are potentially complex rather than real for some roots. In addition, for more than two investors, the equivalent polynomial will in general be of degree 6 or higher – and by the Abel–Ruffini theorem there are no general algebraic solutions to polynomials of degree 5 and above.

Instead, I focus on the more general case of N investors and M assets. In this context I show that equilibrium exists and that it is unique. Furthermore, this equilibrium can be calculated to arbitrary degree of accuracy using the method of successive approximations.

What insights do we gain from the existence of a unique equilibrium? Most obviously each episode of fire sales must end with some set of new prices – and there is only one such set of new prices. Therefore, we need not be concerned with the possibility of multiple equilibria or how to select among them.

1.4.3 Method of successive approximations

Besides proving existence and uniqueness my proof also guarantees that the equilibrium can be calculated using the method of successive approximations. What this means in practice is that we take the initial set of shocked prices and calculate the optimal portfolio sale fractions using equation (1.3.7) *assuming* prices won't change. Then we calculate new prices using equation (1.3.14) based on those portfolio sale fractions. These new prices will be closer to the true equilibrium prices than the initial set of prices. Repeating the process yields prices that converge monotonically to equilibrium prices as the number of iterations grows larger. Figure 1.4.1 shows the evolution of prices using the method of successive approximations for the example discussed below.

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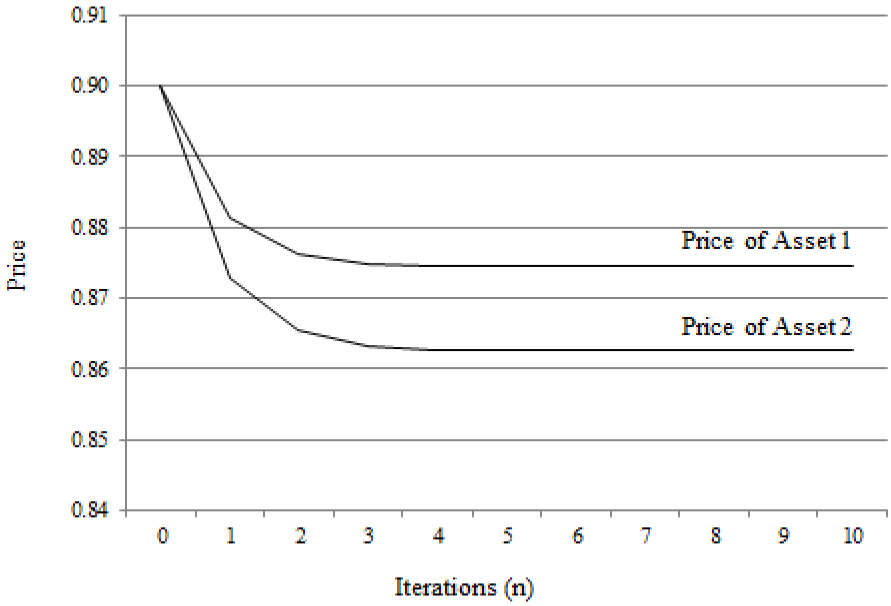
[16] Results available from the author on request

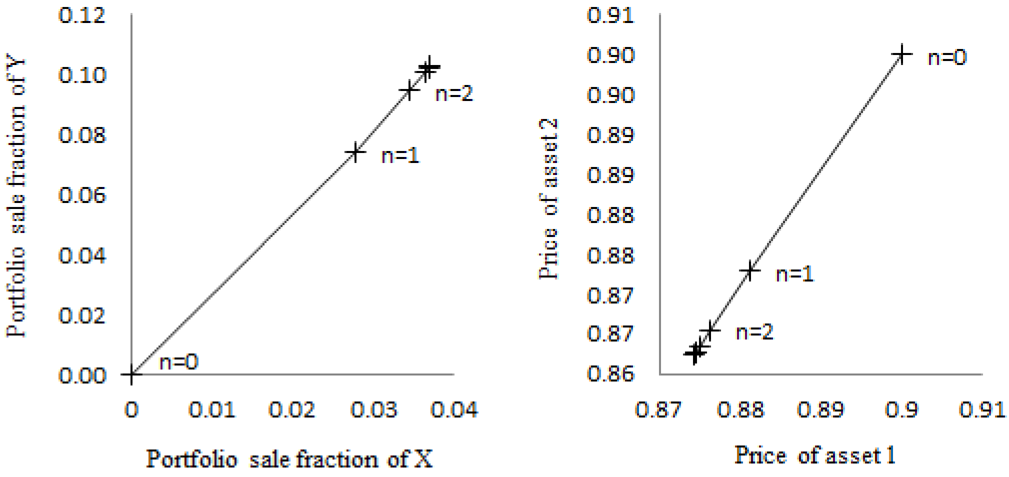
Figure 1.4.1: The evolution of prices using the method of successive approximations



In our example we have two investors: X and Y. Likewise there are two assets, 1 and 2. Investor X has debt corresponding to 20% of assets and has 70% of his portfolio invested in asset 1 and 30% invested in asset 2. By contrast, investor Y has debt corresponding to 40% of assets and has 30% of his portfolio invested in asset 1 and 70% invested in asset 2. The liquidity parameter for both assets is 2 and both assets suffer a 10% exogenous shock so their period one shocked prices are both 0.9. What will be the equilibrium prices and portfolio sale fractions? The method of successive approximations starts by taking the shocked prices of 0.9 for each asset as given. These are the zero order approximations ($n = 0$). I then calculate the optimal portfolio sale fractions for both investors, and using those, calculate a new set of prices (the first order approximations, or $n = 1$). These new prices turn out to be 0.881250 and 0.8729167 for asset 1 and 2 respectively. I continue to calculate new prices in the same way until the prices converge to our desired degree of accuracy. The equilibrium prices in our example turn out to be 0.874404731 and 0.862525028 for asset 1 and 2 respectively – accurate to 9 decimal places and obtained after 15 iterations.

I illustrate the evolution of prices and portfolio share fractions graphically in Figure 1.4.2

Figure 1.4.2: The evolution of portfolio share fractions (LHS) and prices (RHS) using the method of successive approximations



Notably, the bulk of the adjustment from the initial shocked prices towards the final equilibrium prices occur in the first iteration – the first order approximation accounts for more than 70% of the adjustment towards the final equilibrium prices for both prices in the example above (see RHS in Figure 1.4.2 above). This suggests that the first order approximation may be a good estimate for final equilibrium prices, at least in a relatively sense. I explore this idea further in the next section.

1.5 Approximation

1.5.1 An analytical approximation

As we do not have an analytic solution to the general case of N investors and M assets, we need to consider alternative approaches. One such approach is to rely on the fact that the method of successive approximations converge to a unique equilibrium solution, in accordance with our proof. One can then argue that the first order approximation (i.e., the approximate solution obtained from iterating the equilibrium requirements once) should provide a reasonable proxy for the direction and (relative) magnitude of the true equilibrium prices.

I show (in subsection 1.8.2 in the Appendix) that the first order price approximation p'_j of asset j is given by

$$p'_j = s_j - \frac{s_j}{\delta_j} WA_j \left[C_i D_i \right] \qquad (1.5.1)$$

where, for a given asset j, $WA_j[.]$ is the weighted average across investors of an investor specific variable z_i with weights provided by each investor's holdings of asset j. So

$$WA_j\left[z_i\right] \equiv \frac{\sum_{i=1}^{N} h_{i,j} z_i}{\sum_{i=1}^{N} h_{i,j}} = \frac{1}{G_j} \sum_{i=1}^{N} h_{i,j} z_i$$

where

$$G_j \equiv \sum_{i=1}^{N} h_{i,j}$$

is the total supply of asset j. Recall that the initial debt-equity ratio D_i of investor i is denoted by

$$D_i \equiv \frac{d_i}{A_i - d_i} \quad \text{where} \quad A_i \equiv \sum_{j=1}^{M} h_{i,j}$$

I denote by C_i the percentage loss of investor i's portfolio caused by the shocked prices s_j in period one (the loss being expressed as a percentage of the portfolio valued at the *new* shocked prices (S_i))

$$C_i \equiv \frac{A_i - S_i}{S_i} = \frac{A_i}{S_i} - 1 \quad \text{where} \quad S_i \equiv \sum_{j=1}^{M} h_{i,j} s_j$$

This means that the change of price given by the first order approximation will be

$$\begin{aligned}
\Delta p_j' &= p_j' - s_j \\
&= -\frac{s_j}{\delta_j} WA_j\left[C_i D_i\right] \\
&= -\frac{\text{Asset } j \, \text{shocked price}}{\text{Asset } j \, \text{price elasticity of supply}} \\
&\quad \times WA_j \left[(\text{Investor } i \, \text{debt-equity ratio})\,(\text{Investor } i \, \text{portfolio loss } \%)\right]
\end{aligned}$$

For a given asset, the fire sale induced drop in price *beyond* the initial shock is equal to the ratio of the asset's shocked price to its liquidity, multiplied by the weighted average (across all investors according to their initial holdings of the asset) of the product of each investor's debt-equity ratio and their percentage portfolio loss due to the initial shock (the percentage being expressed in terms of the value their post shock portfolio).

The relevance of this approximation becomes more apparent if we consider that the approximate price of an asset can be interpreted as a measure of the fire-sale risk that an asset is exposed to. In other words, it allows us to *quantify* fire-sale risk analytically at the individual asset level. An immediate insight arising from this interpretation is that the fire-sale risk of an asset depends not only on asset specific attributes such as liquidity or asset-specific shock, but also depends crucially on the distribution of asset holdings and debt across *all* investors. All else being equal, an asset disproportionately owned by indebted investors will have a lower fire-sale equilibrium price and a higher fire-sale risk (Wagner (2011) reaches a similar conclusion in a different setting). This has policy implications; if some investors take on more debt, this increases the fire-sale risk of assets owned by those investors. This in turn means that other investors in those assets are now exposed to higher risk – clearly a negative externality.

With a few substitutions we can give the analytical expression for the first order approximation p'_j purely in model parameters

$$\begin{aligned} p'_j &= s_j - \frac{s_j}{\delta_j}\left(\frac{\sum_{i=1}^{N} x_i h_{i,j}}{\sum_{i=1}^{N} h_{i,j}}\right) \\ &= s_j - \frac{s_j}{\delta_j}\left(\frac{\sum_{i=1}^{N} h_{i,j}\left(\left(\frac{\sum_{j=1}^{M} h_{i,j}}{\sum_{j=1}^{M} h_{i,j} p_j} - 1\right) \times \frac{d_i}{\left(\sum_{j=1}^{M} h_{i,j}\right) - d_i}\right)}{\sum_{i=1}^{N} h_{i,j}}\right) \end{aligned}$$

This provides us with an analytical approximation of the true equilibrium price – useful in gaining economic insights into the equilibrium price; *provided* that it is a good approximation. In the next section I numerically validate the accuracy of the first order approximation.

1.5.2 Numerical validation

To test the performance of the first order approximation, I randomly sample the parameter space of our model 10 million times, calculating for each draw both the first order price approximations and the true equilibrium prices (accurate to six decimal place). This allows me to validate the accuracy of the first order approximation by regressing true equilibrium prices on the corresponding first order approximations. To begin with, I consider a setting with two investors and two assets (2×2). This setup requires 10 model parameters[17], which I generate using a (uniformly distributed) random number generator. Table 1.3 below details the range used for each class of model parameter.

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[17] In general, a $N \times M$ setup requires $M \times N + 2M + N$ model parameters, made up of $M \times N$ initial asset holdings, M asset liquidity parameters, M shocked prices and N initial debt levels.

Table 1.3: Model parameter sample ranges

Parameter class	Symbol	Min	Max	Theoretical Range
Asset holdings of asset j by i	$h_{i,j}$	0.01	0.99	$[0, \infty]$
Debt of investor i (where $A_i = \sum_{j=1}^{M} h_{i,j}$)	d_i	0.01	A_i	$[0, A_i)$
Shocked price of asset j	s_j	0.51	0.99	$[0, 1)$
Liquidity of asset j	δ_j	1.01	10.01	$(1, \infty)$

Note that, although the shocked price s_j theoretically ranges from 1 to 0, I instead sample from the range 1 to 0.50 – this corresponds to negative individual asset price shocks between 0% and 50%, which I consider more realistic. I limit the analysis below to asset 1 – because the model is symmetric in assets, these results should equal that obtained using any other asset[18].

Regression (1) in Table 1.4 below sets out the results of a regression of the true equilibrium price p_1^* on the shocked price s_1. This is a benchmark, because s_1 is one of the model parameters rather than an approximation – therefore, any proposed approximation should do better than that. The variation in the shocked price explains 96% of the observed variation in the true equilibrium price. Regression (2) shows the results of regressing the equilibrium price p_1^* on the first order price approximation p_1'. Fully 99% of the variation in the equilibrium price is explained by the first order approximation. However, one may well argue that it is not the *level* of equilibrium prices that are of interest, but the *price adjustment* from the initial shocked price to the final equilibrium price – an argument boosted by the shocked price s_j already explaining so much of the variation in the true equilibrium price. In regression (4) I therefore regress the equilibrium change in price $\Delta p_1^* \equiv p_1^* - s_1$ on the first approximation change in price $\Delta p_1' \equiv p_1' - s_1$. I also include regression (3) – a regression of the equilibrium price adjustment on the initial shocked price – as an analogue to regression (1). The results show that 86% of the change in price (from the initial shocked price to the true equilibrium price) is explained by the difference between the first order price approximation and the shocked price. By comparison, the initial shocked price explains only 42% of the equilibrium price adjustment.

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[18] This is indeed the case in the data.

Table 1.4: Regression results – true equilibrium price (2x2)

The regressions below consider the degree to which approximate equilibrium prices are able to explain exact equilibrium prices using simulated data, assuming two investors and two assets. Regression (1) considers a regression of the true equilibrium price p_1^* on the shocked price s_1. Regression (2) shows the results of regressing the equilibrium price p_1^* on the first order price approximation p_1'. Regression (3) regresses the equilibrium *change* in price $\Delta p_1^* \equiv p_1^* - s_1$ on the the shocked price s_1. Regression (4) regresses the equilibrium *change* in price $\Delta p_1^* \equiv p_1^* - s_1$ on the first approximation *change* in price $\Delta p_1' \equiv p_1' - s_1$. t-Statistics are calculated using White robust OLS standard errors.

	(1)	(2)	(3)	(4)
	$p_1^\star$	$p_1^\star$	$\Delta p_1^\star$	$\Delta p_1^\star$
	β/t-Stat	β/t-Stat	β/t-Stat	β/t-Stat
s_1	0.856***		0.144***	
	(15137.44)		(2546.45)	
p_1'		0.961***		
		(31749.47)		
$\Delta p_1'$				1.280***
				(4381.14)
Adj R^2	0.9625	0.9902	0.4209	0.8647
N	10,000,000	10,000,000	10,000,000	10,000,000

It is reasonable to question whether these results, based as they are on a setup with two investors and two assets, do in fact generalise to a setting with a larger number of investors and assets. As a robustness check, I also consider a setup with five investors and five assets (in Table 1.5) and 10 assets and 10 investors (Table 1.6). The results I obtain are even stronger. For five investors and five assets, the first order approximation explains 96% of the true price adjustment and 99.6% of the true equilibrium price.

Table 1.5: Regression results – true equilibrium price (5x5)

The regressions below consider the degree to which approximate equilibrium prices are able to explain exact equilibrium prices using simulated data, assuming five investors and five assets. Regression (1) considers a regression of the true equilibrium price p_1^* on the shocked price s_1. Regression (2) shows the results of regressing the equilibrium price p_1^* on the first order price approximation p_1'. Regression (3) regresses the equilibrium *change* in price $\Delta p_1^* \equiv p_1^* - s_1$ on the the shocked price s_1. Regression (4) regresses the equilibrium *change* in price $\Delta p_1^* \equiv p_1^* - s_1$ on the first approximation *change* in price $\Delta p_1' \equiv p_1' - s_1$. t-Statistics are calculated using White robust OLS standard errors.

	(1)	(2)	(3)	(4)
	$p_1^\star$	$p_1^\star$	$\Delta p_1^\star$	$\Delta p_1^\star$
	β/t-Stat	β/t-Stat	β/t-Stat	β/t-Stat
s_1	0.854***		0.146***	
	(18423.93)		(3145.89)	
p_1'		0.967***		
		(57638.14)		
$\Delta p_1'$				1.286***
				(6679.31)
Adj R^2	0.9744	0.9963	0.5258	0.9567
N	10,000,000	10,000,000	10,000,000	10,000,000

In a setup with 10 investors and 10 assets, the first order approximation explains 98% of the true price adjustment and 99.8% of the true equilibrium price. Taken together, these results provide substantial support for the accuracy of the first order approximation.

Table 1.6: Regression results – true equilibrium price (10x10)

The regressions below consider the degree to which approximate equilibrium prices are able to explain exact equilibrium prices using simulated data, assuming ten investors and ten assets. Regression (1) considers a regression of the true equilibrium price p_1^* on the shocked price s_1. Regression (2) shows the results of regressing the equilibrium price p_1^* on the first order price approximation p_1'. Regression (3) regresses the equilibrium *change* in price $\Delta p_1^* \equiv p_1^* - s_1$ on the the shocked price s_1. Regression (4) regresses the equilibrium *change* in price $\Delta p_1^* \equiv p_1^* - s_1$ on the first approximation *change* in price $\Delta p_1' \equiv p_1' - s_1$. t-Statistics are calculated using White robust OLS standard errors.

	(1) $p_1^\star$ β/t-Stat	(2) $p_1^\star$ β/t-Stat	(3) $\Delta p_1^\star$ β/t-Stat	(4) $\Delta p_1^\star$ β/t-Stat
s_1	0.854***		0.146***	
	(20613.05)		(3531.04)	
p_1'		0.968***		
		(80969.55)		
$\Delta p_1'$				1.271***
				(10578.79)
Adj R^2	0.9795	0.9980	0.5832	0.9825
N	10,000,000	10,000,000	10,000,000	10,000,000

If anything, the analysis presented here is conservative – for two reasons. First, in this analysis I give equal weight to both median (typical) parameter values and extremal (atypical) parameter values. Since the approximation error is generally larger when confronted with extreme parameter values, this suggests that in an empirical application – where parameter values are likely to be clustered around more typical levels – approximation error would be smaller than that generated by this approach. Second, in any empirical application one would likely be confronted with thousands of assets and thousands of investors. Based on the improvement in accuracy we observe as we scale up to more investors and more assets, an analysis based on a setting with only a handful of investors and assets is likely to be conservative.

1.5.3 Fire-sale multiplier

Earlier I introduced the notion of a fire-sale multiplier, which I define as the ratio between the total change in price to the initial shock. The fire-sale multiplier m_j is given by

$$m_j = \frac{1 - p_j^*}{1 - s_j}$$

The numerical validation we performed earlier provides a useful way to quantify the fire-sale multiplier under the assumption that all combinations of (valid) model parameters are equally likely. For two investors and two assets, the mean value of the fire-sale multiplier is 1.522. This means that fire-sale effects extend the drop due to the exogenous shock by an additional 52% on average. In the 5×5 setup the fire-sale multiplier is 1.515 and in the 10×10 setup it is 1.512, similar to that of the 2×2 setup. This suggests that the average level of the fire-sale multiplier is not particularly sensitive to the number of investors or assets.

Using the first order approximation, we can derive an approximate fire-sale multiplier m'_j defined as

$$m'_j = \frac{1 - p'_j}{1 - s_j}$$

Substituting in p'_j from equation (1.5.1) and simplifying, we obtain

$$m'_j = 1 + \frac{s_j}{1 - s_j} \times \frac{1}{\delta_j} WA_j \left[C_i D_i \right]$$

Note that the second term is always positive so $m'_j \geq 1$. In summary, the fire-sale multiplier for an asset is highest when

- the liquidity (δ_j) is low (a small volume of forced sales induce a relatively large drop in price)
- the expression $\frac{s_j}{1-s_j}$ is large. This will be the case for s_j close to 1, which implies a *small* initial shock to the asset. This may seem counter-intuitive, but recall that the fire-sale multiplier relates the total change in price to the initial shock. If the initial shock is small – relative to the price change driven by fire-sale effects involving this asset and, crucially, all other assets – then the ratio of the total price change to the initial shock will be large.
- The weighted average of (first order) portfolio sale fractions is large. If investors, weighted by their holdings of asset j, are likely to have to sell most of their portfolio then the fire-sale multiplier will also be large. Because the first order portfolio sale fractions are driven by shocks, this means the multiplier will be large when the shocks to *unrelated* assets are large. (This is in contrast to the impact of a same-asset shock – see the previous point.)

1.6 Discussion

A benefit of having an analytic approximation to the equilibrium price is that we can use it to derive analytic statics from which economic content can be extracted. Of course, such statics – based as they are on an approximation – are themselves also approximate. (Derivations of the statics are collected in subsection 1.8.3 in the Appendix). Note that, to avoid confusion, I adopt the convention of using bold-face to indicate a *specific* investor or asset index. For instance, $s_{\mathbf{j}}$ refers to a specific asset $\mathbf{j} \in [1..M]$ while s_j, contained inside a sum over all assets, refers to the shocked price of *each asset in turn*[19].

1.6.1 Debt imposes externalities

Does debt matter? Yes, it does, at least within our model. Consider the first derivative of the approximate price of some asset with regards to the debt of a given investor

$$\frac{\partial p'_{\mathbf{j}}}{\partial d_{\mathbf{i}}} = -\frac{s_{\mathbf{j}}}{\delta_{\mathbf{j}} G_{\mathbf{j}}} h_{\mathbf{i},\mathbf{j}} C_{\mathbf{i}} \frac{A_{\mathbf{i}}}{(A_{\mathbf{i}} - d_{\mathbf{i}})^2}$$

This derivative is weakly negative, and if the investor's debt and holdings of the asset are both strictly positive, the derivative is strictly negative. If an investor has some debt, then an increase in that debt will decrease the approximate equilibrium price of all assets held by this investor. The magnitude of the price change is particularly sensitive to the level of debt the investor already has. Consider the last term in of the expression, $A_{\mathbf{i}} / (A_{\mathbf{i}} - d_{\mathbf{i}})^{-2}$. The quantity $A_{\mathbf{i}} - d_{\mathbf{i}}$ represents the difference between initial assets and initial debt – that is, equity. As initial debt grows close to initial assets, equity tends towards zero and the expression $A_{\mathbf{i}} / (A_{\mathbf{i}} - d_{\mathbf{i}})^{-2}$ tends towards infinity. In economic terms, as the initial equity of *any* investor in an asset approaches zero, the sensitivity of the fire-sale price to that investor's initial debt becomes infinite. Thus an investor's debt imposes externalities on other investors. What this analysis suggests is that the sensitivity of externalities imposed by debt is particularly severe as the level of debt of investors in the asset becomes high relative to equity.

1.6.2 Fire-sale contagion

Consider the price impact of a shock to an unrelated asset. If $p'_{\mathbf{k}}$ is the approximate price of asset $\mathbf{k}$, what is the price impact of a shock to an unrelated asset $\mathbf{j} \neq \mathbf{k}$? The answer is

$$\frac{\partial p'_{\mathbf{k}}}{\partial s_{\mathbf{j}}} = \frac{1}{\delta_{\mathbf{k}}} s_{\mathbf{k}} W A_{\mathbf{k}} \left[\frac{h_{i,\mathbf{j}} D_i A_i}{S_i^2} \right]$$

---

[19] To make this clear, consider the following example

$$\frac{\partial}{\partial s_{\mathbf{j}}} \sum_{j=1}^{M} s_j = \sum_{j=1}^{M} \frac{\partial}{\partial s_{\mathbf{j}}} s_j = \frac{\partial}{\partial s_{\mathbf{j}}} s_{\mathbf{j}} = 1$$

I am able reduce the sum to a single term because each $s_{j \neq \mathbf{j}}$ is a constant for $s_{\mathbf{j}}$ and therefore becomes zero when taking the derivative, leaving only the term $\frac{\partial}{\partial s_{\mathbf{j}}} s_{j=\mathbf{j}} = \frac{\partial}{\partial s_{\mathbf{j}}} s_{\mathbf{j}} = 1$.

The derivative is always positive, so a lower-shocked price $s_{\mathbf{j}}$ (corresponding to a larger negative shock) leads to a lower approximate price $p'_{\mathbf{k}}$. This is pure contagion. A shock to one asset leads to a decrease in the price of unrelated assets. This contagion effect is regulated by several factors. The price impact on asset **k** is reduced if the shocked price of asset **k** itself is lower (a larger shock) or if its liquidity is higher (the effect of additional supply on the price is small). In addition, the price impact depends on the weighted average across all investors (weighted by initial holdings of asset **k**) of the quantity $\frac{h_{i,\mathbf{j}} D_i A_i}{S_i^2}$, which roughly corresponds to scaled leverage. In short, higher levels of average investor leverage results in larger contagion effects.

Putting this together, the contagion effect will be most pronounced when

- System-wide average leverage (D_i) is high, weighted by holdings in the unrelated asset ($h_{i,\mathbf{j}}$)
- System-wide average post-shock portfolio values (S_i) are low, in other words, other assets are also suffering from negative shocks (i.e., a systematic as opposed to idiosyncratic shock)
- the asset in question (asset **k**) has itself experienced a smaller shock (higher $s_{\mathbf{k}}$) or has a low liquidity (small $\delta_{\mathbf{k}}$)
- there is substantial overlap of asset holdings between asset **j** and asset **k** (so large weights in the weighted average $WA_{\mathbf{k}}[]$ coincide with large holdings of asset **j** ($h_{i,\mathbf{j}}$) inside the weighted average)

1.6.3 Liquidity and robust markets

The price impact of a change in same-asset liquidity is given by

$$\frac{\partial p'_{\mathbf{j}}}{\partial \delta_{\mathbf{j}}} = \frac{s_{\mathbf{j}}}{\delta_{\mathbf{j}}^2} WA_{\mathbf{j}}\left[C_i D_i\right]$$

This derivative is always positive, so an increase in liquidity leads to a higher approximate equilibrium price. The intuition behind this result is clear; higher liquidity is equivalent to more buyers willing to buy at a smaller price discount. Higher liquidity weakens the price response to the forced sales of investors seeking to re-establish their leverage ratios. From a policy perspective, this suggests that markets could be made more robust by reducing the barriers facing outside investors in assets in a fire sale, such as information asymmetries and regulatory barriers. It also provides support for the notion of a “buyer of last resort” that can enter markets and provide liquidity when prices become severely dislocated.

1.7 Conclusion

I formulate a model of fire sales consisting of multiple investors investing in multiple assets. Initially investors have arbitrary holdings of assets and heterogeneous leverage profiles. Bankruptcy is endogenous and occurs whenever an investor's debt is greater or equal to his assets. Exogenous individual price shocks interact with the requirement that each investor maintain their original leverage ratio to generate forced asset sales. Simultaneously, a price response mechanism links individual asset prices to the volume of forced sales of that asset. Within this framework I show that there exists set of equilibrium prices that simultaneously satisfies equilibrium requirements. (That is, (1) investors either re-establish their original leverage ratios or become bankrupt and liquidate their entire portfolio and (2) equilibrium asset prices satisfies the price response mechanism given the pattern of investor asset sales). I also show that such an equilibrium set of prices are unique, and that it can be calculated to arbitrary accuracy using the method of successive approximations. Because fire-sale equilibrium prices can proxy for *ex-ante* fire-sale risk, this enables direct testing of the hypothesis that fire-sale risk is priced. In numerical simulations an analytical approximation of equilibrium prices is shown to explain 98% of the variation in true equilibrium prices. I proceed to derive analytical statics for the (approximate) equilibrium prices. This allows the existence of fire-sale contagion effects to be quantified. These contagion effects are entirely due to overlapping patterns of asset holdings and heterogeneous leverage. Further, I show that a change in leverage by a single investor has spillover effects such that risk externalities are imposed on other investors.

1.8 Chapter 1 Appendix

1.8.1 Equilibrium

I show that equilibrium exists, that it is unique, and that it can be calculated to arbitrary accuracy using the method of successive approximations. This is also true for the case of a solvent equilibrium. My proof proceeds by showing that the iterated application of equilibrium conditions gives rise to a sequence of prices that converge to equilibrium prices.

Notation and assumptions

Notation I assume N investors indexed by $i \in [1..N]$ and M assets indexed by $j \in [1..M]$. The initial holdings of asset j by investor i (in units) is denoted by $h_{i,j}$. The price of asset j is denoted by p_j. Let $H_i \equiv \sum_{j=1}^{M} h_{i,j} p_j$ be the dollar value of investor i's portfolio at some set of prices *(before* taking into account any assets sales) and let $A_i \equiv \sum_{j=1}^{M} h_{i,j}$ be the initial portfolio value of investor when $p_j = 1$ for all assets. Initial debt for investor i is denoted by d_i. For asset j the period $t = 1$ shocked price is denoted by s_j and its liquidity is denoted by δ_j. The $t = 2$ period endogenous portfolio sale fraction for investor i is denoted by x_i.

Assumptions I make the following assumptions

Assumption 1. $h_{i,j} \geq 0$

Remark. Initial investor holdings of each asset are non-negative (no short positions).

Assumption 2. $\sum_{i=1}^{N} h_{i,j} = h_{1,j} + h_{2,j} + \cdots + h_{N,j} > 0$

Remark. The total stock of each asset (across the entire set of investors) is strictly positive for each asset. This is necessary for the price response to be well defined for each asset.

Assumption 3. $\delta_j > 1$

Remark. The supply of assets at $t = 2$ is strictly elastic. This rules out zero or negative prices for any level of portfolio sales.

Assumption 4. $0 < s_j \leq 1$ *with* $s_j < 1$ *for at least one* $j \in [1..M]$

Remark. The period $t = 1$ (post-shock) price is strictly positive and less than unity. At least one asset must experience a non-zero price shock, otherwise the model is trivially in equilibrium without requiring any portfolio sales.

Assumption 5. $0 \leq d_i < A_i$

Remark. No negative debt and all investors must be solvent initially.

Equilibrium requirements

Equilibrium portfolio sale fractions (solvent case) Equilibrium portfolio sales in period $t = 2$, such that each investor i re-establishes his initial leverage level, is denoted by x_i^s (solvent portfolio sale fraction for investor i) and is given by

$$x_i^s = f_i^s(p_1, p_2, ..., p_M) = \frac{d_i}{A_i - d_i}\left(\frac{A_i}{H_i} - 1\right) \tag{1.8.1}$$

This equation[20] only holds if every investor is solvent, $H_i = \sum_{j=1}^{M} h_{i,j}p_j > d_i$. Below I consider the more general formulation for the equilibrium sale fraction when the bankruptcy of one or more investors is a possibility.

Equilibrium portfolio sale fractions (general case) An investor is bankrupted if his assets valued at current prices is less than or equal to his debt, $H_i = \sum_{j=1}^{M} h_{i,j}p_j \leq d_i$. In the event of bankruptcy the entire portfolio of the investor is liquidated, so $x_i = 1$. When we specifically address the possibility of bankruptcy we denote the portfolio sale fraction formula by x_i so

$$x_i = f_i(p_1, p_2, ..., p_M) = \begin{cases} f_i^s(p_1, p_2, ..., p_M) & H_i > d_i \quad \text{(Solvent)} \\ 1 & H_i \leq d_i \quad \text{(Bankrupt)} \end{cases} \tag{1.8.2}$$

Note that the portfolio sale fraction equals one whenever $H_i = d_i$ and less than one if $H_i > d_i$[21]. This means that f_i remains a continuous function because the limit of f_i as $H_i \to d_i$ is one and f_i is exactly one for any $H_i \geq d_i$. It also means we can simplify the representation of x_i to

$$x_i = \min(x_i^s, 1)$$

In other words the portfolio sale fraction in the general case is equal to the portfolio sale fraction in the solvent case, but bounded from above at 1.

Equilibrium price response At the same time we require that the price of asset j is given by

$$p_j = g_j(x_1, x_2, ..., x_N) = s_j - \frac{s_j}{\delta_j}\left(\frac{\sum_{i=1}^{N} h_{i,j}(x_i)}{\sum_{i=1}^{N} h_{i,j}}\right) \tag{1.8.3}$$

This can be written more compactly as

$$p_j = g_j(x_1, ..., x_N) = s_j - \frac{s_j}{\delta_j} WA_j[x_i] \tag{1.8.4}$$

---

[20] See subsection 1.3.4 for the derivation

[21] $f_i(p_1, p_2, ..., p_M) = \frac{d_i}{A_i - d_i} \times \frac{A_i - H_i}{H_i} = 1$ if $H_i = d_i$

where

$$WA_j[x_i] \equiv \frac{\sum_{i=1}^{N} h_{i,j} x_i}{\sum_{i=1}^{N} h_{i,j}}$$

is the weighted average of x_i across all investors using their initial holdings of asset j as weights[22].

Equilibrium

Definition 1. I define *equilibrium* as the $N + M$ tuple $(x_1^*, ..., x_N^*, p_1^*, ...p_M^*)$ that simultaneously satisfies the following set of $N + M$ equations.

$$\begin{aligned} x_i^* =& f_i(p_1^*, ...p_M^*) \quad \text{for all } i \in [1..N] \\ p_j^* =& g_j(x_1^*, ..., x_N^*) \quad \text{for all } j \in [1..M] \end{aligned}$$

Remark. This is a broad definition of equilibrium such that the bankruptcy of one or more investors is permitted in equilibrium. This means that it is not a requirement that each investor re-establish their leverage ratio; instead, each investor re-establishes his leverage ratio *if he is able to*, otherwise the entire portfolio of the investor is liquidated. This definition of equilibrium is consistent with the notion of the model as a game between investors, and, therefore, can be interpreted as a non-symmetric static Nash equilibrium in continuous strategies. Strategic interdependence follows from the fact that each investor's choice of portfolio sales impacts the price of one or more assets and therefore impacts the portfolio sale requirements of other investors, and vice versa. The game is non-symmetric because the investors differ in their asset holdings and debt constraints and are therefore *ex-ante* distinct.

Definition 2. K is the closed unit hyper-cube in M-dimensional Euclidean space given by $K \equiv \{(p_1, ..., p_M) \in \mathbb{R}^M \,|\, 0 \leq p_j \leq 1 \text{ for all } j \in [1..M]\}$

Remark. K represents the "price space" in our model.

Definition 3. The map $h{:}K \to \mathbb{R}^M$ is given by the composition of the optimal portfolio sale fractions and price responses consistent with the definition of equilibrium in Definition 1.

$$p' \equiv h(p) = (g_1(x_1, ..., x_N), ..., g_1(x_1, ..., x_N))$$

where

$$x_i = \min[1, f_i(p_1, ..., p_M)]$$

---

[22] See subsection 1.3.5 for a detailed discussion and derivation of the price response.

Remark. The function h transforms a given set of prices into a new set of prices in a two stage process. First, optimal portfolio sale fractions are calculated based on the initial prices. Then those portfolio sale fractions are used to calculate the new prices using the price response formula.

Lemma 1. *Let* $(p'_1, ..., p'_M) = h((p_1, ..., p_M))$ *with* $p \in K$. *Then* $0 < p'_j \leq 1$ *for all* $j \in [1..M]$ *and* $0 < p'_j < 1$ *for at least one* $j \in [1..M]$

Remark. This lemma restricts all prices to the range [0..1] and ensures that at least one of the prices lies within [0..1). The general restriction is needed to show that prices are bounded in the sense that they remain within the unit hyper-cube. The requirement that at least one of the prices is strictly less than unity is needed to ensure that the model converges to an interior equilibrium.

Proof. Case (a): $p'_j > 0$

Consider $p'_j = g_j(x_1, ..., x_N) = \frac{s_j}{\delta_j} \times \frac{\sum_{i=1}^{N}(\delta_j - x_i)h_{i,j}}{\sum_{i=1}^{N} h_{i,j}}$.[23] Let $w_j = \frac{\sum_{i=1}^{N}(\delta_j - x_i)h_{i,j}}{\sum_{i=1}^{N} h_{i,j}}$, so $p'_j = \frac{s_j}{\delta_j} w_j$. Note that $\delta_j - x_i > 0$ for given j and any i follows from $x_i \leq 1$ (because $x_i = \min[1, x_i^s]$) and $\delta_j > 1$ (Assumption 3). In conjunction with $\sum_{i=1}^{N} h_{i,j} > 0$ (Assumption 2) and $h_{i,j} \geq 0$ (Assumption 1) this implies that the numerator of w_j is strictly positive. Also the denominator of w_j is strictly positive (again Assumption 2). Therefore $w_j > 0$. Since $s_j > 0$ (Assumption 4) and $\delta_j > 1$ (Assumption 3), we have $p'_j = \frac{s_j}{\delta_j} w_j > 0$ as required.

Case (b): $p'_j \leq 1$ for all j and $p'_j < 1$ for some j

Consider $p'_j = g_j(x_1, ..., x_N) = s_j \left(1 - \frac{1}{\delta_j} WA_j[x_i]\right)$.[24] First, $x_i = \left(\frac{d_i}{A_i - d_i}\right)\left(\frac{A_i - H_i}{H_i}\right)$ with $H_i = \sum_{j=1}^{M} h_{i,j} p_j$. Now $\frac{d_i}{A_i - d_i} \geq 0$ since $0 \leq d_i < A_i$ (Assumption 5). Because $p \in K$, this means $0 \leq p_j \leq 1$ for all $j \in [1..M]$. Hence $H_i \geq 0$ since $p_j \geq 0$ and $h_{i,j} \geq 0$ (Assumption 1) . And since $p_j \leq 1$, it follows that $H_i \leq A_i$. Therefore $\frac{A_i - H_i}{H_i} \geq 0$ and, as shown earlier, $\frac{d_i}{A_i - d_i} \geq 0$ so $x_i \geq 0$. Now consider the term $\left(1 - \frac{1}{\delta_j} WA_j[x_i]\right)$. Using $x_i \geq 0$, we have $WA_j[x_i] \geq 0$ since the weights are non-negative ($h_{i,j} \geq 0$ by Assumption 1). Therefore $\frac{1}{\delta_j} WA_j[x_i] \geq 0$ since $\delta_j > 1$ (Assumption 3). It follows that $\left(1 - \frac{1}{\delta_j} WA_j[x_i]\right) \leq 1$. Now $s_j \leq 1$ (Assumption 4) so $p'_j = s_j \left(1 - \frac{1}{\delta_j} WA_j[x_i]\right) \leq 1$ for all $j \in [1..M]$ as required. In addition, there exists at least one asset j such that $s_j < 1$ (Assumption 4) so $p'_j = s_j \left(1 - \frac{1}{\delta_j} WA_j[x_i]\right) < 1$ for at least one $j \in [1..M]$ as required. $\square$

Lemma 2. *h is a self-map from K to K, that is $h : K \rightarrow K$*

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[23] With a modest amount of rearranging we can show that this is the same as equation (1.8.3), $\frac{s_j}{\delta_j} \times \frac{\sum_{i=1}^{N}(\delta_j - x_i)h_{i,j}}{\sum_{i=1}^{N} h_{i,j}} = s_j - \frac{s_j}{\delta_j}\left(\frac{\sum_{i=1}^{N} h_{i,j}(x_i)}{\sum_{i=1}^{N} h_{i,j}}\right)$.

[24] Derived from equation (1.8.4) by factoring out s_j

Proof. h maps from K by construction. By Lemma 1 we have $0 < p'_j \leq 1$ for all $j \in [1..M]$ whenever $p' = h(p)$ with $p \in K$. Therefore $h(p) \in K$ for all $p \in K$ and thus h maps to K. $\square$

Definition 4. The M-dimensional price sequence $\{p^{(n)}\}$ for $n \in \mathbb{N}$, $n \geq 0$ is defined as follows: for $n = 0$ all prices are equal to unity, $p^{(0)} = (1, ..., 1)$. For $n = 1$, all prices are equal to the shocked prices, $p^{(1)} = (s_1, ..., s_M)$. For $n \geq 2$, prices are given by the recurrence relation $p^{(n+1)} = h(p^{(n)})$. Because $p^{(n)} = (p_1^{(n)}, ..., p_M^{(n)})$ this gives rise to a sequence of prices $\{p_j^{(n)}\}$ for each asset $j \in [1..M]$.

Remark. $\{p^{(n)}\}$ is the sequence of price tuples generated by the repeated application of the map $h : K \rightarrow K$ for $n \geq 2$. The number n is a count of the repeated iterations of the map h and should not be confused with the different periods in the model.

Definition 5. The M-dimensional price change sequence $\{\Delta p^{(n)}\}$ for $n \geq 1$ is defined by $\Delta p^{(n)} \equiv p^{(n)} - p^{(n-1)}$. Similarly, for a specific asset j the price change sequence $\{\Delta p_j^{(n)}\}$ is defined by $\Delta p_j^{(n)} \equiv p_j^{(n)} - p_j^{(n-1)}$.

Remark. $\{\Delta p^{(n)}\}$ is the differenced price sequence starting at $n = 1$ while the price sequence $\{p_j^{(n)}\}$ starts at $n = 0$.

Definition 6. The N-dimensional portfolio sale fraction sequence $\{x^{(n)}\}$ for $n \in \mathbb{N}$, $n \geq 1$ is defined by $x_i^{(n)} = f_i(p_1^{(n-1)}, ..., p_M^{(n-1)})$. Because $x^{(n)} = (x_1^{(n)}, ..., x_N^{(n)})$ this gives rise to a sequence of portfolio sale fractions $\{x_i^{(n)}\}$ for each investor $i \in [1..N]$.

Remark. Note that each sequence element $x^{(n)}$ is calculated by reference to the prices $p^{(n-1)}$ holding during the previous iteration.

Definition 7. The N-dimensional portfolio sale fraction change sequence $\{\Delta x^{(n)}\}$ for $n \geq 2$ is defined by $\Delta x^{(n)} \equiv x^{(n)} - x^{(n-1)}$. Similarly, for investor i the portfolio sale fraction change sequence $\{\Delta x_i^{(n)}\}$ is defined by $\Delta x_i^{(n)} \equiv x_i^{(n)} - x_i^{(n-1)}$.

