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Manufacturing Conformal Invariants of Hypersurfaces

by

Yuri Vyatkin

A thesis submitted in fulfilment of the requirements
for the degree of *Doctor of Philosophy*
in Mathematics

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Dedicated to my mother

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Preface

This thesis is concerned with an application of the tractor calculus to the constructions of conformal invariants of hypersurfaces. As a motivation, the problem of finding a suitable analogue of the Willmore invariant in higher dimensions is used. The necessary tools of elementary differential geometry such as the Gauss–Codazzi equations for conformal hypersurfaces are prepared and subsequently reformulated in terms of tractors for the convenience of calculations. A development of the tractor calculus on hypersurfaces is undertaken to give the tools for proliferating hypersurface conformal invariants. The standard constructions of conformal invariants are modified accordingly, and some examples are obtained and analyzed. Finally, we demonstrate how this knowledge can be combined with some other methods in conformal geometry to discover a conformally invariant action on 4-dimensional hypersurfaces and analyze its Euler–Lagrange equation.

The conformal structure is an example of a parabolic geometry. Naturally associated to parabolic geometries are the so-called tractor bundles, which provide calculi similar to the tensor calculus in Riemannian geometry. The tractor calculi make dealing with the geometric problems in parabolic geometries computationally efficient. Further details can be found in [17] . The main hope of this thesis is that the application of the conformal tractor calculus can lead to a success in the search of the conformal invariants of hypersurfaces.

One of the sources of motivation for this research was the problem of generalizing the Willmore invariant to higher dimensions. By the Willmore invariant we mean the left hand side of the Euler–Lagrange equation of the Willmore functional. This functional is known to be conformally invariant.

The present thesis has the following structure.

An overview of Riemannian geometry is given in Chapter 1, where we establish the notation and collect the basic facts, which are used throughout the thesis. The exposition in this chapter is rather elementary and can be harmlessly skipped by a well-versed reader. A somewhat original (cf. also [45, Chapter IV] and [63, p.42]) version of the abstract Gauss–Codazzi–Ricci decomposition of the curvature in the direct sum of vector bundles is presented in this chapter. It was inspired by a proof of Bonnet's theorem in [15]. This decomposition turns out to be a unifying framework for the treatment of the usual Gauss and Codazzi equations for hypersurfaces, as well as for the derivation of the tractor versions of these fundamental equations that we perform in Chapter 3. In the end of Chapter 1 we give a fairly complete proof of the known formulas for the variations of geometric quantities on the Riemannian hypersurfaces. These formulas are used in Chapter 5 to find the Euler–Lagrange equation of the Willmore functional.

In Chapter 2 we give a brief introduction, and fix the notation, for the conformal geometry of hypersurfaces. Implementing our methodology to work with vector bundle-valued quantities, we give an exposition of the conformal rescaling rules in conformally weighted vector bundles in Section 2.1.4, which is one of the pivotal observations in this thesis. A derivation of the conformal versions of the equations of Gauss and Codazzi for hypersurfaces, based on the Weyl–Schouten decomposition of the Riemannian curvature (discussed in detain in Chapter 1), is presented. The conformal Gauss and Codazzi equations are essential tools for manufacturing invariants of hypersurfaces and other calculations in submanifold geometry. While these equations are classically known in many appearances (cf. [78], [77], [56], [55], [42]), we have been able to link them to their tractor analogues in Chapter 3. We were also influenced by [16] and [39].

The core of the thesis is Chapter 3 where we give introduction to the conformal tractor calculus. Some parts of our treatment in this chapter are rather terse and require additional sources to get the full appreciation, especially in the first sections. On the other hand, we supply some very explicit computations, which are usually omitted in the existing literature. The tractor calculus for hypersurfaces is presented in greater detail, and the theory of the Gauss–Codazzi–Ricci decomposition in vector bundles is applied to obtain the tractor Gauss and Codazzi equations. After that we show how the conformal Gauss and Codazzi equations follow from the tractor versions. We finish this chapter with introducing modified versions of the intrinsic tractor-D operators, the so-called twisted intrinsic tractor operators, and discuss their properties. These operators are the essential ingredient in the constructions of conformal invariants of hypersurfaces, which are the main subject of study in this thesis.

In the beginning of Chapter 4 we give a simplified version of the constructions of conformal invariants, which are introduced and studied comprehensively in

[27] (see also [26] for a brief overview). These constructions serve as a model for our development of this theory for the case of hypersurfaces. Of course, a definition of hypersurface invariant is needed. This notion is discussed in the second section of this chapter. The definition of hypersurface invariants presented in this chapter is a new contribution, to the best of our knowledge, as it is not found in the literature. Based on this definition, we describe the basic classes of hypersurface conformal invariants, which we can construct using the tractor-D operators introduced in Chapter 3. Making examples of these types of invariants, we begin harvesting our first results.

The culmination of the present thesis is Chapter 5. We start with a brief historical overview of the Willmore functional. In our research we adopt the definition of the Willmore functional as the integral of the square of the length of the umbilicity tensor[1]. In the next sections we find its Euler–Lagrange equation for the case when the surface is embedded into an arbitrary Riemannian (or conformal) background. As a tool for that, we derive a useful formula for the variation of the umbilicity tensor. The left hand side of the Euler–Lagrange equation of the Willmore functional is a conformal invariant of the hypersurface with respect to the rescaling of the ambient conformal structure, which is known as the Willmore invariant (cf.[14]). This invariant was also found by completely different methods in [1]. Our expression for the Willmore invariant allows us to confirm a conjecture of A.R.Gover that this invariant can be obtained using the tractor calculus on hypersurfaces. In order to find an analogue of the Willmore functional in higher dimensions, we combine the methods developed in the previous chapters with the theory of the Branson–Gover operators[2] that reveals a deep connection of the Willmore functional with the celebrated Q-curvature. Using these ideas we construct a conformally invariant action on a 4-dimensional hypersurface and analyze its Euler–Lagrange equation. This discovery opens new perspectives for a subsequent research. It would be interesting to compare our results with the work of J.Guven [36], who has constructed a conformally invariant bending energy for hypersurfaces in $\mathbb{R}^4$ by a direct calculation. His expression differs from the one that we have derived using the tractor calculus. The advantage of our approach is that it works for hypersurfaces in arbitrary 4-dimensional conformally flat manifolds, and also contains the germ of possible generalizations to higher dimensions.

[1] A short name for the trace free part of the second fundamental form.

[2] The author's attention to this possibility was drawn by his supervisor Prof. A.R.Gover.

Appendix A contains the tables of complete and partial contractions of the tractor projectors that are useful in the calculations.

In Appendix B we describe briefly a computer algebra system (CAS) called Cadabra, which we find to be a handy tool for taming hairy computations and making explorations of tractor expressions.

In Appendix C we have collected all the facts related to the symmetric 2-tractors, which are used in the expressions of some examples that we construct.

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$\mathcal{E}^a$	Tangent bundle	6
$\mathcal{E}_a$	Cotangent bundle	6
$t_{(ab)}$	Symmetric part of tensor t_{ab} .	6
$t_{[ab]}$	Antisymmetric part of tensor t_{ab}	6
$\nabla^{\mathcal{F}}$	Connection in vector bundle $\mathcal{F}$	7
$\overset{\oplus}{\nabla}$	Direct sum connection	9
T^{∇}	Torsion of a linear connection ∇	10
Δ	Laplacian	17
$[\Delta]$	Forms Laplacian	18
$h^{\oplus}$	Direct sum metric	26
f_{φ}	Restriction of f along φ	30
$\varphi^* t$	Pullback of a covariant tensor .	30
$\varphi^* \nabla$	Pullback connection	31
W_{abcd}	Weyl tensor	33
P_{ab}	Schouten tensor	33
J	Schouten scalar	33
$\underline{\mathcal{E}}^a$	Ambient tangent bundle	46
$\underline{\nabla}$	Ambient connection	46
N^a	Unit normal vector	47
$\mathrm{N}_a{}^b$	Normal projection operator . .	48
$\Pi_a{}^b$	Projection operator	48
$\overline{\mathcal{E}}^a$	Intrinsic tangent bundle	48
$\bar{g}_{ab}$	Intrinsic metric	48
$\overline{\nabla}$	Intrinsic connection	49
L_{ab}	Shape tensor	50
H	Mean curvature	50
$\underline{\Delta}$	Tangential Laplacian	53
$\overset{\mathrm{NN}}{\mathrm{R}}_{ab}$	$\mathrm{R}_{a,c,b,d}\mathrm{N}^c\mathrm{N}^d$	59
$\overset{\mathrm{NN}}{\mathrm{Ric}}$	$\mathrm{Ric}_{ab}\mathrm{N}^a\mathrm{N}^b$	59
δ_0	Variational derivative	59

$\mathcal{E}[w]$	Conformal densities	67
$\mathbf{g}_{ab}$	Conformal metric	68
$\Box$	Box operator	70
$\mathcal{F}[w]$	Weighted vector bundle	71
Y_{abd}	Cotton tensor	73
$\mathbf{L}_{ab}$	Shape tensor (weighted)	76
$\mathbf{H}$	Mean curvature (weighted)	76
$\mathring{\mathbf{L}}_{ab}$	Umbilicity tensor (weighted)	77
$\mathbb{T}M$	Tractor bundle	86
$\mathcal{T}^A$	Tractor bundle	86
h_{AB}	Tractor metric	87
$\nabla^{\mathcal{T}}$	Tractor connection	87
X_A, $Z_A{}^a$ and Y_A	Standard tractor projectors	89
$\mathcal{A}_{AA'}$	Adjoint tractor bundle	91
$\mathbb{Y}_{AA'}{}^a$,$\mathbb{W}_{AA'}$,$\mathbb{Z}_{AA'}{}^{aa'}$,$\mathbb{X}_{AA'}{}^a$	Adjoint tractor projectors	91
$\mathfrak{D}_A$	Tractor pre-D operator	94
$\mathbb{D}_{AP}$	Double-D operator	94
D_A	Thomas-D operator	95
W_{ABCD}	Weyl tractor	103
B_{bd}	Bach tensor	104
$C_{A'ABCDD'}$	Lifted ... Weyl tensor	107
$C_{A'ABDD'}$	Lifted ... Cotton tensor	107
$\underline{\mathcal{T}}^A$	Ambient tractor bundle	108
$\underline{\mathsf{P}}_a{}^b$	$\Pi^{a'}{}_a \mathsf{P}_{a'}{}^b$	109
$\underline{\delta}_a{}^b$	$\underline{\delta}_a{}^b := \Pi^{a'}{}_a \delta_{a'}{}^b$	110
N^A	Normal tractor	110
$\mathcal{N}^A$	Tractor normal bundle	111
$\nabla^{\mathcal{N}}$	Tractor normal connection	111
$\mathrm{N}_A{}^B$	Tractor normal projection	111
$\Pi^A{}_B$	Tractor projection operator	112
$\overline{\mathcal{T}}^A$	Intrinsic tractor bundle	113
$\bar{Y}^A$, $\bar{Z}^A{}_a$, $\bar{X}^A$	Intrinsic tractor projectors	115
$\bar{h}_{AB}$	Intrinsic tractor metric	115
$\check{\nabla}$	Projected ambient tractor connection	116
$\check{\mathsf{P}}_{ab}$	Schouten–Fialkow tensor	118
$\mathbb{S}_a{}^B{}_C$	Intrinsic tractor contorsion	120

$\mathcal{F}_{ab}$	Fialkow tensor	121
$\mathbb{L}_a{}^B$	Tractor shape 1-form	123
$\overline{\mathbb{D}}_{AP}$	Intrinsic double-D operator	131
$\overline{\mathrm{D}}_A$	Intrinsic Thomas-D operator	131
$\underline{\overline{\mathfrak{D}}}^A$	Twisted intrinsic pre-D operator	132
$\underline{\overline{\mathbb{D}}}_{AP}$	Twisted intrinsic double-D	132
$\underline{\overline{\Box}}$	Twisted intrinsic box operator	133
$\underline{\overline{\mathrm{D}}}_A$	Twisted intrinsic Thomas-D	133
$\underline{\mathfrak{D}}_A$	Ambient pre-D operator	134
$\underset{\mathbb{NN}}{\underline{D}}{}_{ABC}$	Triple-D operator	134
$\overline{\mathrm{W}}_{BD}$	Binormal . . . Weyl tractor	150
$\mathbb{N}_{AA'}$	Adjoint normal tractor	151
$\mathbf{L}_{AB}$	Shape 2-tractor	154
$\mathcal{W}(\Sigma)$	Willmore functional	159
$\mathcal{W}^{(2)}$	Willmore invariant	164
Y_{AB}, $\mathsf{U}_{AB}{}^a$, V_{AB}, $\mathsf{Z}_{AB}{}^{ab}$, $\mathsf{W}_{AB}{}^a$, X_{AB}	Symmetric 2-tractor projectors	182

Chapter 1

Riemannian geometry of hypersurfaces

The main object of study in this thesis are conformal invariants of hypersurfaces. The definition of hypersurface conformal invariants will be discussed more thoroughly in Chapter 4, but at the moment we need to know that they are metric hypersurface invariants with an additional property of suitably transforming with respect to multiplication of the Riemannian metric by a strictly positive smooth function.

In order to understand what metric invariants are, and also for the purposes of our calculations, we need to review the basic facts and definitions of Riemannian geometry, and introduce the notation that will be used later throughout the text. Since the notation is the primary focus of this chapter, the details, proofs and verifications for the standard facts are usually omitted and only those that are not commonly treated in the literature, or that we want to present in an unusual form, are given with proofs and more thorough explanations. In particular, the existence of metrics, connections and other standard objects of differential geometry is assumed to be known and not discussed explicitly.

In the end of this chapter we give a detailed derivation of the well-known identities (see e.g [79] or [38], and also [38]) for the normal variation of the geometric quantities on hypersurfaces, which we use in Chapter 5 to compute the normal variation of the umbilicity tensor (Proposition 5.1.2). Using this identity we express the Euler–Lagrange equation of the Willmore functional in a convenient form. This allows us to identify the Willmore invariant with an invariant that can be obtained using the tractor calculus on hypersurfaces, see the last section of Chapter 4.

The following sources are used as a general reference: [8], [19], [40].

1.1 Background and notation from Riemannian geometry

The main technical framework adopted in this thesis are vector bundles over manifolds. Vector bundles are thought as real vector spaces smoothly parametrized by points of the base manifold. All vector bundles that we use have finite rank.

All manifolds and maps between them are assumed to be smooth, that is C^∞-differentiable.

The base field for all the vector spaces, vector bundles and manifolds used in this thesis is the field of real numbers $\mathbb{R}$.

1.1.1 Manifolds

A manifold is a topological space that locally looks like an open subset of a Euclidean space $\mathbb{R}^n$. Let us recall the precise definitions for the sake of completeness. We follow [37] where the omitted details can be found.

Definition 1.1.1. A topological space M is called a *topological manifold* if M is **Hausdorff** (any two points can be separated by open neighborhoods around them) and **second countable** (there is a countable base of topology in M), and for each point $p \in M$ there is an open subset $U \subseteq M$ containing point p together with a map $\vec{x}\colon U \to \mathbb{R}^n$ such that $\vec{x}(U)$ is open in $\mathbb{R}^n$ equipped with the standard Euclidean topology (open cubes $(-\varepsilon, \varepsilon)^n$ form a base of topology) and the map $\vec{x}$ is a **homeomorphism** (continuous one-to-one with the continuous inverse). Such a pair $(U, \vec{x})$ is called a *chart around point p*, and, if $\vec{x}(p) = 0 \in \mathbb{R}^n$, the chart $(U, \vec{x})$ is called *centered around point p*. No generality is lost if we consider only centered charts, and we do so whenever convenient.

Observe that if there are two charts $(U, \vec{x})$ and $(V, \vec{y})$ containing a point $p \in M$, the maps $\vec{y} \circ \vec{x}^{-1}\colon \vec{x}(U \cap V) \to \vec{y}(U \cap V)$ and $\vec{x} \circ \vec{y}^{-1}\colon \vec{y}(U \cap V) \to \vec{x}(U \cap V)$ are homeomorphisms of open subsets in $\mathbb{R}^n$. These maps are referred to as the *transition maps* between the charts $(U, \vec{x})$ and $(V, \vec{y})$.

A collection of charts $\{(U_\alpha, \vec{x}_\alpha)\}$ with the property $\bigcup_\alpha U_\alpha = M$ is called an *atlas* on the manifold M. If for any two charts in an atlas the transition maps are smooth, that is C^∞-differentiable, the atlas is called smooth. All smooth atlases form a partially ordered set with respect to inclusion, and a maximal atlas is called a *smooth structure* on the manifold. A manifold M equipped with a smooth structure is referred to as a *smooth manifold*.

1.1.2 Vector bundles

Vector bundles arise naturally in differential geometry as tangent and cotangent bundles of manifolds, and then various constructions with them can be used to make a variety of examples. As announced in the beginning of this chapter, we do not attempt to replace the standard sources, but for the sake of completeness let us give the basic definitions following [52]. Further details can be found also in [41], [46] and [49].

Definition 1.1.2. A (real) *vector bundle* is a set of data (V, M, π, F) where V is a manifold called the **total space**, M is another manifold called the **base space**, $\pi\colon V \to M$ is a smooth map called the projection, and F is a vector space called the **typical fiber**, such that the following conditions are satisfied:

1. the map π is surjective;
2. for each point $p \in M$ the set $\pi^{-1}(p)$ is given the structure of a real vector space;
3. each vector space $\pi^{-1}(p)$ is isomorphic to the typical fiber F;
4. for each point $p \in M$ there is an open subset $U \subseteq M$ and a diffeomorphism $\varphi\colon \pi^{-1}(U) \to U \times F$ called a **bundle chart** (also called "local trivialization").

Usually vector bundle referred to by naming its total space, for instance the vector bundle from the definition would be denoted as V. In fact, we shall switch to another convention later, as it will be explained in the subsection on the abstract index notation.

Most of the object that we use or construct will be sections of some vector bundles. Recall that we restrict ourselves to smooth categories.

Definition 1.1.3. Let V be a vector bundle from the previous definition A *section* of a vector bundle E is a map $\xi\colon M \to V$ such that $\pi \circ \xi = \mathrm{id}_M$. In other words, a section maps each point $p \in M$ to the fiber over p, that is to the set $\pi^{-1}(p)$.

The space of all sections of a vector bundle will be denoted by the corresponding calligraphic letter, that is

$$\mathcal{V} := \{s\colon M \to V \mid \pi \circ \xi = \mathrm{id}_M\}$$

We use the symbol $:=$ as a shortcut for "equals by definition".

We shall also deal with vector bundle maps (vector bundle morphisms).

Definition 1.1.4. A *vector bundle map* is a set of data consisting of two vector bundles (V, M, π, F) and (V', M', π', F'), and two maps $F\colon V \to V'$ and $f\colon M \to M'$, such that the following diagram commutes

$$\begin{array}{ccc} V & \overset{F}{\to} & V' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \overset{f}{\to} & M' \end{array}$$

A vector bundle morphism maps the fiber over each point to the fiber over the image of that point under the map between the base manifolds.

One of the most important notion for us that would be fair to mention at least briefly is the one of differential operators. Not going into the full details, we shall understand a *differential operator* as a map between the spaces of sections of two vector bundles such that locally, that is in a choice of bundle charts, this map is represented by a function of partial derivatives of the section that this operator acts on. This description is too broad to be useful, and in fact we shall deal only with linear differential operators (l.d.o.), or, as the most general case, polynomially defined on the partial derivatives of a section. One of the first expositions on differential operators was given by R.Palais in [57], and, remarkably, there only l.d.o. were considered. The differential operators that we work with are *natural* in a certain precise sense. The details on this subject can be found in [41] and references thereof.

1.1.3 The abstract index notation

In order to perform efficient computations in vector bundles we prefer to use Roger Penrose's abstract index notation [62] (see also [72]). This notation is both quite economical and flexible enough to specify sections of vector bundles with rather complicated structure. Using abstract indices the calculations can be carried over without choosing a local or coordinate frame, yet giving a way to compare the formulas stated in either notation.

The essence of the abstract index notation is that the indices are used as labels to indicate both vector spaces and their elements by attaching indices to the symbols. If a symbol is used to denote a vector space, we call it the *core symbol*. The indices are elements of some set called the *index range* which is usually taken to be a subrange of some familiar alphabet. For instance, if V

is the core symbol for a vector space in the consideration, and we have chosen $\alpha, \beta, \gamma, \ldots$ to be the set of indices associated with the vector space V, then writing V^α, V^β and so forth we refer to copies of the same vector space V. It becomes meaningful and convenient when we write down (linear) maps between vector spaces. The first example to look at is the identity map that can be written as $\mathrm{id}_\alpha{}^\beta \colon V^\alpha \to V^\beta$ in the abstract index notation. Furthermore, the elements of a copy V^α of the vector V are labeled with the index referring to that copy, e.g. we write $v^\alpha \in V^\alpha$ and whenever we use v^α later on, we know an element of what vector space it is. Applying a map moves an element from one copy of a vector space to another copy of a (possibly different) vector space. In our example, the identity map $\mathrm{id}_\alpha{}^\beta$ takes $v^\alpha \in V^\alpha$ to the same element labeled differently, that is to $v^\beta \in V^\beta$. An arbitrary linear map between two vector spaces V and W can be denoted by $L_\alpha{}^\beta \colon V^\alpha \to W^\beta$. This is already an example which shows the power of the abstract index notation, because due to the canonical isomorphism $\mathrm{Hom}(V, W) \cong V \otimes W$ the map L can be viewed as an element of the tensor product of spaces V_a and W^a. Here we encounter the convention to change (raise or lower) the position of an index in order to denote the (algebraically) dual vector space. Thus, V_a is the notation for $\mathrm{Hom}(V, \mathbb{R})$. This convention automatically ensures that if the same index is used in an element twice, in the upper and in the lower position, then the natural pairing between the vector space and its dual is assumed. In the cases when we fix an isomorphism between the vector space and its dual, the position of the indices becomes irrelevant, and the contraction rule applies for the indices repeated in any position. Such an isomorphism induces an *inner product* in the vector space, and we usually denote it with the letter g as inspired by Riemannian geometry. If an inner product in the vector space V^α is chosen, the identity map $\mathrm{id}_\alpha{}^\beta$ coincides with the element $g_\alpha{}^\beta \in V_a{}^\beta$. Following the tradition, however, the identity map is usually denoted by $\delta_\alpha{}^\beta$ by analogy with the Kronecker symbol in the concrete index notation. Introducing a basis $\{e^\alpha{}_i\}$ (where $i = 1, \ldots, n = \dim V$ are now the concrete indices) in the space V^α, we can find for any element $v^\alpha \in V^\alpha$ its components v^i, that is the coefficients in the expansion of $v^\alpha = v^i e^\alpha{}_i$ with respect to the basis $\{e^\alpha{}_i\}$. When the concrete indices are repeated, the Einstein summation convention is always used (unless explicitly disabled). For instance, $v^i e^\alpha{}_i = \sum_{i=1}^n v^i e^\alpha{}_i$.

A vector bundle can be thought as a collection of vector spaces attached smoothly to each point of a manifold, all the vector spaces being isomorphic to a vector space called the typical fiber. Thus a vector bundle can be denoted in the same manner as it is done for vector spaces, that is with a Latin capital as the core symbol, accompanied with a choice of the index range.

It is usually safe to abuse the language and notation by denoting vector bundles and spaces of their sections by the same symbol, and we shall do so without further mention. That is, for example, we shall use the symbol $\mathcal{V}$ both for the bundle V and the space of its section, if there is no danger of ambiguity, or, sometimes, the ambiguity is intentional, when either interpretation can be rendered correct. (Typically, this will be the case when we speak about covariant derivatives in the direction X).

When we need to talk about arbitrary vector bundles we shall usually use the letters $\mathcal{F}, \mathcal{G}, \mathcal{V}, \mathcal{W}, \dots$ for the core symbols. The Greek capitals $\Phi, \Psi, \Xi, \dots$ will be used by default for the corresponding index range, e.g. $X^\Phi \in \mathcal{F}^\Phi$ will mean a section X of a vector bundle $\mathcal{F}$. Notice that we may suppress the indices when the context permits.

The tangent $\mathrm{T}M$ and cotangent T^*M bundles of a manifold M are denoted by the symbols $\mathcal{E}^a$ and $\mathcal{E}_a$ respectively. The initial segment of the Latin minuscules $\{a, b, c, \dots\}$ is reserved for the indices of tensor bundles that are referred to as the *tensor indices*. Tensor bundles are various finite tensor products of the tangent and the cotangent bundle of the manifold, and in the notation we use for them the symbol $\mathcal{E}$ adorned with tensor indices in upper and lower positions, so that, for instance, $\mathrm{Hom}(\mathrm{T}M, \mathrm{T}M) \cong \mathrm{T}^*M \otimes \mathrm{T}M$ is denoted by ${\mathcal{E}_a}^b$. A juxtaposition of several indices is understood as the tensor product of the corresponding spaces, e.g. ${\mathcal{E}^{ab}}_{cd} = \mathcal{E}^a \otimes \mathcal{E}^b \otimes \mathcal{E}_c \otimes \mathcal{E}_d$.

The symbol $\mathcal{E}$ alone (without indices) will be used as a synonym for the space $C^\infty(M)$ of smooth functions on M.

As it has been said already, we abuse the notation by denoting the spaces of sections of vector bundles by the same symbols as the vector bundles themselves. For example, $\omega_a \in \mathcal{E}_a$ means $\omega \in \Gamma(\mathrm{T}^*M)$ in the index-free notation.

For a tensor t_{ab} its symmetric part $t_{(ab)}$ is defined as

$$t_{(ab)} \coloneqq \frac{1}{2}\,(t_{ab} + t_{ba})$$

and its skew, or antisymmetric, part $t_{[ab]}$ as

$$t_{[ab]} \coloneqq \frac{1}{2}\,(t_{ab} - t_{ba})$$

This notation is extended to various tensor parts of tensor bundles and, more generally, vector bundles, e.g. constructions such as $\mathcal{F}^{[\Phi\Psi]}$ or $\mathcal{E}_{[ab]} \otimes \mathcal{F}^{[\Phi\Psi]}$ will be used frequently.

In some cases we need to work in local coordinates, and the indices will be concrete. The use of concrete indices will be always declared explicitly, and once the fact has been established using such indices we shall tend to reformulate the results back into abstract indices for the subsequent use. A typical example of this rule is our treatment of the variational formulas for Riemannian hypersurfaces in the last section of this chapter.

1.1.4 Vector-bundle metrics

The *dual bundle* $\mathcal{F}_\Phi$ of a vector bundle $\mathcal{F}^\Phi$ is the bundle of linear mappings $\mathrm{Hom}(\mathcal{F}, \mathbb{R})$. This notation agrees with our use of $\mathcal{E}_a$ for the cotangent bundle.

The dual bundle $\mathcal{F}_\Phi$ is isomorphic (in the category of vector bundles) to the bundle $\mathcal{F}^\Phi$ but this isomorphism requires a choice. This choice can be represented by a smoothly assigned inner product in each fiber of the bundle $\mathcal{F}^\Phi$. Equivalently, this can be stated as fixing a section $h_{\Phi\Psi}$ of the symmetric tensor product $\mathcal{F}_{(\Phi\Psi)}$ that induces an inner product in each fiber of $\mathcal{F}^\Phi$. Such a section $h_{\Phi\Psi}$ is called a *fiber metric* in the vector bundle $\mathcal{F}^\Psi$, or simply a (vector-bundle) metric.

1.1.5 Connections

A connection $\nabla^{\mathcal{F}}$ in a vector bundle $\mathcal{F}$ over manifold M is a linear map

$$\nabla^{\mathcal{F}} \colon \Gamma(\mathcal{F}) \to \mathrm{T}^*M \otimes \Gamma(\mathcal{F})$$

or, using the abstract index notation and our conventions,

$$\nabla^{\mathcal{F}}_a \colon \mathcal{F}^\Phi \to \mathcal{E}_a \otimes \mathcal{F}^\Phi$$

such that for any smooth function $s \in \mathcal{E}$ and any section $f \in \Gamma(\mathcal{F})$ a Leibniz-type identity holds:

$$\nabla^{\mathcal{F}}(sf) = \mathrm{d}s \otimes f + s\,\nabla^{\mathcal{F}} f$$

or, in the abstract index notation,

$$\nabla^{\mathcal{F}}_a(s \otimes f^\Phi) = (\mathrm{d}s)_a \otimes f^\Phi + s \otimes \nabla^{\mathcal{F}}_a f^\Phi \qquad (1.1)$$

If X is a vector field on the manifold M, that is $X^a \in \mathcal{E}^a$, the value

$$\nabla^{\mathcal{F}}_X f \coloneqq X^a \nabla^{\mathcal{F}}_a f$$

is referred to as the *covariant derivative* of the section $f \in \mathcal{F}$ with respect to the connection $\nabla^{\mathcal{F}}_a$ in the direction of the vector field X. Due to the linearity of the connection in the slot a it is safe to speak of the covariant derivative at a point p in the direction of a tangent vector $X_p \in \mathrm{T}_p M$, so we usually refer to X as to the tangent vector in this context.

The definition of connection suggests that on functions $s \in C^\infty(M)$ any connection $\nabla^{\mathcal{F}}$ should be defined to act as

$$\nabla^{\mathcal{F}}_a s \coloneqq (\mathrm{d}s)_a$$

where $\mathrm{d}s$ is the differential of the function s on M.

With this extension the Leibniz rule (1.1) can be written formally as the usual product rule

$$\nabla^{\mathcal{F}}_a (s \otimes f) = \nabla^{\mathcal{F}}_a s \otimes f + s \otimes \nabla^{\mathcal{F}}_a f$$

where $s \otimes f \coloneqq s\, f$ by convention.

In particular, $\nabla_a s$ will be used as an abstract index style of the notation for the differential $\mathrm{d}s$ of a function $s \in \mathcal{E}$.

When a vector bundle and a connection in it are understood, it is customary to write $\nabla_a f$ for the covariant derivative of $f \in \mathcal{F}$ without indicating the bundle in the subscript. Moreover, we may occasionally suppress the tensor index in the notation for the connection, the symbol ∇f being understood as a $\mathcal{F}$-valued 1-form on M.

If a vector bundle is equipped with more than one connection we sometimes use the symbol ∇ with various decorations (accents, underlines, overlines etc) which has to be used with due care because such a notation can easily become ambiguous.

Let $(\mathcal{V}, \nabla^{\mathcal{V}})$ and $(\mathcal{W}, \nabla^{\mathcal{W}})$ be two vector bundles with connections.

The tensor product $\mathcal{V} \otimes \mathcal{W}$ of the bundles $\mathcal{V}$ and $\mathcal{W}$ as a rule is equipped with the tensor product connection, that we usually refer to as the *coupled connection* on $\mathcal{V} \otimes \mathcal{W}$. The bundle $\mathcal{V} \otimes \mathcal{W}$ is generated by finite linear combinations of tensor products $v \otimes w$ where $v \in \mathcal{V}$ and $w \in \mathcal{W}$ are the sections of the respective factors. The action of the coupled connection $\overset{\otimes}{\nabla} \equiv \nabla^{\mathcal{V} \otimes \mathcal{W}}$ on simple sections $v \otimes w$ is defined by insisting that the product rule holds:

$$\overset{\otimes}{\nabla}(v \otimes w) = (\nabla^{\mathcal{V}} v) \otimes w + v \otimes \nabla^{\mathcal{W}} w \tag{1.2}$$

With the convention $\nabla s \equiv \mathrm{d}s$ mentioned above this product rule lies in the very definition of every connection, and predominantly we use the coupled connections on tensor products of vector bundles. In view of this fact the notation $\nabla^{\mathcal{F}}$ and $\overset{\otimes}{\nabla}$ becomes superfluous, and it is customary to use the symbol ∇ without decorations for all the connections involved. For instance, the above product rule can be written as

$$\nabla(v \otimes w) = (\nabla v) \otimes w + v \otimes \nabla w$$

where each ∇ "knows" the bundle it is able to act on, so the meaning of each term is unambiguous.

Remark 1.1.5. The qualification "coupled" is usually applied for connections that arise in tensor products of vector bundles of different nature, say, for instance, if vector bundles have different rank.

For tensor products of the same bundle $\mathcal{V}$ and its dual we usually refer to the whole family of coupled connections in these bundles constructed from a given connection $\nabla^{\mathcal{V}}$ in the bundle $\mathcal{V}$ as to the connection $\nabla^{\mathcal{V}}$, or just ∇, if only one connection in $\mathcal{V}$ is assumed.

Similarly, the direct sum $\mathcal{V} \oplus \mathcal{W}$ of the bundles $\mathcal{V}$ and $\mathcal{W}$ is equipped with the *direct sum connection* $\overset{\oplus}{\nabla}$ that is defined on sections $v + w$ of the $\mathcal{V} \oplus \mathcal{W}$ by the sum rule

$$\overset{\oplus}{\nabla}(v + w) = \nabla^{\mathcal{V}} v + \nabla^{\mathcal{W}} w \tag{1.3}$$

Again, the connection can be selected by the section it acts on, so the above rule is safely presented as

$$\nabla(v + w) = \nabla v + \nabla w$$

if the connections in bundles $\mathcal{V}$ and $\mathcal{W}$ are fixed.

If a connection $\nabla^{\mathcal{F}}$ in a vector bundle $\mathcal{F}^{\Phi}$ is given, it is used to define connections in the dual bundle $\mathcal{F}_{\Phi} = (\mathcal{F}^*)_{\Phi}$ and in all the tensor bundles of a finite number of copies of $\mathcal{F}$ and $\mathcal{F}^*$.

A connection in the tangent bundle (and the whole family of connections induced by that in the tensor bundles) is termed a *linear connection*.

1.1.6 The torsion

Recall that for two vector fields $X, Y \in \mathrm{T}M$ their Lie bracket $[X, Y]$ is defined by its action on smooth functions $\varphi \in C^\infty(M)$ as

$$[X, Y]\varphi = X(Y\varphi) - Y(X\varphi)$$

This condition uniquely defines a vector field called the Lie bracket $[X, Y]$ of vector fields X and Y on M. Notice that only a smooth structure is needed for this definition to make sense.

If ∇ is a connection on $\mathrm{T}M$, the following operator

$$T^\nabla(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$$

is $C^\infty(M)$-bilinear, and thus gives rise to a tensor T^∇ which is called the *torsion* of the connection ∇.

Definition 1.1.6. A linear connection ∇ is called *torsion free* if $T^\nabla = 0$.

Proposition 1.1.7. *For torsion free connections we have*

$$\nabla_a \nabla_b \varphi = \nabla_b \nabla_a \varphi \tag{1.4}$$

where $\varphi \in C^\infty(M)$.

Proof. Let $\varphi \in C^\infty(M)$, and $X, Y \in \Gamma(\mathrm{T}M)$.

Recall that $X\varphi = X^a \nabla_a \varphi$, $Y\varphi = Y^b \nabla_b f$, and similarly $(\nabla_X Y)\varphi = (X^a \nabla_a Y^b)\nabla_b \varphi$ and $(\nabla_Y X)\varphi = (Y^a \nabla_a X^b)\nabla_b \varphi$.

Furthermore,

$$[X, Y]\varphi = X\,Y\varphi - Y\,X\varphi$$

but

$$X\,Y\varphi = X^a \nabla_a (Y^b \nabla_b \varphi) = X^a(\nabla_a Y^b)\nabla_b \varphi + X^a Y^b \nabla_a \nabla_b \varphi$$

and

$$Y\,X\varphi = Y^b \nabla_b (X^a \nabla_a \varphi) = Y^b(\nabla_b X^a)\nabla_a \varphi + Y^b X^a \nabla_b \nabla_a \varphi$$

so we can compute

$$\begin{aligned}T(X,Y)\varphi &= (\nabla_X Y)\varphi - (\nabla_Y X)\varphi - [X,Y]\varphi \\ &= X^a(\nabla_a Y^b)\nabla_b\varphi - Y^b(\nabla_b X^a)\nabla_a\varphi \\ &- X^a(\nabla_a Y^b)\nabla_b\varphi - X^a Y^b \nabla_a\nabla_b\varphi \\ &+ Y^b(\nabla_b X^a)\nabla_a\varphi + Y^b X^a \nabla_b\nabla_a\varphi \\ &= Y^b X^a \nabla_b\nabla_a\varphi - X^a Y^b\nabla_a\nabla_b\varphi\end{aligned}$$

Since X and Y are arbitrary, the result follows. □

From the calculation in the proof of the above proposition we get an expression for the torsion tensor in abstract indices:

$$(T^\nabla)_{ab}{}^c\nabla_c\varphi = \nabla_b\nabla_a\varphi - \nabla_a\nabla_b\varphi \tag{1.5}$$

Using the Poncaré lemma [50, p.23] one can show that the condition (1.4) is also sufficient for the connection to be torsion-free.

1.1.7 Metric vector bundles

A *metric vector bundle* or *Riemannian vector bundle* (see e.g. [42]) is a vector bundle $\mathcal{F}^\Phi$ equipped with a fiber metric $h_{\Phi\Psi}$ and a connection $\nabla^{\mathcal{F}}$ which is compatible with the metric in the sense that

$$\nabla^{\mathcal{F}_{\Phi\Psi}} h_{\Phi\Psi} = 0$$

or, equivalently, dropping the subscripts for the ∇-s,

$$\nabla_X h(v,w) = h(\nabla_X v, w) + h(v, \nabla_X w)$$

where $h(v,w) \coloneqq v^\Phi w^\Psi h_{\Phi\Psi}$.

Each vector bundle with a metric can be equipped with a compatible connection, though such a connection need not to be unique.

Definition 1.1.8. A positive-definite fiber metric g in the tangent bundle $\mathrm{T}M$ of a manifold M is referred to as a *Riemannian metric* on M, and the pair (M,g) is called a *Riemannian manifold*.

1.1.8 The Levi-Civita connection

A torsion-free connection that is compatible with the Riemannian metric in (M, g) is defined uniquely [72, p.35]. Such a unique connection is called the *Riemannian connection*, or the *Levi-Civita connection* of the metric g. Sometimes we express the fact that ∇ is the Levi-Civita connection of a metric g by the symbolic expression $\nabla = \nabla^g$.

Proposition 1.1.9 (Koszul's formula)**.** *The Levi-Civita connection $\nabla = \nabla^g$ on the Riemannian manifold (M, g) is uniquely defined by the identity*

$$\begin{aligned} 2\, g(\nabla_X Y, Z) &= X\, g(Y, Z) + Y\, g(Z, X) - Z\, g(X, Y) \\ &\quad - g(Y, [X, Z]) - g([Y, Z], X) + g(Z, [X, Y]) \end{aligned}$$

Proof. See e.g. [67]. □

1.1.9 Vector-bundle valued forms

From now on we assume, following [12], that sequentially labeled indices as in $\mathcal{E}_{a^1 \ldots a^k}$ or $\mathcal{E}_{a^2 \ldots a^k}$ are skewed over, that is, for instance, $\mathcal{E}_{a^1 \ldots a^k} = \mathcal{E}_{[a^1 \ldots a^k]}$ as vector bundles.

Sections of the bundles $\mathcal{E}_{a^1 a^2 \ldots a^k} \otimes \mathcal{F}$ where $\mathcal{F}$ is a vector bundle over manifold M, are called $\mathcal{F}$-valued k-forms. We shall use the notation $\mathcal{F}^k = \mathcal{E}_{a^1 a^2 \ldots a^k} \otimes \mathcal{F}$ for the bundles of $\mathcal{F}$-valued k-forms. It is convenient to set by definition that $\mathcal{F}^0 = \mathcal{F}$.

In particular, the usual k-forms are section of the bundles $\mathcal{E}^k := \mathcal{E}_{a^1 \ldots a^k}$, and 0-forms are just smooth functions $\mathcal{E}^0 = \mathcal{E}$.

The wedge product of a k-form $\alpha_{a^1 \ldots a^k} \in \mathcal{E}_{a^1 \ldots a^k}$ and a l-form $\beta_{b^1 \ldots b^l} \in \mathcal{E}_{b^1 \ldots b^k}$ is a $k + l$-form $\alpha \wedge \beta$ given by

$$(\alpha \wedge \beta)_{c^1 \ldots c^{k+l}} := \frac{(k+l)!}{k!l!} \alpha_{c^1 \ldots c^k} \beta_{c^{k+1} \ldots c^{k+l}} \tag{1.6}$$

where a juxtaposition of tensors is understood as the tensor product, and in the right hand side the term is skewed over the indices $c^1, c^2, \ldots c^{k+l}$ according to our convention.

In general, the wedge product of vector-bundle valued forms $\alpha \in \mathcal{F}^k$ and $\beta \in \mathcal{G}^l$ is a section of $(\mathcal{F} \otimes \mathcal{G})^{k+l}$ given by the same formula as in (1.6).

When $\mathcal{F} = \mathcal{G}$ and a there is a map $\mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$ with certain properties (for instance, when $\mathcal{F} = \mathcal{G} = \mathrm{Hom}(\mathcal{H}, \mathcal{H})$ and the map is the composition of vector bundle morphisms) the wedge product is composed with this map, yielding again a section of the bundle $\mathcal{F}$. We shall frequently do this implicitly if the context permits.

For a k-form $\omega \in \mathcal{F}^k$ where $\mathcal{F}$ is a vector bundle with a connection ∇ the *covariant exterior derivative* is defined in the index-free notation as

$$\begin{aligned} \mathrm{d}^{\nabla}\omega(X_0, ..., X_k) = &\sum_{0\leq i\leq k} (-1)^i \nabla_{X_i} \left(\omega(X_0, \dots, \hat{X}_i, \dots, X_k)\right) \\ &+ \sum_{0\leq i<j\leq k} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k) \end{aligned} \tag{1.7}$$

This formula specified to the case $f \in \mathcal{F}^0$ becomes

$$\mathrm{d}^{\nabla} f(X) = \nabla_X f$$

In the abstract index notation we can write this equation as

$$(\mathrm{d}^{\nabla} f)_a{}^{\Phi} = \nabla_a f^{\Phi}$$

Using the above definition it is straightforward to verify the following fundamental property of the exterior covariant derivative.

Proposition 1.1.10. *The exterior covariant derivative* $\mathrm{d}^{\nabla^{\mathcal{F}}}$ *is the unique derivation in the algebra* $\Omega^*(T^*M) \otimes \mathcal{F}$ *of differential forms with coefficients in* $\mathcal{F}$ *such that for* 0*-forms we have* $\mathrm{d}^{\nabla^{\mathcal{F}}} f = \nabla f$, *and the graded Leibniz rule holds, viz*

$$\mathrm{d}^{\nabla^{\mathcal{F}}}(\alpha \wedge \beta) = \mathrm{d}^{\nabla^{\mathcal{F}}}\alpha \wedge \beta + (-1)^{\deg(\alpha)}\alpha \wedge \mathrm{d}^{\nabla^{\mathcal{F}}}\beta \tag{1.8}$$

Proof. See e.g. [50, pp.170-171]. □

If a torsion-free connection is chosen on the base manifolds, the exterior covariant derivative can be expressed using abstract indices in the following form

$$\mathrm{d}^{\nabla}\omega_{a^0 a^1 \dots a^k} := (n+1)\nabla_{a^0}\omega_{a^1 \dots a^k} \tag{1.9}$$

It is easy to see (cf. e.g. [72, p. 429]) that the result of the above expression is independent of the choice of a torsion free connection in the right hand side.

If a Riemannian metric on manifold M is given, we have another operation that we will use later, the (covariant exterior) *codifferential* of a $\mathcal{F}$-valued

k-form ω which is given by

$$\delta^{\nabla}\omega_{a^2\dots a^k} := -\nabla^{a^1}\omega_{a^1 a^2\dots a^k} \tag{1.10}$$

1.1.10 The curvature operator

Recall, that the formula (1.7) specified to a section $\omega \in \mathcal{F}^1$ becomes

$$\mathrm{d}^{\nabla}\omega(X,Y) = \nabla_X\omega(Y) - \nabla_Y\omega(X) - \omega([X,Y])$$

If now $\varphi \in \mathcal{F} = \mathcal{F}^0$ we can apply the previous identity to a $\mathcal{F}$-valued 1-form $\mathrm{d}^{\nabla}\varphi = \nabla\varphi$ to get

$$\begin{aligned}\mathrm{d}^{\nabla}\nabla\varphi(X,Y) &= \nabla_X \mathrm{d}^{\nabla}\varphi(Y) - \nabla_Y \mathrm{d}^{\nabla}\varphi(X) - \mathrm{d}^{\nabla}\varphi([X,Y]) \\ &= \nabla_X\nabla_Y\varphi - \nabla_Y\nabla_X\varphi - \nabla_{[X,Y]}\varphi\end{aligned}$$

It is easily verified that the operator $\mathrm{d}^{\nabla} \circ \nabla$ is C^{∞}-linear and therefore gives rise to a section of $\mathcal{E}^2 \otimes \mathrm{End}\,\mathcal{F}$ that is uniquely defined by the connection ∇ in the vector bundle $\mathcal{F}$.

Definition 1.1.11 (Curvature operator)**.** The operator $K^{\nabla} \colon \mathcal{F} \to \mathcal{F}^2$ defined as

$$K^{\nabla} := \mathrm{d}^{\nabla} \circ \nabla \tag{1.11}$$

is called the *curvature operator*, or simply the *curvature*, of the connection ∇ in the vector bundle $\mathcal{F}$.

When using the abstract index notation it is customary to write the second covariant derivative of a section as $\nabla_a\nabla_b X^{\Phi}$, and one must be careful here because this really means $(\nabla\nabla X)_{ab}{}^{\Phi}$ that is defined as

$$(\nabla\nabla f)_{XY}{}^{\Phi} \equiv \nabla^2_{XY} f^{\Phi} := \nabla_X\left(\nabla_Y f^{\Phi}\right) - \nabla_{D_X Y} f^{\Phi}$$

where X, Y are some tangent vector fields on the manifold M, and D is some linear connection on M. The skew part of the second covariant is then given by

$$\nabla^2_{XY} f^{\Phi} - \nabla^2_{YX} f^{\Phi} = \nabla_X\left(\nabla_Y f^{\Phi}\right) - \nabla_X\left(\nabla_Y f^{\Phi}\right) - \nabla_{(D_X Y - D_Y X)} f^{\Phi}$$

but using the torsion $T(X,Y) = D_X Y - D_Y X - [X,Y]$ we can rewrite this as

$$2\nabla^2_{[XY]} f^\Phi = K^\nabla_{XY} f^\Phi - \nabla_{T(X,Y)} f^\Phi$$

When a choice of a torsion free connection D on the base manifold has been made, the expression for the curvature takes a particularly simple form

$$K^\nabla_{a^1 a^2} f^\Phi = 2\nabla_{a^1}\nabla_{a^2} f^\Phi$$

and in what follows we shall always assume that this is the case.

If D is a linear connection, no other choice of a connection on M is needed, and the expression for its curvature operator takes a classical form

$$K^D_{XY} Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z$$

Proposition 1.1.12 (The Leibniz rule for curvature)**.** *Let $\mathcal{V}$ and $\mathcal{W}$ be vector bundles over a manifold M, $\nabla^\mathcal{V}$ and $\nabla^\mathcal{W}$ be connections in the bundles $\mathcal{V}$ and $\mathcal{W}$ respectively, and $v \in \mathcal{V}$ and $w \in \mathcal{W}$ be arbitrary sections. The curvature operator $K^{\mathcal{V}\otimes\mathcal{W}}$ of the coupled connection $\nabla^{\mathcal{V}\otimes\mathcal{W}}$ in the tensor product $\mathcal{V}\otimes\mathcal{W}$ satisfies the following product rule:*

$$K^{\mathcal{V}\otimes\mathcal{W}}_{a^1 a^2}(v\otimes w) = (K^\mathcal{V}_{a^1 a^2} v)\otimes w + v\otimes(K^\mathcal{W}_{a^1 a^2} w) \tag{1.12}$$

where the indices on sections v, w and $v\otimes w$ are suppressed, and $K^\mathcal{V}_{a^1 a^2}$, $K^\mathcal{W}_{a^1 a^2}$ are the curvature operators on $(\mathcal{V},\nabla^\mathcal{V})$, $(\mathcal{W},\nabla^\mathcal{W})$ respectively.

Proof. A straightforward calculation:

$$\begin{aligned} K^{\mathcal{V}\otimes\mathcal{W}}_{ab} v\, w &= \nabla_a\nabla_b(v\, w) - \nabla_b\nabla_a(v\, w) \\ &= \nabla_a(w\nabla_b v - v\nabla_b w) - \nabla_b(w\nabla_a v - w\nabla_a v) \\ &= w\nabla_a\nabla_b v - \cancel{(\nabla_a w)(\nabla_b v)} + \cancel{(\nabla_a v)(\nabla_b w)} + v\nabla_a\nabla_b w \\ &\quad - \cancel{(\nabla_b w)(\nabla_a v)} - w\nabla_b\nabla_a v - \cancel{(\nabla_b v)(\nabla_a w)} - v\nabla_b\nabla_a w \\ &= v(2\nabla_{[a}\nabla_{b]} w) + w(2\nabla_{[a}\nabla_{b]} v) = (K^\mathcal{V}_{ab} v)\, w + v\,(K^\mathcal{W}_{ab} w) \end{aligned}$$ □

Proposition 1.1.13 (The Bianchi symmetry)**.** *If ∇ is a torsion-free connection in $\mathcal{E}^a$ and $K^\nabla_{a^1 a^2}{}^c{}_d$ is the corresponding curvature operator on $\mathcal{E}^a$, then*

$$K^\nabla_{a^1 a^2}{}^c{}_{a^3} = 0$$

Proof. Let ω_a be an arbitrary 1-form on M. Recall that

$$2\,\nabla_{a^1}\nabla_{a^2}\omega_c = -K^{\nabla}_{a^1a^2}{}^{b}{}_{c}\omega_b$$

Locally $\omega_a = \nabla_a\varphi$ for some smooth function φ (The Poincaré Lemma [50, p.23]). Thus,

$$K^{\nabla}_{a^1a^2}{}^{b}{}_{a^3}\omega_b = -2\nabla_{a^1}\nabla_{a^2}\nabla_{a^3}\varphi = \nabla_{a^1}T^{\nabla}_{a^2a^3}{}^{b}\nabla_b\varphi = 0$$ □

Proposition 1.1.14 (Curvature on $\mathcal{F}$-valued k-forms). *Let $\mathcal{F}$ be a v.b. with connection $\nabla^{\mathcal{F}}$ coupled to a torsion-free connection ∇ on the manifold M, and $X_{a^1\ldots a^k}{}^{\Phi} \in \mathcal{F}^k$. Then*

$$2\,\nabla_{a^1}\nabla_{a^2}X_{a^3\ldots a^{k+2}}{}^{\Phi} = K^{\mathcal{F}}_{a^1a^2}{}^{\Phi}{}_{\Psi}X_{a^3\ldots a^{k+2}}{}^{\Psi}$$

Proof. A straightforward calculations that we give only for the case $k = 1$ since the general case is similar but involves more notation.

Let us start with $X_a{}^{\Phi} = \alpha_a V^{\Phi}$

$$\begin{aligned} 2\,\nabla_{a^1}\nabla_{a^2}X_{a^3}{}^{\Phi} &= 2\,\nabla_{a^1}\nabla_{a^2}(\alpha_a V^{\Phi}) \\ &= (2\,\nabla_{a^1}\nabla_{a^2}\alpha_{a^3})V^{\Phi} + \alpha_{a^3}(2\,\nabla_{a^1}\nabla_{a^2}V^{\Phi}) \end{aligned}$$

where the second term in the last line vanishes due to the Bianchi symmetry.

Since every $X_a{}^{\Phi}$ is a finite sum of terms of the form $\alpha_a V^{\Phi}$ and the curvature operator is $\mathbb{R}$-linear, this completes the proof. □

The section $K^{\nabla}_{a^1a^2}X^{\Phi}$ is a $\mathcal{F}$-valued 2-form, and so the exterior covariant derivative of it can be taken again, however the following Proposition shows that the result will be always zero.

Proposition 1.1.15 (The Bianchi identity). *If $\mathcal{F}$ is a vector bundle with the connection ∇ over manifold M with a torsion-free connection D, and K^{∇} is the curvature 2-form of the connection ∇ coupled to the connection D, then*

$$\mathrm{d}^{\nabla}K^{\nabla} = 0$$

or, equivalently,

$$\nabla_{a^0}K_{a^1a^2}{}^{\Phi}{}_{\Psi} = 0$$

Proof. Let X^{Φ} be a section of the vector bundle $\mathcal{F}$. The curvature operator

$K = K^\nabla$ acts on this section as

$$(KX)_{a^1a^2}{}^\Phi = K_{a^1a^2}{}^\Phi{}_\Psi X^\Psi = 2\,\nabla_{a^1}\nabla_{a^2}X^\Phi$$

Computing the action of the covariant exterior derivative $\mathrm{d}^\nabla K$ on the section X^Φ we get

$$\begin{aligned}\left[(\mathrm{d}^\nabla K)_{a^0a^1a^2}{}^\Phi{}_\Psi\right] X^\Psi &= \left[3\,\nabla_{a^0}K_{a^1a^2}{}^\Phi{}_\Psi\right] X^\Psi \\ &= 3\,\nabla_{a^0}(K_{a^1a^2}{}^\Phi{}_\Psi X^\Psi) - 3\,K_{a^1a^2}{}^\Phi{}_\Psi\nabla_{a^0}X^\Psi \\ &= 3\,\nabla_{a^0}(2\,\nabla_{a^1}\nabla_{a^2}X^\Phi) - 3(2\,\nabla_{a^1}\nabla_{a^2}\nabla_{a^0}X^\Phi) - R_{a^1a^2}{}^b{}_{a_0}\nabla_b X^\Phi \\ &= 6\,\nabla_{a^0}\nabla_{a^1}\nabla_{a^2}X^\Phi - 6\,\nabla_{a^0}\nabla_{a^1}\nabla_{a^2}X^\Phi + 0 = 0 \qquad \square\end{aligned}$$

Proposition 1.1.16 (The skew symmetry in Riemannian vector bundles)**.** *If $\mathcal{F}^\Phi$ is a Riemannian vector bundle with a connection $\nabla = \nabla^\mathcal{F}$ and a metric $h_{\Phi\Psi}$ then the curvature operator has the additional symmetry*

$$K_{a^1a^2\Phi\Psi} = -K_{a^1a^2\Psi\Phi}$$

where $K_{a^1a^2\Phi\Psi} \coloneqq h_{\Phi\Xi}K_{a^1a^2}{}^\Xi{}_\Psi$

Proof. A direct application of the Leibniz rule yields

$$0 = 2\nabla_{a^1}\nabla_{a^2}h_{\Phi\Psi} = -K_{a^1a^2}{}^\Xi{}_\Phi h_{\Xi\Psi} - K_{a^1a^2}{}^\Xi{}_\Psi h_{\Phi\Xi} = -K_{a^1a^2\Psi\Phi} - K_{a^1a^2\Phi\Psi} \quad \square$$

1.1.11 The vector-bundle Laplacian

Let $\mathcal{F}$ be a vector bundle over a Riemannian manifold (M, g), and $\nabla \equiv \nabla^\mathcal{F}$ is a connection in $\mathcal{F}$ coupled to the Levi-Civita connection on M.

Definition 1.1.17. The (rough, or connection) Laplacian Δ is a second order differential operator $\Delta\colon \mathcal{F} \to \mathcal{F}$ defined on any section $f \in \mathcal{F}$ as

$$\Delta f \coloneqq g^{ab}\nabla_b\nabla_a f \coloneqq \nabla^a\nabla_a f \tag{1.13}$$

The following easy consequence of the Leibniz rule proves to be useful in calculations:

Proposition 1.1.18 (Quasi-Leibniz rule for the Laplacian)**.** *For any two sections $s, t \in \mathcal{F}$*

$$\Delta(s \otimes t) = s \otimes \Delta t + 2g^{ab}(\nabla_a s) \otimes \nabla_b t + t \otimes \Delta s \tag{1.14}$$

On vector-bundle valued forms there is another second order operator which has an advantage of being formally self adjoint.

Definition 1.1.19. The *forms Laplacian* $[\Delta]$ is a second order differential operator $[\Delta]\colon \mathcal{F}^k \to \mathcal{F}^k$ defined as

$$[\Delta] f_{a^1 \ldots f^k}{}^{\Phi} \coloneqq -\Big((\mathrm{d}^\nabla \delta^\nabla f)_{a^1 \ldots a^k}{}^{\Phi} + (\delta^\nabla \mathrm{d}^\nabla f)_{a^1 \ldots a^k}{}^{\Phi} \Big) \tag{1.15}$$

on any section $f \in \mathcal{F}^k$.

We shall need the following particular case of a well-known fact.

Proposition 1.1.20 (Weitzenböck identity for 1-forms)**.** *The difference between the action of the forms Laplacian and the rough Laplacian on 1-forms is given by the identity*

$$[\Delta] X_a{}^{\Phi} = \Delta X_a{}^{\Phi} - K^b{}_a{}^{\Phi}{}_{\Psi} X_b{}^{\Psi} - \mathrm{Ric}_a{}^b X_b{}^{\Phi} \tag{1.16}$$

where $K_{ab}{}^{\Phi}{}_{\Psi}$ is the curvature operator of the $\mathcal{F}-$connection, and $\mathrm{R}_{ab}{}^c{}_d$ is the usual Riemannian curvature of the metric g, so that Ric_{ab} is the corresponding Ricci tensor.

Proof. By the definition,

$$[\Delta] X_a{}^{\Phi} \coloneqq -\Big(\delta^\nabla \mathrm{d}^\nabla + \mathrm{d}^\nabla \delta^\nabla \Big) X_a{}^{\Phi}$$

The codifferential of the the covariant exterior derivative of $\mathcal{F}$-valued 1-form $X_a{}^{\Phi}$ is

$$\begin{aligned} \delta^\nabla \mathrm{d}^\nabla X_a{}^{\Phi} &= \nabla^b \Big(\nabla_a X_b{}^{\Phi} - \nabla_b X_a{}^{\Phi} \Big) = -\nabla^b \nabla_b X_a{}^{\Phi} + \nabla^b \nabla_a X_b{}^{\Phi} \\ &= -\Delta X_a{}^{\Phi} + \nabla^b \nabla_a X_b{}^{\Phi} \end{aligned}$$

The exterior covariant derivative of the codifferential of $X_a{}^{\Phi}$ is just

$$\mathrm{d}^\nabla \delta^\nabla X_a{}^{\Phi} = -\nabla_a \nabla^b X_b{}^{\Phi}$$

Combining these equations we obtain

$$\begin{aligned} [\Delta] X_a{}^{\Phi} &= -\Big(-\Delta X_a{}^{\Phi} + \nabla^b \nabla_a X_b{}^{\Phi} - \nabla_a \nabla^b X_b{}^{\Phi} \Big) \\ &= \Delta X_a{}^{\Phi} - \nabla^b \nabla_a X_b{}^{\Phi} + \nabla_a \nabla^b X_b{}^{\Phi} \end{aligned}$$

In the second term of the last display we can commute the derivatives

$$\begin{aligned}
\nabla^b \nabla_a X_b{}^\Phi &= g^{bc} \nabla_c \nabla_a X_b{}^\Phi \\
&= g^{bc} \Big(\nabla_a \nabla_c X_b{}^\Phi + K_{ca}{}^\Phi{}_\Psi X_b{}^\Psi - \mathrm{R}_{ca}{}^d{}_b X_d{}^\Phi \Big) \\
&= \nabla_a \nabla^b X_b{}^\Phi + K^b{}_a{}^\Phi{}_\Psi X_b{}^\Psi - \mathrm{R}^b{}_a{}^d{}_b X_d{}^\Phi \\
&= \nabla_a \nabla^b X_b{}^\Phi + K^b{}_a{}^\Phi{}_\Psi X_b{}^\Psi + \mathrm{Ric}_a{}^b X_b{}^\Phi
\end{aligned}$$

Using this we see that

$$[\Delta] X_a{}^\Phi = \Delta X_a{}^\Phi - \Big(\cancel{\nabla_a \nabla^b X_b{}^\Phi} + K^b{}_a{}^\Phi{}_\Psi X_b{}^\Psi + \mathrm{Ric}_a{}^b X_b{}^\Phi \Big) + \cancel{\nabla_a \nabla^b X_b{}^\Phi}$$

so the result follows. □

1.1.12 The difference formulas

Now let us consider the situation when we have two connections ∇ and ∇' in a vector bundle $\mathcal{F}$. Computing the action of $\nabla' - \nabla$ on φf for $\varphi \in \mathcal{E}$ and $f \in \mathcal{F}$ we obtain

$$\nabla'_a(\varphi f) - \nabla_a(\varphi f) = \varphi \nabla'_a f + \cancel{f \nabla'_a \varphi} - \varphi \nabla_a f - \cancel{f \nabla_a \varphi} = \varphi(\nabla'_a f - \nabla_a f)$$

because connections agree on smooth functions.

Therefore the operator $\nabla' - \nabla$ is linear on sections of $\mathcal{F}$ and corresponds to a section $A_a{}^\Phi{}_\Psi$ of the space of End($\mathcal{F}$)-valued 1-forms that we refer to as the *difference operator*, or the *contorsion*, of connections ∇' and ∇, so we can write

$$\nabla'_a f^\Phi = \nabla_a f^\Phi + A_a{}^\Phi{}_\Psi f^\Psi \tag{1.17}$$

or, symbolically, as

$$\nabla' = \nabla + A$$

Expanding the action of the curvature operator $K_{a^1 a^2}{}^\Phi{}_\Psi$ of ∇' on a section X^Φ we get (using that on the base manifold we have only one connection)

$$\begin{aligned}
2\, \nabla'_{a^1} \nabla'_{a^2} X^\Phi &= 2\, \nabla_{a^1} \left(\nabla_{a^2} X^\Phi + A_{a^2}{}^\Phi{}_\Psi X^\Psi \right) + 2\, A_{a^1}{}^\Phi{}_\Psi \left(\nabla_{a^2} X^\Psi + A_{a^2}{}^\Psi{}_\Xi X^\Xi \right) \\
&= 2\, \nabla_{a^1} \nabla_{a^2} X^\Phi + 2\, (\nabla_{a^1} A_{a^2}{}^\Phi{}_\Psi) X^\Psi + \underbrace{2\, A_{a^2}{}^\Phi{}_\Psi \nabla_{a^1} X^\Psi} \\
&\quad + \underbrace{2\, A_{a^1}{}^\Phi{}_\Psi \nabla_{a^2} X^\Psi} + 2\, A_{a^1}{}^\Phi{}_\Psi A_{a^2}{}^\Psi{}_\Xi X^\Xi
\end{aligned}$$

so we have proven the difference of curvatures formula

$$K'_{a^1a^2}{}^{\Phi}{}_{\Psi}X^{\Psi} = K_{a^1a^2}{}^{\Phi}{}_{\Psi}X^{\Psi} + (2\nabla_{a^1}A_{a^2}{}^{\Phi}{}_{\Psi})X^{\Psi} + 2A_{a^1}{}^{\Phi}{}_{\Xi}A_{a^2}{}^{\Xi}{}_{\Psi}X^{\Psi} \tag{1.18}$$

or, succinctly,

$$K' = K + \nabla \wedge A + A \wedge A$$

A similar calculation yields a formula for the difference of vector-bundle Laplacians

$$\Delta' X^{\Phi} = \Delta X^{\Phi} + 2\, A^{a\Phi}{}_{\Psi}\nabla_a X^{\Psi} + X^{\Psi}\nabla^a A_a{}^{\Phi}{}_{\Psi} + A^{a\Phi}{}_{\Xi}A_a{}^{\Xi}{}_{\Psi}X^{\Psi} \tag{1.19}$$

where we assume that M is equipped with a Riemannian metric, and connections ∇' and ∇ are coupled to the corresponding Levi-Civita connection on the tensor indices.

1.1.13 The Gauss–Codazzi–Ricci decomposition

We consider now the situation that we encounter at least twice in this thesis, firstly, when we recapitulate the Riemannian geometry of hypersurfaces, and secondly, when we study connections in the tractor bundles on hypersurfaces. In both cases we have canonical curvature operators in the natural bundles defined along the hypersurface that come in pairs: an ambient bundle and a subbundle in it that is referred to as intrinsic. The intrinsic bundle in both interesting for us cases is the image of a projection operator such as the tangential projection operator on a Riemannian submanifold, or the tractor projection operator on a conformal hypersurface.

Let us formalize this situation as follows. Let $\mathcal{F}$ be a vector bundle on a manifold M. Let ∇ be a connection in the vector bundle $\mathcal{F}$. Furthermore, let $\mathcal{F}^{\top}$ and $\mathcal{F}^{\perp}$ be subbundles of $\mathcal{F}$ such that $\mathcal{F} = \mathcal{F}^{\top} \oplus \mathcal{F}^{\perp}$. Thus any section $\xi \in \mathcal{F}$ is uniquely represented as a sum of two sections $\tau \in \mathcal{F}^{\top}$ and $\nu \in \mathcal{F}^{\perp}$, and we write $\xi = \binom{\tau}{\nu}$ or simply $\xi = \tau + \nu$ to save the space when $\tau \in \mathcal{F}^{\top}$ and $\nu \in \mathcal{F}^{\perp}$ is assumed.

The projection operators $\pi^{\top} \colon \mathcal{F} \to \mathcal{F}$ and $\pi^{\perp} \colon \mathcal{F} \to \mathcal{F}$ are defined on $\xi = \binom{\tau}{\nu} \in \mathcal{F}^{\top} \oplus \mathcal{F}^{\perp}$ as $\pi^{\top}(\xi) = \tau$ and $\pi^{\perp}(\xi) = \nu$. Clearly, $\pi^{\top}$ and $\pi^{\perp}$ are smooth vector bundle endomorphisms, and

$$\begin{aligned} \pi^{\top} \circ \pi^{\top} &= \pi^{\top} & \qquad \pi^{\top} \circ \pi^{\perp} &= 0 \\ \pi^{\perp} \circ \pi^{\perp} &= \pi^{\perp} & \qquad \pi^{\perp} \circ \pi^{\top} &= 0 \end{aligned}$$

so they are indeed can be called projections.

This structure allows us to define connections $\nabla^\top$ and $\nabla^\perp$ in the subbundles $\mathcal{F}^\top$ and $\mathcal{F}^\perp$ respectively by setting

$$\nabla^\top := \pi^\top \circ \nabla \qquad\qquad \nabla^\perp := \pi^\perp \circ \nabla$$

Returning to the bundle $\mathcal{F} = \mathcal{F}^\top \oplus \mathcal{F}^\perp$ we obtain the direct sum connection that is defined (cf. (1.3)) as

$$\overset{\oplus}{\nabla}\xi = \nabla^\top \tau + \nabla^\perp \nu$$

on sections $\xi = \tau + \nu$ of $\mathcal{F}$.

The properties of connections are easily verified for $\nabla^\top$ and $\nabla^\perp$ since $\pi^\top$ and $\pi^\perp$ are linear.

The connection $\overset{\oplus}{\nabla}$ is called the *van der Waerden–Bortolotti connection* corresponding to the connection ∇ in the vector bundle $\mathcal{F}$ equipped with a direct sum decomposition $\mathcal{F} = \mathcal{F}^\top \oplus \mathcal{F}^\perp$.

The bundle $\mathcal{F}$ is now equipped with two connections ∇ and $\overset{\oplus}{\nabla}$, and we can consider their contorsion operator (1.17). Let us refer to $A = \nabla - \overset{\oplus}{\nabla}$ as the *direct sum contorsion operator* in this context.

The difference formula

$$\boxed{\nabla_a V^b = \overset{\oplus}{\nabla}_a V^b + A_a{}^b{}_c V^c}$$

can be thought as the *abstract Gauss formula* for the direct sum decomposition of the vector bundle $\mathcal{F} = \mathcal{F}^\top \oplus \mathcal{F}^\perp$ with the connections ∇ and $\overset{\oplus}{\nabla}$.

Since the operator A is linear, its action on a section $\xi = \tau + \nu$ of $\mathcal{F}$ decomposes as

$$A_X \begin{pmatrix} \tau \\ \nu \end{pmatrix} = A_X \begin{pmatrix} \tau \\ 0 \end{pmatrix} + A_X \begin{pmatrix} 0 \\ \nu \end{pmatrix}$$

that we can also write in the compact form

$$A_X(\tau + \nu) = A_X \tau + A_X \nu$$

It turns out that the right hand side of the last display is already a direct sum decomposition.

Proposition 1.1.21. *The direct sum contorsion operator A restricted to the* *bundle $\mathcal{F}^\top$ takes values in the bundle $\mathcal{F}^\perp$, and vice versa, that is*

$$\pi^\top \circ A|_{\mathcal{F}^\top} = 0 \qquad\qquad \pi^\perp \circ A|_{\mathcal{F}^\perp} = 0$$

Proof. By definitions of $A \coloneqq \nabla - \overset{\oplus}{\nabla}$ and the connections $\nabla^\top$ and $\nabla^\perp$

$$A_X\tau = \nabla_X\tau - \nabla^\top_X\tau = \nabla_X\tau - \pi^\top(\nabla_X\tau) = \pi^\perp\nabla_X\tau \in \mathcal{F}^\perp$$

where $\tau \in \mathcal{F}^\top$, $X \in \mathrm{T}M$, and similarly, for $\nu \in \mathcal{F}\perp$

$$A_X\nu = \nabla_X\nu - \nabla^\perp_X\nu = \nabla_X\tau - \pi^\perp(\nabla_X\nu) = \pi^\top\nabla_X\nu \in \mathcal{F}^\top \qquad \square$$

It is convenient to introduce the operators ${H_a}^\Phi{}_\Psi \in \mathcal{E}_a \otimes \mathcal{F}^{\perp\,\Phi} \otimes \mathcal{F}^\top_\Psi$ and ${S_a}^\Phi{}_\Psi \in \mathcal{E}_a \otimes \mathcal{F}^{\top\,\Phi} \otimes \mathcal{F}^\perp_\Psi$ defined by

$$H_X\tau \coloneqq \pi^\perp(A_X\begin{pmatrix}\tau\\0\end{pmatrix}) \qquad\qquad S_X\nu \coloneqq \pi^\top(A_X\begin{pmatrix}0\\\nu\end{pmatrix})$$

that capture the nontrivial parts of the action of A on the corresponding subbundles.

In terms of these operators we can express the action of the contorsion A on sections $\xi = \binom{\tau}{\nu}$ of $\mathcal{F}$ as

$$A_X\begin{pmatrix}\tau\\\nu\end{pmatrix} = \begin{pmatrix}S_X\nu\\H_X\tau\end{pmatrix} = S_X\begin{pmatrix}\tau\\\nu\end{pmatrix} + H_X\begin{pmatrix}\tau\\\nu\end{pmatrix}$$

In the matrix form the operator A can be now presented as

$$A_X = \begin{pmatrix}0 & S_X\\ H_X & 0\end{pmatrix} \tag{1.20}$$

The operators H_X and S_X can be interpreted as the derivatives of the projection operators $\pi^\top$, $\pi^\perp \in \mathcal{F}^* \otimes \mathcal{F}$ with respect to the connection ∇ in $\mathcal{F}$-bundles. The definition of this connection applied to the operators $\pi^\top$, $\pi^\perp$ can be explicitly written as

$$(\nabla\pi^\top)_X\xi = \nabla_X(\pi^\top(\xi)) - \pi^\top(\nabla_X\xi)$$
$$(\nabla\pi^\perp)_X\xi = \nabla_X(\pi^\perp(\xi)) - \pi^\perp(\nabla_X\xi)$$

where $\xi \in \mathcal{F}$ and $X \in \mathrm{T}M$. Restricting these identities respectively to the bundles $\mathcal{F}^\top$ and $\mathcal{F}^\perp$ we obtain

$$(\nabla\pi^\top)_X\tau = \nabla_X(\pi^\top(\tau)) - \pi^\top(\nabla_X\tau) = \nabla_X\tau - \nabla^\top_X\tau = A_X\tau$$
$$(\nabla\pi^\perp)_X\nu = \nabla_X(\pi^\perp(\nu)) - \pi^\perp(\nabla_X\nu) = \nabla_X\nu - \nabla^\perp_X\nu = A_X\nu$$

where $X \in \mathrm{T}M$, $\tau \in \mathcal{F}^\top$ and $\nu \in \mathcal{F}^\perp$.

These calculations show that

$$\boxed{H = (\nabla\pi^\top)\circ\pi^\top} \qquad\qquad \boxed{S = (\nabla\pi^\perp)\circ\pi^\perp}$$

The above identities can be thought as the *abstract Weingarten equations* for the direct sum decomposition $\mathcal{F} = \mathcal{F}^\top \oplus \mathcal{F}^\perp$. They express the block components H and S of the direct sum contorsion operator A in terms of the derivatives of the projection operators $\pi^\top$ and $\pi^\perp$.

In terms of the operators H and S the difference formula (1.17) between the connections ∇ and $\overset{\oplus}{\nabla}$ in the vector bundle $\mathcal{F} = \mathcal{F}^\top \oplus \mathcal{F}^\perp$ can be presented as

$$\boxed{\nabla_X\xi^\Phi = \overset{\oplus}{\nabla}_X\xi^\Phi + \left(H_X{}^\Phi{}_\Psi + S_X{}^\Phi{}_\Psi\right)\xi^\Psi}$$

This can be viewed as the *abstract Gauss–Weingarten formula* for the direct sum decomposition $\mathcal{F} = \mathcal{F}^\top \oplus \mathcal{F}^\perp$.

The curvature difference formula (1.18) now reads as

$$K = \overset{\oplus}{K} + \overset{\oplus}{\nabla} \wedge A + A \wedge A \tag{1.21}$$

where $A_X = S_X + H_X$, and we are going to decompose this identity with respect to the direct sum structure $\mathcal{F}^\top \oplus \mathcal{F}^\perp$ on $\mathcal{F}$.

The first term in the right hand side of the last display is the curvature operator of the direct sum connection, and it is easy to see that it can be presented in the matrix form as

$$\overset{\oplus}{K} = \begin{pmatrix} K^\top & 0 \\ 0 & K^\perp \end{pmatrix}$$

where $K^\top$ and $K^\perp$ are the curvature operators of the connections $\nabla^\top$ and $\nabla^\perp$ in the bundles $\mathcal{F}^\top$ and $\mathcal{F}^\perp$ respectively.

The term $\overset{\oplus}{\nabla} \wedge A$ is understood as the exterior covariant derivative of the $\mathrm{Hom}(\mathcal{F},\mathcal{F})$-valued form A with respect to the connection $\nabla = \nabla^{cF}$ coupled to the linear connection in the base manifold.

Let for a moment $\mathcal{F}$ be any vector bundle with a connection $\nabla^{\mathcal{F}}$ over a manifold M equipped with a torsion free linear connection ∇, and $A_X\colon \mathcal{F}\to\mathcal{F}$ be a End $\mathcal{F}$-valued 1-form M. Notice that we can regard this 1-form as a section $A_{a\Psi}{}^{\Phi} \in \mathcal{E}_a \otimes \mathcal{F}_{\Psi}{}^{\Phi}$. Then the covariant derivative of A with respect to the coupled connection $\nabla^{\mathcal{F}\otimes\mathcal{E}}$ in the bundle $\mathcal{E}_a \otimes \mathcal{F}_{\Psi}{}^{\Phi}$ can be explicitly represented as

$$(\nabla^{\mathcal{F}\otimes\mathcal{E}}A)_{XY}\xi = \nabla^{\mathcal{F}}_X(A_Y\xi) - A_{\nabla_X Y}\xi - A_X(\nabla^{\mathcal{F}}_Y\xi)$$

where $\xi\in\mathcal{F}$, and $X,Y\in \mathrm{T}M$.

The exterior covariant derivative of A by definition is

$$(\nabla^{\mathcal{F}\otimes\mathcal{E}}\wedge A)_{XY}\xi = (\nabla^{\mathcal{F}\otimes\mathcal{E}}A)_{XY}\xi - (\nabla^{\mathcal{F}\otimes\mathcal{E}}A)_{YX}\xi$$

or, substituting the explicit expressions,

$$(\nabla^{\mathcal{F}\otimes\mathcal{E}}\wedge A)_{XY}\xi = 2\,\nabla^{\mathcal{F}}_{[X}A_{Y]}\xi - A_{[X,Y]}\xi - 2\,A_{[Y}(\nabla^{\mathcal{F}}_{X]}\xi)$$

where in the second term of the right hand side we have also used the torsion-freeness of the linear connection in M, that is $\nabla_X Y - \nabla_Y X = [X,Y]$.

Applying these formulas to $\overset{\oplus}{\nabla}A$ and expressing the appearing terms further using the definition of $\overset{\oplus}{\nabla}$ and the decomposition $A_X(\tau+\nu) = S_X\nu + H_X\tau$, we compute

$$\begin{aligned}(\overset{\oplus}{\nabla}A)_{XY}\xi &= \overset{\oplus}{\nabla}_X(A_Y\xi) - A_{\nabla_X Y}\xi - A_Y(\overset{\oplus}{\nabla}_X\xi)\\ &= \nabla^{\perp}_X S_Y\nu + \nabla^{\top}_X H_Y\tau - S_{\nabla_X Y}\tau - H_{\nabla_X Y}\nu - S_Y\nabla^{\perp}_X\nu - H_Y\nabla^{\top}_X\tau\end{aligned}$$

Passing to the exterior covariant derivative and using the torsion-freeness of ∇ on M, we obtain

$$\begin{aligned}(\overset{\oplus}{\nabla}\wedge A)_{XY}\xi = 2\,\nabla^{\top}_{[X}S_{Y]}\nu - S_{[X,Y]}\nu - 2\,S_{[Y}\nabla^{\perp}_{X]}\nu\\ + 2\,\nabla^{\perp}_{[X}H_{Y]}\tau - H_{[X,Y]}\tau - 2\,H_{[Y}\nabla^{\top}_{Y]}\tau\end{aligned}$$

The two similar expressions in the top and bottoms lines of right hand side of the last display can be interpreted as the exterior covariant derivatives of the homomorphism-valued 1-forms S and H respectively. A suggestive notation for these derivatives can be introduced as follows

$$(\mathrm{d}^{\nabla^\perp,\nabla^\top} H)_{XY}\tau = 2\,\nabla^\perp_{[X} H_{Y]}\tau - H_{[X,Y]}\tau - 2\,H_{[Y}\nabla^\top_{Y]}\tau$$
$$(\mathrm{d}^{\nabla^\top,\nabla^\perp} S)_{XY}\nu = 2\,\nabla^\top_{[X} S_{Y]}\nu - S_{[X,Y]}\nu - 2\,S_{[Y}\nabla^\perp_{X]}\nu$$

using which we are able to express $\overset{\oplus}{\nabla} A$ as

$$\overset{\oplus}{\nabla} A_{XY}\begin{pmatrix}\tau\\ \nu\end{pmatrix} = \begin{pmatrix}(\mathrm{d}^{\nabla^\top,\nabla^\perp} S)_{XY}\nu\\ (\mathrm{d}^{\nabla^\perp,\nabla^\top} H)_{XY}\tau\end{pmatrix}$$

for some section $\xi \in \mathcal{F}$ represented as $\xi = \tau + \nu$ and vectors $X, Y \in \mathrm{T}M$. To emphasize that this is indeed a direct sum decomposition of $\overset{\oplus}{\nabla} A$ we can rewrite this expression informally in the matrix notation as

$$\overset{\oplus}{\nabla} A = \begin{pmatrix} 0 & \mathrm{d}^{\nabla^\top,\nabla^\perp} S\\ \mathrm{d}^{\nabla^\perp,\nabla^\top} H & 0\end{pmatrix}$$

viewing the right hand side as acting on sections $\binom{\tau}{\nu} \in \mathcal{F}^\top \oplus \mathcal{F}^\perp = \mathcal{F}$ by the usual matrix multiplication.

The term $A \wedge A$ in the curvature difference formula (1.21) can be computed in terms of H_X and S_X too. Treating A as a linear operator (1.20) we can present its composition as the matrix product

$$(A \circ A)_{XY} = \begin{pmatrix} 0 & S_X\\ H_X & 0\end{pmatrix}\begin{pmatrix} 0 & S_Y\\ H_Y & 0\end{pmatrix}$$

so that the wedge product is given by the commutator

$$(A \wedge A)_{XY} = (A \circ A)_{XY} - (A \circ A)_{YX}$$

It is easily verified that explicitly the wedge product $A \wedge A$ turns out to be

$$(A \wedge A)_{XY} = \begin{pmatrix} (S \wedge H)_{XY} & 0\\ 0 & (H \wedge S)_{XY}\end{pmatrix} \tag{1.22}$$

where the bundle homomorphisms involved are

$$S_X \colon \mathcal{F}^\perp \to \mathcal{F}^\top \qquad\qquad H_X \colon \mathcal{F}^\top \to \mathcal{F}^\perp$$
$$S \circ H, S \wedge H \colon \mathcal{F}^\top \to \mathcal{F}^\top \qquad\qquad H \circ S, H \wedge S \colon \mathcal{F}^\perp \to \mathcal{F}^\perp$$

so that the composition is well defined and the formula (1.22) agrees with the direct sum decomposition.

Combining the expressions for the terms of the curvature difference formula (1.21) in the matrix form we obtain what we may call the *Gauss–Codazzi–Ricci decomposition* for the direct sum connection in vector bundle

Theorem 1.1.22 (The Gauss–Codazzi–Ricci decomposition)**.** *The curvature K of the connection ∇ in a vector bundle $\mathcal{F}$ decomposed into the direct sum as $\mathcal{F} = \mathcal{F}^\top \oplus \mathcal{F}^\perp$ is expressed in terms of the curvatures of the projected connections on the respective subbundles and the components of the direct sum contorsion as*

$$K = \begin{pmatrix} K^\top + S \wedge H & \mathrm{d}^{\nabla^\top,\nabla^\perp} S \\ \mathrm{d}^{\nabla^\perp,\nabla^\top} H & K^\perp + H \wedge S \end{pmatrix} \tag{1.23}$$

Remark 1.1.23. Traditionally, the Gauss–Codazzi–Ricci equation is stated separately for each subspace in the direct sum, namely as

$$\begin{aligned} \pi^\top(K|_{\mathcal{F}^\top}) &= K^\top + S \wedge H && \text{Gauss} \\ \pi^\perp(K|_{\mathcal{F}^\top}) &= \mathrm{d}^{\nabla^\perp,\nabla^\top} H && \text{Codazzi I} \\ \pi^\top(K|_{\mathcal{F}^\perp}) &= \mathrm{d}^{\nabla^\top,\nabla^\perp} S && \text{Codazzi II} \\ \pi^\perp(K|_{\mathcal{F}^\perp}) &= K^\perp + H \wedge S && \text{Ricci} \end{aligned}$$

Now we turn to the case when the orthogonal decomposition is compatible with the metric h in a Riemannian vector bundle $\mathcal{F}$ with a connection ∇. That is we have $\nabla h = 0$, and the metrics $h^\top$ and $h^\perp$ on the component subbundles $\mathcal{F}^\top$ and $\mathcal{F}^\perp$ are defined as

$$h^\top = h|_{\mathcal{F}^\top} \qquad\qquad h^\perp = h|_{\mathcal{F}^\perp}$$

Now we can define the *direct sum metric* $h^\oplus = \begin{pmatrix} h^\top & 0 \\ 0 & h^\perp \end{pmatrix}$ in the bundle $\mathcal{F}$.

From this point we begin to assume that $h = h^\oplus$. This is equivalent to saying that $\mathcal{F}^\perp$ is the orthogonal complement of $\mathcal{F}^\top$ with respect to the metric h, or simply to the condition

$$h(\tau, \nu) = 0$$

for all $\tau \in \mathcal{F}^\top$, $\nu \in \mathcal{F}^\perp$. Indeed,

$$h(\tau, \nu) = h^\oplus(\tau, \nu) = h^\oplus\left(\begin{pmatrix} \tau \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ \nu \end{pmatrix}\right) = h(\tau, 0) + h(0, \nu) = 0$$

Proposition 1.1.24. *The $h^\top$ and $h^\perp$ on the component subbundles are compatible with the corresponding connections:*

$$\nabla^\top h^\top = 0 \qquad\qquad \nabla^\perp h^\perp = 0$$

and if $h = h^\oplus$ we also have

$$\overset{\oplus}{\nabla} h = 0$$

Proof. Straightforward calculations. □

Proposition 1.1.25. *The operators H_X and S_X are skew-adjoint in the sense that*

$$h^\perp(H_X\tau, \nu) = -h^\top(\tau, S_X\nu) \tag{1.24}$$

for any section $\tau + \nu \in \mathcal{F}^\top \oplus \mathcal{F}^\perp = \mathcal{F}$

Proof. Differentiating $h(\tau, \nu) = 0$ with respect to ∇ we get

$$\begin{aligned} 0 &= h(\nabla_X\tau, \nu) + h(\tau, \nabla_X\nu) \\ &= h(\nabla_X\tau - \nabla^\top_X\tau, \nu) + h(\tau, \nabla_X\nu - \nabla^\perp_X\nu) \\ &= h(H_X\tau, \nu) + h(\tau, S_X\nu) \end{aligned}$$

where $X \in \mathrm{T}M$. □

An important addition to the observations made in this section is to consider a local orthonormal frame $\{N_x\}, x \in 1, \dots, k$ of the subbundle $\mathcal{F}^\perp$. In this thesis we only use the case $k = 1$ but it is natural to give a more general perspective here.

Recall that a frame $\{N_x\}, x \in 1, \dots, k$ is orthonormal if

$$h_{\Phi\Psi} N^\Phi{}_x N^\Phi{}_y = \delta_{xy} := \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

The indices $x, y, \dots = 1, \dots, k$ are not abstract, and we shall identify δ_{xy} with $\delta_x{}^y$ etc, so effectively we do not care about the upper or lower position of these indices. In particular, we can write $h_{\Phi\Psi} N^\Phi{}_x N^\Phi{}_y = N^\Phi{}_x N^y{}_\Phi = \delta_{xy}$.

In the presence of a local orthonormal frame $\{N_x\}, x \in 1, \dots, k$ of the bundle $\mathcal{F}^\perp$ we can write an expression of the "normal" projection operator $\pi^\perp$ as follows:

$$(\pi^\perp)^\Phi{}_\Psi = \sum_{x=1}^{k} N^\Phi{}_x N^x{}_\Psi$$

where $N^x{}_\Psi := h_{\Psi\Xi} N^\Xi{}_x$.

The "tangential" projection operator $\pi^\top$ is then expressed as

$$(\pi^\top)^\Phi{}_\Psi = \delta^\Phi{}_\Psi - \sum_{x=1}^{k} N^\Phi{}_x N^x{}_\Psi$$

where $\delta^\Phi{}_\Psi$ is the abstract index notation for the identity operator $\mathrm{id}\colon \mathcal{F} \to \mathcal{F}$.

It is customary to use the Einstein summation convention here and write $(\pi^\perp)^\Phi{}_\Psi = N^\Phi{}_x N^x{}_\Psi$ and $(\pi^\top)^\Phi{}_\Psi = \delta^\Phi{}_\Psi - N^\Phi{}_x N^x{}_\Psi$ dropping the explicit summation operator, but we shall not do this now for the sake of clarity.

Differentiating the relation $N^\Phi{}_x N^y{}_\Phi = \delta_{xy}$ we obtain the identity

$$N^\Phi{}_x \nabla_a N^y{}_\Phi = -N^y{}_\Phi \nabla_a N^\Phi{}_x$$

which in the case $k = 1$ yields the property (dropping the index x)

$$N^\Phi \nabla_a N_\Phi = 0$$

that we shall frequently use in this thesis.

Now we can take advantage of these preparations and give an expression for the operators S and H in terms of an orthonormal frame N_x in $\mathcal{F}^\perp$.

Recall that $S_a{}^\Phi{}_\Psi := (\nabla\pi^\perp)_a{}^\Phi{}_\Xi (\pi^\perp)^\Xi{}_\Psi$. Substituting the expression for $\pi^\perp$ and performing differentiation we get

$$\begin{aligned}
S_a{}^\Phi{}_\Psi = (\nabla\pi^\perp)_a{}^\Phi{}_\Xi (\pi^\perp)^\Xi{}_\Psi &= \left(\nabla_a\left(\sum_{x=1}^{k} N^\Phi{}_x N^x{}_\Xi\right)\right)\left(\sum_{y=1}^{k} N^\Xi{}_y N^y{}_\Psi\right) \\
&= \left(\sum_{x=1}^{k}\big((\nabla_a N^\Phi{}_x) N^x{}_\Xi + N^\Phi{}_x \nabla_a N^x{}_\Xi\big)\right)\left(\sum_{y=1}^{k} N^\Xi{}_y N^y{}_\Psi\right) \\
&= \sum_{x=1}^{k}\sum_{y=1}^{k}\big((\nabla_a N^\Phi{}_x) N^x{}_\Xi N^\Xi{}_y N^y{}_\Psi + N^\Phi{}_x N^\Xi{}_y N^y{}_\Psi \nabla_a N^x{}_\Xi\big) \\
&= \sum_{x=1}^{k}\sum_{y=1}^{k}\big((\nabla_a N^\Phi{}_x) N^x{}_\Psi - N^\Phi{}_x N^x{}_\Xi N^y{}_\Psi \nabla_a N^\Xi{}_y\big) \\
&= \sum_{x=1}^{k} (\pi^\top)^\Phi{}_\Xi (\nabla_a N^\Phi{}_x) N^x{}_\Psi
\end{aligned}$$

The quantities

$$\mathrm{L}_a{}^\Phi{}_x := (\pi^\top)^\Phi{}_\Xi \nabla_a N^\Xi{}_x$$

deserve the name of the *shape* 1-*forms* for their role in what follows.

Thus we can write ${S_a}^{\Phi}{}_{\Psi} = \sum_{x=1}^{k} (\pi^{\top})^{\Phi}{}_{\Xi} {\mathrm{L}_a}^{\Xi}{}_x N^x{}_{\Psi}$.

Now we can see that ${H_a}^{\Phi}{}_{\Psi} = -h^{\Phi\Theta} h_{\Xi\Psi} {S_a}^{\Xi}{}_{\Theta}$.

In the case $k = 1$ we shall work with the shape 1-form

$$ {\mathrm{L}_a}^{\Phi} := \nabla_a N^{\Phi} $$

dropping the projection on the index Φ due to the identity $N^{\Phi}\nabla_a N_{\Phi} = 0$ and ignoring the index x.

1.1.14 The pull-back connection

The construction of the pullback bundle and the pullbacks of geometric quantities will be used systematically in the rest of this thesis, mainly because this is the way how the submanifolds (and hypersurfaces) inherit the structure from the background. For this reason we give a brief overview of these constructions, mainly following [4, section 1.8] where many subtleties have been brought to our attention. A more formal and detailed treatment can be found in [50].

Definition 1.1.26 (The pullback bundle)**.** Let $\varphi \colon M \to N$ be a smooth map, and $\mathcal{F}$ be a vector bundle over manifold N. The *pullback bundle* $\varphi^*\mathcal{F}$ is defined over manifold M so that the fiber over each point $p \in M$ is the same as the fiber over the point $\varphi(p) \in N$. In other words, the following diagram commutes

$$ \begin{array}{ccc} \varphi^*\mathcal{F} & \longrightarrow & \mathcal{F} \\ \big\downarrow & & \big\downarrow \\ M & \underset{\varphi}{\longrightarrow} & N \end{array} $$

Remark 1.1.27. As it is clear from the diagram, the pullback bundle has the universal property, that allows to establish the following important fact.

Proposition 1.1.28. *"Pullbacks commute with taking duals and tensor products" [4], that is there are canonical isomorphisms such that*

$$ (\varphi^*\mathcal{F})^* \cong \varphi^*(\mathcal{F}^*) \text{ and } (\varphi^*\mathcal{F}_1) \otimes (\varphi^*\mathcal{F}_2) \cong \varphi^*(\mathcal{F}_1 \otimes \mathcal{F}_2) $$

Any section $f \in \mathcal{F}$ gives rise to a section of the pullback bundle $\varphi^*\mathcal{F}$ by taking the composition of f with φ. This operation is called the restriction along the map φ.

Definition 1.1.29 (Restrictions)**.** The *restriction* f_φ of the section f *along* the map φ is defined by

$$f_\varphi \coloneqq f \circ \varphi$$

.

Remark 1.1.30*.* Restrictions of sections of any vector bundle on the target manifold along smooth maps always exist (that is, no requirements to the map are needed in addition to smoothness).

Remark 1.1.31*.* If the map $\varphi\colon M \to N$ is just an inclusion $\iota\colon M \to N$, especially when the manifold M is a submanifold of N, the restriction is usually denoted by the symbol $\cdot|_M$. We shall do so later after the precise definitions related to submanifolds are given. Moreover, when working only with sections on the submanifold, the restriction operation will be often suppressed form the notation.

Definition 1.1.32 (Pushforwards)**.** Recall that the tangent map $\mathrm{T}\varphi\colon \mathrm{T}M \to \mathrm{T}N$ is vector bundle morphism (linear on fibers at each point $p \in M$), and therefore can be viewed as a $\varphi^*\mathrm{T}N$-valued 1-form on M, that is as a section $\varphi_* \in \mathrm{T}_*M \otimes \varphi^*\mathrm{T}N$. This 1-form acts on tangent vectors $X \in \mathrm{T}M$, and the result of this action is called the *pushforward* φ_*X which is an element of $\varphi^*\mathrm{T}N$.

Combining the restriction and the pushforward operation we can define the pullback operation on any $(0,k)$-tensor.

Definition 1.1.33 (The pullback of a covariant tensor)**.** Let t be a $(0,k)$ tensor on N, and $\varphi\colon M \to N$ be a map between manifolds. The pullback φ^*t is a $(0,k)$-tensor on M defined as

$$\varphi^*t(X_1, \dots, X_k) \coloneqq t(\varphi_*X_1, \dots, \varphi_*X_k)$$

It is important for us to notice that the pullback operation is defined for $\mathcal{F}$-valued $(0,k)$-tensors without any change in the above display, because the restrictions along φ can be taken for a section of any vector bundle over N. In other words, if $t \in \mathcal{E}_{ab\dots c} \otimes \mathcal{F}$, both sides of the above display are viewed as taking values in the bundle $\mathcal{F}$.

In particular, the pullback is defined for any vector bundle-valued k-form.

Now let $\varphi\colon M \to N$ be a map between manifolds as before, and $\mathcal{F}$ is a vector bundle over N equipped with a connection ∇. In this situation there is a canonically defined connection in the pullback bundle $\varphi^*\mathcal{F}$ called the pullback connection and denoted by $\varphi^*\nabla$.

Definition 1.1.34 (The pullback connection)**.** Let $\varphi\colon M \to N$ be a map between manifolds, and $\mathcal{F}$ is a vector bundle over N. The pullback connection $\varphi^*\nabla$ on the pullback bundle $\varphi^*\mathcal{F}$ is defined on restrictions f_φ of sections $f \in \mathcal{F}$ along φ by setting

$$(\varphi^*\nabla^{\mathcal{F}})_X f_\varphi \coloneqq \nabla^{\mathcal{F}}_{\varphi_* X} f \tag{1.25}$$

and extending this definition by linearity since the pullback bundle is generated by such restrictions.

Remark 1.1.35. It is instructive to write down the equation (1.25) in abstract indices. In order to do that we need to distinguish the tangent bundles of the manifolds M and N by the indices, so let for the moment the range $\{i, j, k, \dots\}$ be associated with the tangent bundle $\mathcal{E}^i$ of the manifold M, and the range $\{\lambda, \mu, \nu, \dots\}$ be the range associated to the tangent bundle $\mathcal{E}^\lambda$ of the manifold N. The tangent map $\mathrm{T}\varphi$ is represented by a section ${\varphi_i}^\lambda \in \mathcal{E}_i \otimes \varphi^*\mathcal{E}^\lambda$ as we already noted above (using the index-free notation). The pushforward of a vector $X^i \in \mathcal{E}^i$ is then given by the contraction with the differential ${\varphi_i}^\lambda X^i$. Let also the range $\Phi, \Psi, \Xi.\dots$ be associated with the bundle $\mathcal{F}^\Phi$, and notice that we can use this range both for the bundle $\mathcal{F}$ and for the pullback bundle $\varphi^*\mathcal{F}$ because their fibers are isomorphic as vector spaces. Having done these preparations we can now rewrite the equation (1.25) as follows

$$X^i(\varphi_*\nabla_i)f^\Phi|_\varphi = ({\varphi_i}^\lambda X^i)\nabla_\lambda f^\Phi \tag{1.26}$$

where we have used the notation $\cdot|_\varphi$ for the restriction of a section along the map φ in order to distinguish φ from an index.

The pullback connection agrees with taking the duals and tensor products, and also preserves the metric structure and the curvature operator on vector bundles.

Proposition 1.1.36. *Let $\varphi\colon M \to N$ be a map between manifolds, and $\mathcal{F}$ be a Riemannian vector bundle over manifold N with a metric h and a connection ∇. The pullback bundle $\varphi^*\mathcal{F}$ equipped with the pullback metric φ^* and the pullback connection $\varphi^*\nabla$ is again a Riemannian vector bundle.*

Proof. Applying the equation (1.25) we get

$$(\varphi^*\nabla)_Z h_\varphi = \nabla_{\varphi_* Z} h = 0$$

□

Proposition 1.1.37 (The curvature of the pull-back connection)**.** *In the settings of the previous proposition, let* K^{∇} *be the curvature operator of the connection* ∇ *in the bundle* $\mathcal{F}$ *over the manifold* N*, and* $K^{\varphi^*\nabla}$ *be the curvature operator of the pullback connection* $\varphi^*\nabla$ *in the pullback bundle* $\varphi^*\mathcal{F}$*. Then the following equality holds*

$$K^{\varphi^*\nabla^{\mathcal{F}}} = \varphi^* K^{\nabla^{\mathcal{F}}} \tag{1.27}$$

where in the both sides we have $\operatorname{End}(\mathcal{F})$*-valued* 2*-forms on the manifold* M*.*

Proof. It is a straightforward verification that can be done using a local frame. See e.g. [4] or [50, p. 176]. □

The pullback of a linear connection preserves the symmetry of the connection in the following sense.

Proposition 1.1.38 (The symmetry of the pullback connection)**.** *Let* ∇ *be a torsion free connection in the tangent bundle of the manifold* N*, and the map* φ *is as before. The pullback connection satisfies the following identity*

$$(\varphi^*\nabla)_X(\varphi_* Y) - (\varphi^*\nabla)_Y \varphi_* X = \varphi_*([X, Y]) \tag{1.28}$$

Proof. The proof is best done using local coordinates. See e.g. [4]. □

1.1.15 The Riemannian curvature

On a Riemannian manifold (M, g) we have the Levi-Civita connection $\nabla = \nabla^g$ in the tangent bundle $\mathrm{T}M$. This connection is extended to tensor bundles in the usual way (using the dual connection in T^*M and the coupled connections in $\mathrm{T}^*M\otimes\cdots\otimes\mathrm{T}^*M\otimes\mathrm{T}M\otimes\cdots\otimes\mathrm{T}M$), and we refer to this extended connection as to the Levi-Civita connection in tensor bundles.

The curvature operator of the Levi-Civita connection is termed the Riemann curvature operator and denoted by $\mathrm{R}_{ab}{}^{c}{}_{d}$. Thus, by definition

$$\mathrm{R}_{ab}{}^{c}{}_{d} \coloneqq K^{\nabla^g}{}_{ab}{}^{c}{}_{d}$$

Since in the Riemannian structure provides a canonical identification of the tangent and cotangent spaces, it is customary to use it to raise and lower the indices without mentioning this explicitly. In particular, it is convenient to present the Riemann curvature operator in the covariant form. The resulting object is traditionally referred to as the Riemannian curvature tensor, or simply as the Riemannian curvature. Explicitly, the Riemannian curvature is given by the equation

$$\mathrm{R}_{abcd} := g_{ce}\mathrm{R}_{ab}{}^{e}{}_{d}$$

Sometimes, it is convenient to write the Riemannian curvature as $\mathrm{R}_{ab}{}^{cd}$ emphasizing that

$$\mathrm{R}_{ab}{}^{cd} \in \mathcal{E}_{[ab]}{}^{[cd]}$$

or, in other words, that the Riemannian curvature is a $\Lambda^2(\mathrm{T}^*M)$-valued 2-form on M.