Lemma 3. *The sequence $\{p^{(n)}\}$ is bounded by the boundaries of K.*

Proof. By induction on n. For $n = 1$ we have $p_j^{(1)} = s_j$ so $0 < p_j^{(1)} \leq 1$ for all $j \in [1..M]$ because $0 < s_j \leq 1$ (Assumption 4). Thus $p^{(1)} \in K$. Let $n = k$. If $p^{(k)} \in K$ then $p^{(k+1)} \in K$ because $p^{(k+1)} = h(p^{(k)})$ (Definition 4) and h maps from K to K (Lemma 2). Therefore, by induction on n we have $p^{(n)} \in K$ for all $n \geq 1$. For $n = 0$, $p^{(0)} = (1, ..., 1) \in K$ also. Thus $\{p^{(n)}\}$ is bounded by K. $\square$

Lemma 4. *For $n \geq 1$ if $\Delta p_j^{(n)} \leq 0$ for all $j \in [1..M]$ then $\Delta x_i^{(n+1)} \geq 0$ for all $i \in [1..N]$.*

Proof. For $n \geq 0$, the next iteration portfolio sale fraction $x_i^{(n+1)}$ is linked to the previous iteration prices $p_j^{(n)}$ by the following expression for the optimal portfolio sale fraction

$$x_i^{(n+1)} = \frac{d_i}{A_i - d_i}\left(\frac{A_i}{H_i(p^{(n)})} - 1\right) = \frac{d_i}{A_i - d_i}\left(\frac{A_i}{\sum_{j=1}^{M} h_{i,j}p_j^{(n)}} - 1\right)$$

and, therefore, for $n \geq 1$ and for all $i \in [1..N]$ we have

$$\begin{aligned}
\Delta x_i^{(n+1)} &= x_i^{(n+1)} - x_i^{(n)} \\
&= \frac{d_i}{A_i - d_i}\left(\left(\frac{A_i}{\sum_{j=1}^{M} h_{i,j}p_j^{(n)}} - 1\right) - \left(\frac{A_i}{\sum_{j=1}^{M} h_{i,j}p_j^{(n-1)}} - 1\right)\right) \\
&= \frac{A_i d_i}{A_i - d_i}\left(\frac{1}{\sum_{j=1}^{M} h_{i,j}p_j^{(n)}} - \frac{1}{\sum_{j=1}^{M} h_{i,j}p_j^{(n-1)}}\right) \\
&= \frac{A_i d_i}{A_i - d_i}\left(\left(\sum_{j=1}^{M} h_{i,j}p_j^{(n)}\right)^{-1} - \left(\sum_{j=1}^{M} h_{i,j}p_j^{(n-1)}\right)^{-1}\right)
\end{aligned}$$

Now, because $\Delta p_j^{(n)} \leq 0$ we can write $p_j^{(n)} - p_j^{(n-1)} \leq 0$ or $p_j^{(n)} \leq p_j^{(n-1)}$ for all $j \in [1..M]$. So $\sum_{j=1}^{M} h_{i,j}p_j^{(n)} \leq \sum_{j=1}^{M} h_{i,j}p_j^{(n-1)}$. That means $\left(\sum_{j=1}^{M} h_{i,j}p_j^{(n)}\right)^{-1} \geq \left(\sum_{j=1}^{M} h_{i,j}p_j^{(n-1)}\right)^{-1}$ and therefore $\left(\sum_{j=1}^{M} h_{i,j}p_j^{(n)}\right)^{-1} - \left(\sum_{j=1}^{M} h_{i,j}p_j^{(n-1)}\right)^{-1} \geq 0$. And because $0 \leq d_i \leq A_i$ (Assumption 5) it follows that $\frac{A_i d_i}{A_i - d_i} \geq 0$ as well, so $\Delta x_i^{(n+1)} \geq 0$ for all $i \in [1..N]$ as required. ☐

Lemma 5. *For $n \geq 2$ if $\Delta x_i^{(n)} \geq 0$ for all $i \in [1..N]$ then $\Delta p_j^{(n)} \leq 0$ for all $j \in [1..M]$.*

Proof. For $n \geq 1$ prices for a given iteration are determined according to the price response requirement based on the portfolio sale fractions of that iteration

$$p_j^{(n)} = g_j(x_1^{(n)}, ..., x_N^{(n)}) = s_j - \frac{s_j}{\delta_j} WA_j[x_i^{(n)}]$$

and so for $n \geq 2$ and for all $j \in [1..M]$

$$\begin{aligned}
\Delta p_j^{(n)} &= p_j^{(n)} - p_j^{(n-1)} \\
&= \left(s_j - \frac{s_j}{\delta_j} WA_j[x_i^{(n)}]\right) - \left(s_j - \frac{s_j}{\delta_j} WA_j[x_i^{(n-1)}]\right) \\
&= -\frac{s_j}{\delta_j}\left(WA_j[x_i^{(n)}] - WA_j[x_i^{(n-1)}]\right) \\
&= -\frac{s_j}{\delta_j} WA_j[x_i^{(n)} - x_i^{(n-1)}] \\
&= -\frac{s_j}{\delta_j} WA_j[\Delta x_i^{(n)}]
\end{aligned}$$

Now, because $\Delta x_i^{(n)} \geq 0$ for all $i \in [1, .., N]$, we have $WA_j[\Delta x_i^{(n)}] \geq 0$ for any given $j \in [1..M]$. (Recall the weights used in $WA_j[]$ are non-negative – $h_{i,j} \geq 0$ by Assumption 1.) Since $\frac{s_j}{\delta_j} > 0$ (from Assumption 3 and 4), it follows directly from the above that $\Delta p_j^{(n)} \leq 0$ for any given $j \in [1..M]$, as required. $\square$

Lemma 6. *For $n \geq 2$ if $\Delta p_j^{(n)} \leq 0$ for all $j \in [1..M]$ then $\Delta p_j^{(n+1)} \leq 0$ for all $j \in [1..M]$.*

Proof. By Lemma 4 $\Delta p_j^{(n)} \leq 0$ for all $j \in [1..M]$ implies $\Delta x_j^{(n+1)} \geq 0$ for all $i \in [1..N]$. And from Lemma 5 $\Delta x_j^{(n+1)} \geq 0$ for all $i \in [1..N]$ implies $\Delta p_j^{(n+1)} \leq 0$ for all $j \in [1..M]$ as required. $\square$

Lemma 7. *The sequence $\{p_j^{(n)}\}$ is monotonically decreasing for all $j \in [1..M]$.*

Proof. By induction on n. For $n = 1$ we have $\Delta p_j^{(1)} = p_j^{(1)} - p^{(0)} = s_j - 1$ for all $j \in [1..M]$. Since by Assumption 4 we have $0 < s_j \leq 1$ for all $j \in [1..M]$, this means that $\Delta p_j^{(1)} \leq 0$ for all $j \in [1..M]$. Now let $n = k \geq 2$ and assume $\Delta p_j^{(k)} \leq 0$ for all $j \in [1..M]$, then (by Lemma 6) we have $\Delta p_j^{(k+1)} \leq 0$ for all $j \in [1..M]$. Hence by induction we have $\Delta p_j^{(n)} \leq 0$ for all $j \in [1..M]$ and for all $n \geq 1$. Therefore $\{p_j^{(n)}\}$ is monotonically decreasing in n for all $j \in [1..M]$. $\square$

Lemma 8. *The sequence $\{p^{(n)}\}$ converges to $p^\star \in K$*

Proof. By Lemma 7 the sequence $\{p_j^{(n)}\}$ is monotonically decreasing for each $j \in [1..M]$ and by Lemma 3 the sequence $\{p^{(n)}\}$ is bounded by K, which, from the definition of K, implies that $\{p_j^{(n)}\}$ is bounded by $0 \leq p^{(n)} \leq 1$ for each $j \in [1..M]$. By the monotone convergence theorem any monotonic and bounded sequence is also convergent. Therefore, for all $j \in [1..M]$ the sequence $\{p_j^{(n)}\}$ converges to some $p_j^\star \in [0, 1]$. Construct $p^\star = (p_N^\star, ..., p_M^\star)$. Because each sequence $\{p_j^{(n)}\}$ is defined by reference to the single sequence $\{p^{(n)}\}$, if follows that $\{p^{(n)}\}$ converges to $p^\star$. And because $\{p^{(n)}\}$ is bounded by K (Lemma 3) we have $p^\star \in K$. $\square$

Theorem 1. *Equilibrium exists, is unique, and can be calculated by the method of successive approximations.*

Proof. By Lemma 8 the sequence $\{p^{(n)}\}$ converges to $p^\star \in K$. The map h generating $\{p^{(n)}\}$ is formed from the composition of all the equilibrium requirements – h enforces both the equilibrium portfolio sale fraction requirement and the price response requirement for each investor and each asset – and, therefore, the tuple $(f_1(p_1^\star, ..., p_M^\star), ..., f_N(p_1^\star, ..., p_M^\star), p_1^\star, ...p_N^\star)$ constructed from $p^\star$ is an equilibrium in accordance with Definition 1. The convergence point $p^\star$ is approached in the limit by the successive application of the map h, and, therefore, $p^\star$ can be calculated to arbitrary accuracy by the method of successive approximations. Finally, because the limit of a convergent sequence is unique, the equilibrium is also unique. $\square$

1.8.2 Approximation

Substituting $s_j = p_j$ in equation (1.3.7) I obtain

$$x_i = \left(\frac{A_i}{S_i} - 1\right) \times D_i \tag{1.8.5}$$

where A_i

$$A_i \equiv \sum_{j=1}^{M} h_{i,j}$$

is the initial, pre-shock portfolio value of investor i, D_i

$$D_i \equiv \frac{d_i}{A_i - d_i}$$

is investor i's initial debt-equity ratio and S_i

$$S_i \equiv \sum_{j=1}^{M} h_{i,j} s_j$$

is his initial portfolio holdings (before any sales) valued at the period one shocked prices s_j. I can simplify this more by defining C_i

$$C_i \equiv \frac{A_i}{S_i} - 1 = \frac{A_i - S_i}{S_i}$$

so I can write

$$x_i = C_i D_i \tag{1.8.6}$$

that is, the optimal portfolio sale fraction x_i is the product of the investor's percentage portfolio loss due to the period one shock and his initial debt-equity ratio. Having fixed our equilibrium portfolio sale fractions, I consider the (first order) successive price approximation given by $p' = (p'_1, ..., p'_M)$ where

$$p'_j = g(x_1, ..., x_N) = s_j - \frac{s_j}{\delta_j} \left(\frac{\sum_{i=1}^{N} h_{i,j} x_i}{\sum_{i=1}^{N} h_{i,j}} \right) \tag{1.8.7}$$

What does this mean? Consider the expression inside the brackets. This is really just the weighted average of x_i, where the weights $(h_{i,j})$ are the holdings of asset j across the different investors. So denote by $WA_j\left[z_i\right]$ the weighted average of variable z_j across all investors using each investor's holdings of asset j as weights

$$WA_j\left[z_i\right] \equiv \frac{\sum_{i=1}^{N} h_{i,j} z_i}{\sum_{i=1}^{N} h_{i,j}} = \frac{\sum_{i=1}^{N} h_{i,j} z_i}{G_j}$$

where G_j is the total holdings (in units) of asset j by all investors, that is

$$G_j \equiv \sum_{i=1}^{N} h_{i,j}$$

Then I can rewrite the first order approximation as

$$p'_j = s_j - \frac{s_j}{\delta_j} WA_j\left[x_i\right] = s_j - \frac{s_j}{\delta_j} WA_j\left[C_i D_i\right] \tag{1.8.8}$$

The first order change in price is then

$$\begin{aligned}\Delta p'_j &= p'_j - s_j \\ &= -\frac{s_j}{\delta_j} WA_j\left[C_i D_i\right]\end{aligned}$$

I can substitute the expression of $C_i D_i$ in the above to obtain an expression for the successive price approximation p'_j expressed purely in model parameters.

$$\begin{aligned}p'_j &= s_j - \frac{s_j}{\delta_j}\left(\frac{\sum_{i=1}^{N} h_{i,j} C_i D_i}{\sum_{i=1}^{N} h_{i,j}}\right) \\ &= s_j - \frac{s_j}{\delta_j}\left(\frac{\sum_{i=1}^{N} h_{i,j}\left(\left(\frac{A_i}{\sum_{j=1}^{M} h_{i,j} s_j} - 1\right) \times \frac{d_i}{A_i - d_i}\right)}{\sum_{i=1}^{N} h_{i,j}}\right)\end{aligned}$$

1.8.3 Approximate Analytical Statics

Table 1.7 below provides a summary of the key statics derived from our price approximation. I adopt the convention of using bold-face to indicate a *specific* investor or asset index. For instance, $s_{\mathbf{j}}$ refers to a specific asset $\mathbf{j} \in [1..M]$ while s_j, contained inside a sum over all assets, refers to the shocked price of each asset in turn.

Table 1.7: Summary of approximate statics

$$\begin{aligned}
\frac{\partial p'_{\mathbf{k}}}{\partial h_{\mathbf{i},\mathbf{j}}} &= -\frac{s_{\mathbf{k}} h_{\mathbf{i},\mathbf{k}} D_{\mathbf{i}}}{\delta_{\mathbf{k}} G_{\mathbf{k}}} \left(\frac{S_{\mathbf{i}} - s_{\mathbf{j}} A_{\mathbf{i}}}{S_{\mathbf{i}}^2} - \frac{C_{\mathbf{i}}}{A_{\mathbf{i}} - r_{\mathbf{i}}} \right) \\
\frac{\partial p'_{\mathbf{j}}}{\partial h_{\mathbf{i},\mathbf{j}}} &= -\frac{s_{\mathbf{j}}}{\delta_{\mathbf{j}} G_{\mathbf{j}}} \left(h_{\mathbf{i},\mathbf{j}} D_{\mathbf{i}} \left(\frac{S_{\mathbf{i}} - s_{\mathbf{j}} A_{\mathbf{i}}}{S_{\mathbf{i}}^2} - \frac{C_{\mathbf{i}}}{A_{\mathbf{i}} - r_{\mathbf{i}}} \right) - W A_{\mathbf{j}} \left[C_i D_i \right] \right) \\
\frac{\partial p'_{\mathbf{j}}}{\partial r_{\mathbf{i}}} &= -\frac{s_{\mathbf{j}}}{\delta_{\mathbf{j}} G_{\mathbf{j}}} h_{\mathbf{i},\mathbf{j}} C_{\mathbf{i}} \frac{A_{\mathbf{i}}}{\left(A_{\mathbf{i}} - r_{\mathbf{i}}\right)^2} \\
\frac{\partial p'_{\mathbf{j}}}{\partial s_{\mathbf{j}}} &= 1 + \frac{1}{\delta_{\mathbf{j}}} s_{\mathbf{j}} W A_{\mathbf{j}} \left[\frac{h_{i,\mathbf{j}} D_i A_i}{S_i^2} \right] - \frac{1}{\delta_{\mathbf{j}}} W A_{\mathbf{j}} \left[C_i D_i \right] \\
\frac{\partial p'_{\mathbf{k}}}{\partial s_{\mathbf{j}}} &= \frac{1}{\delta_{\mathbf{k}}} s_{\mathbf{k}} W A_{\mathbf{k}} \left[\frac{h_{i,\mathbf{j}} D_i A_i}{S_i^2} \right] \\
\frac{\partial p'_{\mathbf{j}}}{\partial \delta_{\mathbf{j}}} &= \frac{s_{\mathbf{j}}}{\delta_{\mathbf{j}}^2} W A_{\mathbf{j}} \left[C_i D_i \right]
\end{aligned}$$

Investor debt

Select an asset $\mathbf{j} \in [1..M]$. The sensitivity of the approximate price of this asset, $p'_{\mathbf{j}}$, to a change in debt $d_{\mathbf{i}}$ of a specific investor $\mathbf{i} \in [1..N]$ is given by

$$\begin{aligned}
\frac{\partial p'_{\mathbf{j}}}{\partial d_{\mathbf{i}}} &= \frac{\partial}{\partial d_{\mathbf{i}}}\left(s_{\mathbf{j}} - \frac{s_{\mathbf{j}}}{\delta_{\mathbf{j}}} W A_{\mathbf{j}}\left[C_i D_i\right]\right) \\
&= -\frac{s_{\mathbf{j}}}{\delta_{\mathbf{j}}} \frac{\partial}{\partial d_{\mathbf{i}}} W A_{\mathbf{j}}\left[C_i D_i\right] \\
&= -\frac{s_{\mathbf{j}}}{\delta_{\mathbf{j}}} \frac{\partial}{\partial d_{\mathbf{i}}} \frac{\sum_{i=1}^{N} h_{i,\mathbf{j}} C_i D_i}{\sum_{i=1}^{N} h_{i,\mathbf{j}}} \\
&= -\frac{s_{\mathbf{j}}}{\delta_{\mathbf{j}} G_{\mathbf{j}}} \frac{\partial}{\partial d_{\mathbf{i}}} \sum_{i=1}^{N} h_{i,\mathbf{j}} C_i D_i \\
&= -\frac{s_{\mathbf{j}}}{\delta_{\mathbf{j}} G_{\mathbf{j}}} \sum_{i=1}^{N} h_{i,\mathbf{j}} C_i \frac{\partial}{\partial d_{\mathbf{i}}} D_i \quad \text{(differentiation is linear)} \\
&= -\frac{s_{\mathbf{j}}}{\delta_{\mathbf{j}} G_{\mathbf{j}}} h_{\mathbf{i},\mathbf{j}} C_{\mathbf{i}} \frac{\partial}{\partial d_{\mathbf{i}}} D_{\mathbf{i}} \quad \text{(only term } i = \mathbf{i} \text{ is non-zero)} \\
&= -\frac{s_{\mathbf{j}}}{\delta_{\mathbf{j}} G_{\mathbf{j}}} h_{\mathbf{i},\mathbf{j}} C_{\mathbf{i}} \frac{\partial}{\partial d_{\mathbf{i}}} \frac{d_{\mathbf{i}}}{A_{\mathbf{i}} - d_{\mathbf{i}}} \\
&= -\frac{s_{\mathbf{j}}}{\delta_{\mathbf{j}} G_{\mathbf{j}}} h_{\mathbf{i},\mathbf{j}} C_{\mathbf{i}} \frac{A_{\mathbf{i}}}{\left(A_{\mathbf{i}} - r_{\mathbf{i}}\right)^2}
\end{aligned}$$

Shocked price

Price effect of a shock to the same asset Select an asset $\mathbf{j} \in [1..M]$. The sensitivity of the approximate price of this asset, $p'_{\mathbf{j}}$, to a change in the shocked price $s_{\mathbf{j}}$ of the same asset is given by

$$
\begin{aligned}
\frac{\partial p'_{\mathbf{j}}}{\partial s_{\mathbf{j}}} &= \frac{\partial}{\partial s_{\mathbf{j}}}\left(s_{\mathbf{j}} - \frac{s_{\mathbf{j}}}{\delta_{\mathbf{j}}} W A_{\mathbf{j}}\left[C_i D_i\right]\right) \\
&= 1 - \frac{1}{\delta_{\mathbf{j}}}\left(\frac{\partial}{\partial s_{\mathbf{j}}} s_{\mathbf{j}} W A_{\mathbf{j}}\left[C_i D_i\right]\right) \\
&= 1 - \frac{1}{\delta_{\mathbf{j}}}\left(W A_{\mathbf{j}}\left[C_i D_i\right]\frac{\partial}{\partial s_{\mathbf{j}}} s_{\mathbf{j}} + s_{\mathbf{j}}\frac{\partial}{\partial s_{\mathbf{j}}} W A_{\mathbf{j}}\left[C_i D_i\right]\right) \quad \text{(product rule)} \\
&= 1 - \frac{1}{\delta_{\mathbf{j}}}\left(W A_{\mathbf{j}}\left[C_i D_i\right] + s_{\mathbf{j}}\frac{\partial}{\partial s_{\mathbf{j}}}\left(\frac{\sum_{i=1}^{N} h_{i,\mathbf{j}} C_i D_i}{\sum_{i=1}^{N} h_{i,\mathbf{j}}}\right)\right) \\
&= 1 - \frac{1}{\delta_{\mathbf{j}}}\left(W A_{\mathbf{j}}\left[C_i D_i\right] + \frac{s_{\mathbf{j}}}{G_{\mathbf{j}}}\frac{\partial}{\partial s_{\mathbf{j}}}\left(\sum_{i=1}^{N} h_{i,\mathbf{j}} C_i D_i\right)\right) \\
&= 1 - \frac{1}{\delta_{\mathbf{j}}}\left(W A_{\mathbf{j}}\left[C_i D_i\right] + \frac{s_{\mathbf{j}}}{G_{\mathbf{j}}}\left(\sum_{i=1}^{N} h_{i,\mathbf{j}} D_i \frac{\partial}{\partial s_{\mathbf{j}}} C_i\right)\right) \quad \text{(differentiation is linear)} \\
&= 1 - \frac{1}{\delta_{\mathbf{j}}}\left(W A_{\mathbf{j}}\left[C_i D_i\right] + \frac{s_{\mathbf{j}}}{G_{\mathbf{j}}}\left(\sum_{i=1}^{N} h_{i,\mathbf{j}} D_i \frac{\partial}{\partial s_{\mathbf{j}}}\left(\frac{A_i}{S_i} - 1\right)\right)\right) \\
&= 1 - \frac{1}{\delta_{\mathbf{j}}}\left(W A_{\mathbf{j}}\left[C_i D_i\right] + \frac{s_{\mathbf{j}}}{G_{\mathbf{j}}}\left(\sum_{i=1}^{N} h_{i,\mathbf{j}} D_i A_i \frac{\partial}{\partial S_i}\frac{1}{S_i}\frac{\partial}{\partial s_{\mathbf{j}}} S_i\right)\right) \quad \text{(chain rule)} \\
&= 1 - \frac{1}{\delta_{\mathbf{j}}}\left(W A_{\mathbf{j}}\left[C_i D_i\right] - \frac{s_{\mathbf{j}}}{G_{\mathbf{j}}}\left(\sum_{i=1}^{N} h_{i,\mathbf{j}} D_i A_i \frac{1}{S_i^2}\frac{\partial}{\partial s_{\mathbf{j}}}\sum_{j=1}^{M} h_{i,j} s_j\right)\right) \\
&= 1 - \frac{1}{\delta_{\mathbf{j}}}\left(W A_{\mathbf{j}}\left[C_i D_i\right] - \frac{s_{\mathbf{j}}}{G_{\mathbf{j}}}\left(\sum_{i=1}^{N} h_{i,\mathbf{j}} D_i A_i \frac{1}{S_i^2}\frac{\partial}{\partial s_{\mathbf{j}}} h_{i,\mathbf{j}} s_{\mathbf{j}}\right)\right) \quad \text{(only term } j = \mathbf{j} \text{ is non-zero)} \\
&= 1 - \frac{1}{\delta_{\mathbf{j}}}\left(W A_{\mathbf{j}}\left[C_i D_i\right] - \frac{s_{\mathbf{j}}}{G_{\mathbf{j}}}\left(\sum_{i=1}^{N} h_{i,\mathbf{j}} D_i A_i \frac{1}{S_i^2} h_{i,\mathbf{j}}\right)\right) \\
&= 1 - \frac{1}{\delta_{\mathbf{j}}}\left(W A_{\mathbf{j}}\left[C_i D_i\right] - s_{\mathbf{j}}\left(\frac{\sum_{i=1}^{N} h_{i,\mathbf{j}} D_i A_i \frac{1}{S_i^2} h_{i,\mathbf{j}}}{\sum_{i=1}^{N} h_{i,\mathbf{j}}}\right)\right) \quad \text{(expand } G_{\mathbf{j}}\text{)} \\
&= 1 - \frac{1}{\delta_{\mathbf{j}}}\left(W A_{\mathbf{j}}\left[C_i D_i\right] - s_{\mathbf{j}} W A_{\mathbf{j}}\left[\frac{h_{i,\mathbf{j}} D_i A_i}{S_i^2}\right]\right) \\
&= 1 + \frac{1}{\delta_{\mathbf{j}}} s_{\mathbf{j}} W A_{\mathbf{j}}\left[\frac{h_{i,\mathbf{j}} D_i A_i}{S_i^2}\right] - \frac{1}{\delta_{\mathbf{j}}} W A_{\mathbf{j}}\left[C_i D_i\right]
\end{aligned}
$$

Price effect of a shock to a different asset Select an asset $\mathbf{k} \in [1..M]$. The sensitivity of the approximate price of this asset, $p'_{\mathbf{k}}$, to a change in the shocked price $s_{\mathbf{j}}$ of a different asset $\mathbf{j} \neq \mathbf{k}$ is given by

$$
\begin{aligned}
\frac{\partial p'_{\mathbf{k}}}{\partial s_{\mathbf{j}}} &= \frac{\partial}{\partial s_{\mathbf{j}}}\left(s_{\mathbf{k}} - \frac{s_{\mathbf{k}}}{\delta_{\mathbf{k}}} WA_{\mathbf{k}}\left[C_i D_i\right]\right) = -\frac{1}{\delta_{\mathbf{k}}}\left(\frac{\partial}{\partial s_{\mathbf{j}}} s_{\mathbf{k}} WA_{\mathbf{k}}\left[C_i D_i\right]\right) \\
&= -\frac{1}{\delta_{\mathbf{k}}}\left(WA_{\mathbf{k}}\left[C_i D_i\right]\frac{\partial}{\partial s_{\mathbf{j}}} s_{\mathbf{k}} + s_{\mathbf{k}}\frac{\partial}{\partial s_{\mathbf{j}}} WA_{\mathbf{k}}\left[C_i D_i\right]\right) \qquad \text{(product rule)} \\
&= -\frac{1}{\delta_{\mathbf{k}}}\left((0) + s_{\mathbf{k}}\frac{\partial}{\partial s_{\mathbf{j}}}\left(\frac{\sum_{i=1}^{N} h_{i,\mathbf{k}} C_i D_i}{\sum_{i=1}^{N} h_{i,\mathbf{k}}}\right)\right) \\
&= -\frac{1}{\delta_{\mathbf{k}}}\left(\frac{s_{\mathbf{k}}}{G_{\mathbf{k}}}\frac{\partial}{\partial s_{\mathbf{j}}}\left(\sum_{i=1}^{N} h_{i,\mathbf{k}} C_i D_i\right)\right) \\
&= -\frac{1}{\delta_{\mathbf{k}}}\left(\frac{s_{\mathbf{k}}}{G_{\mathbf{k}}}\left(\sum_{i=1}^{N} h_{i,\mathbf{k}} D_i \frac{\partial}{\partial s_{\mathbf{j}}} C_i\right)\right) \qquad \text{(differentiation is linear)} \\
&= -\frac{1}{\delta_{\mathbf{k}}}\left(\frac{s_{\mathbf{k}}}{G_{\mathbf{k}}}\left(\sum_{i=1}^{N} h_{i,\mathbf{k}} D_i \frac{\partial}{\partial s_{\mathbf{j}}}\left(\frac{A_i}{S_i} - 1\right)\right)\right) \\
&= -\frac{1}{\delta_{\mathbf{k}}}\left(\frac{s_{\mathbf{k}}}{G_{\mathbf{k}}}\left(\sum_{i=1}^{N} h_{i,\mathbf{k}} D_i A_i \frac{\partial}{\partial S_i}\frac{1}{S_i}\frac{\partial}{\partial s_{\mathbf{j}}} S_i\right)\right) \qquad \text{(chain rule)} \\
&= \frac{1}{\delta_{\mathbf{k}}}\left(\frac{s_{\mathbf{k}}}{G_{\mathbf{k}}}\left(\sum_{i=1}^{N} h_{i,\mathbf{k}} D_i A_i \frac{1}{S_i^2}\frac{\partial}{\partial s_{\mathbf{j}}}\sum_{j=1}^{M} h_{i,j} s_j\right)\right) \\
&= \frac{1}{\delta_{\mathbf{k}}}\left(\frac{s_{\mathbf{k}}}{G_{\mathbf{k}}}\left(\sum_{i=1}^{N} h_{i,\mathbf{k}} D_i A_i \frac{1}{S_i^2}\frac{\partial}{\partial s_{\mathbf{j}}} h_{i,\mathbf{j}} s_{\mathbf{j}}\right)\right) \qquad \text{(only term } j = \mathbf{j} \text{ is non-zero)} \\
&= \frac{1}{\delta_{\mathbf{k}}}\left(\frac{s_{\mathbf{k}}}{G_{\mathbf{k}}}\left(\sum_{i=1}^{N} h_{i,\mathbf{k}} D_i A_i \frac{1}{S_i^2} h_{i,\mathbf{j}}\right)\right) \\
&= \frac{1}{\delta_{\mathbf{k}}}\left(s_{\mathbf{k}}\left(\frac{\sum_{i=1}^{N} h_{i,\mathbf{k}} D_i A_i \frac{1}{S_i^2} h_{i,\mathbf{j}}}{\sum_{i=1}^{N} h_{i,\mathbf{k}}}\right)\right) \qquad \text{(expand } G_{\mathbf{k}}\text{)} \\
&= \frac{1}{\delta_{\mathbf{k}}} s_{\mathbf{k}} WA_{\mathbf{k}}\left[\frac{h_{i,\mathbf{j}} D_i A_i}{S_i^2}\right]
\end{aligned}
$$

Liquidity

Price effect of the liquidity of the same asset Select an asset $\mathbf{j} \in [1..M]$. The sensitivity of the approximate price of this asset, $p'_{\mathbf{j}}$, to a change in the liquidity $\delta_{\mathbf{j}}$ is given by

$$
\begin{aligned}
\frac{\partial p'_{\mathbf{j}}}{\partial \delta_{\mathbf{j}}} &= \frac{\partial}{\partial \delta_{\mathbf{j}}} \left(s_{\mathbf{j}} - \frac{s_{\mathbf{j}}}{\delta_{\mathbf{j}}} W A_{\mathbf{j}} \left[C_i D_i \right] \right) \\
&= -s_{\mathbf{j}} \left(\frac{\partial}{\partial \delta_{\mathbf{j}}} \frac{1}{\delta_{\mathbf{j}}} W A_{\mathbf{j}} \left[C_i D_i \right] \right) \\
&= -s_{\mathbf{j}} \left(W A_{\mathbf{j}} \left[C_i D_i \right] \frac{\partial}{\partial \delta_{\mathbf{j}}} \frac{1}{\delta_{\mathbf{j}}} + \frac{1}{\delta_{\mathbf{j}}} \frac{\partial}{\partial \delta_{\mathbf{j}}} W A_{\mathbf{j}} \left[C_i D_i \right] \right) \quad \text{(product rule)} \\
&= s_{\mathbf{j}} \left(W A_{\mathbf{j}} \left[C_i D_i \right] \frac{1}{\delta_{\mathbf{j}}^2} + 0 \right) \\
&= \frac{s_{\mathbf{j}}}{\delta_{\mathbf{j}}^2} W A_{\mathbf{j}} \left[C_i D_i \right]
\end{aligned}
$$

Price effect of the liquidity of a different asset Select an asset $\mathbf{k} \in [1..M]$. The sensitivity of the approximate price of this asset, $p'_{\mathbf{k}}$, to a change in the liquidity $\delta_{\mathbf{j}}$ of a different asset $\mathbf{j} \neq \mathbf{k}$ is given by

$$
\begin{aligned}
\frac{\partial p'_{\mathbf{k}}}{\partial \delta_{\mathbf{j}}} &= \frac{\partial}{\partial \delta_{\mathbf{j}}} \left(s_{\mathbf{k}} - \frac{s_{\mathbf{k}}}{\delta_{\mathbf{k}}} W A_{\mathbf{k}} \left[C_i D_i \right] \right) \\
&= -\frac{s_{\mathbf{k}}}{\delta_{\mathbf{k}}} \left(\frac{\partial}{\partial \delta_{\mathbf{j}}} W A_{\mathbf{k}} \left[C_i D_i \right] \right) \\
&= -\frac{s_{\mathbf{k}}}{\delta_{\mathbf{k}}} (0) \\
&= 0
\end{aligned}
$$

Investor holdings

Price effect of asset holdings on a different asset It is more convenient to calculate several preliminary derivatives beforehand. First, $\frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} A_{\mathbf{i}}$.

$$
\begin{aligned}
\frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} A_{\mathbf{i}} &= \frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} \sum_{j=1}^{M} h_{\mathbf{i},j} \\
&= \sum_{j=1}^{M} \frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} h_{\mathbf{i},j} \quad \text{(differentiation is linear)} \\
&= \frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} h_{\mathbf{i},\mathbf{j}} \quad \text{(only term } j = \mathbf{j} \text{ is non-zero)} \\
&= 1
\end{aligned}
$$

Next, I consider $\frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} S_{\mathbf{i}}$.

$$\begin{aligned}
\frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} S_{\mathbf{i}} &= \frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} \sum_{j=1}^{M} h_{\mathbf{i},j} s_j \\
&= \sum_{j=1}^{M} \frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} h_{\mathbf{i},j} s_j \quad \text{(differentiation is linear)} \\
&= \frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} h_{\mathbf{i},\mathbf{j}} s_{\mathbf{j}} \quad \text{(only term } j = \mathbf{j} \text{ is non-zero)} \\
&= s_{\mathbf{j}}
\end{aligned}$$

Now, consider $\frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} x_{\mathbf{i}}$.

$$\begin{aligned}
\frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} x_{\mathbf{i}} &= \frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} C_{\mathbf{i}} D_{\mathbf{i}} \\
&= \left(\frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} C_{\mathbf{i}}\right) D_{\mathbf{i}} + C_{\mathbf{i}} \left(\frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} D_{\mathbf{i}}\right) \\
&= \left(\frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} \left(\frac{A_{\mathbf{i}}}{S_{\mathbf{i}}} - 1\right)\right) D_{\mathbf{i}} + C_{\mathbf{i}} \left(\frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} \left(\frac{r_{\mathbf{i}}}{A_{\mathbf{i}} - r_{\mathbf{i}}}\right)\right) \\
&= \left(\frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} \left(\frac{A_{\mathbf{i}}}{S_{\mathbf{i}}}\right)\right) D_{\mathbf{i}} + C_{\mathbf{i}} r_{\mathbf{i}} \left(\frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} \left(\frac{1}{A_{\mathbf{i}} - r_{\mathbf{i}}}\right)\right) \\
&= \left(\frac{S_{\mathbf{i}} \left(\frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} A_{\mathbf{i}}\right) - \left(\frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} S_{\mathbf{i}}\right) A_{\mathbf{i}}}{S_{\mathbf{i}}^2}\right) D_{\mathbf{i}} + C_{\mathbf{i}} r_{\mathbf{i}} \left(\frac{\partial}{\partial A_{\mathbf{i}}} \left(\frac{1}{A_{\mathbf{i}} - r_{\mathbf{i}}}\right) \frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} A_{\mathbf{i}}\right) \\
&= \left(\frac{S_{\mathbf{i}} (1) - (s_{\mathbf{j}}) A_{\mathbf{i}}}{S_{\mathbf{i}}^2}\right) D_{\mathbf{i}} + C_{\mathbf{i}} r_{\mathbf{i}} \left(\left(-\frac{1}{(A_{\mathbf{i}} - r_{\mathbf{i}})^2}\right) (1)\right) \\
&= \frac{S_{\mathbf{i}} - s_{\mathbf{j}} A_{\mathbf{i}}}{S_{\mathbf{i}}^2} D_{\mathbf{i}} - \frac{C_{\mathbf{i}} r_{\mathbf{i}}}{(A_{\mathbf{i}} - r_{\mathbf{i}})^2} \\
&= \frac{S_{\mathbf{i}} - s_{\mathbf{j}} A_{\mathbf{i}}}{S_{\mathbf{i}}^2} D_{\mathbf{i}} - \frac{C_{\mathbf{i}}}{(A_{\mathbf{i}} - r_{\mathbf{i}})} D_{\mathbf{i}} \\
&= D_{\mathbf{i}} \left(\frac{S_{\mathbf{i}} - s_{\mathbf{j}} A_{\mathbf{i}}}{S_{\mathbf{i}}^2} - \frac{C_{\mathbf{i}}}{A_{\mathbf{i}} - r_{\mathbf{i}}}\right)
\end{aligned}$$

Finally, I calculate $\frac{\partial p'_{\mathbf{k}}}{\partial h_{\mathbf{i},\mathbf{j}}}$.

$$\begin{aligned}\frac{\partial p'_{\mathbf{k}}}{\partial h_{\mathbf{i},\mathbf{j}}} &= \frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}}\left(s_{\mathbf{k}} - \frac{s_{\mathbf{k}}}{\delta_{\mathbf{k}}} W A_{\mathbf{k}}\left[C_i D_i\right]\right)\\ &= -\frac{s_{\mathbf{k}}}{\delta_{\mathbf{k}}}\frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}}\left(W A_{\mathbf{k}}\left[C_i D_i\right]\right)\\ &= -\frac{s_{\mathbf{k}}}{\delta_{\mathbf{k}}}\frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}}\left(\frac{\sum_{i=1}^{N} h_{i,\mathbf{k}} x_i}{\sum_{i=1}^{N} h_{i,\mathbf{k}}}\right)\end{aligned}$$

Then apply the quotient rule

$$\begin{aligned} &= -\frac{s_{\mathbf{k}}}{\delta_{\mathbf{k}}}\left(\frac{\left(\sum_{i=1}^{N} h_{i,\mathbf{k}}\right)\left(\frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}}\sum_{i=1}^{N} h_{i,\mathbf{k}} x_i\right) - \left(\frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}}\sum_{i=1}^{N} h_{i,\mathbf{k}}\right)\left(\sum_{i=1}^{N} h_{i,\mathbf{k}} x_i\right)}{\left(\sum_{i=1}^{N} h_{i,\mathbf{k}}\right)^2}\right)\\ &= -\frac{s_{\mathbf{k}}}{\delta_{\mathbf{k}}}\left(\frac{G_{\mathbf{k}}\left(\frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}}\sum_{i=1}^{N} h_{i,\mathbf{k}} x_i\right) - (0)\left(\sum_{i=1}^{N} h_{i,\mathbf{k}} x_i\right)}{G_{\mathbf{k}}^2}\right)\\ &= -\frac{s_{\mathbf{k}}}{\delta_{\mathbf{k}}}\left(\frac{G_{\mathbf{k}}\left(\frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}}\sum_{i=1}^{N} h_{i,\mathbf{k}} x_i\right)}{G_{\mathbf{k}}^2}\right)\\ &= -\frac{s_{\mathbf{k}}}{\delta_{\mathbf{k}}}\left(\frac{\left(\sum_{i=1}^{N} h_{i,\mathbf{k}}\frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} x_i\right)}{G_{\mathbf{k}}}\right) \qquad \text{(differentiation is linear)}\\ &= -\frac{s_{\mathbf{k}}}{\delta_{\mathbf{k}}}\left(\frac{h_{\mathbf{i},\mathbf{k}}\frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} x_{\mathbf{i}}}{G_{\mathbf{k}}}\right) \qquad \text{(only term } i = \mathbf{i}\text{is non-zero)}\\ &= -\frac{s_{\mathbf{k}}}{\delta_{\mathbf{k}}}\left(\frac{h_{\mathbf{i},\mathbf{k}}\left(D_{\mathbf{i}}\left(\frac{S_{\mathbf{i}} - s_{\mathbf{j}} A_{\mathbf{i}}}{S_{\mathbf{i}}^2} - \frac{C_{\mathbf{i}}}{A_{\mathbf{i}} - r_{\mathbf{i}}}\right)\right)}{G_{\mathbf{k}}}\right)\\ &= -\frac{s_{\mathbf{k}} h_{\mathbf{i},\mathbf{k}} D_{\mathbf{i}}}{\delta_{\mathbf{k}} G_{\mathbf{k}}}\left(\frac{S_{\mathbf{i}} - s_{\mathbf{j}} A_{\mathbf{i}}}{S_{\mathbf{i}}^2} - \frac{C_{\mathbf{i}}}{A_{\mathbf{i}} - r_{\mathbf{i}}}\right)\end{aligned}$$

Price effect of asset holdings on the same asset

$$\begin{aligned}
\frac{\partial p'_{\mathbf{j}}}{\partial h_{\mathbf{i},\mathbf{j}}} &= \frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} \left(s_{\mathbf{j}} - \frac{s_{\mathbf{j}}}{\delta_{\mathbf{j}}} WA_{\mathbf{j}} \left[C_i D_i \right] \right) \\
&= -\frac{s_{\mathbf{j}}}{\delta_{\mathbf{j}}} \frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} \left(WA_{\mathbf{j}} \left[C_i D_i \right] \right) \\
&= -\frac{s_{\mathbf{j}}}{\delta_{\mathbf{j}}} \frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} \left(\frac{\sum_{i=1}^{N} h_{i,\mathbf{j}} x_i}{\sum_{i=1}^{N} h_{i,\mathbf{j}}} \right)
\end{aligned}$$

Then apply the quotient rule

$$\begin{aligned}
&= -\frac{s_{\mathbf{j}}}{\delta_{\mathbf{j}}} \left(\frac{\left(\sum_{i=1}^{N} h_{i,\mathbf{j}} \right) \left(\frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} \sum_{i=1}^{N} h_{i,\mathbf{j}} x_i \right) - \left(\frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} \sum_{i=1}^{N} h_{i,\mathbf{j}} \right) \left(\sum_{i=1}^{N} h_{i,\mathbf{j}} x_i \right)}{\left(\sum_{i=1}^{N} h_{i,\mathbf{j}} \right)^2} \right) \\
&= -\frac{s_{\mathbf{j}}}{\delta_{\mathbf{j}}} \left(\frac{G_{\mathbf{j}} \left(\frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} \sum_{i=1}^{N} h_{i,\mathbf{j}} x_i \right) - (1) \left(\sum_{i=1}^{N} h_{i,\mathbf{j}} x_i \right)}{G_{\mathbf{j}}^2} \right) \\
&= -\frac{s_{\mathbf{j}}}{\delta_{\mathbf{j}} G_{\mathbf{j}}} \left(\frac{G_{\mathbf{j}} \left(\frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} \sum_{i=1}^{N} h_{i,\mathbf{j}} x_i \right) - (1) \left(\sum_{i=1}^{N} h_{i,\mathbf{j}} x_i \right)}{G_{\mathbf{j}}} \right) \\
&= -\frac{s_{\mathbf{j}}}{\delta_{\mathbf{j}} G_{\mathbf{j}}} \left(\frac{\sum_{i=1}^{N} h_{i,\mathbf{j}} \frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} x_i}{1} - \frac{\sum_{i=1}^{N} h_{i,\mathbf{j}} x_i}{G_{\mathbf{j}}} \right) \\
&= -\frac{s_{\mathbf{j}}}{\delta_{\mathbf{j}} G_{\mathbf{j}}} \left(\left(h_{\mathbf{i},\mathbf{j}} \frac{\partial}{\partial h_{\mathbf{i},\mathbf{j}}} x_i \right) - WA_{\mathbf{j}} \left[x_i \right] \right) \\
&= -\frac{s_{\mathbf{j}}}{\delta_{\mathbf{j}} G_{\mathbf{j}}} \left(h_{\mathbf{i},\mathbf{j}} D_{\mathbf{i}} \left(\frac{S_{\mathbf{i}} - s_{\mathbf{j}} A_{\mathbf{i}}}{S_{\mathbf{i}}^2} - \frac{C_{\mathbf{i}}}{A_{\mathbf{i}} - r_{\mathbf{i}}} \right) - WA_{\mathbf{j}} \left[C_i D_i \right] \right)
\end{aligned}$$

1.8.4 Numerical Comparison of Pro rata versus Sequential Asset Sales

In the model I assume that investors sell their assets pro rata to re-establish their original leverage ratios. As pointed out earlier, this is a somewhat restrictive assumption – there is no *a priori* reason to suppose that investors are really bound by such a constraint. Moreover, one might argue that investors may well prefer to sell more of the assets that declined less in value rather than those assets that declined more in value (in expectation that assets may revert to their fundamental value). Therefore, an alternative assumption might be that investors sell their assets sequentially, selling down each asset in turn from highest price to lowest price until they have re-established their leverage ratio[25]. In this section I test (numerically) the impact of these competing assumptions on the resulting equilibrium prices. The numerical simulation setup is identical to that discussed in subsection 1.5.2. First we consider a setup with 2 investors and 2 assets (2×2), then a setup with 5 investors and 5 assets (5×5) and finally a setup with 10 investors and 10 assets (10×10). In each setup we perform 10 million random draws from the model parameter space. For each draw we calculate both the equilibrium prices $\{p_j^\star\}$ based on the pro rata portfolio sale assumption and the equilibrium prices $\{p_j^{seq}\}$ based on the sequential asset sale assumption[26]. We are interested in what impact the different asset sale assumptions have on the numerically calculated equilibrium prices. To test this we regress the equilibrium sequential sale price of asset 1 (p_1^{seq}) on the equilibrium pro rata sale price of asset 1 ($p_1^\star$), that is

$$p_1^{seq} = \beta p_1^\star + \varepsilon$$

If the equilibrium pro rata sale price ($p_1^\star$) is a good predictor of the equilibrium sequential sale price (p_1^{seq}), this would suggest that the equilibrium price in our model is not particularly sensitive to the alternative assumption of sequential asset sales. The results indicate that this is the case (for all regression results refer to Table 1.8 below). Regression (1) in the table corresponds to the regression we discussed above. The equilibrium pro rata sale price $p_1^\star$ explains 99.56% of the variation in the equilibrium sequential sale price in the 2×2 setup (5×5: 99.54%, 10×10: 99.56%). The results are even stronger in regression (2) in which I exclude simulations where sequential sale prices did not converge to an equilibrium solution.

Again, one might argue that the more relevant metric is not the equilibrium price, but rather the equilibrium price *adjustment* from the initial shocked price, that is

---

[25] Recall that prices are normalised to 1 initially. So decreasing price order is the same as ordering assets from least affected to most affected by the negative shock.

[26] Note that prices do not always converge when using the sequential asset sales assumption. In the simulation I limit the number of iterations used to calculate equilibrium prices to 1000. In every single instance I am able to obtain equilibrium prices assuming pro rata sales. Assuming sequential sales, however, often results in prices that fail to converge (typically, prices and sale fractions get stuck in a repeating cycle).

$\Delta p_j^\star \equiv s_j - p_j^\star$ for pro rata sales and, equivalently, $\Delta p_j^{seq} \equiv s_j - p_j^{seq}$ for sequential sales. This suggests a regression in the form

$$\Delta p_1^{seq} = \beta \Delta p_1^\star + \varepsilon$$

in which the price adjustment under the assumption of sequential sales is regressed on the price adjustment under the assumption of pro rata sales. The results are shown as regression (3) in the table. The pro rata sales price adjustment $\Delta p_1^\star$ explains 93.6% of the variation in the sequential sales price adjustment Δp_1^{seq} in the 2×2 setup (5×5: 91.6%, 10×10: 91.5%). Again, the results are substantially improved when I exclude simulations where sequential sale prices did not converge to an equilibrium solution, shown as regression (4) in the tables.

Table 1.8: Numerical simulation regression results

2 Investors and 2 assets (2×2)

	(1)	(2)	(3)	(4)
	p_1^{seq}	p_1^{seq}	Δp_1^{seq}	Δp_1^{seq}
	β/t-Stat	β/t-Stat	β/t-Stat	β/t-Stat
$p_1^\star$	1.002***	1.000***		
	(64871.53)	(73042.83)		
$\Delta p_1^\star$			0.934***	0.954***
			(4803.41)	(6183.69)
Adj R^2	0.9959	0.9972	0.9360	0.9588
N	10,000,000	8,603,942	10,000,000	8,603,942

5 Investors and 5 assets (5×5)

	(1)	(2)	(3)	(4)
	p_1^{seq}	p_1^{seq}	Δp_1^{seq}	Δp_1^{seq}
	β/t-Stat	β/t-Stat	β/t-Stat	β/t-Stat
$p_1^\star$	1.006***	1.004***		
	(53965.15)	(51683.84)		
$\Delta p_1^\star$			0.870***	0.910***
			(5345.95)	(5859.76)
Adj R^2	0.9954	0.9973	0.9160	0.9612
N	10,000,000	4,981,221	10,000,000	4,981,221

10 Investors and 10 assets (10×10)

	(1)	(2)	(3)	(4)
	p_1^{seq}	p_1^{seq}	Δp_1^{seq}	Δp_1^{seq}
	β/t-Stat	β/t-Stat	β/t-Stat	β/t-Stat
$p_1^\star$	1.008***	1.005***		
	(54051.05)	(32848.62)		
$\Delta p_1^\star$			0.832***	0.888***
			(5978.95)	(4327.30)
Adj R^2	0.9956	0.9975	0.9150	0.9669
N	10,000,000	1,859,749	10,000,000	1,859,749

The results of this numerical study suggests that equilibrium prices in my model are not particularly sensitive to the assumption of pro rata sales, as compared against an assumption of sequential asset sales in reverse price order.

Chapter 2

Fire Sales – Evidence

Abstract

I estimate fire-sale risk using a multi-asset, multi-investor fire-sale model driven mainly by overlapping investor holdings and investor-level leverage constraints. Model fire-sale returns predict the cross-section of US stock returns during times of market distress. However, this risk does not appear to be priced *ex-ante*. These findings have implications for investors, risk managers and regulators. In particular, it suggests that overlapping asset holdings and investor-level constraints may be important determinants of stock returns during episodes of market distress.

2.1 Introduction

I use a multi-asset, multi-investor model that yields equilibrium fire-sale prices based on the pattern of asset holdings and estimated investor-level leverage constraints as described in detail in chapter 1. Conditional on *market distress* – which I define as a decrease of 10% or more in the S&P 500 index over a calendar quarter – model fire-sale returns positively and significantly predict the cross-section of realised US equity returns. This conditional predictability persists after controlling for stock-level characteristics such as size, book-to-market, CAPM beta, illiquidity and historical return volatility.

However, this predictability does not hold unconditionally. Although there is some evidence that model fire-sale risk[1] is priced in panel data, both cross-sectional and sorted portfolio tests fail to support the hypothesis that fire-sale return is priced.

A substantial literature on fire sales has developed over the past 30 years and the recent financial crises has provided new impetus for work in this area. Shleifer and Vishny (2010a) is a recent survey of the fire-sale literature. They provide the following working definition of a fire sale:

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[1] In this paper I use model-derived hypothetical fire-sale returns as a measure of *ex ante* fire-sale risk. Therefore, I use the terms “model fire-sale return” and “fire-sale risk” interchangeably.

> The term "fire sale" has been around since the nineteenth century to describe firms selling smoke-damaged merchandise at cut-rate prices in the aftermath of a fire. But what are fire sales in broad financial markets with hundreds of participants? How can fire sales matter for generic goods, such as airplanes or financial securities? In modern financial research, the term "fire sale" has acquired a different meaning. As we suggested in a 1992 paper, a fire sale is essentially a *forced sale* of an asset at a *dislocated price.* (Shleifer and Vishny (2010a) p. 3, emphasis added)

The two key elements in this definition of a fire sale are "forced sale" and "dislocated price". In the model presented here investors become forced sellers when an exogenous negative price shock decreases the value of their assets and they have to liquidate part of their portfolio to meet a binding leverage ratio requirement. Similarly, I model the dislocated price element via prices that decrease linearly in the volume of forced sales. Therefore, the greater the proportion of an asset put up for sale by forced sellers, the greater the decrease in price.

Fire sales may also have implications for asset pricing. Wagner (2011) proposes a model in which investors demand a premium for investing in assets held by "liquidation-prone" investors. In distressed markets, liquidation-prone investors tend to liquidate their assets at the same time. This gives rise to fire-sale prices in those assets predominantly held by liquidation-prone investors. Ex ante, investors rationally demand a premium to compensate them for suffering large losses from holding such assets in the event of market distress. The work by Wagner (2011) thus serves as a motivation in this paper for considering whether fire sale returns might be priced.

The empirical impact of overlapping holdings in assets subjected to forced sales has been shown recently by Hau and Lai (2012). They show that non-bank stocks held by funds with large exposures to bank stocks suffered disproportionate negative returns during the banking-stock led market downturn of 2008. This is stark evidence that the pattern of ownership can have a significant impact on realised returns during episodes of market distress.

For a detailed review of the literature on fire sales and how the model used here fits within that literature the reader is referred to subsection 1.1 in chapter 1.

In the next section I provide a summary of the model used to calculate stock-level fire-sale returns. Section 2.3 outlines the data used, while section 2.4 considers the hypothesis that fire-sale returns conditionally predict realised returns in the cross-section. In section 2.5 I consider whether model fire-sale risk is priced. Section 2.6 concludes and section 2.7 contains the Appendix for this chapter.

2.2 Model

What follows is a high-level overview of the fire-sale model used to calculate fire-sale returns (for a detailed exposition, the reader is referred to chapter 1). Broadly, I model equilibrium fire-sale prices in a setting where multiple investors holding multiple assets are subjected to an exogenous negative price shock. Each investor has a given level of leverage initially. After the negative price shock, this leverage increases. It is assumed that each investor faces a binding requirement to re-establish their pre-shock leverage ratio. This requires investors to sell a fraction (hereafter the *portfolio sale fraction*) of their assets in the market – the "forced sale" element. Crucially, the prices investors obtain are in turn linked to the volume of forced asset sales, such that the price of each asset is linearly decreasing in the volume of fire sales of that asset – this provides the "dislocated price" element (hereafter the *price response*). This means that investors are affected not only by their own sale of assets, but also by the sale of assets by other investors, and *vice versa*. Therefore, this is a situation where the optimal actions of investors are interlinked; in other words this model constitutes a game in the game-theoretic sense. I sketch out the model more formally below.

2.2.1 Notation and setup

This is essentially a two period model. Initial conditions are fixed at $t = 0$; at $t = 1$ an exogenous negative price shock occurs, which disrupts the initial equilibrium and at $t = 2$ investors re-establish equilibrium at a new set of prices through portfolio sales.

I start with N investors indexed by $i \in [1..N]$ with access to M assets indexed by $j \in [1..M]$. The units of asset j held by investor i is denoted by $h_{i,j}$. Each investor i initially has debt of d_i. For convenience, let $A_i \equiv \sum_{j=1}^{M} h_{i,j}$ be the value of each investor's portfolio at $t = 0$ prices (without loss of generality I normalise the $t = 0$ prices of assets to unity). I also define $H_i \equiv \sum_{j=1}^{M} h_{i,j}p_j$ – this might be thought of as the valuation of the *initial* portfolio holdings $\{h_{i,1}, ..., h_{i,M}\}$ of investor i valued at the *new* $t = 2$ prices $\{p_1, ..., p_M\}$. I enforce a positive equity constraint initially; assets exceeds debt for all investors; $0 \leq d_i < A_i$.

Asset holdings are infinitely divisible and non-negative – short positions are not allowed. Therefore, $h_{i,j} \geq 0$ for all $i \in [1..N]$ and for all $j \in [1..M]$.[2] The initial leverage ratio of investor i is given by $L_i(0) = \frac{d_i}{A_i}$. At $t = 1$ there occurs an exogenous negative shock to assets, such that prices reduce to s_j with $0 < s_j \leq 1$. At $t = 2$ investors re-establish their original leverage ratios by each selling (pro rata) a fraction x_i of their portfolio at prices p_j.[3] The post-sale leverage ratio is thus given by $L_i(2) =$

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[2] Hereafter a subscript i will be taken to mean $i \in [1..N]$ (i.e., applicable to all investors) and similarly a subscript j will be taken to mean $j \in [1..M]$ (i.e., applicable to all assets).

[3] If debt exceeds assets at the new prices for any investor (before considering any sales), that investor is bankrupted and their entire portfolio is liquidated. Otherwise the investor re-establishes his original leverage ratio by selling a portion of his assets pro rata. The assumption that investors

$\frac{Original\,Debt - Sales\,Proceeds}{Original\,Portfolio - Sales\,Proceeds} = \frac{d_i - x_i H_i}{(1-x_i)H_i}$.

2.2.2 Portfolio sale fractions

Each investor has to sell a fraction x_i of his portfolio such that his original leverage ratio is re-established. That is, x_i has to satisfy $L_i(2) = L_i(0) \implies \frac{d_i - x_i H_i}{(1-x_i)H_i} = \frac{d_i}{A_i}$. Solving for x_i yields the portfolio sale fraction

$$x_i = \min\left[\frac{A_i - H_i}{H_i} \times \frac{d_i}{A_i - d_i}, 1\right] \text{ where } H_i \equiv \sum_{j=1}^{M} h_{i,j} p_j \text{ and } A_i \equiv \sum_{j=1}^{M} h_{i,j} \qquad (2.2.1)$$

Note that we impose a minimum portfolio sale fraction of 1, because in our model an investor cannot sell more than 100% of his portfolio.

2.2.3 Price response

Equilibrium prices in turn depend on the volume of fire sales. The percentage change in price for each asset is linearly decreasing in the percentage of each asset put up for sale by all investors in aggregate. The slope of the linear price response is regulated by a liquidity parameter δ_j for each asset. That is, $\delta_j = \frac{\%\Delta Q_j}{\%\Delta p_j}$.[4] I interpret the change in price in the usual way as $\%\Delta p_j = \frac{p_j - s_j}{s_j}$. The change in quantity is taken to mean the percentage of each asset put up for sale by all investors in aggregate, so $\%\Delta Q_j = -\frac{\sum_{i=1}^{N} h_{i,j} x_i}{\sum_{i=1}^{N} h_{i,j}}$. Solving for p_j I obtain the price response

$$p_j = s_j - \frac{s_j}{\delta_j}\left(\frac{\sum_{i=1}^{N} h_{i,j} x_i}{\sum_{i=1}^{N} h_{i,j}}\right) \qquad (2.2.2)$$

Note that the price response depends on the level of equilibrium portfolio sales, and *vice versa*.

2.2.4 Equilibrium fire-sale prices

Equilibrium in this model is a set of portfolio sale fractions $\{x_i\}$ and prices $\{p_j\}$ that satisfies *simultaneously* the leverage requirement embodied by the portfolio sale fraction

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sell their assets pro rata can be challenged. Hau and Lai (2012) report that investors suffering losses during the financial crises of 2008 tended to liquidate their least affected assets over assets that suffered larger losses. However, in previous work (Geertsema, 2011) I considered an alternative specification in which investors liquidate assets in their portfolio in descending order of the percentage losses suffered on each asset. (Investors first sell the asset with the smallest loss, then the asset with the second-smallest loss, and so forth). Using simulated data I demonstrated that the numerical equilibrium prices obtained using each of the two approaches are similar (correlations between the prices exceeded 98% in the simulated data). Therefore, in this paper I retain the assumption that the assets in each investor's portfolio are sold pro rata.

[4] This definition is congruent with the price elasticity concept from economics

equation (2.2.1) for each investor *and* the price response requirement for each asset as per equation (2.2.2). In earlier work (see chapter 1) I proved that such an equilibrium exists, that it is unique and that it can be calculated numerically using the method of successive approximations. In this setting, that means starting with the initial set of shocked prices and then iteratively applying first the portfolio sale fraction formula and then the price response formula until convergence is achieved (convergence being guaranteed by the proof).

It should be stressed that this model is a *conditional* model, not a general asset pricing model. The model assumes that an exogenous price shock creates conditions under which leveraged investors become forced sellers. Equilibrium fire-sale prices reflect the pattern of asset holdings, the leverage of investors and the price response of assets.

In short, the model reviewed here yields hypothetical prices that would obtain in a setting where a negative market shock caused investors to become forced sellers. At the same time prices are decreasing in the volume of forced assets put up for sale. The interaction between forced selling and negative price response yields the model equilibrium prices. The obvious question is – does the model work? Is there a relationship between model fire-sale returns and realised returns? To answer that question I first need to calculate model fire-sale prices; the next section explains this step in more detail.

2.3 Data

2.3.1 Model inputs

To test the model I need to calculate model fire-sale prices. This requires knowledge of the pattern of asset ownership, investor leverage and asset liquidity. The model we consider is essentially a single period multi-asset model. It generates a vector of model prices at a given point in time, based on the model inputs. It is understood that all model inputs can change over time, even if they are not explicitly time subscripted.

Asset holdings

For asset holdings I turn to the Thompson Reuters 13F Institutional Ownership database (previously the Spectrum database, hereafter the 13F data). The 13F data is based on mandatory quarterly SEC form 13F fillings. Under US securities law institutional investors with holdings of US equity securities in excess of USD 100mn have to declare their holding on a quarterly basis. This requirement applies even if the investor is not resident in the United States; as long as the investor conducts business in the US the requirement applies. The dataset starts in 1980 and continues to the present.

Inevitably, the 13F dataset is not ideal. Most glaringly, it only covers some investors (institutions with more than USD100mn of US equities under management) and some assets (US equities). In contrast, the model theoretically applies to all investors and all assets. That said, the data can be considered reasonably reliable, if incomplete, given that filling 13F forms is a statutory requirement for those investors subjected to its requirements. This also allays potential concern about the reporting bias that plague some other databases. There are, nonetheless, a number of housekeeping issues that need to be addressed before the 13F data can be used – these are discussed in more detail in subsection 2.7.1 in the Appendix.

Debt

It is more difficult to obtain data about the level of debt of each investor. I take a pragmatic approach and estimate an "imputed" level of debt for each investor based on their historical behaviour. If an investor tends to sell as markets decline we take the view that they are behaving "as if" they are leveraged, whether or not they actually are. This interpretation is consistent with the role of debt in the theoretical model as a mechanism by which a negative price shock translates into forced sales. Such an approach also allows me to sidestep the problem of how to take into account derivatives (which can incorporate high levels of economic leverage without showing up as debt on a balance sheet). I explain the process of estimating imputed debt from investor behaviour in detail in the Appendix in subsection 2.7.2. For robustness checks, see subsection 2.4.5.

Liquidity

As for the price response, I estimate the liquidity parameter (δ) for each quarter based on prices and volumes over the past 250 trading days. The detail of the estimation is set out in subsection 2.7.3 in the Appendix. It is well-known that liquidity is a predictor of expected return in the cross-section (see for instance, Amihud, Mendelson and Pedersen (2005)). Therefore, there is a possibility that any predictability we observe using model fire-sale prices might be driven, at least in part, by the estimated liquidity used as a model input. I deal with this concern in two ways. First, in robustness checks I use a constant liquidity parameter in calculating model fire-sale prices.[5] It turns out that the results are essentially the same when I use a constant liquidity parameter. Second, I include Amihud illiquidity as a control variable in my tests. Again, the results that I obtain (including those where I use a constant liquidity parameter) are robust to the inclusion of Amihud illiquidity as a control variable.

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[5] In robustness checks I consider constant liquidity parameters $\delta = 1.5$, $\delta = 2$, $\delta = 2.5$ and $\delta = 5$. See subsection 2.4.5.

Shocked price

In the model assets are subjected to a negative exogenous shock such that prices are reduced to their shocked level $s_j \lneq 1$.[6] I impose a common shock across assets and time. For simplicity I assume $s = 0.9$; this equates to a 10% negative price shock given that all initial prices are normalised to unity in the model.[7] The use of a constant shock means that model fire-sale prices should be interpreted as the hypothetical model price that would obtain *if* all assets suffered the same negative return.

Model fire-sale prices and returns

I calculate model fire-sale prices at each quarter for each asset based on the 13F holdings data for that quarter and the imputed debt levels calculated for each investor based on their behaviour over the previous 4 years (16 quarters).[8] Fire-sale prices are then calculated numerically using the model parameters (for more detail, refer to subsections 2.7.1 to 2.7.3 in the Appendix):

$$\textit{model}\left[\{H_{i,j}(t)\}, \{\hat{d}_i(t)\}, \{\hat{\delta}_j(t)\}, \{s_j = 0.9\}\right] \to \{p^{FS}_{j,t}\}$$

I define the model fire-sale return of an asset as the hypothetical instantaneous raw return that would obtain if the current price $p_{j,t}$ of asset j dropped to the fire-sale price $p^{FS}_{j,t}$, so $r^{FS}_{j,t} \equiv \frac{p^{FS}_{j,t} - 1}{1} = p^{FS}_{j,t} - 1$ (recall that initial model prices are normalised to unity). I then subtract the annualised risk free rate r^f for the quarter (divided by 4) so the model fire-sale return is expressed as a quarterly excess return. Therefore, $r^{FS}_{j,t} \equiv p^{FS}_{j,t} - 1 - \frac{r^f}{4}$. Note that the model fire-sale return $r^{FS}_{j,t}$ incorporates only information up to time t. (In this chapter all returns are excess returns, unless otherwise noted).

2.3.2 Independent variables and controls

The dataset consists of 112 quarters (from 1982Q1 to 2010Q4) with an average of c. 7000 stocks in each quarter. This gives around 780,000 realised return observations in total. The principal variables of interest are described below.

The realised quarterly excess total return ("**return**") is calculated from CRSP monthly total returns compounded over each quarter, less the quarterly risk free rate $[r_{j,t} \equiv r^{totalreturn}_{j,t} - \frac{r^f_t}{4}]$. The model fire-sale return ("**fsreturn**") is the instantaneous excess return implied by the calculated model fire-sale price $[r^{FS}_{j,t} \equiv \frac{p^{FS}_{j,t} - 1}{1} - \frac{r^f}{4} = p^{FS}_{j,t} - 1 - \frac{r^f_t}{4}]$. The Amihud illiquidity measure is denoted "**ailliq**" (see Amihud,

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[6] The notation $\lneq$ indicates that $s_j \leq 1$ for all j and $s_j < 1$ for at least some j. Recall that initial prices are normalised to 1 in the model.

[7] In my robustness checks I also consider negative shocks equivalent to price drops of 5%, 15% and 30%. See subsection 2.4.5.

[8] In robustness checks I also consider 8 quarter and 24 quarter estimation windows. See subsection 2.4.5.

Mendelson and Pedersen (2005)). This is equal to the average (over 250 trading days) of the absolute daily total return divided by the dollar daily trading volume. [$ailliq_t = \frac{1}{250}\sum_{s=t-250}^{t} \frac{|r_s|}{Volume_s \times p_s}$]. Volatility ("**volatility"**) is the realised daily return volatility (not annualised) over the quarter. The CAPM beta ("**capmbeta**") is estimated using an OLS regression of excess security returns on excess market returns using monthly data and a 60 month rolling window. Market capitalisation ("**mktcap**") is the capitalisation (in USD billions) of the firm at the end of the quarter [$mktcap_t = p_t \times SharesOutstanding(bn)_t$]. The book-to-market ratio is denoted by "**b2m**". I use reported equity as the measure of book value, then calculate the book to market ratio as the ratio of firm equity to market capitalisation [$b2m_t = \frac{BookEquity_t}{MarketCapitalisation_t}$]. Weighted average investor debt ("**wadebt**") is calculated as the average debt-to-asset (namely, leverage) ratio for a given asset, weighted by the holdings of each investor in that asset [$wadebt_{t,j} = \frac{\sum_{i=1}^{N} L_i(0)h_{i,j}}{\sum_{i=1}^{N} h_{i,j}}$]. The weighted average investor leverage is used as a control variable to address a potential endogenous clientele effect (discussed in more detail later). Finally, the estimated liquidity parameter δ_t ("**liquidity**") is calculated as explained in the Appendix in sub-section 2.7.3.

2.3.3 Model fire-sale returns

Summary statistics are provided in Table 2.1. Note that explanatory and control variables are lagged by one period. Using lagged explanatory variables allows me to sidestep potential endogeneity issues and to characterise the cross-sectional relationship between lagged variables and realised returns as predictive rather than explanatory.

Table 2.1: Summary statistics

	count	mean	sd	min	max
r_t	786,602	0.0309	0.3241	-1.0004	18.3317
r_{t-1}	756,402	0.0336	0.3187	-0.9855	18.3318
r_t^{FS}	755,510	-0.1354	0.0479	-0.9935	-0.1000
r_{t-1}^{FS}	726,686	-0.1359	0.0481	-0.9935	-0.1000
$capmbeta_{t-1}$	495,882	0.9830	0.6744	-0.9264	4.0454
$mktcap_{t-1}$	756,402	1.0251	3.6909	0.0005	52.2902
$b2m_{t-1}$	644,862	1.1377	3.7408	-5.1535	104.1262
$ailliq_{t-1}$	687,917	0.0000	0.0000	0.0000	0.0004
$volatility_{t-1}$	756,221	0.0339	0.0251	0.0027	0.2366
$wadebt_{t-1}$	756,402	0.4124	0.3902	0.0000	0.9000
$liquidity_{t-1}(\delta_{t-1})$	756,402	5.1753	7.4262	1.0100	20.0000

The pairwise correlations between the variables are set out in Table 2.2. A few are worth discussing. Note the high positive correlation (0.7) between fire-sale return and lagged fire-sale return. This suggests that model fire-sale return is highly persistent over time, unlike realised return[9]. Persistence in model fire-sale return is a mechanical consequence of persistence in the model inputs. Asset holdings typically evolve slowly and imputed debt is estimated using an 16-quarter rolling window. Also of interest is the negative correlation between model fire-sale return and weighted average investor debt. This suggests that higher average investor debt is associated with lower (more negative) model fire-sale returns – consistent with the fire-sale model discussed earlier.

A word about weighted average investor debt: it is plausible that high-leverage investors might endogenously elect to invest in high risk, high expected return assets (a *clientele* effect). If so one might question whether any predictability attributed to model fire-sale returns are not in fact due to this clientele effect. I address this concern by including weighted average investor debt as a control variable. This provides some comfort that the results I obtain are not merely driven by clientele effects.

Figure 2.3.1 shows a histogram of model fire-sale returns over the entire sample. The setup of the model (strictly negative shocks, no short sales, positive asset holdings only) means that model fire-sale returns are always negative and are bounded above by the initial shock of -10%.

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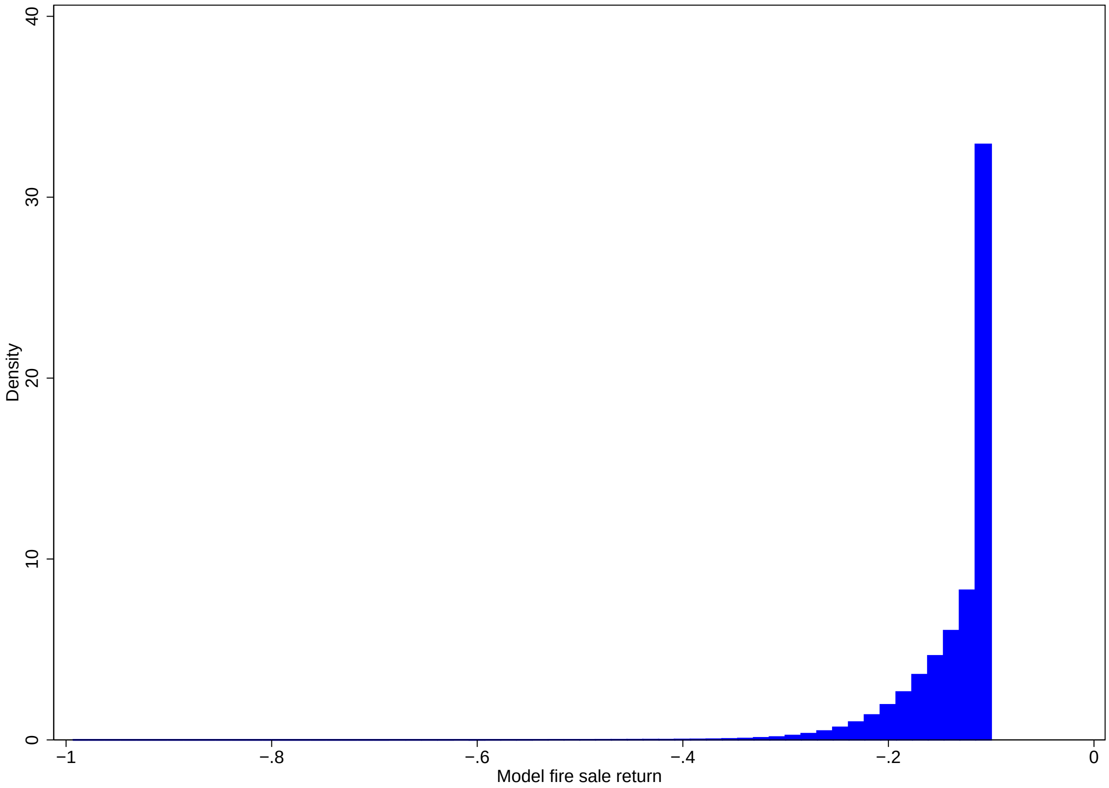
[9] As expected, the time-series variation in fire-sale returns is relatively low. The mean time-series standard deviation of fire-sale returns (averaged across stocks) is 0.02 compared to the mean fire-sale return of -0.13.

Table 2.2: Correlation matrix – realised returns, explanatory variables and controls

	r_t	r_t^{FS}	r_{t-1}^{FS}	$capmbeta_{t-1}$	$mktcap_{t-1}$	$b2m_{t-1}$	$ailliq_{t-1}$	$volatility_{t-1}$	$wadebt_{t-1}$	$liquidity_{t-1}(\delta_{t-1})$
r_t	1.00									
r_t^{FS}	0.02	1.00								
r_{t-1}^{FS}	-0.01	0.70	1.00							
$capmbeta_{t-1}$	0.00	-0.01	-0.01	1.00						
$mktcap_{t-1}$	-0.01	-0.18	-0.18	-0.07	1.00					
$b2m_{t-1}$	0.02	-0.05	-0.06	-0.03	-0.04	1.00				
$ailliq_{t-1}$	0.05	0.12	0.11	0.01	-0.06	0.04	1.00			
$volatility_{t-1}$	0.02	0.11	0.09	0.33	-0.13	0.01	0.42	1.00		
$wadebt_{t-1}$	0.00	-0.09	-0.11	-0.00	0.03	-0.02	-0.06	-0.12	1.00	
$liquidity_{t-1}(\delta_{t-1})$	0.00	0.13	0.31	0.01	-0.08	-0.00	0.10	0.00	-0.00	1.00

Figure 2.3.1: Histogram of model fire-sale return (r^{FS})

The model fire-sale return – r^{FS} – is calculated according to the model outlined in section 2.2 and discussed in detail in chapter 1. Because of the model set up and associated assumptions, model fire-sale prices will range between the initial shock (-10%) and total loss (-100%). The histogram below shows the distribution of the model fire-sale return across the entire dataset, comprising of 755,510 observations.



2.4 Conditional predictability

2.4.1 Hypothesis

Ideally our model of fire sales should have something to say about the realised pattern of fire sales. However, our model cannot predict *when* a fire-sale episode will occur. It simply predicts what would have happen *if* a fire-sale episode occurred. So the model is *conditional* in the sense that it only holds in the context of a systemic negative price shock. To make this operational, I assume that a large market downturn (defined as a 10% quarter-to-quarter drop or more in the S&P 500 index) is a sufficient negative systemic price shock.[10] At the same time the multi-asset nature of the model means that it is essentially a model of the cross-section of asset returns. Therefore, we have a *conditional cross-sectional* model that naturally leads to a conditional cross-sectional hypothesis.

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[10] Hereafter I use the terms *market downturn* or *market distress* as a shorthand for “a quarter during which the S&P 500 index suffered a negative total return of 10% or more”.

Hypothesis 1
Model fire-sale returns predict the cross-section of realised returns during a market downturn.

Perhaps the most straightforward approach to testing this hypothesis[11] is to conduct cross-sectional regressions during each market downturn to see how well model fire-sale returns predict realised returns; the results of this approach are set out below.

2.4.2 Cross-sectional tests

I consider the following cross-sectional regression at time t, *conditional* on a market downturn in quarter t

$$r_{j,t} = \alpha + \beta_t^{FS} r_{j,t-1}^{FS} + [controls_{j,t-1}] + \varepsilon_{j,t}$$

Under the null hypothesis that model fire-sale returns have no explanatory power for subsequent realised returns, one would expect to find $\beta_t^{FS} = 0$.

As noted earlier, I restrict myself to examining those quarters in which a market downturn took place. For example, the S&P 500 suffered a drop of 16% over 2001Q3. Thus I regress the realised excess return of each asset for the quarter ended 2001Q3 against the model fire-sale return for that asset as calculated based on asset holdings data for the previous quarter (2001Q2). Loosely speaking, this is a test of the ability of the fire-sale model to predict the cross-section of future returns conditional on market distress in the future. In the period covered by the data I identify 10 market distress quarters.[12] The results of these conditional cross-sectional regressions are set out in Table 2.3 (Panel A). I find that the model fire-sale coefficient (β_t^{FS}) is significant at a 1% confidence level in 9 out of 10 of these quarters (the remaining observation is significant at a 10% confidence level). Save for the first quarter, the model fire-sale coefficient is consistently positive, suggesting a positive relationship between model fire-sale returns and realised returns over the subsequent quarter. This relationship is economically significant, with the model fire-sale coefficient averaging 0.30 across all the 9 positive coefficient quarters[13]. Put differently, a 10% cross-sectional difference in the model fire-sale return predicts a 3% cross-sectional difference in realised return over the subsequent distressed quarter.

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[11] In keeping with the approach commonly adopted in the empirical finance literature, I state the alternative hypothesis (that there is predictability) rather than the null hypothesis (that there is no predictability). Hence my empirical tests are conducted under the null hypothesis that fire-sale returns do not predict subsequent realised returns.

[12] For the record, these 10%+ market distress quarters are 1987Q4, 1990Q3, 1998Q3, 2001Q1, 2001Q3, 2002Q2, 2002Q3, 2008Q4, 2009Q1 and 2010Q2.

[13] While the R^2 statistics are low, that does not in itself preclude economic significance. This is borne out by the results in the next section, where I show that a hedge portfolio comprising of a long position in fire-sale safe securities and a short position in fire-sale risky securities generates excess returns of between 5% and 7% on a quarterly basis.

Panels B to D in Tables 2.3 to 2.5 include the Fama & French factors as well as the Amihud illiquidity measure, realised historical volatility and weighted average investor debt as controls. When including all the controls in Panel D (see Table 2.5) the coefficient on the lagged model fire-sale return is still positive in 9 out of 10 regressions and positive and significant in 8 out of 10 regressions.

Table 2.3: Conditional cross-sectional regression results

The results below relate to cross-sectional regressions (across stocks) performed in each of the 10 quarters in our dataset during which the S&P500 index decreased by more than 10%, indicative of general market distress. The dependent variable is the excess return of each stock in that quarter, $r_{j,t}$. The explanatory variable is the model fire-sale price of each stock in the *prior* quarter, $r^{FS}_{j,t-1}$. In Panel A I consider the simple predictive regression $r_{j,t} = \alpha + \beta^{FS}_t r^{FS}_{j,t-1} + \varepsilon_{j,t}$ without additional controls. In Panel B I control for the Fama and French factors; the CAPM beta (estimated over a 60-month rolling window), size (in billions) and the book to market ratio. All controls are lagged one quarter (and so are measured at the same time as the model fire-sale price) and are winsorized at the 1% level to account for possible outliers. I report t-statistics based on White (1980) heteroskedasticity adjusted standard errors. Significance levels are indicated with * (p<0.1), ** (p<0.05) and *** (p<0.01).

Panel A: $r_{j,t} = \alpha + \beta^{FS}_t r^{FS}_{j,t-1} + \varepsilon_{j,t}$

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
	t1987Q4	t1990Q3	t1998Q3	t2001Q1	t2001Q3	t2002Q2	t2002Q3	t2008Q4	t2009Q1	t2010Q2
	b/t	b/t	b/t	b/t	b/t	b/t	b/t	b/t	b/t	b/t
r^{FS}_{t-1}	-0.450***	0.297***	0.111*	0.849***	0.321***	0.289***	0.355***	0.194***	0.436***	0.165***
	(-6.60)	(4.28)	(1.88)	(11.50)	(5.80)	(4.65)	(7.21)	(4.83)	(9.20)	(3.00)
Const	-0.354***	-0.175***	-0.201***	0.140***	-0.131***	-0.034***	-0.128***	-0.251***	0.032***	-0.056***
	(-37.48)	(-17.31)	(-24.05)	(11.35)	(-14.16)	(-3.27)	(-15.82)	(-30.93)	(3.06)	(-6.90)
Adj R-sqr	0.0060	0.0019	0.0002	0.0129	0.0035	0.0022	0.0069	0.0032	0.0107	0.0012
N	5,792	5,851	8,416	7,689	7,316	7,005	6,917	6,690	6,569	6,311

Panel B: $r_{j,t} = \alpha + \beta^{FS}_t r^{FS}_{j,t-1} + \beta^{capmbeta}_t capmbeta_{j,t-1} + \beta^{size}_t size_{j,t-1} + \beta^{booktomarket}_t b2m_{j,t-1} + \varepsilon_{j,t}$

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
	t1987Q4	t1990Q3	t1998Q3	t2001Q1	t2001Q3	t2002Q2	t2002Q3	t2008Q4	t2009Q1	t2010Q2
	b/t	b/t	b/t	b/t	b/t	b/t	b/t	b/t	b/t	b/t
r^{FS}_{t-1}	-0.368***	0.219**	0.113	0.948***	0.408***	0.286***	0.339***	0.100*	0.470***	0.088
	(-4.47)	(2.49)	(1.44)	(10.20)	(5.94)	(3.82)	(5.70)	(1.68)	(6.40)	(1.12)
$capmbeta_{t-1}$	-0.098***	-0.110***	-0.067***	-0.033***	-0.135***	-0.143***	-0.084***	-0.081***	0.086***	-0.057***
	(-17.95)	(-10.20)	(-13.51)	(-3.89)	(-25.36)	(-24.45)	(-15.69)	(-11.67)	(6.00)	(-9.68)
$mktcap_{t-1}$	0.010***	0.012***	0.006***	-0.006***	-0.000	-0.005***	0.001	0.002***	0.000	-0.002***
	(7.14)	(6.72)	(8.59)	(-9.50)	(-0.42)	(-9.80)	(1.14)	(5.71)	(0.81)	(-7.04)
$b2m_{t-1}$	0.001	0.008**	0.001	0.005***	-0.000	-0.000	-0.000	0.000	0.001	-0.002
	(0.29)	(2.00)	(0.70)	(3.23)	(-0.72)	(-0.16)	(-0.48)	(0.16)	(0.68)	(-1.40)
Const	-0.233***	-0.068***	-0.126***	0.225***	0.038***	0.123***	-0.057***	-0.208***	-0.069***	-0.006
	(-16.08)	(-3.62)	(-10.39)	(14.50)	(3.27)	(9.15)	(-5.45)	(-14.79)	(-3.89)	(-0.45)
Adj R-sqr	0.1364	0.0555	0.0532	0.0435	0.1629	0.1169	0.0763	0.0430	0.0244	0.0300
N	2,998	3,264	4,277	4,259	4,292	4,333	4,404	3,882	3,843	3,754

Table 2.4: Conditional cross-sectional regression results (continued 1)

The results below relate to cross-sectional regressions (across stocks) performed in each of the 10 quarters in our dataset during which the S&P500 index decreased by more than 10%, indicative of general market distress. The dependent variable is the excess return of each stock in that quarter, $r_{j,t}$. The explanatory variable is the model fire-sale price of each stock in the *prior* quarter, $r^{FS}_{j,t-1}$. Panel C controls for additional stock characteristics that might be expected to predict returns – Amihud illiquidity ($ailliq_{j,t-1}$), realised daily return volatility in the quarter ($volatility_{j,t-1}$) and weighted average debt of investors holding the stock ($wadebt_{j,t-1}$). As in Table 2.3, all controls are lagged by one quarter relative to the dependent variable and are winsorized at the 1% level. I report t-statistics based on White (1980) heteroskedasticity adjusted standard errors. Significance levels are indicated with * (p<0.1), ** (p<0.05) and *** (p<0.01).

Panel C: $r_{j,t} = \alpha + \beta^{FS}_t r^{FS}_{j,t-1} + \beta^{ailliq}_t ailliq_{j,t-1} + \beta^{volatility}_t volatility_{j,t-1} + \beta^{wadebt}_t wadebt_{j,t-1} + \varepsilon_{j,t}$

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
	t1987Q4	t1990Q3	t1998Q3	t2001Q1	t2001Q3	t2002Q2	t2002Q3	t2008Q4	t2009Q1	t2010Q2
	b/t	b/t	b/t	b/t	b/t	b/t	b/t	b/t	b/t	b/t
r^{FS}_{t-1}	-0.074	0.464***	0.385***	0.930***	0.437***	0.473***	0.458***	0.232***	0.448***	0.112*
	(-1.02)	(5.63)	(5.76)	(10.30)	(6.63)	(7.10)	(8.54)	(5.18)	(8.20)	(1.94)
$ailliq_{t-1}$	1844.954***	1789.432***	5912.020***	9151.479***	6574.172***	4060.157***	2052.418***	519.059	628.484*	1225.144***
	(4.27)	(6.90)	(7.25)	(5.46)	(8.63)	(8.40)	(6.30)	(1.64)	(1.76)	(4.64)
$volatility_{t-1}$	-3.996***	-2.356***	-3.158***	-0.123	-4.338***	-4.614***	-2.973***	-3.436***	0.448**	-1.864***
	(-20.42)	(-12.30)	(-10.82)	(-0.62)	(-28.32)	(-16.58)	(-15.35)	(-18.73)	(2.01)	(-6.47)
$wadebt_{t-1}$	0.067***	-0.026	0.032***	0.154***	-0.025	0.056***	0.003	0.044***	0.035*	0.049***
	(2.72)	(-0.87)	(2.84)	(3.18)	(-0.73)	(2.79)	(0.16)	(3.35)	(1.82)	(4.98)
Const	-0.206***	-0.085***	-0.078***	0.157***	0.054***	0.129***	-0.019*	-0.140***	-0.030	-0.062***
	(-18.46)	(-6.77)	(-6.86)	(10.13)	(5.10)	(11.32)	(-1.94)	(-8.66)	(-1.21)	(-4.98)
Adj R-sqr	0.1253	0.0593	0.0720	0.0329	0.1752	0.0959	0.0801	0.0959	0.0171	0.0250
N	5,123	5,537	7,627	6,987	6,868	6,731	6,653	6,287	6,296	5,974

Table 2.5: Conditional cross-sectional regression results (continued 2)

The results below relate to cross-sectional regressions (across stocks) performed in each of the 10 quarters in our dataset during which the S&P500 index decreased by more than 10%, indicative of general market distress. The dependent variable is the excess return of each stock in that quarter, $r_{j,t}$. The explanatory variable is the model fire-sale price of each stock in the *prior* quarter, $r^{FS}_{j,t-1}$. Panel D controls for both the Fama and French factors (CAPM beta, size and book-to-market) and additional stock characteristics that might be expected to predict returns – Amihud illiquidity ($ailliq_{j,t-1}$), realised daily return volatility ($volatility_{j,t-1}$) and weighted average debt of investors holding the stock ($wadebt_{j,t-1}$). As before, all controls are lagged by one quarter relative to the dependent variable and are winsorized at the 1% level. I report t-statistics based on White (1980) heteroskedasticity adjusted standard errors. Significance levels are indicated with * (p<0.1), ** (p<0.05) and *** (p<0.01).

Panel D: $r_{j,t} = \alpha + \beta^{FS}_t r^{FS}_{j,t-1} + \beta^{capmbeta}_t capmbeta_{j,t-1} + \beta^{size}_t size_{j,t-1} + \beta^{booktomarket}_t b2m_{j,t-1} + \beta^{ailliq}_t ailliq_{j,t-1} + \beta^{volatility}_t volatility_{j,t-1} + \beta^{wadebt}_t wadebt_{j,t-1} + \varepsilon_{j,t}$

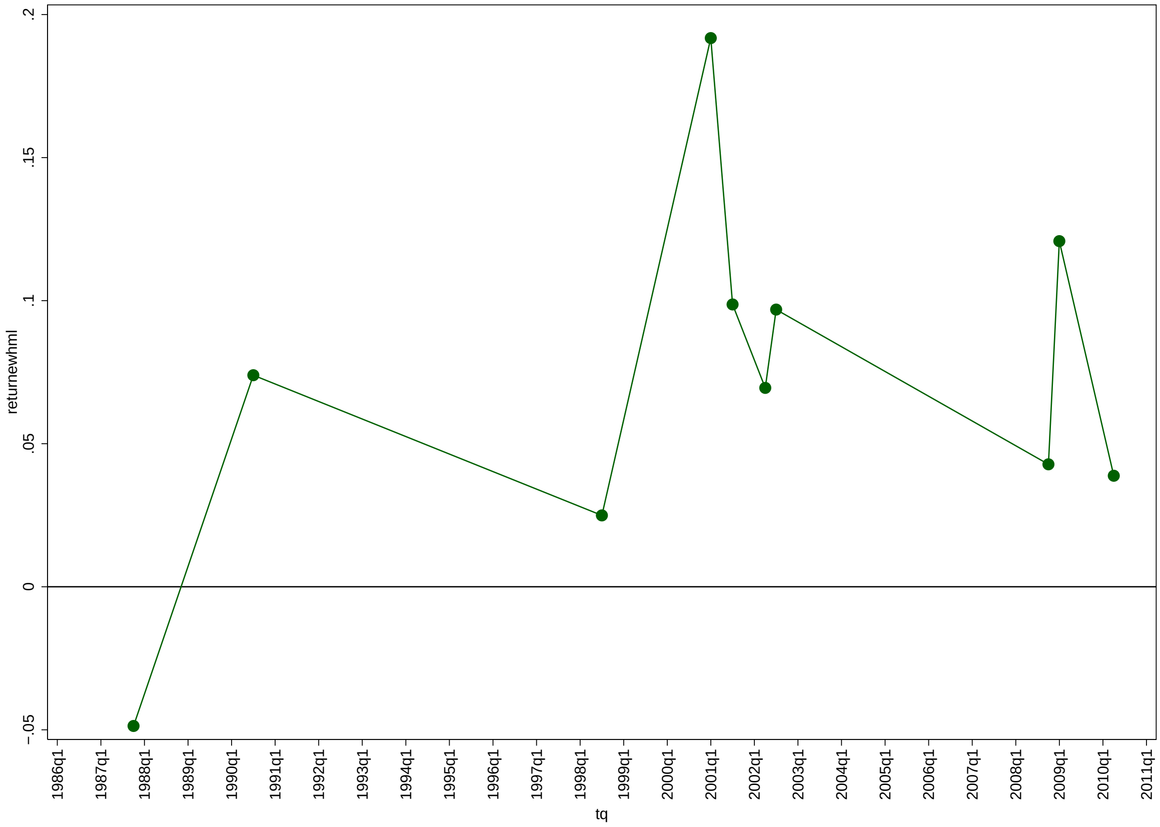
	(1) t1987Q4 b/t	(2) t1990Q3 b/t	(3) t1998Q3 b/t	(4) t2001Q1 b/t	(5) t2001Q3 b/t	(6) t2002Q2 b/t	(7) t2002Q3 b/t	(8) t2008Q4 b/t	(9) t2009Q1 b/t	(10) t2010Q2 b/t
r^{FS}_{t-1}	-0.139	0.286***	0.212**	0.752***	0.426***	0.343***	0.353***	0.177***	0.400***	0.105
	(-1.61)	(2.89)	(2.57)	(7.20)	(4.97)	(4.37)	(5.51)	(2.85)	(5.31)	(1.41)
$capmbeta_{t-1}$	-0.075***	-0.090***	-0.053***	-0.056***	-0.097***	-0.120***	-0.070***	-0.061***	0.096***	-0.044***
	(-13.30)	(-7.94)	(-8.10)	(-4.98)	(-13.31)	(-13.41)	(-10.43)	(-8.55)	(5.95)	(-6.14)
$mktcap_{t-1}$	0.005***	0.009***	0.005***	-0.005***	-0.001*	-0.005***	0.000	0.001***	0.001	-0.003***
	(3.91)	(4.82)	(6.85)	(-8.52)	(-1.72)	(-10.87)	(0.35)	(3.10)	(1.09)	(-8.74)
$b2m_{t-1}$	0.000	0.007*	-0.000	0.004***	-0.001	-0.000	-0.000	0.001	0.001	-0.002
	(0.02)	(1.79)	(-0.11)	(2.88)	(-1.54)	(-0.43)	(-0.75)	(0.46)	(0.60)	(-1.60)
$ailliq_{t-1}$	1246.751***	1190.750***	3405.456***	2383.391	3167.645***	1897.597***	1222.252***	-73.205	1134.227***	834.407***
	(2.62)	(4.08)	(2.63)	(1.38)	(4.84)	(3.05)	(2.95)	(-0.19)	(2.83)	(2.68)
$volatility_{t-1}$	-3.049***	-1.419***	-1.562**	1.200***	-1.977***	-1.754***	-1.057***	-2.356***	-0.212	-1.477***
	(-11.56)	(-4.88)	(-2.57)	(3.37)	(-7.18)	(-3.13)	(-3.19)	(-9.38)	(-0.59)	(-3.32)
$wadebt_{t-1}$	0.035	-0.011	0.007	0.006	0.007	0.022	0.004	0.015	0.029	0.053***
	(1.26)	(-0.31)	(0.53)	(0.10)	(0.17)	(1.04)	(0.20)	(1.01)	(1.33)	(4.56)
Const	-0.152***	-0.042**	-0.084***	0.157***	0.074***	0.159***	-0.036***	-0.118***	-0.108***	-0.020
	(-10.16)	(-2.24)	(-4.76)	(8.32)	(5.18)	(10.43)	(-2.84)	(-5.52)	(-3.44)	(-1.26)
Adj R-sqr	0.1930	0.0762	0.0659	0.0549	0.1811	0.1249	0.0830	0.0799	0.0278	0.0406
N	2,998	3,264	4,277	4,259	4,292	4,333	4,404	3,882	3,843	3,754

2.4.3 Sorted portfolio tests

Next I turn to sorted portfolios. Again, I consider only market distress quarters. For each quarter t I rank stocks on their lagged $(t-1)$ model fire-sale returns and sort them into five quintiles, such that quintile 1 contains the lowest (most negative) 20% of model fire-sale returns and quintile 5 contains the highest (least negative) 20% of model fire-sale returns. I then take the equal-weighted average of realised quarter t returns to construct a portfolio return for each quintile. This allows me to construct a "high minus low" portfolio that is long the quintile 5 portfolio and short the quintile 1 portfolio. Portfolio 5 corresponds to those stocks with the least negative model fire-sale returns - the "safest" stocks. Similarly, portfolio 1 corresponds to those stocks with the most negative model fire-sale returns, namely, the "riskiest" stocks. Thus we would expect portfolio 5 (the safe portfolio) to outperform portfolio 1 (the risky portfolio) in market downturns. Therefore, the "high minus low" portfolio formed by going long the safe portfolio 5 and shorting the risky portfolio 1 should produce positive returns in market downturns. This turns out to be the case in 9 out 10 quarters experiencing market distress, as shown in Figure 2.4.1.