Notice that we refer to the tensor R_{abcd} and all its index variations ($\mathrm{R}_{ab}{}^{c}{}_{d}$, $\mathrm{R}_{ab}{}^{cd}$) as the Riemannian curvature, and use the upright capital letter R for it.

In addition to the symmetries of the curvature operator in vector bundles, that is the skew symmetry

$$\mathrm{R}_{abcd} = -\mathrm{R}_{bacd}$$

and the Bianchi symmetry

$$\mathrm{R}_{[abc]d} = 0$$

the Riemannian curvature has also the swap symmetry

$$\mathrm{R}_{abcd} = \mathrm{R}_{cdab}$$

The Bianchi identity for the Riemannian curvature

$$\nabla_{[a}\mathrm{R}_{bc]de} = 0 \tag{1.29}$$

is a consequence of a more general fact established above, see Prop. 1.1.15.

The Riemannian curvature decomposes (if $n \geq 3$) orthogonally into the totally trace-free part called the *Weyl tensor* W_{abcd} and the trace part that can be written as

$$\mathrm{R}_{ab}{}^{cd} = \mathrm{W}_{ab}{}^{cd} + 4\,\delta_{[a}{}^{[c}\mathsf{P}_{b]}{}^{d]} \tag{1.30}$$

where the *Schouten tensor* P_{ab} and the *Schouten scalar* J are defined by

$$\mathrm{Ric}_{ab} = (n-2)\mathsf{P}_{ab} + \mathsf{J}g_{ab}, \quad \mathsf{J} := g^{ab}\mathsf{P}_{ab} \tag{1.31}$$

A proof of (1.30) will be given in a more general setting in Section 1.1.17.

1.1.16 Algebraic curvature tensors (ACT)

In this subsection we work over an inner product space (V, g) where the inner product g may have an arbitrary signature.

We use the inner product g to identify V and V^* without mention, so that the *position* of abstract indices is chosen according to convenience (an thus has no particular meaning).

In the abstract index notation the inner product is written either as $g_{ab} \in V_{ab}$ or $\delta_a{}^b \in V_a{}^b$, and we do not distinguish them really by virtue of the aforementioned identification $V \equiv V^*$.

Tensors will be elements of the tensor products $\otimes^k V$ of the vector space V, and the inner product is extended to these tensor products as usual, e.g. for $t, s \in \otimes^2 V$ we have their inner product defined as $t^{ab} s_{ab}$.

Definition 1.1.39. A tensor $\mathcal{R} \in \otimes^4 V$ is called an *algebraic curvature tensor* (ACT, or Bianchi tensor), if it has the algebraic symmetries of the Riemannian curvature

$$\mathcal{R}_{abcd} = -\mathcal{R}_{bacd} = \mathcal{R}_{cdab} \text{ and } \mathcal{R}_{[abc]d} = 0 \tag{1.32}$$

It is convenient to regard algebraic curvature tensors as elements of $\Lambda^2(V) \otimes \Lambda^2(V)$, that is as exterior 2-forms with values in the space of exterior 2-forms. This is a special case of so-called *double forms*, the technique that was originated by J.A.Thorpe in [71], and then used extensively by A.Gray [35], R.Kulkarni [43], K.Nomizu, and many others. Currently this technique is actively developed by M.L.Labbi, see e.g. [44]. We only use this idea to get some insights, but won't go into the full implementation of double forms here. It is enough for us to observe that 2-tensors can be also seen as double (1,1)-forms due to the trivial isomorphism $\otimes^2 V \cong \Lambda^1(V) \otimes \Lambda^1(V)$. The wedge product of double forms gets a classical appearance as the Kulkarni–Nomizu product of 2-tensors.

Definition 1.1.40. The *Kulkarni–Nomizu product* of tensors $t, s \in \otimes^2 V$ is denoted by $t \owedge s$ and defined to be

$$(t \owedge s)_{ab}{}^{cd} := 4\, t_{[a}{}^{[c} s_{b]}{}^{d]} \tag{1.33}$$

Writing all the indices in the lower position, the Kulkarni–Nomizu product is given by

$$(t \owedge s)_{abcd} = t_{ac} s_{bd} - t_{bc} s_{ad} + t_{bd} s_{ac} - t_{ad} s_{bc} \tag{1.34}$$

The Kulkarni-Nomizu product of 2-tensors has the following properties.

Proposition 1.1.41. *The Kulkarni–Nomizu product of 2-tensors is a commutative operation:*

$$t \owedge s = s \owedge t \tag{1.35}$$

for $t, s \in \otimes^2 V$.

Proof. This follows easily from (1.34):

$$\begin{aligned}(t \owedge s)_{abcd} &= t_{ac}s_{bd} - t_{bc}s_{ad} + t_{bd}s_{ac} - t_{ad}s_{bc} \\ &= s_{ac}t_{bd} - t_{ad}s_{bc} + s_{bd}t_{ac} - t_{bc}s_{ad} = (s \owedge t)_{abcd}\end{aligned}$$

Notice that no symmetry of 2-tensors t and s is used here. □

Remark 1.1.42. Classically, the double forms are required to have the swap symmetry, but we need to relax on this for some applications where we use similar ideas for $\mathcal{F}$-valued forms where $\mathcal{F}$ is a vector bundle, that is the first group of indices acts on tensor bundles, the second group of indices acts on the $\mathcal{F}$-bundles, and within each group we have the skew symmetry.

Proposition 1.1.43. *The Kulkarni–Nomizu product* $t \owedge s$ *of two symmetric 2-tensors* $t, s \in \odot^2 V$ *is an ACT*

Proof. The swap symmetry $(t \owedge s)_{abcd} = (t \owedge s)_{cdab}$ follows from the commutativity.

The skew symmetry $(t \owedge s)_{abcd} = -(t \owedge s)_{bacd}$ is obvious from the definition.

It remains to verify the Bianchi symmetry. Since this result is well known and has been proved in many various ways mathematically, we take a chance to demonstrate how this can be established in an experimental manner using a computer-algebra system. We have used Cadabra, see Appendix B for more information. The use of a computer system is just a handy tool for a manual computation since once can check the full listing step by step.

This simple and self-explanatory code in Cadabra

```
{a,b,c,d,e,e#}::Indices(vector);
KN := t_{a c} s_{b d} - t_{b c} s_{a d} + t_{b d} s_{a c} - t_{a d} s_{b c};
@asym!(%){_{a}, _{b}, _{c} };
@collect_terms!(%);
```

produces the following output

$$KN := \frac{1}{3}\, t_{ac}s_{bd} - \frac{1}{3}\, t_{ab}s_{cd} - \frac{1}{3}\, t_{bc}s_{ad} + \frac{1}{3}\, t_{ba}s_{cd} + \frac{1}{3}\, t_{cb}s_{ad} - \frac{1}{3}\, t_{ca}s_{bd}$$
$$+ \frac{1}{3}\, t_{bd}s_{ac} - \frac{1}{3}\, t_{cd}s_{ab} - \frac{1}{3}\, t_{ad}s_{bc} + \frac{1}{3}\, t_{cd}s_{ba} + \frac{1}{3}\, t_{ad}s_{cb} - \frac{1}{3}\, t_{bd}s_{ca}$$

from which one can see that if 2-tensors t and s are symmetric, all the terms cancel out. □

Remark 1.1.44. The above computation certainly can be done by hand. The purpose is to show that a computer algebra system may prove useful in practical discovering of tensor identities. Of course, the code can be improved to give the zero result automatically.

The g-traces of algebraic curvature tensors are the Ricci and the Scalar parts.

Definition 1.1.45. The Ricci part $\mathrm{Ric}(\mathcal{R})$ of an ACT $\mathcal{R}$ is defined as

$$\mathrm{Ric}(\mathcal{R})_b{}^d := \delta_c{}^a \mathcal{R}_{ab}{}^{cd} \tag{1.36}$$

Definition 1.1.46. The scalar part $\mathrm{Scal}(\mathcal{R})$ of an ACT $\mathcal{R}$ is the next contraction:

$$\mathrm{Scal}(\mathcal{R}) := \delta_d{}^b \mathrm{Ric}(\mathcal{R})_b{}^d \tag{1.37}$$

Proposition 1.1.47. *For a $2-$tensor t over an inner product space (V, g) of dimension n we have*

$$\mathrm{Ric}(g \owedge t) = (n-2)\, t + (\mathrm{tr}_g\, t)\, g \tag{1.38}$$

and

$$\mathrm{Scal}(g \owedge t) = 2(n-1)\ \mathrm{tr}_g\, t \tag{1.39}$$

Proof. Indeed,

$$\delta_c{}^a(\delta_a{}^c t_b{}^d - \delta_b{}^c t_a{}^d + \delta_b{}^d t_a{}^c - \delta_a{}^d t_b{}^c) \quad = \quad n\, t_b{}^d - t_b{}^d + (\mathrm{tr}_g\, t)\delta_b{}^d - t_b{}^d$$

Notice that tensor $t \in \otimes^2 V$ does not have to be symmetric. □

Proposition 1.1.48. *Let $t \in \otimes^2 V$. The map $t \mapsto g \owedge t$ is injective for $n > 2$*

Proof. Suppose that $g \owedge t = 0$ for some $t \in \otimes^2 V$. Using equation (1.38) we get $0 = \mathrm{Ric}(g \owedge t) = (n-2)t + (\mathrm{tr}_g\, t)\, g$ and so $t = -\frac{\mathrm{tr}_g\, t}{n-2} g$. Taking the trace yields $\mathrm{tr}_g\, t = -\frac{\mathrm{tr}_g\, t}{n-2} n$, or $(1 + \frac{n}{n-2})\, \mathrm{tr}_g\, t = 0$, so the claim follows. □

Definition 1.1.49. The *composition* $t \circ s$ of two $2-$tensors t and s over an inner product space (V, g) is defined as just as the composition of the corresponding endomorphisms

$$(t \circ s)_a{}^b := t_a{}^c s_c{}^b \tag{1.40}$$

that is just as the matrix product of t and s if they are are interpreted as matrices.

Remark 1.1.50. It is well known that this operation is **not symmetric** in general , even if both tensors t and s are symmetric.

Remark 1.1.51. The operation of composition of $2-$tensors is **associative**:

$$(t \circ s) \circ r = t \circ (s \circ r) \tag{1.41}$$

since the composition of functions is associative.

Definition 1.1.52. The *double contraction* $t : s$ of two $2-$tensors t and s over an inner product space (V, g) is defined as

$$t : s := t_a{}^c s_c{}^a = \operatorname{tr}_g(t \circ s) \tag{1.42}$$

Proposition 1.1.53. *The double contraction of 2-tensors is a commutative operation:*

$$t : s = s : t$$

for any $t, s \in \otimes^2 V$.

Proof. From the definition (1.42) by renaming the indices: $t_a{}^c s_c{}^a = s_a{}^c t_c{}^a$. ☐

Proposition 1.1.54. *For a* $2-$*tensor* t *over an inner product space* (V, g) *the double contraction of it with itself is also known as the* square of its length

$$t : t = |t|^2 \tag{1.43}$$

Proof. Obvious. ☐

Definition 1.1.55. The symmetric product $t \odot s$ of two $2-$tensors t and s over an inner product space (V, g) is defined as

$$t \odot s := \tfrac{1}{2}\Big(t \circ s + s \circ t\Big) \tag{1.44}$$

Remark 1.1.56. This is just an index-free notation for the symmetrization of the composition

$$(t \odot s)_{ab} = (t \circ s)_{(ab)}$$

Definition 1.1.57. For $k \geq 0$ the $k-$th power of a $2-$tensor t over an inner product space (V, g) is defined recurrently as

$$t^0 := \mathrm{id}$$

$$t^{k+1} := t^k \circ t$$

In particular,

$$t^2 := t \circ t$$

Proposition 1.1.58. *For two symmetric $2-$tensors t and s over an inner product space (V, g) of dimension n we have*

$$\mathrm{Ric}(t \owedge s) = (\mathrm{tr}_g\, s)\, t + (\mathrm{tr}_g\, t)\, s - 2\, t \odot s$$

and

$$\mathrm{Scal}(t \owedge s) = 2(\mathrm{tr}_g\, t)(\mathrm{tr}_g\, s) - 2\, t : s$$

Proof. Compute

$$\delta_c{}^a(t_a{}^c s_b{}^d - t_b{}^c s_a{}^d + t_b{}^d s_a{}^c - t_a{}^d s_b{}^c) = (\mathrm{tr}_g\, t) s_b{}^d - (t \circ s)_b{}^d + (\mathrm{tr}_g\, s) t_b{}^d - (s \circ t)_b{}^d$$

and

$$\begin{aligned} &\delta_d{}^b\Big((\mathrm{tr}_g\, t) s_b{}^d - (t \circ s)_b{}^d + (\mathrm{tr}_g\, s) t_b{}^d - (s \circ t)_b{}^d\Big) \\ &= (\mathrm{tr}_g\, t)(\mathrm{tr}_g\, s) - (t \circ s)_b{}^d + (\mathrm{tr}_g\, s)(\mathrm{tr}_g\, t) - (s \circ t)_b{}^d \end{aligned}$$

□

Corollary 1.1.59. *For a symmetric $2-$tensor t the following holds:*

$$\mathrm{Ric}(t \owedge t) = 2(\mathrm{tr}_g\, t)t - 2\, t^2$$

and

$$\mathrm{Scal}(t \owedge t) = 2(\mathrm{tr}_g\, t)^2 - 2\, |t|^2$$

Corollary 1.1.60. *For an inner product space (V, g)*

$$\mathrm{Ric}(g \owedge g) = 2(n-1)g$$

and

$$\mathrm{Scal}(g \owedge g) = 2(n-1)n$$

1.1.17 The Weyl–Schouten decomposition of ACT

Definition 1.1.61. A $(4,0)-$tensor $\mathcal{W}$ is called the *Weyl part* of an ACT $\mathcal{R}$ if it is totally trace free and there exist a $(2,0)-$tensor $\mathcal{P}$ called the *Schouten part* of $\mathcal{R}$ such that

$$\mathcal{R} = \mathcal{W} + g \owedge \mathcal{P} \tag{1.45}$$

We use the notation $\mathbb{W}(\mathcal{R}) \coloneqq \mathcal{W}$ and $\mathbb{P}(\mathcal{R}) \coloneqq \mathcal{P}$ for the Weyl part and the Schouten part of an ACT $\mathcal{R}$

As the Schouten part $\mathbb{P}(\mathcal{R})$ of an ACT is a symmetric tensor, we can define its trace

$$\mathbb{J}(\mathcal{R}) \coloneqq \mathrm{tr}_g \, \mathbb{P}(\mathcal{R})$$

that we term the Schouten scalar of $\mathcal{R}$.

Proposition 1.1.62. *For any ACT $\mathcal{R}$ there exist and defined uniquely the Weyl part $\mathcal{W} = \mathbb{W}(\mathcal{R})$ and the Schouten part $\mathcal{P} = \mathbb{P}(\mathcal{R})$ so that the equation (1.45) holds. In fact,*

$$\mathbb{P}(\mathcal{R}) = \tfrac{1}{n-2}\Big(\mathrm{Ric}(\mathcal{R})_b{}^d - \tfrac{\mathrm{Scal}(\mathcal{R})}{2(n-1)}\delta_b{}^d\Big) \tag{1.46}$$

and $\mathbb{W}(\mathcal{R}) = \mathcal{R} - g \owedge \mathbb{P}(\mathcal{R})$.

The Schouten scalar is then given by

$$\mathbb{P}(\mathcal{R}) = \tfrac{1}{2(n-1)}\mathrm{Scal}(\mathcal{R}) \tag{1.47}$$

Proof. For the proof it is enough to compute:

$$\begin{aligned}
\mathrm{Ric}(\mathcal{R})_b{}^d &= \delta_c{}^a \mathcal{R}_{ab}{}^{cd} \\
&= \delta_c{}^a(\mathcal{W} + \delta_a{}^c\mathcal{P}_b{}^d - \delta_b{}^c\mathcal{P}_a{}^d + \delta_b{}^d\mathcal{P}_a{}^c - \delta_a{}^d\mathcal{P}_b{}^c) \\
&= 0 + n\,\mathcal{P}_b{}^d - \mathcal{P}_b{}^d + \delta_b{}^d\mathcal{J} - \mathcal{P}_b{}^d \\
&= (n-2)\mathcal{P}_b{}^d + \delta_b{}^d\mathcal{J}
\end{aligned}$$

where we denoted

$$\mathcal{J} \coloneqq \delta_c{}^a\mathcal{P}_a{}^c \tag{1.48}$$

Contracting again we get

$$\begin{aligned}\mathrm{Scal}(\mathcal{R}) &= \delta_d{}^b \mathrm{Ric}(\mathcal{R})_b{}^d \\ &= (n-2)\mathcal{J} + n\mathcal{J} = 2(n-1)\mathcal{J}\end{aligned}$$

Now it is easy to solve the equations:

$$\mathcal{J} = \tfrac{1}{2(n-1)}\mathrm{Scal}(\mathcal{R})$$

and

$$\begin{aligned}\mathcal{P}_b{}^d &= \tfrac{1}{n-2}\Big(\mathrm{Ric}(\mathcal{R})_b{}^d - \delta_b{}^d\mathcal{J}\Big) \\ &= \tfrac{1}{n-2}\Big(\mathrm{Ric}(\mathcal{R})_b{}^d - \tfrac{\mathrm{Scal}(\mathcal{R})}{2(n-1)}\delta_b{}^d\Big)\end{aligned}$$

□

There a further decomposition of the Ricci part $\mathrm{Ric}(\mathcal{R})$ into the the symmetric trace-free part $\mathring{Ric}(\mathcal{R})$ and the trace part $\frac{\mathrm{Scal}(\mathcal{R})}{n}g$, that is

$$\mathrm{Ric}(\mathcal{R}) = \mathring{Ric}(\mathcal{R}) + \tfrac{\mathrm{Scal}(\mathcal{R})}{n}g \tag{1.49}$$

Some more auxiliary lemmas will be useful in what follows.

Lemma 1.1.63. *For two symmetric* $2-$*tensors* t *and* s *over an inner product space* (V, g) *of dimension* n *the Schouten part of their Kulkarni-Nomizu product is given by*

$$\mathbb{P}(t \owedge s) = \tfrac{1}{n-2}\Big((\mathrm{tr}_g\, s)\, t + (\mathrm{tr}_g\, t)\, s - 2\, t \odot s - \tfrac{(\mathrm{tr}_g\, t)(\mathrm{tr}_g\, s) - t:s}{(n-1)}\, g\Big)$$

The Schouten scalar part is then

$$\mathbb{J}(t \owedge s) = \tfrac{n}{n-1}\Big(2(\mathrm{tr}_g\, t)\,(tr_g s) - 2\, t : s\Big)$$

Proof. We just compute:

$$\begin{aligned}\mathbb{P}(t \owedge s) &= \tfrac{1}{n-2}\Big(\mathrm{Ric}(t \owedge s) - \tfrac{\mathrm{Scal}(t \owedge s)}{2(n-1)}\, g\Big) \\ &= \tfrac{1}{n-2}\Big((\mathrm{tr}_g\, s)\, t + (\mathrm{tr}_g\, t)\, s - 2\, t \odot s - \tfrac{2(\mathrm{tr}_g\, t)(\mathrm{tr}_g\, s) - 2\, t:s}{2(n-1)}\, g\Big)\end{aligned}$$

Taking the traces we get

$$\begin{aligned}
\mathbb{J}(t \owedge s) &= \tfrac{1}{n-2}\Big((\mathrm{tr}_g\, s)\,(\mathrm{tr}_g\, t) + (\mathrm{tr}_g\, t)\,(tr_g s) - 2\, t : s - \tfrac{(\mathrm{tr}_g\, t)(\mathrm{tr}_g\, s) - t:s}{(n-1)}\, n\Big) \\
&= \tfrac{1}{n-2}\left(\tfrac{(n-1)(2(\mathrm{tr}_g\, t)\,(tr_g s) - 2\, t:s) - \Big((\mathrm{tr}_g\, t)(\mathrm{tr}_g\, s) - t:s\Big)}{(n-1)}\, n\right) \\
&= \tfrac{n}{n-1}\Big(2(\mathrm{tr}_g\, t)\,(tr_g s) - 2\, t : s\Big)
\end{aligned}$$

□

Corollary 1.1.64. *For a symmetric* $2-$*tensor* t *over an inner product space* (V, g) *of dimension* n *the Schouten part of its Kulkarni-Nomizu square* $(t \owedge t)$ *is computed as*

$$\mathbb{P}(t \owedge t) = \tfrac{1}{n-2}\Big(2\,(\mathrm{tr}_g\, t)\, t - 2\, t^2 - \tfrac{(\mathrm{tr}_g\, t)^2 - |t|^2}{(n-1)}\, g\Big)$$

The Schouten scalar part is then

$$\mathbb{J}(t \owedge t) = \tfrac{n}{n-1}\Big(2(\mathrm{tr}_g\, t)^2 - 2\,|t|^2\Big)$$

Corollary 1.1.65. *For a symmetric* $2-$*tensor* t *over an inner product space* (V, g) *of dimension* n

$$\mathbb{P}(g \owedge t) = t$$

and

$$\mathbb{J}(g \owedge t) = \mathrm{tr}_g\, t$$

From this one derives that for any such t

$$\mathbb{W}(g \owedge t) = 0$$

From this we easily derive that operations $\mathbb{W}$ *and* $g \owedge \mathbb{P}\cdot$ *are projectors in the corresponding spaces.*

1.2 Hypersurfaces in Riemannian manifolds

The main goal of this section is to make an overview of Riemannian geometry of hypersurfaces and to prepare the notation for the remaining parts of this thesis.

Using the general Gauss–Codazzi–Ricci decomposition from Section 1.1.13 applied to the direct sum splitting of the pullback bundle along the hypersurface, we recover the Gauss and Codazzi equations for hypersurfaces. In Chapter 2 we apply the Weyl-Schouten decomposition (1.45) to these equations and obtain their forms, which are convenient for the conformal geometry of hypersurfaces.

These constructions, including the derivation of the Gauss and Codazzi equations, are paralleled in Chapter 3. In particular, we formulate the tractor analogues of these equations. This leads to an alternative way to obtain the conformal Gauss and Codazzi equations from Chapter 2.

1.2.1 Submanifolds

Definition 1.2.1. A map $f\colon N \to M$ between two smooth manifolds is called an *immersion* if for any $p \in N$ the differential $\mathrm{T}_p f\colon \mathrm{T}_p N \to \mathrm{T}_p M$ is injective.

Immersions can have a rather complicated behavior, for instance, nothing stops an immersion from having self intersections or an awkward topology.

Definition 1.2.2. An *embedding* is an immersion $f\colon M \to N$ which is also a homeomorphism onto its image.

Informally, a submanifold $\bar{M}$ is a subset of a manifold M such that locally looks like an affine subspace in the Euclidean space $\mathbb{R}^n$.

Definition 1.2.3. Let M be a smooth manifold. A subset $\bar{M} \subseteq M$ is an *embedded submanifold* in M if for any point $p \in \bar{M}$ there is a chart $(U, \vec{x})$ in M centered around point p such that $\bar{M} \cap U = \vec{x}((\mathbb{R}^m \times 0) \cap \vec{x}(U))$ where $\mathbb{R}^m \times 0 = \{(x^1, \dots, x^m, 0, \dots, 0) \in \mathbb{R}^n | x^1, \dots, x^m \in \mathbb{R}\}$. A chart with this property is called a *slice chart*, and the corresponding coordinates are called *slice coordinates* of the submanifold. The manifold M in this context will be called the *background*.

In other words, locally the inclusion $\bar{M} \subset M$ it looks like the standard embedding

$$\mathbb{R}^m \to \mathbb{R}^n\colon (x^1, \dots, x^m) \mapsto (x^1, \dots, x^m, 0, \dots, 0)$$

In slice coordinates the submanifold is given locally as the zero locus of the last $n - m$ coordinate functions. The converse is a consequence of the implicit function theorem that we state in the form given in [10]

Theorem 1.2.4 (The Implicit Function Theorem)**.** *If $s^1, \dots, s^k$ are smooth functions with $\mathrm{d}s^1 \wedge \dots \wedge \mathrm{d}s^k \neq 0$ at $p \in M$, then they can be completed to a coordinate system near p by the addition of $m = n - k$ functions $s^{k+1}, \dots, s^n$.*

It is convenient to have a global version of this fact.

Theorem 1.2.5. *A subset $\bar{M} \subseteq M$ is an embedded submanifold if and only if $\bar{M}$ is the image of an embedding.*

Proof. See [37, p.21]. □

If we set $\bar{M} = \{p \in M | s^1(p) = \cdots = s^k(p) = 0\}$ we get a subset, which is called the zero locus $\mathcal{Z}(s^1, \ldots, s^k)$ of a system of functions $(s^1, \ldots, s^k)$.

Definition 1.2.6. If a submanifold $\bar{M} \subseteq M$ is represented as the zero locus $\mathcal{Z}(s^1, \ldots, s^k)$ of a system of real valued functions $(s^1, \ldots, s^k)$ with the property $\mathrm{d}s^1 \wedge \cdots \wedge \mathrm{d}s^k \neq 0$, the functions $s^1, \ldots, s^k$ are called the *defining functions* of the submanifold $\bar{M}$.

Proposition 1.2.7. *Any embedded submanifold $\bar{M}$ can be locally represented as the zero locus of a system of defining functions.*

Proof. This is a direct consequence of our definition of embedded submanifolds and the implicit function theorem. □

The number $k = n - m$ is called the codimension $\operatorname{codim}_M(\bar{M})$ of the submanifold $\bar{M}$ in the manifold M. Usually the ambient manifold is fixed, and we denote the codimension simply as $\operatorname{codim} \bar{M}$.

Embedded submanifolds have the following useful properties.

For any smooth function $\bar{f}$ on $\bar{M}$ there is a smooth function f on M such that $\bar{f} = f|_{\bar{M}}$. This ensures the existence of extensions off an embedded submanifold. This is proved easily by using slice coordinates.

Tangent vectors to $\bar{M}$ are precisely those tangent vectors X to M at points on $\bar{M}$ that act on smooth functions f on M so that $Xf = 0$ whenever $f|_{\bar{M}} \equiv 0$. (See e.g. [49], p. 178). This allows to identify naturally the tangent space at a point of $\bar{M}$ with a subspace of the tangent space at that point to M. In other words, the inclusion map $\iota \colon \bar{M} \to M$ is an immersion.

1.2.2 Hypersurfaces

By a *hypersurface* we mean an embedded submanifold of codimension 1. For certainty, we assume that we are given a closed hypersurface Σ in a connected Riemannian manifold (M, g). Recall, that a manifold is called *closed* if it is compact and has no boundary. Both the background manifold M and the hypersurface Σ are assumed to be oriented. This restriction is consistent with the local character of our considerations.

As we discussed earlier, locally a regular embedded hypersurface $\Sigma \subset M$ can be presented as the zero locus $\Sigma = \mathcal{Z}(s) = \{p \in M | s(p) = 0\}$ of a smooth function $s\colon M \to \mathbb{R}$ such that the differential $\mathrm{d}s \neq 0$ at all the points along Σ. We refer to such s as a local defining function of hypersurface Σ.

The freedom in the choice of defining functions can be locally described as multiplication by a smooth nowhere vanishing function.

Proposition 1.2.8. *Let $s, s'\colon U \to \mathbb{R}$ be two local defining functions of the hypersurface Σ around point $p \in \Sigma$. Then there is a smooth function $\lambda\colon U \to \mathbb{R}$ such that $s' = \lambda \cdot s$ on U, and λ is nowhere zero on U.*

Proof. This is a consequence of the following modification of the Taylor theorem (also known as the Hadamard lemma):

Claim. *Let $(U, \{s^i\}_{i=1,\dots,n})$ be a slice coordinate system of the hypersurface Σ, that is s^n is a defining function of Σ, and $f\colon U \to \mathbb{R}$ be any smooth function. Then there exist a smooth function $g\colon U \to \mathbb{R}$ such that for all $x \in U$*

$$f(x^1, \dots, x^n) = f(x^1, \dots, x^{n-1}, 0) + x^n \cdot g(x^1, \dots, x^n)$$

where $x^i \coloneqq s^i(x)$, $i = 1, \dots, n$ and the usual identifications are applied.

Proof of the claim. Pick $x \in U$ and define a function $\varphi\colon [0, 1] \to \mathbb{R}$ by

$$\varphi(t) \coloneqq f(x^1, \dots, tx^n)$$

By the fundamental theorem of calculus

$$\begin{aligned}&f(x^1, \dots, x^n) - f(x^1, \dots, x^{n-1}, 0)\\&= \varphi(1) - \varphi(0) = \int_0^1 \frac{d\varphi}{dt} dt = x^n \int_0^1 \frac{\partial f}{\partial x^n}(x^1, \dots, tx^n) dt\end{aligned}$$

where the function

$$g(x) \coloneqq \int_0^1 \frac{\partial f}{\partial x^n}(x^1, \dots, tx^n) dt$$

is clearly smooth. This confirms the claim.

Now we can complete the proof of the Proposition. Since s is smooth and $\mathrm{d}s \neq 0$ along Σ we can complete it to a slice coordinate system

$$\{s^1, \dots, s^{n-1}, s^n = s\}$$

and using the Claim in these coordinates we get

$$s' = x^n \cdot g =: s \cdot \lambda$$

everywhere on U. Clearly, $\lambda \neq 0$ everywhere off the hypersurface Σ otherwise we get a contradiction with the fact that s' is a defining function of Σ, but because

$$\mathrm{d}s' = \lambda \mathrm{d}s + s\mathrm{d}\lambda$$

becomes

$$\mathrm{d}s' = \lambda \mathrm{d}s$$

along Σ and both $\mathrm{d}s'$ and $\mathrm{d}s$ are non-degenerate linear operators at all the points of Σ, we get $\lambda \neq 0$ everywhere on U. □

Oriented defining functions

In view of our main goal we shall be interested in the local properties of hypersurfaces that are independent of the choice of a defining function. There is an issue with the orientation that is easy to eliminate in the case of hypersurfaces. Observe that if s is a defining function, its opposite $-s$ is a defining function again, but only one of the functions s, $-s$ will be oriented in the following sense. Recall that we assume that our hypersurface Σ and the background manifold M are both assumed to be oriented, that is there is a class of positive volume forms chosen on each of them. Since the differential $\mathrm{d}s$ of the defining function is a nowhere vanishing 1-form at all the points of the hypersurface, it can be used to choose an orientation on Σ.

A defining function s of the hypersurface Σ is called *oriented* if it induces the given orientation on Σ, that is $\epsilon_\Sigma \wedge \mathrm{d}s = \epsilon_M$ for some strictly positive smooth function α, where ϵ_M is a positive n-form on M and ϵ_Σ is a positive $(n-1)$-form on Σ.

1.2.3 The ambient tensor bundles

The tangent bundle $\mathrm{T}M$ of the background manifold M is denoted in the abstract index manner by $\mathcal{E}^a$, and we call it the *background tangent bundle*. The restriction of the background tangent bundle onto Σ can be identified with the pullback bundle $\iota^*\mathrm{T}M$ along the embedding $\iota \colon \Sigma \to M$. The pullback bundle $\iota^*\mathrm{M} \equiv \mathrm{T}M|_\Sigma$ will be called the *ambient tangent bundle* of the hypersurface Σ, or simply the ambient bundle. According to the abstract index notation we may use the same indices $\{a, b, c, \dots\}$ for the ambient bundle since on the fiber level we have isomorphic vector spaces, but the core symbol should be different. In lines with our notational policy, we use the symbol $\underline{\mathcal{E}}^a$ for the ambient bundle on Σ. In most cases we shall view the bundles as defined along the hypersurface, and therefore a special notation will be redundant, so we may (slightly abusively) suppress the restriction operation, and write simply $\mathcal{E}^a$ for the ambient bundle of the hypersurface.

The dual of the bundle $\mathrm{T}M|_\Sigma$ is the pullback of the cotangent bundle T^*M due to Proposition 1.1.28. Therefore, we denote the bundle $\mathrm{T}^*M|_\Sigma$ by $\underline{\mathcal{E}}_a$ or just by $\mathcal{E}_a$. It can be called the dual ambient bundle, or the ambient cotangent bundle.

The tensor products of finite number of copies of $\underline{\mathcal{E}}^a$ and $\underline{\mathcal{E}}_a$ agree with the restrictions (pullbacks) of the background tensor bundles, and are termed the *ambient tensor bundles*.

The restriction of the background metric g_{ab} on M onto the ambient bundle $\underline{\mathcal{E}}^a$ is called the *ambient metric* on Σ and denoted by the same symbol g_{ab}.

The pullback connection $\iota^*\nabla$ of the Levi-Civita connection on M is called the *ambient connection* on the hypersurface Σ and denoted by the symbol $\underline{\nabla}$. The purpose of the underline is to remind that this connection is defined along the hypersurface, and thus, in the index notation, is a map $\underline{\nabla}_a \colon \underline{\mathcal{E}}^b \to \overline{\mathcal{E}}_a \otimes \underline{\mathcal{E}}^b$, so in the calculations we know that the index a in $\underline{\nabla}_a$ is tangential, that is $\mathrm{N}^a \underline{\nabla}_a f = 0$ for any ambient tensor section f.

1.2.4 The unit normal of a hypersurface

Since the hypersurface is supposed to be oriented, the unit section of its normal bundle is uniquely defined and called the unit normal field.

Using an oriented defining function we can actually compute the unit normal, and the result will be independent of the choice of the defining function along the hypersurface.

More specifically, let s be a local defining function of Σ in an open set $U \subseteq M$. We can define a co-vector field N_a on U by the formula

$$\mathrm{N}_a = |\nabla s|^{-1} \nabla_a s$$

Using the metric on M it is easy to see that $\mathrm{N}_a \mathrm{N}^a = 1$ in the points of U where $|\nabla s|$ is defined (it suffices for us that this holds in a tubular neighborhood of Σ in M).

The unit normal vector field N^a will be then obtained from N_a using the metric on the manifold M as $\mathrm{N}^a = g^{ab}\mathrm{N}_b$.

We show now that this construction gives (co-)vector fields on M that agree at all the points along the hypersurface Σ.

Proposition 1.2.9. *Let s and s' be smooth oriented local defining functions of the hypersurface Σ on an open subset U in the manifold M. The covectors $\mathrm{N}_a = |\nabla s|^{-1}\nabla_a s$ and $\mathrm{N}'_a = |\nabla s'|^{-1}\nabla_a s'$ are equal at all the points of $\Sigma \cap U$.*

Proof. If functions s and s' are local defining functions of Σ on U, using Proposition 1.2.8 we can write

$$s' = \lambda s$$

for some smooth function λ, nowhere vanishing on U.

At all the points of $\Sigma \cap U$ we have

$$\nabla_a s' = s\nabla_a\lambda + \lambda\nabla_a s = \lambda\nabla_a s \qquad \text{along } \Sigma$$

Since both defining functions s and s' are oriented, λ must be positive everywhere on U. Thus $|\nabla s'| = \lambda|\nabla s|$.

Therefore

$$\mathrm{N}'_a = \frac{\lambda\,\nabla_a s}{\lambda|\nabla s|} = \frac{\nabla_a s}{|\nabla s|} = \mathrm{N}_a \qquad \text{along } \Sigma$$ □

The point is that along the hypersurface the unit normal is a section of the ambient bundle $\mathcal{E}^a$. Using the defining function s of the hypersurface Σ we can think of the field $\mathrm{N}_a = \frac{1}{|\nabla s|}\nabla_a s$ as of an extension of the unit normal off the hypersurface. The proposition ensures that at the point of the hypersurface all such extensions agree. The general facts that we have for the pullback connections ensure that the ambient derivatives of the unit normal along the hypersurface are also independent of extensions. These observations are important for our discussion of the hypersurface invariants in Chapter 4.

The normal bundle

The normal bundle $\mathcal{N}^a$ of the hypersurface Σ has rank 1, and therefore the normal connection $\nabla^{\mathcal{N}}$, that is the normal projected ambient connection $\nabla^{\perp} := \pi^{\perp} \circ \underline{\nabla}$, is just the trivial (flat) connection d on the bundle $\mathcal{N}^a$.

1.2.5 The projection operators on hypersurface

Using the unit conormal field N_a and the ambient metric in $\underline{\mathcal{E}}^a$ we can give explicit expressions for the projection operators $\pi^\top$ and $\pi^\perp$ in the ambient bundle $\underline{\mathcal{E}}^a$, as is discussed in the end of Section 1.1.13. We rewrite the observations made there as follows.

Proposition 1.2.10. *The* normal projection operator $\pi^\perp\colon \mathrm{T}M|_\Sigma \to \mathrm{T}M|_\Sigma$ *can be identified with a section of* $\underline{\mathcal{E}}_a{}^b$ *denoted by* $\mathrm{N}_a{}^b$ *and is given by the expression*

$$\mathrm{N}_a{}^b := \mathrm{N}_a\mathrm{N}^b$$

The (tangential) projection operator $\pi^\top\colon \mathrm{T}M|_\Sigma \to \mathrm{T}M|_\Sigma$ *can be identified with a section of* $\underline{\mathcal{E}}_a{}^b$ *denoted by* $\Pi_a{}^b$ *and is given by the expression*

$$\Pi_a{}^b := \delta_a{}^b - \mathrm{N}_a{}^b$$

The projection operators are idempotent

$$\mathrm{N}_a{}^b\mathrm{N}_a{}^b = \mathrm{N}_a{}^b \text{ and } \Pi_a{}^b\Pi_a{}^b = \Pi_a{}^b$$

and satisfy the property $\Pi_a{}^b\mathrm{N}_b{}^c = 0$.

1.2.6 The intrinsic tensor bundles

The tangent bundle $\mathrm{T}\Sigma$ of the hypersurface Σ can be identified with the annihilator of the unit conormal field N_a. This bundle is termed the *intrinsic tangent bundle* of the hypersurface and denoted by the symbol $\overline{\mathcal{E}}^a$ according to our conventions. The the intrinsic cotangent bundle $\mathrm{T}^*\Sigma$ (the dual of $\mathrm{T}\Sigma$) is denoted as $\overline{\mathcal{E}}_a$ in the abstract index manner.

The intrinsic tensor bundles are the tensor products of a finite number of copies of the bundles $\overline{\mathcal{E}}^a$ and $\overline{\mathcal{E}}_a$. A. usual, we denote them by adding several indices in the upper and lower positions to the symbol $\overline{\mathcal{E}}$, e.g. $\overline{\mathcal{E}}_{(ab)}$ stands for $\odot^2\mathrm{T}\Sigma$.

The intrinsic metric

The induced Riemannian metric (the first fundamental form) $\bar{g} = \iota^* g$ on the intrinsic bundle is just the restriction of the ambient metric onto the intrinsic bundle viewed as a subbundle of the ambient bundle of the hypersurface. This metric will be termed the *intrinsic metric* and denoted by the symbol $\bar{g}_{ab}$.

Using the projection operator $\Pi_a{}^b$ we can express the intrinsic metric as

$$\bar{g}_{ab} = \Pi_a{}^{a'}\Pi_a{}^{a'} g_{a'b'} \tag{1.50}$$

This is equivalent to the identity

$$\bar{g}_{ab} = g_{ab} - \mathrm{N}_a\mathrm{N}_b \tag{1.51}$$

The intrinsic connection

The Levi-Civita connection of the intrinsic metric $\bar{g}_{ab}$ will be called the *intrinsic connection* on hypersurface, and denoted by the symbol $\overline{\nabla}$.

In the direct sum decomposition $\underline{\mathcal{E}}^a = \overline{\mathcal{E}}^a \oplus \mathcal{N}^a$ we also have the (tangential) projected ambient connection $\nabla^\top := \pi^\top \circ \underline{\nabla}$, that is, for an intrinsic tangent vector $V^a \in \overline{\mathcal{E}}^a$, we have

$$\nabla^\top_a V^b := \Pi_a{}^{a'}\Pi_{b'}{}^b \nabla_{a'} V^{b'}$$

A fundamental fact of Riemannian geometry of hypersurfaces (in fact, submanifolds) is the following.

Theorem 1.2.11 (The Gauss theorem)**.** *The connection $\nabla^\top$ in the intrinsic tensor bundle $\overline{\mathcal{E}}^a$ of hypersurface coincides with the Levi-Civita connection of the intrinsic metric $\bar{g}$:*

$$\nabla^\top \equiv \overline{\nabla}$$

Proof. Using using (1.51) it is straightforward to verify that $\nabla^\top \bar{g} = 0$. The fact $T^{\nabla^\top} = 0$ follows immediately from the "torsion-freeness" of the pull-back of a torsion free connection as in Proposition 1.1.38, but it is also can be seen directly: $\Pi_a{}^{a'}\Pi_b{}^{b'}\nabla_{a'}\nabla_{b'} f = \Pi_a{}^{a'}\Pi_b{}^{b'}\nabla_{b'}\nabla_{a'} f$. The uniqueness of the Levi-Civita connection (Proposition 1.1.9) then implies the claim. □

1.2.7 The Weingarten equations for hypersurfaces

The ambient bundle $\underline{\mathcal{E}}^a$ of the hypersurface is presented now as the direct sum $\underline{\mathcal{E}}^a = \overline{\mathcal{E}}^a \oplus \mathcal{N}^a$ of the intrinsic bundle $\overline{\mathcal{E}}^a$ and the normal bundle $\mathcal{N}^a$.

A section $V^a \in \underline{\mathcal{E}}^a$ can be written uniquely as $V^a = \tau^a + \nu^a$ for some $\tau \in \overline{\mathcal{E}}^a$ and $\tau^a \in \mathcal{N}^a$. It is convenient to use a vector like notation $V^a = \binom{\tau^a}{\nu^a}$ for such sections, as we do in Section 1.1.13. Notice that $\nu^a = \nu \mathrm{N}^a$ for $\nu = \mathrm{N}_a V^a$.

Since by the Gauss theorem $\nabla^\top = \overline{\nabla}$, and we denote $\nabla^\perp$ as $\nabla^\mathcal{N}$, the direct sum connection $\overset{\oplus}{\nabla} = \nabla^\top \oplus \nabla^\perp$ can presented as

$$\overset{\oplus}{\nabla} = \begin{pmatrix} \overline{\nabla} & 0 \\ 0 & \nabla^\mathcal{N} \end{pmatrix} \tag{1.52}$$

The direct sum connection $\overset{\oplus}{\nabla}$ can be extended to ambient tensor bundles in the usual way (as the dual and tensor product connection). It is traditionally called the *van der Waerden–Bortolotti* connection in the literature (see e.g. [63] where this construction is used for submanifolds of arbitrary codimension).

Using the general observations made in Section 1.1.13, the difference operator $A_a{}^b{}_c$ between the ambient connection $\underline{\nabla}$ and the direct sum connection $\overset{\oplus}{\nabla}$ is expressed as $A = S + H$ where the operators $H_a{}^b{}_c$ and $S_a{}^b{}_c$ are defined by

$$H_a{}^b{}_c = (\underline{\nabla}_a \Pi^b{}_d)\Pi^d{}_c$$

and

$$S_a{}^b{}_c = (\underline{\nabla}_a \mathrm{N}^b{}_d)\mathrm{N}^d{}_c$$

The projection operators $\Pi^a{}_b$ and $\mathrm{N}^a{}_b$ are viewed as sections of the ambient tensor bundles, and thus can be differentiated with respect to the ambient covariant derivative.

If we compute these derivatives, we get, for instance,

$$\underline{\nabla}_a \Pi_b{}^c = \underline{\nabla}_a \Big(\delta_b{}^c - \mathrm{N}_b \mathrm{N}^c\Big) = -(\underline{\nabla}_a \mathrm{N}_b)\mathrm{N}^c - \mathrm{N}_b \underline{\nabla}_a \mathrm{N}^c$$

The quantity $\underline{\nabla}_a \mathrm{N}_b$ plays an important role in the geometry of hypersurfaces.

Definition 1.2.12. The *shape tensor* L_{ab} is the ambient covariant derivative of the unit normal along the hypersurface:

$$\mathrm{L}_{ab} \coloneqq \underline{\nabla}_a \mathrm{N}_b \tag{1.53}$$

The *mean curvature* H of hypersurface Σ is defined as

$$\mathrm{H} \coloneqq \bar{n}^{-1} \mathbf{g}^{ab} \mathrm{L}_{ab} \tag{1.54}$$

where $\bar{n} = \dim \Sigma$.

It is easy now to express the operators $H_a{}^b{}_c$ and $S_a{}^b{}_c$ in terms of the shape tensor.

Proposition 1.2.13 (The Weingarten equations). *The operators* ${H_a}^b{}_c$ *and* ${S_a}^b{}_c$ *are given by the formulas*

$$
\begin{aligned}
{S_a}^b{}_c &= \mathrm{L}_a{}^b \mathrm{N}_c \\
{H_a}^b{}_c &= -\mathrm{N}^b \mathrm{L}_{ac}
\end{aligned}
\tag{1.55}
$$

Proof. Straightforward computations, which we have partly done above. □

In practice, the Weingarten equations are often used in the form of the following identity:

$$\underline{\nabla}_a \Pi_b{}^c = -\mathrm{L}_{ab}\mathrm{N}^c - \mathrm{N}_b \mathrm{L}_a{}^c \tag{1.56}$$

1.2.8 The Gauss–Weingarten formula on hypersurfaces

Theorem 1.2.14 (The Gauss–Weingarten formula). *The difference between the ambient connection* $\underline{\nabla}$ *and the van der Waerden–Bortolotti connection* $\overset{\oplus}{\nabla}$ *in the ambient bundle* $\underline{\mathcal{E}}^a$ *of the hypersurface* Σ *is given by the following expression:*

$$\underline{\nabla}_a V^b = \overset{\oplus}{\nabla}_a V^b + (\mathrm{N}_c \mathrm{L}_a{}^b - \mathrm{N}^b \mathrm{L}_{ac}) V^c \tag{1.57}$$

where V^b *is an* ***ambient*** *vector along the hypersurface* Σ, *that is* $V^a \in \underline{\mathcal{E}}^a$.

Proof. This is just a restatement of the abstract Gauss formula $\nabla = \overset{\oplus}{\nabla} + (S + H)$ from Section 1.1.13 for the case of the direct sum decomposition $\underline{\mathcal{E}}^a = \overline{\mathcal{E}}^a \oplus \mathcal{N}^a$ using the Gauss theorem $\nabla^\top = \overline{\nabla}$ and the Weingarten equations (1.55). □

In practical computations this formula is used separately for the intrinsic and normal fields.

Corollary (The Gauss formula). *The tangential covariant derivative* $\underline{\nabla}_a V^b$ *of an* ***intrinsic*** *vector* $V^a \in \overline{\mathcal{E}}^a$ *is given by the sum of the intrinsic covariant derivative* $\overline{\nabla}_a V^b$ *of this vector and the second fundamental form* ${H_a}^b{}_c$ *actinv on this vector, that is*

$$\underline{\nabla}_a V^b = \overline{\nabla}_a V^b - \mathrm{N}^b \mathrm{L}_{ac} V^c \tag{1.58}$$

Corollary (The Weingarten formula). *This is simply the fact that* $\underline{\nabla}_a \mathrm{N}_b = \mathrm{L}_{ab}$, *which is the definition of the shape tensor* (1.53) *in our approach.*

1.2.9 The equations of Gauss and Codazzi

Theorem 1.2.15 (The Gauss equation for Riemannian hypersurfaces)**.** *The the intrinsic Riemannian curvature of the hypersurface is expressed in terms of the curvature of the ambient connection and the shape tensor as*

$$\overline{\mathrm{R}} = \Pi_\Sigma \underline{\mathrm{R}} + \mathrm{L} \wedge \mathrm{L} \tag{1.59}$$

or, explicitly,

$$\overline{\mathrm{R}}_{ab}{}^{c}{}_{d} = \Pi^{a'}{}_{a}\Pi^{b'}{}_{b}\Pi^{c}{}_{c'}\Pi^{d'}{}_{d}\mathrm{R}_{a'b'}{}^{c'}{}_{d'} + \mathrm{L}_{a}{}^{c}\mathrm{L}_{bd} - \mathrm{L}_{b}{}^{c}\mathrm{L}_{ad} \tag{1.60}$$

The contracted forms of this equation are also referred to as the Gauss equations, and we record them here for the completeness sake:

$$\overline{\mathrm{Ric}}_{bd} = \Pi_\Sigma \mathrm{Ric}_{bd} - \overset{\mathrm{NN}}{\mathrm{R}}_{bd} + \bar{n}\mathrm{H}\mathrm{L}_{bd} - (\mathrm{L}^2)_{bd} \tag{1.60-I}$$

and

$$\overline{\mathrm{Scal}} = \mathrm{Scal} - 2\,\overset{\mathrm{NN}}{\mathrm{Ric}} + \bar{n}^2\mathrm{H}^2 - |\mathrm{L}|^2 \tag{1.60-II}$$

where the symbols $\overset{\mathrm{NN}}{\mathrm{R}}_{bd}$ *and* $\overset{\mathrm{NN}}{\mathrm{Ric}}_{bd}$ *are defined on page 59.*

Theorem 1.2.16 (The Codazzi equation for Riemannian hypersurfaces)**.** *The intrinsic exterior covariant derivative of the shape tensor viewed as a* 1*-form on the hypersurface in terms of the curvature of the ambient connection is given by the equation*

$$\overline{\nabla} \wedge \mathrm{L} = \underline{\mathrm{R}}\ \cdot \mathrm{N} \tag{1.61}$$

explicitly

$$\overline{\nabla}_a \mathrm{L}_b{}^{c} - \overline{\nabla}_b \mathrm{L}_a{}^{c} = \Pi^{a'}{}_{a}\Pi^{b'}{}_{b}\mathrm{R}_{a'b'}{}^{c}{}_{d}\mathrm{N}^{d} \tag{1.62}$$

from what we can also obtain the contracted form of the Codazzi equation:

$$\overset{\mathrm{N}}{\mathrm{Ric}}_b = \overline{\nabla}^a \mathrm{L}_{ab} - \bar{n}\,\overline{\nabla}_b \mathrm{H} \tag{1.62-I}$$

where $\overset{\mathrm{N}}{\mathrm{Ric}}_b \coloneqq \Pi^{b'}{}_{b}\mathrm{Ric}_{b'd}N^d$.

Proof of Theorems 1.2.15 and 1.2.16. On can prove these theorems as an application of Theorem 1.1.22, using Proposition 1.1.25 and Equation 1.55.