Figure 2.4.1: High minus low (portfolio 5 - portfolio 1) realised portfolio returns (sorted by lagged model fire-sale return, market distress quarters only)

This graph plots the return, in each of the 10 quarters during which the S&P500 index decreased by 10% or more, of a portfolio formed by going long the 20% of stocks with the lowest model fire-sale returns and shorting the 20% of stocks with the lowest model fire-sale return in the *prior* quarter. Therefore, the portfolio is essentially long "model fire-sale safe" stocks and short "model fire-sale risky" stocks.



This pattern is reinforced by the mean portfolio returns reported in Table 2.6. The mean portfolio realised returns are monotonically increasing in the lagged model fire-sale return quintiles, suggesting that lagged model fire-sale returns are indeed positively related to subsequent realised returns during market downturns. The "safe" stocks of portfolio 5 outperform the "risky" stocks of portfolio 1 by around 7% per quarter on average over the 10 market distress quarters (5.3% per quarter on a market weighted basis). The difference between the quintile 5 and quintile 1 mean returns are significant at the 1% level for both equal weighted and market weighted portfolios[14]. Interestingly, most of the difference is between portfolios 4 and 5, particularly in the case of value weighted portfolios. This suggest that only the very "fire-sale safest" securities confer significant protection, or alternatively, that even a modest amount of fire-sale risk translates into realised losses during distressed market episodes.

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[14] Note, while I show the significance of the returns for portfolios 1 to 5, I do not rely on those results. Rather, I focus on the difference between the lowest and highest quintiles, where testing against a null of zero difference is economically sensible.

Table 2.6: Conditional HML excess returns – sorted by lagged model fire-sale return

I report statistics relating to the model fire-sale return portfolio, calculated using both equal weights and market weights. The model fire-sale return portfolio is formed by going long the 20% of stocks with the highest model fire-sale returns and shorting the 20% of stocks with the lowest model fire-sale returns in the *prior* quarter. In this table only the 10 quarters in which the S&P500 index decreased by 10% of more are considered. Significance levels are calculated using Newey West HAC adjusted standard errors and are indicated with * (p<0.1), ** (p<0.05) and *** (p<0.01).

Equal weighted lfsreturn HML portfolio timeseries statistics

	mean	sd	n	skew	min	max
1	-0.1800***	0.0855	10	0.0381	-0.3112	-0.0588
2	-0.1732***	0.0906	10	0.2021	-0.2914	-0.0356
3	-0.1674***	0.1050	10	0.1267	-0.3095	-0.0059
4	-0.1461***	0.1167	10	0.4510	-0.3061	0.0718
5	-0.1090**	0.1347	10	0.1340	-0.3204	0.1330
hml	0.0710***	0.0639	10	0.0282	-0.0487	0.1918

Market weighted lfsreturn HML portfolio timeseries statistics

	mean	sd	n	skew	min	max
1	-0.1260***	0.0556	10	-0.8383	-0.2256	-0.0541
2	-0.1189***	0.0501	10	-0.4769	-0.2137	-0.0468
3	-0.1124***	0.0691	10	-0.1528	-0.2330	-0.0243
4	-0.1101***	0.0724	10	0.4533	-0.2268	0.0301
5	-0.0731***	0.0747	10	-0.6114	-0.2128	0.0172
hml	0.0529***	0.0391	10	0.3811	0.0044	0.1149

In short the sorted portfolio evidence supports the notion that model fire-sale returns predict realised returns in a conditional sense. Combined with the cross-sectional results discussed earlier, this suggests that model fire-sale returns are conditionally predictive of the cross-section of realised returns in US stocks.

2.4.4 Power and size of tests

Overall the results appear to be significant and suggests that our tests are not lacking in power. However, we still need to consider the size of our tests. That is, can our tests properly discriminate against the null hypothesis of no relationship between the explanatory variables and realised returns. If not, then we cannot rely on these tests to reject the null.

To address this concern, I perform a Monte Carlo study. The approach and detailed results are set out in the Appendix. In short I find that all of the tests are of sufficient size to reject the null hypothesis when it is true.

2.4.5 Robustness

In calculating model fire-sale prices, there are a number of model parameters that I either fix or estimate. To ensure that the results obtained are not driven by a particular choice of model parameter, I recalculate model fire-sale prices using several alternate parameter values. In short, I find that the results do not qualitatively change based on the choice of model parameter values. Choices of parameter values are set out in Table 2.7, each identified by a unique “Run” name. I start by specifying a base set of parameter values (see the first line of Table 2.7) – the “Base” run. The parameters for the “Base” run use a 10% negative shock (so $s_j = 0.9$ for all assets, see “Shock” in the table), an estimated liquidity factor (see “Liquidity” in the table, described more fully in subsection 2.7.3 in the Appendix), a 16 period moving average for estimating imputed debt (“DebtMA” in the table, described more fully in subsection 2.7.2 in the Appendix) and zero debt for the “rest-of-the-world” assets not owned by investors covered by the 13F data set (“DebtRoW” in the table). I then systematically choose higher and lower values for each of these parameters in alternative model fire-sale calculation runs. In one case (run “b_rowe”) I estimate the rest-of-the-world debt level just as I do for individual investors, rather than assuming a fixed leverage percentage.

Table 2.7: Robustness – choices of model parameters

Run	Shock	Liquidity	DebtMA	DebtRoW	Comment
b	0.9	EST	16	0	Base run
b_s70	0.7	EST	16	0	Shocked price = 0.70 (vs 0.90 in Base)
b_s85	0.85	EST	16	0	Shocked price = 0.85 (vs 0.90 in Base)
b_s95	0.95	EST	16	0	Shocked price = 0.95 (vs 0.90 in Base)
b_l15	0.9	1.5	16	0	Liquidity = 1.5 (vs estimated in Base)
b_l20	0.9	2	16	0	Liquidity = 2 (vs estimated in Base)
b_l25	0.9	2.5	16	0	Liquidity = 2.5 (vs estimated in Base)
b_l50	0.9	5	16	0	Liquidity = 5 (vs estimated in Base)
b_row2	0.9	EST	16	0.2	Rest-of-world debt is 20% of assets (vs 0 in Base)
b_ma8	0.9	EST	8	0	Debt estimated with 8 quarters (vs 16 in Base)
b_ma24	0.9	EST	24	0	Debt estimated with 24 quarters (vs 16 in Base)

High-level results for each of the robustness checks in are summarised in Table 2.8. Qualitatively the results presented earlier continue to hold. To help in the interpretation of these results, I discuss the results for run “b_s70” in detail - this corresponds to the forth row in Table 2.8. In this run, we consider a systematic shocked price of 0.7 instead of 0.9 as in the base case. Put another way, we impose an exogenous negative shock of 30% instead of the 10% assumed throughout this paper and in the base scenario. The remaining cells in the forth row of Table 2.8 provide summaries of key results presented earlier in the paper. The second column (labelled "Cross-sectional coefficients") contains a count of the positive and significant coefficients on the lagged fire-sale return in each of 10 cross-sectional regressions documented in Tables 2.3 to 2.5. There are four numbers, each corresponding to a particular regression specification. The first number, 10, indicates that the regression coefficient on the lagged fire-sale return was positive and significant in 10 out of 10 cross-sectional regressions following the specification as set out in panel A of Table 2.3. The last number, 7, indicates that the regression coefficient on the lagged fire-sale return was positive and significant in 7 out of 10 cross-sectional regression following the specification as set out in panel D of Table 2.5.

The rest of Table 2.8 reports on the results of equal weighted portfolios sorted into quintiles using the lagged fire-sale return, considering only distressed quarters. The first of these remaining columns, labelled "Mean of sorted portfolio returns", contain the mean realised excess returns of the portfolios. It shows that, using the fire-sale return calculated on the assumption of a negative shock of 30%, the 20% of stocks with the lowest fire-sale return (namely, the most "risky" stocks) suffered a loss of 18% subsequently; on the other hand, the 20% of stocks with the highest fire-sale return (namely, the "safest" stocks) suffered a loss of 10.9% by comparison. The next column, labelled "Monotonic?", simply records whether the mean sorted portfolio returns are monotonic. A monotonic pattern in mean realised returns of sorted portfolios is generally taken as evidence that the sorting variable has explanatory power with regards to the realised returns. The next column, labelled "Mean" records the mean realised excess return to a portfolio that is long the top 20% of stocks and short the bottom 20% of stocks, when sorted on the lagged fire-sale return. This means that, over the 10 distressed quarters in our dataset, investors holding the portfolio just described would have earned a return of 7.05% per quarter (not annualised) on average. The last column, labelled "Positive (of 10)" counts the number of distressed quarters (out of a total of 10) during which the same portfolio would have generated a positive return.

Empirical results at the individual stock level might be driven by small market capitalisation stocks, which collectively make up only a very small fraction of the total stock market capitalisation. To verify that my results are not driven by these "micro" stocks, I re-perform my analysis after removing any stocks below the bottom 20% NYSE market capitalisation breakpoint for each quarter. I find that the results are not significantly affected when removing these small stocks from the sample.

It is interesting to consider how similar the model fire-sale returns calculated under different assumptions are. To answer this question, I considered the pairwise correlation between base fire-sale return and the fire-sale returns generated under each of the ten alternative model input assumptions as described in Table 2.7. The result of this

Table 2.8: Robustness – summary of results using choices of fire-sale model parameters

Run	Cross-sectional coefficients	Lagged model fire-sale sorted equal-weighted portfolios (market distress quarters only)			
	Count of positive and significant Panel [A B C D] (out of 10)	Mean of sorted portfolio returns [P1 P2 P3 P4 P5]	Monotonic? [Yes / No]	High-minus-low portfolio return	
				Mean	Positive (of 10)
b	[9 7 9 8]	[-0.180 -0.173 -0.167 -0.146 -0.109]	Yes	7.07%	9
b_s85	[9 7 9 8]	[-0.180 -0.175 -0.167 -0.145 -0.109]	Yes	7.05%	9
b_s95	[9 7 9 8]	[-0.180 -0.173 -0.168 -0.147 -0.108]	Yes	7.24%	9
b_s70	[10 7 9 7]	[-0.180 -0.175 -0.167 -0.145 -0.109]	Yes	7.07%	9
b_l15	[9 8 9 7]	[-0.179 -0.173 -0.165 -0.148 -0.110]	Yes	6.85%	9
b_l20	[9 8 9 8]	[-0.179 -0.173 -0.165 -0.149 -0.111]	Yes	6.81%	9
b_l25	[9 8 9 8]	[-0.179 -0.173 -0.165 -0.149 -0.110]	Yes	6.92%	9
b_l50	[9 8 9 8]	[-0.178 -0.173 -0.165 -0.149 -0.110]	Yes	6.83%	9
b_row2	[9 8 8 8]	[-0.178 -0.172 -0.155 -0.145 -0.126]	Yes	5.22%	9
b_ma8	[8 6 9 6]	[-0.177 -0.175 -0.167 -0.145 -0.112]	Yes	6.44%	9
b_ma24	[8 6 9 7]	[-0.179 -0.176 -0.167 -0.145 -0.109]	Yes	6.92%	9
Result Ref→	*Tables 2.3 to* 2.5	*Table* 2.6	*Table* 2.6	*Table* 2.6	*Figure* 2.4.1

analysis is presented in Table 2.9. The first column shows the correlation of the fire-sale return generated under the base model assumptions (run “b”) with the fire-sale return generated under each of the alternate robustness assumptions (runs “b_s70” to “b_ma24”). The correlations range from 0.99 (for negative shocks of 5% and 15% instead of the 10% negative shock in the base scenario) to 0.81 (when estimating the fire-sale price using debt estimated over 8 quarters instead of 16 quarters in the base scenario). Although the model fire-sale return is clearly sensitive to the exact model assumptions used in calculating it, it does appear to share a substantial commonality with fire-sale returns calculated using different assumptions.

Table 2.9: Correlation matrix – fire-sale returns under alternate robustness assumptions

	b	b_s70	b_s85	b_s95	b_l15	b_l20	b_l25	b_l50	b_row2	b_ma8	b_ma24
b	1.00										
b_s70	0.96	1.00									
b_s85	0.99	0.98	1.00								
b_s95	0.99	0.92	0.98	1.00							
b_l15	0.84	0.80	0.84	0.84	1.00						
b_l20	0.84	0.79	0.83	0.84	1.00	1.00					
b_l25	0.84	0.78	0.83	0.85	1.00	1.00	1.00				
b_l50	0.84	0.77	0.82	0.85	0.98	0.99	1.00	1.00			
b_row2	0.95	0.91	0.94	0.94	0.72	0.72	0.72	0.71	1.00		
b_ma8	0.81	0.85	0.82	0.78	0.66	0.65	0.65	0.64	0.76	1.00	
b_ma24	0.87	0.83	0.86	0.87	0.75	0.75	0.75	0.75	0.80	0.69	1.00

2.5 Asset pricing tests

2.5.1 Hypothesis

If we accept that assets with high *ex ante* model fire-sale risk experience larger losses during market downturns, it is reasonable to ask if model fire-sale risk is priced. There is theoretical support for the idea that assets subjected to higher model fire-sale risk should yield a higher expected return (namely, that model fire-sale risk should be priced). In the Wagner (2011) model investors are reluctant to invest in assets held by “liquidation-prone” investors. In times of market distress liquidation-prone investors tend to all have to sell assets at the same time, giving rise to a fire sale in such assets. Therefore, investors rationally demand a premium *ex-ante* for holding assets disproportionately held by liquidation-prone investors. In contrast to my conditional model of fire sales, the Wagner (2011) model is a fully fledged unconditional asset pricing model in which the “fire-sale premium” arises endogenously. Hence there is an argument that fire-sale return should be a priced risk, both on theoretical grounds and based on the empirical evidence presented earlier, showing that assets with high model fire-sale risk suffer disproportionately in market downturns. That leads us to the second hypothesis:

Hypothesis 2
Model fire-sale returns are priced.

2.5.2 Cross-sectional tests

If model fire-sale return is a priced risk factor, then stocks with low fire-sale returns (high model fire-sale risk) should have comparatively higher expected returns. Assuming average realised return is a proxy for expected return, that implies low model fire-sale returns should be associated with high excess realised returns. If model fire-sale return is a priced risk, we would expect to find a negative and significant coefficient on the lagged fire-sale return, that is, $\beta_t^{FS} < 0$ in the cross-sectional predictive regression below.

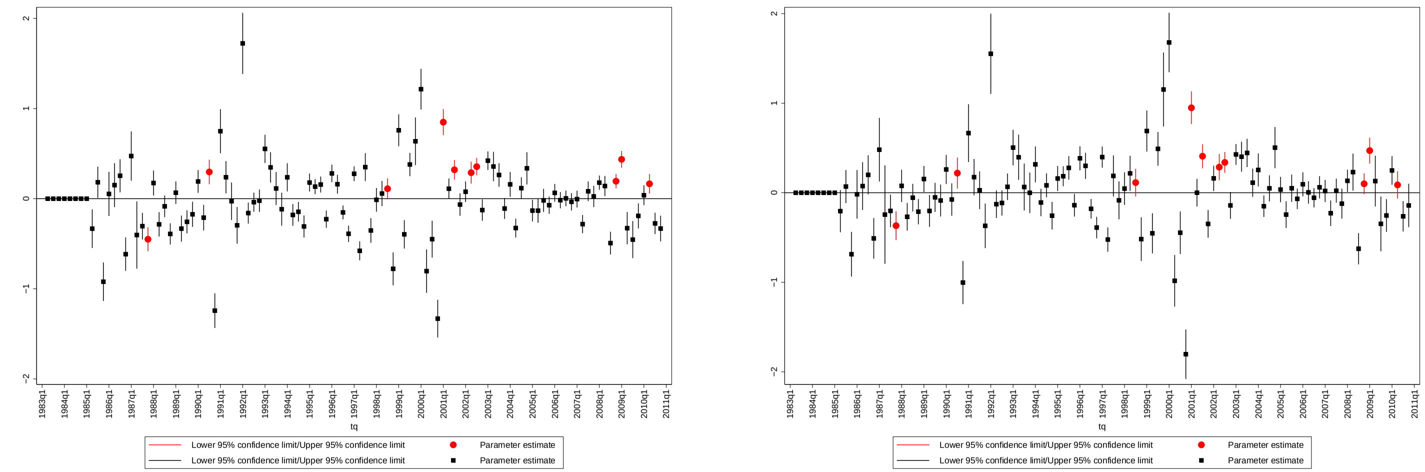
$$r_{j,t} = \alpha + \beta_t^{FS} r_{j,t-1}^{FS} + [controls_{j,t-1}] + \varepsilon_{j,t}$$

As in the conditional predictability section, in different specifications I control for the Fama & French characteristics as well as Amihud illiquidity, historical volatility and weighted average investor debt. The results are set out in Figure 2.5.1. To aid interpretation, the coefficients associated with market distress quarters are indicated by a large dot, while routine quarters are indicated by a square. A cursory examination reveals that there appears no consistent or dominant pattern in the cross-sectional lagged model fire-sale return coefficient. Although the model fire-sale return coefficient is mostly significant in specification 1 ($r_{j,t} = \alpha + \beta_t^{FS} r_{j,t-1}^{FS} + \varepsilon_{j,t}$), the sign of the coefficient is unstable. In specification 4 I include all the controls – when I do so far fewer lagged model fire-sale return coefficients remain significant. Among those that do remain significant there does not appear to be a dominant sign. On the whole the cross-sectional evidence does not support the notion of model fire-sale returns as a priced risk.

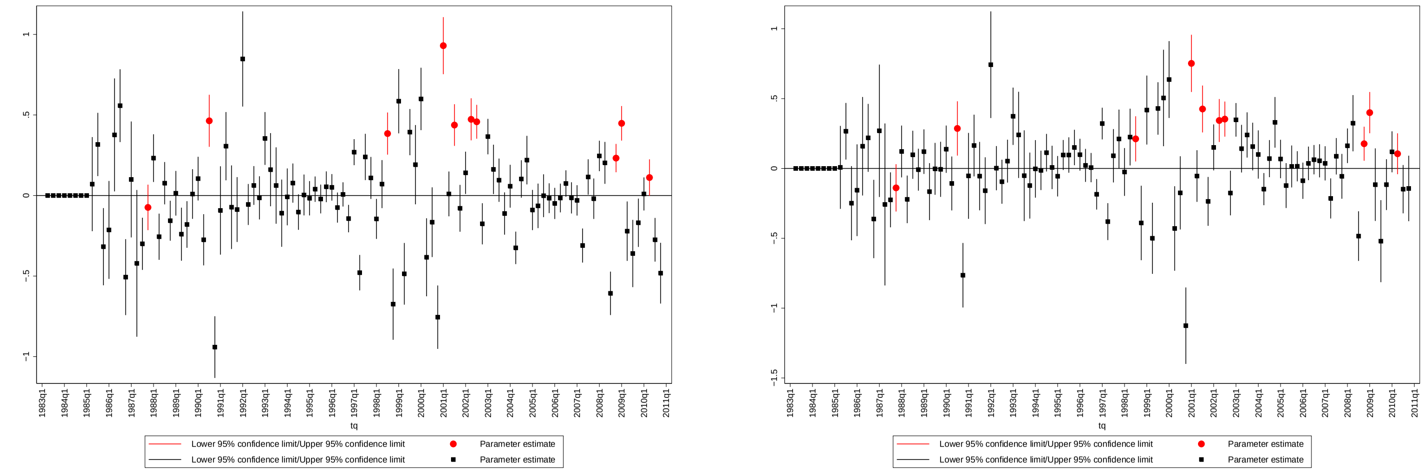
Figure 2.5.1: β_t^{FS}: Unconditional cross-sectional coefficient of $r_{j,t}$ regressed on $r_{j,t-1}^{FS}$

The charts below plot the coefficient β_t^{FS}, estimated by regressing the lagged model fire-sale return $r_{j,t-1}^{FS}$ of each stock on the cross-section of realised excess stock returns $r_{j,t}$ (Specification 1). In Specification 2 I additionally control for the Fama and French factors; the CAPM beta (estimated over a 60-month rolling window), size (in billions) and the book to market ratio – $[FF3] = \beta_t^{capmbeta} capmbeta_{j,t-1} + \beta_t^{size} size_{j,t-1} + \beta_t^{booktomarket} b2m_{j,t-1}$. In Specification 3 I control for other stock characteristics that might be expected to predict returns – Amihud illiquidity ($ailliq_{j,t-1}$), realised daily return volatility in the quarter ($volatility_{j,t-1}$) and weighted average debt of investors holding the stock ($wadebt_{j,t-1}$) – $[OTHER] = \beta_t^{ailliq} ailliq_{j,t-1} + \beta_t^{volatility} volatility_{j,t-1} + \beta_t^{wadebt} wadebt_{j,t-1}$. Specification 4 combines the controls of Specification 2 and Specification 3. Distressed quarter observations are indicated with a (red) dot instead of a cross. All controls are lagged one quarter (so are measured at the same time as the model fire-sale price) and are winsorized at the 1% level to account for possible outliers. I report t-statistics based on White (1980) heteroskedasticity adjusted standard errors. Significance levels are indicated with * (p<0.1), ** (p<0.05) and *** (p<0.01).

Specification 1: $r_{j,t} = \alpha + \beta_t^{FS} r_{j,t-1}^{FS} + \varepsilon_{j,t}$ Specification 2: $r_{j,t} = \alpha + \beta_t^{FS} r_{j,t-1}^{FS} + [FF3] + \varepsilon_{j,t}$



Specification 3: $r_{j,t} = \alpha + \beta_t^{FS} r_{j,t-1}^{FS} + [OTHER] + \varepsilon_{j,t}$ Specification 4: $r_{j,t} = \alpha + \beta_t^{FS} r_{j,t-1}^{FS} + [FF3] + [OTHER] + \varepsilon_{j,t}$



2.5.3 Panel data tests

Next I turn to panel data tests. The basic regression specification is

$$r_{j,t} = \alpha + \beta^{FS} r^{FS}_{j,t-1} + [controls_{j,t-1}] + \varepsilon_{j,t}$$

If model fire-sale risk is a priced risk factor, then the coefficient β^{FS} should be negative and significant. (A *low* fire-sale return implies *high* fire-sale risk and thus ought to be compensated for by a *high* realised return on average.)

As before, I consider alternate specifications in which I control for the Fama & French characteristics (CAPM beta, size, book-to-market) as well as Amihud illiquidity, historical volatility and weighted average investor debt. I also consider the effect of adding a dummy variable (*crash*), set equal to 1 if the S&P500 index decreased by more than 10% in a given quarter. Finally, I include an interaction term consisting of the product of the lagged model fire-sale return and the crash dummy variable. The intent is to better understand how return predictability is conditioned on market distress. After including these terms the specification becomes:

$$r_{j,t} = \alpha + \beta^{FS} r^{FS}_{j,t-1} + [controls_{j,t-1}] + \beta^{Crash} crash_t + \beta^{FSCrash} \left(crash_t \times r^{FS}_{j,t-1} \right) + \varepsilon_{j,t}$$

I follow the guidance in Petersen (2008), adjusting standard errors by clustering along both time and stock dimensions using the method set out in Thompson (2011) (see also Cameron, Gelbach and Miller (2011) for a more general discussion of multi-way clustered standard errors). The results are set out in Table 2.10.

I find that the lagged model fire-sale return coefficient is not significant in any of the first four specifications, and in addition is positive rather than negative as would be expected if model fire-sale returns were priced as a risk factor. The next four specifications include the crash dummy variable and the crash×fire-sale return interaction term. The model fire-sale return is negative in all four specifications and significant (at the 5% level) in three of the specifications – consistent with model fire-sale risk being priced. At the same time, the crash×fire-sale return interaction term is positive and significant (at the 10% level) in all four specifications – consistent with the earlier evidence that model fire-sale risk predicts realised returns conditional on market distress.

Taken together, the results suggests that model fire-sale risk *might* be priced (the negative coefficient for $r^{FS}_{j,t-1}$ in the last four specifications) but that the premium investors demand is insufficient to compensate fully for the actual losses incurred during market downturns (the positive coefficient for the $crash_t \times r^{FS}_{j,t-1}$ interaction term). This could explain why the model fire-sale return is negative and not significant in the first four specifications, in which I do not explicitly control for the crash dummy or the crash×fire-sale interaction term.

In short, the panel data evidence for model fire-sale return as a priced risk factor is suggestive but not persuasive.

Table 2.10: Panel regression of realised return on lagged model fire-sale return and lagged controls

This table shows the results of panel data regressions at the individual stock level. The basic regression, applicable to specifications (1) through (4), is $r_{j,t} = \alpha + \beta^{FS} r^{FS}_{j,t-1} + [controls_{j,t-1}] + \varepsilon_{j,t}$. I regress stock realised excess return ($r_{j,t}$) on the previous quarter model fire-sale return for that stock ($r^{FS}_{j,t-1}$). Specifications (2) to (4) add stock-level controls to specification (1). (The control variables are discussed in detail in section 2.3.) Specification (5) extends specification (1) by adding the crash dummy variable ($crash_t$), equal to 1 if the S&P500 decreased by 10% or more in the quarter and otherwise is zero. I also interact the lagged model fire-sale return ($r^{FS}_{j,t-1}$) with the crash dummy ($crash_t$) in specification (5). Specifications (6) to (8) add controls to specification (5). Thus the regression model for specifications (5) to (8) can be summarised as $r_{j,t} = \alpha + \beta^{FS} r^{FS}_{j,t-1} + [controls_{j,t-1}] + \beta^{Crash} crash_t + \beta^{FSCrash}(crash_t \times r^{FS}_{j,t-1}) + \varepsilon_{j,t}$. All controls are lagged by one quarter relative to the dependent variable and winsorized at the 1% level to account for the effect of possible outliers. Significance levels are indicated with * (p<0.1), ** (p<0.05) and *** (p<0.01) and are based on 2-way clustered standard errors as per Thompson (2011), following the guidance in Petersen (2008).

	(1) spec1 b/t	(2) spec2 b/t	(3) spec3 b/t	(4) spec4 b/t	(5) spec1c b/t	(6) spec2c b/t	(7) spec3c b/t	(8) spec4c b/t
r^{FS}_{t-1}	-0.059	-0.015	-0.094	-0.063	-0.206**	-0.165	-0.265***	-0.242**
	(-0.50)	(-0.11)	(-0.78)	(-0.47)	(-2.10)	(-1.53)	(-2.66)	(-2.20)
$capmbeta_{t-1}$		-0.000		-0.004		-0.001		-0.008
		(-0.07)		(-0.69)		(-0.12)		(-1.47)
$mktcap_{t-1}$		-0.001***		-0.001**		-0.001***		-0.000
		(-2.70)		(-2.23)		(-2.68)		(-1.42)
$b2m_{t-1}$		0.002***		0.002***		0.002***		0.002***
		(2.97)		(2.91)		(3.72)		(3.69)
$ailliq_{t-1}$			699.021***	584.411**			560.371**	411.899*
			(2.71)	(2.56)			(2.22)	(1.91)
$volatility_{t-1}$			0.207	0.326			0.454	0.689*
			(0.52)	(0.73)			(1.19)	(1.72)
$wadebt_{t-1}$			-0.001	0.001			-0.010	-0.009
			(-0.02)	(0.06)			(-0.45)	(-0.44)
$crash_t$					-0.139***	-0.130**	-0.129**	-0.129**
					(-2.77)	(-2.32)	(-2.52)	(-2.39)
$r^{FS}_{t-1} \times crash_t$					0.437*	0.450*	0.513**	0.504*
					(1.89)	(1.67)	(2.17)	(1.96)
Const	0.021	0.034*	0.008	0.016	0.021	0.031*	0.000	0.006
	(1.28)	(1.89)	(0.33)	(0.53)	(1.35)	(1.94)	(0.01)	(0.23)
Adj R-sqr	0.0001	0.0008	0.0028	0.0038	0.0338	0.0359	0.0374	0.0406
N	726,686	420,822	670,151	420,815	726,686	420,822	670,151	420,815

2.5.4 Sorted portfolio tests

An alternative approach for testing whether model fire-sale return is a priced risk is to create quintile portfolios sorted on the lagged model fire-sale return. Below I show the next period portfolio returns for portfolios sorted on the model fire-sale return in the current period. The mean realised returns for each of the sorted portfolios are tabulated in Table 2.11. Recall that portfolio 1 refers to the portfolio formed from the 20% of stocks with the most negative model fire-sale returns (therefore, these are the "fire-sale risky" stocks); similarly portfolio 5 is formed from the 20% of assets with the least negative model fire-sale returns (and thus these are the "fire-sale safe" stocks). If model fire-sale risk is priced, we would expect investors to require an additional premium for holding those stocks. Again assuming realised returns proxy for expected returns on average, this should result in portfolio 1 having the highest mean realised return and portfolio 5 having the lowest mean realised return. Thus, the model fire-sale return high-minus-low portfolio should generate a negative return if model fire-sale return is a priced risk. This is not what I find; instead, the model fire-sale return high-minus-low portfolio mean return is positive for equal weighted and market weighted portfolios, as is evident in Table 2.11 (column "mean")

Table 2.11: Lagged fire-sale return quintile portfolio excess returns

I report statistics relating to model fire-sale return portfolios, calculated using both equal weights and market weights. The high-minus-low model fire-sale return portfolio is formed by going long the 20% of stocks with the highest model fire-sale returns and shorting the 20% of stocks with the lowest model fire-sale returns in the *prior* quarter. This table is based on the full dataset. The last four columns report the alpha of time series regressions controlling for common time-series risk factors. The time series regressions are in the form $r_t^{FireSaleHML} = \alpha + [TimeSeriesControls_t] + \varepsilon_t$; note that $r_t^{FireSaleHML}$ denotes the realised return of the model fire-sale portfolio in the current quarter t but is formed on the basis of the model fire-sale return in the prior quarter $t-1$. "CAPM-α" controls for the contemporaneous market excess return, "3F-α" additionally controls for the Small-minus-Big and High-minus-Low portfolio returns of Fama and French (1993), "4F-α" additionally controls for the Up-minus-Down momentum portfolio (as described in Carhart (1997)). Finally, "All-α" controls for all the controls in 4F-α as well as long-term reversal, short-term reversal and Pástor and Stambaugh (2003) liquidity innovations. Significance levels are calculated using Newey West HAC adjusted standard errors and are indicated with * (p<0.1), ** (p<0.05) and *** (p<0.01).

Equal weighted lfsreturn HML portfolio timeseries statistics

	mean	sd	n	skew	min	CAPM-α	3F-α	4F-α	All-α
1	0.0299***	0.1117	111	-0.1626	-0.3112	0.0086	0.0055***	0.0094***	0.0104***
2	0.0310***	0.1177	103	-0.0218	-0.2914	0.0086	0.0068***	0.0110***	0.0113***
3	0.0313***	0.1248	103	0.0456	-0.3095	0.0084	0.0077***	0.0099***	0.0091***
4	0.0300***	0.1200	103	0.0579	-0.3061	0.0091	0.0080***	0.0110***	0.0100***
5	0.0313***	0.1182	100	0.0660	-0.3204	0.0138	0.0123***	0.0116**	0.0096*
hml	0.0009	0.0716	100	0.5457	-0.1972	0.0045	0.0063	0.0014	-0.0014

Market weighted lfsreturn HML portfolio timeseries statistics

	mean	sd	n	skew	min	CAPM-α	3F-α	4F-α	All-α
1	0.0467***	0.0878	111	-0.2203	-0.2256	0.0292***	0.0285***	0.0286***	0.0294***
2	0.0515***	0.0882	103	-0.1676	-0.2137	0.0337***	0.0340***	0.0341***	0.0335***
3	0.0694***	0.1327	103	3.0010	-0.2330	0.0475***	0.0532***	0.0439***	0.0437***
4	0.0598***	0.1060	103	0.9203	-0.2268	0.0402***	0.0420***	0.0390***	0.0382***
5	0.0609***	0.1135	100	2.4531	-0.2128	0.0439***	0.0459***	0.0375***	0.0372***
hml	0.0146*	0.0726	100	3.5918	-0.1424	0.0150**	0.0175**	0.0089*	0.0084

The sorted portfolio approach might be misleading if the sorting variable (in this case lagged model fire-sale risk) is correlated with other potential pricing factors. In that case the sorted portfolios might have realised returns that appear to be due to the sorting variable while the difference in realised returns could in fact be explained by differences in some unmodelled risk factor between the portfolios. I deal with this possibility by conducting a time-series regression of the model fire-sale high minus low portfolio returns on a range of standard time series risk factor returns. First, I control for the market excess return using the specification

$$r_t^{FireSaleHML} = \alpha + \beta^{MKT} MKT_t + \varepsilon_t$$

where $r_t^{FireSaleHML}$ represents the excess return of the model fire-sale high minus low portfolio. The alpha is positive for equal weighted and market weighted portfolios (see column "CAPM-α" in Table 2.11). Because a priced risk factor should generate a negative alpha for our portfolio, this does not support a priced risk interpretation. In addition, I control also for the Fama and French three factor model ("3F-α" in Table 2.11) and the Carhart four factor model ("4F-α" in Table 2.11). Again, alphas are positive for equal weighted and market weighted portfolios. Finally, column "All-α" in Table 2.11 controls for all four of the Carhart factors and long-term reversal, short-term reversal and Pástor and Stambaugh (2003) liquidity innovations. The alpha is negative for equal weighted portfolios, but at 0.14% is both economically and statistically insignificant. For market weighted portfolios, the alpha is positive.

Based on the sorted portfolio evidence model fire-sale risk does not appear to be a priced risk; at the very least, any premium does not appear to compensate, on average, for realised losses incurred in episodes of market distress. I conclude that there is insufficient evidence to support the hypothesis that model fire-sale return is a priced risk factor.

2.6 Conclusion

Two stylised facts emerge from the evidence presented in this chapter. First, model fire-sale return appears to be a valid proxy for fire-sale risk. It positively predicts realised returns conditional on market distress. Second, although fire-sale risk appears real, I am unable to find substantial evidence of it being priced in the market. There are several reasons why this might be the case. Participants in the market might not be able to assess fire-sale risk at the individual asset level (perhaps they are not aware of it; perhaps the cost of estimating the risk outweighs the perceived benefits). Or possibly fire-sale risk is priced, but we lack sufficient data to detect the effect, which would suggest that even if priced, the "market price of fire-sale risk" might be small.

Of equal importance is the finding that the pattern of asset holdings and investor constraints, as incorporated in the model fire-sale return, translates into realised returns during times of market distress. This has implications for investors, risk managers and regulators. Investors need to be aware that, in a distressed market, much of the price dynamics of assets might be driven by ownership patterns and investor constraints. It appears that, historically, this risk has not been priced. But perhaps it should be? Similarly, risk managers might want to reflect on the possibility that the overall "riskiness" of an asset might be determined not only by the inherent riskiness of the asset, but also by the "riskiness" of the investors holding that asset. This argues for vigilance when investing in "crowded" asset classes or trading strategies, particularly those that hold appeal to leveraged investors. Finally, regulators have an interest in preserving the integrity of markets – which includes reducing the frequency and severity of fire-sale episodes. The findings in this chapter suggest that a combination of overlapping holdings and binding investor constraints (such as might be induced by excessive leverage) can exacerbate market downturns and lead to additional fire-sale driven losses. Markets might be made more robust by policy measures that reduce forced sales by investors during market downturns. For instance, lower leverage, higher liquidity buffers, longer term financing and redemption lockout provisions might all help reduce forced investor selling in market downturns.

2.7 Chapter 2 Appendix

2.7.1 Thompson Reuters 13F data problems

There are several problems that need to be addressed before the 13F data can be used. I summarise here the steps I took to address these problems.

Manager number's are recycled

Thompson Reuters reuses manager numbers. This means that manager numbers cannot be relied on to uniquely identify an investor through time. However, Brian Bushee's website[15] provides hand-crafted permanent id's (at an annual frequency), matched to the Thomson Reuters manager numbers. This allows researchers to substitute permanent id's for the unreliable Thompson Reuters manager numbers.

Legacy CUSIP

Thompson Reuters uses legacy CUSIP's - so where CUSIP's have changed, this is not reflected in the records. This means the Thomson Reuters CUSIP cannot be used to match to other datasets where the CUSIP identifiers are kept current (such as CRSP). To handle this I use the historical CUSIP in CRSP – this can be used to link to the current CUSIP, which can then be used to link to CRSP market data.

Manager / Quarter combination can have several entries

This reflects the fact that managers can file updates and corrections for a given reporting date. Therefore, for a given combination of Manager and reporting quarter (RDATE), there might be several filings (FDATE entries). I use only the first filing date holdings. Usually the first filing will contain the bulk of the holdings data, with subsequent filings used to correct mistakes.

Stock splits are not correctly accounted for

RDATE is the actual reporting date, while FDATE is the date that the report is filed. If a stock split happens between RDATE and FDATE, holdings will be misstated. To deal with this I use the CRSP shareholdings as of FDATE.

Reported prices for the same CUSIP are not the same across all managers in the same quarter

Different prices may be recorded in the 13F data set for the same assets at the same point in time by different managers. Note, if an asset did not trade, the manager has some discretion as to where the asset is marked. To ensure consistent prices, I use the

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[15] http://acct.wharton.upenn.edu/faculty/bushee/IIclass.html

CRSP closing price instead of the individually disclosed prices as per the Form 13F fillings.

2.7.2 Investor debt estimation

The approach is to impute a level of “debt” to each investor based on their historical behaviour. The drawback of this approach is that there is no guarantee the imputed level of debt has any bearing on actual debt levels. That said, the purpose of debt in the theoretical model is to regulate the degree to which a negative portfolio return is translated into fractional asset sales. Because we are modelling behaviour, perhaps it is not unreasonable to calibrate our model to past behaviour. Of course, we are likely to end with fairly noisy estimates of debt. The benefits of this approach is that it can be implemented with the data available, it generates a debt measure that varies across time and cross-sectionally and it sidesteps problems that otherwise bedevil estimates of leverage (such as embedded leverage in derivatives such as swaps, futures, options, high-beta stocks). In addition it encompasses behaviour consistent with (but distinct from) leverage, such as sales driven by investor fund withdrawals, changes in funding conditions, etc.

Going back to the theory, investors must sell a sufficient fraction of their portfolio to re-establish their original leverage, that is $L_i(2) = L_i(0)$

or, recalling that $L_i(0) = \frac{d_i}{A_i}$ and $L_i(2) = \frac{d_i - f_i^s H_i}{(1-f_i^s)H_i}$

$$\frac{d_i - f_i H_i}{H_i - f_i H_i} = \frac{d_i}{A_i}$$

Recall that $H_i \equiv \sum_{j=1}^{M} h_{i,j}(0)p_j(1)$ is the value of the investor’s *old* (pre-sales) portfolio at *new* prices and $A_i \equiv \sum_{j=1}^{M} h_{i,j}(0)p_j(0)$ is the original value of the investor’s portfolio at the old prices. Rewrite by introducing G_i as the total value realised from selling assets at the new prices $p_j(1)$

$$G_i \equiv f_i H_i = \sum_{j=1}^{M} \left(h_{i,j}(1) - h_{i,j}(0)\right) p_j(1)$$

Therefore, we have

$$\frac{d_i - G_i}{H_i - G_i} = \frac{d_i}{A_i}$$

Solving for the level of debt d_i we obtain

$$d_i = \frac{G_i A_i}{A_i - H_i + G_i}$$

In the rest of the discussion we drop the investor subscript i (it being understood that we are in each case estimating the debt of a specific investor) and add a time subscript t to emphasise the time dependence of our calculated debt d_t.

$$d_t = \frac{G_t A_t}{A_t - H_t + G_t}$$

I then estimate $\hat{d_t}$ by taking the average of d_t over a rolling 16 quarter window[16].

$$\hat{d_t} \equiv \frac{1}{16} \sum_{\tau=1}^{16} d_{t-\tau+1}$$

The calculated $\hat{d_t}$ theoretically range from negative infinity to positive infinity. As a practical matter, I use the truncated debt $\overline{d_t}$ such that debt is truncated at zero and 90% of the investor's assets[17].

$$\overline{d_t} = max(0, min(0.9A_i, \hat{d_t}))$$

2.7.3 Liquidity estimation

In the broadest sense, the liquidity of a security relates its change in price (linearly) to the change in quantity supplied. I equate "quantity supplied" with the volume traded in the market. I start with the notion of liquidity as a price elasticity, given by the basic relationship

$$\delta_j = -\frac{\%\Delta Q_j}{\%\Delta p_j}$$

Note that $\%\Delta p_j$ is just the one-period price return of asset j, which we denote as r_j. Because all initial prices are normalised to unity, it follows that the associated trading volume should be measured in dollars. Therefore, the change in quantity can be interpreted in this context as the dollar volume of shares that traded, so $\Delta Q_j = DollarVolume_j$ and by extension, $\%\Delta Q_j = \frac{DollarVolume_j}{MarketCap_j}$. This yields (after some rearranging)

$$\delta_j = -\frac{r_j}{DollarVolume_j} \times \frac{1}{MarketCap_j}$$

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[16] In robustness checks I also consider an 8 quarter (2 year) rolling window and a 24 quarter (6 year) rolling window.

[17] It seems reasonable to insist that there is no such thing as negative debt. In addition, if an investor is presently part of the set of investors, it appears reasonable to assume such an investor is not currently bankrupt.

To obtain an estimate of the liquidity parameter I take a rolling window average of the last 250 trading days (approximately one year)[18]. Dropping the asset subscript j and introducing a time series subscript t yields

$$\hat{\delta}_t \equiv \frac{1}{250}\sum_{\tau=1}^{250}\delta_{t-\tau+1}$$

I then truncate these estimates to lie within the range $[1.1, 10]$. (The theoretical model requires $\delta_j \geq 1$)

$$\overline{\delta_t} = max(1.1, min(10, \hat{\delta}_t))$$

2.7.4 Monte Carlo study

In this sub-section I consider the "size" of the conditional predictability tests conducted in section 2.4. In short, I conduct a Monte Carlo study to create a distribution of test outcomes under the null hypothesis that realised returns are random noise uncorrelated with any explanatory variables. I then compare the results of the tests obtained using real data to see if it is significantly different from that expected under the null.

Given that all the tests are conditional on market distress, I model the uncorrelated random returns by drawing them from a normal distribution with the same mean and standard deviation as those of realised excess return in downturns. (In downturns the mean realised excess return is -15.25% with a standard deviation of 0.2954.) Our simulated returns under the null hypothesis then follows:

$$r_i = \varepsilon,\, \varepsilon \sim N(\mu = -0.15, \sigma = 0.29)$$

In each simulation I replace the actual realised excess returns in the full sample of downturn quarters (comprising of 73,666 observations) with returns drawn from the normal distribution described above. I then proceed to perform the tests in exactly the same manner as before. I complete a total of 1,000 simulations: the results are set out in table 2.12. For comparison I also include the equivalent metrics obtained by applying the same tests to actual data. As demonstrated in the table, the actual results exceed the 95th percentile of simulated results by a wide margin. This provides evidence that the tests of section 2.4 has sufficient size to reject the null hypothesis.

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[18] Since the supply of stock could in principle exceed the actual amount traded (volume) the estimated liquidity parameter δ_j might be under-estimated. However, I show in the robustness section that the main results continue to hold even if we use a constant liquidity parameter.

Table 2.12: Monte Carlo study results

	Cross-section coefficients (count of 10)				Portfolio returns by quintile					monotonic?	HML Avg	HML Positive
	Spec. A	Spec. B	Spec. C	Spec. D	P1	P2	P3	P4	P5			
Simulations												
Mean	0.545	0.525	0.504	0.545	-15.1%	-15.0%	-15.1%	-15.1%	-15.0%	0.001	0.00%	4.949
SD	0.728	0.732	0.697	0.735	0.3%	0.3%	0.3%	0.3%	0.2%	0.032	0.36%	1.595
Min	0	0	0	0	-15.9%	-15.8%	-15.8%	-15.9%	-16.1%	0	-1.34%	0
Max	4	5	4	5	-14.1%	-14.1%	-14.3%	-14.3%	-14.3%	1	1.15%	9
5th pctile	0	0	0	0	-15.5%	-15.5%	-15.5%	-15.5%	-15.5%	0	-0.63%	2
95th pctile	2	2	2	2	-14.6%	-14.6%	-14.7%	-14.6%	-14.6%	0	0.58%	8
Real data												
Actual data	9	7	9	8	-18.0%	-17.3%	-16.7%	-14.6%	-10.9%	1	7.07%	9
Actual > 95pct	Yes	Yes	Yes	Yes						Yes	Yes	Yes

Chapter 3

Extreme Stocks

Abstract

Autocorrelation of log stock prices predicts stock returns in US data. Stocks with autocorrelation coefficients substantially above or below unity outperform the market in the subsequent month by 40-50bp. This outperformance is robust to a wide range of previously documented risk factors and stock characteristics, including the Fama & French factors, long and short-term reversal and momentum. The outperformance of stocks with autocorrelation substantially below unity is highly persistent, earning statistically significant excess returns in every subsequent month up to 25 years post-formation. Analysis of returns around earnings announcements suggest that this persistence cannot be attributed to biased investor expectations.

3.1 Introduction

Autocorrelation of log-prices predicts future realised returns in US stocks. The top and bottom deciles of stocks sorted on prior period autocorrelation[1] generate mean excess returns of 125 basis points and 118 basis points per month respectively, compared with a mean excess return of 71 basis points for the remaining middle quintiles. This outperformance is statistically significant and remains so after controlling for standard time series risk factors such as the Fama & French factors, long and short-term reversal, liquidity and momentum. I find similar statistical and economic significance in predictive panel data regressions at the individual stock level, even after controlling for up to 12 stock-level characteristics in various specifications. Intriguingly, this outperformance is highly persistent for stocks with low autocorrelations but not for stocks with high autocorrelations. On average the lowest decile autocorrelation stocks continue to outperform in *every* post-formation month, up to a horizon of 25 years. This remains the case even after controlling for the common time-series risk factors mentioned above. Such a persistent return premium is suggestive of a potential priced risk factor not captured by the other standard time series risk factors. However, preliminary work has not uncovered a convincing relationship with macroeconomic state variables that might motivate such an interpretation.

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[1] In this paper, "autocorrelation" means the first order autoregressive coefficient estimated on the natural logarithm of prices using the time series OLS regression $ln(P_t) = \beta ln(P_{t-1}) + \varepsilon_t$.

This chapter is inspired by recent work on the dating of financial asset bubbles. In particular, Phillips, Wu and Yu (2011) (hereafter PWY) develop a new econometric approach for dating the origination and collapse of asset bubbles. In doing so the authors are able to accurately date the bubble phase in the NASDAQ index as originating in July 1995 and collapsing in March 2001. In a related paper Phillips and Yu (2011) apply a similar approach to dating bubbles in house prices, crude oil prices and bond spreads in the run-up to the sub-prime crises.

If it is indeed possible to accurately identify bubbles in real time it raises the question whether such knowledge can be used to engage in profitable trading. This intuition is pursued by Guenster, Kole and Jacobsen (2012); their bubble signal is based on prolonged and accelerating deviations of realised industry returns from the returns predicted by standard models such as the Fama & French 3-factor model (Fama and French, 1993) and the Carhart 4-factor model (Carhart, 1997). They show that "riding bubbles", as they term it, is a profitable strategy when applied to US industry portfolio returns, even after allowing for a reasonable degree of investor risk aversion.

I consider whether an approach in the spirit of PWY can be used to predict returns at the individual stock level. In so doing I deviate from PWY in a number of important respects. First, the motivation in PWY is to conduct statistical inference on a particular price time series with a view to date the inception and collapse of a bubble. By contrast I do not consider whether the autocorrelations I estimate at the individual stock level are statistically significant; instead, I am interested in whether such point estimates of autocorrelation have predictive power in the cross-section of stock returns. Second, I deviate from PWY in that my estimate of autocorrelation corresponds to a substantially simpler equivalent unit root test than is used by PWY.

It is tempting to argue that the predictive power of autocorrelation is based on some underlying economic foundation. For instance, one might argue that deviations from the canonical random walk paradigm might signal to investors that such (deviant) stocks are somehow unusual and therefore more "risky". The problem is that it is difficult to see how a simple economic story can simultaneously account for the fact that upward deviations from the pure random walk norm are transient while downward deviations are highly persistent. This suggests that the true economic explanation may in fact be different for upwards and downwards deviations from the pure random walk norm. Rather than trying to fit a "pro-forma" theory to match the empirical findings in this Chapter, I instead opt to delve deeper into the characteristics of those stocks with estimated autocorrelations outside the normal range. As I discuss in sub-section 3.3.4, stocks with relatively low autocorrelation look a lot like "distressed value" stocks while stocks with relatively high autocorrelation look a lot like "risky growth" stocks. This intuition may help guide further research into the ultimate economic foundation of the predictability documented here.

The approach taken in this chapter is therefore decidedly empirical. In this respect I am following a well-established tradition in the finance literature of documenting empirical "puzzles", "anomalies" and "regularities". Much of this empirical finance literature is not concerned with direct statistical tests of specific theoretical models. Nonetheless, these findings highlight features in the data not accounted for by existing theory. It also serves as an impetus for the development of new theory, capable of explaining such anomalies and regularities. One of the premier examples in the empirical finance literature is momentum, as first documented by Jegadeesh and Titman (1993). In documenting the momentum effect Jegadeesh and Titman (1993) did not rely on hypothesis derived from any particular theory. But their work has motivated a large body of subsequent theoretical literature that seeks to explain their results. A more recent example of this empirical approach is the work of Han and Zhou (2012). In their article, the authors show that a trend factor – based on a signal derived from historical price moving averages – has explanatory power in the cross-section and is robust to the usual factors including momentum.

The relevance of the momentum effect to this paper is conceptual rather than empirical; I show that the predictive power of autocorrelation is independent of, and robust to, momentum.[2] The conceptual similarity between autocorrelation and momentum derives from the fact that both factors can be estimated directly from past price history with relative ease. Whereas momentum is essentially the average geometric return based on past prices, my estimate of autocorrelation is a measure of the deviation from a random walk in logged prices.

The rest of this chapter is set out as follows: in section 3.2 I define autocorrelation, section 3.3 describes the data, section 3.4 presents sorted portfolio evidence while section 3.5 presents panel data evidence. In section 3.6 I present the evidence on return persistence for low autocorrelation stocks. Section 3.7 discusses the results of various robustness checks, while section 3.8 concludes. The Appendix is contained in section 3.9.

3.2 Autocorrelation

In this section I define the term autocorrelation as used in this paper; I also explain how autocorrelation is related to unit root tests, random walks and exponential price growth.

---

[2] As shown in Table 3.2, the pair-wise correlation between autocorrelation and 6-month momentum is 0.15, which suggest that they constitute distinct empirical effects.

Let P_t be the price of some asset at discrete time intervals indexed by t and denote by $p_t \equiv ln(P_t)$ the natural logarithm of the price of such an asset at time t. Now consider some time-series window such that we cover w time periods[3] starting at t_{k-w+1} and ending at t_k inclusive, such that $t_{k-w+1} \leq t \leq t_k$. The autocorrelation coefficient ρ over that window can then be estimated by the following OLS regression

$$p_t = \rho p_{t-1} + \varepsilon_t \quad \text{for} \quad t_{k-w+1} \leq t \leq t_k \tag{3.2.1}$$

Because this is a univariate OLS regression without an intercept, the autocorrelation coefficient ρ can be calculated directly using the following formula (Hamilton (1994), page 475)

$$\widehat{\rho} = \frac{\sum\limits_{s=t_{k-w+1}}^{t_k} p_{s-1}p_s}{\sum\limits_{s=t_{k-w+1}}^{t_k} p_{s-1}^2} \tag{3.2.2}$$

In the special case of $\rho = 1$ we obtain a pure random walk in log-prices:

$$p_t = p_{t-1} + \varepsilon_t$$

Therefore the estimated autocorrelation coefficient ρ can be interpreted as a measure of the degree to which a log-price series over a given window follows a pure random walk. If $\rho > 1$ then the price series will be increasing exponentially (and hence is termed *explosive*) while $\rho < 1$ implies exponential decay, both of which are deviations from a pure random walk. This insight is, of course, the basis of unit root testing. To see this consider the following regression specification derived from equation (3.2.1)

$$\begin{aligned} p_t &= \rho p_{t-1} + \varepsilon_t \\ p_t - p_{t-1} &= \rho p_{t-1} - p_{t-1} + \varepsilon_t \\ \Delta p_t &= (\rho - 1)p_{t-1} + \varepsilon_t \\ \Delta p_t &= \beta p_{t-1} + \varepsilon_t \text{ where } \beta \equiv \rho - 1 \end{aligned} \tag{3.2.3}$$

The regression specification derived above is in fact an augmented Dicky-Fuller (ADF) test for unit roots on the log-price time series where we assume a zero intercept, no deterministic time trend and no lags (see Dickey and Fuller (1979))[4].

---

[3] In estimating autocorrelation I use $w = 24$ or 24 monthly observations. I consider alternate windows and observation frequencies in the robustness section.

[4] To make this clear, consider the general specification of the ADF test for a time series $\{y_t\}$. The augmented Dicky-Fuller test fits the following regression using OLS: $\Delta y_t = \alpha + \beta y_{t-1} + \delta t + \zeta_1 \Delta y_{t-1} +$

Consistent with Phillips, Wu and Yu (2011) I use the natural logarithm of prices (or more concisely, log-prices). To see what this means for prices, it is instructive to re-cast equation (3.2.1) in pre-log prices P_t (ignoring the error term ε_t)

$$\begin{aligned} p_t &= \rho p_{t-1} \\ e^{p_t} &= e^{\rho p_{t-1}} \\ e^{ln(P_t)} &= e^{\rho ln(P_{t-1})} \\ P_t &= e^{ln(P_{t-1}^{\rho})} \\ P_t &= P_{t-1}^{\rho} \end{aligned}$$

Contrast this with the usual definition of a price return

$$P_t = (1+r)P_{t-1}$$

In other words the estimated autocorrelation coefficient ρ induces growth in prices P_t by raising the prior price P_{t-1} by a power equal to ρ. If $\rho = 1$ the price does not change - this corresponds to $r = 0$. Thus while conventional returns induce geometric growth in prices, the estimated autocorrelation coefficient ρ induces exponential growth in prices.

3.3 Data and descriptive statistics

My primary dataset consists of all US stocks contained in the CRSP monthly data file between 1926m1 and 2013m12 inclusive, comprising of 3,871,825 non-missing return observations. Prior to performing subsequent analysis (and consistent with standard practise in the empirical finance literature) I drop all observations where a) the security is not a domestic US common stock (CRSP share code 10 or 11) in the previous month, b) the price is below \$5 dollars at the end of the previous month and c) the market capitalisation of the security is below the 5% NYSE size percentile at the end of the previous month. This leaves 2,028,477 non-missing return observations.

3.3.1 Autocorrelation

To calculate autocorrelation I apply equation (3.2.2) to individual stocks over 24-month rolling windows[5]. Prior to doing so, I adjust all stock prices by dividing them by the

---

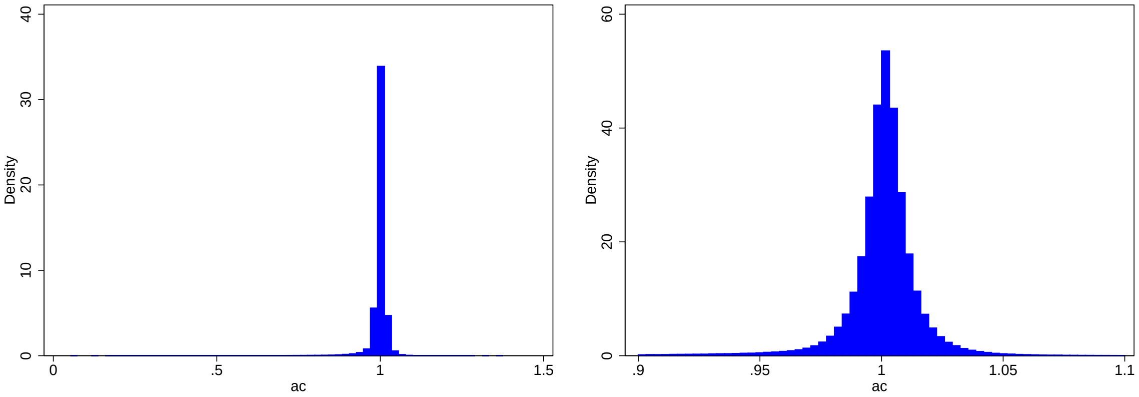
$... + \zeta_k \Delta y_{t-k} + \varepsilon_t$. Assume that there is no intercept ($\alpha = 0$), no deterministic time trend ($\delta = 0$) and zero lags ($\zeta_k = 0$ for all k), then the regression becomes $\Delta y_t = \beta y_{t-1} + \varepsilon_t$. Now if we assume the time series under consideration is the natural logarithm of the asset price such that $y_t = p_t = ln(P_t)$, then we obtain $\Delta p_t = \beta p_{t-1} + \varepsilon_t$ as per equation 3.2.3.

[5] I consider alternative windows and frequencies in the robustness section

CRSP cumulative factor to adjust for prices (CFACPR) – this eliminates price discontinuities because of stock-splits and other corporate actions. After the filtering described above, this yields 1,750,404 estimates of autocorrelation (sometimes abbreviated as "**ac**" in the results). These estimates start in January 1928 and end in June 2013 (inclusive). As expected, autocorrelation is centred on unity with a sample mean of 0.999 and standard deviation of 0.027; its distribution approximates the standard bell curve shape of the normal distribution (see Figure 3.3.1). Autocorrelation is itself highly auto-correlated, with pair-wise correlations around 0.959; this is not surprising, given that it is estimated by overlapping rolling-window regressions.[6] As a matter of convenience, I use the term *"explosive"* to describe stocks with a comparatively high level of autocorrelation (at an average of 1.017 in the top quintile) and the term *"decaying"* to describe stocks with a comparatively low level of autocorrelation (at an average of 0.973 in the bottom quintile).

Figure 3.3.1: Histogram of estimated autocorrelation

Histogram of autocorrelation ("ac"), as described in more detail in sub-section 3.3.3. The histogram on the right excludes all autocorrelations outside the interval [0.9, 1.1] in order to provide more detail on the central part of the distribution. This summary excludes any stocks below $5, stocks below the 5% NYSE cut-off and any stocks that are not common domestic US stocks (i.e., stocks that do not have a stock code of 10 or 11 in CRSP) in the prior month. Monthly estimates of autocorrelation start in January 1928 and end in June 2013 (inclusive).



3.3.2 Realised returns

The dependent variable in my tests of return predictability is the realised excess stock return ("**eret**"), which I calculate as the raw monthly CRSP total return (RET) less the monthly risk free rate (see sub-section "Time-series variables" below). In the robustness section I also consider alternate measures of realised return. These are raw total returns ("**ret**"), total returns excluding dividends ("**retx**"), excess returns excluding dividends ("**eretx**") and delisting adjusted excess returns ("**dleret**", calculated following the approach in Beaver, McNichols and Price (2007)).

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[6] In robustness tests I also calculate monthly autocorrelation using daily data in each month, so monthly autocorrelation in that setup is calculated using non-overlapping estimation windows. Then I find that autocorrelation has a pair-wise autocorrelation of 0.306.

3.3.3 Control variables

A summary of autocorrelation and the various control variables is presented in Table 3.1, hereafter referred to simply as "the dataset".

Table 3.1: Summary data

Summary data for excess returns ("eret"), autocorrelation ("ac") and the control variables described in more detail in sub-section 3.3.3. This summary excludes any stocks below $5, stocks below the 5% NYSE cut-off and any stocks that are not common domestic US stocks (i.e., stocks that do not have a stock code of 10 or 11 in CRSP) in the prior month. Note that both ac and the control variables are lagged by one month relative to excess returns, the dependent variable. The variables from capmbeta to coskew (inclusive) are winsorized at the 1% level to account for the influence of possible outliers. Monthly estimates of autocorrelation start in January 1928 and end in June 2013 (inclusive).

	count	mean	sd	min	max
ac	1,750,404	0.9986	0.0273	0.0522	1.3759
ret	2,028,477	0.0102	0.1266	-0.9813	6.4074
eret	2,023,094	0.0066	0.1267	-0.9830	6.4030
capmbeta	1,380,321	1.0859	0.5905	-1.6287	4.7171
size	2,036,592	1.2127	4.1678	0.0002	56.5078
b2m	1,368,809	0.7637	0.8267	0.0127	56.7900
momentum	1,949,745	0.0103	0.0492	-0.3627	0.4545
pe	1,393,013	14.1429	43.9122	-399.7877	456.1538
turnover	1,930,732	0.9117	1.4058	0.0000	48.7967
illiq	1,818,550	0.0000	0.0000	0.0000	0.0073
ivol	2,073,785	0.0212	0.0166	0.0000	0.3652
coskew	1,380,321	-0.0221	0.2491	-1.2437	1.6078
rating	376,234	9.8341	3.7377	1.0000	23.0000
prevret	2,014,953	0.0090	0.1250	-0.8940	7.9231
prev24mret	1,735,354	0.0136	0.0256	-0.1234	0.5856
l2price	1,954,986	38.4028	947.9842	5.0000	159,000.0000

The following is a concise description of the stock level control variables I use.

The CAPM beta ("**capmbeta**") is estimated using an OLS regression of monthly excess stock returns on the market excess return over rolling 60-month windows.

Size ("**size**") is the market capitalisation calculated as the end of month stock price times the number of shares outstanding, in billions of US dollars.

Book-to-market or "**b2m**" is the ratio of the book value of equity to the market capitalisation of the firm. To calculate book value, I follow the definition from Kenneth

French's website[7]. That is, I calculate the difference between quarterly total assets (ATQ)[8] and total liabilities (LTQ), subtract preferred and preference stocks (PSTKRQ) (if not missing), then add deferred taxes and investment tax credits (TXDITCQ) (again, if not missing). Finally, I set all negative resulting book values equal to missing.

Jegadeesh and Titman (1993) momentum is calculated as the monthly compounded total return over the previous six months, and is denoted, predictably, as **"momentum"** in the results.

The price-to-equity ratio or "**pe**" is calculated as the end of month stock price divided by the most recent earnings per share. I calculate earnings per share by dividing the sum of the trailing 4 quarters of income (IBQ) by the total number of shares issued as of the most recent quarter (CSHOQ).

Trading turnover ("**turnover**") is the monthly volume of shares traded (CRSP "VOL", in units) divided by the number of shares outstanding (CRSP "SHROUT") at the end of the month. In other words, it is the percentage of outstanding shares that traded in that month.

The Amihud measure of illiquidity or "**illiq**" (Amihud, 2002) is the average over the preceding 12 months of daily data (sourced from the CRSP daily file) of the ratio of the daily absolute return over the daily dollar volume of trading. More succinctly, $illiq = \Sigma \frac{|r_t|}{P_t \cdot volume_t}$. I discard any measures of illiquidity estimated with less than 120 daily ratios over the past 12 months.

Idiosyncratic volatility ("**ivol**") is calculated following Ang et al. (2006) as the standard deviation of the residuals of a Fama & French style OLS regression ($r_t - r_t^f = \beta^{MktRf} \cdot MktRf_t + \beta^{HML} \cdot SMB_t + \beta^{HML} \cdot HML_t + \varepsilon_t$) on daily excess returns in each month. I discard estimates of idiosyncratic volatility where I have fewer than 10 observations in a given month.

Following Harvey and Siddique (2000) coskewness ("**coskew**") is calculated over a rolling 60-month window as

$$coskew = \frac{E\left[\epsilon^{CAPM} \cdot \epsilon^{Market}\right]}{\sqrt{E\left[\left(\epsilon^{CAPM}\right)^2\right] \cdot E\left[\left(\epsilon^{Market}\right)^2\right]}}$$

where the $E[\cdot]$ operator represents the time-series mean over the window, ϵ_t^{CAPM} is the CAPM monthly OLS regression residual and $\epsilon_t^{Market} \equiv r_t^{Market} - E[r^{Market}]$ is the demeaned market return.

Ratings ("**rating**") are sourced from the COMPUSTAT long term S&P issuer rating ("SPLTICRM") and is transformed into an ordinal number such that "AAA" → 1, "AA+" → 2 and so on, all the way down to "D" → 23 (full default). Note that I am able to obtain ratings for only 376,234 observations (post filtering) – however, that accounts for approximately 70% of market capitalisation across all time periods and stocks in the data.

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[7] http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

[8] I include the relevant COMPUSTAT variables in parenthesis where relevant.

The previous period excess return is indicated by "**prevret**" and "**prev24mret**" denotes the average excess return over the previous 24 months.

Finally, "**l2price**" is the unadjusted[9] price of the stock, twice lagged. (Excess return, my dependent variable in tests of return predictability, is calculated directly by reference to the current price and the once-lagged price – to avoid econometric issues, I therefore use the twice-lagged price as a proxy for the general price level of a stock.)

The following variables are winsorized at 1%: capmbeta, size, b2m, momentum, pe, turnover, illiq, ivol and coskew. *In return predictability tests all explanatory and control variables in this chapter are lagged by one month relative to realised excess return, both in panel data regressions and sorted portfolio tests.* This allows me to characterise the relationship between the explained variable and the explanatory variables as "predictive" – it also mitigates endogeneity issues that might otherwise arise.

In Table 3.2 I present the correlation matrix of the variables discussed above. Excess returns ("eret") exhibit low correlation with both autocorrelation and the stock-level control variables. Autocorrelation is also, on the whole, weakly correlated with the control variables. The correlations between autocorrelation and the control variables are less than 10% in absolute terms, with the exception of momentum (0.15) and book-to-market (-0.13). (Again note that autocorrelation and all the stock-level control variables are lagged by one month relative to excess returns.)

3.3.4 Stock characteristics of decaying and explosive stocks

It might be instructive to consider how the different stock characteristics relate to autocorrelation. In Table 3.3 I summarise various stock characteristics after pre-sorting the dataset into autocorrelation quintiles. Doing so allow me to investigate relationships that might not be be revealed by simple pair-wise correlations. Quintile 1 stocks, which I label "decaying", consist of those stocks with the lowest 20% of autocorrelations in a particular month. These decaying stocks have an average autocorrelation of 0.973 versus a mean of 0.999 across all stocks. Quintile 5 stocks are labelled "explosive"; these are the 20% of stocks with the highest autocorrelations in a particular month. Excess returns ("eret") are highest for decaying and explosive stocks – a result I will discuss in more detail in the next section.

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[9] Note, unlike the prices used to estimate autocorrelation, this price is *not* adjusted for corporate actions using the CRSP cumulative factor to adjust prices ("CFACPR").

Table 3.2: Correlation matrix (pair-wise)

Pair-wise correlations for excess returns (“eret”), autocorrelation (“ac”) and the control variables described in more detail in sub-section 3.3.3. The data this table is based on excludes any stocks below $5, stocks below the 5% NYSE cut-off and any stocks that are not common domestic US stocks (i.e., stocks that do not have a stock code of 10 or 11 in CRSP) in the prior month. Note that both autocorrelation and the control variables are lagged by one month relative to excess returns, the dependent variable. The variables from capmbeta to coskew (inclusive) are winsorized at the 1% level to account for the influence of possible outliers.

	ac	ret	eret	capmbeta	size	b2m	momentum	pe	turnover	illiq	ivol	coskew	rating
ac	1.00												
ret	-0.01	1.00											
eret	-0.01	1.00	1.00										
capmbeta	-0.06	-0.01	-0.01	1.00									
size	0.04	-0.01	-0.00	-0.05	1.00								
b2m	-0.13	0.03	0.03	-0.06	-0.09	1.00							
momentum	0.15	0.01	0.01	0.00	0.02	-0.14	1.00						
pe	0.04	-0.00	0.00	-0.02	0.04	-0.07	0.04	1.00					
turnover	0.07	-0.02	-0.01	0.26	0.13	-0.15	0.06	0.01	1.00				
illiq	-0.05	0.00	0.00	0.01	-0.02	0.14	0.02	-0.02	-0.04	1.00			
ivol	-0.03	-0.02	-0.02	0.23	-0.10	-0.08	-0.07	-0.04	0.36	0.16	1.00		
coskew	-0.05	-0.01	-0.01	0.19	0.05	-0.01	0.01	0.00	0.05	0.01	-0.04	1.00	
rating	-0.06	-0.01	-0.01	0.31	-0.40	0.05	0.01	-0.06	0.27	0.09	0.39	-0.09	1.00

Table 3.3: Mean (equal-weighted) stock characteristics by autocorrelation quintile

This table presents mean stock characteristics of interest in each of five quintiles sorted on autocorrelation (“ac”). The stock characteristic variables are described in more detail in sub-section 3.3.3. The data this table is based on excludes any stocks below $5, stocks below the 5% NYSE cut-off and any stocks that are not common domestic US stocks (i.e., stocks that do not have a stock code of 10 or 11 in CRSP) in the prior month. Note that both ac and the control variables are lagged by one month relative to excess returns (“eret”). The variables from capmbeta to coskew (inclusive) are winsorized at the 1% level to account for the influence of possible outliers.

	Autocorrelation ("ac") quintile				
Stock characteristic	Decaying	2	3	4	Explosive
ac	0.9730	0.9970	1.0012	1.0052	1.0168
eret	0.0087	0.0065	0.0065	0.0068	0.0103
capmbeta	1.2510	1.0391	0.9799	1.0239	1.1556
size	0.6662	1.4427	1.7693	1.6952	1.1455
b2m	0.9666	0.9189	0.8149	0.6948	0.5328
momentum	-0.0050	-0.0012	0.0072	0.0164	0.0373
pe	9.1457	13.3076	14.5015	15.5845	18.2598
turnover	0.9942	0.7504	0.7027	0.7950	1.1908
illiq	1.3600	1.2900	1.1300	1.0500	1.0500
ivol	0.0230	0.0182	0.0165	0.0172	0.0208
coskew	-0.0111	-0.0147	-0.0176	-0.0289	-0.0388
rating	11.6582	9.1255	8.4880	9.0629	11.0529
l2price	23.1986	44.7597	53.1847	42.9041	37.2458

The results in Table 3.3 can be summarised as follows: momentum and P/E ratio are *increasing* in autocorrelation while book-to-market, illiquidity and coskewness are *decreasing* in autocorrelation. The remaining stock characteristics do not follow a simple linear relationship. As seen earlier, autocorrelation-sorted excess returns follow a “U”-shaped pattern with the decaying and explosive stocks generating higher excess returns than the middle quintiles. The same “U”-shaped pattern also holds for CAPM beta, rating, turnover and idiosyncratic volatility. As for size, the decaying and explosive stocks tend to be smaller than average – an inverted “U”-shaped pattern. This also applies to twice-lagged price; the stocks in the highest and lowest autocorrelation deciles have lower stock prices than average. These patterns hint at something – decaying and explosive stocks are “extreme” stocks exactly because they stray from the middle-of-the-road random walk. Those stocks most likely to stray are smaller stocks (size), stocks sensitive to market shocks (capmbeta), volatile stocks (ivol) and financially/-operationally leveraged or distressed stocks (rating). However, decaying and explosive stocks are not extreme in the same way; decaying stocks have low P/E and high book-to-market ratios while the reverse holds for explosive stocks. In other words, decaying stocks look a lot like distressed “value” stocks while explosive stocks look a lot like risky “growth” stocks.

3.3.5 Time-series variables

In time-series regressions I control for seven factors, all sourced from Kenneth French's data library[10]. The market excess return ("MktRf") is the return on the market less the risk free rate. The Fama & French factors and the momentum factor retain their conventional names ("SMB" is Small-Minus-Big, "HML" is High-Minus-Low and "UMD" is Up-Minus-Down). I also control for Pástor and Stambaugh (2003) liquidity innovations ("PSLiq"), short term reversal ("STRev") and long term reversal ("LTRev").

3.3.6 Macroeconomic variables

In panel regressions I consider the following macroeconomic indicators, all at a monthly frequency: investor sentiment (orthogonalised), as per Baker and Wurgler (2006), NBER recessions (coded as a monthly dummy variable: 1 = recession), corporate bond credit spreads (Aaa - Baa), term spread (10-year treasury yield minus the 1-year treasury yield), US industrial production growth and US unemployment (all sourced from the Federal Reserve Bank of St. Louis, save for sentiment).

3.4 Portfolio sorts

3.4.1 Single sorts

Can stock-level autocorrelation predict realised returns? To answer this question I turn to sorted portfolios; the results are summarised in Figure 3.4.1. I sort stocks into decile portfolios (cross-sectionally in each month) using the *prior* month autocorrelation estimate for each stock. Strikingly, the decile 1 and decile 10 portfolios show much higher realised excess returns and Sharpe ratio's than the remaining middle decile portfolios. The value-weighted mean portfolio excess return for deciles 1 and 10 are around 100 basis points per month; this contrasts with the remaining deciles, which average around 40 to 70 basis points per month. (The results for equal-weighted portfolios are similar, if slightly stronger). The results in Figure 3.4.1 suggest that extreme autocorrelations in past log price history – whether low or high – are associated with higher realised excess returns in the following period.

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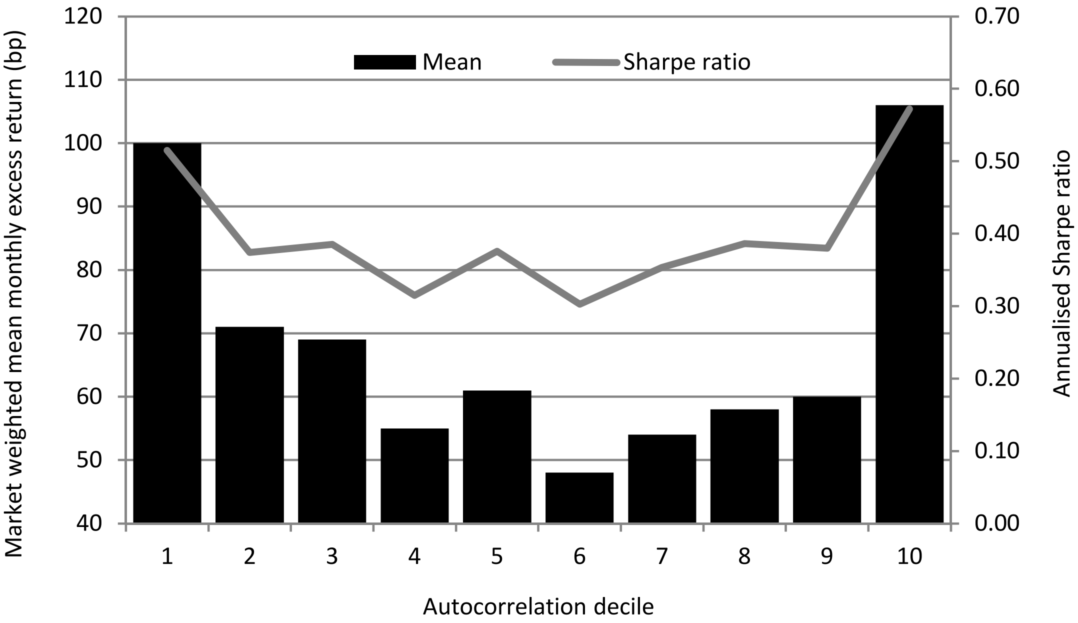
[10] http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

Figure 3.4.1: Decile portfolio mean excess returns, sorted on lagged autocorrelation (value weighted)

This figure shows value-weighted mean monthly excess return (in basis points) for decile portfolios sorted on autocorrelation (lagged one month) and annualised Sharpe ratios calculated from the mean and standard deviation of monthly excess returns.



Of course, this pattern may simply result from systematic differences in risk between decile portfolios. Perhaps autocorrelation, as I estimate it, is somehow correlated with other priced risk factors? If so, this might at least partially explain the pattern of excess returns observed in Figure 3.4.1. To isolate the influence of known risk factors from autocorrelation I turn to formal tests of time-series alphas. The results are summarised in Table 3.4.

My methodology follows the standard approach in the literature. I calculate monthly excess returns for each decile portfolio sorted on prior month autocorrelation. Portfolio 1 corresponds to decaying stocks (low autocorrelation) and portfolio 10 corresponds to explosive stocks (high autocorrelation). The "middle" portfolio consists of deciles 2 through to 9 inclusive. The *decaying hedge portfolio* or "p1mm" (portfolio 1 minus middle) is formed by going long portfolio 1 and going short the middle portfolio in equal measure. Similarly, the *explosive hedge portfolio* or "p10mm" (portfolio 10 minus middle) is formed by going long portfolio 10 and going short the middle portfolio in equal measure. Table 3.4 reports summary statistics (mean, standard deviation, Sharpe ratio, skewness and minima) for each decile portfolio and for the hedge portfolios. I calculate mean returns using equal weights (first panel) and market weights (second panel). My decision to include results for both equal-weighted and value-weighted portfolios is motivated by the finding of Plyakha, Uppal and Vilkov (2012) that equal-weighted portfolios tend to outperform value-weighted portfolios because equal-weighted portfolios have a higher exposure to the market, size and book-to-market factors, as well as requiring more frequent rebalancing, thereby inducing a higher exposure to the short-term reversal effect.

Both the decaying ("p1mm") and explosive ("p10mm") hedge portfolios generate excess returns between 40 and 54 basis points per month relative to the middle portfolio. This outperformance is significant at the 1% confidence level based on Newey-West HAC standard errors.

To disentangle the potential influence of known risk factors, I perform time-series regressions of the decile and hedge portfolio returns against the returns of other known time-series risk factors. The column labelled "CAPM-α" in Table 3.4 controls for the market excess return ("MktRf"). The resulting alphas range between 31 and 48 basis points per month and remain statistically significant. I perform a similar analysis using the Fama & French factors (market excess return, SMB and HML) and the Carhart four factor model which adds momentum (UMD) to the Fama & French factors. The resulting alphas remain positive, and (with a single exception), statistically significant. Finally, in the last column of Table 3.4 I control for the four Carhart factors and short-term reversal, long term reversal and Pástor and Stambaugh liquidity innovations – and find that the hedge portfolio alphas range from 37 basis points per month to 57 basis points per month, all statistically significant at the 1% level. (Detailed regression results can be found in Tables 3.16 and 3.17 in the Appendix.) This suggests that the returns earned by the decaying and explosive hedge portfolios cannot be attributed to the standard time-series risk factors alone. The effect is economically significant: the value-weighted decaying hedge portfolio earns an annualised return of 7.1% (t-statistic of 3.29) *after* accounting for all seven time-series risk factors. The value-weighted explosive hedge portfolio earns an annual excess return of 4.5% (t-statistic of 3.25), again after accounting for all seven time-series risk factors.

It is instructive to visualise how the decaying and explosive hedge portfolios have performed over time. In Figure 3.4.2 I plot a twenty-four-month trailing moving average of the value weighted excess returns of the decaying and explosive hedge portfolios. It is immediately apparent that both hedge portfolios have become more volatile in the last two decades. This observation might be related to the well-documented fact that stocks have become significantly more volatile in recent times (see for instance, Campbell et al. (2001) and Wei and Zhang (2006)). To the extent that both hedge portfolios will tend to capture those stocks that deviate most from a random walk, increasing stock volatility would be expected to disproportionately impact the returns of the hedge portfolios. Less strikingly, but no less interesting, is the fact that the two hedge portfolios appear to be fairly uncorrelated. The correlation between the value-weighted monthly excess returns of the decaying hedge portfolio and the explosive hedge

Table 3.4: Decile portfolio monthly excess returns (bp), sorted on lagged autocorrelation

I report statistics relating to autocorrelation portfolios, calculated using equal weights and market weights. The portfolios in the table below are formed by sorting stocks into deciles using their prior period autocorrelation estimates. Portfolio 1 corresponds to decaying stocks (low autocorrelation) and portfolio 10 corresponds to explosive stocks (high autocorrelation). The "middle" portfolio consists of deciles 2 through to 9 inclusive. The decaying hedge portfolio "p1mm" (portfolio 1 minus middle) is formed by going long portfolio 1 and going short the middle portfolio in equal measure. Similarly, the explosive hedge portfolio "p10mm" (portfolio 10 minus middle) is formed by going long portfolio 10 and going short the middle portfolio in equal measure. The last four columns report the alpha of time series regressions controlling for common time-series risk factors. The time series regressions are in the form $r_t^{Portfolio} = \alpha + [TimeSeriesControls_t] + \varepsilon_t$; note that $r_t^{Portfolio}$ denotes the realised return of the decile portfolio in the current month t but is formed on the basis of the autocorrelation in the prior month $t-1$. "CAPM-α" controls for the contemporaneous market excess return, "3F-α" additionally controls for the Small-Minus-Big and High-Minus-Low portfolio returns of Fama and French (1993), "4F-α" additionally controls for the Up-Minus-Down momentum portfolio (following Carhart (1997)). Finally, "All-α" controls for all the controls in 4F-α as well as long-term reversal, short-term reversal and Pástor and Stambaugh (2003) liquidity innovations. Significance levels are calculated using Newey West HAC adjusted standard errors and are indicated with * ($p<0.1$), ** ($p<0.05$) and *** ($p<0.01$).

Equal weighted acdecile HML portfolio timeseries statistics

	mean	sd	n	sharpe	skew	min	CAPM-α	3F-α	4F-α	All-α
1	0.0118***	0.0775	1,025	0.5642	0.2134	-0.3269	0.0039***	0.0026***	0.0040***	0.0034***
2	0.0080***	0.0766	1,025	0.3763	0.5256	-0.3480	0.0000	-0.0016**	0.0008	-0.0007
3	0.0076***	0.0726	1,025	0.3772	0.8069	-0.3516	0.0000	-0.0017***	0.0006	-0.0001
4	0.0069***	0.0688	1,025	0.3587	1.1404	-0.3137	-0.0004	-0.0021***	-0.0004	-0.0010**
5	0.0073***	0.0670	1,025	0.3926	1.2794	-0.3594	0.0002	-0.0015***	-0.0001	-0.0008
6	0.0064***	0.0614	1,025	0.3734	0.0490	-0.3665	-0.0001	-0.0014***	-0.0012**	-0.0010*
7	0.0064***	0.0593	1,025	0.3853	-0.1529	-0.2848	0.0000	-0.0008	-0.0016**	-0.0014**
8	0.0068***	0.0582	1,025	0.4201	-0.5997	-0.2830	0.0006	0.0003	-0.0016**	-0.0010
9	0.0078***	0.0606	1,025	0.4669	-0.4237	-0.3018	0.0015*	0.0014**	-0.0014**	-0.0011*
10	0.0125***	0.0729	1,025	0.6388	0.1893	-0.3122	0.0051***	0.0048***	0.0019***	0.0033***
middle	0.0071***	0.0636	1,025	0.4039	0.2623	-0.3144	0.0003	-0.0009***	-0.0006	-0.0009**
p1mm	0.0047***	0.0247	1,025	0.6753	0.5069	-0.1262	0.0036***	0.0035***	0.0046***	0.0043***
p10mm	0.0054***	0.0299	1,025	0.6446	0.9970	-0.1428	0.0048***	0.0058***	0.0025***	0.0042***

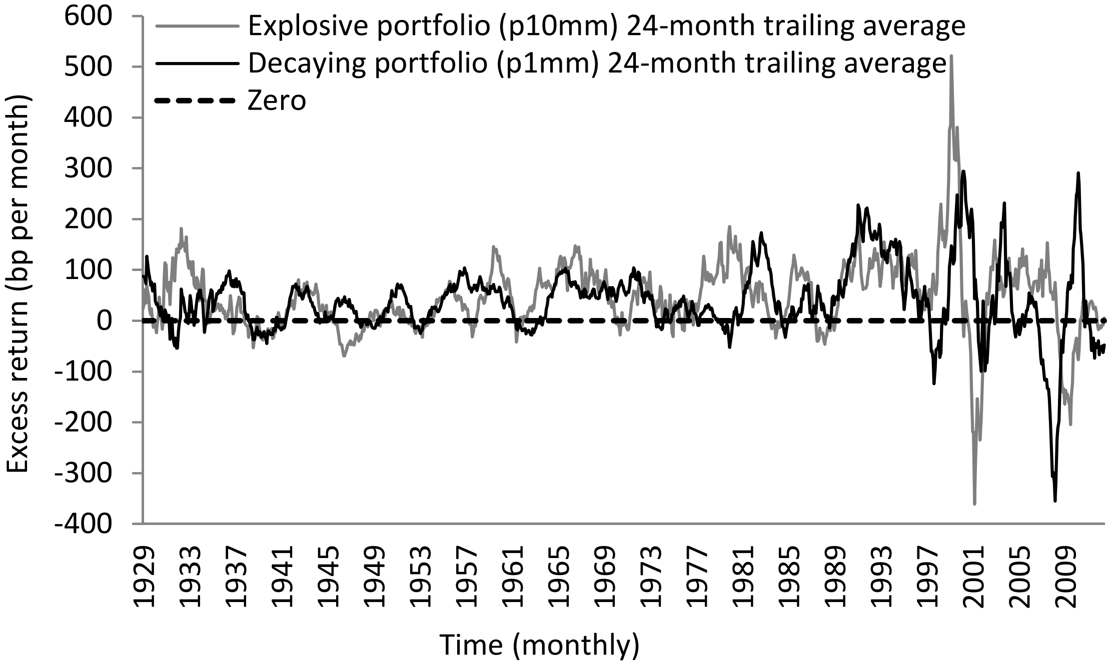
Value weighted acdecile HML portfolio timeseries statistics

	mean	sd	n	sharpe	skew	min	CAPM-α	3F-α	4F-α	All-α
1	0.0100***	0.0709	1,026	0.5147	0.1644	-0.3203	0.0028**	0.0027**	0.0049***	0.0055***
2	0.0071***	0.0688	1,026	0.3741	0.3499	-0.3345	0.0001	-0.0004	0.0028***	0.0014
3	0.0069***	0.0640	1,026	0.3854	0.8085	-0.3011	0.0001	-0.0007	0.0020***	0.0013
4	0.0055***	0.0620	1,026	0.3148	0.9126	-0.3256	-0.0011	-0.0020***	0.0002	-0.0004
5	0.0061***	0.0582	1,026	0.3757	0.4784	-0.3572	-0.0001	-0.0009	0.0006	0.0001
6	0.0048***	0.0567	1,026	0.3027	0.1708	-0.3264	-0.0013**	-0.0019***	-0.0012**	-0.0009
7	0.0054***	0.0546	1,026	0.3534	-0.1903	-0.3015	-0.0005	-0.0008	-0.0012**	-0.0015**
8	0.0058***	0.0537	1,026	0.3862	-0.4031	-0.2433	0.0000	0.0002	-0.0014***	-0.0004
9	0.0060***	0.0565	1,026	0.3798	-0.4009	-0.2662	0.0001	0.0006	-0.0017***	-0.0014**
10	0.0106***	0.0680	1,026	0.5724	0.0844	-0.3058	0.0037***	0.0044***	0.0012*	0.0035***
middle	0.0059***	0.0564	1,026	0.3779	0.1764	-0.2986	-0.0003	-0.0007**	0.0000	-0.0002
p1mm	0.0040***	0.0315	1,026	0.4512	1.0741	-0.1733	0.0031***	0.0034***	0.0048***	0.0057***
p10mm	0.0047***	0.0362	1,026	0.4575	0.5893	-0.2130	0.0040***	0.0051***	0.0012	0.0037***

portfolio is only 10.3%; for the equal-weighted hedge portfolios it is 1.2%. Given that both hedge portfolios individually generate substantial outperformance, the fact that they appear to be largely uncorrelated suggests that a combination of these portfolios might generate particularly favourable risk adjusted returns[11].

Figure 3.4.2: Decaying and explosive portfolio monthly excess returns (value-weighted trailing 24-month average)

The figure below plots the 24-month trailing average of the excess returns earned by the decaying hedge portfolio and the explosive hedge portfolio respectively (both value-weighted). The decaying hedge portfolio is formed by going long portfolio 1 and going short the middle portfolio in equal measure. Similarly, the explosive hedge portfolio is formed by going long portfolio 10 and going short the middle portfolio in equal measure. Portfolio 1 corresponds to the lowest decile of stocks sorted on autocorrelation while portfolio 10 corresponds to those stocks in the highest decile.