Of course, it is not difficult to derive the Gauss and Codazzi equations directly, using the difference formula (1.18) for the curvatures. The difference operator in $\underline{\nabla} = \overset{\oplus}{\nabla} + A$ takes the form $A_a{}^{b}{}_{c} = \mathrm{L}_a{}^{b}\mathrm{N}_c - \mathrm{L}_{ac}\mathrm{N}^b$, according to the

Gauss–Weingarten formula (1.57), and therefore

$$
\begin{aligned}
(\overset{\oplus}{\nabla} \wedge A)_{[ab]}{}^c{}_d = 2\overset{\oplus}{\nabla}_{[a} A_{b]}{}^c{}_d = \overset{\oplus}{\nabla}_a (\mathrm{L}_b{}^c N_d - \mathrm{L}_{bd} N^c) - (a \leftrightarrow b) \\
= (\overline{\nabla}_a \mathrm{L}_b{}^c) N_d + \cancelto{0}{\mathrm{L}_b{}^c \nabla^{\mathcal{N}}_a N_d} - (\overline{\nabla}_a \mathrm{L}_{bd}) N^c - \cancelto{0}{\mathrm{L}_{bd} \nabla^{\mathcal{N}}_a N^c} - (a \leftrightarrow b) \\
= 2\overline{\nabla}_{[a} \mathrm{L}_{b]}{}^c N_d - 2\overline{\nabla}_{[a} \mathrm{L}_{b]d} N^c
\end{aligned}
$$

$$
\begin{aligned}
(A \wedge A)_{[ab]}{}^c{}_d = A_a{}^c{}_e A_b{}^e{}_d - (a \leftrightarrow b) \\
= (\mathrm{L}_a{}^c N_e - \mathrm{L}_{ae} N^c)(\mathrm{L}_b{}^e N_d - \mathrm{L}_{bd} N^e) - (a \leftrightarrow b) \\
= -\mathrm{L}_a{}^c \mathrm{L}_{bd} - \underbrace{\mathrm{L}_{ae} \mathrm{L}_b{}^e N^c N_d}_{\text{symmetric in a and b}} - (a \leftrightarrow b) \\
= -2\mathrm{L}_{[a}{}^c \mathrm{L}_{b]d}
\end{aligned}
$$

Taking the corresponding orthogonal parts of $\underline{\mathrm{R}} = \overset{\oplus}{\mathrm{R}} + \overset{\oplus}{\nabla} \wedge A + A \wedge A$ we obtain the results. □

1.2.10 The tangential Laplacian on hypersurfaces

Here we give an application for the difference of the Laplacians formula (1.19) to the orthogonal splitting of the ambient bundle along the hypersurface.

The *tangential Laplacian* $\underline{\Delta}$ is the Laplacian of the ambient connection $\underline{\nabla}$ in the ambient bundle $\underline{\mathcal{E}}^a = \overline{\mathcal{E}}^a \oplus \mathcal{N}^a$. We can compare it with the Laplacian of the van der Waerden–Bortolotti connection using the Gauss–Weingarten formula.

Indeed, let us pick an ambient vector field $V^a \in \underline{\mathcal{E}}^a = \overline{\mathcal{E}}^a \oplus \mathcal{N}^a$ that we can present as $V^a = \binom{\tau^a}{\nu^a} = \tau^a + \nu^a$. Let $\nu := \mathrm{N}_c \nu^c$ so that $\nu^a = \nu \mathrm{N}^a$.

The difference of Laplacians formula (1.19) from section 1.1.12 takes the form

$$
\overline{\Delta} V^b = \overset{\oplus}{\Delta} V^b + 2A_a{}^b{}_c \overset{\oplus}{\nabla}{}^a V^c + \left(\overset{\oplus}{\nabla}{}^a A_a{}^b{}_c\right) V^c + A_a{}^b{}_c A^{ac}{}_d
$$

The Laplacian $\overset{\oplus}{\Delta}$ of the van der Waerden–Bortolotti connection $\overset{\oplus}{\nabla}$ is easy to compute: $\overset{\oplus}{\Delta} V^b = \overline{\Delta} \tau^b + \Delta^{\mathcal{N}} \nu^b$. It is also clear that $\overset{\oplus}{\nabla}{}^a V^c = \overline{\nabla}^a \tau^c + \nabla^{\mathcal{N}a} \nu^c$. Differentiating with $\overset{\oplus}{\nabla}$ the direct sum difference operator $A_a{}^b{}_c = \mathrm{L}_a{}^b \mathrm{N}_c - \mathrm{N}^b \mathrm{L}_{ac}$ we obtain the expression $\overset{\oplus}{\nabla}{}^a A_a{}^b{}_c = \left(\overline{\nabla}^a \mathrm{L}_a{}^b\right) \mathrm{N}_c - \mathrm{N}^b \overline{\nabla}^a \mathrm{L}_{ac}$. It remains to compute $A_a{}^b{}_c A^{ac}{}_d = -(\mathrm{L}^2)^b{}_d - |\mathrm{L}|^2 \mathrm{N}^b \mathrm{N}_d$, and we can substitute these expressions into the difference of Laplacians formula to get

$$\underline{\Delta}\begin{pmatrix}\tau^b\\ \nu^b\end{pmatrix} = \begin{pmatrix}\overline{\Delta}\tau^b + 2\,\mathrm{L}_a{}^b\nabla^a\nu + \nu\overline{\nabla}^a\mathrm{L}_a{}^b - (\mathrm{L}^2)^b_c\tau^c\\ \mathrm{N}^b\left(\Delta\nu - \mathrm{L}_{ac}\overline{\nabla}^a\tau^c - \tau^c\overline{\nabla}^a\mathrm{L}_{ac} - \nu|\mathrm{L}|^2\right)\end{pmatrix}$$

Proposition 1.2.17. *The tangential Laplacian of an intrinsic vector field* $V^a \in \overline{\mathcal{E}}^a$ *along the hypersurface* Σ *is expressed via the intrinsic Laplacian as follows:*

$$\underline{\Delta}V^a = \overline{\Delta}V^a - (\mathrm{L}^2)^a{}_bV^b - \mathrm{N}^a\overline{\nabla}^c(\mathrm{L}_{cb}V^b) \tag{1.63}$$

where $(\mathrm{L}^2)_a{}^b := \mathrm{L}_a{}^c\mathrm{L}_c{}^b$.

Proof. From the calculation of $\underline{\Delta}\binom{\tau^b}{\nu^b}$ made above we can deduce that

$$\underline{\Delta}\begin{pmatrix}\tau^b\\ 0\end{pmatrix} = \begin{pmatrix}\overline{\Delta}\tau^b - (\mathrm{L}^2)^b{}_c\tau^c\\ -\mathrm{N}^b\left(\mathrm{L}_{ac}\overline{\nabla}^a\tau^c + \tau^c\overline{\nabla}^a\mathrm{L}_{ac}\right)\end{pmatrix}$$

which is equivalent to the claim. □

Similarly, we obtain an expression for the tangential Laplacian acting on the normal fields.

Proposition 1.2.18. *The tangential Laplacian acting on a normal field* $V^a \in \mathcal{N}^a$ *along the hypersurface* Σ *is given by the expression*

$$\underline{\Delta}V^a = 2\,\mathrm{L}_b{}^a\nabla^b\left(\mathrm{N}_cV^c\right) + \left(\mathrm{N}_cV^c\right)\overline{\nabla}^b\mathrm{L}_b{}^a + \mathrm{N}^a\left(\Delta\left(\mathrm{N}_bV^b\right) - \mathrm{N}_bV^b|\mathrm{L}|^2\right) \tag{1.64}$$

In particular, the tangential Laplacian of the unit normal N^a *along the hypersurface* Σ *has the following expression in terms of the shape tensor:*

$$\underline{\Delta}\mathrm{N}^a = \overline{\nabla}^b\mathrm{L}_b{}^a - \mathrm{N}^a|\mathrm{L}|^2$$

Proof. From the formula for $\underline{\Delta}\binom{\tau^b}{\nu^b}$ computed above we get an expression

$$\underline{\Delta}\begin{pmatrix}0\\ \nu^b\end{pmatrix} = \begin{pmatrix}2\,\mathrm{L}_a{}^b\nabla^a\nu + \nu\overline{\nabla}^a\mathrm{L}_a{}^b\\ \mathrm{N}^b\left(\Delta\nu - \nu|\mathrm{L}|^2\right)\end{pmatrix}$$

from which the claim follows. □

Remark 1.2.19. It is possible to make analogous identities for the orthogonal parts of the Laplacian in vector bundles in the spirit of the section 1.1.13.

1.2.11 Simons's identity on hypersurfaces

Sometimes we need to commute the derivatives of the shape tensor, and the identities of this were proposed in James Simons's paper [65], and then were given in a modified form in Huisken and Polden's lectures [38]. These identities are an essential tool in many calculations of concrete examples of hyperusrface invariants, and we give their derivation in our notation for the sake of completeness and for the future references.

Proposition 1.2.20 (Simons's identity)**.** *The second covariant derivative of the shape tensor has the following commutation property:*

$$\begin{aligned}\overline{\nabla}_c\overline{\nabla}_d\mathrm{L}_{ab} = \overline{\nabla}_a\overline{\nabla}_b\mathrm{L}_{cd} + \Pi_\Sigma\nabla_a\mathrm{R}_{cbdN} + \Pi_\Sigma\nabla_c\mathrm{R}_{dabN}\\ - \mathrm{L}_{ab}\overset{\mathrm{N}}{\mathrm{R}}_{cd} + \mathrm{L}_{cd}\overset{\mathrm{N}}{\mathrm{R}}_{ab} - 2\Pi_\Sigma\mathrm{R}_{caf(d}\mathrm{L}^f{}_{b)} + 4\mathrm{L}_{f[a}\mathrm{L}_{c](d}\mathrm{L}^f{}_{b)}\end{aligned}$$

Proof. We start with writing the Codazzi equation in the following form

$$\overline{\nabla}_d\mathrm{L}_{ab} - \overline{\nabla}_a\mathrm{L}_{db} = \underline{\mathrm{R}}_{dabe}N^e$$

and differentiate it to get

$$\overline{\nabla}_c\overline{\nabla}_d\mathrm{L}_{ab} = \overline{\nabla}_c\overline{\nabla}_a\mathrm{L}_{db} + \overline{\nabla}_c(\underline{\mathrm{R}}_{dabe}N^e)$$

In the second term of the above display we can commute the derivatives

$$\overline{\nabla}_c\overline{\nabla}_a\mathrm{L}_{db} = \overline{\nabla}_a\overline{\nabla}_c\mathrm{L}_{db} - \overline{\mathrm{R}}_{ca}{}^f{}_d\mathrm{L}_{fb} - \overline{\mathrm{R}}_{ca}{}^f{}_b\mathrm{L}_{df}$$

Using the Codazzi equation again

$$\overline{\nabla}_c\mathrm{L}_{bd} - \overline{\nabla}_b\mathrm{L}_{cd} = \underline{\mathrm{R}}_{cbde}N^e$$

and taking the derivative

$$\overline{\nabla}_a\overline{\nabla}_c\mathrm{L}_{db} = \overline{\nabla}_a\overline{\nabla}_b\mathrm{L}_{cd} + \overline{\nabla}_a(\underline{\mathrm{R}}_{cbde}N^e)$$

we arrive to the equation

$$\overline{\nabla}_c\overline{\nabla}_d\mathrm{L}_{ab} = \overline{\nabla}_a\overline{\nabla}_b\mathrm{L}_{cd} + \overline{\nabla}_a(\underline{\mathrm{R}}_{cbde}N^e) + \overline{\nabla}_c(\underline{\mathrm{R}}_{dabe}N^e) - \overline{\mathrm{R}}_{ca}{}^f{}_d\mathrm{L}_{fb} - \overline{\mathrm{R}}_{ca}{}^f{}_b\mathrm{L}_{df}$$

We can write the Gauss formula as

$$\overline{\nabla}_a V_b = \underline{\nabla}_a V_b + N_b \mathrm{L}_a{}^c V_c$$

where $V_b \in \overline{\mathcal{E}}_b$, i.e. $V_b N^b = 0$.

Using the Gauss formula we compute

$$\begin{aligned}\overline{\nabla}_a(\underline{\mathrm{R}}_{cbde}N^e) &= \underline{\nabla}_a(\underline{\mathrm{R}}_{cbde}N^e) + N_c\mathrm{L}_a{}^f(\underline{\mathrm{R}}_{fbde}N^e)\\ &\quad + N_b\mathrm{L}_a{}^f(\underline{\mathrm{R}}_{cfde}N^e) + N_d\mathrm{L}_a{}^f(\underline{\mathrm{R}}_{cbfe}N^e)\end{aligned}$$

The first term in the right hand side of the last display is now in the form that we can differentiate using the ambient derivative:

$$\begin{aligned}\underline{\nabla}_a(\underline{\mathrm{R}}_{cbde}N^e) &= \underline{\nabla}_a(\Pi^{c'}{}_c\Pi^{b'}{}_b\Pi^{d'}{}_d\mathrm{R}_{cbde}N^e)\\ &= (\underline{\nabla}_a\Pi^{c'}{}_c)\Pi^{b'}{}_b\Pi^{d'}{}_d\mathrm{R}_{c'b'd'e}N^e + \Pi^{c'}{}_c(\underline{\nabla}_a\Pi^{b'}{}_b)\Pi^{d'}{}_d\mathrm{R}_{c'b'd'e}N^e\\ &\quad + \Pi^{c'}{}_c\Pi^{b'}{}_b(\underline{\nabla}_a\Pi^{d'}{}_d)\mathrm{R}_{c'b'd'e}N^e + \Pi^{c'}{}_c\Pi^{b'}{}_b\Pi^{d'}{}_d\underline{\nabla}_a(\mathrm{R}_{c'b'd'e}N^e)\\ &= (-N^{c'}\mathrm{L}_{ac} - N_c\mathrm{L}_a{}^{c'})\Pi^{b'}{}_b\Pi^{d'}{}_d\mathrm{R}_{c'b'd'e}N^e + \Pi^{c'}{}_c(-N^{b'}\mathrm{L}_{ab} - N_b\mathrm{L}_a{}^{b'})\Pi^{d'}{}_d\mathrm{R}_{c'b'd'e}N^e\\ &\quad + \Pi^{c'}{}_c\Pi^{b'}{}_b(-N^{d'}\mathrm{L}_{ad} - N_d\mathrm{L}_a{}^{d'})\mathrm{R}_{c'b'd'e}N^e + \Pi^{c'}{}_c\Pi^{b'}{}_b\Pi^{d'}{}_d\underline{\nabla}_a(\mathrm{R}_{c'b'd'e}N^e)\end{aligned}$$

where we used

$$\underline{\nabla}_a\Pi^{b'}{}_b = -N^{b'}\mathrm{L}_{ab} - N_b\mathrm{L}_a{}^{b'}$$

Making simplifications and using a compact notation

$$\Pi_\Sigma\mathrm{R}_{NbdN} = N^{c'}\Pi^{b'}{}_b\Pi^{d'}{}_d\mathrm{R}_{c'b'd'e}N^e$$

we further get

$$\begin{aligned}\underline{\nabla}_a(\underline{\mathrm{R}}_{cbde}N^e) &= -\mathrm{L}_{ac}\Pi_\Sigma\mathrm{R}_{NbdN} - N_c\mathrm{L}_a{}^f\underline{\mathrm{R}}_{fbdN} - \mathrm{L}_{ab}\Pi_\Sigma\mathrm{R}_{cNdN} - N_b\mathrm{L}_a{}^f\underline{\mathrm{R}}_{cfdN}\\ &\quad - \cancel{N^{d'}\mathrm{L}_{ad}\Pi^{c'}{}_c\Pi^{b'}{}_b\mathrm{R}_{c'b'd'e}N^e} - N_d\mathrm{L}_a{}^f\underline{\mathrm{R}}_{cbfN} + \Pi_\Sigma\underline{\nabla}_a\mathrm{R}_{cbdN}\end{aligned}$$

Using the shorthand notation $\overset{NN}{\mathrm{R}}_{bd} := \Pi_\Sigma\mathrm{R}_{NbNd}$ (see page 59), we rewrite

$$\underline{\nabla}_a(\underline{\mathrm{R}}_{cbdN}) = \Pi_\Sigma\underline{\nabla}_a\mathrm{R}_{cbdN} + 2\,\mathrm{L}_{a[c}\overset{NN}{\mathrm{R}}_{b]d} - N_c\mathrm{L}_a{}^f\underline{\mathrm{R}}_{fbdN} - N_b\mathrm{L}_a{}^f\underline{\mathrm{R}}_{cfdN} - N_d\mathrm{L}_a{}^f\underline{\mathrm{R}}_{cbfN}$$

Returning to

$$\overline{\nabla}_a(\underline{\mathrm{R}}_{cbdN}) = \underline{\nabla}_a(\underline{\mathrm{R}}_{cbdN}) + N_c\mathrm{L}_a{}^f\underline{\mathrm{R}}_{fbdN} + N_b\mathrm{L}_a{}^f\underline{\mathrm{R}}_{cfdN} + N_d\mathrm{L}_a{}^f\underline{\mathrm{R}}_{cbfN}$$

we get

$$\overline{\nabla}_a(\underline{\mathrm{R}}_{cbdN}) = \Pi_\Sigma \underline{\nabla}_a \mathrm{R}_{cbdN} + 2\,\mathrm{L}_{a[c}\overset{NN}{\mathrm{R}}_{b]d} - \cancelto{1}{N_c \mathrm{L}_a{}^f \underline{\mathrm{R}}_{fbdN}} - \cancelto{2}{N_b \mathrm{L}_a{}^f \underline{\mathrm{R}}_{cfdN}} - \cancelto{3}{N_d \mathrm{L}_a{}^f \underline{\mathrm{R}}_{cbfN}}$$
$$+ \cancelto{1}{N_c \mathrm{L}_a{}^f \underline{\mathrm{R}}_{fbdN}} + \cancelto{2}{N_b \mathrm{L}_a{}^f \underline{\mathrm{R}}_{cfdN}} + \cancelto{3}{N_d \mathrm{L}_a{}^f \underline{\mathrm{R}}_{cbfN}}$$

Thus we obtain the identity

$$\overline{\nabla}_a \underline{\mathrm{R}}_{cbdN} = \Pi_\Sigma \underline{\nabla}_a \mathrm{R}_{cbdN} + 2\,\mathrm{L}_{a[c}\overset{NN}{\mathrm{R}}_{b]d}$$

Similarly, we get also

$$\overline{\nabla}_c \underline{\mathrm{R}}_{dabN} = \Pi_\Sigma \underline{\nabla}_c \mathrm{R}_{dabN} + 2\,\mathrm{L}_{c[d}\overset{NN}{\mathrm{R}}_{a]b}$$

Using these results we can complete the calculation:

$$\overline{\nabla}_c \overline{\nabla}_d \mathrm{L}_{ab} = \overline{\nabla}_a \overline{\nabla}_b \mathrm{L}_{cd} + \overline{\nabla}_a(\underline{\mathrm{R}}_{cbde} N^e) + \overline{\nabla}_c \overline{\mathrm{R}}_{cafb} = (\underline{\mathrm{R}}_{dabe} N^e) - \overline{\mathrm{R}}_{cafd} \mathrm{L}^f{}_b - \overline{\mathrm{R}}_{cafb} \mathrm{L}^f{}_d$$
$$= \overline{\nabla}_a \overline{\nabla}_b \mathrm{L}_{cd} + \Pi_\Sigma \underline{\nabla}_a \mathrm{R}_{cbdN} + 2\,\mathrm{L}_{a[c}\overset{NN}{\mathrm{R}}_{b]d} + \Pi_\Sigma \underline{\nabla}_c \mathrm{R}_{dabN} + 2\,\mathrm{L}_{c[d}\overset{NN}{\mathrm{R}}_{a]b}$$
$$- (\Pi_\Sigma \mathrm{R}_{cafd} + \mathrm{L}_{cf}\mathrm{L}_{ad} - \mathrm{L}_{cd}\mathrm{L}_{af})\mathrm{L}^f{}_b - (\Pi_\Sigma \mathrm{R}_{cafb} + \mathrm{L}_{cf}\mathrm{L}_{ab} - \mathrm{L}_{cb}\mathrm{L}_{af})\mathrm{L}^f{}_d$$
$$= \overline{\nabla}_a \overline{\nabla}_b \mathrm{L}_{cd} + \Pi_\Sigma \underline{\nabla}_a \mathrm{R}_{cbdN} + \Pi_\Sigma \underline{\nabla}_c \mathrm{R}_{dabN} + \cancel{\mathrm{L}_{ac}\overset{NN}{\mathrm{R}}_{bd}} - \mathrm{L}_{ab}\overset{NN}{\mathrm{R}}_{cd} + \mathrm{L}_{cd}\overset{NN}{\mathrm{R}}_{ab} - \cancel{\mathrm{L}_{ca}\overset{NN}{\mathrm{R}}_{db}}$$
$$- \Pi_\Sigma \mathrm{R}_{cafd}\mathrm{L}^f{}_b - \mathrm{L}_{cf}\mathrm{L}_{ad}\mathrm{L}^f{}_b + \mathrm{L}_{cd}\mathrm{L}_{af}\mathrm{L}^f{}_b - \Pi_\Sigma \mathrm{R}_{cafb}\mathrm{L}^f{}_d - \mathrm{L}_{cf}\mathrm{L}_{ab}\mathrm{L}^f{}_d + \mathrm{L}_{cb}\mathrm{L}_{af}\mathrm{L}^f{}_d$$

This is equivalent to the claim. □

The Simons's identity has a useful contracted form.

Proposition 1.2.21 (Contracted Simons's identity)**.** *The intrinsic Laplacian of the shape tensor has the following expression*

$$\overline{\Delta}\mathrm{L}_{ab} = (n-1)\overline{\nabla}_a \overline{\nabla}_b H + \Pi_\Sigma \nabla_a \mathrm{Ric}_{bN} + \Pi_\Sigma \nabla^c \mathrm{R}_{cabN} - \mathrm{L}_{ab}\mathrm{Ric}_{NN} + (n-1)H\overset{NN}{\mathrm{R}}_{ab}$$
$$+ \Pi_\Sigma \mathrm{Ric}_{af}\mathrm{L}^f{}_b - \Pi_\Sigma \mathrm{R}_{cafb}\mathrm{L}^{cf} + (n-1)H\mathrm{L}^2_{ab} - \mathrm{L}_{ab}|\mathrm{L}|^2$$

Proof. Contracting (with $\bar{g}^{ab}$) the indices a and b in the Simons's identity we see that

$$
\begin{aligned}
(n-1)\overline{\nabla}_c\overline{\nabla}_d H = {} & \overline{\Delta}\mathrm{L}_{cd} + \bar{g}^{ab}\Pi_{abcd}^{a'b'c'd'}\nabla_{a'}\mathrm{R}_{c'b'd'N} + \bar{g}^{ab}\Pi_{abcd}^{a'b'c'd'}\underline{\nabla}_{c'}\mathrm{R}_{d'a'b'N} \\
& - (n-1)H\overset{\mathrm{N}}{\mathrm{R}}_{cd} + \mathrm{L}_{cd}\mathrm{Ric}_{NN} \\
& - \bar{g}^{ab}\Pi_{abcd}^{a'b'c'd'}\mathrm{R}_{c'a'fd'}\mathrm{L}^f{}_{b'} - \bar{g}^{ab}\Pi_{abcd}^{a'b'c'd'}\mathrm{R}_{c'a'fb'}\mathrm{L}^f{}_d \\
& + \bar{g}^{ab}\Big(\mathrm{L}_{fa}\mathrm{L}_{cd}\mathrm{L}^f{}_b + \mathrm{L}_{fa}\mathrm{L}_{cb}\mathrm{L}^f{}_d - \mathrm{L}_{fc}\mathrm{L}_{ad}\mathrm{L}^f{}_b - \mathrm{L}_{fc}\mathrm{L}_{ab}\mathrm{L}^f{}_d\Big)
\end{aligned}
$$

and using

$$\bar{g}^{ab}\Pi_a^{a'}\Pi_b^{b'} = \Pi_a^{a'}\Pi_b^{b'}g^{ab}$$

we obtain

$$
\begin{aligned}
(n-1)\overline{\nabla}_c\overline{\nabla}_d H = {} & \overline{\Delta}\mathrm{L}_{cd} + \Pi_\Sigma\nabla^a\mathrm{R}_{cadN} - \Pi_\Sigma\nabla_c\mathrm{Ric}_{dN} - (n-1)H\overset{\mathrm{NN}}{\mathrm{R}}_{cd} + \mathrm{L}_{cd}\mathrm{Ric}_{NN} \\
& - \Pi_\Sigma\mathrm{R}_{cafd}\mathrm{L}^{fa} - \Pi_\Sigma\mathrm{Ric}_{cf}\mathrm{L}^f{}_d \\
& + \mathrm{L}_{fa}\mathrm{L}_{cd}\mathrm{L}^{fa} + \cancel{\mathrm{L}_{fa}\mathrm{L}_c{}^a\mathrm{L}^f{}_d} - \cancel{\mathrm{L}_{fc}\mathrm{L}_{ad}\mathrm{L}^{fa}} - (n-1)\mathrm{L}_{fc}H\mathrm{L}^f{}_d
\end{aligned}
$$

that completes the proof. □

1.3 Variations of Riemannian hypersurfaces

In Chapter 5 we shall examine a conformal invariant of hypersurfaces that arises from a variational problem. In this section we prepare the necessary background for that. This material is known classically, see e.g. [79], [38]. We reformulate the identities in the abstract index notation and derive a formula for the variation of the umbilicity tensor (Proposition 5.1.2), that to our best knowledge is not found in the literature, at least in the form that we need.

1.3.1 Variations of embeddings

Let Σ be a hypersurface in a Riemannian manifold (M, g), and let Σ be represented as an image of some embedding $f\colon \Sigma \to M$.

Definition 1.3.1. A *variation* of an embedding $f\colon \Sigma \to M$ is a smooth 1-parametric family of embeddings $f^{(t)}\colon \Sigma \to M$ such that $f^{(0)} = f$. Here I is an open interval $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.

The image of Σ under $f^{(t)}$ will be denote by either Σ^t or Σ_t whatever is convenient. The definition requires that Σ^t is an embedded hypersurface in M for all t, and $\Sigma^0 = \Sigma$.

Each point $p \in \Sigma$ gives rise to a curve $p(t) := f^{(t)}(p)$ in M. Let $W_p := \dot{p} = \frac{d}{dt}p$ be the velocity field of the curve p(t). Fixing t and letting p vary along Σ we obtain a vector field $W(t)$ along the hypersurface Σ^t. We refer to these vector fields as the *velocity* fields of the corresponding hypersurfaces. For the initial hypersurface $\Sigma = \Sigma^0$ the velocity field will be denoted by $V = W(0)$. The vector field V is commonly referred to as the *variation(al) vector field* of the variation f^t.

Definition 1.3.2. The variation $f^{(t)}$ of the hypersurface Σ is called *normal* if $V = \varphi \mathrm{N}$ along Σ for some smooth function $\varphi\colon \Sigma \to \mathbb{R}$. In other words, we require the velocity of the initial hypersurface Σ be either orthogonal to Σ, or vanish.

Clearly, the velocity V of a normal variation vanishes at the points $p \in \Sigma$ where φ does, and the closure of the complement of all such points is the support $\operatorname{supp}(\varphi)$ of the function φ.

1.3.2 Normal variations of Riemannian hypersurfaces

For each point $p \in \Sigma$ we can consider restrictions of the tensorial quantities onto the curve p^t and thus we can talk about their covariant derivative D_t along this curve (see [48], p.57, or [21], p.50).

For the sake of readability of calculations we use a shorthand notation for the binormal part $\overset{\mathrm{NN}}{\mathrm{R}}_{ab}$ of the background Riemannian curvature:

$$\overset{\mathrm{NN}}{\mathrm{R}}_{ab} = \mathrm{R}_{a,c,b,d}\mathrm{N}^c\mathrm{N}^d \tag{1.65}$$

and for the binormal part $\overset{\mathrm{NN}}{\mathrm{Ric}}$ of the background Ricci tensor

$$\overset{\mathrm{NN}}{\mathrm{Ric}} = \mathrm{Ric}_{ab}\mathrm{N}^a\mathrm{N}^b \tag{1.66}$$

Notice that due to the symmetries of the Riemann tensor the quantity $\overset{\mathrm{NN}}{\mathrm{R}}_{ab}$ is symmetric and has both indices tangential, that is $\overset{\mathrm{NN}}{\mathrm{R}}_{ab}\mathrm{N}^b = 0$, so it can be identified with a section of the intrinsic tensor bundle $\overline{\mathcal{E}}_{(ab)}$.

We shall also use the notation $\mathrm{L}^2_{ab} := \mathrm{L}_a{}^c\mathrm{L}_{cb}$.

It is be convenient to use the symbol of "variational derivative" δ_0, which, when applied to the tensorial objects, is understood as the covariant derivative D_t along the curve p^t at $t = 0$, but when it acts on the components of tensors it will be a synonym for $\frac{d}{dt}\big|_{t=0}$.

Theorem 1.3.3. *Let Σ^t be a normal variation of the hypersurface Σ in the manifold M with the velocity vector field V along $\Sigma = \Sigma^0$ such that $V^a = \varphi \mathrm{N}^a$. Each of the quantities $\bar{g}_{ab}$, $\mathrm{d}\Sigma$, N^a, L_{ab} and H are defined for all $t \in I$ at the points $p^t \in \Sigma^t$, and give rise to tensor fields along the curves p^t.*

The covariant derivatives $\delta_0 := D_t|_{t=0}$ of these quantities (induced by the background Levi-Civita connection) along the curve p^t at $t = 0$ are given by the following expressions:

$$\delta_0\, \bar{g}_{ab} = 2\varphi \mathrm{L}_{ab} \tag{1.67}$$

$$\delta_0\, \bar{g}^{ab} = -2\varphi \mathrm{L}^{ab} \tag{1.68}$$

$$\delta_0\, \mathrm{d}\Sigma = \varphi \bar{n} H \,\mathrm{d}\Sigma \tag{1.69}$$

$$\delta_0\, \mathrm{N}^a = -\overline{\nabla}^a \varphi \tag{1.70}$$

$$\delta_0\, \mathrm{L}_{ab} = -\overline{\nabla}_a \overline{\nabla}_b \varphi + \varphi\Big(\mathrm{L}^2_{ab} - \overset{\mathrm{NN}}{\mathrm{R}}_{ab}\Big) \tag{1.71}$$

$$\delta_0\, \bar{n} H = -\overline{\Delta}\varphi - \varphi\Big(|\mathrm{L}|^2 + \overset{\mathrm{NN}}{\mathrm{Ric}}\Big) \tag{1.72}$$

These identities are well known in the theory of geometric evolution equations and in the geometry of minimal submanifolds.

Proofs using the normal coordinates can be found in [38] or, in an invariant notation, in [2] and [3].

Yano in [79] has developed the theory of variations of Riemannian submanifolds in the full generality. His results agree with the stated, at least for the case of hypersurfaces (he uses different sign conventions, however).

For the proof we need to recall two elementary facts from the differential geometry that will facilitate our calculations.

Lemma 1.3.4. *For two vector fields X and Y defined along a curve p^t in a Riemannian manifold (M, g) we have*

$$\tfrac{d}{dt} g(X, Y) = g(D_t X, Y) + g(X, D_t Y) \tag{1.73}$$

Proof. This is the compatibility of the Levi-Civita connection with the metric applied to the pullback connection D_t on the curve p^t.

See [48], p.57, or [21], p.50. □

Recall also that $D_t X = \nabla_W X$ for $W = \dot{p}$ if the field X along the curve p^t is extendible.

Lemma 1.3.5 (Symmetry lemma). *Let $F\colon I \times J \to M$ be a smooth map such that for each $t \in I$ the map $F_t = F(t, \cdot)$ is a regular curve, and for each $s \in J$ the map $F_s = F(\cdot, s)$ is again an regular curve, see Fig. 1.1. Then*

$$D_t \frac{\partial F_t}{\partial s} = D_s \frac{\partial F_s}{\partial t}$$

Proof. This is essentially the torsion-freeness of the background Levi-Civita connection applied to the pullback connection D_t along a curve $\gamma(t)$, cf. Proposition 1.1.38.

An elementary proof can be found in [48], p.97, or [21], p.68. □

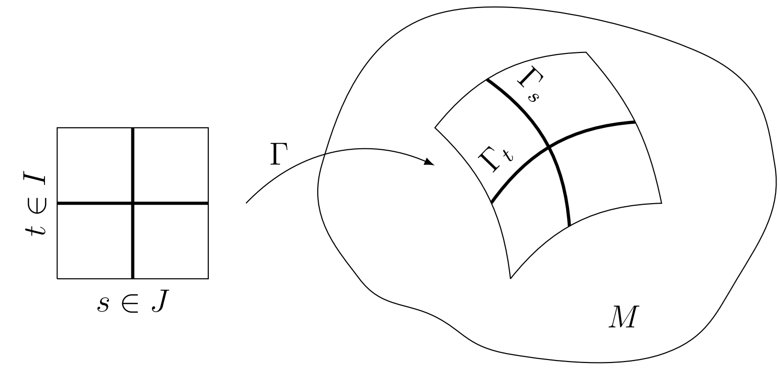

Figure 1.1: Parametrized surface in a manifold

We shall also need to differentiate the determinant of the intrinsic metric, so the we give the required formula here.

Lemma 1.3.6. *Let $A(t)$ be a smooth 1-parametric family of nondegenerate $n \times n$-matrices $A\colon I \to M_n(\mathbb{R})$, where $I = (-\epsilon, \epsilon)$ for some $\epsilon > 0$. Then the following identity holds*

$$\tfrac{d}{dt} \det(A) = \det(A) \operatorname{tr}(A^{-1} \tfrac{d}{dt} A) \tag{1.74}$$

Proof. This result is standard, and a proof can be found in many sources. One of the shortest ways to show this is to observe that if $A(0) = I$ with $I = I_n$ being $n \times n$-unit matrix, then $A = I + tB$ for some other smooth family of matrices $B(t)$, and

$$\det(I + tB) = I + t \operatorname{tr}(B) + O(t^2) \tag{1.75}$$

To see that one may use the formula for the determinant

$$\det(A) = \sum_{\sigma \in S_n} (-1)^{\mathrm{sgn}(\sigma)} \prod_{i=1}^{n} a_{i,\sigma(i)} \tag{1.76}$$

The equation (1.75) is a first-order Taylor expansion that immediately yields

$$\tfrac{d}{dt} \det(A) = \mathrm{tr}(\tfrac{d}{dt} A), \text{ for } A(0) = I \tag{1.77}$$

The general case $A(t) = A(0) + tB$ is obtained form the previous one by rewriting

$$\det(A + tB) = \det(A) \det(I + tA^{-1}B)$$

that yields the claim since $B = \frac{d}{dt} A$. □

Remark 1.3.7. Before we start the proof let us notice that at the points where the function φ vanishes, all the identities in Theorem 1.3.3 hold trivially, so we shall assume that $\varphi(p) \neq 0$ everywhere in the proof.

Proof of Theorem 1.3.3. We can use the fact that all the quantities are tensorial (the mean curvature is even a scalar), and for a given point $p \in \Sigma$ they are defined along the curve p^t that arises form the variation.

Choosing some coordinates y^{la}, $\lambda = 1, \dots, n$ in a neighborhood $U \subseteq M$ around p we can assume that the embedding $f \colon \Sigma \to M$ is locally represented as $f \colon \bar{U} \to U$ for some $\bar{U} \in \Sigma$ with local coordinates x^i, $i = 1, \dots, \bar{n}$ on it. The coordinate vector fields ∂_i push forward by f^t on all the hypersurfaces Σ^t, so we get fields $f_i = f_*(\partial_i)$ along each Σ. In particular, the fields f_i are defined along the curve p^t for each $p \in \Sigma$.

We can notice that the construction of the fields f_i^t is smooth and therefore they form a set of vector fields on the whole U.

The unit normal N^t to each Σ^t is given as the unique solution of the system

$$\begin{cases} g(\mathrm{N}^t, f_i^t) = 0 & i = 1, \dots, \bar{n} \\ g(\mathrm{N}^t, \mathrm{N}^t) = 1 \end{cases} \tag{1.78}$$

compatible with the orientations on Σ and M (locally there is no loss of generality here, of course).

The construction of the fields N^t is again defined smoothly, so we can assume that the unit normal field N along the hypersurface is extended onto U this way.

From now on we shall drop the label t from the notation for f_i and N assuming that the context clarifies when they depend on the parameter $t \in I$.

On each Σ^t the intrinsic metric induced by the embedding f^t is given now by the components in the frame $\{f_i\}$ as

$$\bar{g}_{ij} = g(f_i, f_j)$$

Regarding these components as scalar fields along p^t we can calculate

$$\begin{aligned} \delta_0\, \bar{g}_{ij} &= \delta_0\, g(f_i, f_j) = 2\, g(\delta_0 f_i, f_j) \\ &= 2\, g(\nabla_V f_i, f_j) = 2\, g(\nabla_{f_i}(\varphi \mathrm{N}), f_j), f_j) \\ &= 2\, g(\nabla_{f_i}(\varphi \mathrm{N}), f_j) = 2\, g((\nabla_{f_i}\varphi)\mathrm{N} + \varphi \nabla_{f_i}\mathrm{N}, f_j) \\ &= 2\, \varphi \mathrm{L}_{ij} \end{aligned}$$

We have used the defining identities (1.78) for the unit normals N^t, the Weingarten equation represented in the components with respect to the coordinate frame $\{f_i\}$

$$L_{ij} = g(\nabla_{f_i}\mathrm{N}, f_j), \tag{1.79}$$

and the fact that for a fixed i the fields f_i and W arise as the velocity fields of a family of curves on a parametrized surface as in Fig. 1.1: we take Γ_t to be p^t, and Γ_s to be $\gamma_i(s)$, the coordinate curve with the velocity f_i. The Symmetry Lemma now allows to commute the derivatives $\nabla_V f_i = \nabla_{f_i} V$.

To differentiate the components of the inverse intrinsic metric $\bar{g}^{ij}$ we use the identity

$$\bar{g}^{ij}\bar{g}_{jk} = \delta^i{}_k$$

where on the right hand side we have the Kronecker symbol (the components of the identity map), and the Einstein summation is used.

Differentiating the above display we get

$$0 = (\delta_0\, \bar{g}^{ij})\bar{g}_{jk} + \bar{g}^{ij}\delta_0\, \bar{g}_{jk} = (\delta_0\, \bar{g}^{ij})\bar{g}_{jk} + 2\, \varphi \mathrm{L}_{jk}\bar{g}^{ij}$$

from where we obtain

$$(\delta_0\, \bar{g}^{ij}) = -2\, \varphi \mathrm{L}^{ij}$$

The Riemannian volume form $\mathrm{d}\Sigma$ of the intrinsic metric $\bar{g}$ on each Σ^t is given in the coordinates as

$$\mathrm{d}\Sigma = \sqrt{\det \bar{g}}\mathrm{d}x^i \wedge \cdots \wedge \mathrm{d}x^{\bar{n}}$$

Since we have fixed the coordinate system on $\bar{U} \subseteq \Sigma$ and pushed it forward on Σ^t, the form $\mathrm{d}x^i \wedge \cdots \wedge \mathrm{d}x^{\bar{n}}$ is constant on p^t for all t, and we only need to differentiate $\sqrt{\det \bar{g}}$. The components $\bar{g}_{ij}$ form a non-degenerate matrix, so we are in the conditions of Lemma 1.3.6

$$\delta_0 \sqrt{\det \bar{g}} = \frac{\delta_0 \det \bar{g}}{2\sqrt{\det \bar{g}}} = \frac{\det \bar{g}\,\mathrm{tr}(\bar{g}^{ik}\delta_0 \bar{g}_{kj})}{2\sqrt{\det \bar{g}}} = \det \bar{g}\tfrac{1}{2}(\bar{g}^{ij} 2\,\varphi \mathrm{L}_{ij}) = \varphi \bar{n} H \det \bar{g}$$

The unit normal N is determined by the equations (1.78) and the choice of orientations on M and Σ so that the frame $\{f_1, \ldots, f_{\bar{n}}, \mathrm{N}$ is positively oriented in $\mathrm{T}_p M$ at each point p where they are defined.

Differentiating the equation $g(\mathrm{N}, \mathrm{N}) = 1$ we obtain $2\,g(\nabla_V \mathrm{N}, \mathrm{N})$, that is $\nabla_V \mathrm{N}$ is tangent to Σ^t, so we can write

$$\nabla_V \mathrm{N} = \xi^i f_i$$

for some functions ξ^i that are components of vector $\nabla_V \mathrm{N}$ in the frame $\{f_i\}$.

To determine these ξ^i we differentiate the equation $g(\mathrm{N}, f_j)$ and get

$$\begin{aligned}
0 = \delta_0 g(\mathrm{N}, f_j) &= g(\nabla_V \mathrm{N}, f_j) + g(\mathrm{N}, \nabla_V f_j)\\
&= g(\xi^i f_i, f_j) + g(\mathrm{N}, \nabla_{f_j} V) = g(\xi^i f_i, f_j) + g(\mathrm{N}, \nabla_{f_j}(\varphi \mathrm{N}))\\
&= g(\xi^i f_i, f_j) + g\Big(\mathrm{N}, (\nabla_{f_j}\varphi)\mathrm{N} + \varphi \nabla_{f_j}\mathrm{N}\Big) = \xi_j + \nabla_{f_j}\varphi
\end{aligned}$$

Notice that $\nabla_{f_j}\varphi = \overline{\nabla}_{f_j}\varphi$, and that the function φ can be extended arbitrarily off the initial Σ for this purpose.

Thus, we get

$$\nabla_V \mathrm{N} = -(\overline{\nabla}^i \varphi) f_i$$

To make the calculations more readable let us write ∇_i for ∇_{f_i}.

In this notation the Gauss equation implies

$$\nabla_i f_j = -\mathrm{N}\mathrm{L}_{ij}$$

and the Weingarten equation becomes

$$\nabla_i \mathrm{N} = \mathrm{L}_i{}^j f_j$$

The components of the shape tensor L_{ij} in the coordinate frame $\{f_i\}$ are given by (1.79). Differentiating these components we compute

$$\begin{aligned}
\delta_0 \mathrm{L}_{ij} &= \delta_0 g(\nabla_i \mathrm{N}, f_j) = g(\nabla_V \nabla_i \mathrm{N}, f_j) + g(\nabla_i \mathrm{N}, \nabla_V f_j) \\
&= g\left(\nabla_i \nabla_V \mathrm{N} + \mathrm{R}(V, f_i)\mathrm{N}, f_j\right) + g(\nabla_i \mathrm{N}, \nabla_j V) \\
&= -g\left(\nabla_i((\overline{\nabla}^k \varphi) f_k), f_j\right) + \mathrm{R}(\varphi \mathrm{N}, f_i, f_j, \mathrm{N}) + g(\nabla_i \mathrm{N}, \nabla_j(\varphi \mathrm{N})) \\
&= -g\left(f_k \nabla_i \overline{\nabla}^k \varphi, f_j\right) - g\left((\overline{\nabla}^k \varphi)\nabla_i f_k, f_j\right) - \varphi \mathrm{R}(f_i, \mathrm{N}, f_j, \mathrm{N}) \\
&\quad + g\left(\mathrm{L}_i{}^k f_k, \mathrm{N}\nabla_j \varphi + \varphi \nabla_j \mathrm{N}\right) \\
&= -\overline{\nabla}_i \overline{\nabla}_j \varphi + g\left((\overline{\nabla}^k \varphi)\mathrm{N}\mathrm{L}_{ik}, f_j\right) - \varphi \mathrm{R}(f_i, \mathrm{N}, f_j, \mathrm{N}) \\
&\quad + \varphi g\left(\mathrm{L}_i{}^k f_k, \mathrm{L}_j{}^l f_l\right) \\
&= -\overline{\nabla}_i \overline{\nabla}_j \varphi - \varphi \mathrm{R}(f_i, \mathrm{N}, f_j, \mathrm{N}) + \varphi \mathrm{L}_i{}^k \mathrm{L}_{kj}
\end{aligned}$$

It is remaining now to apply the equations we have already received to the definition of the mean curvature:

$$\bar{n} H = \bar{g}^{ij} \mathrm{L}_{ij}$$

Differentiating, we get

$$\begin{aligned}
\delta_0 \bar{n} H &= (\delta_0 \bar{g}^{ij})\mathrm{L}_{ij} + \bar{g}^{ij} \delta_0 \mathrm{L}_{ij} \\
&= -2\,\varphi \mathrm{L}^{ij} \mathrm{L}_{ij} + \bar{g}^{ij}\left(-\overline{\nabla}_i \overline{\nabla}_j \varphi - \varphi \mathrm{R}(f_i, \mathrm{N}, f_j, \mathrm{N}) + \varphi \mathrm{L}_i{}^k \mathrm{L}_{kj}\right) \\
&= \overline{\Delta} \varphi - \varphi \mathrm{Ric}(\mathrm{N}, \mathrm{N}) - \varphi \mathrm{L}^{ij} \mathrm{L}_{ij}
\end{aligned}$$

Rewriting the results of these calculations using the abstract index notation and our conventions (such that identification of $\mathrm{T}\Sigma$ with a subspace of $\mathrm{T}M$ and using the same indices in both bundles), we obtain the expressions as claimed. □

These identities will be used in Chapter 5 after we have discussed the conformal geometry of hypersurfaces. In particular, we shall re-interpret the quantities in Theorem 1.3.3 as having a certain conformal weight.

Chapter 2

Conformal geometry of hypersurfaces

2.1 Background and notation from conformal geometry

From now on we start assuming that M is a smooth manifold with dimension $\dim M = n \geq 3$.

2.1.1 Conformal structure

Two Riemannian metrics $\widehat{g}$ and g are said to be *conformally equivalent*, or *conformal* to each other, if there exist a smooth strictly positive (nowhere vanishing) function Ω such that $\widehat{g} = \Omega^2 g$.

It is straightforward to verify that this is indeed an equivalence relation. An equivalence class is called a *conformal structure* on manifold M. A manifold M equipped with a conformal structure c is termed a *conformal manifold* (M, c).

We only deal with the case of Riemannian conformal structures in this thesis, but clearly these definitions work for pseudo-Riemannian metrics since multiplying by a positive function does not change the signature.

2.1.2 Conformal densities

A conformal structure is equivalent to fixing a $\mathbb{R}_+$-ray subbundle $\mathcal{G}$ in the space of all Riemannian metrics $\mathcal{M}_{ab} \subset \mathcal{E}_{(ab)}$. A choice of scale $g \in c$ is then equivalent to a section of $\mathcal{G}$.

The bundle $\mathcal{G}$ is a $\mathbb{R}^+$-principal bundle, and it is convenient to pass to the associated vector bundles $\mathcal{E}[w]$ that arise via representations $\lambda \in \mathbb{R}_+ \mapsto (r \in$

$\mathbb{R} \to \lambda^{-w/2}r)$. We refer to sections of $\mathcal{E}[w]$ as *densities* of conformal weight w. They can be seen as functions $f\colon \mathcal{G} \to \mathbb{R}$ that are homogeneous in the sense that $f(x, \Omega^2 g) = \Omega^w f(x, g)$ where $x \in M$.

Alternatively, we can start with the determinant line bundle $\Lambda^n(\mathrm{T}M)$ of manifold M and consider its tensor square $(\Lambda^n(\mathrm{T}M))^2$ which is a line bundle again. This way we eliminate the need to distinguish cases when manifold M is oriented or not because $(\Lambda^n(\mathrm{T}M))^2$ is always trivializable. Thereofre there exists its $2n$-th root, $((\Lambda^n(\mathrm{T}M))^2)^{1/2n}$, and we refer to it as the bundle of densities with conformal weight 1 and denote it by $\mathcal{E}[1]$. Taking w^{th} powers of $\mathcal{E}[1]$ we obtain the bundles $\mathcal{E}[w]$ of densities with conformal weight w. A detailed discussion see e.g. in [17].

A choice of a metric g in TM induces a volume form ϵ_g that trivializes $(\Lambda^n(\mathrm{T}M))^2$ and its powers. Locally, in a choice of coördinates, the volume form is given by $\epsilon_g = \sqrt{|g|}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ where $|g|$ is the determinant of the matrix (g_{ij}) in these coördinates. One can observe now that for a conformally rescaled metric $\widehat{g} = \Omega^2 g$ the representative of section $f \in \mathcal{E}[1]$ in the new induced trivialization becomes $\widehat{f} = \Omega f$ whereas sections f of conformal weight w rescale as $\widehat{f} = \Omega^w f$.

The conformal class c on manifold M determines the tautological section $\mathbf{g}_{ab} \in \mathcal{E}_{(ab)}[2]$ that we call the *conformal metric*. Its inverse $\mathbf{g}^{ab}$ is a section of $\mathcal{E}^{(ab)}[-2]$.

The conformal metric $\mathbf{g}_{ab}$ induces the isomorphism $\mathcal{E}^a \cong \mathcal{E}_a[2]$ and from now on we use it to raise and lower indices without mention. For instance, $R_{ab}{}^c{}_d \in \mathcal{E}_{[ab]} \otimes \mathcal{E}^c{}_d$ has no weight (i.e.$w = 0$) but $R_{abcd} = \mathbf{g}_{ce} R_{ab}{}^e{}_d$ has the weight 2.

2.1.3 Conformal rescaling rules

The bundle $\mathcal{E}[w]$ is trivialized (symbolically, $\mathcal{E}[w] \stackrel{g}{=} M \times \mathbb{R}$) by any choice of a metric $g \in c$, and this metric induces the canonical connection $\nabla = \nabla^g$. When a new representative $\widehat{g}_{ab} = \Omega^2 g$ in the conformal class is chosen, the connection ∇ on $\mathcal{E}[w]$ rescales as

$$\widehat{\nabla}_a f = \nabla_a f + \Upsilon_a w f \tag{2.1}$$

for $f \in \mathcal{E}[w]$. Here

$$\Upsilon_a \coloneqq \nabla_a \log \Omega \tag{2.2}$$

Let us give a quick justification for (2.1). The density bundle $\mathcal{E}[w]$ acquires a connection associated with a metric g which is induced by the Levi-Civita connection of g on the n-forms. The choice of a metric also trivializes the bundle $\mathcal{E}[w]$ so a section $s \in \mathcal{E}[w]$ becomes a function $[s]_g \in \mathcal{E}$, and because all connections agree on functions, we can define $[\nabla_a s]_g := \mathrm{d}_a[s]_g$ using the differential d_a. For another choice of scale $\widehat{g} = \Omega^2 g$ the function representing section s becomes $[s]_{\widehat{g}} = \Omega^w [s]_g$. Computing the connection in the new scale we get

$$\begin{aligned}[\nabla_a s]_{\widehat{g}} &= (\mathrm{d}[s]_{\widehat{g}})_a = \mathrm{d}_a(\Omega^w [s]_g) = \Omega^w \mathrm{d}_a[s]_g + [s]_g w\, \Omega^{w-1}\mathrm{d}_a\Omega \\ &= \Omega^w\Big(\mathrm{d}_a[s]_g + (\Omega^{-1}\mathrm{d}_a\Omega) w\,[s]_g\Big) = \Omega^w\Big(\mathrm{d}_a[s]_g + \Upsilon_a w\,[s]_g\Big)\end{aligned}$$

Returning to the densities we confirm the rescaling rule (2.1).

The Levi-Civita connection on tensor bundles rescales according to the following rules (can be proved by the Koszul formula, Proposition 1.1.9):

$$\widehat{\nabla}_a X^b = \nabla_a X^b + \Upsilon_a X^b - X_a \Upsilon^b + \Upsilon_c X^c \delta_a{}^b \tag{2.3}$$

$$\widehat{\nabla}_a X_b = \nabla_a X_b - \Upsilon_a X_b - \Upsilon_b X_a + \Upsilon^c X_c g_{ab} \tag{2.4}$$

Using the Leibniz rule one can obtain a more general formula (cf. [64])

$$\begin{aligned}\widehat{\nabla}_a f_{b_1\dots b_k} = &\nabla_a f_{b_1\dots b_k} + (w-k)\Upsilon_a f_{b_1\dots b_k} \\ &- \Upsilon_{b_1} f_{ab_2\dots b_k} - \dots - \Upsilon_{b_k} f_{b_1\dots b_{k-1}a} \\ &+ \Upsilon^c f_{cb_2\dots b_k}\mathbf{g}_{ab_1} + \dots + \Upsilon^c f_{b_1\dots b_{k-1}c}\mathbf{g}_{ab_k}\end{aligned} \tag{2.5}$$

where $f_{b_1\dots b_k} \in \mathcal{E}_{b_1\dots b_k}[w]$.

We can rewrite (2.3) as

$$\widehat{\nabla}_a V^b = \nabla_a V^b + A_a{}^b{}_c V^c$$

where

$$A_a{}^b{}_c = \delta_a{}^b \Upsilon_c + \delta_c{}^b \Upsilon_a - \Upsilon^b g_{ac}$$

and then use $R^{\widehat{\nabla}} = R^{\nabla} + \nabla \wedge A + A \wedge A$ to compute the conformal rescaling of the Riemannian curvature.

Indeed,

$$(\nabla \wedge A)_{ab}{}^c{}_d = \delta_b{}^c \nabla_a \Upsilon_d + \cancelto{0}{\delta_d{}^c \nabla_a \Upsilon_b} - (\nabla_a \Upsilon^c) g_{bd} - (a \leftrightarrow b)$$

$$(A \wedge A)_{ab}{}^c{}_d = (\delta_a{}^c \Upsilon_e + \delta_e{}^c \Upsilon_a - \Upsilon^c g_{ae})(\delta_b{}^e \Upsilon_d + \delta_d{}^e \Upsilon_b - \Upsilon^e g_{bd}) - (a \leftrightarrow b)$$

so expanding, raising index d and simplifying we get

$$\widehat{R}_{ab}{}^{cd} = R_{ab}{}^{cd} - 4\,\nabla_{[a}\Upsilon^{[c}\delta_{b]}{}^{d]} + 4\,\delta_{[a}{}^{[c}\Upsilon_{b]}\Upsilon^{d]} - 4\,\delta_{[a}{}^{[c}\delta_{b]}{}^{d]}\frac{\Upsilon_e\Upsilon^e}{2}$$

Introducing the notation

$$\Lambda_{ab} \coloneqq -\nabla_a\Upsilon_b + \Upsilon_a\Upsilon_b - \tfrac{1}{2}\Upsilon_c\Upsilon^c\mathbf{g}_{ab} \tag{2.6}$$

we can represent the Riemannian curvature's rescaling rule as

$$\widehat{\mathrm{R}}_{ab}{}^{cd} = \mathrm{R}_{ab}{}^{cd} + 4\,\delta_{[a}{}^{[c}\Lambda_{b]}{}^{d]} \tag{2.7}$$

This implies that the Weyl tensor (appropriately weighted) is conformally invariant

$$\widehat{\mathrm{W}}_{abcd} = \mathrm{W}_{abcd} \tag{2.8}$$

while the Schouten tensor rescales as

$$\widehat{\mathsf{P}}_{ab} = \mathsf{P}_{ab} + \Lambda_{ab} \tag{2.9}$$

The Schouten scalar's rescaling rule will be useful too:

$$\widehat{\mathsf{J}} = \mathsf{J} - \nabla\cdot\Upsilon + (1 - \tfrac{n}{2})|\Upsilon|^2 \tag{2.10}$$

The Laplacian on $\mathcal{E}[w]$ rescales according to the formula

$$\widehat{\Delta} f = \Delta f + (n+2w-2)\Upsilon\cdot\nabla f + w(\nabla\cdot\Upsilon)f + w(n+w-2)\Upsilon\cdot\Upsilon f \tag{2.11}$$

There is a curvature modified Laplacian, the so called "box" operator $\square$,

$$\square f \coloneqq \Delta f + w\mathsf{J}f \tag{2.12}$$

that has a more suggestive transformation rule

$$\widehat{\square} f = \square f + (n+2w-2)\Big(\Upsilon\cdot\nabla f + w\tfrac{|\Upsilon|^2}{2}f\Big) \tag{2.13}$$

This equation shows that on $\mathcal{E}[\frac{2-n}{2}]$ the box operator is conformally invariant and is known as the conformal Laplacian, or the *Yamabe* operator.

2.1.4 Twisted versions of the conformal rescaling rules

If we have a vector bundle $\mathcal{F}$ over a manifold M with a connection $\nabla^{\mathcal{F}}$, and the manifold M is equipped with a conformal class c, for each choice of a metric $g \in c$ it makes sense to speak about the connection $\nabla^{\mathcal{F}} \otimes \nabla^{g}$ in the bundle $\mathcal{F}[w] \coloneqq \mathcal{F} \otimes \mathcal{E}[w]$ that we shall refer to as the (conformally) *weighted bundle* $\mathcal{F}[w]$ with conformal weight w.

The connection $\nabla^{\mathcal{F}} \otimes \nabla^{g}$ is the connection $\nabla^{\mathcal{F}}$ coupled to the Levi-Civita connection ∇^{g}, and we shall denote this coupled connection by the same symbol $\nabla^{\mathcal{F}}$, assuming the choice of a metric implicitly. As usual, if another metric $\widehat{g} = \Omega^2 g$ has been chosen in the conformal class c, the corresponding (rescaled) connection will be denoted by $\widehat{\nabla^{\mathcal{F}}}$.

The bundle $\mathcal{F}[w]$ can also be seen as the bundle $\mathcal{E}[w]$ *twisted* with a vector bundle $\mathcal{F}$. If the connection in the bundle $\mathcal{F}$ is conformally invariant (independent from the conformal structure), the rescaling rules stated above have their twisted analogues that remain formally unchanged, the connection being replaced with the $\mathcal{F}$-twisted Levi-Civita connection $\nabla^{\mathcal{F}}$.

Indeed, for any $f \in \mathcal{F}[w]$ the $\mathcal{F}$-connection twisted to the Levi-Civita connection on $\mathcal{E}[w]$ rescales as

$$\widehat{\nabla^{\mathcal{F}}_a} f = \nabla^{\mathcal{F}}_a f + \Upsilon_a w\, f \tag{2.1-$\otimes$}$$

To verify this we apply the Leibniz rule to a decomposable section $f^{\Phi} = \varphi \otimes \xi^{\Phi}$ where $\varphi \in \mathcal{E}[w]$ and $\xi^{\Phi} \in \mathcal{F}^{\Phi}$, so we have

$$\begin{aligned}
\widehat{\nabla^{\mathcal{F}}_a f} &= (\widehat{\nabla}_a \varphi) \otimes \xi^{\Phi} + \varphi \otimes \nabla^{\mathcal{F}}_a \xi^{\Phi} = (\nabla_a \varphi + \Upsilon_a w\, \varphi) \otimes \xi^{\Phi} + \varphi \otimes \nabla^{\mathcal{F}}_a \xi^{\Phi} \\
&= (\nabla_a \varphi) \otimes \xi^{\Phi} + \Upsilon_a w\, \varphi \otimes \xi^{\Phi} + \varphi \otimes \nabla^{\mathcal{F}}_a \xi^{\Phi} = \nabla^{\mathcal{F}}_a (\varphi \otimes \xi^{\Phi}) + \Upsilon_a w\, (\varphi \otimes \xi^{\Phi}) \\
&= \nabla^{\mathcal{F}}_a f^{\Phi} + \Upsilon_a w\, f^{\Phi}
\end{aligned}$$

and then extend this property to arbitrary fields in $\mathcal{F}[w]$ by the linearity of connections.

Using the same technique we can establish the connection rescaling rules for $\mathcal{F}$-twisted tangent and cotangent bundles with conformal weight w, that is

$$\widehat{\nabla^{\mathcal{F}}_a X^b} = \nabla^{\mathcal{F}}_a X^b + (w+1)\Upsilon_a X^b - X_a \Upsilon^b + \Upsilon_c X^c \delta_a{}^b \tag{2.3-$\otimes$}$$

for a section $X^a \in \mathcal{E}^a \otimes \mathcal{F}[w]$, and

$$\widehat{\nabla^{\mathcal{F}}_a X_b} = \nabla^{\mathcal{F}}_a X_b + (w-1)\Upsilon_a X_b - \Upsilon_b X_a + \Upsilon^c X_c g_{ab} \tag{2.4-$\otimes$}$$

for $X_a \in \mathcal{E}_a \otimes \mathcal{F}[w]$, where we have suppressed the $\mathcal{F}$-indices (i.e. $\Phi, \Psi, \ldots$).

It is useful to write down the rescaling identities for the divergences of the weighted (co-)vectors with values in $\mathcal{F}$, as we shall see in the next few paragraphs, and we do so:

$$\widehat{\nabla^{\mathcal{F}}_a X^a} = \nabla^{\mathcal{F}}_a X^a + (n+w)\Upsilon_a X^a$$

where $X^a \in \mathcal{E}^a \otimes \mathcal{F}[w]$, and

$$\widehat{\nabla^{\mathcal{F}a} X_a} = \nabla^{\mathcal{F}a} X_a + (n+w-2)\Upsilon^a X_a$$

for $X_a \in \mathcal{E}_a \otimes \mathcal{F}[w]$, and we have omitted the label $\mathcal{F}$ from the notation for a better readability.

The Laplacian of the twisted connection still retains the conformal rescaling rule of the usual Laplacian, so if $f \in \mathcal{F}[w]$,

$$\widehat{\Delta^{\mathcal{F}} f} = \Delta^{\mathcal{F}} f + (n+2w-2)\Upsilon \cdot \nabla^{\mathcal{F}} f + w(\nabla \cdot \Upsilon) f + w(n+w-2)|\Upsilon|^2 f \quad (2.11\text{-}\otimes)$$

The usual calculation is repeated literally step by step, and we give it here in the full form because this observation is crucial for our constructions of invariant differential operators in Chapter 3. Let us compute,

$$\begin{aligned}\widehat{\Delta^{\mathcal{F}}} f =& \widehat{\nabla}^{\mathcal{F}a} \widehat{\nabla}^{\mathcal{F}}_a f = \widehat{\nabla}^{\mathcal{F}a}\Big(\nabla^{\mathcal{F}}_a f + w\Upsilon_a f\Big) = \widehat{\nabla}^{\mathcal{F}a} \underbrace{\nabla^{\mathcal{F}}_a f}_{[w]} + wf\widehat{\nabla}^a \underbrace{\Upsilon_a}_{[0]} + w\Upsilon_a \widehat{\nabla}^{\mathcal{F}a} f \\ =& \nabla^{\mathcal{F}a}\nabla^{\mathcal{F}}_a f + (n+w-2)\Upsilon^a \nabla^{\mathcal{F}}_a f + wf\Big(\nabla^a \Upsilon_a + (n-2)\Upsilon^a\Upsilon_a\Big) \\ &+ w\Upsilon_a\Big(\nabla^{\mathcal{F}a} f + w\Upsilon^a f\Big)\end{aligned}$$

(we have indicated the weights of some terms by underbracing them).

Distributing the factors and simplifying the expressions we get

$$\widehat{\Delta^{\mathcal{F}}} f = \Delta^{\mathcal{F}} f + (n+w-2)\Upsilon \cdot \nabla^{\mathcal{F}} f + wf\nabla \cdot \Upsilon + wf(n-2)|\Upsilon|^2 + w\Upsilon \cdot \nabla^{\mathcal{F}} f + w^2|\Upsilon|^2 f$$

that confirms the formula (2.11-$\otimes$).

This result is so important for us that we state it separately but formulate for the twisted box operator $\Box^{\mathcal{F}}$ that plays a distinguished rôle in the theory of invariant operators. Notice that we have relied on the fact that we do not commute the covariant derivatives, and the curvature of $\nabla^{\mathcal{F}}$ is not involved.

Proposition 2.1.1. *The twisted box operator* $\Box^{\mathcal{F}} \colon \mathcal{F}[w] \to \mathcal{F}[w-2]$ *rescales* *according to the identity* (2.13) *where the operators* $\Box$ *and* ∇ *are replaced by their twisted counterparts, that is*

$$\widehat{\Box^{\mathcal{F}}} f = \Box^{\mathcal{F}} f + (n+2w-2)\Big(\Upsilon\cdot\nabla^{\mathcal{F}} f + w\tfrac{|\Upsilon|^2}{2} f\Big) \tag{2.13-$\otimes$}$$

Proof. As usual, applying the rescaling rules for $\Delta^{\mathcal{F}}$ and J, we have

$$\begin{aligned}\widehat{\Box^{\mathcal{F}}} f =& \widehat{\Delta^{\mathcal{F}}} f + \widehat{\mathsf{J}} w f \\ &= \Delta^{\mathcal{F}} f + (n+2w-2)\Upsilon\cdot\nabla^{\mathcal{F}} f + w(\nabla^{\mathcal{F}}\cdot\Upsilon) f + w(n+w-2)|\Upsilon|^2 f \\ &+ \mathsf{J} w\, f - w\, f\nabla\cdot\Upsilon + (1-\tfrac{n}{2})|\Upsilon|^2 w\, f\end{aligned}$$

and after some simplifications the result follows. $\Box$

2.1.5 The conformal Bianchi identities

For the sake of completeness and for the future references we give the derivation of the versions of the Bianchi identities suitable for the conformal geometry. As usual, we assume that $\dim M \geq 3$.

Definition 2.1.2. The *Cotton tensor* Y_{abd} is defined as

$$\mathrm{Y}_{ab}{}^{c} := 2\nabla_{[a}\mathsf{P}_{b]}{}^{c} \tag{2.14}$$

Proposition 2.1.3 (The Bianchi identity for the Weyl tensor)**.** *The Weyl tensor satisfies the following identity*

$$\nabla_{[p}\mathrm{W}_{ab]}{}^{cd} = 4\,\delta_{[p}{}^{[c}\mathrm{Y}_{ab]}{}^{d]} \tag{2.15}$$

Proof. Using equations (1.29) and (1.30) we compute

$$\begin{aligned}0 &= 3\,\nabla_{[p}\mathrm{R}_{ab]}{}^{cd} \\ &= \nabla_p \mathrm{R}_{ab}{}^{cd} + \nabla_a \mathrm{R}_{bp}{}^{cd} + \nabla_b \mathrm{R}_{pa}{}^{cd} \\ &= \nabla_p(\mathrm{W}_{ab}{}^{cd} + 4\,\delta_{[a}{}^{[c}\mathsf{P}_{b]}{}^{d]}) + \nabla_a(\mathrm{W}_{bp}{}^{cd} + 4\,\delta_{[b}{}^{[c}\mathsf{P}_{p]}{}^{d]}) + \nabla_b(\mathrm{W}_{pa}{}^{cd} + 4\,\delta_{[p}{}^{[c}\mathsf{P}_{a]}{}^{d]}) \\ &= 3\,\nabla_{[p}\mathrm{W}_{ab]}{}^{cd} + 4\,\delta_{[a|}{}^{[c}\nabla_p\mathsf{P}_{|b]}{}^{d]} + 4\,\delta_{[b|}{}^{[c}\nabla_a\mathsf{P}_{|p]}{}^{d]} + 4\,\delta_{[p|}{}^{[c}\nabla_b\mathsf{P}_{|a]}{}^{d]} \\ &= 3\,\nabla_{[p}\mathrm{W}_{ab]}{}^{cd} + \delta_a{}^c\nabla_p\mathsf{P}_b{}^d - \delta_b{}^c\nabla_p\mathsf{P}_a{}^d + \delta_b{}^d\nabla_p\mathsf{P}_a{}^c - \delta_a{}^d\nabla_p\mathsf{P}_b{}^c \\ &\qquad +\delta_b{}^c\nabla_a\mathsf{P}_p{}^d - \delta_p{}^c\nabla_a\mathsf{P}_b{}^d + \delta_p{}^d\nabla_a\mathsf{P}_b{}^c - \delta_b{}^d\nabla_a\mathsf{P}_p{}^c \\ &\qquad +\delta_p{}^c\nabla_b\mathsf{P}_a{}^d - \delta_a{}^c\nabla_b\mathsf{P}_p{}^d + \delta_a{}^d\nabla_b\mathsf{P}_p{}^c - \delta_p{}^d\nabla_b\mathsf{P}_a{}^c\end{aligned}$$

Rearranging we get

$$\begin{aligned}
0 = 3\,\nabla_{[p}\mathrm{R}_{ab]}{}^{cd} &= 3\,\nabla_{[p}\mathrm{W}_{ab]}{}^{cd} - \delta_p{}^c\nabla_a\mathsf{P}_b{}^d + \delta_p{}^c\nabla_b\mathsf{P}_a{}^d + \delta_p{}^d\nabla_a\mathsf{P}_b{}^c - \delta_p{}^d\nabla_b\mathsf{P}_a{}^c \\
&\quad -\delta_a{}^c\nabla_b\mathsf{P}_p{}^d + \delta_a{}^c\nabla_p\mathsf{P}_b{}^d + \delta_a{}^d\nabla_b\mathsf{P}_p{}^c - \delta_a{}^d\nabla_p\mathsf{P}_b{}^c \\
&\quad -\delta_b{}^c\nabla_p\mathsf{P}_a{}^d + \delta_b{}^c\nabla_a\mathsf{P}_p{}^d + \delta_b{}^d\nabla_p\mathsf{P}_a{}^c - \delta_b{}^d\nabla_a\mathsf{P}_p{}^c \\
&= 3\,\nabla_{[p}\mathrm{W}_{ab]}{}^{cd} - 4\,\delta_p{}^{[c}\nabla_{[a}\mathsf{P}_{b]}{}^{d]} - 4\,\delta_a{}^{[c}\nabla_{[b}\mathsf{P}_{p]}{}^{d]} - 4\,\delta_b{}^{[c}\nabla_{[p}\mathsf{P}_{a]}{}^{d]}
\end{aligned}$$

or

$$3\,\nabla_{[p}\mathrm{W}_{ab]}{}^{cd} = 12\,\delta_{[p}{}^{[c}\nabla_a\mathsf{P}_{b]}{}^{d]}$$

that proves the claim. □

Corollary 2.1.4 (The once contracted Bianchi identity for the Weyl tensor)**.**

$$\nabla_c\mathrm{W}_{ab}{}^{cd} = (n-3)\mathrm{Y}_{ab}{}^d \tag{2.16}$$

Proof. Rewriting the Bianchi identity for the Weyl tensor (2.15) as

$$3\,\nabla_{[p}\mathrm{W}_{ab]}{}^{cd} = 3\,\delta_{[p}{}^c\mathrm{Y}_{ab]}{}^d - 3\,\delta_{[p}{}^d\mathrm{Y}_{ab]}{}^c$$

and then as

$$\nabla_p\mathrm{W}_{ab}{}^{cd} + \nabla_a\mathrm{W}_{bp}{}^{cd} + \nabla_b\mathrm{W}_{pa}{}^{cd} = \delta_p{}^c\mathrm{Y}_{ab}{}^d + \delta_a{}^c\mathrm{Y}_{bp}{}^d + \delta_b{}^c\mathrm{Y}_{pa}{}^d - \delta_p{}^d\mathrm{Y}_{ab}{}^c - \delta_a{}^d\mathrm{Y}_{bp}{}^c - \delta_b{}^d\mathrm{Y}_{pa}{}^c$$

we can contract the indices p and c to get

$$\nabla_c\mathrm{W}_{ab}{}^{cd} + \cancel{\nabla_a\mathrm{W}_{bc}{}^{cd}} + \cancel{\nabla_b\mathrm{W}_{ca}{}^{cd}} = \delta_c{}^c\mathrm{Y}_{ab}{}^d + \delta_a{}^c\mathrm{Y}_{bc}{}^d + \delta_b{}^c\mathrm{Y}_{ca}{}^d - \delta_c{}^d\mathrm{Y}_{ab}{}^c - \cancel{\delta_a{}^d\mathrm{Y}_{bc}{}^c} - \cancel{\delta_b{}^d\mathrm{Y}_{ca}{}^c}$$

where some terms obviously vanish due to trace-freeness of the Weyl and Cotton tensors. Continuing we obtain

$$\nabla_c\mathrm{W}_{ab}{}^{cd} = n\,\mathrm{Y}_{ab}{}^d + \mathrm{Y}_{ba}{}^d + \mathrm{Y}_{ba}{}^d - \mathrm{Y}_{ab}{}^d$$

which is equivalent to the claim. □

Corollary 2.1.5 (The twice contracted Bianchi identity for the Weyl tensor)**.**

$$\nabla^a\mathsf{P}_{ab} = \nabla_b\mathsf{J} \tag{2.17}$$

Proof.

$$0 = \mathrm{Y}_{ab}{}^a = 2\,\nabla_{[a}\mathsf{P}_{b]}{}^a = \nabla_a\mathsf{P}_b{}^a - \nabla_b\mathsf{P}_a{}^a$$

□

Corollary 2.1.6. *The Cotton tensor is divergence-free:*

$$\nabla_d \mathrm{Y}_{ab}{}^d = 0 \tag{2.18}$$

Proof. Indeed,

$$\begin{aligned}
\nabla_d \mathrm{Y}_{ab}{}^d &= 2\,\nabla_d \nabla_{[a} \mathsf{P}_{b]}{}^d \\
&= \nabla_d \nabla_a \mathsf{P}_b{}^d - \nabla_d \nabla_b \mathsf{P}_a{}^d \\
&= \nabla_a \nabla_d \mathsf{P}_b{}^d + \mathrm{R}_{da}{}^d{}_e \mathsf{P}_b{}^e - \mathrm{R}_{da}{}^e{}_b \mathsf{P}_e{}^d - \nabla_b \nabla_d \mathsf{P}_a{}^d - \mathrm{R}_{db}{}^d{}_e \mathsf{P}_a{}^e + \mathrm{R}_{db}{}^e{}_a \mathsf{P}_e{}^d \\
&= \nabla_a \nabla_b \mathsf{J} - \nabla_b \nabla_a \mathsf{J} + \mathrm{Ric}_{ae} \mathsf{P}_b{}^e - \mathrm{Ric}_{be} \mathsf{P}_a{}^e - \mathrm{R}_{daeb} \mathsf{P}^{ed} + \mathrm{R}_{dbea} \mathsf{P}^{ed} \\
&= 0
\end{aligned}$$

due to torsion-freeness of the Levi-Civita connection and the “swap” symmetry of the Riemann tensor. □

2.2 Hypersurfaces in conformal geometry

Let now M be a manifold of dimension $n \geq 3$ endowed with a conformal structure c, and Σ be an embedded hypersurface.

2.2.1 The induced conformal structure

For each choice of a metric g from the conformal class c the hypersurface inherits the induced metric $\bar{g}$, and for another metric $\widehat{g} = \Omega^2 g$ from the conformal class c, the induced metric agrees with the rescaling of the metric $\bar{g}$ in the conformal class of the metric $\bar{g}$ on the hypersurface:

$$\widehat{\bar{g}} = (\Omega|_\Sigma)^2 \bar{g}$$

2.2.2 Densities on hypersurfaces

For each choice of a scale the densities on M are represented by functions, and therefore can be restricted onto Σ. Regarding the densities as homogeneous functions on the ray bundle of metrics representing the conformal structure ensures that the restriction of the bundle of densities of weight w descends to the bundle of densities of the same weight on the hypersurface.

2.2.3 Canonical sections along hypersurfaces

Let Σ be a hypersurface represented as the zero locus of a defining function $s\colon M \to \mathbb{R}$ with $\mathrm{d}s \neq 0$ along Σ.

From the point of view adopted in conformal geometry, the unit normal

$$\mathrm{N}^a = |\nabla s|^{-1} \nabla_a s$$

has a conformal weight. Indeed, since $\mathbf{g}^{ab} \in \mathcal{E}^{ab}[-2]$, the quantity

$$|\nabla s| = \sqrt{\mathbf{g}^{ab}(\nabla_a s)\nabla_b s}$$

must have the weight -1. This implies that $\mathrm{N}_a \in \mathcal{E}_a[1]$. This, of course, agrees with the fact that

$$|\mathrm{N}|^2 = \mathbf{g}^{ab}\mathrm{N}_a\mathrm{N}_b = 1 \tag{2.19}$$

where 1 is seen as a constant section of $\mathcal{E} \equiv \mathcal{E}[0]$.

Given a conformal weight the unit (co)normal is conformally invariant along the hypersurface

$$\widehat{\mathrm{N}_a} = \mathrm{N}_a$$

Extending the Levi-Civita connection to act on $\mathcal{E}_a[1]$ and considering the restricted bundle $\mathcal{E}_a[1]|_\Sigma$ with the pullback connection $\underline{\nabla}_a\colon \mathcal{E}_b[1]|_\Sigma \to \overline{\mathcal{E}}_a \otimes \mathcal{E}_b[1]|_\Sigma$ we can differentiate the section $\mathrm{N}_b \in \mathcal{E}_b[1]|_\Sigma$ and get

$$\mathbf{L}_{ab} \coloneqq \underline{\nabla}_a \mathrm{N}_b$$

As usual, differentiating (2.19) we see that $\mathbf{L}_{ab}N^b = 0$ and thus $\mathbf{L}_{ab}$ can be seen as a section of $\overline{\mathcal{E}}_{(ab)}[1]$ (i.e. with values in the intrinsic bundles). This is the conformally weighted shape tensor (we have distinguished it from the shape tensor used in the Riemannian geometry with the boldfaced letter “L”).

The mean curvature can be seen as a density of weight -1 along the hypersurface

$$\mathbf{H} \coloneqq \bar{n}^{-1}\, \bar{\mathbf{g}}^{ab}\mathbf{L}_{ab}$$

and for this reason we use the boldface letter “H” instead of the usual mean curvature $H = (\dim\Sigma)^{-1}\bar{g}^{ab}\mathrm{I\!I}_{ab}$ from the Riemannian geometry of hypersurfaces. We could refer to $\mathbf{H}$ as the mean curvature density, but it is customary to speak about the mean curvature, while the precise meaning should be clear from the context.

Proposition 2.2.1. *The conformal rescaling properties of the (conformally weighted) shape tensor and the mean curvature are*

$$\widehat{\mathbf{L}_{ab}} = \mathbf{L}_{ab} + \Upsilon \cdot \mathrm{N}\, \bar{\mathbf{g}}_{ab} \tag{2.20}$$

$$\widehat{\mathbf{H}} = \mathbf{H} + \Upsilon \cdot \mathrm{N} \tag{2.21}$$

Proof. Using (2.4-⊗) and the definitions, we compute along the hypersurface:

$$\widehat{\mathring{\mathbf{L}}_{ab}} = \widehat{\underline{\nabla}_a \mathrm{N}_b} = \Pi_a{}^{a'} \nabla_{a'} \mathrm{N}_b = \Pi_a{}^{a'} \Big(\nabla_{a'} \mathrm{N}_b - \Upsilon_b \mathrm{N}_{a'} + \Upsilon^c \mathrm{N}_c \mathbf{g}_{a'b} \Big) = \mathring{\mathbf{L}}_{ab} + \Upsilon^c \mathrm{N}_c \bar{\mathbf{g}}_{ab}$$

and contracting (2.20) yields (2.21). □

Corollary 2.2.2. *The trace free part* $\mathring{\mathbf{L}}_{ab} \equiv \mathbf{L}_{(ab)\circ}$ *of the (conformally weighted) shape tensor*

$$\mathring{\mathbf{L}}_{ab} := \mathbf{L}_{ab} - \mathbf{H}\bar{\mathbf{g}}_{ab} \tag{2.22}$$

is conformally invariant: $\widehat{\mathring{\mathbf{L}}_{ab}} = \mathring{\mathbf{L}}_{ab}$.

We prefer to term $\mathring{\mathbf{L}}_{ab}$ the *umbilicity tensor* for the sake of brevity. This is a fundamental conformal invariant of hypersurfaces, the precise meaning of that is discussed in Chapter 4.

2.2.4 The conformal Gauss equations

We can decompose the ambient Riemannian curvature R and the intrinsic Riemannian curvature $\overline{\mathrm{R}}$ Riemannian curvatures in the Gauss equation (1.59) according to (1.45) to get

$$\mathrm{R} = \mathrm{W} + g \owedge \mathsf{P} \tag{2.23}$$

and

$$\overline{\mathrm{R}} = \overline{\mathrm{W}} + \bar{g} \owedge \overline{\mathsf{P}} \tag{2.24}$$

One has to be careful with these equations because the tensors in (2.23) live in vector bundle of different ranks to those in (2.24) This is emphasized with the overlines, however.

Since the total tangential projection Π_Σ is a linear operator, we may apply it to (2.23) and write

$$\Pi_\Sigma \mathrm{R} = \Pi_\Sigma \mathrm{W} + \bar{g} \owedge \Pi_\Sigma \mathsf{P} \tag{2.25}$$

Indeed, $\Pi_\Sigma g = \bar{g}$.

Now both equations (2.24) and (2.25) live in the same dimension, so we can legitimately perform algebraic operations with them.

Our goal now is to decompose into Weyl and Schouten parts all the terms of the Gauss equation. At the same time, we want to decompose the shape tensors into the trace-free and the trace part, which is logically the same process as the Weyl-Schouten decomposition.

The Kulkarni-Nomizu product is bilinear, so we can write

$$\mathrm{L} \owedge \mathrm{L} = \mathring{\mathrm{L}} \owedge \mathring{\mathrm{L}} + 2\, H\bar{g} \owedge \mathring{\mathrm{L}} + H^2 \bar{g} \owedge \bar{g} \tag{2.26}$$

The two latter terms in this equation are already in the pure trace form, that is their trace-free part is zero . The first term, however, is not trace-free yet, as we can see:

$$\mathrm{Ric}(\mathring{\mathrm{L}} \owedge \mathring{\mathrm{L}}) = \mathrm{tr}(\mathring{\mathrm{L}} \owedge \mathring{\mathrm{L}}) = -2\, \mathring{\mathrm{L}} \circ \mathring{\mathrm{L}}$$

Indeed, explicitly,

$$\bar{\delta}_c{}^a (2\, \mathring{\mathrm{L}}_a{}^c \mathring{\mathrm{L}}_b{}^d - 2\, \mathring{\mathrm{L}}_b{}^c \mathring{\mathrm{L}}_a{}^d) = -2\, \mathring{\mathrm{L}}_b{}^a \mathring{\mathrm{L}}_a{}^d$$

The scalar part of $\mathring{\mathrm{L}} \owedge \mathring{\mathrm{L}}$ is then

$$\mathrm{Scal}(\mathring{\mathrm{L}} \owedge \mathring{\mathrm{L}}) = -2\, |\mathring{\mathrm{L}}|^2$$

so we can compute the Schouten part of $\mathring{\mathrm{L}} \owedge \mathring{\mathrm{L}}$ as

$$\mathbb{P}(\mathring{\mathrm{L}} \owedge \mathring{\mathrm{L}}) = \tfrac{1}{n-3}\Big(-2\mathring{\mathrm{L}} \circ \mathring{\mathrm{L}} - \tfrac{-2|\mathring{\mathrm{L}}|^2}{2(n-2)} \bar{\mathbf{g}} \Big) = \tfrac{1}{n-3}\Big(\tfrac{|\mathring{\mathrm{L}}|^2}{(n-2)} \bar{\mathbf{g}} - 2\mathring{\mathrm{L}} \circ \mathring{\mathrm{L}} \Big)$$

Let us temporarily write the decomposition of $\mathring{\mathrm{L}} \owedge \mathring{\mathrm{L}}$ as

$$\mathring{\mathrm{L}} \owedge \mathring{\mathrm{L}} = \mathbb{W}(\mathring{\mathrm{L}} \owedge \mathring{\mathrm{L}}) + \bar{\mathbf{g}} \owedge \mathbb{P}(\mathring{\mathrm{L}} \owedge \mathring{\mathrm{L}})$$

So far, we can represent the Gauss equation as follows

$$\Pi_\Sigma \mathrm{W} + \bar{\mathbf{g}} \owedge \Pi_\Sigma \mathsf{P} = \overline{\mathrm{W}} + \bar{\mathbf{g}} \owedge \overline{\mathsf{P}} - \tfrac{1}{2}\Big(\mathbb{W}(\mathring{\mathrm{L}} \owedge \mathring{\mathrm{L}}) + \bar{g} \owedge \mathbb{P}(\mathring{\mathrm{L}} \owedge \mathring{\mathrm{L}}) + 2\, H\bar{\mathbf{g}} \owedge \mathring{\mathrm{L}} + H^2 \bar{\mathbf{g}} \owedge \bar{\mathbf{g}} \Big)$$

Examining the terms we find that $\Pi_\Sigma \mathrm{W}$ is not trace-free or pure trace yet, but has a decomposition

$$\Pi_\Sigma \mathrm{W} = \mathbb{W}(\Pi_\Sigma \mathrm{W}) + \bar{\mathbf{g}} \owedge \mathbb{P}(\Pi_\Sigma \mathrm{W})$$

where

$$\mathbb{P}(\Pi_\Sigma \mathrm{W}) = -\tfrac{1}{n-3}\overset{\mathrm{NN}}{\mathrm{W}} \tag{2.27}$$

Here we use the notation for the "bi-normal" part of the ambient Weyl tensor

$$\overset{\mathrm{NN}}{\mathrm{W}}_{bd} := \Pi_b{}^{b'}\Pi_d{}^{d'}\mathrm{W}_{a'b'c'd'}N^{a'}N^{c'} \tag{2.28}$$

Indeed, here is the calculation

$$\begin{aligned}
\delta_c{}^a\Pi^{a'}{}_a\Pi^{b'}{}_b\Pi^c{}_{c'}\Pi^d{}_{d'}\mathrm{W}_{ab}{}^{cd} &= \delta_c{}^a(\delta_a{}^{a'} - N_aN^{a'})(\delta_{c'}{}^c - N_{c'}N^c)\Pi^{b'}{}_b\Pi^d{}_{d'}\mathrm{W}_{a'b'}{}^{c'd'}\\
&= (\delta_a{}^{a'} - N_aN^{a'})(\delta_{c'}{}^a - N_{c'}N^a)\Pi^{b'}{}_b\Pi^d{}_{d'}\mathrm{W}_{a'b'}{}^{c'd'}\\
&= (\delta_{c'}{}^{a'} - N_{c'}N^{a'} - N_{c'}N^{a'} + N_{c'}N^{a'})\Pi^{b'}{}_b\Pi^d{}_{d'}\mathrm{W}_{a'b'}{}^{c'd'}\\
&= -\Pi^{b'}{}_b\Pi^d{}_{d'}\mathrm{W}_{a'b'}{}^{c'd'}N^{a'}N_{c'}
\end{aligned}$$

Observe that in $\mathrm{W}_{ab}{}^{cd}N^aN_c$ indices b and d are both tangential in the sense that tensoring them to the unit normal yields zero. Thus we may identify $\overset{\mathrm{N}}{\mathrm{W}}_{bd}$ as defined above with $\mathrm{W}_{ab}{}^{cd}N^aN_c$. Formally,

$$\mathrm{W}_{abcd}N^aN^c = \Pi_b{}^{b'}\Pi_d{}^{d'}\mathrm{W}_{a'b'c'd'}N^{a'}N^{c'}$$

since we identify the intrinsic tensor bundle with the annihilator of the unit normal.

Thus, we have

$$\mathrm{Ric}(\Pi_\Sigma \mathrm{W}) = -\overset{\mathrm{N}}{\mathrm{W}} \tag{2.29}$$

and also

$$\mathrm{Scal}(\Pi_\Sigma \mathrm{W}) = 0 \tag{2.30}$$

that proves (2.27) by an application of (1.46) in the dimension $\bar{n} = n-1$.

Collecting the bits together we obtain the equation

$$\begin{aligned}
&\mathbb{W}(\Pi_\Sigma \mathrm{W}) - \tfrac{1}{n-3}\bar{\mathbf{g}} \owedge \overset{\mathrm{N}}{\mathrm{W}} + \bar{\mathbf{g}} \owedge \Pi_\Sigma \mathsf{P} =\\
&\overline{\mathrm{W}} + \bar{\mathbf{g}} \owedge \overline{\mathsf{P}} - \tfrac{1}{2}\Big(\mathbb{W}(\mathring{\mathrm{L}} \owedge \mathring{\mathrm{L}}) + \bar{\mathbf{g}} \owedge \mathbb{P}(\mathring{\mathrm{L}} \owedge \mathring{\mathrm{L}}) + 2\,H\bar{\mathbf{g}} \owedge \mathring{\mathrm{L}} + H^2\bar{\mathbf{g}} \owedge \bar{\mathbf{g}}\Big)
\end{aligned}$$

that we rewrite by moving the Weyl parts to the left hand side, and the Schouten parts to the right hand side, and thus get

$$\mathbb{W}\Big(\Pi_\Sigma \mathrm{W} - \overline{\mathrm{R}} + \tfrac{1}{2}\mathring{\mathrm{L}} \owedge \mathring{\mathrm{L}}\Big) = \bar{\mathbf{g}} \owedge \Big(\tfrac{1}{n-3}\overset{\mathrm{N}}{\mathrm{W}} - \Pi_\Sigma \mathsf{P} + \overline{\mathsf{P}} - \tfrac{1}{2}\mathbb{P}(\mathring{\mathrm{L}} \owedge \mathring{\mathrm{L}}) - H\mathring{\mathrm{L}} - \tfrac{H^2}{2}\bar{\mathbf{g}}\Big)$$

Since the LHS is trace-free and the RHS is a pure trace, this can only hold when both sides are zero.

We must also state that this equation only makes sense in dimensions $n > 3$.

When $n > 3$, or equivalently $\bar{n} > 2$, the map $\bar{g} \owedge \cdot$ is injective, so we obtain the following equation

$$\Pi_\Sigma \mathsf{P} - \overline{\mathsf{P}} = \tfrac{1}{n-3}\overset{\mathrm{N}}{\mathrm{W}} - \tfrac{1}{2(n-3)}\left(\tfrac{|\mathring{\mathrm{L}}|^2}{(n-2)}\bar{g} - 2\mathring{\mathrm{L}} \circ \mathring{\mathrm{L}}\right) - H\mathring{\mathrm{L}} - \tfrac{H^2}{2}\bar{g}$$

that we can represent in the following form

$$\Pi_\Sigma \mathsf{P} - \overline{\mathsf{P}} = \tfrac{1}{n-3}\overset{\mathrm{N}}{\mathrm{W}} + \tfrac{1}{n-3}\mathring{\mathrm{L}} \circ \mathring{\mathrm{L}} - H\mathring{\mathrm{L}} - \tfrac{|\mathring{\mathrm{L}}|^2}{2(n-3)(n-2)}\bar{g} - \tfrac{H^2}{2}\bar{g} \tag{2.31}$$

Rearranging as

$$\Pi_\Sigma \mathsf{P} - \overline{\mathsf{P}} + H\mathring{\mathrm{L}} + \tfrac{H^2}{2}\bar{g} = \tfrac{1}{n-3}\left(\overset{\mathrm{N}}{\mathrm{W}} + \mathring{\mathrm{L}} \circ \mathring{\mathrm{L}} - \tfrac{|\mathring{\mathrm{L}}|^2}{(n-2)}\bar{g}\right)$$

we obtain an explicit expression for the so called Fialkow tensor

$$\mathcal{F} := \Pi_\Sigma \mathsf{P} - \overline{\mathsf{P}} + H\mathring{\mathrm{L}} + \tfrac{H^2}{2}\bar{g}$$

and immediately see it conformal invariance. This tensor has been studied by R.Stafford [68], A.Juhl[1] [39] and S.Curry [20].

We want to have another consequence of these results, namely an expression for $\Pi_\Sigma \mathrm{W} - \overline{\mathrm{W}}$ that would be a more suitable candidate for the role of the conformal Gauss equation.

Subtracting from the tangential projection of the Weyl-Schouten decomposition of the ambient Riemannian curvature (2.25) the decomposition for the intrinsic Riemannian curvature (2.24) we get

$$\Pi_\Sigma \mathrm{R} - \overline{\mathrm{R}} = \Pi_\Sigma \mathrm{W} - \overline{\mathrm{W}} + \bar{\mathbf{g}} \owedge \left(\Pi_\Sigma \mathsf{P} - \overline{\mathsf{P}}\right)$$

Now we can apply the Gauss equation to the left hand side, and use (2.26) to get

$$-\tfrac{1}{2}\left(\mathring{\mathrm{L}} \owedge \mathring{\mathrm{L}} + 2\,H\bar{\mathbf{g}} \owedge \mathring{\mathrm{L}} + H^2\bar{\mathbf{g}} \owedge \bar{\mathbf{g}}\right) = \Pi_\Sigma \mathrm{W} - \overline{\mathrm{W}} + \bar{\mathbf{g}} \owedge \left(\Pi_\Sigma \mathsf{P} - \overline{\mathsf{P}}\right)$$

[1]It was A.Juhl who attributed this tensor to A. Fialkow, and this suggested to us the terminology.

Substituting (2.31)

$$-\frac{1}{2}\left(\mathring{\mathrm{L}} \owedge \mathring{\mathrm{L}} + 2\,H\bar{\mathbf{g}} \owedge \mathring{\mathrm{L}} + H^2\bar{\mathbf{g}} \owedge \bar{\mathbf{g}}\right) =$$
$$\Pi_\Sigma \mathrm{W} - \overline{\mathrm{W}} + \bar{\mathbf{g}} \owedge \left(\tfrac{1}{n-3}\overset{\mathrm{NN}}{\mathrm{W}} + \tfrac{1}{n-3}\mathring{\mathrm{L}} \circ \mathring{\mathrm{L}} - H\mathring{\mathrm{L}} - \tfrac{|\mathring{\mathrm{L}}|^2}{2(n-3)(n-2)}\bar{\mathbf{g}} - \tfrac{H^2}{2}\bar{\mathbf{g}}\right)$$

and rearranging as

$$\Pi_\Sigma \mathrm{W} - \overline{\mathrm{W}} = -\tfrac{1}{2}\mathring{\mathrm{L}} \owedge \mathring{\mathrm{L}} - \bar{\mathbf{g}} \owedge \left(\tfrac{1}{n-3}\overset{\mathrm{NN}}{\mathrm{W}} + \tfrac{1}{n-3}\mathring{\mathrm{L}} \circ \mathring{\mathrm{L}} - \cancel{H\mathring{\mathrm{L}}} - \tfrac{|\mathring{\mathrm{L}}|^2}{2(n-3)(n-2)}\bar{\mathbf{g}} - \cancel{\tfrac{H^2}{2}\bar{\mathbf{g}}} + \cancel{H\mathring{\mathrm{L}}} + \cancel{\tfrac{H^2}{2}\bar{\mathbf{g}}}\right)$$

after some cancellations we arrive to

$$\overline{\mathrm{W}} - \Pi_\Sigma \mathrm{W} = \tfrac{1}{2}\mathring{\mathrm{L}} \owedge \mathring{\mathrm{L}} + \bar{\mathbf{g}} \owedge \tfrac{1}{n-3}\left(\overset{\mathrm{NN}}{\mathrm{W}} + \mathring{\mathrm{L}}^2 - \tfrac{|\mathring{\mathrm{L}}|^2}{2(n-2)}\bar{\mathbf{g}}\right) \tag{2.32}$$

that we refer to as *the conformal Gauss equation.*

Remark 2.2.3. One has to be careful when interpreting the conformal Gauss equation in the proposed form since it is not a Weyl-Schouten decomposition: some terms are not totally-trace free!

Remark 2.2.4. Rewriting (2.32) as

$$\overline{\mathrm{W}} = \Pi_\Sigma \mathrm{W} + \tfrac{1}{2}\mathring{\mathrm{L}} \owedge \mathring{\mathrm{L}} + \bar{g} \owedge \tfrac{1}{n-3}\left(\overset{\mathrm{NN}}{\mathrm{W}} + \mathring{\mathrm{L}}^2 - \tfrac{|\mathring{\mathrm{L}}|^2}{2(n-2)}\bar{g}\right) \tag{2.33}$$

we can state it as that the intrinsic Weyl curvature of a hypersurface is determined by the totally projected ambient Weyl curvature and the “umbilic” curvature up to a trace term (which has some normal part of the ambient curvature in it). The trace term is an effect of the traces of the totally projected Weyl and the “umbilic” curvatures, since the intrinsic Weyl tensor is totally trace free. Passing to the trace free parts we can modify this statement to

$$\overline{\mathrm{W}} = (\Pi_\Sigma \mathrm{W})_\circ + \left(\tfrac{1}{2}\mathring{\mathrm{L}} \owedge \mathring{\mathrm{L}}\right)_\circ \tag{2.34}$$

where by $(\cdot)_\circ$ we have denoted the trace-free part of the embraced tensor.

As an immediate consequence of (2.34) we recover a result that according to Yano [78] is due to Schouten:

Proposition 2.2.5. *A totally umbilic hypersurface in a conformally flat background is again conformally flat with respect to the induced conformal structure.*

2.2.5 The conformal Codazzi equations

The Weyl-Schouten decomposition of the Riemannian curvature proves to be also useful for the representation of the Codazzi equation in the form suitable for conformal geometry. The calculations and the statements are less involved then for the Gauss equations but we need to use some new notation to expose the real meaning of the terms.

Proposition 2.2.6 (The conformal Codazzi equation)**.** *The intrinsic exterior covariant derivative of the umbilicity tensor* $\mathring{\mathbf{L}}_a{}^c$ *viewed as a* 1*-form on the hypersurface* Σ *is expressed in terms of the ambient curvatures as*

$$2\overline{\nabla}_{[a}\mathring{\mathrm{L}}_{b]}{}^c = \underline{\mathrm{W}}_{ab}{}^{cd}N_d + 2\,\overline{\delta}_{[a}{}^c\overline{\nabla}_{b]}H + 2\,\overline{\delta}_{[a}{}^c\underline{\mathsf{P}}_{b]}{}^dN_d \tag{2.35}$$

Proof. Rewrite the Codazzi equation for hypersurfaces as

$$\Pi^{a'}{}_a\Pi^{b'}{}_b\mathrm{R}_{a'b'}{}^{cd}N_d = \overline{\nabla}_a\mathrm{L}_b{}^c - \overline{\nabla}_b\mathrm{L}_a{}^c$$

Using the Weyl-Schouten decomposition of the Riemannian curvature

$$\mathrm{R}_{ab}{}^{cd} = \mathrm{W}_{ab}{}^{cd} + 4\delta_{[a}{}^{[c}\mathsf{P}_{b]}{}^{d]}$$

we rewrite the Codazzi equation as

$$\Pi^{a'}{}_a\Pi^{b'}{}_b(\mathrm{W}_{a'b'}{}^{cd} + 4\delta_{[a'}{}^{[c}\mathsf{P}_{b']}{}^{d]})N_d = \overline{\nabla}_a(\mathrm{L}_b{}^c + H\overline{\delta}_b{}^c) - \overline{\nabla}_b(\mathrm{L}_a{}^c + H\overline{\delta}_a{}^c)$$

Expanding all the terms in the above display we arrive to

$$\begin{aligned}\Pi^{a'}{}_a\Pi^{b'}{}_b(\mathrm{W}_{a'b'}{}^{cd}N_d + \delta_{a'}{}^c\mathsf{P}_{b'}{}^dN_d - \delta_{b'}{}^c\mathsf{P}_{a'}{}^dN_d + \cancel{\delta_{b'}{}^d\mathsf{P}_{a'}{}^eN_d} - \cancel{\delta_{a'}{}^d\mathsf{P}_{b'}{}^eN_d})\\ = \overline{\nabla}_a\mathring{\mathrm{L}}_b{}^c + \overline{\delta}_b{}^c\overline{\nabla}_aH - \overline{\nabla}_b\mathring{\mathrm{L}}_a{}^c - \overline{\delta}_a{}^c\overline{\nabla}_bH\end{aligned}$$

After simplifications we get

$$\underline{\mathrm{W}}_{ab}{}^{cd}N_d + 2\,\underline{\delta}_{[a}{}^c\underline{\mathsf{P}}_{b]}{}^dN_d = 2\overline{\nabla}_{[a}\mathring{\mathrm{L}}_{b]}{}^c - 2\,\overline{\delta}_{[a}{}^c\overline{\nabla}_{b]}H$$

where we have used the notation

$$\underline{\mathrm{W}}_{ab}{}^{cd} := \Pi^{a'}{}_a\Pi^{b'}{}_b\mathrm{W}_{a'b'}{}^{cd}$$

$$\underline{\delta}_a{}^b := \Pi^{a'}{}_a\delta_{a'}{}^b$$

and

$$\underline{\mathsf{P}}_a{}^b = \Pi^{a'}{}_a \mathsf{P}_{a'}{}^b$$

The underline reminds that these quantities are seen as k-forms on Σ with values in the ambient bundles.