I also consider whether the returns generated by the decaying and explosive hedge portfolios are related to the macroeconomic state variables described in the data section, namely, sentiment, NBER recessions, Aaa-Baa bond credit spread, 10-year treasury term spread, US industrial production growth and US unemployment. US industrial production growth appears to positively explain some of the returns to the equal weighted decaying hedge portfolio (see Table 3.35 in the Appendix), while the other macroeconomic variables are not significant. The NBER recession dummy has a negative and significant coefficient for both the value-weighted and equal-weighted explosive hedge portfolio (see Table 3.36 in the Appendix). No other macroeconomic variables are significant in explaining the returns of the explosive hedge portfolio. In other words, the decaying hedge portfolio appears to do better when US industrial production is growing, while the explosive hedge portfolio seems to performs better in NBER expansions.

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[11] This might well be a fruitful avenue for future research (or trading)

3.4.2 Double sorts

To account for the potential influence of any cross-sectional risk factors based on stock characteristics I consider simple cross-tabulations. I use independent sorts to construct quintile based cross-tabulations of autocorrelation against CAPM beta, size, book-to-market and momentum; the results are contained in Table 3.5. Cells in the table contain the mean (equal-weighted) excess return. Rows labelled "HMM" provide the difference between the explosive quintile (quintile 5) and the middle quintiles (quintiles 2 to 4). Similarly, "LMM" provide the difference between the decaying quintile (quintile 1) and the middle quintiles. Statistical significance is calculated using a two-sample t-test with unequal variances. The evidence in Table 3.5 suggests that the predictive power of autocorrelation is reasonably robust to pre-sorting by CAPM beta, size and book-to-market (Panels A to C); although more so for explosive stocks than decaying stocks. The predictive power of autocorrelation is also robust to momentum (Panel D), although in that case decaying stocks fare better than explosive stocks.

The results of two-way tabulation of autocorrelation against all 12 stock characteristics are summarised more concisely in Table 3.6. Panel A details the mean (equal-weighted) excess return of the explosive hedge portfolio after being presorted into quintiles based on the stock characteristics listed in the row of that panel. Panel B provides the same information, but for the decaying hedge portfolio. (Detailed results of cross-tabulations using other stock characteristics are relegated to the Appendix in Tables 3.18 to 3.20).

In general the explosive hedge portfolio appears somewhat more robust to different stock characteristics than the decaying portfolio. The explosive hedge portfolio generates positive mean excess returns in all but one out of 60 cases (12 stock characteristics times 5 quintiles), and is positive and significant at the 5% level in 58 of those cases. By comparison, the decaying portfolio generates positive mean excess returns in all but 5 cases and is positive and significant at the 5% level in 48 out of 60 cases.

A notable result is that the explanatory power of both decaying and explosive stocks appears weaker for some higher ratings categories. The explosive hedge portfolio generate statistically significant positive excess returns in every ratings category save for category 2 (stocks rated A- to A+), which produce an insignificant negative return of -8 basis points. The decaying hedge portfolio generates positive excess returns in each ratings category save for the highest (corresponding to stocks rated AA- or higher), where it loses 12 basis points per month. The mean excess return earned by the decaying hedge portfolio remains statistically significant in three of the five ratings groups. This pattern is reminiscent of the finding in Avramov et al. (2013) that a range of anomalies in the finance literature is concentrated in stocks with low credit ratings.

Table 3.5: Two-way tabulations showing mean monthly excess return (bp)

Each panel below shows the result of two-way tabulations of mean monthly excess returns (in basis points). The rows in each panel relate to the different quintiles of autocorrelation, while the columns relate to quintiles of a stock characteristic. Rows labelled LMM ("Low Minus Middle") provide statistics for decaying hedge portfolios and rows labelled HMM ("High Minus Middle") do the same for explosive hedge portfolios. The last row in each panel, labelled "All", provide the comparable mean excess returns for the entire dataset. Significance levels are indicated by * for $p < 0.1$, ** for $p < 0.05$ and *** for $p < 0.01$ and are computed using a two-sample t-test with unequal variances.

Panel A

	CAPM Beta quintiles						
ac quintiles	Low	2	3	4	High	HML	All
Decaying	65	89	100	104	103	38***	87
2	55	73	75	66	66	11	65
3	67	75	76	69	42	-25***	65
4	78	84	78	68	43	-35***	68
Explosive	94	111	121	107	78	-16**	103
HMM	27***	34***	45***	39***	26***		38***
LMM	-2	11*	24***	36***	51***		21***
All	71	85	88	83	72	1	66

Panel B

	Size (Market Cap) quintiles						
ac quintiles	Small	2	3	4	Big	HML	All
Decaying	92	83	90	89	75	-17**	87
2	74	59	68	68	57	-17***	65
3	71	77	69	59	55	-16***	65
4	90	77	73	63	50	-40***	68
Explosive	138	119	107	89	71	-67***	103
HMM	60***	49***	37***	26***	17***		38***
LMM	14**	13**	20***	26***	21***		21***
All	73	67	69	65	57	-16***	66

Panel C

	Book-to-market quintiles						
ac quintiles	Low	2	3	4	High	HML	All
Decaying	16	56	74	90	97	81***	87
2	-2	34	37	64	93	95***	65
3	2	37	50	66	104	102***	65
4	11	32	57	89	122	111***	68
Explosive	66	79	112	135	183	117***	103
HMM	61***	45***	63***	63***	80***		38***
LMM	11	23**	26***	18***	-6		21***
All	17	39	60	78	105	88***	66

Panel D

	Momentum (prev. 6 months) quintiles						
ac quintiles	Down	2	3	4	Up	HML	All
Decaying	66	75	92	99	139	73***	87
2	74	62	67	49	70	-4	65
3	70	74	64	53	60	-10	65
4	44	75	72	74	61	17**	68
Explosive	87	72	96	104	116	29***	103
HMM	20**	3	28***	42***	53***		38***
LMM	0	6	25***	38***	77***		21***
All	49	64	69	71	94	45***	66

Table 3.6: Summary of two-way sorts showing mean monthly excess return

The panels below summarise the (equal-weighted) mean monthly excess returns earned by the autocorrelation sorted hedge portfolios after pre-sorting by various stock characteristics. The explosive hedge portfolio is long the highest quintile of stocks and short the middle quintiles (quintiles 2 to 4) in equal measure, while the decaying hedge portfolio is long the lowest quintile of stocks and short the middle quintiles (quintiles 2 to 4) in equal measure. The stock characteristics are described in more detail in subsection 3.3.3. All quintiles are formed on the basis of one month lagged variables. For ease of interpretation ratings are categorised into ratings categories, rather than strict quintiles, on the following basis: 1 = AAA and AA, 2 = A, 3 = BBB, 4 = BB, 5 = below BB. Significance levels are indicated by * for $p < 0.1$, ** for $p < 0.05$ and *** for $p < 0.01$ and are computed using a two-sample t-test with unequal variances.

Panel A: Explosive hedge portfolio (high-minus-middle)

	Quintile of primary sorting variable				
Primary sorting variable	1	2	3	4	5
capmbeta	27***	34***	45***	39***	26***
size	60***	49***	37***	26***	17***
b2m	61***	45***	63***	63***	80***
momentum	20**	3	28***	42***	53***
pe	59***	63***	42***	46***	65***
rating	36**	-8	22**	24**	88***
turnover	36***	31***	34***	40***	52***
illiq	27***	25***	31***	45***	54***
ivol	38***	35***	40***	44***	62***
coskew	38***	34***	35***	29***	20***
l2price	52***	54***	38***	29***	27***
nysesize	66***	43***	28***	25***	13***

Panel B: Decaying hedge portfolio (Low-minus-middle)

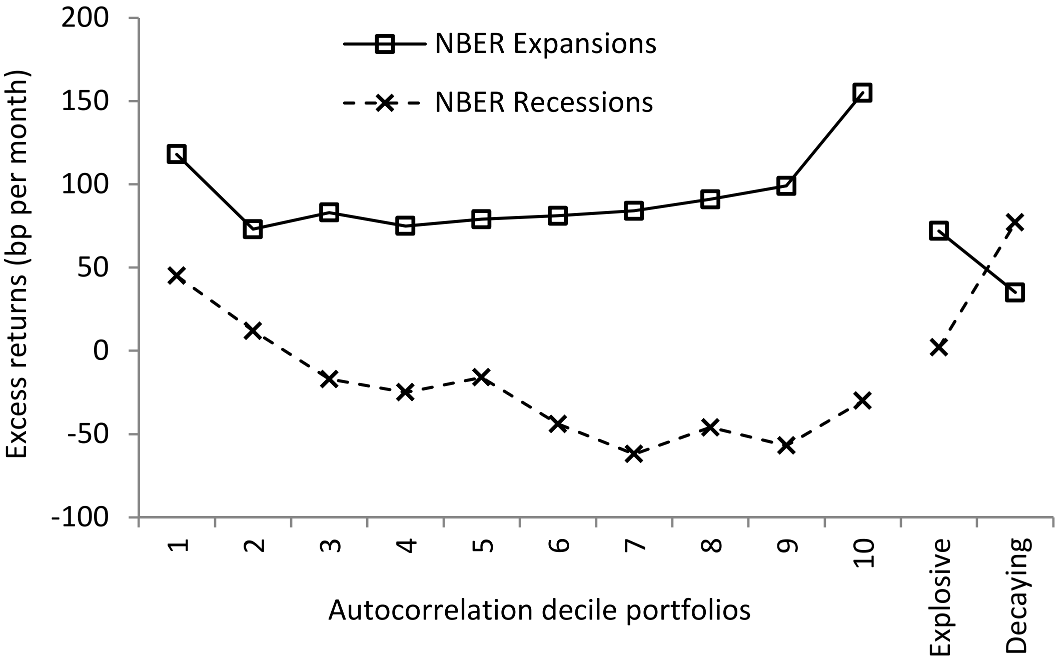
	Quintile of primary sorting variable				
Primary sorting variable	1	2	3	4	5
capmbeta	-2	11*	24***	36***	51***
size	14**	13**	20***	26***	21***
b2m	11	23**	26***	18***	-6
momentum	0	6	25***	38***	77***
pe	-28***	31***	65***	68***	76***
rating	-12	34**	43***	11	26*
turnover	-4	12**	19***	28***	38***
illiq	15***	20***	8	16***	21***
ivol	20***	22***	34***	43***	32***
coskew	28***	21***	21***	28***	33***
l2price	10*	27***	30***	37***	50***
nysesize	13***	17***	28***	25***	27***

An interesting pattern reveals itself when we consider NBER expansions and recessions separately. (See Figure 3.4.3 below – the last two data points represents the excess returns of the hedge portfolios.) In expansions both the explosive hedge portfolio and the decaying hedge portfolio generate positive excess returns (significant at the 1% level); however, the returns earned by the decaying portfolio (35bp per month) is much smaller than that earned by the explosive portfolio (72bp per month). In

NBER recession months, the decaying hedge portfolio generates much higher positive returns of 77bp per month (significant at the 1% level). By contrast, the explosive hedge portfolio generates only 2 basis points per month in NBER recessions and is not statistically significant. In other words, explosive stocks generate their excess returns almost exclusively in expansions while decaying stocks generate excess returns in both expansions and recessions, but much more so in recessions. These findings resonate with our earlier characterisation of explosive stocks as risky "growth" stocks and decaying stocks as distressed "value" stocks.

Figure 3.4.3: Monthly excess return by lagged autocorrelation decile in NBER expansions and recessions

The figure below shows the monthly excess returns generated by decile portfolios sorted on autocorrelation in the previous month in NBER expansions and recessions.



3.4.3 Double-sorted time-series tests

Cross-tabulations are a simple but useful way to control for stock characteristics that might be priced. However, a more stringent approach would be to first sort by a stock characteristic and *then* apply time-series regressions to hedge portfolios created *within* each of the characteristic-sorted portfolios, to additionally control for time-series risk factors[12]. In fact it may be an overly stringent approach. Running asset pricing tests within pre-sorted portfolios introduces a bias in favour of rejecting the model, as has been shown by Berk (2000)[13], particularly if the two sorting variables are correlated. In our data momentum exhibits the highest correlation (0.15) with autocorrelation among all the stock characteristics controlled for, and, therefore, is most likely to be subjected to such a bias. For this reason I illustrate this approach using momentum as the primary sorting variable[14] – see Table 3.7. The results obtained by pre-sorting by other stock characteristics are contained in the Appendix in Tables 3.21 to 3.34 and is summarised in Figure 3.4.4.

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[12] In the previous subsection I used independent sorts to generate the quintiles for both autocorrelation and the various stock characteristics considered. However, in this subsection I first sort stocks into quintiles by the relevant stock characteristic – the primary sorting variable – then sort stocks into deciles *separately within* each of the primary quintiles. This approach ensures that, within each primary quintile, an equal number of stocks are allocated to each of the 10 autocorrelation deciles used to create the hedge portfolios. It also means that, in contrast to the cross-tabulations presented earlier, the autocorrelation decile cut-points will likely differ between different primary variable quintiles. This approach is consistent with underlying motivation of applying time-series tests on an equal basis within each of the primary variable quintiles.

[13] "It is shown that this empirical procedure biases the results in favor of rejecting whatever asset pricing model is being tested" (Berk (2000), page 1).

[14] In the next section I show that autocorrelation actually subsumes momentum in panel data regressions.

Table 3.7: Hedge portfolio time-series alphas within momentum sorted quintiles

The panels below show the alphas obtained from time-series regressions of the mean (equal-weighted) excess returns of autocorrelation hedge portfolios. The columns contain the results of alternate time-series regression specifications (described in detail in Table 3.4), while the different momentum quintiles are detailed in the rows, first using equal weights, then using value weights. The primary sorting variable, momentum, is calculated over a 6-month horizon and is lagged one month relative to excess returns. Panel A contains the results for the explosive hedge portfolio (quintile 10 minus quintiles 2 to 9, in equal measure) while panel B contains the results for the decaying hedge portfolio (quintile 1 minus quintiles 2 to 9, in equal measure). Significance levels are calculated using Newey-West HAC adjusted standard errors (at a maximum lag length of 12 months) and are indicated with * ($p<0.1$), ** ($p<0.05$) and *** ($p<0.01$).

Panel A: Explosive hedge portfolio time-series alphas

Portfolio (qmomentum)	p10mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	0.0004	0.0004	0.0013	-0.0015	-0.0014
Equal weighted: quintile 2	0.0003	0.0003	0.0015	-0.0013*	-0.0011
Equal weighted: quintile 3	0.0025**	0.0015	0.0020**	0.0008	-0.0001
Equal weighted: quintile 4	0.0052***	0.0046***	0.0055***	0.0039***	0.0055***
Equal weighted: quintile 5	0.0081***	0.0075***	0.0084***	0.0070***	0.0087***
Value weighted: quintile 1	0.0007	0.0007	0.0015	-0.0016	-0.0019
Value weighted: quintile 2	-0.0000	-0.0005	0.0007	-0.0023**	0.0001
Value weighted: quintile 3	0.0020	0.0008	0.0014	-0.0001	0.0008
Value weighted: quintile 4	0.0053***	0.0043***	0.0052***	0.0038***	0.0063***
Value weighted: quintile 5	0.0088***	0.0077***	0.0085***	0.0069***	0.0110***

Panel B: Decaying hedge portfolio time-series alphas

Portfolio (qmomentum)	p1mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	0.0050***	0.0044***	0.0045***	0.0049***	0.0050***
Equal weighted: quintile 2	0.0033***	0.0023**	0.0022**	0.0026***	0.0010
Equal weighted: quintile 3	0.0022**	0.0012	0.0012	0.0020**	0.0022**
Equal weighted: quintile 4	0.0046***	0.0035***	0.0029***	0.0041***	0.0028**
Equal weighted: quintile 5	0.0068***	0.0060***	0.0059***	0.0063***	0.0062***
Value weighted: quintile 1	0.0044***	0.0032**	0.0031**	0.0038***	0.0042**
Value weighted: quintile 2	0.0030**	0.0016	0.0018	0.0029**	0.0044***
Value weighted: quintile 3	0.0025*	0.0014	0.0014	0.0025*	0.0013
Value weighted: quintile 4	0.0043***	0.0033**	0.0030**	0.0041***	0.0032
Value weighted: quintile 5	0.0070***	0.0061***	0.0060***	0.0072***	0.0085***

Table 3.7 shows the alphas from various regressions specifications (detailed in the columns) within different momentum quintiles (detailed in the rows), considering first equal-weighted means, then value-weighted means. Panel A presents the results for the explosive hedge portfolio while panel B presents the results for the decaying hedge portfolio. The first column ("p10mm Mean" and "p1mm Mean" in the two panels) shows mean excess return for the hedge portfolios. The remaining columns all show time series alphas after controlling for different risk factors. The "CAPM-α" column contains the alpha after controlling for the market return ("MktRf"). The next column shows results controlling for the three Fama & French time-series factors – the market return, SMB and HML. The column headed "4F-α" adds the momentum factor

("UMD") in addition to the Fama & French factors. Finally, the last column controls for all the time-series risk factors. The regression for the last column is given below:

$$\begin{aligned} r_t^{HedgePortfolio} &= \alpha + \beta_1 MktRf_t + \beta_2 SMB_t + \beta_3 HML_t + \beta_4 UMD_t \\ &\quad + \beta_5 STRev_t + \beta_6 LTRev_t + \beta_7 PSLiq_t + \varepsilon_t \end{aligned}$$

The right-hand variables are all the time-series control variables discussed in the data section earlier. I calculate significance levels for the alphas (shown in Table 3.7 in parenthesis) using Newey-West HAC adjusted standard errors with a maximum lag length of 12 months.

Based on the evidence in panel A of Table 3.7 the explosive hedge portfolio excess returns are quite robust to pre-sorting by momentum, generally faring better in the high momentum quintiles. The results are stronger for the decaying hedge portfolio. As is evident in panel B of Table 3.7 the equal-weighted decaying hedge portfolio remains positive in every momentum quintile and regression specification, and significantly so in 22 out of 25 cases. The evidence is slightly weaker in the value-weighted case, with significant alphas in 19 out of 25 cases, although the alphas remain positive everywhere. Overall it seems that the predictive power of autocorrelation is mostly contained in the higher momentum quintiles and less pronounced in the lower momentum quintiles.

A high-level summary of the double sorted time-series alpha tests is contained in Figure 3.4.4. The results indicate that the predictive power of autocorrelation is generally robust to time series risk factors after pre-sorting by other stock characteristics. For the explosive hedge portfolio, the stock characteristics momentum and rating do the most to diminish the explanatory power of autocorrelation. The stock characteristics that do the most to diminish the explanatory power of autocorrelation for decaying stocks are book-to-market and rating. Stocks with high book-to-market ratios are often the same firms that might be experiencing financial distress and thus low ratings. This suggests that, for decaying stocks, the explanatory power of autocorrelation is most diminished when faced with firms in relative distress. This result is consistent with our earlier characterisation of decaying stocks as distressed "value" stocks.

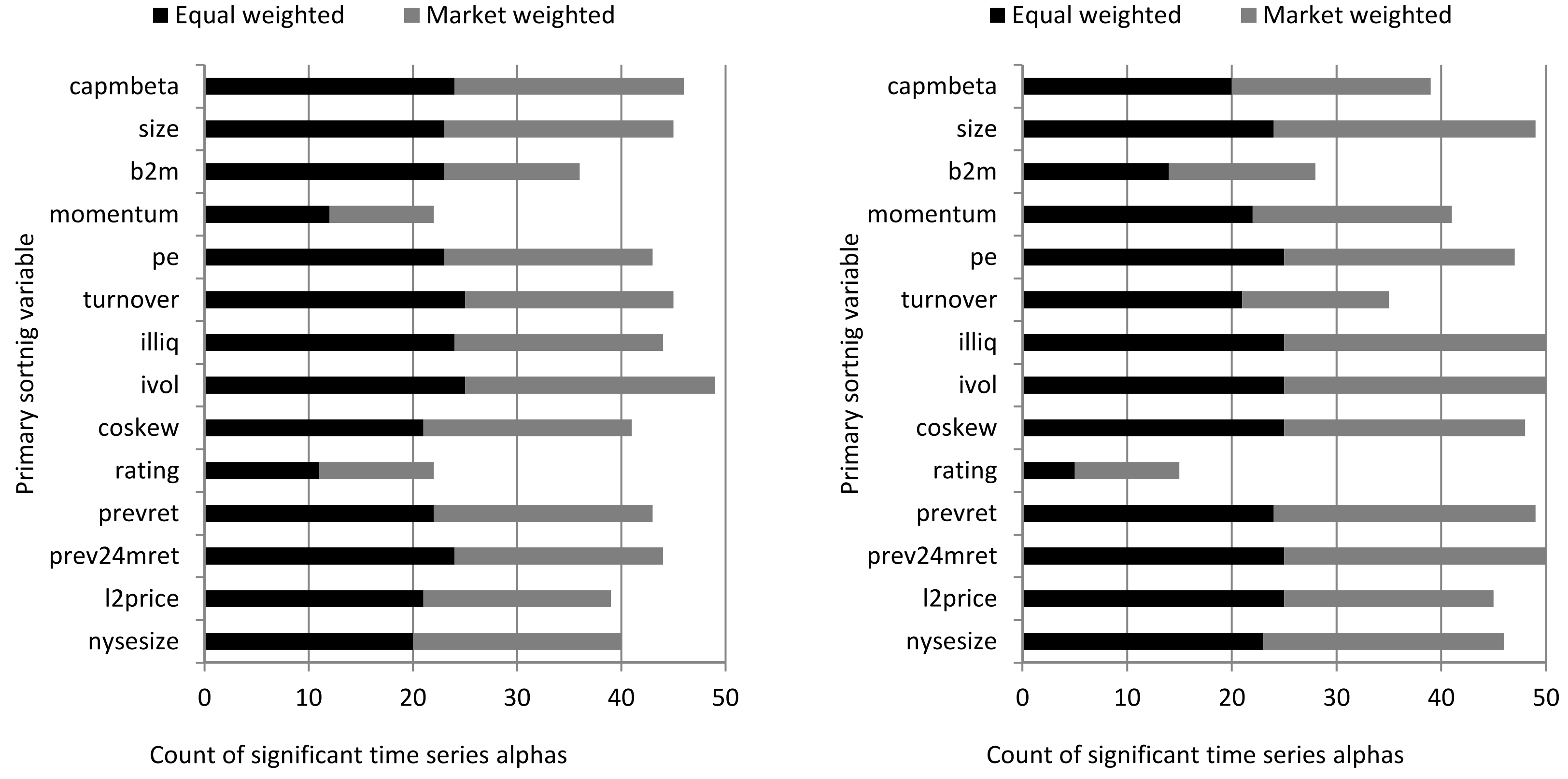
In short, the predictive power of autocorrelation seems reasonably robust to a large range of stock characteristics, even after controlling for time series risk factors within each stock characteristic quintile. The single exception is ratings in the case of decaying stocks, which warrants a closer look in the next subsection.

3.4.4 Ratings and autocorrelation

Pre-sorting by ratings results in a total of 15 significant alpha's from 50 separate time series regressions for the decaying hedge portfolio. Why does ratings do so much to reduce the apparent predictive power of autocorrelation? The detailed results (see Table

Figure 3.4.4: Counts of significant alphas ($p<0.1$) in double-sorted time-series tests

The graphs below show the count of significant time-series alphas for pre-sorted portfolios (out of a maximum of 50 per primary sorting variable, as explained below). I first sort stocks into quintiles using the primary sorting variable, here listed along the vertical axis of the graphs below. I then sort stocks into prior month autocorrelation deciles *within* each primary variable quintile. I form the explosive hedge portfolio (within each primary variable quintile) as the mean excess return in the highest autocorrelation decile less the mean excess return in the middle autocorrelation deciles (quintiles 2 to 9 inclusive). Similarly, I form the decaying hedge portfolio (within each primary variable quintile) as the mean excess return in the lowest autocorrelation decile less the mean return in the middle autocorrelation deciles. I then conduct time-series regressions using the returns of the explosive and decaying hedge portfolios as constructed above using 5 different specifications (described in detail in Table 3.4) to obtain time series alphas. Because each primary variable is sorted into 5 quintiles, and within each quintile I run time series regressions using 5 different specifications and using 2 weighting schemes (equal-weighted and value-weighted), there are a total of $5 \times 5 \times 2 = 50$ time series regressions performed for each primary sorting variable. I count the number of those alphas significant at the 10% level or better and report that number in the graphs below as horizontal bars. Significance levels are computed using Newey-West HAC adjusted standard errors with a maximum lag length of 12 months.



Panel A: Explosive hedge portfolio (P10-minus-middle)

Panel B: Decaying hedge portfolio (P1-minus-middle)

3.30 in the Appendix) reveal that most of these significant alpha's are concentrated in the lower ratings categories 4 and 5, corresponding to firms rated below investment grade (that is, below BBB-). As noted earlier, this pattern is consistent with the finding by Avramov et al. (2013) that a wide range of anomalies appears to be concentrated in the lower ratings groups. This suggests that ratings and autocorrelation may be systematically related. While the pair-wise correlation between ratings and autocorrelation is only -0.06 in the dataset (see Table 3.2), a simple frequency table reveals a richer structure. In panel A of Table 3.8 I record, for each ratings category, the percentage of stocks associated with each autocorrelation decile (based on an independent sort). Highly rated stocks appear clustered around the middle autocorrelation deciles, while the lower-rated stocks are over-represented in the extreme autocorrelation deciles. This gives rise to a subtle effect when pre-sorting by ratings category, then again sorting by autocorrelation *within* each ratings category. In the high ratings group very few stocks have extreme autocorrelation estimates compared to the dataset as a whole. As a consequence, the median level of autocorrelation in the extreme autocorrelation deciles are actually far less extreme for the higher ratings categories than they are for the lower ratings categories – see panel B in Table 3.8. This results in hedge portfolios with a much smaller range of autocorrelation estimates in the higher ratings than in the lower ratings. Thus, even if autocorrelation does predict returns at the individual stock level, this predictability might not be apparent in the higher ratings categories for the simple reason that highly rated stocks tends to have very few extreme autocorrelation estimates. In fact, at the individual stock level autocorrelation does predict realised returns and this predictability is robust to the inclusion of stock-level rating as a control variable, as I demonstrate in the next section.

Table 3.8: Autocorrelation and ratings

The tables below show aggregate statistics of (rated) stocks grouped by ratings category. Panel A shows the percentage of stocks allocated to each autocorrelation decile based on a full dataset sort, that is, sorted independent of ratings. Panel B shows the median autocorrelation estimate of each autocorrelation decile based on *separate* sorts on autocorrelation *within* each ratings category.

Panel A: Percentage allocated to independent autocorrelation deciles, by ratings group

Full dataset	Ratings category				
autocorrelation decile	AAA/AA	A	BBB	BB	Below BB
1	1%	2%	4%	11%	20%
2	4%	6%	9%	12%	13%
3	10%	11%	12%	11%	9%
4	15%	15%	13%	9%	7%
5	17%	16%	13%	9%	6%
6	17%	16%	13%	9%	6%
7	14%	14%	13%	9%	7%
8	12%	11%	11%	10%	8%
9	7%	6%	8%	11%	10%
10	3%	3%	3%	8%	15%
All	100%	100%	100%	100%	100%
Number of Stocks	27,224	80,482	99,582	87,955	61,927

Panel B: Median autocorrelation by autocorrelation deciles sorted within ratings groups

Within ratings group	Ratings category				
autocorrelation decile	AAA/AA	A	BBB	BB	Below BB
1	0.9963	0.9942	0.9912	0.9851	0.9778
2	0.9990	0.9981	0.9969	0.9927	0.9893
3	1.0001	0.9995	0.9988	0.9963	0.9941
4	1.0009	1.0006	1.0000	0.9987	0.9974
5	1.0017	1.0014	1.0011	1.0007	1.0000
6	1.0024	1.0022	1.0021	1.0025	1.0023
7	1.0033	1.0030	1.0031	1.0042	1.0047
8	1.0044	1.0040	1.0044	1.0062	1.0078
9	1.0060	1.0056	1.0063	1.0091	1.0126
10	1.0099	1.0098	1.0112	1.0160	1.0230

3.5 Panel data regressions

Portfolio based tests have many benefits, such as conceptual simplicity and robustness. However, panel data tests have complementary strengths. First, it makes use of all observations in a direct way. Second, perhaps more important, it allows one to simultaneously control for multiple variables. Finally, it makes use of both cross-sectional and time-series variation in the explained and explanatory variables. I follow the guidance in Petersen (2008) regarding the use of finance panel datasets. Petersen (2008) shows that two-way clustered standard errors (see Thompson (2011) and Cameron, Gelbach and Miller (2011)) are relatively robust to a range of issues typically affecting finance panel datasets compared to other common approaches[15].

In Table 3.9 I show the results of various specifications in which I directly test the predictive power of autocorrelation for subsequent realised excess returns, controlling for a range of other variables known to have predictive power for stock returns. (These variables are discussed in subsection 3.3.3)

To distinguish between the relative contribution of explosive and decaying stocks, I generate two explanatory variables for use in panel regressions. This is necessary because the expected sign of the coefficient depends on whether autocorrelation is above or below unity. From the earlier sorted portfolio evidence we know that as autocorrelation increases above unity, returns in the subsequent month tends to increase (implying a positive coefficient for autocorrelation). That is, explosive stocks appear to earn a premium. However, as autocorrelation decreases below unity, returns in the subsequent month also increases (implying a negative coefficient for autocorrelation). I address this by decomposing autocorrelation into two explanatory variables. The two explanatory variables – *explosive* and *decaying* – are derived from the stock-level autocorrelation estimate as follows: $explosive = max(1, ac)$ and $decaying = min(1, ac)$ where ac is the autocorrelation estimate for a particular stock at a particular time. In other words, *explosive* is autocorrelation floored at unity, while *decaying* is autocorrelation capped at unity. Thus the regression specification provides for asymmetric coefficients on autocorrelation.

The general panel regression specification is given below

$$r_{i,t} - r_t^f = \alpha + \beta_e \cdot explosive_{i,t-1} + \beta_d \cdot decaying_{i,t-1} + [\Sigma_c \beta_c \cdot control_{c,i,t-1}] + \varepsilon_{i,t}$$

In summary, I regress the excess return of stock i at time t on the *explosive* and *decaying* measures of stock i as estimated at time $t-1$ as well as as on a number of stock specific controls, also measured at time $t-1$.

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[15] Thompson (2011) (page 10) comes to the same conclusion: "Both the statistical theory and the Monte Carlo results suggest that simultaneously clustering by firms and time leads to significantly more accurate inference in finance panels."

Specification (1) in Table 3.9 considers the predictive power of *explosive* and *decaying*. *Explosive* is positive and significant at the 10% level while decaying is negative and significant at the 1% level. The opposite signs on *explosive* and *decaying* validate the decision to decompose autocorrelation into two parts and is consistent with the findings from sorted portfolios presented earlier.

The predictability is also economically significant. Autocorrelation has a standard deviation of 0.027 in our dataset – a change of that magnitude results in a change in monthly realised returns of 42 basis points for *explosive* stocks and 23 basis points for *decaying* stocks. (The effect is even stronger when I control for additional stock characteristics: using the coefficients in specification (6) generates a change of 127 basis points per month for *explosive* stocks and 74 basis points for *decaying* stocks, based on the same one standard deviation shock.)

In specification (2) I control for the size and book-to-market characteristics. (While Fama and French (1993) claim that it is the covariance with the Small-Minus-Big and High-Minus-Low portfolios that is priced, Daniel and Titman (1997) argue that it is the characteristics (stock level size and book-to-market ratio), rather than the covariance, that is priced). The results suggest that the predictive power of the *explosive* and *decaying* metrics are robust to the inclusion of size and book-to-market ratio – the coefficients on both *explosive* and *decaying* are largely unchanged and remain statistically significant. Adding the CAPM beta in specification (3) does not change things significantly, nor does adding momentum in specification (4).

In specification (5) I add all the stock characteristics as controls, save for rating. I (initially) exclude rating since adding it would reduce the sample size from c. 943,000 to c. 311,000. I find that *explosive* and *decaying* remain significant and of comparable magnitude to the original specification (1) results. Specification (6) adds rating as a control - this results in the coefficients on both *explosive* and *decaying* roughly doubling while statistical significance is unaffected. This suggests that the predictive power of the *explosive* and *decaying* metrics are robust to the inclusion of a range of stock characteristics considered to have predictive power in the cross section; size, book-to-market, CAPM beta, momentum, price-earnings ratio, Amihud illiquidity, idiosyncratic volatility, stock turnover and rating.

Specification (7) controls for a number of simple historical return metrics to rule out the possibility that the predictive power of autocorrelation might be driven by those metrics instead. I control for the previous period return (prevret) in case my results are driven by a short term continuation or reversal effect. I also additionally control for the average return over the previous 24 months (prev24mret). Since I estimate autocorrelation over 24 months, I want to ensure that my results are not simply driven by the average level of historical returns over the same sample window. Finally, I control for the twice lagged[16] stock price (l2price). Because autocorrelation is calculated using past prices I want to ensure that my results are not simply driven by the price level of individual stocks. The results suggest that the predictive power of both the *explosive* and *decaying* metrics are independent of simple past returns or stock price levels.

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[16] As noted in the data section, I lag the stock price twice since the dependent variable – excess

In short, panel data tests provide substantial evidence that autocorrelation has predictive power at the individual stock level. This predictive power is robust to a range of different stock characteristics and regression specifications.

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return – directly depends on the current stock price and the once lagged stock price. Using the twice-lagged price allows me to avoid potential econometric issues while still controlling for the overall price level of the stock.

Table 3.9: Panel regression of excess monthly realised returns on lagged autocorrelation

The table below shows the results from individual stock-level panel regressions. The dependent variable is realised excess return. The two explanatory variables – *explosive* and *decaying* – are derived from the stock-level autocorrelation estimate as follows: $explosive = max(1, ac)$ and $decaying = min(1, ac)$. The general regression equation can be written as $r_{i,t} - r_t^f = \alpha + \beta_e \cdot explosive_{i,t-1} + \beta_d \cdot decaying_{i,t-1} + [\beta_c controls_{i,t-1}] + \varepsilon_t$. The stock-level control variables are described in more detail in subsection 3.3.3. All explanatory variables and controls are lagged by one month relative to the dependent variable. t-Statistics are shown in parenthesis below their associated coefficients. Significance levels are indicated by * for $p < 0.1$, ** for $p < 0.05$ and *** for $p < 0.01$ and are based on two-way clustered standard errors (clustered by individual stock and by time).

	(1) r1 b/t	(2) r2 b/t	(3) r3 b/t	(4) r4 b/t	(5) r5 b/t	(6) r6 b/t	(7) r7 b/t
explosive	0.1560*	0.2173*	0.2149*	0.2494**	0.2803***	0.4326**	0.3333***
	(1.70)	(1.88)	(1.90)	(2.32)	(2.59)	(2.10)	(3.03)
decaying	-0.0846***	-0.1115***	-0.1102***	-0.1111***	-0.1224***	-0.2746***	-0.0831***
	(-3.88)	(-4.63)	(-5.17)	(-5.19)	(-5.54)	(-3.46)	(-3.93)
size		-0.0000	-0.0001	-0.0001	-0.0001	-0.0002*	
		(-0.35)	(-0.91)	(-0.89)	(-1.38)	(-1.73)	
b2m		0.0045***	0.0051***	0.0050***	0.0048***	0.0028***	
		(3.57)	(3.72)	(3.73)	(3.81)	(2.71)	
capmbeta			-0.0020	-0.0020	-0.0010	0.0008	
			(-1.26)	(-1.30)	(-0.71)	(0.42)	
momentum				-0.0164	-0.0217	-0.0366	
				(-0.38)	(-0.51)	(-0.65)	
pe					-0.0000	-0.0000	
					(-0.44)	(-0.79)	
illiq					79.4918	-713.5042	
					(0.34)	(-1.59)	
ivol					-0.1642	-0.2582	
					(-1.13)	(-1.35)	
turnover					-0.0003	-0.0008	
					(-0.23)	(-0.57)	
coskew					-0.0031	-0.0025	
					(-1.48)	(-1.06)	
rating						-0.0002	
						(-0.47)	
prevret							0.0090
							(0.75)
prev24mret							-0.1104
							(-1.51)
l2price							-0.0000
							(-0.69)
const	-0.0650	-0.1041	-0.1010	-0.1346	-0.1518	-0.1484	-0.2432**
	(-0.69)	(-0.90)	(-0.89)	(-1.26)	(-1.38)	(-0.65)	(-2.29)
Adj R-sqr	0.0004	0.0012	0.0015	0.0015	0.0018	0.0022	0.0008
N	1,740,062	1,198,872	963,093	963,093	943,333	311,878	1,684,956

Based on panel data tests, there is limited evidence linking autocorrelation with macro-economic state variables. Classic financial economics suggest that pricing factors ought to be related to aggregate consumption. In this section, I consider the relationship between autocorrelation and various macro-economic indicators typically associated with systematic risk (and therefore, potentially related to aggregate consumption risk).

I consider the following indicators, all at a monthly frequency: investor sentiment (orthogonalised), as per Baker and Wurgler (2006), NBER recessions (coded as a monthly dummy variable: 1 = recession), corporate bond spreads (Aaa - Baa) and US personal consumption growth (the last two indicators are sourced from the Federal Reserve Bank of St. Louis).

As before, I regress excess returns on the *explosive* and *decaying* metrics, as well as on macroeconomic state variables. To gauge the marginal impact of macroeconomic state variables on the predictive power of the *explosive* and *decaying* metrics, I add interaction terms. Specifically, I interact both the *explosive* and *decaying* metrics with various macroeconomic state variables ($macro_t$) , as shown below:

$$\begin{aligned} r_{i,t} - r_t^f &= \alpha + \beta_e \cdot explosive_{i,t-1} + \beta_d \cdot decaying_{i,t-1} + \beta_m \cdot [macro_t] \\ &\quad + \beta_{me} \cdot [macro_t] \times explosive_{i,t-1} + \beta_{md} \cdot [macro_t] \times decaying_{i,t-1} + \varepsilon_t \end{aligned}$$

This specification allows me to test whether a particular macroeconomic state variable acts to regulate the influence of either of the *explosive* and *decaying* metrics against the null hypothesis that the macroeconomic state variable does not regulate the predictive power of the two metrics. The results are set out in Table 3.10.

Table 3.10: Panel regression of excess monthly realised returns on lagged autocorrelation and lagged macroeconomic state variables and interaction terms

The table below shows the result of individual stock-level panel regressions. The dependent variable is realised excess returns. The two explanatory variables – *explosive* and *decaying* – are derived from the stock-level autocorrelation estimate as follows: $explosive = max(1, ac)$ and $decaying = min(1, ac)$. Sentiment is the investor sentiment (orthogonalised) as per Baker and Wurgler (2006). NBER recessions are coded as a monthly dummy variable (1 = recession). Corporate bond spreads (Aaa - Baa) and US personal consumption growth are sourced from the Federal Reserve Bank of St. Louis. The general regression equation can be written as $r_{i,t} - r^f_t = \alpha + \beta_e \cdot explosive_{i,t-1} + \beta_d \cdot decaying_{i,t-1} + \beta_m \cdot [macro_t] + \beta_{me} \cdot [macro_t] \times explosive_{i,t-1} + \beta_m \cdot [macro_t] \times decaying_{i,t-1} + \varepsilon_t$. In other words, I interact a number of macroeconomic state variables with *explosive* and *decaying*. All explanatory variables including the macroeconomic state variables are lagged by one month relative to the dependent variable. t-Statistics are shown in parenthesis below their associated coefficients. Significance levels are indicated by * for $p < 0.1$, ** for $p < 0.05$ and *** for $p < 0.01$ and are based on two-way clustered standard errors (clustered by individual stock and by time).

	(1) r1 b/t	(2) r2 b/t	(3) r3 b/t	(4) r4 b/t	(5) r5 b/t	(6) r6 b/t
explosive (exp)	0.1790	0.1233	0.1395	0.0828	0.1672*	0.5936
	(1.61)	(1.35)	(0.73)	(0.57)	(1.76)	(1.47)
decaying (dec)	-0.1097***	-0.0739***	-0.0027	-0.0805***	-0.0862***	0.0279
	(-5.35)	(-3.65)	(-0.04)	(-3.44)	(-3.59)	(0.47)
sentiment	0.0673					
	(0.65)					
exp X sentiment	-0.0491					
	(-0.49)					
dec X sentiment	-0.0222					
	(-1.44)					
NBER dummy		-0.1479				
		(-0.46)				
exp X NBER		0.2109				
		(0.64)				
dec X NBER		-0.0674				
		(-0.94)				
creditspread			3.4384			
			(0.21)			
exp X Aaa-Baa spread			2.3820			
			(0.15)			
dec X Aaa-Baa spread			-5.3574			
			(-0.96)			
termspread				-3.2518		
				(-0.37)		
exp X 10y - 1y spread				7.0383		
				(0.80)		
dec X 10y - 1y spread				-3.4856**		
				(-2.11)		
indprodgrowth					4.5565	
					(0.76)	
exp X indprodgrowth					-5.0764	
					(-0.87)	
dec X indprodgrowth					0.6517	
					(0.34)	
unemployment						9.7846
						(1.58)
exp X unemployment						-7.6254
						(-1.26)
dec X unemployment						-1.8294*
						(-1.80)
Const	-0.0635	-0.0424	-0.1352	0.0008	-0.0750	-0.6347
	(-0.56)	(-0.45)	(-0.69)	(0.01)	(-0.77)	(-1.56)
Adj R-sqr	0.0014	0.0005	0.0010	0.0016	0.0005	0.0022
N	1,330,987	1,717,531	1,717,531	1,536,858	1,717,531	1,590,273

Two significant results emerge: *decaying* is negatively significant when interacted with the treasury term spread and is also negatively significant when interacted with unemployment. However decreasing unemployment is associated with *lower* macro-economic risk, while decreasing term spread is commonly a precursor to lower economic growth and hence is associated with *higher* macro-economic risk. Thus the signs on these two interaction terms are contradictory, frustrating any attempt to motivate the outperformance of decaying stocks as some sort of premium for bearing macro-economic risk.

3.6 Persistence

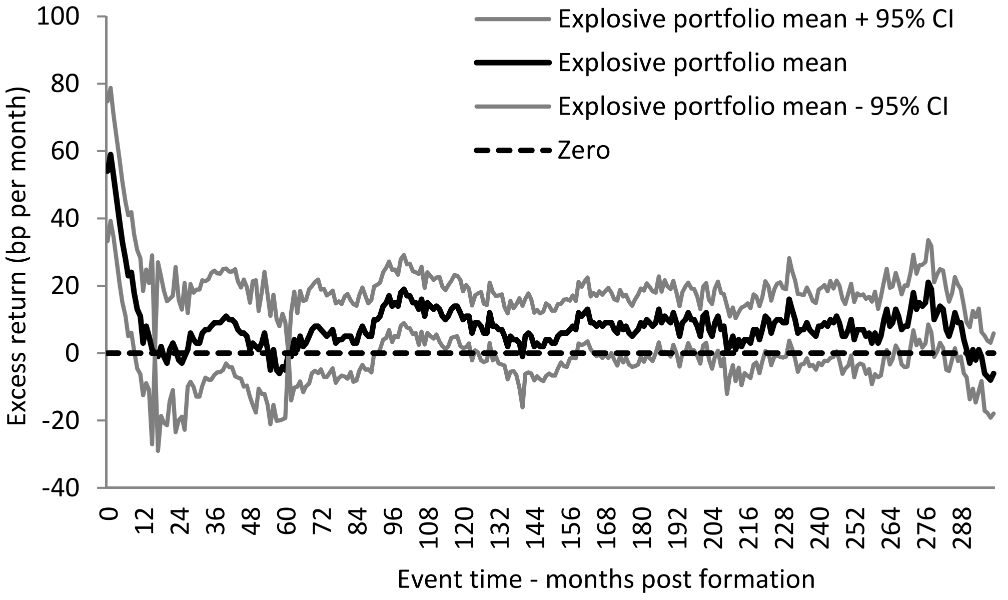
3.6.1 Persistence of explosive and decaying stocks

The excess returns accruing to decaying stocks are persistent over very long horizons, but this is not the case for explosive stocks. To investigate the return persistence of explosive and decaying stocks, I consider the excess returns accruing to the explosive and decaying hedge portfolios up to 300 months post-formation. I form the explosive and decaying hedge portfolios at month t using the autocorrelation estimates as of month $t-1$. I then calculate the average excess returns across all those portfolios (keeping their composition unchanged) for months t, $t+1$ and so forth, up to month $t+300$ (that is, 25 years later). I illustrate the average hedge portfolio excess return at each month post-formation in Figure 3.6.1, along with 95% confidence bounds. As is evident from panel A, the explosive hedge portfolio generates excess returns that peak in the second month post-formation and becomes statistically insignificant 9 months post-formation. Informally, this suggests that whatever predictability is contained in explosive stocks is most effective at a 2-month lag and becomes “stale” within 9 months. By contrast, returns to the decaying hedge portfolio are highly persistent. As shown in panel B of Figure 3.6.1, the decaying hedge portfolio generates monthly excess returns that actually increases post-formation to a maximum of 66bp per month at 15 months post formation. At longer post-formation lags the monthly excess returns become smaller but remain positive and statistically significant. Impressively, the decaying hedge portfolio generates these positive and statistically significant excess returns in *every* post-formation month for up to 25 years.