□

The corollary is much wider known than the full form from the previous proposition.

Corollary 2.2.7 (The contracted conformal Codazzi equation)**.** *The exterior covariant divergence of the umbilicity tensor* $\mathring{\mathbf{L}}_{ab}$ *viewed as a* 1*-form on the hypersurface* Σ *is given by*

$$\overline{\nabla}^a \mathring{\mathrm{L}}_{ab} = (\bar{n} - 1)\left[\underline{\mathsf{P}}_b{}^d N_d + \overline{\nabla}_b H\right] \tag{2.36}$$

Proof. Rewriting the conformal Codazzi equation in the expanded form

$$\underline{\mathrm{W}}_{ab}{}^{cd} N_d = \overline{\nabla}_a \mathring{\mathrm{L}}_b{}^c - \overline{\nabla}_b \mathring{\mathrm{L}}_a{}^c - \bar{\delta}_a{}^c \overline{\nabla}_b H + \bar{\delta}_b{}^c \overline{\nabla}_a H - \bar{\delta}_a{}^c \underline{\mathsf{P}}_b{}^d N_d + \bar{\delta}_b{}^c \underline{\mathsf{P}}_a{}^d N_d$$

we can contract (with the intrinsic metric) the indices a and c to get

$$0 = \overline{\nabla}_a \mathring{\mathrm{L}}_b{}^a - 0 - \bar{n}\overline{\nabla}_b H + \overline{\nabla}_b H - \bar{n}\underline{\mathsf{P}}_b{}^d N_d + \underline{\mathsf{P}}_b{}^d N_d$$

where $\bar{n} = n - 1$ is the dimension of the hypersurface Σ. □

Chapter 3

Invariant calculus for conformal hypersurfaces

In the first section of this chapter we give a fairly brief overview of conformal tractor calculus. The emphasis is made on the ways of obtaining invariant operators acting on tractor bundles. Some useful formulas are presented, such as explicit expressions for the action of the box operator and the Thomas-D operator on the standard tractors, as well as on the sections of the adjoint tractor bundles. In Appendix C one can find the similar facts related to the symmetric 2-tractor bundle.

The the second section is devoted to a detailed treatment of the tractor calculus of hypersurfaces. We define the tractor bundles that arise on a hypersurface in a conformal manifold, and describe the metric connections that are available in these bundles.

In the culmination of this chapter we introduce twisted versions of the intrinsic tractor operators, and also construct a specific, so called triple-D operator, and discuss their properties.

3.1 The conformal tractor calculus

A conformal manifold (M, c) can be regarded as a parabolic geometry with the structure group $SO(n+1, 1)$ and a certain parabolic subgroup P (the stabilizer of the unit ray in the flat model). As it is shown in [18] such geometries possess a canonical vector bundle that can be identified with the associated bundle of the Cartan geometry.

3.1.1 The standard conformal tractor bundle

To equip a conformal manifold (M, c) with a canonical tractor bundle $\mathbb{T}M$ one can consider the following construction (see [17]).

The density bundle $\mathcal{E}[1]$ has the 2-jet exact sequence (see e.g. [66])

$$0 \to \mathcal{E}_{(ab)}[1] \to J^2(\mathcal{E}[1]) \to J^1(\mathcal{E}[1]) \to 0$$

where the bundle $\mathcal{E}_{(ab)}[1]$ splits by the conformal metric $\mathbf{g}^{ab}$ into the trace-free and the trace part as $\mathcal{E}_{(ab)}[1] = \mathcal{E}_{(ab)\circ}[1] \otimes \mathcal{E}[-1]$, and thus the bundle $\mathcal{E}_{(ab)\circ}[1]$ turns out to be a smooth subbundle of $J^2(\mathcal{E}[1])$. The standard conformal co-tractor bundle is then defined as the quotient $\mathcal{T}_A \coloneqq J^2(\mathcal{E}[1])/\mathcal{E}_{(ab)\circ}[1]$. Taking into consideration the 1-jet exact sequence of $\mathcal{E}[1]$

$$0 \to \mathcal{E}_a[1] \to J^1(\mathcal{E}[1]) \to \mathcal{E}[1] \to 0$$

we obtain a composition series for $\mathcal{T}_A$ that (cf. [12], see also [6, pp.2-3]) can be expressed as

$$\mathcal{T}_A \coloneqq \mathcal{E}[1] \oplus\!\!\!\!+ \mathcal{E}_a[1] \oplus\!\!\!\!+ \mathcal{E}[-1]$$

where the semidirect product $\oplus\!\!\!\!+$ is used to show that the presentation of a section changes when the metric is moved within the conformal class.

The bundle $\mathcal{T}_A$ has a metric and compatible connection $\nabla^{\mathcal{T}}$, which are invariant with respect to the underlying conformal structure.

To keep things simple, we adopt (as in [22]) the following

Definition 3.1.1. The *standard conformal tractor bundle* $\mathbb{T}M$ or $\mathcal{T}^A$ in the index notation, on the conformal manifold (M, c) is defined for any choice of metric g from the conformal class c by

$$\mathcal{T}^A \stackrel{g}{=} \mathcal{E}[1] \oplus \mathcal{E}^a[-1] \oplus \mathcal{E}[-1]$$

so that if $V^A \in \mathcal{T}^A$ is an arbitrary section that in a choice of a metric g can be written as

$$[V^A]_g = \begin{pmatrix} \sigma \\ \mu^a \\ \rho \end{pmatrix} \tag{3.1}$$

but when the metric is moved within the conformal class to $\widehat{g} = \Omega^2 g$, the presentation of section V^A changes to

$$[V^A]_{\widehat{g}} = \begin{pmatrix} \sigma \\ \mu^a + \Upsilon^a \sigma \\ \rho - \Upsilon_b \mu^b - \frac{1}{2} \Upsilon_b \Upsilon^b \sigma \end{pmatrix} \tag{3.2}$$

Remark 3.1.2. There are reasons to denote the tractor bundle by symbol $\mathcal{E}^A$. For instance, this would emphasize the affinity of the tractor bundle both with bundles $\mathcal{E}^a$ and $\mathcal{E}[w]$. We choose to use a separate letter ($\mathcal{T}$) for the tractor bundles to stick closer to our conventions.

The tractor bundle is equipped with a metric h_{AB} that we call the *tractor metric* . For any two tractors U^A and V^B and a choice of a metric $g \in c$ so that

$$U^A = \begin{pmatrix} \sigma \\ \mu^a \\ \rho \end{pmatrix} \text{ and } V^B = \begin{pmatrix} \sigma' \\ \mu'^b \\ \rho' \end{pmatrix}$$

the metric is given by

$$h_{AB} U^A V^B := \sigma \rho' + g_{ab} \mu^a \mu'^b + \rho \sigma' \tag{3.3}$$

It is easily checked that this definition is invariant with respect to conformal rescalings.

The tractor metric h_{AB} is assigned the conformal weight 0 and allows to identify the bundles $\mathcal{T}_A$ and $\mathcal{T}^A$, so we no longer distinguish the co-tractor and tractor bundles in the sequel.

More importantly, the tractor bundle possesses a connection $\nabla^{\mathcal{T}}$ that in any choice of metric $g \in c$ and for any $V^A \in \mathcal{T}^A$ as above is defined by

$$\nabla^{\mathcal{T}}_a V^B \stackrel{g}{=} \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu^b + \delta_a{}^b \rho + \mathsf{P}_a{}^b \sigma \\ \nabla_a \rho - \mathsf{P}_{ab} \mu^b \sigma \end{pmatrix} \tag{3.4}$$

where ∇_a in the right hand side is the Levi-Civita connection of the metric g, and P_{ab} is the corresponding Schouten tensor. In the standard way, this connection is extended to all tensor products of tractor and tensor bundles, the resulting (coupled [1]) connection being referred to as the *tractor connection* throughout this thesis.

The tractor connection is compatible with the tractor metric:

[1]In the terminology from page 71 this is the Levi-Civita-twisted-to-tractor connection.

Proposition 3.1.3. *The standard conformal tractor bundle $\mathcal{T}^A$ equipped with the tractor metric h_{AB} and the tractor connection $\nabla^{\mathcal{T}}$ is a Riemannian tractor bundle in the sense that $\nabla^{\mathcal{T}}_a h_{BC} = 0$.*

A proof will be given shortly in the next section after we develop the necessary tools for calculations.

3.1.2 The standard tractor projectors

The usual vector-like notation quickly becomes inconvenient when the number of tractor indices grows. In order to facilitate calculations we introduce (as in [32]) the following operators

$$\begin{aligned} &X_A\colon \mathcal{E}^A \to \mathcal{E}[1] \\ &Z_A{}^a\colon \mathcal{E}^A \to \mathcal{E}^a[-1] \\ &Y_A\colon \mathcal{E}^A \to \mathcal{E}[-1] \end{aligned} \tag{3.5}$$

These operators act on an arbitrary tractor V^A, as in (3.1), by the rules

$$\begin{aligned} X_A V^A &= \sigma \\ Z_A{}^a V^A &= \mu^a \\ Y_A V^A &= \rho \end{aligned} \tag{3.6}$$

The operators (3.5) can be seen as sections of the corresponding bundles:

$$\begin{aligned} &X_A \in \mathcal{E}^A[1] \\ &Z_A{}^a \in \mathcal{E}_A{}^a[-1] \\ &Y_A \in \mathcal{E}_A[-1] \end{aligned} \tag{3.7}$$

Using the inverse tractor (h^{AB}) and the conformal ($\mathbf{g}_{ab}$) metrics we can construct sections $Y^A \in \mathcal{E}^A[-1]$, $Z^A{}_a \in \mathcal{E}^A{}_a[1]$ and $X^A \in \mathcal{E}^A[1]$.

These sections allow us to write any tractor $V^A \in \mathcal{T}^A$ as

$$V^A = Y^A\sigma + Z^A{}_a\mu^a + X^A\rho \tag{3.8}$$

The operator X^A is invariant, however Y^A and $Z^A{}_a$ transform under rescal-

	X^A	$Z^A{}_b$	Y^A
X_A	0	0	1
Z_{Aa}	0	$\mathbf{g}_{ab}$	0
Y_A	1	0	0

Table 3.1: Contractions of the standard tractor projectors

ing of the metric according to the rules

$$\begin{aligned} \widehat{X^A} &= X^A \\ \widehat{Z^A{}_a} &= Z^A{}_a + X^A \Upsilon_a \\ \widehat{Y^A} &= Y^A - Z^A{}_a \Upsilon^a - \tfrac{1}{2}\Upsilon^a \Upsilon_a X^A \end{aligned} \tag{3.9}$$

These identities easily follow from the tractor rescaling rule (3.2).

The operators X_A, $Z_A{}^a$ and Y_A, as well as X^A, $Z^A{}_a$, Y^A etc, will be referred to as the *standard (conformal) tractor projectors*.

The tractor metric (3.3) can be expressed in terms of the standard tractor projectors by the following identity:

$$h_{AB} = Y_A X_B + Z_A{}^a Z_B{}^b \mathbf{g}_{ab} + X_A Y_B \tag{3.10}$$

When performing calculations, it is convenient to use the tractor metric in the form of a multiplication table (cf. [32]), see Table 3.1.

The tractor connection (3.4) can be expressed (or even defined) in terms of its action on the standard tractor projectors, as it is given below:

$$\begin{aligned} \nabla_b X^A &= Z^A{}_b \\ \nabla_b Z^A{}_a &= -Y^A \mathbf{g}_{ba} - X^A \mathsf{P}_{ba} \\ \nabla_b Y^A &= Z^A{}_a \mathsf{P}_b{}^a \end{aligned} \tag{3.11}$$

We can now demonstrate the power of the tractor projectors.

Proof of Proposition 3.1.3. A straightforward calculation:

$$\begin{aligned}
\nabla_c h_{AB} &= \nabla_c(Y_A X_B + Z_A{}^a Z_B{}^b \mathbf{g}_{ab} + X_A Y_B) \\
&= (\nabla_c Y_A) X_B + Y_A \nabla_c X_B + (\nabla_c Z_A{}^a) Z_B{}^b \mathbf{g}_{ab} \\
&\quad + Z_A{}^a (\nabla_c Z_B{}^b) \mathbf{g}_{ab} + (\nabla_c X_A) Y_B + X_A (\nabla_c Y_B) \\
&= Z_A{}^a \mathsf{P}_{ac} X_B + Y_A Z_B{}^b \mathbf{g}_{bc} + (-Y_A \delta_c{}^a - X_A \mathsf{P}_c{}^a) Z_B{}^b \mathbf{g}_{ab} \\
&\quad + Z_A{}^a (-Y_B \delta_c{}^b - X_B \mathsf{P}_c{}^b) \mathbf{g}_{ab} + Z_A{}^a \mathbf{g}_{ac} Y_B + X_A Z_B{}^b \mathsf{P}_{bc} \\
&= \cancelto{1}{Z_A{}^a X_B \mathsf{P}_{ac}} + \cancelto{2}{Y_A Z_{Bc}} - \cancelto{2}{Y_A Z_{Bc}} - \cancelto{3}{X_A Z_B{}^b \mathsf{P}_{cb}} \\
&\quad - \cancelto{4}{Z_A{}^a Y_B \mathbf{g}_{ac}} - \cancelto{1}{Z_A{}^a X_B \mathsf{P}_{ca}} + \cancelto{4}{Z_A{}^a Y_B \mathbf{g}_{ac}} + \cancelto{3}{X_A Z_B{}^b \mathsf{P}_{bc}} = 0
\end{aligned}$$

where we have used the identities (3.11), the Leibniz rule and the symmetry of the Schouten tensor $\mathsf{P}_{ab} = \mathsf{P}_{ba}$. $\square$

The tractor Laplacian $\Delta^{\mathcal{T}} := \bar{g}^{ab} \nabla^{\mathcal{T}}_a \nabla^{\mathcal{T}}_b$, in the following denoted by Δ, acts on the standard tractor projectors by the rules

$$\begin{aligned}
\Delta X^A &= -Y^A\, n - X^A\, \mathsf{J} \\
\Delta Z^A{}_a &= -2\, Z^A{}_b \mathsf{P}^b{}_a - X^A \nabla^b \mathsf{P}_{ab} \\
\Delta Y^A &= -Y^A\, \mathsf{J} + Z^A{}_a \nabla^b \mathsf{P}_b{}^a - X^A |\mathsf{P}|^2
\end{aligned} \tag{3.12}$$

These formulas easily follow from (3.11), the Leibniz rule and the conformal Bianchi identity (2.17).

The action of the tractor box operator $\square := \Delta + \mathsf{J}w$ on the standard tractor projectors is then given by the identities

$$\begin{aligned}
\square X^A &= -Y^A\, n \\
\square Z^A{}_a &= -2\, Z^A{}_b \mathsf{P}^b{}_a - X^A \nabla_a \mathsf{J} + \mathsf{J} Z^A{}_a \\
\square Y^A &= -2\, Y^A\, \mathsf{J} + Z^A{}_a \nabla^a \mathsf{J} - X^A |\mathsf{P}|^2
\end{aligned} \tag{3.13}$$

Using the Leibniz rule, the identities (3.13), (3.12) and (3.13) one can compute the action of the operators Δ and $\square$ on any tractor. Sometimes the quasi-Leibniz-type rules (as in (1.14)) help to organize the calculations.

Now we introduce the notation for some special tractor bundles that prove to be quite useful for our discussion. In fact, these bundles play an important role in the invariant theory, but we only will be able to mention this in passing.

3.1.3 The adjoint tractor bundle

For our purposes we only need the following simple definition.

Definition 3.1.4. The adjoint tractor bundle $\mathcal{A}_{AA'}$ is defined as the antisymmetric part of the tensor square of the standard tractor bundle, that is

$$\mathcal{A}_{AA'} := \mathcal{T}_{[AA']} \equiv \mathcal{T}_{[A} \otimes \mathcal{T}_{A']}$$

Remark 3.1.5. The use of the *primed* indices in this context is intentional: a pair AA' will be considered as skew. A notation that is more suitable for general k-form tractors can be used as well, namely sequentially numbered indices are assumed to be skewed over. See e.g. [12] or [64] for more details.

Definition 3.1.6. The *adjoint tractor projectors* $\mathbb{Y}_{AA'}{}^{a}$,$\mathbb{W}_{AA'}$,$\mathbb{Z}_{AA'}{}^{aa'}$,$\mathbb{X}_{AA'}{}^{a}$ are the sections defined by the following identities:

$$\begin{aligned} \mathbb{Y}_{AA'}{}^{a} &:= 2Y_{[A}Z_{A']}{}^{a} \\ \mathbb{W}_{AA'} &:= 2Y_{[A}X_{A']} \\ \mathbb{Z}_{AA'}{}^{aa'} &:= 2Z_{[A}{}^{a}Z_{A']}{}^{a'} \\ \mathbb{X}_{AA'}{}^{a} &:= 2X_{[A}Z_{A']}{}^{a} \end{aligned} \tag{3.14}$$

Remark 3.1.7. In the index-free manner this can be stated as

$$\begin{aligned} \mathbb{Y} &= Y \underline{\wedge} Z \\ \mathbb{W} &= Y \underline{\wedge} X \\ \mathbb{Z} &= Z \underline{\wedge} Z \\ \mathbb{X} &= X \underline{\wedge} Z \end{aligned}$$

where $\underline{\wedge}$ in this case denotes the wedge product in the tractor bundle indices (as in [12]).

An arbitrary adjoint tractor $V_{AA'} \in \mathcal{A}_{AA'}$ of weight 0 can be represented as

$$V_{AA'} \stackrel{g}{=} \mathbb{Y}_{AA'}{}^{a}\upsilon_a + \mathbb{W}_{AA'}\omega + \mathbb{Z}_{AA'}{}^{aa'}\zeta_{aa'} + \mathbb{X}_{AA'}{}^{a}\xi_a \tag{3.15}$$

where $\upsilon_a \in \mathcal{E}_a[2]$, $\omega \in \mathcal{E}$, $\zeta_{aa'} \in \mathcal{E}_{[aa']}[2]$ and $\xi \in \mathcal{E}_a[-2]$.

Proposition 3.1.8. *The adjoint projectors rescale conformally as*

$$\begin{aligned}
\widehat{\mathbb{Y}}_{AA'}{}^{a'} &= \mathbb{Y}_{AA'}{}^{a'} + \mathbb{W}_{AA'}\Upsilon^{a'} - \mathbb{Z}_{AA'}{}^{aa'}\Upsilon_a + \mathbb{X}_{AA'}{}^{a}\Big(\Upsilon_a\Upsilon^{a'} - \delta_a{}^{a'}\tfrac{|\Upsilon|^2}{2}\Big)\\
\widehat{\mathbb{W}}_{AA'} &= \mathbb{W}_{AA'} + \mathbb{X}_{AA'}{}^{a}\Upsilon_a\\
\widehat{\mathbb{Z}}_{AA'}{}^{aa'} &= \mathbb{Z}_{AA'}{}^{aa'} + \mathbb{X}_{AA'}{}^{b}(\delta_b{}^{a'}\Upsilon^a - \delta_b{}^{a}\Upsilon^{a'})\\
\widehat{\mathbb{X}}_{AA'}{}^{a'} &= \mathbb{X}_{AA'}{}^{a'}
\end{aligned} \tag{3.16}$$

where, as usual, $\Upsilon_a := \nabla_a \log\Omega$ *for the rescaling* $\widehat{g} = \Omega^2 g$*.*

Proof. Straightforward calculations using (3.9) and (3.14). □

Remark 3.1.9. From these identities we can recover the composition series for the adjoint tractor bundle:

$$\mathcal{A} = \mathcal{E}_a[2] \,\text{⦁}\!\!+ \begin{array}{c}\mathcal{E}\\ \oplus \\ \mathcal{E}_{[aa']}[2]\end{array} \text{⦁}\!\!+ \mathcal{E}_a[-2]$$

where $V_{AA'}$ as above is regarded as

$$V \stackrel{g}{=} \begin{pmatrix} & w & \\ y_a & & x_a \\ & z_{aa'} & \end{pmatrix}$$

Definition 3.1.10 (Adjoint tractor metric)**.** The tractor metric h_{AB} induces a metric $\mathbb{h}_{[AA'][BB']}$ in the adjoint tractor bundle that can be represented as

$$\mathbb{h}_{[AA'][BB']} \stackrel{g}{=} \tfrac{1}{2}\mathbb{Y}_{AA'}{}^{a}\mathbb{X}_{BB'}{}^{b}\mathbf{g}_{ab} - \tfrac{1}{2}\mathbb{W}_{AA'}\mathbb{W}_{AA'} + \tfrac{1}{4}\mathbb{Z}_{AA'}{}^{aa'}\mathbb{Z}_{BB'}{}^{bb'}\mathbf{g}_{ab}\mathbf{g}_{a'b'} + \tfrac{1}{2}\mathbb{X}_{AA'}{}^{a}\mathbb{Y}_{BB'}{}^{b}\mathbf{g}_{ab}$$

The standard tractor connection extends to a connection in the adjoint tractor bundle via the Leibniz rule, and it can be expressed by its action on the adjoint tractor projectors as follows:

Proposition 3.1.11.

$$\begin{aligned}
\nabla_b\mathbb{Y}_{AA'}{}^{a'} &= \mathbb{Z}_{AA'}{}^{aa'}\mathsf{P}_{ab} - \mathbb{W}_{AA'}\mathsf{P}_b{}^{a'}\\
\nabla_b\mathbb{W}_{AA'} &= \mathbb{Y}_{AA'}{}^{a}\mathbf{g}_{ab} - \mathbb{X}_{AA'}{}^{a}\mathsf{P}_{ab}\\
\nabla_b\mathbb{Z}_{AA'}{}^{aa'} &= 2\,\mathbb{Y}_{AA'}{}^{[a}\delta_b{}^{a']} + 2\,\mathbb{X}_{AA'}{}^{[a}\mathsf{P}_b{}^{a']}\\
\nabla_b\mathbb{X}_{AA'}{}^{a'} &= \mathbb{Z}_{AA'}{}^{aa'}\mathbf{g}_{ab} + \mathbb{W}_{AA'}\delta_b{}^{a'}
\end{aligned} \tag{3.17}$$

Proof. Straightforward calculations using (3.11) and (3.14). □

It will be useful in our calculations to have the explicit identities for the action of the tractor Laplacian on the adjoint projectors, so collect them here.

Proposition 3.1.12.

$$\begin{aligned}
\Delta\mathbb{Y}_{AA'}{}^{a'} &= -\mathbb{Y}_{AA'}{}^{a'}\mathsf{J} - \mathbb{W}_{AA'}\nabla^{a'}\mathsf{J} + \mathbb{Z}_{AA'}{}^{aa'}\nabla_a\mathsf{J} + \mathbb{X}_{AA'}{}^{a}\Big[2\,\mathsf{P}^{2a'}_{a} - \delta_a{}^{a'}|\mathsf{P}|^2\Big]\\
\Delta\mathbb{W}_{AA'} &= -2\,\mathbb{W}_{AA'}\mathsf{J} - \mathbb{X}_{AA'}{}^{a}\nabla^b\mathsf{P}_{ab}\\
\Delta\mathbb{Z}_{AA'}{}^{aa'} &= 4\,\mathbb{Z}_{AA'}{}^{c[a}\mathsf{P}_c{}^{a']} + 2\,\mathbb{X}_{AA'}{}^{[a}\nabla^{a']}\mathsf{J}\\
\Delta\mathbb{X}_{AA'}{}^{a'} &= \mathbb{Y}_{AA'}{}^{a'}(2-n) - \mathbb{X}_{AA'}{}^{a'}\mathsf{J}
\end{aligned} \tag{3.18}$$

3.1.4 Invariant tractor operators

We briefly remind (see [26] and [27]) the construction of the basic invariant operators of the conformal tractor calculus.

We have already defined some invariant tractor operators of zeroth differential order, namely $\mathrm{X}_A\colon \mathcal{E}[w] \to \mathcal{T}_A[w]$, $\mathbb{X}_{AA'}{}^{a}\colon \mathcal{T}_a[w] \to \mathcal{A}\otimes\mathcal{T}[w]$ and $\mathsf{X}_{AB}\colon \mathcal{T}[w] \to \mathcal{T}_{(AB)}\otimes\mathcal{T}[w]$. These are the simplest examples of the so called bottom operators (see e.g. [22] and [64]). Informally we say that they place a section into the bottom slot of the resulting tractor.

Operators on densities

Let us introduce the following operator [27]. It can be viewed as an invariant operator on densities of zeroth order.

Definition 3.1.13. The *weight operator* $\mathbf{w}\colon \mathcal{E}[w] \to \mathcal{E}[w]$ is defined as

$$\mathbf{w}(f) = wf \tag{3.19}$$

The significance of the weight operator is based on the following elementary but rather useful fact.

Proposition 3.1.14. *The weight operator satisfies the Leibniz rule:*

$$\mathbf{w}(f_1 f_2) = f_1\mathbf{w}(f_2) + f_2\mathbf{w}(f_1) \tag{3.20}$$

Examining the rescaling behavior of the tractor expression for the covariant derivative of a weighted section $\mathrm{Z}_A{}^a\nabla_a f$. using (2.1) and (3.9), one can come up with the following.

Definition 3.1.15. The *pre-D operator* $\mathfrak{D}_A$ acts on $\mathcal{E}[w]$ by the rule defined in a choice of scale as

$$[\mathfrak{D}^A f]_g = Y^A \mathbf{w} f + Z^A{}_a \nabla^a f \tag{3.21}$$

Remark 3.1.16. This operator was denoted by $\tilde{D}_A$ in [26] and [27]. We have changed the notation in order to have accented versions of it that we use for hypersurface versions of the tractor operators.

Proposition 3.1.17. *The pre-D operator* $\mathfrak{D}^A \colon \mathcal{E}[w] \to \mathcal{E}[w-1]$

1. *satisfies the Leibniz rule:* $\mathfrak{D}_A(f_1 f_2) = f_1 \mathfrak{D}_A f_2 + f_2 \mathfrak{D}_A f_1$
2. *rescales as*

$$\widehat{\mathfrak{D}^A f} = \mathfrak{D}^A f + X^A \left(\Upsilon^a \nabla_a f + \tfrac{|\Upsilon|^2}{2} w f \right) \tag{3.22}$$

Having the conformal deformation accumulated in the bottom slot (i.e. in the X_A term), we can easily construct an invariant operator by "wedging out" the deformation.

Definition 3.1.18. The *double-D operator* $\mathbb{D}_{AP}$ acts on $\mathcal{E}[w]$ according to the formula

$$\mathbb{D}_{AP} f \coloneqq 2 X_{[P} \mathfrak{D}_{A]} f \tag{3.23}$$

Proposition 3.1.19. *The double-D operator* $\mathbb{D}_{A'} \colon \mathcal{E}[w] \to \mathcal{E}[w]$

- *satisfies the Leibniz rule:* $\mathbb{D}_{AP}(f_1 f_2) = f_1 \mathbb{D}_{AP}(f_2) + f_2 \mathbb{D}_{AP}(f_1)$
- *is conformally invariant.*

One minor inconvenience of the double-D operator is that the result of its action has two more tractor indices. This, actually, is an advantage since these indices can be interpreted in terms on the adjoint tractor bundle. More precisely, the double-D operator can be seen as $\mathbb{D} \colon \mathcal{T} \to \mathcal{A} \otimes \mathcal{T}$.

Proposition 3.1.20. *The double-D operator in terms of the adjoint tractor projectors can be expressed in a choice of scale as*

$$\mathbb{D}_{AA'} f \overset{g}{=} \mathbb{W}_{AA'} \mathbf{w} f - \mathbb{X}_{AA'}{}^a \nabla_a f \tag{3.24}$$

Remark 3.1.21. If we adopt the expression from this proposition as the definition of the double-D operator, then we can prove its invariance using the formulas for the conformal rescaling of the adjoint tractor projectors (Proposition 3.1.8):

$$\begin{aligned}\widehat{\mathbb{D}}_{AA'}f =& \widehat{\mathbb{W}}_{AA'}wf - \widehat{\mathbb{X}}_{AA'}{}^{a}\widehat{\nabla}_{a}f \\ =& \mathbb{W}_{AA'}wf + \mathbb{X}_{AA'}{}^{a}\Upsilon_{a}wf - \mathbb{X}_{AA'}{}^{a}(\nabla_{a}f + \Upsilon_{a}wf) = \mathbb{D}_{AA'}f\end{aligned}$$

We can interpret the double-D operator as placing the derivative of f into the bottom slot of an adjoint tractor and compensating the conformal transformation by another slot.

It is possible to construct another invariant operator on densities that adds only one tractor index to the result.

The key idea is to notice that the pre-D operator rescales (3.22) up to a constant coefficient by the same quantity as the box operator on densities 2.13 does. This immediately shows leads to the following

Definition 3.1.22. The *Thomas-D* (or tractor-D) *operator* D_A is defined on $\mathcal{E}[w]$ as

$$\mathrm{D}_A f \coloneqq (n + 2w - 2)\mathfrak{D}_A f - X_A \Box f \tag{3.25}$$

We have recovered the celebrated Thomas-D operator from [6] or [69].

Even it is obvious in our approach, we record its important property.

Proposition 3.1.23. *The Thomas-D operator* $\mathrm{D}_A \colon \mathcal{E}[e] \to \mathcal{E}[w-1]$ *is conformally invariant.*

The operators that we have just described can be extended to a much larger class of domains.

Twisted operators

Let $(\mathcal{F}, D)$ be a vector bundle on M with an invariant connection D.

Definition 3.1.24. A conformally invariant differential operator P defined on $\mathcal{E}[w]$ is said to be *strongly invariant* if it can be extended to act on the weighted bundle $\mathcal{F}[w]$ equipped with the twisted Levi-Civita connection ∇^D so that the new operator denoted as $P^{\mathcal{F}}$ or P^D is again conformally invariant.

Theorem 3.1.25. *The operators* $\mathbf{w}$*,* $\mathbb{D}_{AP}$ *and* D_A *are strongly invariant.*

Proof. The invariance operator $\mathbf{w}$ is obvious. The double-D operator is of the first differential order, and its strong invariance is immediate. The strong invariance of the Thomas-D operator

$$\mathrm{D}^{\mathcal{F}}_{A} f = (n + 2w - 2)\mathfrak{D}^{\mathcal{F}}_{A} f - X_A \Box^{\mathcal{F}} f \tag{3.26}$$

where $\mathfrak{D}^{\mathcal{F}}_{A}$ is the pre-D operator twisted with $\mathcal{F}$, and $\Box^{\mathcal{F}}$ is the twisted box operator, follows from (2.1-⊗) and Proposition 2.1.1. □

Invariant operators on tractors

It was noticed by M.Eastwood that the Thomas-D operator can act on sections of any tractor bundle as the tractor derivative on any weighted tractor rescales by the same rule as the Levi-Civita on densities. T.P.Branson and A.R.Gover [13] have extended the notion of strong invariance to operators acting on sections twisted with arbitrary vector bundles equipped with an invariant connection. This is the point of view, which is extensively used in the present thesis.

Since the tractor connection (3.4) is conformally invariant, the operators $\mathbf{w}$, $\mathbb{D}_{AP}$ and D_A can act invariantly on tractor bundles with any conformal weight.

3.1.5 The action of the tractor operators on the tractor projectors

In explicit calculations it is convenient to have the formulas for action of the operators $\mathfrak{D}_A$, $\mathbb{D}_{AA'}$ and D_A handy. We collect them in the present section. Of course, they are just consequences of the definitions and the Leibniz rule.

The pre-D operator on the tractor projectors

The action of the pre-D operator on the standard tractor projectors is given by the identities

$$\mathfrak{D}_A Y_B = -Y_A Y_B + Z_A{}^a Z_B{}^b \mathsf{P}_{ab} \tag{3.27a}$$

$$\mathfrak{D}_A Z_B{}^b = -2Y_{(A} Z_{B)}{}^b - Z_A{}^a X_B \mathsf{P}_a{}^b \tag{3.27b}$$

$$\mathfrak{D}_A X_B = Y_A X_B + Z_A{}^a Z_{Ba} = h_{AB} - X_A Y_B \tag{3.27c}$$

It is also quite useful to observe that

$$\mathfrak{D}^A Y_A = \mathsf{J} \tag{3.28a}$$
$$\mathfrak{D}^A Z_A{}^b = 0 \tag{3.28b}$$
$$\mathfrak{D}^A X_A = n+1 \tag{3.28c}$$

and

$$Y^A \mathfrak{D}_A f = 0 \tag{3.29a}$$
$$Z^A{}_b \mathfrak{D}_A f = \nabla_b f \tag{3.29b}$$
$$X^A \mathfrak{D}_A f = wf \tag{3.29c}$$

Example 3.1.26. These identities can be used to compute the iterated action of the pre-D operator on a weighted section f of some natural bundle (scalars, tensors, tractors) with weight w (that is tensored with $\mathcal{E}[w]$).

Here is a complete calculation.

$$\begin{aligned}
\mathfrak{D}_B \mathfrak{D}_A f &= \mathfrak{D}_B (Y_A wf + Z_A{}^a \nabla_a f) \\
&= (\mathfrak{D}_B Y_A) wf + Y_A w (\mathfrak{D}_B f) + (\mathfrak{D}_B Z_A{}^a) \nabla_a f + Z_A{}^a \mathfrak{D}_B \nabla_a f \\
&= (-Y_B Y_A + Z_B{}^b Z_A{}^a \mathsf{P}_{ba}) wf + Y_A w (Y_B wf + Z_B{}^b \nabla_b f) \\
&\quad + (-2Y_{(B} Z_{A)}{}^a - Z_B{}^b X_A \mathsf{P}_b{}^a) \nabla_a f + Z_A{}^a (Y_B w \nabla_a f + Z_B{}^b \nabla_b \nabla_a f) \\
&= Y_B Y_A (-wf + w^2 f) \\
&\quad + Y_A Z_B{}^b w \nabla_b f - 2Y_{(B} Z_{A)}{}^a \nabla_a f + Z_A{}^a Y_B w \nabla_a f \\
&\quad + Z_B{}^b Z_A{}^a (\nabla_b \nabla_a f + w \mathsf{P}_{ba} f) \\
&\quad - Z_B{}^b X_A \mathsf{P}_b{}^a \nabla_a f
\end{aligned} \tag{3.30}$$

After some simplifications we get

$$\begin{aligned}
\mathfrak{D}_B \mathfrak{D}_A f = Y_B Y_A (w-1) wf + 2Y_{(B} Z_{A)}{}^a (w-1) \nabla_a f \\
+ Z_B{}^b Z_A{}^a (\nabla_b \nabla_a f + w \mathsf{P}_{ab} f) - Z_B{}^b X_A \mathsf{P}_b{}^a \nabla_a f
\end{aligned} \tag{3.31}$$

As an easy corollary we immediately obtain

$$\mathfrak{D}^A \mathfrak{D}_A f = \Box f \tag{3.32}$$

The double-D operator on the tractor projectors

The following identities will be useful for the concrete calculations in Chapter 4.

The action of the double-D operator on the standard tractor projectors is given by the following identities

$$\mathbb{D}_{AP}Y_B = 2X_{[A}Y_{P]}Y_B - 2X_{[A}Z_{P]}{}^a Z_B{}^b \mathsf{P}_{ab} \tag{3.33a}$$

$$\mathbb{D}_{AP}Z_B{}^b = 2X_{[A}Y_{P]}Z_B{}^b + 2X_{[A}Z_{P]}{}^b Y_B + 2X_{[A}Z_{P]}{}^a X_B \mathsf{P}_a{}^b \tag{3.33b}$$

$$\mathbb{D}_{AP}X_B = 2X_{[P}h_{A]B} = -2X_{[A}h_{P]B} \tag{3.33c}$$

In fact, it is more convenient to have this action expressed in terms of the adjoint tractor projectors, so we also record the following formulæ, that are easily obtained using (3.24):

$$\begin{aligned}
\mathbb{D}_{AA'}Y_B &= -\mathbb{W}_{AA'}Y_B - \mathbb{X}_{AA'}{}^a Z_B{}^b \mathsf{P}_{ab} \\
\mathbb{D}_{AA'}Z_B{}^b &= -\mathbb{W}_{AA'}Z_B{}^b + \mathbb{X}_{AA'}{}^a Y_B \delta_a{}^b + \mathbb{X}_{AA'}{}^a X_B \mathsf{P}_a{}^b \\
\mathbb{D}_{AA'}X_B &= \mathbb{W}_{AA'}X_B - \mathbb{X}_{AA'}{}^a Z_B{}^b \mathbf{g}_{ab}
\end{aligned} \tag{3.34}$$

The double-D operator of the adjoint tractor projectors can be expressed purely in terms of the adjoint tractor projectors. This is again a direct consequence of (3.24) and the definitions (3.14).

$$\begin{aligned}
\mathbb{D}_{AA'}\mathbb{Y}_{BB'}{}^{b'} &= -2\,\mathbb{W}_{AA'}\mathbb{Y}_{BB'}{}^{b'} - \mathbb{X}_{AA'}{}^a \mathbb{Z}_{BB'}{}^{bb'}\mathsf{P}_{ab} + \mathbb{X}_{AA'}{}^a \mathbb{W}_{BB'}\mathsf{P}_a{}^{b'} \\
\mathbb{D}_{AA'}\mathbb{W}_{BB'} &= -\mathbb{X}_{AA'}{}^a \mathbb{Y}_{BB'}{}^b \mathbf{g}_{ab} + \mathbb{X}_{AA'}{}^a \mathbb{X}_{BB'}{}^b \mathsf{P}_{ab} \\
\mathbb{D}_{AA'}\mathbb{Z}_{BB'}{}^{bb'} &= -2\,\mathbb{W}_{AA'}\mathbb{Z}_{BB'}{}^{bb'} + 2\,\mathbb{X}_{AA'}{}^{[b}\mathbb{Y}_{BB'}{}^{b']} - 2\,\mathbb{X}_{AA'}{}^a \mathbb{X}_{BB'}{}^{[b}\mathsf{P}_a{}^{b']} \\
\mathbb{D}_{AA'}\mathbb{X}_{BB'}{}^{b'} &= -\mathbb{X}_{AA'}{}^a \mathbb{W}_{BB'}\delta_a{}^{b'} - \mathbb{X}_{AA'}{}^a \mathbb{Z}_{BB'}{}^{bb'}\mathbf{g}_{ab}
\end{aligned} \tag{3.35}$$

The Thomas-D operator on the tractor projectors

The action of the Thomas-D operator on the standard tractor projectors has quite involved explicit form. We record it here for the sake of completeness and for future references.

Proposition 3.1.27. *The Thomas-D operator acts on the standard tractor projectors according to the following identities:*

$$
\begin{aligned}
\mathrm{D}_A Y_B = &-(n-4)Y_A Y_B + 2X_A Y_B \mathsf{J} + (n-4) Z_A{}^a Z_B{}^b \mathsf{P}_{ab} \\
&- X_A Z_B{}^b \nabla_b \mathsf{J} - X_A X_B |\mathsf{P}|^2 \\
\mathrm{D}_A Z_B{}^b = &-2(n-4) Y_{(A} Z_{B)}{}^b - (n-4) Z_A{}^a X_B \mathsf{P}_a{}^b \\
&+ 2X_A Z_B{}^c (\mathsf{P}_c{}^b + \delta_c{}^b \mathsf{J}) + X_A X_B \nabla^b \mathsf{J} \\
\mathrm{D}_A X_B = &\, n h_{AB}
\end{aligned}
\tag{3.36}
$$

Proof. Using the definition (3.25), the identities(3.27), (3.13) and that the weights of Y_A, $Z_A{}^a$ and X_A are -1, -1 and 1, respectively, we compute:

$$
\begin{aligned}
\mathrm{D}_A Y_B &= (n-2-2)\mathfrak{D}_A Y_B - X_A \Box Y_B \\
&= (n-4)(-Y_A Y_B + Z_A{}^a Z_B{}^b \mathsf{P}_{ab}) - X_A(-2\, Y_B\, \mathsf{J} + Z_B{}^b \nabla_b \mathsf{J} - X_B |\mathsf{P}|^2) \\
&= -(n-4)Y_A Y_B + 2X_A Y_B \mathsf{J} + (n-4) Z_A{}^a Z_B{}^b \mathsf{P}_{ab} - X_A Z_B{}^b \nabla_b \mathsf{J} - X_A X_B |\mathsf{P}|^2
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{D}_A Z_B{}^b &= (n-2-2)\mathfrak{D}_A Z_B{}^b - X_A \Box Z_B{}^b \\
&= (n-4)(-2Y_{(A} Z_{B)}{}^b - Z_A{}^a X_B \mathsf{P}_a{}^b) - X_A(-2\, Z_B{}^c \mathsf{P}_c{}^b - X_B \nabla^b \mathsf{J} - \mathsf{J} Z_B{}^b) \\
&= -2(n-4) Y_{(A} Z_{B)}{}^b - (n-4) Z_A{}^a X_B \mathsf{P}_a{}^b + 2X_A Z_B{}^c (\mathsf{P}_c{}^b + \delta_c{}^b \mathsf{J}) + X_A X_B \nabla^b \mathsf{J}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{D}_A X_B &= n\mathfrak{D}_A X_B - X_A \Box X_B \\
&= n(h_{AB} - X_A Y_B) - X_A(-Y_B n) = n h_{AB}
\end{aligned}
$$
□

Corollary 3.1.28. *The tractor-contracted action of the Thomas-D operator of the standard tractor projectors is given by these expressions*

$$
\begin{aligned}
\mathrm{D}^A Y_A &= (n-2)\mathsf{J} \\
\mathrm{D}^A Z_A{}^b &= 0 \\
\mathrm{D}^A X_A &= n(n+1)
\end{aligned}
\tag{3.37}
$$

The Thomas-D operator does not enjoy the Leibniz rule, so sometimes it is useful to have a replacement that could be called the quasi-Leibniz rule.

Proposition 3.1.29. *For sections φ and ψ with weights w_φ and w_ψ respectively the action of the Thomas-D on their tensor product (denoted by juxta-*

position) satisfies the following identity:

$$\mathrm{D}_A(\varphi\,\psi) = \psi\mathrm{D}_A\varphi + \varphi\mathrm{D}_A\psi + 2\Big(w_\psi\psi\mathfrak{D}_A\varphi + w_\varphi\varphi\mathfrak{D}_A\psi - X_A(\nabla_b\varphi)\nabla^b\psi\Big) \tag{3.38}$$

Proof. A straightforward calculation using the definitions:

$$\begin{aligned}
\mathrm{D}_A(\varphi\,\psi) &= (n+2(w_\varphi+w_\psi)-2)\mathfrak{D}_A(\varphi\,\psi) - X_A\Box(\varphi\,\psi)\\
&= (n+2(w_\varphi+w_\psi)-2)(\psi\mathfrak{D}_A\varphi + \varphi\mathfrak{D}_A\psi) - X_A(\psi\Box\varphi + 2(\nabla_b\varphi)\nabla^b\psi + \varphi\Box\psi)\\
&= \psi\Big((n+2w_\varphi-2)+2w_\psi\Big)\mathfrak{D}_A\varphi - \psi X_A\Box\varphi\\
&\quad + \varphi\Big((n+2w_\psi-2)+2w_\varphi\Big)\mathfrak{D}_A\psi - \varphi X_A\Box\psi - 2X_A(\nabla_b\varphi)\nabla^b\psi
\end{aligned}$$ □

In the real world calculations, for example, in the procedures that we describe in Chapter 4, more useful are the the following identities, that can be verified either directly, using the definition (3.25), or by an application of the contracted versions (3.37) of the identities from Proposition 3.1.27 and the quasi-Leibniz rule (3.38).

Proposition 3.1.30. *Let f be a section of weight w. The following formulas are useful for the calculations of the contracted action of the Thomas-D operator:*

$$\begin{aligned}
\mathrm{D}^A Y_A f &= (n+2w-2)\mathrm{J}f - \Box f\\
\mathrm{D}^A Z_A{}^a f &= (n+2w-2)\nabla^a f\\
\mathrm{D}^A X_A f &= (n+2w+2)(n+w)f
\end{aligned} \tag{3.39}$$

We may informally refer to an application of the identities (3.39) as an *elimination* of tractor indices with the Thomas-D operator.

3.1.6 The curvature tractors

In this section we collect the definitions and some useful identities related to the tractor curvature, the Weyl tractor and the related quantities.

We also give very explicit calculations, which may be useful in mastering the tractor calculus techniques.

The tractor curvature

Definition 3.1.31. The tractor curvature $\Omega_{ab}{}^{C}{}_{D}V^{D}$ is the curvature of the tractor connection (3.4). It is given by the equation

$$\Omega_{ab}{}^{C}{}_{D}V^{D} = 2\nabla_{[a}\nabla_{b]}V^{C} \tag{3.40}$$

To express the tractor curvature explicitly we begin with the following

Lemma 3.1.32. *The action of the tractor curvature $\Omega_{ab}{}^{C}{}_{D}$ on the standard tractor projectors is given by the identities*

$$\begin{aligned}
&\Omega_{ab}{}^{C}{}_{D}Y^{D} = Z^{C}{}_{c}\mathrm{Y}_{ab}{}^{c}\\
&\Omega_{ab}{}^{C}{}_{D}Z^{D}{}_{c} = -Z^{C}{}_{d}\Big(2\,\mathsf{P}^{d}{}_{[a}\mathbf{g}_{b]c} + 2\,\delta^{d}{}_{[a}\mathsf{P}_{b]c}\Big) - X^{C}\mathrm{Y}_{abc}\\
&\Omega_{ab}{}^{C}{}_{D}X^{D} = 0
\end{aligned} \tag{3.41}$$

where $\mathrm{Y}_{ab}{}^{d}$ *is the Cotton tensor* (2.14).

Proof. Straightforward computations:

$$\begin{aligned}
2\nabla_{[a}\nabla_{b]}Y^{C} &= \nabla_{a}\nabla_{b}Y^{C} - (a \leftrightarrow b) = \nabla_{a}(Z^{C}{}_{c}\mathsf{P}^{c}{}_{b}) - (a \leftrightarrow b)\\
&= (\nabla_{a}Z^{C}{}_{c})\mathsf{P}^{c}{}_{b} + Z^{C}{}_{c}\nabla_{a}\mathsf{P}^{c}{}_{b} - (a \leftrightarrow b)\\
&= (-Y^{C}\mathbf{g}_{ac} - X^{C}_{C}\mathsf{P}_{ac})\mathsf{P}^{c}{}_{b} + Z^{C}{}_{c}\nabla_{a}\mathsf{P}^{c}{}_{b} - (a \leftrightarrow b)\\
&= \cancel{-Y^{C}}\mathsf{P}_{ab} - \cancel{X^{C}\mathsf{P}_{ac}\mathsf{P}^{c}{}_{b}} + 2\,Z^{C}{}_{c}\nabla_{[a}\mathsf{P}^{c}{}_{b]}
\end{aligned}$$

$$\begin{aligned}
2\nabla_{[a}\nabla_{b]}Z^{C}{}_{c} &= \nabla_{a}\nabla_{b}Z^{C}{}_{c} - (a \leftrightarrow b)\\
&= \nabla_{a}(-Y^{C}\mathbf{g}_{bc} - X^{C}\mathsf{P}_{bc}) - (a \leftrightarrow b)\\
&= -(\nabla_{a}Y^{C})\mathbf{g}_{bc} - (\nabla_{a}X^{C})\mathsf{P}_{bc} - X^{C}\nabla_{a}\mathsf{P}_{bc} - (a \leftrightarrow b)\\
&= -Z^{C}{}_{d}\mathsf{P}^{d}{}_{a}\mathbf{g}_{bc} - Z^{C}{}_{d}\delta^{d}{}_{a}\mathsf{P}_{bc} - X^{C}\nabla_{a}\mathsf{P}_{bc} - (a \leftrightarrow b)\\
&= -Z^{C}{}_{d}\Big(2\,\mathsf{P}^{d}{}_{[a}\mathbf{g}_{b]c} + 2\,\delta^{d}{}_{[a}\mathsf{P}_{b]c}\Big) - 2\,X^{C}\nabla_{[a}\mathsf{P}_{b]c}
\end{aligned}$$

$$\begin{aligned} 2\nabla_{[a}\nabla_{b]}X^C &= \nabla_a\nabla_b X^C - (a \leftrightarrow b) = \nabla_a Z^C{}_c\delta^c{}_b - (a \leftrightarrow b) \\ &= (-Y^C\mathbf{g}_{ac} - X^C\mathsf{P}_{ac})\delta^c{}_b - (a \leftrightarrow b) \\ &= -Y^C\mathbf{g}_{ac}\delta^c{}_b - X^C\mathsf{P}_{ac}\delta^c{}_b - (a \leftrightarrow b) \\ &= -\underbrace{Y^C\mathbf{g}_{ab}}_{\text{sym. in a,b}} - \underbrace{X^C\mathsf{P}_{ab}}_{\text{sym. in a,b}} - (a \leftrightarrow b) = 0 \end{aligned}$$

□

Theorem 3.1.33. *Tractor curvature is explicitly given in a choice of scale by the expression*

$$\Omega_{ab}{}^C{}_D = Z^C{}_c X_D\,\mathrm{Y}_{ab}{}^c + Z^C{}_c Z_D{}^d \mathrm{W}_{ab}{}^c{}_d - X^C Z_D{}^d \mathrm{Y}_{abd} \tag{3.42}$$

where W_{abcd} *is the Weyl tensor, and* Y_{abd} *is the Cotton tensor* (2.14).