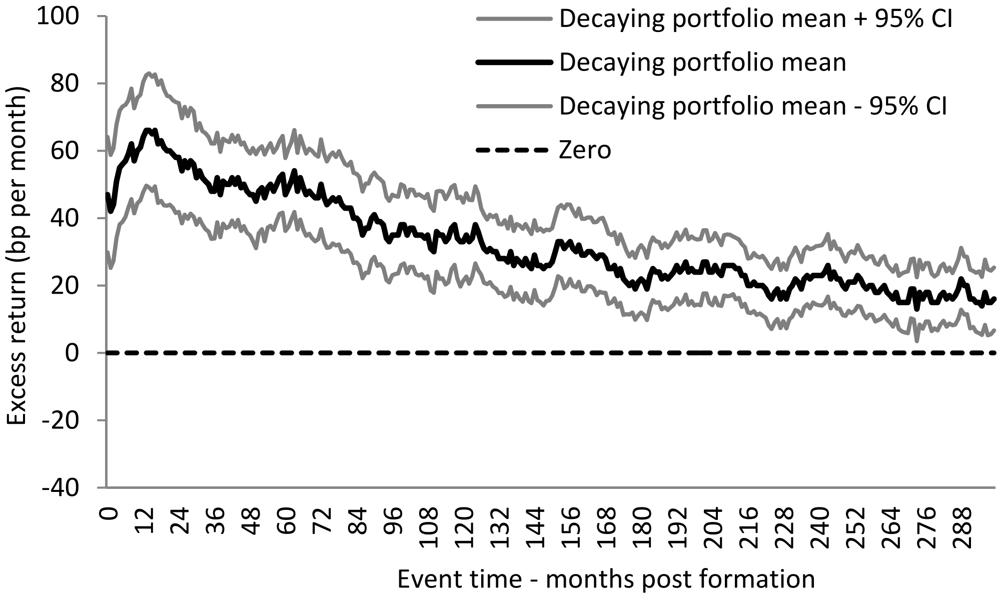
Figure 3.6.1: Mean portfolio excess return post-formation

The graphs below plot the average of monthly excess returns generated by hedge portfolios (across each possible formation month) at k months post-formation, with 95% confidence intervals. The hedge portfolios are formed from decile sorts using lagged autocorrelation. The decaying hedge portfolio is formed by going long the decile 1 stocks and going short the stocks in deciles 2 to 9 in equal measure. Similarly, the explosive hedge portfolio is formed by going long the decile 10 stocks and going short the stocks in deciles 2 to 9 in equal measure. Confidence intervals are computed using Newey-West HAC adjusted standard errors with a maximum lag length of 12 months, across each possible formation month in the dataset.

Panel A: Explosive hedge portfolio



Panel B: Decaying hedge portfolio



3.6.2 Controlling for time-series risk factors

Such a degree of return persistence is unusual. One potential explanation could be that decaying stocks are differentially exposed to some priced risk factor. To consider whether this is the impetus behind the return persistence observed in the decaying hedge portfolio, I consider time-series alphas instead of excess returns. To calculate these alphas I run the following time series regression for each lag $0 \leq k \leq 300$, where $r_t^{hedge(k)}$ denotes the excess return earned by a hedge portfolio formed k months earlier at time $t - k$.

$$\begin{aligned} r_t^{hedge(k)} &= \alpha_{(k)} + \beta_1 MktRf_t + \beta_2 SMB_t + \beta_3 HML_t + \beta_4 UMD_t \\ &\quad + \beta_5 STRev_t + \beta_6 LTRev_t + \beta_7 PSLiq_t + \varepsilon_t \end{aligned}$$

This generates a sequence of alphas, one for each lag k, which I plot in Figure 3.6.2 for both the explosive and decaying hedge portfolios. Controlling for additional time-series risk factors does not alter the picture significantly. The decaying hedge portfolio continues to generate positive excess returns in every month up to 25 years post-formation (and is significant in 287 out of 300 months); these excess returns cannot be explained by the time-series risk factors controlled for[17]. It is tempting to speculate whether this persistence is driven by some priced risk factor associated with decaying stocks. If so, such a novel risk factor would have to be unrelated to size, book-to-market, momentum, short-term reversal, long-term reversal, or liquidity. Alternatively, behavioural or institutional factors might explain the outperformance generated by decaying stocks. But any explanation that relies on a behavioural or institutional story would still have to account for this long-term persistence[18]. It is not clear why investors would continue to demand a premium for holding stocks that experienced a "decaying" episode more than two decades ago, if not as compensation for some manner of fundamental risk associated with those stocks. As noted in section 3.5, the potential motivation for such a priced risk factor remains elusive.

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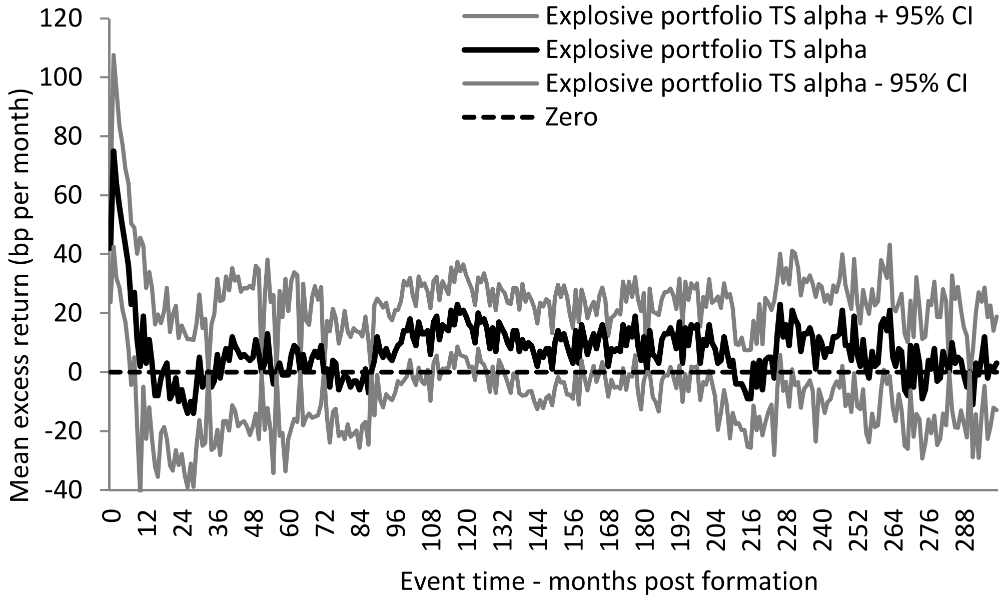
[17] The results presented in Figures 3.6.2 and 3.6.1 are essentially unchanged when using delisting adjusted returns (calculated in accordance with Beaver, McNichols and Price (2007))

[18] I show that returns around earnings announcements does not appear consistent with biased investor expectations of future earnings (see subsection 3.6.4)

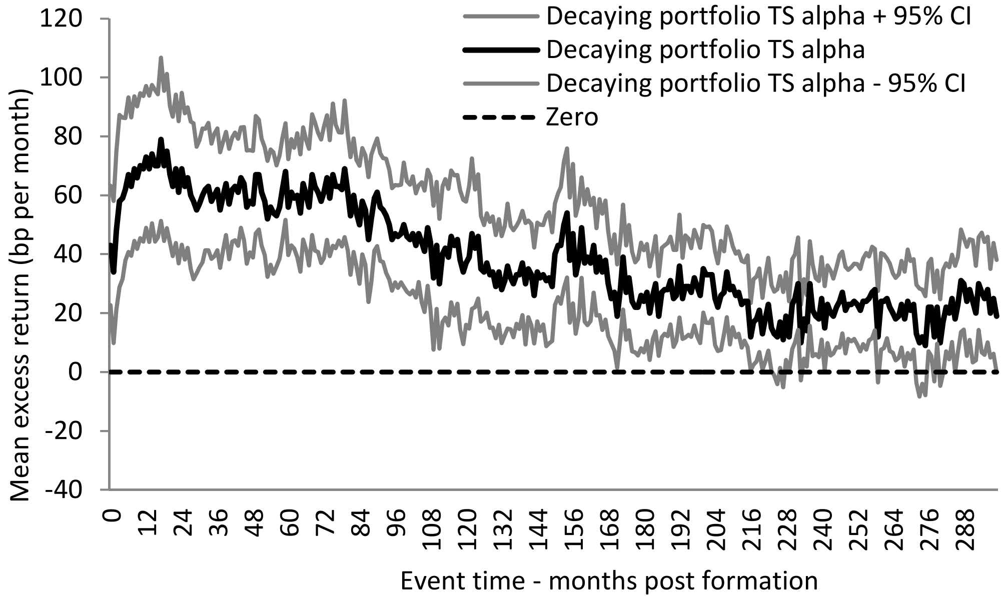
Figure 3.6.2: Portfolio time-series alphas post-formation

The graphs below plot the alpha obtained from regressing the monthly excess return generated by hedge portfolios formed k months earlier on a set of time-series risk factors, with 95% confidence intervals for the alpha estimates. The hedge portfolios are formed from decile sorts using autocorrelation from k months previously. The decaying hedge portfolio is formed by going long the decile 1 stocks and going short the stocks in deciles 2 to 9 in equal measure. Similarly, the explosive hedge portfolio is formed by going long the decile 10 stocks and going short the stocks in deciles 2 to 9 in equal measure. The time-series regression controls for the market return, SMB portfolio return, HML portfolio return, momentum portfolio return, short-term reversal return, long-term reversal return and liquidity innovations (as described more fully in subsection 3.3.5). The time series regression specification is, for each post-formation lag k contemplated: $r_t^{hedge(k)} = \alpha_{(k)} + \beta_1 mktrf_t + \beta_2 SMB_t + \beta_3 HML_t + \beta_4 UMD_t + \beta_5 STRev_t + \beta_6 LTRev_t + \beta_7 PSLiq_t + \varepsilon_t$. Confidence intervals are computed using Newey-West HAC adjusted standard errors with a maximum lag length of 12 months.

Panel A: Explosive portfolio



Panel B: Decaying portfolio



3.6.3 Survivorship and potential delisting bias

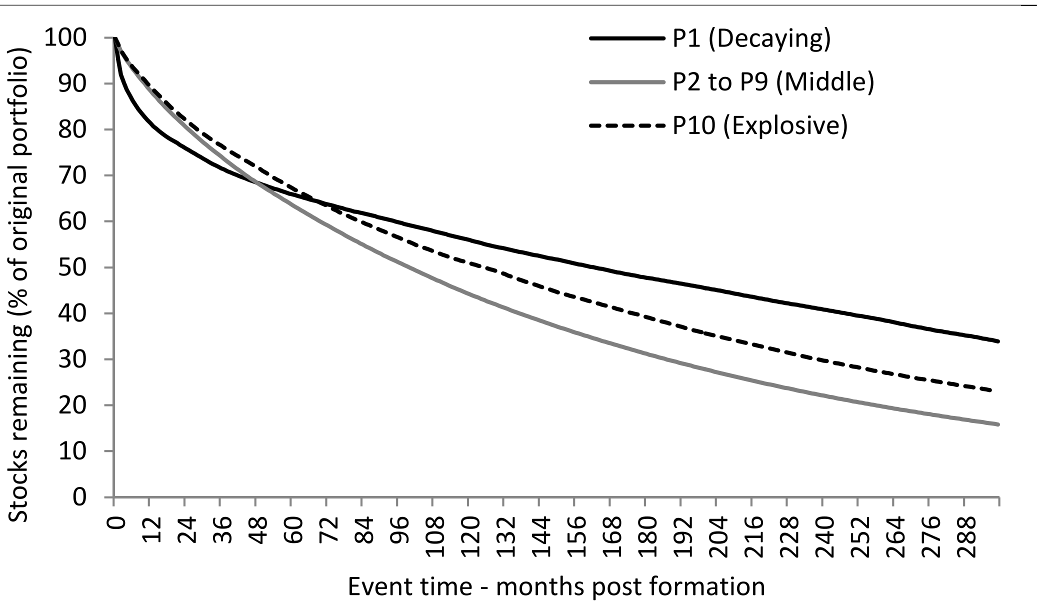
In any static portfolio stocks are likely to drop out of the portfolio over time as stocks delist from the exchange. Stocks usually delist for one of two reasons: they are acquired by other firms or they fail and are subsequently liquidated. In panel A of Table 3.6.3 I show the percentage of surviving stocks for different autocorrelation decile portfolios in event time, with the event being the original formation of the portfolio. The decaying portfolio P1 initially exhibits a high attrition rate, which subsequently moderates. By contrast, the explosive portfolio exhibits comparatively low attrition rates from the beginning. After about seven years post-formation both the decaying portfolio P1 and the explosive portfolio P10 have a higher percentage of surviving stocks than the middle portfolio consisting of deciles 2 to 9 inclusive. This is notable, considering that the stocks in both the decaying and explosive portfolios are smaller and more lowly rated than the average stock.

The differential survival experience of the autocorrelation decile portfolios points to a potential measurement problem: delisting returns. The more a portfolio suffers attrition, the more it will be affected by the omission of delisting returns. To address this concern, I construct delisting adjusted excess returns, following the approach set out in Beaver, McNichols and Price (2007). At a high level this means substituting the average delisting return for any missing delisting returns, for each delisting code in turn. These augmented delisting returns are the combined with the normal CRSP returns to create delisting adjusted excess returns. Using delisting adjusted returns instead of the standard CRSP returns yield essentially the same result.

Figure 3.6.3: Stock performance and survival (in event time)

The graph below plots the percentage of surviving stocks in event time for autocorrelation portfolios. The relevant event is the formation, in each month, of static portfolios sorted on prior month autocorrelation. Portfolio P1 (decaying stocks) consists of the lowest decile of stocks, portfolio P10 (explosive stocks) consists of the highest decile of stocks and the middle portfolio consists of all the remaining stocks (in deciles 2 to 9 inclusive).

Surviving stocks as a percentage of original portfolio



3.6.4 Returns around earnings announcements

Are decaying portfolio excess returns so persistent because investors collectively suffer from systematic negative biases when assessing the expected return of these stocks? Or are those persistent excess returns the result of investors collectively demanding compensation for some manner of risk that happens to be prevalent in the decaying portfolio? One way to distinguish between these alternative (but not mutually exclusive) explanations is to consider the abnormal returns of stocks around earnings announcements (see for instance, Jegadeesh and Titman (1993)). If investors are systematically negatively biased in forming expectations about the future earnings of decaying stocks, then we would expect them to react strongly and positively to future earnings announcements that exceed their biased expectations. In other words, if the bulk of the decaying portfolio returns occurs around earnings announcements, that could be interpreted as support an explanation involving investor biases. Alternatively, if most of the decaying portfolio returns are realised on non-earnings announcement days, that could be viewed as support for a risk premium explanation.

I start by defining the abnormal return of a stock as its CRSP total return less the return on the CRSP value-weighted index. The abnormal returns earned by the different autocorrelation portfolios are illustrated in panel A of Figure 3.6.4. Note, all the abnormal returns are positive; this is because the market return proxy I use is value-weighted, while the portfolios are formed using equal weights[19]. As before, both the decaying (P1) and explosive (P10) autocorrelation deciles outperform the middle portfolio initially. The decaying portfolio continues to generate substantial abnormal returns relative to the middle portfolios while the outperformance of the explosive portfolio is much more modest over horizons longer than one year. Panel B performs the same analysis, but restricted to abnormal returns earned by stocks in the three days centred on the earnings announcement date, which I term abnormal earnings announcement returns. Abnormal earnings announcement returns are calculated as the cumulative abnormal daily return of a stock from the trading day preceding an earnings announcement to the trading day following that earnings announcement (for a total of three trading days). It is immediately apparent that the magnitude of earnings announcement abnormal returns is much lower than that of abnormal returns generally. The mean abnormal return across all stocks in the dataset is 12.6bp per month; the mean abnormal earnings announcement return is 3.6bp per month, or roughly 28% of the total abnormal monthly return.[20]

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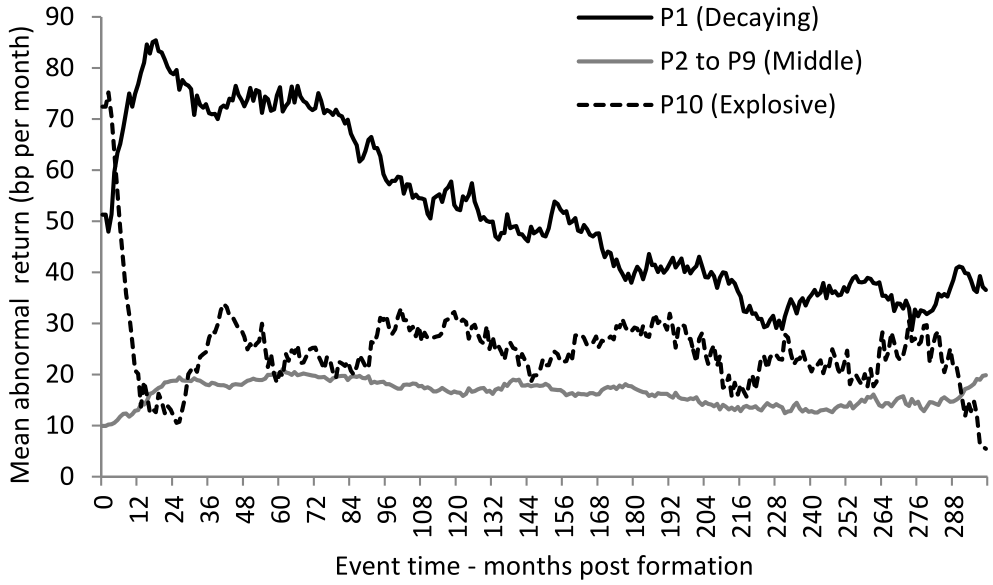
[19] As pointed out in Plyakha, Uppal and Vilkov (2012) equal weighted portfolios tend to outperform value weighted portfolios; their randomly constructed portfolio of 100 stocks generate equal weighted returns of 271 bp per annum above the market weighted returns over the last forty years. That is roughly 22 bp per month, consistent with the level of abnormal returns earned by the middle deciles (deciles 2 to 9) in panel A.

[20] Of course, the abnormal earnings announcement return is earned over 1 day per month on average for each stock (based on 3 days each quarter) while the abnormal return is usually calculated over 20 to 21 trading days per month. Therefore, the actual daily return on earnings announcements will typically dwarf the daily returns on other days.

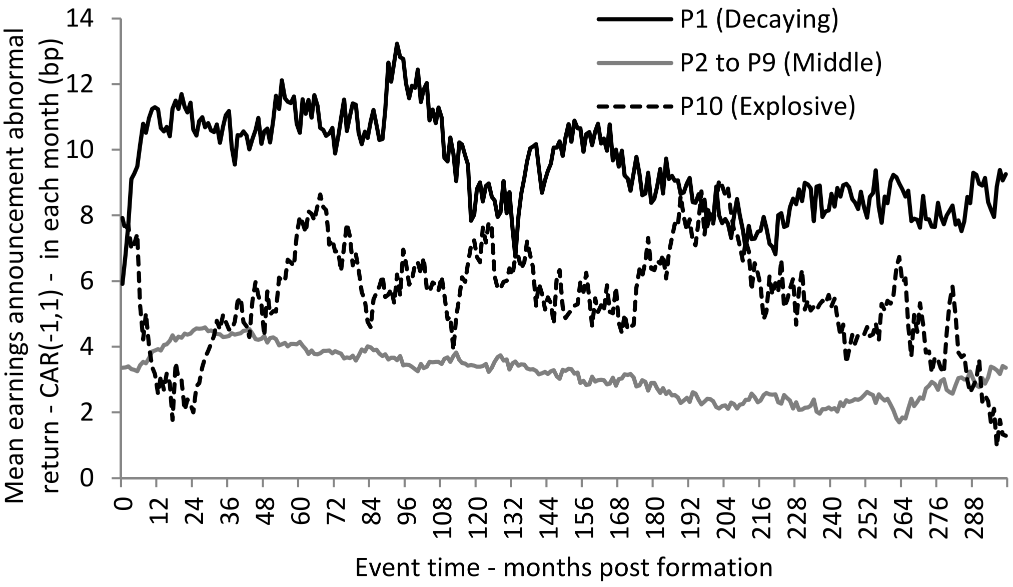
Figure 3.6.4: Stock abnormal returns around earnings announcements (in event time)

The graphs below plots mean abnormal returns in event time for autocorrelation portfolios. The relevant event is the formation, in each month, of static portfolios sorted on prior month autocorrelation. Portfolio P1 (decaying stocks) consists of the lowest decile of stocks, portfolio P10 (explosive stocks) consists of the highest decile of stocks and the middle portfolio consists of all the remaining stocks (in deciles 2 to 9 inclusive). Panel A plots mean abnormal returns where abnormal return is the monthly total return of each stock less the CRSP value weighted market return. Panel B plots mean abnormal earnings announcement returns. Abnormal earnings announcement returns are calculated as the cumulative abnormal return (as defined above) of a stock over the three trading days centred on the date of its earnings announcement (returns on other days being ignored), that is CAR(-1,1) returns.

Panel A: Mean abnormal returns



Panel B: Mean abnormal earnings announcement returns

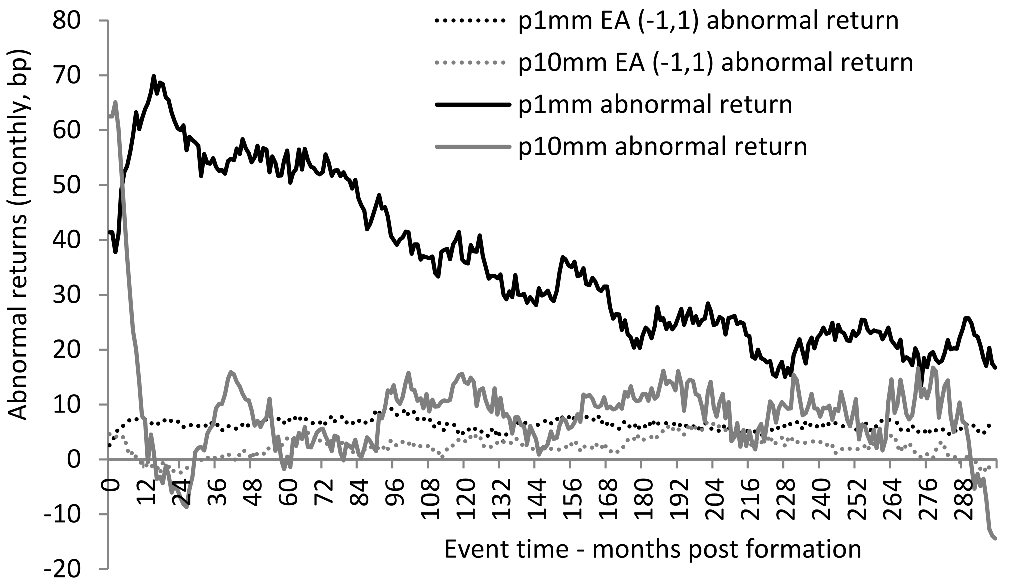


To return to our original question, how much of the return persistence observed in the decaying portfolio can be attributed to returns around earnings announcements? To directly address this question I plot both abnormal returns and abnormal earnings announcement returns for the hedge portfolios in panel A of Figure 3.6.5. Since I am considering the hedge portfolios the abnormal returns are those over and above the abnormal returns earned by the middle portfolio. In the first year post formation abnormal earnings announcement returns form only a small fraction of the total abnormal return; about 10%. This argues against an investor bias explanation, at least over short horizons. Over longer horizons the decaying hedge portfolio continues to earn most of its abnormal returns outside earnings announcements dates although the gap tightens over time. Over all 300 post formation months considered, the abnormal earnings announcement return of the decaying hedge portfolio makes up 21% of the total total abnormal returns earned by the decaying hedge portfolio. This is reassuringly close to, if somewhat below, the 28% ratio observed for stocks generally. In panel B I make the comparison even more explicit by calculating abnormal earnings announcements as a percentage of total abnormal returns for the autocorrelation deciles, again in event time. The resulting percentages are mostly contained in the 15% to 35% range. Overall, there is a tendency for the middle portfolio to lag the decaying and explosive portfolio as the post formation horizon becomes longer. On the whole this analysis tends to favours a risk based rather than an investor bias based explanation for the persistent returns earned by the decaying portfolio.

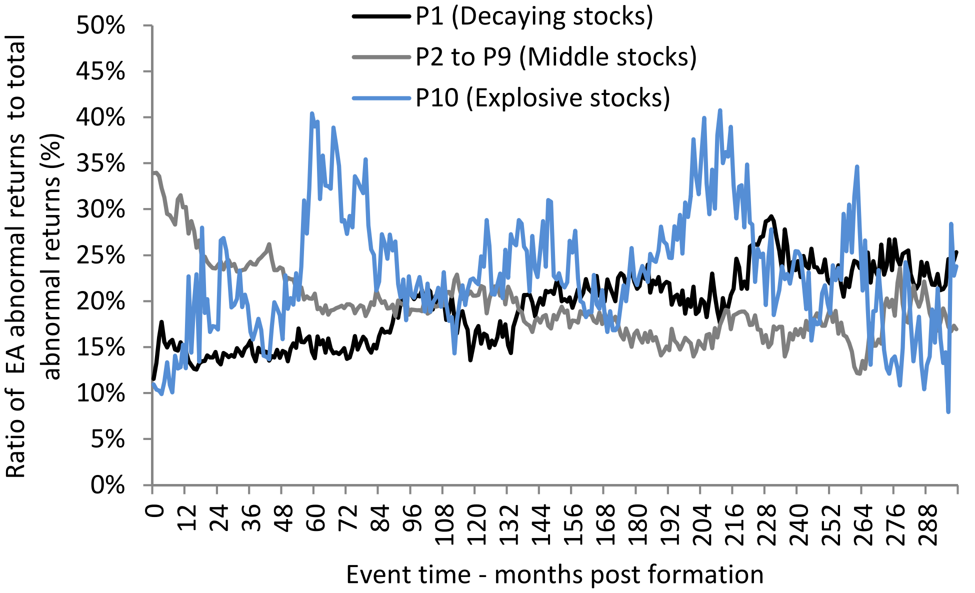
Figure 3.6.5: Relative contribution of earnings announcement days to abnormal returns

The graph in panel A below plots the mean abnormal hedge portfolio return in event time for autocorrelation portfolios. Abnormal return is the monthly total return of each stock less the CRSP value weighted market return. The relevant event is the formation, in each month, of static portfolios sorted on prior month autocorrelation. Portfolio p1mm (the decaying hedge portfolio) is long the lowest decile of stocks and short the middle deciles (2 to 9 inclusive) in equal measure. Portfolio p10mm (the explosive hedge portfolio) is long the highest decile of stocks and short the middle deciles (2 to 9 inclusive) in equal measure. Panel B plots mean earnings announcement abnormal returns as a percentage of total abnormal returns for portfolio P1 (decile 1 stocks), P2 to P9 (stocks in the middle deciles, 2 to 9 inclusive) and P10 (decile 10 stocks). Abnormal earnings announcement returns are calculated as the cumulative abnormal return (as defined above) of a stock over the three trading days centred on the date of its earnings announcement (returns on other days being ignored), that is CAR(-1,1) returns.

Panel A: Abnormal hedge portfolio returns – EA vs All



Panel B: Earnings announcement contribution to abnormal returns



3.7 Robustness

In the previous sections I have demonstrated that the predictive ability of autocorrelation is robust to a number of known priced risk factors, both in time-series and cross-sectionally. The sorted portfolio tests as per Table 3.4 and the panel data regressions as per Table 3.9 (hereafter the “main results”) are also robust to different estimation windows, data frequencies, sample periods and realised return definitions, as I show below.

3.7.1 Sample periods

To test whether my main results also hold in sub-samples, I partition my dataset chronologically into four sample periods. The first sample period runs from 1923 to the end of 1945, the second sample period from the beginning of 1946 to the end of 1965, the third sample period from the beginning of 1966 to the end of 1985 and the last sample period from the beginning of 1986 to the end of 2013. The results are summarised in Table 3.11. Because of data constraints[21] I only run panel data tests for the last two sample periods (from 1966 onwards). The mean return of the explosive hedge portfolio (p10mm) is positive and significant at the 5% level in each of the four chronological sample periods. The decaying hedge portfolio is positive in each of the four chronological sample periods and significant at the 10% level in all sample periods. Panel data regressions yield positive coefficients for the *explosive* metric in each of the last two sample periods. However, these coefficients are statistically significant only in the last sample period (1986 to 2013). The *decaying* metric is negative and significant at the 1% level in every specification in both of the two most recent sample periods.

Overall the evidence suggests that the predictive power of decaying and explosive stocks is not restricted to a single sample period. While the sorted portfolio evidence for a predictive effect for decaying stocks in the period 1986 to 2013 is weak in comparison to other sample periods, this is countered by strong evidence of predictability for the *decaying* metric in panel data over the same period.

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[21] Most of the panel data specifications require stock characteristics that are based on accounting data sourced from Compustat. Quarterly compustat data are only available from 1961 onwards.

Table 3.11: Sample period robustness results

The table below summarises the results of robustness checks relating to sample period. The first column identifies the range of the relevant sample. The second two columns, labelled "Mean" and "Count Sig α", details the excess returns accruing to a value-weighted portfolio and the number of time-series regressions resulting in an alpha that is significant at the 10% level (out of 10; based on 5 time series regression specifications times 2 portfolio weighting schemes). Columns 2 and 3 relate to the explosive hedge portfolio while columns 4 and 5 relate to the decaying hedge portfolio. Both hedge portfolios are formed on the basis of decile sorts, as explained in Table 3.4. The final two columns record the number of significant coefficients (at the 10% level) obtained for each of the *explosive* and *decaying* metrics from seven panel data regressions, as explained more fully in Table 3.9. Entries marked "n/a" could not be calculated due to a lack of data. The pre-1946 time series alpha tests do not include Pástor and Stambaugh (2003) liquidity since it is not available; hence the significance counts are scored out of 8 rather than 10.

	Sorted Portfolio Excess Returns (VW, bp)				Panel Regressions	
	Explosive		Decaying		*Explosive*	*Decaying*
Sample window	Mean	Count Sig α	Mean	Count Sig α	Count Sig	Count Sig
1923-2013	47***	9/10	40***	10/10	7/7	7/7
Pre-1946	36**	7/8	29**	7/8	n/a	n/a
1946-1965	29**	6/10	35***	9/10	n/a	n/a
1966-1985	57***	8/10	41***	10/10	1/7	7/7
1985-2013	60**	9/10	51*	5/10	6/7	7/7

3.7.2 Estimation windows and data frequencies

The main results are robust to different estimation windows and data frequencies in the calculation of autocorrelation. I estimated autocorrelation using a 24-month rolling estimation window and monthly data. As a robustness check I reproduce the main results using autocorrelation calculated with both monthly and daily data. For autocorrelation calculated on monthly data I recalculate the results using rolling windows of 6, 12, 36, 48 and 60 months in addition to the 24 months used before. The detailed results are summarised in Table 3.12 below.

Table 3.12: Estimation window robustness results (monthly data)

The table below summarises the results of robustness checks relating to estimation windows. The first column specifies the estimation window used. The second two columns, labelled "Mean" and "Count Sig α", details the excess returns accruing to a value-weighted portfolio and the number of time-series regressions resulting in an alpha that is significant at the 10% level (out of 10; based on 5 time series regression specifications times 2 portfolio weighting schemes). Columns 2 and 3 relate to the explosive hedge portfolio while columns 4 and 5 relate to the decaying hedge portfolio. Both hedge portfolios are formed on the basis of decile sorts, as explained in Table 3.4. The final two columns record the number of significant coefficients (at the 10% level) obtained for each of the *explosive* and *decaying* metrics from seven panel data regressions, as explained more fully in Table 3.9.

	Sorted Portfolio Excess Returns (VW, bp)				Panel Regressions	
	Explosive		Decaying		*Explosive*	*Decaying*
Estimation window	Mean	Count Sig α	Mean	Count Sig α	Count Sig	Count Sig
6 Months	41***	9/10	23**	6/10	7/7	7/7
12 Months	68***	10/10	24**	7/10	7/7	7/7
24 Months	47***	9/10	40***	10/10	7/7	7/7
36 Months	29**	9/10	48***	10/10	5/7	7/7
48 Months	23*	7/10	49***	10/10	0/7	7/7
60 Months	11	2/10	52***	10/10	0/7	7/7

The predicative power of explosive stocks holds up well for shorter estimation windows but tends to decrease as the window length increases. This may be because the predictability due to explosive stocks is relatively short term, peaking at 2 months post formation. By contrast, the evidence supporting predictability due to decaying stocks tends to increase as the window length is increased. The fact that returns generated by decaying stocks are highly persistent may help make them less sensitive to different estimation windows. Overall the main results appear robust to different estimation windows.

Turning to daily data, in Table 3.13 I summarise the results of autocorrelation estimated over different monthly windows using *daily* data. As before explosive stocks do better with shorter estimation windows while decaying stocks do (slightly) better with longer estimation windows. Overall, the results suggest that the main results are robust to a range of different estimation windows, although explosive stocks and decaying stocks show different patterns in predictive ability as window lengths change.

Table 3.13: Estimation window robustness results (using daily data, estimated each month)

The table below summarises the results of robustness checks relating to estimation windows, using daily data. The first column specificities the estimation window used. The second two columns, labelled "Mean" and "Count Sig α", details the excess returns accruing to a value-weighted portfolio and the number of time-series regressions resulting in an alpha that is significant at the 10% level (out of 10; based on 5 time series regression specifications times 2 portfolio weighting schemes). Columns 2 and 3 relate to the explosive hedge portfolio while columns 4 and 5 relate to the decaying hedge portfolio. Both hedge portfolios are formed on the basis of decile sorts, as explained in Table 3.4. The final two columns record the number of significant coefficients (at the 10% level) obtained for each of the *explosive* and *decaying* metrics from seven panel data regressions, as explained more fully in Table 3.9.

	Sorted Portfolio Excess Returns (VW, bp)				Panel Regressions	
	Explosive		Decaying		*Explosive*	*Decaying*
Estimation window	Mean	Count Sig α	Mean	Count Sig α	Count Sig	Count Sig
1 Months	12	2/10	35***	9/10	6/7	6/7
2 Months	23**	4/10	29***	10/10	7/7	6/7
3 Months	33***	9/10	28***	9/10	7/7	7/7
6 Months	43***	9/10	26**	7/10	7/7	7/7
12 Months	71***	10/10	26**	7/10	7/7	7/7
24 Months	44***	9/10	42***	10/10	5/7	7/7
36 Months	27**	9/10	45***	10/10	3/7	7/7
48 Months	18	6/10	51***	10/10	0/7	7/7
60 Months	10	2/10	52***	10/10	0/7	7/7

3.7.3 Different realised return measures

In the results presented earlier I performed tests using realised excess returns (the monthly raw CRSP total return ("ret") less the risk free rate). I also consider whether the main results are robust to using unadjusted monthly CRSP returns. I find that they are. The same result holds when using realised returns excluding dividends ("retx") and when using excess realised returns excluding dividends ("retx" less "rf", to be precise). This suggests that my main results are not driven by a particular choice of realised return. Also, since the results continue to hold after excluding dividends from realised returns, it suggests that systematic differences in the underlying dividend cash flow of different stocks are not the main driver of the main results. This finding may be of interest to those who would like to link autocorrelation to bubbles.[22]

To ensure that my results are not tainted by missing delisting returns (see Shumway (1997)) I follow the approach in Beaver, McNichols and Price (2007); I fill in missing delisting returns by using the average delisting return for each delisting code. I then combine these augmented delisting returns with normal returns to generate delisting adjusted excess returns ("dleret"). Using delisting adjusted returns as the dependent variable does not materially alter the main results. Table 3.14 summarises these results.

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[22] I prefer to remain agnostic on the matter, particularly because I am not formally presenting or testing a theory of bubbles.

Table 3.14: Realised return robustness results

The table below summarises the results of robustness checks relating to the realised return used. The first column specificities the realised return metric used. The second two columns, labelled "Mean" and "Count Sig α", details the excess returns accruing to a value-weighted portfolio and the number of time-series regressions resulting in an alpha that is significant at the 10% level (out of 10; based on 5 time series regression specifications times 2 portfolio weighting schemes). Columns 2 and 3 relate to the explosive hedge portfolio while columns 4 and 5 relate to the decaying hedge portfolio. Both hedge portfolios are formed on the basis of decile sorts, as explained in Table 3.4. The final two columns record the number of significant coefficients (at the 10% level) obtained for each of the *explosive* and *decaying* metrics from seven panel data regressions, as explained more fully in Table 3.9.

	Sorted Portfolio Excess Returns (VW, bp)				Panel Regressions	
	Explosive		Decaying		*Explosive*	*Decaying*
Realised return	Mean	Count Sig α	Mean	Count Sig α	Count Sig	Count Sig
eret (base case)	46***	9/10	40***	10/10	7/7	7/7
ret	55***	10/10	47***	10/10	7/7	7/7
retx	55***	10/10	47***	10/10	7/7	7/7
eretx	47***	9/10	40***	10/10	7/7	7/7
dleret	46***	9/10	40***	10/10	7/7	7/7

3.7.4 Other robustness checks

The main results continue to hold when winsorizing autocorrelation at the 1% level. This suggests that the main results are not driven by outliers. I also re-estimate autocorrelation using prices constructed from CRSP total returns[23]. This way I incorporate non-price related returns such as dividends and other distributions directly into the price series used to estimate autocorrelation. The results are summarised in Table 3.15

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[23] I start with the price at the start of the window, then inflate it using the CRSP total return ("ret") to generate a price series over the relevant window. These synthetic "total return" prices are then used to estimate autocorrelation in the same way as before.

Table 3.15: Other robustness results

The table below summarises the results of two further robustness checks. The row first considers the impact of winsorizing autocorrelation. The second row uses an estimate of autocorrelation based on synthetic prices constructed from realised total returns instead of actual prices (which would not include dividends and other distributions). The second two columns, labelled "Mean" and "Count Sig α", details the excess returns accruing to a value-weighted portfolio and the number of time-series regressions resulting in an alpha that is significant at the 10% level (out of 10; based on 5 time series regression specifications times 2 portfolio weighting schemes). Columns 2 and 3 relate to the explosive hedge portfolio while columns 4 and 5 relate to the decaying hedge portfolio. Both hedge portfolios are formed on the basis of decile sorts, as explained in Table 3.4. The final two columns record the number of significant coefficients (at the 10% level) obtained for each of the *explosive* and *decaying* metrics from seven panel data regressions, as explained more fully in Table 3.9.

	Sorted Portfolio Excess Returns (VW, bp)				Panel Regressions	
	Explosive		Decaying		*Explosive*	*Decaying*
	Mean	Count Sig α	Mean	Count Sig α	Count Sig	Count Sig
Winsorized	47***	9/10	40***	10/10	5/7	7/7
Total return prices	52***	10/10	41***	10/10	7/7	7/7

3.8 Conclusion

I introduce autocorrelation as a new predictor of realised returns. Broadly speaking, autocorrelation at the stock level captures past deviations from a pure random walk in log-stock prices. In both portfolio tests and panel data regressions, autocorrelation is a statistically and economically significant predictor of subsequent realised returns. This predictability appears to be distinct from a range of time-series risk factors and cross-sectional stock characteristics. The predictability due to low autocorrelations – decaying stocks – is particularly persistent; a "decaying" hedge portfolio formed by investing in stocks with low historical autocorrelation and shorting "middle" stocks continues (on average) to generate statistically significant monthly excess returns even at a lag of more than two decades post-formation.

An analysis of returns around earnings announcements does not support biased investor expectations as an explanation for this persistence. This suggests a risk factor explanation; however, preliminary work has not uncovered a convincing relationship with macroeconomic state variables that might motivate such an interpretation. Much work remains to be done to understand why autocorrelation predicts returns, and why excess returns accruing to decaying stocks are so persistent.

3.9 Appendix

3.9.1 Portfolio time-series alphas (common risk factors)

Table 3.16: Explosive portfolio (P10 minus middle portfolio) excess returns, regressed on common risk factors

Panel A: Equal weighted portfolios

	(1)	(2)	(3)	(4)	(5)	(6)
	mean	market	sizebook	FF3	Carhart4	all
	b/t	b/t	b/t	b/t	b/t	b/t
MktRf		0.0952**		0.1110***	0.1828***	0.1418***
		(2.56)		(3.98)	(5.98)	(5.15)
SMB			0.2565***	0.1993**	0.2198***	0.3407***
			(2.65)	(2.02)	(2.68)	(7.30)
HML			-0.3547***	-0.3873***	-0.2422***	-0.4020***
			(-3.74)	(-4.58)	(-3.32)	(-4.37)
MOM					0.3224***	0.4485***
					(5.08)	(13.24)
STRev						-0.1377***
						(-3.55)
LTRev						0.0348
						(0.45)
PSLiq						-0.0088
						(-0.39)
Const	0.0054***	0.0048***	0.0062***	0.0058***	0.0025***	0.0042***
	(5.10)	(4.62)	(6.74)	(6.58)	(3.03)	(4.49)
Adj R-sqr	-0.0000	0.0290	0.2325	0.2666	0.4730	0.7318
N	1,025	1,025	1,025	1,025	1,025	605

Panel B: Value weighted portfolios

	(1)	(2)	(3)	(4)	(5)	(6)
	mean	market	sizebook	FF3	Carhart4	all
	b/t	b/t	b/t	b/t	b/t	b/t
MktRf		0.1062**		0.1337***	0.2201***	0.1330***
		(2.47)		(3.64)	(5.56)	(3.61)
SMB			0.2502**	0.1813	0.2061**	0.3477***
			(2.21)	(1.55)	(2.12)	(5.92)
HML			-0.4031***	-0.4423***	-0.2676***	-0.5112***
			(-3.66)	(-4.52)	(-3.02)	(-4.24)
MOM					0.3884***	0.5343***
					(5.83)	(10.83)
STRev						-0.1546***
						(-2.71)
LTRev						-0.0482
						(-0.50)
PSLiq						-0.0339
						(-1.20)
Const	0.0047***	0.0040***	0.0056***	0.0051***	0.0012	0.0037***
	(3.95)	(3.38)	(5.44)	(5.15)	(1.36)	(3.25)
Adj R-sqr	-0.0000	0.0246	0.1896	0.2234	0.4286	0.6671
N	1,026	1,025	1,025	1,025	1,025	605

Note: Significance levels are indicated by * for $p < 0.1$, ** for $p < 0.05$ and *** for $p < 0.01$ and are computed using Newey-West HAC adjusted standard errors with a maximum lag length of 12 months.

Table 3.17: Decaying portfolio (P1 minus middle) excess returns, regressed on common risk factors

Panel A: Equal weighted portfolios

	(1) mean b/t	(2) market b/t	(3) sizebook b/t	(4) FF3 b/t	(5) Carhart4 b/t	(6) all b/t
MktRf		0.1741***		0.1293***	0.1046***	0.1898***
		(4.70)		(4.38)	(4.54)	(4.72)
SMB			0.3448***	0.2781***	0.2710***	0.2789***
			(10.34)	(10.03)	(11.11)	(6.80)
HML			-0.0417	-0.0796*	-0.1294**	-0.1188*
			(-0.99)	(-1.68)	(-2.52)	(-1.92)
MOM					-0.1106**	-0.2594***
					(-2.00)	(-5.08)
STRev						0.0384
						(0.82)
LTRev						0.1851***
						(3.17)
PSLiq						-0.0160
						(-0.84)
Const	0.0047***	0.0036***	0.0040***	0.0035***	0.0046***	0.0043***
	(5.38)	(4.32)	(5.21)	(4.54)	(5.30)	(4.16)
Adj R-sqr	-0.0000	0.1466	0.2109	0.2797	0.3149	0.5255
N	1,025	1,025	1,025	1,025	1,025	605

Panel B: Value weighted portfolios

	(1) mean b/t	(2) market b/t	(3) sizebook b/t	(4) FF3 b/t	(5) Carhart4 b/t	(6) all b/t
MktRf		0.1517***		0.1434***	0.1118***	0.1955***
		(3.24)		(3.54)	(3.19)	(3.39)
SMB			0.2405***	0.1665***	0.1575**	0.2548***
			(4.04)	(2.68)	(2.44)	(3.04)
HML			-0.1334**	-0.1755***	-0.2393***	-0.3050***
			(-2.55)	(-3.42)	(-3.87)	(-3.16)
MOM					-0.1419**	-0.3461***
					(-2.00)	(-4.80)
STRev						-0.0026
						(-0.05)
LTRev						0.3300***
						(3.28)
PSLiq						-0.0345
						(-1.08)
Const	0.0040***	0.0031***	0.0040***	0.0034***	0.0048***	0.0057***
	(3.58)	(2.79)	(3.53)	(3.07)	(3.74)	(3.29)
Adj R-sqr	-0.0000	0.0677	0.0772	0.1288	0.1642	0.4057
N	1,026	1,025	1,025	1,025	1,025	605

Note: Significance levels are indicated by * for $p < 0.1$, ** for $p < 0.05$ and *** for $p < 0.01$ and are computed using Newey-West HAC adjusted standard errors with a maximum lag length of 12 months.

3.9.2 Double sorts

Table 3.18: Two-way tabulations showing mean monthly excess return (bp)

Panel A

	P/E ratio quintiles						
ac quintiles	Low	2	3	4	High	HML	All
Decaying	15	150	134	97	60	45***	87
2	25	119	76	25	-5	-30***	65
3	47	114	66	30	-22	-69***	65
4	66	126	66	31	-23	-89***	68
Explosive	101	183	111	75	49	-52***	103
HMM	59***	63***	42***	46***	65***		38***
LMM	-28***	31***	65***	68***	76***		21***
All	28	132	84	47	12	-16***	66

Panel B

	Major rating class (AAA/AA, A, BBB, BB,)						
ac quintiles	High	2	3	4	Low	HML	All
Decaying	55	108	111	70	44	-11	87
2	60	88	71	51	23	-37**	65
3	71	72	62	57	20	-51***	65
4	66	60	70	68	13	-53***	68
Explosive	103	65	89	83	106	3	103
HMM	36**	-8	22**	24**	88***		38***
LMM	-12	34**	43***	11	26*		21***
All	69	76	75	62	45	-24***	66

Panel C

	Turnover quintiles						
ac quintiles	Low	2	3	4	High	HML	All
Decaying	54	79	94	97	80	26***	87
2	52	61	73	68	57	5	65
3	56	68	77	66	34	-22***	65
4	67	74	75	71	32	-35***	68
Explosive	94	99	109	109	93	-1	103
HMM	36***	31***	34***	40***	52***		38***
LMM	-4	12**	19***	28***	38***		21***
All	56	67	76	75	63	8**	66

Panel D

	Amihud illiquidity quintiles						
ac quintiles	Liquid	2	3	4	Illiquid	HML	All
Decaying	68	77	70	85	102	34***	87
2	56	62	53	67	76	20***	65
3	53	55	66	66	78	25***	65
4	50	55	67	74	90	40***	68
Explosive	80	82	93	114	135	55***	103
HMM	27***	25***	31***	45***	54***		38***
LMM	15***	20***	8	16***	21***		21***
All	59	63	65	77	91	32***	66

Note: Significance levels are indicated by * for $p < 0.1$, ** for $p < 0.05$ and *** for $p < 0.01$ and are computed using a two-sample t-test with unequal variances.

Table 3.19: Two-way tabulations showing mean monthly excess return (bp)

Panel A

	Idiosyncratic volatility quintiles						
ac quintiles	Decaying	2	3	4	Volatile	HML	All
Decaying	92	98	112	102	34	-58***	87
2	65	74	81	68	6	-59***	65
3	69	75	72	58	2	-67***	65
4	81	80	81	49	-2	-83***	68
Explosive	110	112	119	103	64	-46***	103
HMM	38***	35***	40***	44***	62***		38***
LMM	20***	22***	34***	43***	32***		21***
All	73	81	87	70	3	-70***	66

Panel B

	60-month coskewsness quintiles						
ac quintiles	Low	2	3	4	High	HML	All
Decaying	107	94	90	93	91	-16*	87
2	71	70	64	64	65	-6	65
3	78	69	69	66	58	-20***	65
4	88	79	74	65	50	-38***	68
Explosive	117	107	104	94	78	-39***	103
HMM	38***	34***	35***	29***	20***		38***
LMM	28***	21***	21***	28***	33***		21***
All	92	84	80	76	68	-24***	66

Panel C

	2 period lagged price quintiles						
ac quintiles	Low	2	3	4	High	HML	All
Decaying	67	90	102	106	113	46***	87
2	53	67	73	66	63	10	65
3	53	57	74	67	64	11	65
4	69	67	69	71	62	-7	68
Explosive	109	117	111	98	91	-18**	103
HMM	52***	54***	38***	29***	27***		38***
LMM	10*	27***	30***	37***	50***		21***
All	50	66	76	71	71	21***	66

Panel D

	NYSE size quintiles						
ac quintiles	Small	2	3	4	Big	HML	All
Decaying	81	85	98	92	84	3	87
2	56	60	74	78	58	2	65
3	65	69	67	64	57	-8	65
4	82	73	70	60	53	-29***	68
Explosive	133	110	99	92	69	-64***	103
HMM	66***	43***	28***	25***	13***		38***
LMM	13***	17***	28***	25***	27***		21***
All	64	67	71	68	59	-5**	66

Note: Significance levels are indicated by * for $p < 0.1$, ** for $p < 0.05$ and *** for $p < 0.01$ and are computed using a two-sample t-test with unequal variances.

Table 3.20: Two-way tabulations showing mean monthly excess return (bp)

Panel A

	NBER recession Dummy			
ac quintiles	Expansion	Recession	HML	All
Decaying	96	28	-68***	87
2	79	-21	-100***	65
3	80	-30	-110***	65
4	87	-54	-141***	68
Explosive	127	-44	-171***	103
HMM	45***	-9		38***
LMM	14***	63***		21***
All	84	-45	-129***	66

Note: Significance levels are indicated by * for p < 0.1, ** for p < 0.05 and *** for p < 0.01 and are computed using a two-sample t-test with unequal variances.

3.9.3 Double-sorted time series alphas

Table 3.21: Alpha's sorted on momentum

Portfolio (qmomentum)	p1mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	0.0050***	0.0044***	0.0045***	0.0049***	0.0050***
Equal weighted: quintile 2	0.0033***	0.0023**	0.0022**	0.0026***	0.0010
Equal weighted: quintile 3	0.0022**	0.0012	0.0012	0.0020**	0.0022**
Equal weighted: quintile 4	0.0046***	0.0035***	0.0029***	0.0041***	0.0028**
Equal weighted: quintile 5	0.0068***	0.0060***	0.0059***	0.0063***	0.0062***
Value weighted: quintile 1	0.0044***	0.0032**	0.0031**	0.0038***	0.0042**
Value weighted: quintile 2	0.0030**	0.0016	0.0018	0.0029**	0.0044***
Value weighted: quintile 3	0.0025*	0.0014	0.0014	0.0025*	0.0013
Value weighted: quintile 4	0.0043***	0.0033**	0.0030**	0.0041***	0.0032
Value weighted: quintile 5	0.0070***	0.0061***	0.0060***	0.0072***	0.0085***

Portfolio (qmomentum)	p10mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	0.0004	0.0004	0.0013	-0.0015	-0.0014
Equal weighted: quintile 2	0.0003	0.0003	0.0015	-0.0013*	-0.0011
Equal weighted: quintile 3	0.0025**	0.0015	0.0020**	0.0008	-0.0001
Equal weighted: quintile 4	0.0052***	0.0046***	0.0055***	0.0039***	0.0055***
Equal weighted: quintile 5	0.0081***	0.0075***	0.0084***	0.0070***	0.0087***
Value weighted: quintile 1	0.0007	0.0007	0.0015	-0.0016	-0.0019
Value weighted: quintile 2	-0.0000	-0.0005	0.0007	-0.0023**	0.0001
Value weighted: quintile 3	0.0020	0.0008	0.0014	-0.0001	0.0008
Value weighted: quintile 4	0.0053***	0.0043***	0.0052***	0.0038***	0.0063***
Value weighted: quintile 5	0.0088***	0.0077***	0.0085***	0.0069***	0.0110***

Note: Significance levels are indicated by * for p < 0.1, ** for p < 0.05 and *** for p < 0.01 and are computed using Newey-West HAC adjusted standard errors with a maximum lag length of 12 months.

Table 3.22: Alpha's sorted on CAPM beta

Portfolio (qcapmbeta)	p1mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	0.0009	0.0001	-0.0003	-0.0000	-0.0006
Equal weighted: quintile 2	0.0037***	0.0028***	0.0026***	0.0031***	0.0018*
Equal weighted: quintile 3	0.0049***	0.0044***	0.0039***	0.0041***	0.0031**
Equal weighted: quintile 4	0.0032***	0.0028***	0.0029***	0.0042***	0.0054***
Equal weighted: quintile 5	0.0069***	0.0064***	0.0062***	0.0070***	0.0079***
Value weighted: quintile 1	0.0007	-0.0004	-0.0005	0.0010	0.0016
Value weighted: quintile 2	0.0041***	0.0036***	0.0033**	0.0048***	0.0027
Value weighted: quintile 3	0.0034**	0.0030**	0.0029**	0.0035**	0.0042**
Value weighted: quintile 4	0.0039**	0.0030*	0.0029*	0.0054***	0.0058***
Value weighted: quintile 5	0.0076***	0.0068***	0.0063***	0.0077***	0.0098***

Portfolio (qcapmbeta)	p10mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	0.0037***	0.0030***	0.0036***	0.0014*	0.0022***
Equal weighted: quintile 2	0.0037***	0.0034***	0.0041***	0.0019**	0.0026**
Equal weighted: quintile 3	0.0047***	0.0049***	0.0058***	0.0028***	0.0039***
Equal weighted: quintile 4	0.0041***	0.0043***	0.0055***	0.0020**	0.0036***
Equal weighted: quintile 5	0.0046***	0.0043***	0.0049***	0.0018	0.0054***
Value weighted: quintile 1	0.0028**	0.0022**	0.0030***	0.0007	0.0018
Value weighted: quintile 2	0.0038***	0.0035***	0.0040***	0.0017*	0.0023*
Value weighted: quintile 3	0.0060***	0.0059***	0.0067***	0.0031***	0.0046***
Value weighted: quintile 4	0.0052***	0.0052***	0.0060***	0.0024**	0.0044***
Value weighted: quintile 5	0.0058***	0.0054***	0.0062***	0.0022	0.0065***

Note: Significance levels are indicated by * for $p < 0.1$, ** for $p < 0.05$ and *** for $p < 0.01$ and are computed using Newey-West HAC adjusted standard errors with a maximum lag length of 12 months.

Table 3.23: Alpha's sorted on size

Portfolio (qsize)	p1mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	0.0046***	0.0040***	0.0044***	0.0042***	0.0038***
Equal weighted: quintile 2	0.0050***	0.0038***	0.0040***	0.0048***	0.0047***
Equal weighted: quintile 3	0.0053***	0.0038***	0.0037***	0.0053***	0.0063***
Equal weighted: quintile 4	0.0041***	0.0028**	0.0027**	0.0050***	0.0047***
Equal weighted: quintile 5	0.0025**	0.0017	0.0020*	0.0038***	0.0036**
Value weighted: quintile 1	0.0046***	0.0038***	0.0042***	0.0040***	0.0038***
Value weighted: quintile 2	0.0051***	0.0038***	0.0040***	0.0049***	0.0051***
Value weighted: quintile 3	0.0053***	0.0038***	0.0037***	0.0053***	0.0063***
Value weighted: quintile 4	0.0042***	0.0029**	0.0027**	0.0051***	0.0049***
Value weighted: quintile 5	0.0031***	0.0022*	0.0023**	0.0044***	0.0044**

Portfolio (qsize)	p10mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	0.0033**	0.0029**	0.0038***	0.0012	0.0033***
Equal weighted: quintile 2	0.0057***	0.0055***	0.0061***	0.0030***	0.0044***
Equal weighted: quintile 3	0.0050***	0.0048***	0.0059***	0.0025**	0.0043***
Equal weighted: quintile 4	0.0055***	0.0046***	0.0054***	0.0021	0.0034**
Equal weighted: quintile 5	0.0046***	0.0038***	0.0049***	0.0016*	0.0043***
Value weighted: quintile 1	0.0040***	0.0037***	0.0045***	0.0019	0.0035***
Value weighted: quintile 2	0.0056***	0.0054***	0.0061***	0.0030***	0.0040***
Value weighted: quintile 3	0.0051***	0.0049***	0.0061***	0.0025**	0.0044***
Value weighted: quintile 4	0.0054***	0.0045***	0.0052***	0.0019	0.0036**
Value weighted: quintile 5	0.0043***	0.0034***	0.0043***	0.0012	0.0040***

Note: Significance levels are indicated by * for $p < 0.1$, ** for $p < 0.05$ and *** for $p < 0.01$ and are computed using Newey-West HAC adjusted standard errors with a maximum lag length of 12 months.

Table 3.24: Alpha's sorted on book-to-market

Portfolio (qb2m)	p1mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	0.0034	0.0024	0.0014	0.0039*	0.0032
Equal weighted: quintile 2	0.0055***	0.0040**	0.0040**	0.0058***	0.0054***
Equal weighted: quintile 3	0.0041**	0.0026	0.0026	0.0048***	0.0048***
Equal weighted: quintile 4	0.0051***	0.0034**	0.0035**	0.0061***	0.0063***
Equal weighted: quintile 5	0.0017	-0.0002	-0.0009	0.0008	0.0010
Value weighted: quintile 1	0.0088***	0.0073**	0.0064**	0.0096***	0.0092***
Value weighted: quintile 2	0.0055***	0.0041*	0.0035*	0.0057***	0.0055***
Value weighted: quintile 3	0.0034	0.0016	0.0007	0.0049**	0.0052**
Value weighted: quintile 4	0.0032	0.0013	0.0018	0.0052**	0.0049**
Value weighted: quintile 5	0.0026	0.0002	-0.0005	0.0020	0.0024

Portfolio (qb2m)	p10mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	0.0119***	0.0112***	0.0123***	0.0084***	0.0091***
Equal weighted: quintile 2	0.0091***	0.0086***	0.0090***	0.0054***	0.0059***
Equal weighted: quintile 3	0.0069***	0.0064***	0.0069***	0.0040***	0.0039***
Equal weighted: quintile 4	0.0049***	0.0046***	0.0051***	0.0021**	0.0025**
Equal weighted: quintile 5	0.0029**	0.0029**	0.0039***	0.0010	0.0016
Value weighted: quintile 1	0.0108***	0.0093***	0.0116***	0.0074***	0.0084***
Value weighted: quintile 2	0.0071***	0.0062***	0.0071***	0.0023	0.0028
Value weighted: quintile 3	0.0046***	0.0038**	0.0042***	0.0005	0.0004
Value weighted: quintile 4	0.0030	0.0026	0.0034**	-0.0004	-0.0001
Value weighted: quintile 5	0.0030	0.0027	0.0042**	0.0004	0.0014

Note: Significance levels are indicated by * for $p < 0.1$, ** for $p < 0.05$ and *** for $p < 0.01$ and are computed using Newey-West HAC adjusted standard errors with a maximum lag length of 12 months.

Table 3.25: Alpha's sorted on price-earnings ratio

Portfolio (qpe)	p1mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	0.0062***	0.0051***	0.0044**	0.0062***	0.0065***
Equal weighted: quintile 2	0.0044***	0.0030**	0.0023*	0.0039***	0.0040***
Equal weighted: quintile 3	0.0079***	0.0068***	0.0060***	0.0083***	0.0079***
Equal weighted: quintile 4	0.0075***	0.0065***	0.0059***	0.0079***	0.0076***
Equal weighted: quintile 5	0.0092***	0.0085***	0.0079***	0.0107***	0.0106***
Value weighted: quintile 1	0.0056**	0.0043*	0.0031	0.0052**	0.0056**
Value weighted: quintile 2	0.0036*	0.0021	0.0014	0.0043**	0.0042**
Value weighted: quintile 3	0.0076***	0.0065***	0.0055***	0.0084***	0.0086***
Value weighted: quintile 4	0.0059***	0.0048***	0.0041**	0.0075***	0.0071***
Value weighted: quintile 5	0.0089***	0.0077***	0.0074***	0.0116***	0.0112***

Portfolio (qpe)	p10mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	0.0087***	0.0083***	0.0101***	0.0054***	0.0067***
Equal weighted: quintile 2	0.0048***	0.0039***	0.0046***	0.0013	0.0016
Equal weighted: quintile 3	0.0066***	0.0056***	0.0061***	0.0038***	0.0042***
Equal weighted: quintile 4	0.0094***	0.0087***	0.0098***	0.0068***	0.0067***
Equal weighted: quintile 5	0.0105***	0.0099***	0.0119***	0.0078***	0.0081***
Value weighted: quintile 1	0.0060***	0.0054**	0.0080***	0.0029	0.0049**
Value weighted: quintile 2	0.0042**	0.0031*	0.0040**	0.0002	0.0008
Value weighted: quintile 3	0.0044**	0.0034*	0.0043**	0.0007	0.0012
Value weighted: quintile 4	0.0077***	0.0069***	0.0084***	0.0044***	0.0046***
Value weighted: quintile 5	0.0110***	0.0103***	0.0127***	0.0085***	0.0092***

Note: Significance levels are indicated by * for $p < 0.1$, ** for $p < 0.05$ and *** for $p < 0.01$ and are computed using Newey-West HAC adjusted standard errors with a maximum lag length of 12 months.

Table 3.26: Alpha's sorted on turnover

Portfolio (qturnover)	p1mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	0.0030**	0.0017	0.0015	0.0023**	-0.0006
Equal weighted: quintile 2	0.0028***	0.0017*	0.0012	0.0018*	0.0020*
Equal weighted: quintile 3	0.0054***	0.0041***	0.0037***	0.0047***	0.0053***
Equal weighted: quintile 4	0.0040***	0.0032***	0.0033***	0.0051***	0.0058***
Equal weighted: quintile 5	0.0071***	0.0067***	0.0066***	0.0080***	0.0081***
Value weighted: quintile 1	0.0017	-0.0003	-0.0006	0.0013	0.0003
Value weighted: quintile 2	0.0020	0.0008	0.0005	0.0017	0.0022
Value weighted: quintile 3	0.0052***	0.0043***	0.0042***	0.0054***	0.0071***
Value weighted: quintile 4	0.0026*	0.0021	0.0026*	0.0042***	0.0061***
Value weighted: quintile 5	0.0084***	0.0075***	0.0073***	0.0093***	0.0095***

Portfolio (qturnover)	p10mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	0.0031***	0.0027***	0.0032***	0.0017**	0.0032***
Equal weighted: quintile 2	0.0036***	0.0032***	0.0037***	0.0018**	0.0025***
Equal weighted: quintile 3	0.0050***	0.0048***	0.0055***	0.0027***	0.0041***
Equal weighted: quintile 4	0.0068***	0.0068***	0.0077***	0.0052***	0.0047***
Equal weighted: quintile 5	0.0082***	0.0083***	0.0091***	0.0055***	0.0069***
Value weighted: quintile 1	0.0026**	0.0017	0.0019*	0.0006	0.0015
Value weighted: quintile 2	0.0030***	0.0022***	0.0027***	0.0005	0.0016
Value weighted: quintile 3	0.0052***	0.0049***	0.0057***	0.0027***	0.0043***
Value weighted: quintile 4	0.0064***	0.0059***	0.0066***	0.0034***	0.0042***
Value weighted: quintile 5	0.0087***	0.0085***	0.0095***	0.0059***	0.0083***

Note: Significance levels are indicated by * for $p < 0.1$, ** for $p < 0.05$ and *** for $p < 0.01$ and are computed using Newey-West HAC adjusted standard errors with a maximum lag length of 12 months.

Table 3.27: Alpha's sorted on Amihud illiquidity

Portfolio (qilliq)	p1mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	0.0029***	0.0021*	0.0024**	0.0048***	0.0046***
Equal weighted: quintile 2	0.0046***	0.0034***	0.0034***	0.0055***	0.0059***
Equal weighted: quintile 3	0.0044***	0.0031***	0.0032***	0.0050***	0.0047***
Equal weighted: quintile 4	0.0041***	0.0031***	0.0032***	0.0039***	0.0035**
Equal weighted: quintile 5	0.0052***	0.0042***	0.0040***	0.0040***	0.0035**
Value weighted: quintile 1	0.0035***	0.0026**	0.0028**	0.0053***	0.0055***
Value weighted: quintile 2	0.0043***	0.0033***	0.0035***	0.0052***	0.0060***
Value weighted: quintile 3	0.0042***	0.0027**	0.0025**	0.0041***	0.0047***
Value weighted: quintile 4	0.0043***	0.0029***	0.0029***	0.0032***	0.0030**
Value weighted: quintile 5	0.0071***	0.0052***	0.0049***	0.0048***	0.0040**

Portfolio (qilliq)	p10mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	0.0051***	0.0042***	0.0052***	0.0019*	0.0041***
Equal weighted: quintile 2	0.0059***	0.0053***	0.0063***	0.0026**	0.0033***
Equal weighted: quintile 3	0.0054***	0.0052***	0.0060***	0.0027**	0.0036***
Equal weighted: quintile 4	0.0049***	0.0048***	0.0059***	0.0025**	0.0053***
Equal weighted: quintile 5	0.0036***	0.0029**	0.0036***	0.0010	0.0032***
Value weighted: quintile 1	0.0044***	0.0035***	0.0044***	0.0014	0.0038***
Value weighted: quintile 2	0.0050***	0.0043***	0.0053***	0.0017	0.0032**
Value weighted: quintile 3	0.0045***	0.0038***	0.0044***	0.0014	0.0027**
Value weighted: quintile 4	0.0040**	0.0039***	0.0049***	0.0011	0.0040***
Value weighted: quintile 5	0.0045***	0.0037**	0.0040***	0.0014	0.0030*

Note: Significance levels are indicated by * for $p < 0.1$, ** for $p < 0.05$ and *** for $p < 0.01$ and are computed using Newey-West HAC adjusted standard errors with a maximum lag length of 12 months.

Table 3.28: Alpha's sorted on idiosyncratic volatility

Portfolio (qivol)	p1mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	0.0029***	0.0021***	0.0020***	0.0025***	0.0028***
Equal weighted: quintile 2	0.0032***	0.0024***	0.0025***	0.0034***	0.0030***
Equal weighted: quintile 3	0.0061***	0.0054***	0.0053***	0.0062***	0.0056***
Equal weighted: quintile 4	0.0057***	0.0053***	0.0053***	0.0063***	0.0074***
Equal weighted: quintile 5	0.0077***	0.0074***	0.0074***	0.0079***	0.0076***
Value weighted: quintile 1	0.0036***	0.0027***	0.0027***	0.0036***	0.0038***
Value weighted: quintile 2	0.0032***	0.0025**	0.0028***	0.0041***	0.0046***
Value weighted: quintile 3	0.0070***	0.0062***	0.0058***	0.0077***	0.0078***
Value weighted: quintile 4	0.0056***	0.0051***	0.0049***	0.0059***	0.0073***
Value weighted: quintile 5	0.0093***	0.0084***	0.0082***	0.0087***	0.0084***

Portfolio (qivol)	p10mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	0.0044***	0.0037***	0.0041***	0.0028***	0.0035***
Equal weighted: quintile 2	0.0051***	0.0046***	0.0053***	0.0030***	0.0043***
Equal weighted: quintile 3	0.0064***	0.0061***	0.0070***	0.0040***	0.0051***
Equal weighted: quintile 4	0.0057***	0.0055***	0.0063***	0.0027***	0.0058***
Equal weighted: quintile 5	0.0060***	0.0061***	0.0072***	0.0032**	0.0074***
Value weighted: quintile 1	0.0040***	0.0032***	0.0039***	0.0020***	0.0043***
Value weighted: quintile 2	0.0055***	0.0049***	0.0056***	0.0028**	0.0035***
Value weighted: quintile 3	0.0066***	0.0066***	0.0074***	0.0036***	0.0054***
Value weighted: quintile 4	0.0072***	0.0066***	0.0073***	0.0030**	0.0071***
Value weighted: quintile 5	0.0068***	0.0068***	0.0079***	0.0026	0.0082***

Note: Significance levels are indicated by * for $p < 0.1$, ** for $p < 0.05$ and *** for $p < 0.01$ and are computed using Newey-West HAC adjusted standard errors with a maximum lag length of 12 months.

Table 3.29: Alpha's sorted on 60-month coskewness

Portfolio (qcoskew)	p1mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	0.0040***	0.0030***	0.0030***	0.0037***	0.0035***
Equal weighted: quintile 2	0.0042***	0.0026**	0.0018*	0.0039***	0.0041***
Equal weighted: quintile 3	0.0043***	0.0036***	0.0036***	0.0044***	0.0028**
Equal weighted: quintile 4	0.0040***	0.0034***	0.0035***	0.0043***	0.0049***
Equal weighted: quintile 5	0.0042***	0.0033***	0.0034***	0.0034**	0.0044***
Value weighted: quintile 1	0.0047***	0.0034**	0.0036**	0.0050***	0.0036*
Value weighted: quintile 2	0.0039**	0.0023	0.0017	0.0049***	0.0063***
Value weighted: quintile 3	0.0056***	0.0051***	0.0052***	0.0065***	0.0056***
Value weighted: quintile 4	0.0056***	0.0043**	0.0043**	0.0066***	0.0079***
Value weighted: quintile 5	0.0038**	0.0026*	0.0027*	0.0033*	0.0050**

Portfolio (qcoskew)	p10mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	0.0036***	0.0037***	0.0046***	0.0014	0.0025**
Equal weighted: quintile 2	0.0039***	0.0037***	0.0046***	0.0013	0.0029**
Equal weighted: quintile 3	0.0040***	0.0039***	0.0050***	0.0016*	0.0039***
Equal weighted: quintile 4	0.0044***	0.0034***	0.0041***	0.0011	0.0025*
Equal weighted: quintile 5	0.0035***	0.0037***	0.0052***	0.0012	0.0022**
Value weighted: quintile 1	0.0033***	0.0031**	0.0038***	0.0005	0.0029**
Value weighted: quintile 2	0.0045***	0.0038***	0.0041***	0.0009	0.0023*
Value weighted: quintile 3	0.0052***	0.0045***	0.0052***	0.0015	0.0043***
Value weighted: quintile 4	0.0046***	0.0035***	0.0039***	0.0006	0.0029**
Value weighted: quintile 5	0.0033**	0.0034**	0.0050***	0.0012	0.0030*

Note: Significance levels are indicated by * for $p < 0.1$, ** for $p < 0.05$ and *** for $p < 0.01$ and are computed using Newey-West HAC adjusted standard errors with a maximum lag length of 12 months.

Table 3.30: Alpha's sorted on rating level

Portfolio (qrating)	p1mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	-0.0039	-0.0062	-0.0073	-0.0038	-0.0039
Equal weighted: quintile 2	0.0035	0.0014	-0.0000	0.0032	0.0030
Equal weighted: quintile 3	0.0017	-0.0006	-0.0021	0.0018	0.0016
Equal weighted: quintile 4	0.0048*	0.0030	0.0017	0.0041	0.0041
Equal weighted: quintile 5	0.0085**	0.0069*	0.0051	0.0071**	0.0074**
Value weighted: quintile 1	-0.0021	-0.0045	-0.0058	-0.0015	-0.0016
Value weighted: quintile 2	0.0045*	0.0025	0.0016	0.0054**	0.0051*
Value weighted: quintile 3	0.0018	-0.0007	-0.0018	0.0025	0.0025
Value weighted: quintile 4	0.0080**	0.0062	0.0048	0.0076**	0.0077**
Value weighted: quintile 5	0.0099**	0.0084*	0.0050	0.0080**	0.0085**

Portfolio (qrating)	p10mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	-0.0003	-0.0014	0.0001	-0.0021	-0.0018
Equal weighted: quintile 2	0.0007	-0.0000	0.0018	-0.0012	-0.0011
Equal weighted: quintile 3	0.0036	0.0029	0.0044**	0.0010	0.0014
Equal weighted: quintile 4	0.0088***	0.0088***	0.0106***	0.0066***	0.0070***
Equal weighted: quintile 5	0.0114***	0.0112***	0.0127***	0.0076***	0.0082***
Value weighted: quintile 1	0.0006	-0.0004	0.0011	-0.0013	-0.0009
Value weighted: quintile 2	0.0015	0.0008	0.0028	-0.0001	0.0000
Value weighted: quintile 3	0.0055*	0.0043*	0.0062**	0.0031	0.0034
Value weighted: quintile 4	0.0105***	0.0099***	0.0122***	0.0079***	0.0085***
Value weighted: quintile 5	0.0074**	0.0076**	0.0082**	0.0040	0.0047

Note: Significance levels are indicated by * for $p < 0.1$, ** for $p < 0.05$ and *** for $p < 0.01$ and are computed using Newey-West HAC adjusted standard errors with a maximum lag length of 12 months.

Table 3.31: Alpha's sorted on previous month excess return

Portfolio (qprevret)	p1mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	0.0065***	0.0058***	0.0055***	0.0060***	0.0074***
Equal weighted: quintile 2	0.0034***	0.0022**	0.0018*	0.0025**	0.0018
Equal weighted: quintile 3	0.0040***	0.0031***	0.0032***	0.0037***	0.0038***
Equal weighted: quintile 4	0.0036***	0.0025**	0.0026***	0.0041***	0.0037***
Equal weighted: quintile 5	0.0049***	0.0045***	0.0047***	0.0055***	0.0045***
Value weighted: quintile 1	0.0056***	0.0046***	0.0043***	0.0057***	0.0073***
Value weighted: quintile 2	0.0033**	0.0026*	0.0029**	0.0042**	0.0054***
Value weighted: quintile 3	0.0044***	0.0033**	0.0032**	0.0047***	0.0069***
Value weighted: quintile 4	0.0057***	0.0042***	0.0041***	0.0063***	0.0066***
Value weighted: quintile 5	0.0059***	0.0055***	0.0057***	0.0066***	0.0069***

Portfolio (qprevret)	p10mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	0.0034***	0.0032***	0.0037***	0.0007	0.0026**
Equal weighted: quintile 2	0.0040***	0.0034***	0.0043***	0.0014	0.0015
Equal weighted: quintile 3	0.0051***	0.0045***	0.0054***	0.0022***	0.0038***
Equal weighted: quintile 4	0.0060***	0.0055***	0.0064***	0.0038***	0.0053***
Equal weighted: quintile 5	0.0108***	0.0100***	0.0108***	0.0081***	0.0087***
Value weighted: quintile 1	0.0038***	0.0033**	0.0037***	0.0009	0.0030*
Value weighted: quintile 2	0.0041***	0.0029**	0.0038***	0.0005	0.0017
Value weighted: quintile 3	0.0037***	0.0030**	0.0043***	0.0009	0.0032***
Value weighted: quintile 4	0.0053***	0.0047***	0.0056***	0.0028***	0.0042***
Value weighted: quintile 5	0.0093***	0.0087***	0.0097***	0.0057***	0.0083***

Note: Significance levels are indicated by * for $p < 0.1$, ** for $p < 0.05$ and *** for $p < 0.01$ and are computed using Newey-West HAC adjusted standard errors with a maximum lag length of 12 months.

Table 3.32: Alpha's sorted on mean 24 month previous excess return

Portfolio (qprev24mret)	p1mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	0.0032***	0.0027***	0.0031***	0.0028**	0.0032**
Equal weighted: quintile 2	0.0048***	0.0036***	0.0040***	0.0040***	0.0027**
Equal weighted: quintile 3	0.0037***	0.0025**	0.0027**	0.0032***	0.0027*
Equal weighted: quintile 4	0.0033***	0.0023*	0.0026**	0.0033**	0.0031**
Equal weighted: quintile 5	0.0067***	0.0057***	0.0059***	0.0072***	0.0066***
Value weighted: quintile 1	0.0038***	0.0033***	0.0040***	0.0040***	0.0056***
Value weighted: quintile 2	0.0062***	0.0052***	0.0058***	0.0061***	0.0069***
Value weighted: quintile 3	0.0036***	0.0026**	0.0032**	0.0043***	0.0049***
Value weighted: quintile 4	0.0040***	0.0030**	0.0034**	0.0048***	0.0054***
Value weighted: quintile 5	0.0082***	0.0073***	0.0074***	0.0086***	0.0097***

Portfolio (qprev24mret)	p10mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	0.0022**	0.0033***	0.0039***	0.0027***	0.0014
Equal weighted: quintile 2	0.0019**	0.0027***	0.0031***	0.0030***	0.0018**
Equal weighted: quintile 3	0.0042***	0.0050***	0.0053***	0.0052***	0.0047***
Equal weighted: quintile 4	0.0052***	0.0060***	0.0064***	0.0055***	0.0060***
Equal weighted: quintile 5	0.0079***	0.0083***	0.0090***	0.0071***	0.0099***
Value weighted: quintile 1	0.0001	0.0010	0.0016*	0.0001	-0.0005
Value weighted: quintile 2	0.0016*	0.0022***	0.0026***	0.0017**	0.0010
Value weighted: quintile 3	0.0029***	0.0032***	0.0036***	0.0031***	0.0038***
Value weighted: quintile 4	0.0045***	0.0049***	0.0053***	0.0045***	0.0056***
Value weighted: quintile 5	0.0089***	0.0088***	0.0093***	0.0079***	0.0114***

Note: Significance levels are indicated by * for $p < 0.1$, ** for $p < 0.05$ and *** for $p < 0.01$ and are computed using Newey-West HAC adjusted standard errors with a maximum lag length of 12 months.

Table 3.33: Alpha's sorted on 2-period lagged price

Portfolio (ql2price)	p1mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	0.0061***	0.0058***	0.0062***	0.0064***	0.0072***
Equal weighted: quintile 2	0.0049***	0.0040***	0.0040***	0.0049***	0.0040***
Equal weighted: quintile 3	0.0049***	0.0039***	0.0041***	0.0047***	0.0043***
Equal weighted: quintile 4	0.0043***	0.0038***	0.0042***	0.0053***	0.0045***
Equal weighted: quintile 5	0.0036***	0.0031***	0.0030***	0.0043***	0.0048***
Value weighted: quintile 1	0.0053***	0.0051***	0.0055***	0.0054***	0.0077***
Value weighted: quintile 2	0.0037***	0.0030**	0.0028**	0.0039***	0.0033*
Value weighted: quintile 3	0.0021	0.0009	0.0011	0.0022	0.0024
Value weighted: quintile 4	0.0034***	0.0027***	0.0029***	0.0044***	0.0032**
Value weighted: quintile 5	0.0037***	0.0032***	0.0030***	0.0049***	0.0051***

Portfolio (ql2price)	p10mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	0.0036**	0.0037**	0.0043***	0.0012	0.0018
Equal weighted: quintile 2	0.0039***	0.0037***	0.0046***	0.0016	0.0030***
Equal weighted: quintile 3	0.0051***	0.0044***	0.0050***	0.0016	0.0032***
Equal weighted: quintile 4	0.0052***	0.0043***	0.0052***	0.0025***	0.0037***
Equal weighted: quintile 5	0.0059***	0.0051***	0.0061***	0.0031***	0.0064***
Value weighted: quintile 1	0.0030*	0.0026	0.0028*	-0.0012	0.0012
Value weighted: quintile 2	0.0035**	0.0038***	0.0046***	0.0009	0.0022
Value weighted: quintile 3	0.0034**	0.0025*	0.0032***	-0.0009	0.0021
Value weighted: quintile 4	0.0057***	0.0049***	0.0057***	0.0021*	0.0040***
Value weighted: quintile 5	0.0060***	0.0050***	0.0060***	0.0031***	0.0063***

Note: Significance levels are indicated by * for $p < 0.1$, ** for $p < 0.05$ and *** for $p < 0.01$ and are computed using Newey-West HAC adjusted standard errors with a maximum lag length of 12 months.

Table 3.34: Alpha's sorted on NYSE size quintile breakpoints

Portfolio (qnysesize)	p1mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	0.0053***	0.0049**	0.0054***	0.0056***	0.0043***
Equal weighted: quintile 2	0.0042***	0.0032***	0.0033***	0.0044***	0.0039***
Equal weighted: quintile 3	0.0046***	0.0036***	0.0041***	0.0057***	0.0058***
Equal weighted: quintile 4	0.0028**	0.0015	0.0014	0.0032**	0.0033**
Equal weighted: quintile 5	0.0028***	0.0020**	0.0020**	0.0043***	0.0041***
Value weighted: quintile 1	0.0046**	0.0041**	0.0046**	0.0049**	0.0045***
Value weighted: quintile 2	0.0045***	0.0034***	0.0035***	0.0048***	0.0043***
Value weighted: quintile 3	0.0048***	0.0037***	0.0042***	0.0060***	0.0060***
Value weighted: quintile 4	0.0028**	0.0016	0.0014	0.0034**	0.0035**
Value weighted: quintile 5	0.0031***	0.0022**	0.0021*	0.0046***	0.0044**

Portfolio (qnysesize)	p10mm Mean	CAPM-α	3F-α	4F-α	All-α
Equal weighted: quintile 1	0.0013	0.0012	0.0021	-0.0014	0.0034***
Equal weighted: quintile 2	0.0055***	0.0049***	0.0058***	0.0032***	0.0052***
Equal weighted: quintile 3	0.0046***	0.0046***	0.0057***	0.0020*	0.0037***
Equal weighted: quintile 4	0.0053***	0.0047***	0.0058***	0.0022*	0.0032**
Equal weighted: quintile 5	0.0043***	0.0034***	0.0043***	0.0010	0.0037***
Value weighted: quintile 1	0.0019	0.0018	0.0027	-0.0010	0.0034***
Value weighted: quintile 2	0.0053***	0.0046***	0.0055***	0.0030***	0.0050***
Value weighted: quintile 3	0.0047***	0.0048***	0.0059***	0.0022**	0.0037***
Value weighted: quintile 4	0.0054***	0.0047***	0.0058***	0.0022*	0.0033**
Value weighted: quintile 5	0.0040***	0.0030**	0.0039***	0.0009	0.0034**

Note: Significance levels are indicated by * for $p < 0.1$, ** for $p < 0.05$ and *** for $p < 0.01$ and are computed using Newey-West HAC adjusted standard errors with a maximum lag length of 12 months.

3.9.4 Portfolio time-series alphas (macroeconomic risk factors)

Table 3.35: Autocorrelation decaying hedge portfolio (P1 minus middle) excess returns, regressed on macroeconomic risk factors

Panel A: Equal weighted portfolios

	(1) nber b/t	(2) indprodgrowth b/t	(3) unemployment b/t	(4) sentiment b/t	(5) creditspread b/t	(6) termspread b/t	(7) all b/t
NBER recessions (=1)	0.0019						0.0011
	(0.88)						(0.23)
Industrial Production Growth Rate		0.1088*					-0.3150
		(1.78)					(-1.62)
Unemployment Rate			0.0708				0.1375
			(1.08)				(0.67)
Baker - Wurgler (2006) Sentiment Index				-0.0007			-0.0008
				(-0.70)			(-0.70)
Baa-Aaa Bond Spread					0.1116		-0.1748
					(0.66)		(-0.24)
Treasury 10Yr Less 1Yr						0.0434	0.0122
						(0.37)	(0.07)
Const	0.0043***	0.0044***	0.0002	0.0051***	0.0034*	0.0040***	-0.0009
	(4.62)	(5.08)	(0.06)	(3.68)	(1.92)	(4.07)	(-0.15)
Adj R-sqr	-0.0001	0.0056	0.0009	-0.0013	0.0001	-0.0011	0.0008
N	1,025	1,025	785	546	1,025	722	546

Panel B: Value weighted portfolios

	(1) nber b/t	(2) indprodgrowth b/t	(3) unemployment b/t	(4) sentiment b/t	(5) creditspread b/t	(6) termspread b/t	(7) all b/t
NBER recessions (=1)	-0.0022						-0.0026
	(-0.72)						(-0.47)
Industrial Production Growth Rate		0.0191					-0.2385
		(0.53)					(-0.90)
Unemployment Rate			0.0991				0.3220
			(1.14)				(1.38)
Baker - Wurgler (2006) Sentiment Index				0.0001			0.0008
				(0.06)			(0.53)
Baa-Aaa Bond Spread					-0.0770		-0.7804
					(-0.48)		(-0.98)
Treasury 10Yr Less 1Yr						0.1231	0.0271
						(0.78)	(0.12)
Const	0.0044***	0.0040***	-0.0014	0.0053***	0.0049***	0.0035**	-0.0052
	(3.75)	(3.43)	(-0.28)	(2.72)	(2.65)	(2.49)	(-0.62)
Adj R-sqr	-0.0003	-0.0009	0.0011	-0.0018	-0.0007	0.0001	0.0002
N	1,026	1,026	786	546	1,026	723	546

Note: Significance levels are indicated by * for $p < 0.1$, ** for $p < 0.05$ and *** for $p < 0.01$ and are computed using Newey-West HAC adjusted standard errors with a maximum lag length of 12 months.

Table 3.36: Autocorrelation explosive hedge portfolio (P10 minus middle) excess returns, regressed on macroeconomic risk factors

Panel A: Equal weighted portfolios

	(1) nber b/t	(2) indprodgrowth b/t	(3) unemployment b/t	(4) sentiment b/t	(5) creditspread b/t	(6) termspread b/t	(7) all b/t
NBER recessions (=1)	-0.0047*						-0.0089*
	(-1.78)						(-1.94)
Industrial Production Growth Rate		0.0160					-0.1147
		(0.32)					(-0.39)
Unemployment Rate			-0.0932				-0.0727
			(-1.09)				(-0.53)
Baker - Wurgler (2006) Sentiment Index				-0.0024			-0.0020
				(-1.39)			(-1.14)
Baa-Aaa Bond Spread					-0.0194		-0.2131
					(-0.12)		(-0.51)
Treasury 10Yr Less 1Yr						-0.1186	-0.0745
						(-1.09)	(-0.65)
Const	0.0063***	0.0054***	0.0111**	0.0069***	0.0056***	0.0072***	0.0159*
	(5.36)	(4.99)	(2.05)	(3.93)	(2.72)	(4.13)	(1.91)
Adj R-sqr	0.0026	-0.0009	0.0012	0.0029	-0.0010	0.0003	0.0073
N	1,025	1,025	785	546	1,025	722	546

Panel B: Value weighted portfolios

	(1) nber b/t	(2) indprodgrowth b/t	(3) unemployment b/t	(4) sentiment b/t	(5) creditspread b/t	(6) termspread b/t	(7) all b/t
NBER recessions (=1)	-0.0061**						-0.0089
	(-2.19)						(-1.47)
Industrial Production Growth Rate		0.0687					-0.1202
		(1.46)					(-0.33)
Unemployment Rate			-0.0566				0.0315
			(-0.61)				(0.20)
Baker - Wurgler (2006) Sentiment Index				-0.0034			-0.0027
				(-1.52)			(-1.21)
Baa-Aaa Bond Spread					-0.0585		-0.4912
					(-0.32)		(-1.09)
Treasury 10Yr Less 1Yr						-0.0864	-0.0816
						(-0.69)	(-0.64)
Const	0.0058***	0.0045***	0.0085	0.0063***	0.0053**	0.0064***	0.0120
	(4.34)	(3.69)	(1.40)	(3.15)	(2.18)	(3.03)	(1.27)
Adj R-sqr	0.0033	0.0002	-0.0006	0.0043	-0.0008	-0.0008	0.0042
N	1,026	1,026	786	546	1,026	723	546

Note: Significance levels are indicated by * for p < 0.1, ** for p < 0.05 and *** for p < 0.01 and are computed using Newey-West HAC adjusted standard errors with a maximum lag length of 12 months.

Part III

Conclusion

I would like to use the conclusion to direct attention to what I consider the major contributions of this thesis to the wider literature. In the introduction I motivate my focus on extreme markets by arguing that extremes are relevant to the day-to-day pricing and risk management of financial assets. I present a model of fire sales in chapter 1. The model incorporates multiple investors with heterogeneous leverage, each investing in multiple assets with heterogeneous liquidity. Due to an exogenous price shock investors are compelled to strategically sell assets to maintain a target leverage ratio. I introduce an equilibrium concept from game theory, the non-symmetric static Nash equilibrium in continuous strategies, and show formally that such equilibrium exists, is unique, and can be calculated using the method of successive approximations within the framework of the model. I then derive an analytical approximation for equilibrium fire-sale prices and validate the accuracy of the approximation through numerical simulation. From this approximation I derive several (approximate) statics amenable to economic interpretation. In particular, I demonstrate and quantify fire-sale contagion effects that emerge endogenously from my model. I go on to quantify the spillover effects of leverage, showing that the leverage decisions of some investors impose externalities on other investors.

In chapter 2 I explain how the model parameters of chapter 1 can be estimated using market prices and incomplete data on asset holdings. I then test the hypothesis that model fire-sale prices predict realised returns in distressed markets, thus providing empirical support for the model postulated in chapter 2.

In chapter 3 I show that autocorrelation – a straightforward estimate of the deviation from a pure random walk in log prices – predicts realised returns. This predictability remains after controlling for market, size, book-to-market, momentum, short-term reversal, long-term reversal and liquidity in sorted portfolio time-series tests. Moreover autocorrelation remains a statistically and economically significant predictor of realised excess returns in stock level panel data regressions, even after controlling for a wide range of stock characteristics in various specifications. Stocks with low estimates of autocorrelation, which I term decaying stocks, generate unusually persistent excess returns. A zero-cost hedge portfolio of decaying stocks generates statistically and economically significant excess returns on average in every month post portfolio formation for horizons up to 25 years. An analysis of abnormal returns around earnings announcement dates suggests that the outperformance of decaying stocks cannot be easily explained by biased investor expectations of future earnings. This indicates a possible alternative: the interpretation of autocorrelation as a novel priced risk factor, or, perhaps more likely, as correlated with a still unknown but priced risk factor.

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