Viewing the tractor curvature as a 2-form with values in the adjoint tractor bundle, we can present (3.42) *as*

$$\Omega_{abCD} = \tfrac{1}{2}\mathbb{Z}_{CD}{}^{cd}\mathrm{W}_{abcd} - \mathbb{X}_{CD}{}^c\mathrm{Y}_{abc} \tag{3.43}$$

Proof. Let us consider an arbitrary tractor $V^C \stackrel{g}{=} Y^C y + Z^C{}_c z^c + X^C x$. Using the Leibniz rule for the curvature (1.12) and that the Levi-Civita connection is torsion free, $2\nabla_{[a}\nabla_{b]}f = 0$, we compute

$$\begin{aligned} 2\nabla_{[a}\nabla_{b]}V^C &= 2y\,\nabla_{[a}\nabla_{b]}Y^C + (2\nabla_{[a}\nabla_{b]}Z^C{}_c)z^c + Z^C{}_c 2\nabla_{[a}\nabla_{b]}z^c + 2x\,\nabla_{[a}\nabla_{b]}X^C \\ &= y\,Z^C{}_c\mathrm{Y}_{ab}{}^c + Z^C{}_c\Big(\mathrm{R}_{ab}{}^c{}_d - 2\,\mathsf{P}^c{}_{[a}\mathbf{g}_{b]d} - 2\,\delta^c{}_{[a}\mathsf{P}_{b]d}\Big)z^d - 2z^d\,X^C\mathrm{Y}_{abd} \end{aligned}$$

Noticing that $\mathrm{R}_{ab}{}^c{}_d - 2\,\mathsf{P}^c{}_{[a}\mathbf{g}_{b]d} - 2\,\delta^c{}_{[a}\mathsf{P}_{b]d} = \mathrm{W}_{ab}{}^c{}_d z^d$ is the Weyl tensor by (1.30), we get

$$\Omega_{ab}{}^C{}_D V^D = Z^C{}_c\,y\,\mathrm{Y}_{ab}{}^c + Z^C{}_c\,z^d\mathrm{W}_{ab}{}^c{}_d - X^C\,z^d\mathrm{Y}_{abd}$$

By (3.6) we have $y = X^D V_D$; $z_d = Z^D{}_d V_D$; $x = Y^D V_D$

Thus we can write

$$\Omega_{ab}{}^C{}_D V^D = Z^C{}_c X_D\,\mathrm{Y}_{ab}{}^c V^D + Z^C{}_c Z_D{}^d\mathrm{W}_{ab}{}^c{}_d V^D - X^C Z_D{}^d\mathrm{Y}_{abd}V^D$$

which is equivalent to the claim.

Rewriting this as

$$\begin{aligned}\Omega_{abCD} &= Z_C{}^c X_D \,\mathrm{Y}_{abc} + Z_C{}^c Z_D{}^d \mathrm{W}_{abcd} - X_C Z_D{}^c \mathrm{Y}_{abc}\\ &= Z_{[C}{}^c Z_{D]}{}^d \mathrm{W}_{abcd} + 2\, Z_{[C}{}^c X_{D]}\, \mathrm{Y}_{abc}\end{aligned}$$

and using (3.14) we obtain (3.43). □

Proposition 3.1.34. *The tractor curvature is conformally invariant*

Proof. The curvature of a conformally invariant connection is indeed invariant.

Alternatively, we may check this by a direct calculation:

$$\begin{aligned}\widehat{\Omega_{abCD}} &= \tfrac{1}{2}\widehat{\mathbb{Z}_{CD}{}^{cd}}\widehat{\mathrm{W}_{abcd}} - \widehat{\mathbb{X}_{CD}{}^{c}}\widehat{\mathrm{Y}_{abc}}\\ &= \tfrac{1}{2}(\mathbb{Z}_{CD}{}^{cd} - 2\,\mathbb{X}_{CD}{}^{[c}\Upsilon^{d]})\mathrm{W}_{abcd} - \mathbb{X}_{CD}{}^{c}(\mathrm{Y}_{abc} - \mathrm{W}_{abcd}\Upsilon^{d})\\ &= \tfrac{1}{2}\mathbb{Z}_{CD}{}^{cd}\mathrm{W}_{abcd} - \cancel{\mathbb{X}_{CD}{}^{c}\mathrm{W}_{abcd}\Upsilon^{d}} - \mathbb{X}_{CD}{}^{c}\mathrm{Y}_{abc} + \cancel{\mathbb{X}_{CD}{}^{c}\mathrm{W}_{abcd}\Upsilon^{d}}\end{aligned}$$

□

The Weyl tractor

The tractor curvature $\Omega_{ab}{}^{C}{}_{D}$ has tensor indices, but we would like to have an invariant section with only tractor indices, which represents the curvature. This would allow us to apply the invariant tractor operators to this section and obtain more invariant sections. Such a tractor W_{ABCD} was proposed by A.R.Gover in [26] and [27] without an explicit derivation. Here we present our version of the calculations based on the adjoint tractor calculus.

Definition 3.1.35. The *tractor expression of the tractor curvature* is

$$\Omega_{ABCD} := Z_A{}^a Z_B{}^b \Omega_{abCD} = \tfrac{1}{2}\mathbb{Z}_{AB}{}^{ab}\Omega_{abCD}$$

Proposition 3.1.36. Ω_{ABCD} *rescales as*

$$\widehat{\Omega_{ABCD}} = \Omega_{ABCD} - \mathbb{X}_{AB}{}^{a}\Upsilon^{b}\Omega_{abCD}$$

Proof. By Proposition 3.1.8, $\frac{1}{2}\widehat{\mathbb{Z}_{AB}{}^{ab}}\Omega_{abCD} = \frac{1}{2}(\mathbb{Z}_{AB}{}^{ab} - 2\,\mathbb{X}_{AB}{}^{[a}\Upsilon^{b]})\Omega_{abCD}$. □

Corollary 3.1.37. *The lifted tractor expression* $3X_{[A'}\Omega_{AB]CD}$ *is invariant.*

Definition 3.1.38. The *Weyl tractor* W_{ABCD} is defined by

$$\mathrm{W}_{ABCD} = \tfrac{3}{n-2}\mathrm{D}^{A'} X_{[A'}\Omega_{AB]CD} \tag{3.44}$$

Definition 3.1.39. The *Bach tensor* B_{bd} is defined by the identity

$$\mathrm{B}_{bd} := \mathsf{P}^{ac}\mathrm{W}_{abcd} + \nabla^a \mathrm{Y}_{abd} \tag{3.45}$$

Proposition 3.1.40. *Explicitly the Weyl tractor is given in a choice of scale by*

$$\begin{aligned}\mathrm{W}_{ABCD} =& \tfrac{n-4}{4}\Big(\mathbb{Z}_{AB}{}^{ab}\mathbb{Z}_{CD}{}^{cd}\mathrm{W}_{abcd} - \mathbb{Z}_{AB}{}^{ab}\mathbb{X}_{CD}{}^{c}\mathrm{Y}_{abc} - \mathbb{X}_{AB}{}^{b}\mathbb{Z}_{CD}{}^{cd}\mathrm{Y}_{cdb}\Big) \\ &+ \mathbb{X}_{AB}{}^{b}\mathbb{X}_{CD}{}^{c}\mathrm{B}_{cb}\end{aligned} \tag{3.46}$$

Corollary 3.1.41. *The Bach tensor is conformally invariant in dimension* $\dim M = 4$.

Proof of Proposition 3.1.40. Using the Thomas-D operator we can eliminate the auxiliary index A' in the lifted tractor expression $3X_{[A'}\Omega_{AB]CD}$

$$\begin{aligned}\mathrm{D}^{A'}3X_{[A'}\Omega_{AB]CD} &= \mathrm{D}^{A'}\Big(X_{A'}\Omega_{ABCD} + X_A\Omega_{BA'CD} + X_B\Omega_{A'ACD}\Big) \\ &= \mathrm{D}^{A'}\Big(X_{A'}\Omega_{ABCD}\Big) + \mathrm{D}^{A'}\Big(X_A\Omega_{BA'CD}\Big) + \mathrm{D}^{A'}\Big(X_B\Omega_{A'ACD}\Big) \\ &= \mathrm{D}^{A'}\Big(X_{A'}\Omega_{ABCD}\Big) - \mathrm{D}^{A'}\Big(\Omega_{A'BCD}X_A\Big) + \mathrm{D}^{A'}\Big(\Omega_{A'ACD}X_B\Big)\end{aligned}$$

Substituting the definition of Ω_{ABCD} we get

$$\begin{aligned}&\mathrm{D}^{A'}3X_{[A'}\Omega_{AB]CD} \\ &= \mathrm{D}^{A'}\Big(X_{A'}\Omega_{ABCD}\Big) - \tfrac{1}{2}\mathrm{D}^{A'}\Big(\mathbb{Z}_{A'B}{}^{ab}\Omega_{abCD}X_A\Big) + \tfrac{1}{2}\mathrm{D}^{A'}\Big(\mathbb{Z}_{A'A}{}^{ab}\Omega_{abCD}X_B\Big) \\ &= \mathrm{D}^{A'}\Big(X_{A'}\Omega_{ABCD}\Big) - \tfrac{1}{2}\mathrm{D}^{A'}\Big(2\,Z_{[A'}{}^{a}Z_{B]}{}^{b}\Omega_{abCD}X_A\Big) + \tfrac{1}{2}\mathrm{D}^{A'}\Big(2\,Z_{[A'}{}^{a}Z_{A]}{}^{b}\Omega_{abCD}X_B\Big) \\ &= \mathrm{D}^{A'}\Big(X_{A'}\Omega_{ABCD}\Big) - \tfrac{1}{2}\mathrm{D}^{A'}\Big(2\,Z_{A'}{}^{a}Z_{B}{}^{b}\Omega_{abCD}X_A\Big) + \tfrac{1}{2}\mathrm{D}^{A'}\Big(2\,Z_{A'}{}^{a}Z_{A}{}^{b}\Omega_{abCD}X_B\Big) \\ &= \mathrm{D}^{A'}\Big(X_{A'}\Omega_{ABCD}\Big) - \mathrm{D}^{A'}\Big(Z_{A'}{}^{a}(X_A Z_B{}^{b} - Z_A{}^{b}X_B)\Omega_{abCD}\Big) \\ &= \mathrm{D}^{A'}\Big(X_{A'}\Omega_{ABCD}\Big) - \mathrm{D}^{A'}\Big(Z_{A'}{}^{a}\mathbb{X}_{AB}{}^{b}\Omega_{abCD}\Big)\end{aligned}$$

Using (3.39) and noticing that $\mathbf{w}(\Omega_{ABCD}) = -2\,\Omega_{ABCD}$ and $\mathbf{w}(\mathbb{X}_{AB}{}^{b}) = 0$

we can write

$$\begin{aligned}
\mathrm{D}^{A'}3X_{[A'}\Omega_{AB]CD} &= \mathrm{D}^{A'}\Big(X_{A'}\Omega_{ABCD}\Big) - \mathrm{D}^{A'}\Big(Z_{A'}{}^{a}\mathbb{X}_{AB}{}^{b}\Omega_{abCD}\Big) \\
&= (n-4+2)(n-2)\Omega_{ABCD} - (n-2)\nabla^{a}(\mathbb{X}_{AB}{}^{b}\Omega_{abCD}) \\
&= (n-2)^2\Omega_{ABCD} - (n-2)(\nabla^{a}\mathbb{X}_{AB}{}^{b})\Omega_{abCD} - (n-2)\mathbb{X}_{AB}{}^{b}\nabla^{a}\Omega_{abCD}
\end{aligned}$$

Let us now compute $\nabla^{a}\Omega_{abCD}$ from the last term in the above display:

$$\begin{aligned}
\nabla^{a}\Omega_{abCD} &= \nabla^{a}\Big(\tfrac{1}{2}\mathbb{Z}_{CD}{}^{cd}\mathrm{W}_{abcd} - \mathbb{X}_{CD}{}^{c}\mathrm{Y}_{abc}\Big) \\
&= \tfrac{1}{2}(\nabla^{a}\mathbb{Z}_{CD}{}^{cd})\mathrm{W}_{abcd} + \tfrac{1}{2}\mathbb{Z}_{CD}{}^{cd}\nabla^{a}\mathrm{W}_{abcd} - (\nabla^{a}\mathbb{X}_{CD}{}^{c})\mathrm{Y}_{abc} - \mathbb{X}_{CD}{}^{c}\nabla^{a}\mathrm{Y}_{abc}
\end{aligned}$$

Using (3.17) we continue

$$\begin{aligned}
\nabla^{a}\Omega_{abCD} &= \tfrac{1}{2}(2\,\mathbb{Y}_{CD}{}^{[c}\mathbf{g}^{d]a} + 2\,\mathbb{X}_{CD}{}^{[c}\mathsf{P}^{d]a})\mathrm{W}_{abcd} + \tfrac{1}{2}\mathbb{Z}_{CD}{}^{cd}\nabla^{a}\mathrm{W}_{abcd} \\
&\quad -(\mathbb{Z}_{CD}{}^{ac} + \mathbb{W}_{CD}\mathbf{g}^{ac})\mathrm{Y}_{abc} - \mathbb{X}_{CD}{}^{c}\nabla^{a}\mathrm{Y}_{abc} \\
&= \cancel{\mathbb{Y}_{CD}{}^{c}\mathbf{g}^{da}\mathrm{W}_{abcd}} + \mathbb{X}_{CD}{}^{c}\mathsf{P}^{da}\mathrm{W}_{abcd} + \tfrac{1}{2}\mathbb{Z}_{CD}{}^{cd}\nabla^{a}\mathrm{W}_{abcd} \\
&\quad -\mathbb{Z}_{CD}{}^{ac}\mathrm{Y}_{abc} - \cancel{\mathbb{W}_{CD}\mathbf{g}^{ac}\mathrm{Y}_{abc}} - \mathbb{X}_{CD}{}^{c}\nabla^{a}\mathrm{Y}_{abc}
\end{aligned}$$

where some terms go away since W_{abcd} and Y_{abc} are totally trace free.

The conformal Bianchi identity (2.15) implies that

$$\nabla^{a}\mathrm{W}_{abcd} = 2(n-3)\nabla_{[c}\mathsf{P}_{d]b} = (n-3)\mathrm{Y}_{cdb}$$

so we get

$$\begin{aligned}
\nabla^{a}\Omega_{abCD} &= \tfrac{1}{2}(n-3)\mathbb{Z}_{CD}{}^{cd}\mathrm{Y}_{cdb} - \mathbb{Z}_{CD}{}^{cd}\mathrm{Y}_{cbd} + \mathbb{X}_{CD}{}^{c}\Big(\mathsf{P}^{da}\mathrm{W}_{abcd} - \nabla^{a}\mathrm{Y}_{abc}\Big) \\
&= \tfrac{1}{2}(n-3)\mathbb{Z}_{CD}{}^{cd}\mathrm{Y}_{cdb} - \mathbb{Z}_{CD}{}^{cd}\mathrm{Y}_{cbd} - \mathbb{X}_{CD}{}^{c}\mathrm{B}_{cb}
\end{aligned}$$

The symmetries $\mathrm{Y}_{[abc]} = 0$ and $\mathrm{Y}_{abc} = -\mathrm{Y}_{bac}$ of the Cotton tensor imply that

$$\mathrm{Y}_{abc} + \mathrm{Y}_{bca} + \mathrm{Y}_{cab} = 0$$

and thus

$$\mathrm{Y}_{cbd} - \mathrm{Y}_{dbc} = \mathrm{Y}_{cbd} + \mathrm{Y}_{bdc} = -\mathrm{Y}_{dcb} = \mathrm{Y}_{cdb}$$

Using this we rewrite the term $\mathbb{Z}_{CD}{}^{cd}\mathrm{Y}_{cbd}$ as

$$\begin{aligned}\mathbb{Z}_{CD}{}^{cd}\mathrm{Y}_{cbd} &= 2Z_{[C}{}^{c}Z_{D]}{}^{d}\mathrm{Y}_{cbd} = Z_{C}{}^{c}Z_{D}{}^{d}2\mathrm{Y}_{[c|b|d]} \\ &= Z_{C}{}^{c}Z_{D}{}^{d}\Big(\mathrm{Y}_{cbd} - \mathrm{Y}_{dbc}\Big) = Z_{C}{}^{c}Z_{D}{}^{d}\mathrm{Y}_{cdb} = \tfrac{1}{2}\mathbb{Z}_{CD}{}^{cd}\mathrm{Y}_{cdb}\end{aligned}$$

Hence we get

$$\nabla^{a}\Omega_{abCD} = \tfrac{1}{2}(n-3)\mathbb{Z}_{CD}{}^{cd}\mathrm{Y}_{cdb} - \tfrac{1}{2}\mathbb{Z}_{CD}{}^{cd}\mathrm{Y}_{cdb} - \mathbb{X}_{CD}{}^{c}\mathrm{B}_{cb}$$

and thus

$$\nabla^{a}\Omega_{abCD} = \tfrac{1}{2}(n-4)\mathbb{Z}_{CD}{}^{cd}\mathrm{Y}_{cdb} - \mathbb{X}_{CD}{}^{c}\mathrm{B}_{cb}$$

Substituting our intermediate results into all the terms of the expression for $\mathrm{D}^{A'}3X_{[A'}\Omega_{AB]CD}$, which we have obtained above. we see that

$$\begin{aligned}3\mathrm{D}^{A'}X_{[A'}\Omega_{AB]CD} = (n-2)^{2}\Omega_{ABCD} &- (n-2)(\mathbb{Z}_{AB}{}^{ab} + \cancel{\mathbb{W}_{AB}\mathbf{g}^{ab}})\Omega_{abCD} \\ &- (n-2)\mathbb{X}_{AB}{}^{b}\Big[\tfrac{1}{2}(n-4)\mathbb{Z}_{CD}{}^{cd}\mathrm{Y}_{cdb} - \mathbb{X}_{CD}{}^{c}\mathrm{B}_{cb}\Big]\end{aligned}$$

Collecting the terms and simplifying, we get

$$\mathrm{W}_{ABCD} = (n-4)\Omega_{ABCD} - \tfrac{n-4}{4}\mathbb{X}_{AB}{}^{b}\mathbb{Z}_{CD}{}^{cd}\mathrm{Y}_{cdb} + \mathbb{X}_{AB}{}^{b}\mathbb{X}_{CD}{}^{c}\mathrm{B}_{cb}$$

Recalling that $\Omega_{abCD} = \tfrac{1}{2}\mathbb{Z}_{CD}{}^{cd}\mathrm{W}_{abcd} - \mathbb{X}_{CD}{}^{c}\mathrm{Y}_{abc}$ and thus

$$\Omega_{ABCD} = \tfrac{1}{2}\mathbb{Z}_{AB}{}^{ab}\Omega_{abCD} = \tfrac{1}{4}\mathbb{Z}_{AB}{}^{ab}\mathbb{Z}_{CD}{}^{cd}\mathrm{W}_{abcd} - \tfrac{1}{2}\mathbb{Z}_{AB}{}^{ab}\mathbb{X}_{CD}{}^{c}\mathrm{Y}_{abc}$$

we recover the sought expression (3.46) for the the Weyl tractor. □

Lifted tractor expressions

A (tractor-valued) quantity with some tensor indices can be embedded into a section of a pure tractor bundle by contracting each tensor index, say, a with the corresponding tractor projector, $Z_{A}{}^{a}$ in our case. The resulting tractor expression need not to be invariant, but because the projector $Z_{A}{}^{a}$ rescales through adding $X_{A}\Upsilon^{a}$, the tractor expression can be made invariant by additionally skewing over the index A (in the considered case) with one more $X_{A'}$ (where we have used the primed A just to keep the correspondence with the original index). In fact, if the tensor has additional tensor symmetries, it may require less X-s to eliminate the non-invariance. The invariant tractors constructed this way are called the *lifted tractor expressions* for the corresponding tensors, that may have values in tractor bundles.

This procedure is better explained for the concrete examples.

We have already seen the tractor expression Ω_{ABCD} for the tractor curvature Ω_{abCD}, and the lifted tractor expression $3X_{[A'}\Omega_{AB]CD}$ for it.

The tractor expression for the Weyl tensor W_{abcd} is

$$C_{ABCD} = Z_A{}^a Z_B{}^b Z_C{}^c Z_D{}^d \mathrm{W}_{abcd}$$

,

Using the curvature tensor symmetries we see that the tractor $C_{A'ABCDD'}$ defined as

$$C_{A'ABCDD'} = 9\, X_{[A'}C_{AB]\,[CD}X_{D']} \tag{3.47}$$

is invariant. It is called the lifted tractor expression for the Weyl tensor.

Similarly, for the Cotton tensor Y_{abd} we form the tractor expression

$$C_{ABD} = Z_A{}^a Z_B{}^b Z_D{}^d \mathrm{Y}_{abd}$$

and the symmetries of Y_{abd} ensure that the *lifted tractor expression for the Cotton tensor* $C_{A'ABDD'}$ defined as

$$C_{A'ABDD'} = 6\, X_{[A'}C_{AB]\,[D}X_{D']} \tag{3.48}$$

is invariant.

We shall use these lifted tractor expressions in the procedure of constructing the so-called quasi-Weyl invariants in Chapter 4.

3.2 Tractor calculus on hypersurfaces

For a hypersurface Σ in a manifold M of dimension $\dim M > 3$ we can consider two bundles that arise as the standard conformal tractor bundles of the respective conformal structures on M and Σ. Recall that we use the induced conformal structure on the hypersurface.

The first bundle is the restriction onto the hypersurface Σ of the tractor bundle $\mathbb{T}M$ of the background manifold M. We call this bundle the ambient tractor bundle. It is defined whenever $\dim M \geq 3$.

The second bundle is the standard tractor bundle $\mathbb{T}\Sigma$ of the hypersurface itself, and for it to be defined defined we need to require that $\dim \Sigma \geq 3$, or, equivalently, $\dim M > 3$.

When the background dimension $\dim M = 3$ there is no conformal structure on the hypersurface Σ because $\dim \Sigma = 2$, but the embedding provides enough additional structure to define a suitable bundle as a replacement of the intrinsic tractor bundle [11]. In fact, we shall use the same construction to define the intrinsic tractor bundle of the hypersurface in both cases $\dim M = 3$ and $\dim M > 3$.

3.2.1 The ambient tractor bundle

As we said earlier, we identify the hypersurface with the image of the inclusion $\iota\colon \Sigma \to M$ which is according to our assumptions is an embedding. The pullback bundle $\iota^*\mathcal{T}$ along the embedding is identified with the restriction $\mathcal{T}|_\Sigma$ of the background bundle $\mathcal{T}$ onto the hypersurface Σ. In the spirit of our conventions we may use $\underline{\mathcal{T}}$ as the core symbol for the ambient tractor bundle using the underline to emphasize the relation to the pullback construction.

As we have already noticed in the section on the pullback connection in Chapter 1, the index range for the pullback bundle can be reused because the fibers of both bundles are isomorphic as vector spaces and, in particular, have the same rank. Thus, we continue using the initial Latin capitals $\{A, B, C, \dots\}$ as the index range for the ambient tractor bundle.

Definition 3.2.1. The pullback bundle $\underline{\mathcal{T}}^A$ of the background tractor bundle $\mathcal{T}^A$ along the inclusion $\iota\colon \Sigma \to M$ is termed the *ambient tractor bundle* of the hypersurface Σ. It is well defined for M of $\dim M \geq 3$.

In practice, when working with hypersurfaces, we often suppress the restriction (pullback) from the notation, so that eventually $\mathcal{T}^A$ may be used instead of $\mathcal{T}^A|_\Sigma$ or $\underline{\mathcal{T}}^A$. The actual meaning of the symbol $\mathcal{T}^A$ should be then clear from the context.

The ambient tractor bundle $\underline{\mathcal{T}}^A$ receives the pullback connection $\iota^*\nabla^{\mathcal{T}}$ denoted again by $\underline{\nabla}_a$. Here $\nabla^{\mathcal{T}}$ means the tractor connection of the background tractor bundle $\mathcal{T}^A$.

Definition 3.2.2. The pullback connection $\underline{\nabla}_a$ in the ambient tractor bundle $\underline{\mathcal{T}}^A$ is called the *ambient tractor connection* along the hypersurface.

The ambient connection is thus a map $\underline{\nabla}_b\colon \underline{\mathcal{T}}^A \to \overline{\mathcal{E}}_b \otimes \underline{\mathcal{T}}^A$. We always view the ambient connection as coupled to the Levi-Civita connection of of the intrinsic metric on the hypersurface, so it is really defined in a choice of a scale. Nevertheless, since the background tractor connections is invariant with respect to the conformal rescalings of the background metric, and the tangential projection operator $\Pi_a{}^b$ is conformally invariant, we obviously have the following property.

Proposition 3.2.3. *The ambient tractor connection $\underline{\nabla}_a$ along the hypersurface is conformally invariant when acting on ambient tractors of zero conformal weight.*

The ambient tractor bundle $\underline{\mathcal{T}}^A$ is equipped with the restriction onto Σ of the metric in the background tractor bundle $\mathcal{T}^A$.

Definition 3.2.4. The restriction of the background tractor metric h_{AB} onto the hypersurface is a metric in the ambient tractor bundle and is termed the *ambient tractor metric*. Again, we suppress references to the restriction from the notation.

When working with hypersurfaces, we shall treat the standard tractor projectors Y^A, $Z^A{}_a$ and X^A as sections of the ambient tractor bundle $\underline{\mathcal{T}}^A \equiv \mathcal{T}^A|_\Sigma$ corresponding to a choice of the ambient scale.

Definition 3.2.5. In this context Y^A, $Z^A{}_a$ and X^A will be termed the *ambient tractor projectors*.

The ambient tractor connection acts on the ambient tractor by the same formulas as in (3.11) but the index b is additionally projected to the hypersurface.

More precisely, let us introduce the notation $\underline{\mathsf{P}}_a{}^b$ for the ambient Schouten tensor with the first index projected tangentially onto the hypersurface:

$$\underline{\mathsf{P}}_a{}^b \coloneqq \Pi^{a'}{}_a \mathsf{P}_{a'}{}^b \tag{3.49}$$

When we use the notation $\underline{\mathsf{P}}$ and the like, we always regard the underlined quantities as the ambient (tensor-, tractor-)valued k-forms on the hypersurface, so in this case $\underline{\mathsf{P}}_a{}^b$ is seen as a section of $\overline{\mathcal{E}}_a \otimes \mathcal{E}^b$.

Using this notation we record the action of the $\underline{\nabla}_a$ on X^B, $Z^B{}_b$ and Y^B as follows.

Proposition 3.2.6. *The action of the ambient tractor connection $\underline{\nabla}$ on the standard ambient tractor projectors is given by the following identities:*

$$\begin{aligned}
\underline{\nabla}_a X^B &= Z^B{}_b \Pi^b{}_a \\
\underline{\nabla}_a Z^B{}_b &= -Y^B \bar{\mathbf{g}}_{ab} - X^B \underline{\mathsf{P}}_{ab} \\
\underline{\nabla}_a Y^B &= Z^B{}_b \underline{\mathsf{P}}_a{}^b
\end{aligned} \tag{3.50}$$

The intrinsic tractor metric $\bar{\mathbf{g}}_{ab}$ in the above identities should be viewed as a section of $\overline{\mathcal{E}}_a \otimes \mathcal{E}^b$ too, but because $\Pi^{a'}{}_a \mathbf{g}_{a'b} \mathrm{N}^b = \Pi^{a'}{}_a \mathrm{N}_{a'} = 0$, both indices of $\Pi^{a'}{}_a \mathbf{g}_{a'b}$ are tangential, so no new notation such as $\underline{\bar{\mathbf{g}}}_{ab}$ is needed. Nevertheless, sometimes it is convenient to have a symbol for $\Pi^{a'}{}_a \delta_{a'}{}^b$ in calculations. We shall define

$$\underline{\delta}_a{}^b \coloneqq \Pi^{a'}{}_a \delta_{a'}{}^b \tag{3.51}$$

and this notation will be frequently used later. We view $\underline{\delta}_a{}^b$ as a section $\underline{\delta}_a{}^b \in \overline{\mathcal{E}}_a \otimes \mathcal{E}^b$. The ambiguity of this notation is inevitable, and a due care is needed when dealing with it. We shall always comment on the meaning in danger of confusion.

Even though the properties of the ambient tractor connection $\underline{\nabla}_a$ and the ambient tractor metric h_{AB} are now immediate consequences of the standard constructions in the pullback bundle, we shall record them for the future references.

Proposition 3.2.7. *The ambient tractor bundle $\underline{\mathcal{T}}^A$ equipped with the ambient tractor connection $\underline{\nabla}_a$ and the ambient tractor metric h_{AB} is a Riemannian tractor bundle in the sense that the metric is compatible with the connection,*

$$\underline{\nabla}_a h_{BC} = 0$$

3.2.2 The normal tractor

The hypersurface Σ inherits a conformal structure from the background manifold M and so it has its own tractor bundle $\mathbb{T}\Sigma$ which has the rank $\bar{n}+2 = n+1$. Since the rank of $\mathbb{T}M$ is $n+2$, the relation between these bundles is not immediately clear. It would be desirable to have a way to identify the hypersurface tractor bundle with a subbundle of the ambient tractor bundle $\underline{\mathcal{T}} \equiv \mathbb{T}M|_\Sigma$ similar to what we have for the tangent bundle of the hypersurface. A solution that we adopt here was first suggested by T.P.Branson and A.R.Gover in [11].

Definition 3.2.8. The *normal tractor* N^A (introduced in [6]) is a section of the ambient tractor bundle along the hypersurface that in a choice of the ambient scale is given by

$$\mathrm{N}^A \overset{g}{=} Z^A{}_a \mathrm{N}^a - X^A \mathbf{H} \tag{3.52}$$

where N^a is the unit normal along the hypersurface regarded as a section of $\mathcal{E}^a[-1]$, and $\mathbf{H}$ is the mean curvature density.

Notice that by the definition the normal tractor (3.52) has the conformal weight zero.

Proposition 3.2.9. *The normal tractor* N^A *is conformally invariant along the hypersurface,* $\widehat{\mathrm{N}^A} = \mathrm{N}^A$*, and has the unit length with respect to the ambient tractor metric,* $\mathrm{N}_A\mathrm{N}^A = 1$.

Proof. The invariance is verified by a quick calculation

$$\widehat{\mathrm{N}^A} = \widehat{Z^A{}_a}\mathrm{N}^a - X^A\widehat{\mathbf{H}} = (Z^A{}_a + X^A\Upsilon_a)\mathrm{N}^a - X^A(\mathbf{H} + \Upsilon_a\mathrm{N}^a) = \mathrm{N}^A$$

Similarly, we check the length

$$(Z_A{}^b\mathrm{N}_b - X_A\mathbf{H})(Z^A{}_a\mathrm{N}^a - X^A\mathbf{H}) = \delta_a{}^b\mathrm{N}_b\mathrm{N}^a = \mathrm{N}_a\mathrm{N}^a = 1 \qquad \square$$

Definition 3.2.10. The line bundle generated by the normal tractor N^A is called the *tractor tractor bundle* $\mathcal{N}^A$ along the hypersurface.

As in the case of the (tensor) normal bundle $\mathcal{N}^a$ of the hypersurface, the tractor normal bundle of the hypersurface is equipped with the flat connection, called the *tractor normal connection* and denoted again by the symbol $\nabla^{\mathcal{N}}$.

3.2.3 The tractor projection operators

Following our usual strategy, which we have described in Section 1.1.13, we define the projection operators acting in the ambient tractor bundle of Σ.

Definition 3.2.11. The *tractor normal projection operator* $\pi^\perp$ is denoted by $\mathrm{N}_A{}^B$ and defined as

$$\mathrm{N}_A{}^B \stackrel{g}{=} \mathrm{N}_A\mathrm{N}^B \tag{3.53}$$

Proposition 3.2.12. *The tractor normal projection operator on* $\underline{\mathcal{T}}$ *is a linear conformally invariant operator of the conformal weight zero. It is idempotent*

$$\mathrm{N}_A{}^B\mathrm{N}_B{}^C = \mathrm{N}_A{}^C$$

and its image is the normal tractor bundle $\mathcal{N}$.

The following expressions of the action of the tractor normal projection operator on the ambient tractor projectors are useful in the calculations.

Proposition 3.2.13. *The action of the tractor normal projection operator on the ambient standard conformal tractor projectors is given by the following identities*

$$\begin{aligned} \mathrm{N}_A{}^B Y^A &= -\mathbf{H}\mathrm{N}^B \\ \mathrm{N}_A{}^B Z^A{}_a &= \mathrm{N}_a\mathrm{N}^B \\ \mathrm{N}_A{}^B X^A &= 0 \end{aligned} \tag{3.54}$$

Proof. Straightforward calculations using Table 3.1:

$$\begin{aligned} \mathrm{N}_A{}^B Y^A &= \mathrm{N}_A\mathrm{N}^B Y^A = (Z_A{}^{a'}\mathrm{N}_{a'} - X_A\mathbf{H})\mathrm{N}^B Y^A = -\mathbf{H}\mathrm{N}^B \\ \mathrm{N}_A{}^B Z^A{}_a &= (Z_A{}^{a'}\mathrm{N}_{a'} - X_A\mathbf{H})\mathrm{N}^B Z^A{}_a = \mathrm{N}_a\mathrm{N}^B \\ \mathrm{N}_A{}^B X^A &= (Z_A{}^{a'}\mathrm{N}_{a'} - X_A\mathbf{H})\mathrm{N}^B X^A = 0 \end{aligned}$$ □

A complimentary operator to the normal tractor projection operator $\mathrm{N}_A{}^B$ is the tractor projection operator $\Pi^A{}_B$.

Definition 3.2.14. The *tractor projection operator* $\pi^\top$ on the ambient tractor bundle $\underline{\mathcal{T}}$ is denoted by $\Pi^A{}_B$ and defined as

$$\Pi^A{}_B \coloneqq \delta^A{}_B - \mathrm{N}^A{}_B \tag{3.55}$$

where $\delta^A{}_B$ is the identity operator $\delta^A{}_B \colon \underline{\mathcal{T}}^B \to \underline{\mathcal{T}}^A$.

Proposition 3.2.15. *The tractor projection operator is a linear conformally invariant operator of zero conformal weight. It is idempotent*

$$\Pi^A{}_B\Pi^B{}_C = \Pi^A{}_C$$

and annihilates the normal tractor bundle

$$\Pi^A{}_B\mathrm{N}^A = 0$$

.

Proposition 3.2.16. *The action of the tractor projection operator $\Pi^A{}_B$ on the ambient standard conformal tractor projectors is given by the following* *identities*

$$\begin{aligned}\Pi_A{}^B Y^A &= Y^B + Z^B{}_b \mathbf{H}\mathrm{N}^b - X^B \mathbf{H}^2 \\ \Pi_A{}^B Z^A{}_a &= Z^B{}_b \Pi^b{}_a + X^B \mathbf{H}\mathrm{N}_a \\ \Pi_A{}^B X^A &= X^B\end{aligned} \tag{3.56}$$

Proof. Straightforward calculations using (3.54) yield

$$\begin{aligned}\Pi_A{}^B Y^A &= Y^B + \mathbf{H}\mathrm{N}^B = Y^B + Z^B{}_b \mathbf{H}\mathrm{N}^b - X^B \mathbf{H}^2 \\ \Pi_A{}^B Z^A{}_a &= Z^B{}_a - \mathrm{N}_a \mathrm{N}^B = Z^B{}_a - Z^B{}_b \mathrm{N}^b \mathrm{N}_a + X^B \mathbf{H}\mathrm{N}_a = Z^B{}_b \Pi^b{}_a + X^B \mathbf{H}\mathrm{N}_a \\ \Pi_A{}^B X^A &= X^B\end{aligned}$$

□

3.2.4 The intrinsic tractor bundle

As we have noticed above, the image of the tractor normal projection operator is the tractor normal bundle of the hypersurface. It turns out that the image of the tractor projection operator can be identified with the tractor bundle of the hypersurface by a conformally invariant isometric isomorphism.

Theorem 3.2.17. *The image $\Pi^{\mathcal{T}}(\underline{\mathcal{T}})$ of the tractor projection operator is isometrically isomorphic to the hypersurface tractor bundle $\mathbb{T}\Sigma$' where the isomorphism $\mathbb{T}\Sigma \to (\mathrm{N}^A)^\perp \equiv \overline{\mathcal{T}}^A$ given by*

$$\begin{aligned} Y^A_\Sigma &\mapsto \bar{Y}^A = Y^A + Z^A{}_a \mathbf{N}^a \mathbf{H} - \tfrac{\mathbf{H}^2}{2} X^A \\ Z^A_{\Sigma a} &\mapsto \bar{Z}^A{}_a = Z^A{}_b \Pi^b{}_a \\ X^A_\Sigma &\mapsto \bar{X}^A = X^A \end{aligned} \tag{3.57}$$

This theorem justifies the following definition

Definition 3.2.18. The *intrinsic tractor bundle* $\overline{\mathcal{T}}^A$ of the hypersurface Σ is a subbundle of the ambient tractor bundle of Σ defined as the image of the tractor projection operator $\Pi_A{}^B \colon \underline{\mathcal{T}}^A \to \underline{\mathcal{T}}^B$.

Equivalently, we may define the intrinsic tractor bundle as the annihilator $(\mathrm{N}^A)^\perp := \{V^A \in \underline{\mathcal{T}}^A | \mathrm{N}_A V^A = 0\}$ of the normal tractor with respect to the ambient tractor metric h_{AB}.

Proof of Theorem 3.2.17. This result is essentially due to Branson and Gover [11]. We give an elementary proof following [34] and [68].

Since the normal tractor bundle $\mathcal{N}^A$ has the rank 1, the direct sum decomposition $\underline{\mathcal{T}}^A = \overline{\mathcal{T}}^A \oplus \mathcal{N}^A$ shows that the image of the tractor projector has the same rank as the hypersurface tractor bundle $\mathbb{T}\Sigma$, that is $n + 1$.

In order to construct an isomorphism that preserves the tractor metric we may try to find sections $\bar{Y}^A$, $\bar{Z}^A a$ and $\bar{X}^A$ that generate the image of $\Pi_B{}^A$ in the sense that any V^A with $\mathrm{N}_A V^A = 0$ is represented as

$$V^A = \bar{Y}^A \sigma + \bar{Z}^A a \mu^a + \bar{X}^A \rho$$

for some $\sigma \in \mathcal{E}[1]$, $\mu^a \in \mathcal{E}^a[-1]$ and $\rho \in \mathcal{E}[-1]$, and such that the map

$$Y^A_\Sigma \mapsto \bar{Y}^A, Z^A_\Sigma a \mapsto \bar{Z}^A a, X^A_\Sigma \mapsto \bar{X}^A$$

is injective, whereas $\bar{Y}^A$, $\bar{Z}^A a$ and $\bar{X}^A$ satisfy the same properties as the hypersurface tractor projectors Y^A_Σ, $Z^A_\Sigma a$ and X^A_Σ. More precisely, we wish that the contractions of the $\bar{Y}^A$, $\bar{Z}^A a$ and $\bar{X}^A$ were given by a table similar to the Table 3.1 with the intrinsic quantities used, and the conformal rescaling rules of $\bar{Y}^A$, $\bar{Z}^A a$ and $\bar{X}^A$ agreed with the corresponding rules for Y^A_Σ, $Z^A_\Sigma a$, X^A_Σ.

Examining the identities (3.56) we see that X^A and $Z^A{}_b \Pi^b a$ are suitable candidates for $\bar{X}^A$ and $\bar{Z}^A a$ respectively.

To find a section $\bar{Y} A$ we may assume that

$$\bar{Y}^A = Y^A \alpha + Z^A{}_a \beta^a + X^A \gamma$$

The condition $\bar{X}_A \bar{Y}^A = 1$ yields $\alpha = 1$. From $\bar{Z}_{Ab} \bar{Y}^A = 0$ we get $\beta^a = \beta \mathrm{N}^a$ for some $\beta \in \mathcal{E}[-1]$. Plugging all that in the condition $\bar{Y}_A \bar{Y}^A = 0$ we see that

$$(Y_A + Z_{Aa} \beta \mathrm{N}^a + X_a \gamma)(Y^A + Z^A{}_a \beta \mathrm{N}^a + X^A \gamma) = 0$$

or, $\gamma = -\frac{\beta^2}{2}$.

It remains to notice that

$$\mathrm{N}_A \bar{Y}^A = (Z_A{}^{a'} \mathrm{N}_{a'} - X_A \mathbf{H})(Y^A + Z^A{}_a \beta \mathrm{N}^a - X^A \frac{\beta^2}{2}) = 0$$

implies $\beta = \mathbf{H}$.

The mapping (3.57) is clearly is linear and injective.

Straightforward calculations using (3.9) and $\Pi^a{}_b \Upsilon^b = \overline{\Upsilon}^a$ where $\Upsilon_a = \nabla_a \log \Omega$

and $\overline{\Upsilon}_a = \overline{\nabla} \log \overline{\Omega}$ for $\overline{\Omega} = \Omega|_\Sigma$, and $\widehat{g_{ab}} = \Omega^2 g_{ab}$, show that

$$\begin{aligned} \widehat{\bar{Y}^A} &= \bar{Y}^A - \bar{Z}^A{}_a \overline{\Upsilon}^a - \bar{X}^A \tfrac{|\overline{\Upsilon}|^2}{2} \\ \widehat{\bar{Z}^A{}_a} &\mapsto \bar{Z}^A{}_a + \bar{X}^A \overline{\Upsilon}_a \\ \widehat{\bar{X}^A} &\mapsto \bar{X}^A \end{aligned} \tag{3.58}$$

so that the rescaling rules of $\bar{Y}^A$, $\bar{Z}^A{}_a$ and $\bar{X}^A$ agree with those for the tractor projectors Y^A_Σ, $Z^A_{\Sigma\,a}$ and X^A_Σ in the "genuine" tractor bundle $\mathbb{T}\Sigma$. □

Remark 3.2.19. From now on we shall *identify* the hypersurface tractor bundle $\mathbb{T}\Sigma$ with the bundle $(\mathrm{N}^A)^\perp$ using the map (3.57).

Definition 3.2.20. The sections $\bar{Y}^A$, $\bar{Z}^A{}_a$, $\bar{X}^A$ defined in the right hand side of (3.57) will be termed the *intrinsic tractor projectors*.

Using the isomorphism from Theorem 3.2.17 we translate the tractor metric and the tractor connection from $\bar{T}\Sigma$ into the bundle $\overline{\mathcal{T}}^A$ to act on the intrinsic tractor projector by the same formulas as we have for Y^A_Σ, $Z^A_{\Sigma\,a}$. We record this as the following definitions.

Definition 3.2.21. The *intrinsic tractor metric* $\bar{h}_{AB}$ is a section of $\mathcal{T}_{(AB)}$ that in a choice of the ambient scale $g \in c$ is given by the formula

$$\bar{h}_{AB} = \bar{Y}_A \bar{X}_B + \bar{Z}_A{}^a \bar{Z}_B{}^b \bar{\mathbf{g}}_{ab} + \bar{X}_A \bar{Y}_B$$

It is clear that actually $\bar{h}_{AB} \in \overline{\mathcal{T}}_{(AB)}$ where $\overline{\mathcal{T}}_{(AB)} \leq \mathcal{T}_{(AB)}$.
Likewise, we define the tractor connection in $\overline{\mathcal{T}}^A$.

Definition 3.2.22. The *intrinsic tractor connection* $\overline{\nabla}_a$ in $\overline{\mathcal{T}}^A$ is given in a choice of scale by its action on the intrinsic tractor projectors as follows:

$$\begin{aligned} \overline{\nabla}_b \bar{Y}^A &= \bar{Z}^A{}_a \overline{\mathsf{P}}^a{}_b \\ \overline{\nabla}_b \bar{Z}^A{}_a &= -\bar{Y}^A \bar{\mathbf{g}}_{ba} - \bar{X}^A \overline{\mathsf{P}}_{ba} \\ \overline{\nabla}_b \bar{X}^A &= \bar{Z}^A{}_b \end{aligned} \tag{3.59}$$

where $\overline{\mathsf{P}}_{ab}$ is the Schouten tensor of the intrinsic metric $\bar{g}_{ab}$ on the hypersurface Σ.

Proposition 3.2.23. *The bundle $\overline{\mathcal{T}}^A$ equipped with the metric $\bar{h}_{AB}$ and the connection $\overline{\nabla}_a$ as above is a Riemannian vector bundle over the hypersurface Σ in a conformal manifold (M, c).*

Proof. We only need to verify that $\overline{\nabla}_a \bar{h}_{BC} = 0$, but the proof of Proposition 3.1.3 on page 89 is literally repeated with the corresponding intrinsic quantities used. □

The following expressions are particularly useful for our further investigations.

Proposition 3.2.24. *The action of the tractor projector operator on the ambient tractor projectors is given in terms of the intrinsic tractor projectors by the identities:*

$$\begin{aligned} \Pi^A{}_B X^B &\stackrel{\bar{g}}{=} \bar{X}^A \\ \Pi^A{}_B Z^B{}_b &\stackrel{\bar{g}}{=} \bar{Z}^A{}_a + \bar{X}^A \, \mathbf{H} \mathrm{N}_b \\ \Pi^A{}_B Y^B &\stackrel{\bar{g}}{=} \bar{Y}^A - \bar{X}^A \, \tfrac{\mathbf{H}^2}{2} \end{aligned} \tag{3.60}$$

where we use the scale afforded by the metric $\bar{g}$ induced from a choice of scale g in the ambient conformal structure.

Proof. The proof is immediate from (3.56) and (3.57). □

Using these identities it is easy to verify that $\Pi^{A'}{}_A \Pi^{B'}{}_B h_{A'B'} = \bar{h}_{AB}$.

3.2.5 The projected ambient tractor connection

As Branson and Gover observed in [11], the projected connection $\nabla^{\top}$ does not agree with the intrinsic tractor connection $\overline{\nabla}$, so for the tractor bundles along the hypersurface the usual Gauss theorem does not hold for the tractor bundles, The difference of the connections connection in the intrinsic tractor bundle is examined in the next section.

Definition 3.2.25. The *projected ambient tractor connection* $\nabla^{\top}$ in the intrinsic tractor bundle $\overline{\mathcal{T}}$ is denoted by $\check{\nabla}$ and defined on a tractor $V^A \in \overline{\mathcal{T}}^A$ as

$$\check{\nabla}_b V^A := \Pi^A{}_{A'} \underline{\nabla}_b V^{A'} \tag{3.61}$$

Proposition 3.2.26. *The intrinsic bundle equipped with the intrinsic tractor metric $\bar{h}_{AB}$ and the projected ambient tractor connection $\check{\nabla}$ is a Riemannian vector bundle in the sense that $\check{\nabla}_a \bar{h}_{BC} = 0$.*

Proof. Straightforward calculations. □

Proposition 3.2.27. *The action of the projected ambient tractor connection* $\check{\nabla}$ *on the intrinsic tractor projectors is given by the following identities*

$$\begin{aligned} \check{\nabla}_b \bar{Y}^A &= \bar{Z}^A{}_a \check{\mathsf{P}}^a{}_b \\ \check{\nabla}_b \bar{Z}^A{}_a &= -\bar{Y}^A \bar{\mathbf{g}}_{ba} - \bar{X}^A \check{\mathsf{P}}_{ba} \\ \check{\nabla}_b \bar{X}^A &= \bar{Z}^A{}_b \end{aligned} \tag{3.62}$$

where

$$\check{\mathsf{P}}_{ab} := \Pi_\Sigma \mathsf{P}_{ab} + \mathbf{H}\mathring{\mathbf{L}}_{ab} + \tfrac{\mathbf{H}^2}{2}\bar{\mathbf{g}}_{ab} \tag{3.63}$$

Proof. Applying the identities (3.50) for the action of the ambient tractor derivatives to the explicit expressions of $\bar{Y}^A$, $\bar{Z}^A{}_a$, $\bar{X}^A$ as in (3.57) we compute:

$$\begin{aligned} \underline{\nabla}_b \bar{Y}^A &= \underline{\nabla}_b (Y^A + Z^A{}_a \mathrm{N}^a \mathbf{H} - X^A \tfrac{\mathbf{H}^2}{2}) \\ &= \underline{\nabla}_b Y^A + (\underline{\nabla}_b Z^A{}_a)\mathrm{N}^a\mathbf{H} + Z^A{}_a(\underline{\nabla}_b \mathrm{N}^a)\mathbf{H} + Z^A{}_a \mathrm{N}^a \underline{\nabla}_b \mathbf{H} - (\underline{\nabla}_b X^A)\tfrac{\mathbf{H}^2}{2} \\ &\quad - X^A \underline{\nabla}_b \tfrac{\mathbf{H}^2}{2} \\ &= Z^A{}_a \underline{\mathsf{P}}_b{}^a + (-Y^A \bar{\mathbf{g}}_{ba} - X^A \underline{\mathsf{P}}_{ba})\mathrm{N}^a\mathbf{H} + Z^A{}_a \mathbf{L}_b{}^a \mathbf{H} + Z^A{}_a \mathrm{N}^a \overline{\nabla}_b \mathbf{H} \\ &\quad - Z^A{}_a \Pi^a{}_b \tfrac{\mathbf{H}^2}{2} - X^A \mathbf{H}\overline{\nabla}_b \mathbf{H} \\ &= Z^A{}_a \underline{\mathsf{P}}_b{}^a + Z^A{}_a \mathbf{L}_b{}^a \mathbf{H} + Z^A{}_a \mathrm{N}^a \overline{\nabla}_b \mathbf{H} - Z^A{}_a \Pi^a{}_b \tfrac{\mathbf{H}^2}{2} - X^A \underline{\mathsf{P}}_{ba} \mathrm{N}^a \mathbf{H} \\ &\quad - X^A \mathbf{H} \overline{\nabla}_{\underline{b}} \mathbf{H} \\ &= Z^A{}_a \Pi^{b'}{}_b \Big(\Pi^a{}_{a'} \big(\mathsf{P}^{a'}{}_{b'} + \mathbf{H} L^{a'}{}_{b'} - \delta^{a'}{}_{b'} \tfrac{\mathbf{H}^2}{2} \big) + \mathrm{N}^a \mathrm{N}_{a'} \mathsf{P}^{a'}{}_{b'} + \mathrm{N}^a \overline{\nabla}_{b'} \mathbf{H} \Big) \\ &\quad - X^A \Pi^{b'}{}_b \Big(\mathbf{H} \overline{\nabla}_{b'} \mathbf{H} + \mathsf{P}_{b'a} \mathrm{N}^a \mathbf{H} \Big) \end{aligned}$$

$$\begin{aligned} \underline{\nabla}_b \bar{Z}^A{}_a &= \underline{\nabla}_b (Z^A{}_{a'} \Pi^{a'}{}_a) = (\underline{\nabla}_b Z^A{}_{a'}) \Pi^{a'}{}_a + Z^A{}_{a'} \underline{\nabla}_b \Pi^{a'}{}_a \\ &= \Pi^{a'}{}_a \Pi^{b'}{}_b (-Y^A \mathbf{g}_{b'a'} - X^A \mathsf{P}_{b'a'}) + Z^A{}_{a'}(-\mathrm{N}_a \mathbf{L}_b{}^{a'} - \mathrm{N}^{a'} \mathbf{L}_{ba}) \\ &= -Y^A \Pi^{a'}{}_a \Pi^{b'}{}_b \mathbf{g}_{b'a'} - Z^A{}_{a'} \Pi^{b'}{}_b \Big(\mathrm{N}^{a'} \mathbf{L}_{b'a} + \mathrm{N}_a \mathbf{L}_{b'}{}^{a'} \Big) - X^A \Pi^{a'}{}_a \Pi^{b'}{}_b \mathsf{P}_{b'a'} \end{aligned}$$

$$\underline{\nabla}_b \bar{X}^A = \underline{\nabla}_b X^A = Z^A{}_a \Pi^a{}_b = \bar{Z}^A{}_b$$

Evaluating the tractor projector operator on the ambient tractor projectors using (3.60), we express the result in terms of the intrinsic tractor projectors and, after some simplifications, obtain the desired result:

$$
\begin{aligned}
\Pi^A{}_{A'}\underline{\nabla}_b\bar{Y}^{A'} &= \Pi^A{}_{A'}Z^{A'}{}_a\Pi^{b'}{}_b\Bigg(\Pi^a{}_{a'}\Big(\mathsf{P}^{a'}{}_{b'} + \mathbf{H}L^{a'}{}_{b'} - \delta^{a'}{}_{b'}\tfrac{\mathbf{H}^2}{2}\Big)\\
&\qquad + \mathrm{N}^a\mathrm{N}_{a'}\mathsf{P}^{a'}{}_{b'} + \mathrm{N}^a\overline{\nabla}_{b'}\mathbf{H}\Bigg) - \Pi^A{}_{A'}X^{A'}\Pi^{b'}{}_b\Big(\mathbf{H}\overline{\nabla}_{b'}\mathbf{H} + \mathsf{P}_{b'a}\mathrm{N}^a\mathbf{H}\Big)\\
&= (\bar{Z}^A{}_a + \bar{X}^A\mathbf{H}\mathrm{N}_a)\Pi^{b'}{}_b\Bigg(\Pi^a{}_{a'}\Big(\mathsf{P}^{a'}{}_{b'} + \mathbf{H}\mathbf{L}^{a'}{}_{b'} - \delta^{a'}{}_{b'}\tfrac{\mathbf{H}^2}{2}\Big)\\
&\qquad + \mathrm{N}^a\mathrm{N}_{a'}\mathsf{P}^{a'}{}_{b'} + \mathrm{N}^a\overline{\nabla}_{b'}\mathbf{H}\Bigg) - X^A\Pi^{b'}{}_b\Big(\mathbf{H}\overline{\nabla}_{b'}\mathbf{H} + \mathsf{P}_{b'a}\mathrm{N}^a\mathbf{H}\Big)\\
&= \bar{Z}^A{}_a\Bigg(\Pi^{b'}{}_b\Pi^a{}_{a'}\Big(\mathsf{P}^{a'}{}_{b'} + \mathbf{H}\mathbf{L}^{a'}{}_{b'} - \delta^{a'}{}_{b'}\tfrac{\mathbf{H}^2}{2}\Big)\Bigg) + \cancel{\bar{X}^A\mathbf{H}\mathrm{N}_{a'}\Pi^{b'}{}_b\mathsf{P}^{a'}{}_{b'}}\\
&\qquad + \cancel{\bar{X}^A\mathbf{H}\Pi^{b'}{}_b\overline{\nabla}_{b'}\mathbf{H}} - \cancel{\bar{X}^A\Pi^{b'}{}_b\mathbf{H}\overline{\nabla}_{b'}\mathbf{H}} - \cancel{\bar{X}^A\Pi^{b'}{}_b\mathsf{P}_{b'a}\mathrm{N}^a\mathbf{H}}\\
&= \bar{Z}^A{}_a\Bigg(\Pi^{b'}{}_b\Pi^a{}_{a'}\Big(\mathsf{P}^{a'}{}_{b'} + \mathbf{H}L^{a'}{}_{b'} - \delta^{a'}{}_{b'}\tfrac{\mathbf{H}^2}{2}\Big)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
\Pi^A{}_{A'}\underline{\nabla}_b\bar{Z}^{A'}{}_a &= -\Pi^A{}_{A'}Y^{A'}\Pi^{a'}{}_a\Pi^{b'}{}_b\mathbf{g}_{b'a'}\\
&\quad - \Pi^A{}_{A'}Z^{A'}{}_{a'}\Pi^{b'}{}_b\Big(\mathrm{N}^{a'}\mathbf{L}_{b'a} + \mathrm{N}_a\mathbf{L}_{b'}{}^{a'}\Big)\\
&\quad - \Pi^A{}_{A'}X^A\Pi^{a'}{}_a\Pi^{b'}{}_b\mathsf{P}_{b'a'}\\
&= -(\bar{Y}^A - \frac{H^2}{2}\bar{X}^A)\bar{\mathbf{g}}_{ba}\\
&\quad - (\bar{Z}^A{}_{a'} + \bar{X}^A\mathbf{H}\mathrm{N}_{a'})\Pi^{b'}{}_b\Big(\mathrm{N}^{a'}\mathbf{L}_{b'a} + \underbrace{\mathrm{N}_a\mathbf{L}_{b'}{}^{a'}}\Big)\\
&\quad - \bar{X}^A\Pi^{a'}{}_a\Pi^{b'}{}_b\mathsf{P}_{b'a'}\\
&= -\bar{Y}^A\bar{\mathbf{g}}_{ba} - \bar{Z}^A{}_{a'}\mathrm{N}_a\mathbf{L}_b{}^{a'}\\
&\quad - \bar{X}^A(\Pi^{a'}{}_a\Pi^{b'}{}_b\mathsf{P}_{b'a'} - \frac{\mathbf{H}^2}{2}\bar{X}^A\bar{\mathbf{g}}_{ba} + \mathbf{H}\mathbf{L}_{ba})
\end{aligned}
$$

$$
\Pi^A{}_{A'}\underline{\nabla}_b\bar{X}^{A'} = \Pi^A{}_{A'}Z^{A'}{}_a\Pi^a{}_b = (\bar{Z}^A{}_a + \bar{X}^A\mathbf{H}\mathrm{N}_a)\Pi^a{}_b = \bar{Z}^A{}_a
$$

Denoting $\check{\mathsf{P}}_{ab} \coloneqq \Pi^{a'}{}_a\Pi^{b'}{}_b\mathsf{P}_{b'a'} - \frac{\mathbf{H}^2}{2}\bar{X}^A\bar{\mathbf{g}}_{ba} + \mathbf{H}\mathbf{L}_{ba}$ and using $\mathbf{L}_{ab} = \mathring{\mathbf{L}}_{ab} + \mathbf{H}\bar{\mathbf{g}}_{ab}$ we confirm the claim. □

Definition 3.2.28. Tensor $\check{\mathsf{P}}_{ab}$ defined by (3.63) in Proposition 3.2.27 will be referred to as the *Schouten–Fialkow tensor*.

An interesting property of the tensor $\check{\mathsf{P}}_{ab}$ is that it rescales under conformal changes of the ambient metric by the same rule as the intrinsic Schouten tensor $\overline{\mathsf{P}}_{ab}$ does (whenever it is defined).

Proposition 3.2.29. *The Schouten–Fialkow tensor $\check{\mathsf{P}}_{ab}$ under a conformal rescaling of the ambient metric transforms as*

$$\widehat{\check{\mathsf{P}}_{ab}} = \check{\mathsf{P}}_{ab} + \overline{\Lambda}_{ab}$$

where $\overline{\Lambda}_{ab} = -\overline{\nabla}_a \overline{\Upsilon}_b + \overline{\Upsilon}_a \overline{\Upsilon}_b - \frac{|\overline{\Upsilon}|^2}{2}\bar{\mathbf{g}}_{ab}$, that is by the same rule as the intrinsic Schouten tensor $\overline{\mathsf{P}}_{ab}$ (when $\dim M > 3$).

Proof. According to (2.9), the intrinsic Schouten tensor $\overline{\mathsf{P}}_{ab}$ rescales under the change of the intrinsic metric $\widehat{\bar{g}_{ab}} = \overline{\Omega}^2 \bar{g}_{ab}$ as $\widehat{\overline{\mathsf{P}}_{ab}} = \overline{\mathsf{P}}_{ab} + \overline{\Lambda}_{ab} = \overline{\mathsf{P}}_{ab} - \overline{\nabla}_a \overline{\Upsilon} + \overline{\Upsilon}_a \overline{\Upsilon}_b - \frac{|\overline{\Upsilon}|^2}{2}\bar{\mathbf{g}}_{ab}$. Here $\overline{\Omega} = \Omega|_\Sigma$ (we consider the conformal structure induced from the background manifold), and $\overline{\Upsilon} = \overline{\nabla}_a \log \overline{\Omega}$ is the 1-form (2.2) that contributes to the conformal rescaling (2.3) of the intrinsic Levi-Civita connection $\overline{\nabla}_a$ on the hypersurface Σ. The chain rule implies that $\overline{\Upsilon}_a = \Pi_a{}^{a'} \Upsilon_{a'}$.

In the decomposition $\underline{\mathcal{E}}_a = \overline{\mathcal{E}}_a \oplus \mathcal{N}_a$ we can write $\Upsilon_a = \overline{\Upsilon}_a + (\Upsilon \cdot \mathrm{N})\mathrm{N}_a$ and therefore $|\Upsilon|^2 = |\overline{\Upsilon}|^2 + (\Upsilon \cdot \mathrm{N})^2$.

Rescaling the totally projected ambient Schouten tensor $\Pi_\Sigma \mathsf{P}_{ab}$, we compute:

$$\begin{aligned}\widehat{\Pi_\Sigma \mathsf{P}_{ab}} &= \Pi_a{}^{a'} \Pi_b{}^{b'} \Big(\mathsf{P}_{a'b'} - \nabla_{a'} \Upsilon_{b'} + \Upsilon_{a'} \Upsilon_{b'} - \tfrac{|\Upsilon|^2}{2} \mathbf{g}_{a'b'} \Big) \\ &= \Pi_\Sigma \mathsf{P}_{ab} - \Pi_b{}^{b'} \Big(\underline{\nabla}_a \overline{\Upsilon}_{b'} + \mathrm{N}_{b'} \underline{\nabla}_a (\Upsilon \cdot \mathrm{N}) + (\Upsilon \cdot \mathrm{N}) \underline{\nabla}_a \mathrm{N}_{b'} \Big) + \overline{\Upsilon}_a \overline{\Upsilon}_b - \tfrac{|\Upsilon|^2}{2} \bar{\mathbf{g}}_{ab}\end{aligned}$$

Noticing that by the Gauss formula $\Pi_b{}^{b'} \underline{\nabla}_a \overline{\Upsilon}_{b'} = \Pi_b{}^{b'} (\overline{\nabla}_a \overline{\Upsilon}_{b'} - \mathrm{N}_{b'} \mathbf{L}_a{}^c \overline{\Upsilon}_c) = \overline{\nabla}_a \overline{\Upsilon}_b$, and using $\Pi_b{}^{b'} \underline{\nabla}_a \mathrm{N}_{b'} = \mathbf{L}_{ab}$, we rewrite the above display as

$$\widehat{\Pi_\Sigma \mathsf{P}_{ab}} = \Pi_\Sigma \mathsf{P}_{ab} - \overline{\nabla}_a \overline{\Upsilon}_b - (\Upsilon \cdot \mathrm{N}) \mathbf{L}_{ab} + \overline{\Upsilon}_a \overline{\Upsilon}_b - \tfrac{|\Upsilon|^2}{2} \bar{\mathbf{g}}_{ab}$$

Substituting $\mathbf{L}_{ab} = \mathring{\mathbf{L}}_{ab} + \mathbf{H} \bar{\mathbf{g}}_{ab}$ and $|\Upsilon|^2 = |\overline{\Upsilon}|^2 + (\Upsilon \cdot \mathrm{N})^2$, we obtain

$$\widehat{\Pi_\Sigma \mathsf{P}_{ab}} = \Pi_\Sigma \mathsf{P}_{ab} + \overline{\Lambda}_{ab} - (\Upsilon \cdot \mathrm{N}) \mathring{\mathbf{L}}_{ab} - (\Upsilon \cdot \mathrm{N}) \mathbf{H} \bar{\mathbf{g}}_{ab} - \tfrac{(\Upsilon \cdot \mathrm{N})^2}{2} \bar{\mathbf{g}}_{ab}$$

The last three terms in the right hand side of the above equation turn out to be the conformal deformation of $\mathbf{H} \mathring{\mathbf{L}}_{ab} + \frac{\mathbf{H}^2}{2} \bar{\mathbf{g}}_{ab}$. Indeed, using $\widehat{\mathbf{H}} = \mathbf{H} + (\Upsilon \cdot \mathrm{N})$

and $\widehat{\mathbf{H}^2} = \mathbf{H}^2 + 2(\Upsilon \cdot \mathrm{N})\mathbf{H} + (\Upsilon \cdot \mathrm{N})^2$, we compute

$$\widehat{\mathbf{H}}\mathring{\mathbf{L}}_{ab} + \tfrac{\widehat{\mathbf{H}^2}}{2}\bar{\mathbf{g}}_{ab} = \mathbf{H}\mathring{\mathbf{L}}_{ab} + (\Upsilon \cdot \mathrm{N})\mathring{\mathbf{L}}_{ab} + \tfrac{\mathbf{H}^2}{2}\bar{\mathbf{g}}_{ab} + (\Upsilon \cdot \mathrm{N})\mathbf{H}\bar{\mathbf{g}}_{ab} + \tfrac{(\Upsilon \cdot \mathrm{N})^2}{2}\bar{\mathbf{g}}_{ab}$$

Adding up, we obtain the claim. □

Remark 3.2.30. The advantage of $\check{\mathsf{P}}_{ab}$ is that it is defined on hypersurfaces in 3-dimensional background manifolds (when $\dim M = 3$). Following [11] we *define* $\overline{\mathsf{P}}_{ab} := \check{\mathsf{P}}_{ab}$ in this dimension, so Proposition 3.2.29 formally holds for $m = \dim M \geq 3$. Moreover, as noticed in *loc.cit.*, this allows to define the intrinsic tractor connection on 2-dimensional hypersurfaces as $\overline{\nabla} := \check{\nabla}$ consistently with the definition of the intrinsic Schouten tensor in this case.

3.2.6 The intrinsic tractor contorsion

As we see, there are two connections $\overline{\nabla}$ and $\check{\nabla}$ that are both naturally defined on the intrinsic tractor bundle $\overline{\mathcal{T}}^A$. Each of them has its own *raison d'être*. Arguably, $\check{\nabla}$ has some advantages compared to $\overline{\nabla}$, one of them is that it is defined in all dimensions $\dim M \geq 3$, and another one is that the tractor Gauss formula for $\check{\nabla}$ is just its definition, so the Gauss–Codazzi decomposition of the corresponding curvature has simpler expressions. On the other hand, the intrinsic tractor connection $\overline{\nabla}$ reflects the induced conformal structure on the hypersurface. The dimensional issue can be resolved as proposed by Branson and Gover in [11] by simply defining $\overline{\nabla}$ to be equal to $\check{\nabla}$ when $\dim M = 3$.

Definition 3.2.31. The *intrinsic tractor contorsion* $\mathbb{S}_a{}^B{}_C$ is defined by the equation

$$\check{\nabla}_a V^B = \overline{\nabla}_a V^B - \mathbb{S}_a{}^B{}_C V^C \tag{3.64}$$

where $V^B \in \overline{\mathcal{T}}^B$ is an intrinsic tractor.

Remark 3.2.32. The sign in the equation has been chosen for at least two reasons, the first one is to keep consistent with the notation in [68], and the second one is the equation $\underline{\nabla}_a V^B = \check{\nabla}_a V^B - \mathrm{N}^B \mathbb{L}_{aC} V^C$ for $V^B \in \overline{\mathcal{T}}^B$.

Using Proposition 3.2.27 it is not difficult now to give an explicit formula for the intrinsic tractor contorsion $\mathbb{S}_a{}^B{}_C$.

Proposition 3.2.33. *The intrinsic tractor contorsion is given explicitly in a choice of scale by the equation*

$$\mathbb{S}_a{}^B{}_C \stackrel{\bar{g}}{=} \bar{X}^B \bar{Z}_C{}^c \mathcal{F}_{ac} - \bar{Z}^B{}_b \bar{X}_C \mathcal{F}_a{}^b \tag{3.65}$$

where

$$\mathcal{F}_{ab} := \begin{cases} \check{\mathsf{P}}_{ab} - \overline{\mathsf{P}}_{ab} & \textit{when } \dim M > 3 \\ 0 & \textit{when } \dim M = 3 \end{cases} \tag{3.66}$$

Proof. Pick an intrinsic tractor $V^A \in \overline{\mathcal{T}}^A$ presented in a choice of scale as

$$V^A = \bar{Y}^A \sigma + \bar{Z}^A{}_a \mu^a + \bar{X}^A \rho \tag{3.67}$$

The projected ambient tractor connection acts on the intrinsic tractor projectors by the equations (3.62), and is coupled to the intrinsic Levi-Civita connection when acting on the tensor indices.

Differentiating the tractor (3.67) by $\check{\nabla}_b$ using equations (3.62) and the Leibniz rule we compute

$$\begin{aligned}
\check{\nabla}_b V^A &= (\check{\nabla}_b \bar{Y}^A)\sigma + \bar{Y}^A \overline{\nabla}_b \sigma + (\check{\nabla}_b \bar{Z}^A{}_a)\mu^a + \bar{Z}^A{}_a \overline{\nabla}_b \mu^a + (\check{\nabla}_b \bar{X}^A)\rho + \bar{X}^A \overline{\nabla}_b \rho \\
&= \bar{Z}^A{}_a \check{\mathsf{P}}^a{}_b \sigma + \bar{Y}^A \overline{\nabla}_b \sigma + (-\bar{Y}^A \bar{\mathbf{g}}_{ba} - \bar{X}^A \check{\mathsf{P}}_{ba})\mu^a + \bar{Z}^A{}_a \overline{\nabla}_b \mu^a + \bar{Z}^A{}_b \rho + \bar{X}^A \overline{\nabla}_b \rho \\
&= \bar{Y}^A \overline{\nabla}_b \sigma - \bar{Y}^A \bar{\mathbf{g}}_{ba} \mu^a + \bar{Z}^A{}_a \overline{\nabla}_b \mu^a + \bar{Z}^A{}_a \bar{\delta}^a{}_b \rho + \bar{Z}^A{}_a \check{\mathsf{P}}^a{}_b \sigma + \bar{X}^A \overline{\nabla}_b \rho - \bar{X}^A \check{\mathsf{P}}_{ba} \mu^a \\
&= \bar{Y}^A \Big(\overline{\nabla}_b \sigma - \bar{\mathbf{g}}_{ba} \mu^a \Big) + \bar{Z}^A{}_a \Big(\overline{\nabla}_b \mu^a + \bar{\delta}^a{}_b \rho + \check{\mathsf{P}}^a{}_b \sigma \Big) + \bar{X}^A \Big(\overline{\nabla}_b \rho - \check{\mathsf{P}}_{ba} \mu^a \Big)
\end{aligned}$$

Similarly, using (3.59), we have

$$\overline{\nabla}_b V^A = \bar{Y}^A \Big(\overline{\nabla}_b \sigma - \bar{\mathbf{g}}_{ba} \mu^a \Big) + \bar{Z}^A{}_a \Big(\overline{\nabla}_b \mu^a + \bar{\delta}^a{}_b \rho + \overline{\mathsf{P}}^a{}_b \sigma \Big) + \bar{X}^A \Big(\overline{\nabla}_b \rho - \overline{\mathsf{P}}_{ba} \mu^a \Big)$$

Subtracting these expressions we get

$$\check{\nabla}_b V^A - \overline{\nabla}_b V^A = \bar{Z}^A{}_a (\check{\mathsf{P}}^a{}_b - \overline{\mathsf{P}}^a{}_b)\sigma - \bar{X}^A (\check{\mathsf{P}}_{ba} - \overline{\mathsf{P}}_{ba})\mu^a$$

but $\sigma = \bar{X}_A V^A$ and $\mu^a = \bar{Z}_A{}^a V^A$, so we can rewrite the above display, at the same time renaming the indices, as

$$\begin{aligned}
\check{\nabla}_a V^B - \overline{\nabla}_a V^B &= \bar{Z}^B{}_b (\check{\mathsf{P}}^b{}_a - \overline{\mathsf{P}}^b{}_a) \bar{X}_C V^C - \bar{X}^B (\check{\mathsf{P}}_{ac} - \overline{\mathsf{P}}_{ac}) \bar{Z}_C{}^c V^C \\
&= \bar{X}_C \bar{Z}^B{}_b \mathcal{F}^b{}_a V^C - \bar{X}^B \bar{Z}_C{}^c \mathcal{F}_{ac} V^C
\end{aligned}$$

where we have used the notation $\mathcal{F}_{ab}$ as defined in the Proposition. □

Definition 3.2.34 (Fialkow tensor)**.** We refer to the tensor $\mathcal{F}_{ab}$ in equation (3.66) as the *Fialkow tensor*. For the motivation of this name see [39].

Proposition 3.2.35. *The Fialkow tensor $\mathcal{F}_{ab}$ is hypersurface conformal invariant of zero conformal weight*

Proof. This immediately follows from the conformal invariance of the projected ambient tractor connection $\check{\nabla}$ and the intrinsic tractor connection $\overline{\nabla}$.

Another way to prove this is to use Proposition 3.2.29. This approach was essentially done in [68]. □

We can also give an explicit expression for the Fialkow tensor for hypersurfaces

Proposition 3.2.36. *When* $n = \dim M > 3$ *the Fialkow tensor is a linear combination of the canonical hypersurface invariants, the ambient Weyl tensor, the unit normal and the umbilicity tensor:*

$$\mathcal{F}_{ab} = \tfrac{1}{\bar{n}-2}\Big(\overset{\mathrm{N}}{\mathrm{W}}_{ab} + \mathring{\mathrm{L}}^2_{ab} - \tfrac{|\mathring{\mathrm{L}}|^2}{\bar{n}-1}\bar{\mathbf{g}}_{ab}\Big) \tag{3.68}$$

In $\dim M = 3$ *it is defined to be zero by* (3.66)*.*

Proof. Follows directly from (2.31). □

This shows, in particular, that $\mathcal{F}_{ab}$ is not trivial in general (for $\dim M > 3$).

Remark 3.2.37. Using the notation for the adjoint tractors we can represent the intrinsic tractor contorsion as

$$\mathbb{S}_{aBC} = \mathbb{X}_{BC}{}^{c}\mathcal{F}_{ac} \tag{3.69}$$

where $\mathbb{X}_{BC}{}^{c}$ is as in (3.14).

This again justifies the choice of the sign for $\mathbb{S}_a{}^B{}_C$.

Even though it is now obvious for many reasons, we record the following important property of the intrinsic tractor contorsion.

Proposition 3.2.38. *The intrinsic tractor contorsion* $\mathbb{S}_a{}^B{}_C$ *is a conformally invariant* $\mathrm{End}(\overline{\mathcal{T}})$*-valued* 1*-form of weight* 0 *on the hypersurface* Σ *in a conformal manifold* M *of dimension* $n = \dim M \geq 3$*. It vanishes identically when* $\dim M = 3$.

3.2.7 The tractor Weingarten equations

The ambient tractor bundle $\underline{\mathcal{T}}^A$ of the hypersurface Σ is now represented as a direct sum of subbundles

$$\underline{\mathcal{T}}^A = \overline{\mathcal{T}}^A \oplus \mathcal{N}^A$$

and the projection operators $\pi^{\top}$ and $\pi^{\perp}$ are expressed using the normal tractor, so we are in the situation $\mathcal{F} = \mathcal{F}^{\top} \oplus \mathcal{F}^{\perp}$ examined in Section 1.1.13.

We introduce the tractor direct sum connection (the tractor van der Waerden–Bortolotti connection)

$$\overset{\oplus}{\nabla} = \begin{pmatrix} \check{\nabla} & 0 \\ 0 & \nabla^{\mathcal{N}} \end{pmatrix}$$

where now $\check{\nabla} \neq \overline{\nabla}$ (except $\bar{n} = 2$) and $\nabla^{\mathcal{N}}$ denotes the connection in the tractor normal bundle $\mathcal{N}^A$.

Using the general observations made in Section 1.1.13, the difference operator $A_a{}^B{}_C$ between the ambient tractor connection $\underline{\nabla}$ and the tractor direct sum connection $\overset{\oplus}{\nabla}$ is expressed as $A = S + H$ where the operators $H_a{}^B{}_C$ and $S_a{}^B{}_C$ are now defined by

$$H_a{}^B{}_C = (\underline{\nabla}_a \Pi^B{}_D)\Pi^D{}_C$$

and

$$S_a{}^B{}_C = (\underline{\nabla}_a \mathrm{N}^B{}_D)\mathrm{N}^D{}_C$$

Computing these derivatives, we get, for instance

$$\underline{\nabla}_a \Pi_B{}^C = \underline{\nabla}_a\Big(\delta_B{}^C - \mathrm{N}_B\mathrm{N}^C\Big) = -(\underline{\nabla}_a\mathrm{N}_B)\mathrm{N}^C - \mathrm{N}_B\underline{\nabla}_a\mathrm{N}^C$$

As before, we introduce the shape 1-form in the tractor bundle.

Definition 3.2.39. The *tractor shape* 1*-form* $\mathbb{L}_a{}^B$ is defined as

$$\mathbb{L}_a{}^B := \underline{\nabla}_a \mathrm{N}^B \tag{3.70}$$

Proposition 3.2.40 (The tractor Weingarten equations)**.** *The operators* $H_a{}^B{}_C$ *and* $S_a{}^B{}_C$ *are given by the formulas*

$$\begin{aligned} S_a{}^B{}_C &= \mathbb{L}_a{}^B\mathrm{N}_C \\ H_a{}^B{}_C &= -\mathrm{N}^B\mathbb{L}_{aC} \end{aligned} \tag{3.71}$$

Proof. Straightforward computations, or an application of the abstract Weingarten equations from Section 1.1.13. □

Proposition 3.2.41. *The tractor shape* 1*-form can be identified with a purely intrinsic section:*

$$\mathbb{L}_a{}^B \equiv \Pi_a{}^c\Pi_D{}^B\mathbb{L}_c{}^D$$

Proof. Indeed, $0 = \underline{\nabla}_a(\mathrm{N}_B\mathrm{N}^B) = 2\,\mathrm{N}_B\underline{\nabla}_a\mathrm{N}^B$ and the index a in the ambient tractor connection is tangential. □

Proposition 3.2.42. *Explicitly the tractor shape 1-form viewed as an intrinsic section can be expressed as*

$$\mathbb{L}_{aB} \stackrel{\bar{g}}{=} \bar{Z}_B{}^b \mathring{\mathbf{L}}_{ab} - \bar{X}_B \left(\overset{\mathrm{N}}{\underline{\mathsf{P}}}_a + \overline{\nabla}_a \mathbf{H} \right) \tag{3.72}$$

where $\overset{\mathrm{N}}{\underline{\mathsf{P}}}_a := \Pi_a{}^b \mathsf{P}_{ac} \mathbf{N}^c$.

Proof. A straightforward calculation:

$$\begin{aligned}
\mathbb{L}_{aB} &= \underline{\nabla}_a \mathrm{N}_B = \underline{\nabla}_a \left(Z_B{}^b \mathbf{N}_b - X_B \mathbf{H} \right) \\
&= \left(\underline{\nabla}_a Z_B{}^b \right) \mathbf{N}_b + Z_B{}^b \left(\underline{\nabla}_a \mathbf{N}_b \right) - \left(\underline{\nabla}_a X_B \right) \mathbf{H} - X_B \underline{\nabla}_a \mathbf{H} \\
&= \left(-Y_B \underline{\delta}_a{}^b - X_B \underline{\mathsf{P}}_a{}^b \right) \mathbf{N}_b + Z_B{}^b \mathbf{L}_{ab} - Z_B{}^b \bar{\mathbf{g}}_{ab} \mathbf{H} - X_B \overline{\nabla}_a \mathbf{H} \\
&= \cancel{-Y_B \underline{\delta}_a{}^b \mathbf{N}_b} + Z_B{}^b \mathbf{L}_{ab} - Z_B{}^b \bar{\mathbf{g}}_{ab} \mathbf{H} - X_B \left(\underline{\mathsf{P}}_a{}^b \mathbf{N}_b + \overline{\nabla}_a \mathbf{H} \right)
\end{aligned}$$

where we have used the notation $\underline{\delta}_a{}^b := \Pi_a{}^{a'} \delta_{a'}{}^b \equiv \Pi_a{}^b$ and $\underline{\mathsf{P}}_a{}^b := \Pi_a{}^{a'} \mathsf{P}_{a'}{}^b$. Now the claim is clear. □

Remark 3.2.43. Using the contracted conformal Codazzi equation (2.36), we can give a nicer expression (3.83) for the tractor shape 1-form. The formula (3.72) allows us to derive the conformal Codazzi equations from the tractor versions of the Gauss and Codazzi equations independently from our previous computations. We will show this shortly.

3.2.8 The tractor Gauss–Weingarten formula

Theorem 3.2.44 (The tractor Gauss–Weingarten formula)**.**

$$\underline{\nabla}_a V^B = \overset{\oplus}{\nabla}_a V^B + (\mathrm{N}_C \mathbb{L}_a{}^B - \mathrm{N}^B \mathbb{L}_{aC}) V^C \tag{3.73}$$

for an ambient tractor $V^B \in \underline{\mathcal{T}}^B$.

Proof. A direct consequence of the definitions $\underline{\nabla} = \overset{\oplus}{\nabla} + A$, $A = S + H$ and the tractor Weingarten equations (3.71). □

Similar to the usual hypersurface geometry, in practice the tractor Gauss–Weingarten formula is used separately for intrinsic and normal tractors.

Corollary 3.2.45 (The tractor Gauss formula)**.**

$$\underline{\nabla}_a V^B = \check{\nabla}_a V^B - \mathrm{N}^B \mathbb{L}_{aC} V^C \tag{3.74}$$

Proof. This follows immediately from (3.73) if $V^B \in \overline{\mathcal{T}}^B$. □

Corollary 3.2.46 (The tractor Weingarten formula)**.** *This is just the definition* (3.70) *of the tractor shape* 1*-form in our approach.*

Corollary 3.2.47 (The Gauss-Stafford formula)**.** *The difference between the ambient* $\underline{\nabla}_a$ *and intrinsic* $\overline{\nabla}_a$ *tractor connections acting on an intrinsic tractor* $V^B \in \overline{\mathcal{T}}^B$ *can be now represented by the following equation*

$$\underline{\nabla}_a V^B = \overline{\nabla}_a V^B - (\mathbb{S}_a{}^B{}_C + \mathrm{N}^B \mathbb{L}_{aC}) V^C \tag{3.75}$$

Proof. This formula first appeared in R.Stafford's thesis (see [68, p.35]). It follows immediately from the definition (3.64) of the intrinsic tractor contorsion $\mathbb{S}_a{}^B{}_C$ and the tractor Gauss formula (3.74). □

Proposition 3.2.48. *The difference between* $2\,\check{\nabla}_{[a}\mathbb{L}_{b]}{}^C$ *and* $2\,\overline{\nabla}_{[a}\mathbb{L}_{b]}{}^C$ *is given by*

$$2\,\check{\nabla}_{[a}\mathbb{L}_{b]}{}^C = 2\,\overline{\nabla}_{[a}\mathbb{L}_{b]}{}^C + \bar{X}^C 2\,\mathcal{F}_{c[a}\mathring{\mathbf{L}}_{b]}{}^c$$

Proof. Using the formula (3.75), we compute

$$\begin{aligned}
\check{\nabla}_a \mathbb{L}_b{}^C &= \overline{\nabla}_a \mathbb{L}_b{}^C - \left(\mathrm{N}^C \mathbb{L}_{aD} + \mathbb{S}_a{}^C{}_D\right) \mathbb{L}_b{}^D \\
&= \overline{\nabla}_a \mathbb{L}_b{}^C - \mathrm{N}^C \mathbb{L}_{aD}\mathbb{L}_b{}^D - \mathbb{S}_a{}^C{}_D \mathbb{L}_b{}^D
\end{aligned}$$

and thus $2\,\check{\nabla}_{[a}\mathbb{L}_{b]}{}^C = 2\,\overline{\nabla}_{[a}\mathbb{L}_{b]}{}^C - 2\,\mathrm{N}^C \mathbb{L}_{[a}{}^D \mathbb{L}_{b]D} - 2\,\mathbb{S}_{[a}{}^{CD}\mathbb{L}_{b]D}$.

But $2\,\mathbb{L}_{[a}{}^D \mathbb{L}_{b]D} = 0$ and

$$\begin{aligned}
2\,\mathbb{S}_{[a}{}^{CD}\mathbb{L}_{b]D} = \left(\bar{X}^C \bar{Z}_D{}^d \mathcal{F}_{ad} - \bar{Z}^C{}_c \bar{X}_D \mathcal{F}_a{}^c\right) \left(\bar{Z}^D{}_e \mathring{\mathbf{L}}_b{}^e - \bar{X}^D \left(\overset{\mathrm{N}}{\underline{\mathrm{P}}}_b + \overline{\nabla}_b \mathbf{H}\right)\right) \\
- (a \leftrightarrow b) = \bar{X}^C 2\,\mathcal{F}_{d[a}\mathring{\mathbf{L}}_{b]}{}^d
\end{aligned}$$

□

3.2.9 The tractor Gauss and Codazzi equations

The connections $\underline{\nabla}_a$, $\check{\nabla}_a$ and $\overline{\nabla}_a$ have the corresponding curvature operators denoted by $\underline{\Omega}_{ab}{}^C{}_D$, $\check{\Omega}_{ab}{}^C{}_D$ and $\overline{\Omega}_{ab}{}^C{}_D$ respectively.

Notice that in $\underline{\Omega}_{ab}{}^C{}_D$ the first two indices are (tensor-) tangential, and the second pair are the ambient tractor indices. It is clear that

$$\underline{\Omega}_{ab}{}^C{}_D = \Pi^{a'}{}_a \Pi^{a'}{}_a \Omega_{ab}{}^C{}_D$$

where $\Omega_{ab}{}^C{}_D$ is the restriction to Σ of the tractor curvature on manifold M. The notation $\underline{\Omega}_{ab}{}^C{}_D$ agrees with the convention that the underlined objects represent ambient-valued forms on Σ (that is the tensor indices are tangential in this case).

The quantities $\check{\Omega}_{ab}{}^{C}{}_{D}$ and $\overline{\Omega}_{ab}{}^{C}{}_{D}$ have only (tensor or tractor) tangential indices.

Decomposing orthogonally (with respect to the normal tractor) the difference of the curvatures $\underline{\Omega} = K^{\underline{\nabla}}$ and $\overset{\oplus}{\Omega} = K^{\overset{\oplus}{\nabla}}$ using the tractor Gauss–Weingarten formula (3.73), and taking tractor-orthogonal parts we get tractor analogue of the Gauss equation

The unit normal tractor and the tractor projection operator allow us to apply the general Gauss–Codazzi–Ricci decomposition of the curvature, so that we can express the ambient tractor curvature acting on the intrinsic tractors in terms of the tractor shape form and the projected ambient tractor connection.

Theorem 3.2.49 (The tractor Gauss equation)**.** *Along the hypersurface Σ, the curvature $\check{\Omega}$ of the projected ambient tractor connection is equal to the totally tractor-projected part of the ambient curvature $\Pi_\Sigma \underline{\Omega}$ plus the exterior square $\mathbb{L} \wedge \mathbb{L}$ of the tractor shape 1-form:*

$$\check{\Omega}_{ab}{}^{C}{}_{D} = \Pi^{C}{}_{C'} \Pi_{D}{}^{D'} \underline{\Omega}_{ab}{}^{C'}{}_{D'} + 2\, \mathbb{L}_{[a}{}^{C} \mathbb{L}_{b]D} \tag{3.76}$$

or, succinctly,

$$\check{\Omega} = \Pi_\Sigma \underline{\Omega} + \mathbb{L} \wedge \mathbb{L} \tag{3.77}$$

Theorem 3.2.50 (The tractor Codazzi equation)**.** *The intrinsic exterior derivative of the shape tractor 1-form is the normal part of the ambient tractor curvature along the hypersurface, that is*

$$2\, \check{\nabla}_{[a} \mathbb{L}_{b]}{}^{C} = \underline{\Omega}_{ab}{}^{C}{}_{D} \mathrm{N}^{D} \tag{3.78}$$

or, with the indices suppressed,

$$\check{\nabla} \wedge \mathbb{L} = \underline{\Omega} \bullet \mathrm{N} \tag{3.79}$$

where $\bullet$ denotes the algebraic action of the ambient tractor curvature on tractors (that is the contraction of a tractor with $\underline{\Omega}$ in the last index).

Proof of Theorems 3.2.49 and 3.2.50. Follows directly from Theorem 1.1.22 Theorem 1.1.22, using Proposition 1.1.25 and Equation 3.71.

Alternatively, the calculation from the proof of Theorem 1.2.15 can be repeated with the corresponding changes.

This time, we have the difference of connections $\underline{\nabla} = \overset{\oplus}{\nabla} + A$ with $A_a{}^B{}_C = \mathrm{N}_C \mathbb{L}_a{}^B - \mathrm{N}^B \mathbb{L}_{aC}$.

Computing

$$
\begin{aligned}
(\overset{\oplus}{\nabla} \wedge A)_{[ab]}{}^C{}_D &= 2\overset{\oplus}{\nabla}_{[a} A_{b]}{}^C{}_D = \overset{\oplus}{\nabla}_a (\mathrm{N}_D \mathbb{L}_b{}^C - \mathrm{N}^C \mathbb{L}_{bD}) - (a \leftrightarrow b) \\
&= (\check{\nabla}_a \mathbb{L}_b{}^C)\mathrm{N}_D + \cancelto{0}{\mathbb{L}_b{}^C \nabla^{\mathcal{N}}_a \mathrm{N}_D} - (\check{\nabla}_a \mathbb{L}_{bD})\mathrm{N}^C - \cancelto{0}{\mathbb{L}_{bD} \nabla^{\mathcal{N}}_a \mathrm{N}^C} - (a \leftrightarrow b) \\
&= 2\check{\nabla}_{[a} \mathbb{L}_{b]}{}^D \mathrm{N}_D - 2\check{\nabla}_{[a} \mathbb{L}_{b]D} \mathrm{N}^C
\end{aligned}
$$

and

$$
\begin{aligned}
(A \wedge A)_{[ab]}{}^C{}_D &= A_a{}^C{}_E A_b{}^E{}_D - (a \leftrightarrow b) \\
&= (\mathrm{N}_E \mathbb{L}_a{}^C - \mathrm{N}^C \mathbb{L}_{aE})(\mathrm{N}_D \mathbb{L}_b{}^E - \mathrm{N}^E \mathbb{L}_{bD}) - (a \leftrightarrow b) \\
&= -\mathbb{L}_a{}^C \mathbb{L}_{bD} - \underbrace{\mathbb{L}_{aE} \mathbb{L}_b{}^E \mathrm{N}^C \mathrm{N}_D}_{\text{symmetric in a and b}} - (a \leftrightarrow b) \\
&= -2\mathbb{L}_{[a}{}^C \mathbb{L}_{b]D}
\end{aligned}
$$

and extracting the orthogonal parts of $\underline{\Omega} = \overset{\oplus}{\Omega} + \overset{\oplus}{\nabla} \wedge A + A \wedge A$, we recover the claim. □

It is interesting to notice, that the tractor Gauss (3.77) and Codazzi (3.79) equations imply the conformal Gauss (2.34) and Codazzi equations (2.35).

The conformal Gauss equation revisited

Proposition 3.2.51. *The curvature $\check{\Omega}_{ab}{}^C{}_D$ of the projected ambient tractor connection $\check{\nabla}_a$ is expressed explicitly in a choice of intrinsic scale $\bar{g} \in \bar{c}$ as*

$$\check{\Omega}_{ab}{}^{CD} \overset{\bar{g}}{=} \bar{Z}^C{}_c \bar{Z}^D{}_d \check{\mathrm{W}}_{ab}{}^{cd} - 2\bar{X}^{[C} \bar{Z}^{D]}{}_d \check{\mathrm{Y}}_{ab}{}^d \tag{3.80}$$

where

$$\check{\mathrm{W}}_{ab}{}^{cd} := \overline{\mathrm{R}}_{ab}{}^{cd} - 4\bar{\delta}_{[a}{}^{[c} \check{\mathsf{P}}_{b]}{}^{d]} \tag{3.81}$$

and

$$\check{\mathrm{Y}}_{ab}{}^d = 2\overline{\nabla}_{[a} \check{\mathsf{P}}_{b]}{}^d \tag{3.82}$$

Proposition 3.2.52. *The curvature $\underline{\Omega}_{ab}{}^C{}_D$ of the ambient tractor connection $\underline{\nabla}_a$ is given explicitly by*

$$\underline{\Omega}_{ab}{}^C{}_D = \Pi_a{}^e \Pi_b{}^f \Omega_{ef}{}^C{}_D$$

where $\Omega_{ab}{}^{C}{}_{D}$ is the background tractor curvature. This can be further specified to

$$\underline{\Omega}_{ab}{}^{CD} = Z^{C}{}_{c}Z^{D}{}_{d}\underline{\mathrm{W}}_{ab}{}^{cd} - 2X^{[C}Z^{D]}{}_{d}\underline{\mathrm{Y}}_{ab}{}^{d}$$

where $\underline{\mathrm{W}}_{ab}{}^{c}{}_{d} = \Pi_{a}{}^{e}\Pi_{b}{}^{f}\mathrm{W}_{ef}{}^{c}{}_{d}$ and $\underline{\mathrm{Y}}_{ab}{}^{d} = \Pi_{a}{}^{e}\Pi_{b}{}^{f}\mathrm{Y}_{ef}{}^{d}$.

Proposition 3.2.53. *The tractor projected part of the ambient tractor curvature $(\Pi_{\Sigma}\underline{\Omega})_{ab}{}^{CD} := \Pi^{C}{}_{E}\Pi^{D}{}_{F}\underline{\Omega}_{ab}{}^{EF}$ is explicitly given in an intrinsic scale $\bar{g} \in \bar{c}$ by*

$$(\Pi_{\Sigma}\underline{\Omega})_{ab}{}^{CD} \overset{\bar{g}}{=} \bar{Z}^{C}{}_{c}\bar{Z}^{D}{}_{d}(\Pi_{\Sigma}W)_{ab}{}^{cd} - 2\,\bar{X}^{[C}\bar{Z}^{D]}{}_{d}\left(\overset{\mathrm{N}}{\underline{\mathrm{W}}}_{ab}{}^{d}\mathbf{H} + (\Pi_{\Sigma}\mathrm{Y})_{ab}{}^{d}\right)$$

where

$$\overset{\mathrm{N}}{\underline{\mathrm{W}}}_{ab}{}^{d} := \underline{\mathrm{W}}_{ab}{}^{de}\mathbf{N}_{e}$$

Proof. We use the expressions (3.60) to compute:

$$\begin{aligned}
\Pi^{C}{}_{E}\Pi^{D}{}_{F}\underline{\Omega}_{ab}{}^{EF} &= \Pi^{C}{}_{E}\Pi^{D}{}_{F}\left(Z^{E}{}_{e}Z^{F}{}_{f}\underline{\mathrm{W}}_{ab}{}^{ef} - X^{E}Z^{F}{}_{f}\underline{\mathrm{Y}}_{ab}{}^{f} + X^{F}Z^{E}{}_{e}\underline{\mathrm{Y}}_{ab}{}^{e}\right)\\
&= (\bar{Z}^{C}{}_{e} + \bar{X}^{C}\mathbf{H}\mathbf{N}_{e})(\bar{Z}^{D}{}_{f} + \bar{X}^{D}\mathbf{H}\mathbf{N}_{f})\underline{\mathrm{W}}_{ab}{}^{ef}\\
&\quad - \bar{X}^{C}(\bar{Z}^{D}{}_{f} + \bar{X}^{D}\mathbf{H}\mathbf{N}_{f})\underline{\mathrm{Y}}_{ab}{}^{f} + \bar{X}^{D}(\bar{Z}^{C}{}_{e} + \bar{X}^{C}\mathbf{H}\mathbf{N}_{e})\underline{\mathrm{Y}}_{ab}{}^{e}\\
&= \bar{Z}^{C}{}_{e}\bar{Z}^{D}{}_{f}\underline{\mathrm{W}}_{ab}{}^{ef} + \bar{X}^{C}\bar{Z}^{D}{}_{f}\mathbf{H}\mathbf{N}_{e}\underline{\mathrm{W}}_{ab}{}^{ef}\\
&\quad + \bar{Z}^{C}{}_{e}\bar{X}^{D}\mathbf{H}\mathbf{N}_{f}\underline{\mathrm{W}}_{ab}{}^{ef} + \cancel{\bar{X}^{C}\bar{X}^{D}\mathbf{H}^{2}\mathbf{N}_{e}\mathbf{N}_{f}\underline{\mathrm{W}}_{ab}{}^{ef}}\\
&\quad - \bar{X}^{C}\bar{Z}^{D}{}_{f}\underline{\mathrm{Y}}_{ab}{}^{f} - \cancel{\bar{X}^{C}\bar{X}^{D}\mathbf{H}\mathbf{N}_{f}\underline{\mathrm{Y}}_{ab}{}^{f}} + \bar{X}^{D}\bar{Z}^{C}{}_{e}\underline{\mathrm{Y}}_{ab}{}^{e} + \cancel{\bar{X}^{D}\bar{X}^{C}\mathbf{H}\mathbf{N}_{e}\underline{\mathrm{Y}}_{ab}{}^{e}}
\end{aligned}$$

Observing that $\bar{Z}^{C}{}_{e}\bar{Z}^{D}{}_{f}\underline{\mathrm{W}}_{ab}{}^{ef} = \bar{Z}^{C}{}_{c}\bar{Z}^{D}{}_{d}(\Pi_{\Sigma}W)_{ab}{}^{cd}$ and $\bar{Z}^{D}{}_{f}\underline{\mathrm{Y}}_{ab}{}^{f} = \bar{Z}^{D}{}_{d}(\Pi_{\Sigma}\mathrm{Y})_{ab}{}^{d}$, we get the claim. □

Proposition 3.2.54. *The exterior square of the tractor shape 1-form is given explicitly by*

$$2\,\mathbb{L}_{[a}{}^{C}\mathbb{L}_{b]}{}^{D} = \bar{Z}^{C}{}_{c}\bar{Z}^{D}{}_{d}2\,\mathring{\mathbf{L}}_{[a}{}^{c}\mathring{\mathbf{L}}_{b]}{}^{d} - 2\,\bar{Z}^{[C}{}_{c}\bar{X}^{D]}\left(2\,\mathring{\mathbf{L}}_{[a}{}^{c}\overset{\mathrm{N}}{\underline{\mathrm{P}}}_{b]} + 2\,\mathring{\mathbf{L}}_{[a}{}^{c}\overline{\nabla}_{b]}\mathbf{H}\right)$$

Proof. We compute:

$$
\begin{aligned}
2\,\mathbb{L}_{[a}{}^{C}\mathbb{L}_{b]}{}^{D} &= \left(\bar{Z}^{C}{}_{c}\mathring{\mathbf{L}}_{a}{}^{c} - \bar{X}^{C}\left(\overset{\mathrm{N}}{\underline{\mathsf{P}}}_{a} + \overline{\nabla}_{a}\mathbf{H}\right)\right)\left(\bar{Z}^{D}{}_{d}\mathring{\mathbf{L}}_{b}{}^{d} - \bar{X}^{D}\left(\overset{\mathrm{N}}{\underline{\mathsf{P}}}_{b} + \overline{\nabla}_{b}\mathbf{H}\right)\right) \\
&\quad - (a \leftrightarrow b) \\
&= \bar{Z}^{C}{}_{c}\bar{Z}^{D}{}_{d} 2\,\mathring{\mathbf{L}}_{[a}{}^{c}\mathring{\mathbf{L}}_{b]}{}^{d} - \bar{Z}^{C}{}_{c}\bar{X}^{D}\left(2\,\mathring{\mathbf{L}}_{[a}{}^{c}\overset{\mathrm{N}}{\underline{\mathsf{P}}}_{b]} + 2\,\mathring{\mathbf{L}}_{a}{}^{c}\overline{\nabla}_{b}\mathbf{H}\right) \\
&\quad - \bar{X}^{C}\bar{Z}^{D}{}_{d}\left(2\,\mathring{\mathbf{L}}_{[b}{}^{d}\overset{\mathrm{N}}{\underline{\mathsf{P}}}_{a]} + 2\,\mathring{\mathbf{L}}_{[b}{}^{d}\overline{\nabla}_{a]}\mathbf{H}\right) \qquad \square
\end{aligned}
$$

Comparing the expressions for $\check{\Omega}_{ab}{}^{CD}$, $(\Pi_\Sigma\underline{\Omega})_{ab}{}^{CD}$ and $2\,\mathbb{L}_{[a}{}^{C}\mathbb{L}_{b]}{}^{D}$ and equating the coefficients at $\bar{Z}^{C}{}_{c}\bar{Z}^{D}{}_{d}$ we obtain from the tractor Gauss equation 3.76 that

$$\overline{\mathrm{W}}_{ab}{}^{cd} - 4\bar{\delta}_{[a}{}^{[c}\mathcal{F}_{b]}{}^{d]} = (\Pi_\Sigma W)_{ab}{}^{cd} + 2\,\mathring{\mathbf{L}}_{[a}{}^{c}\mathring{\mathbf{L}}_{b]}{}^{d}$$

or

$$\overline{\mathrm{W}}_{ab}{}^{cd} = (\Pi_\Sigma W)_{ab}{}^{cd} + 2\,\mathring{\mathbf{L}}_{[a}{}^{c}\mathring{\mathbf{L}}_{b]}{}^{d} + 4\bar{\delta}_{[a}{}^{[c}\mathcal{F}_{b]}{}^{d]}$$

This is clearly a version of the conformal Gauss-Weyl equation (2.32).

Taking the totally trace-free parts (denoted by the subscript $\circ$) and suppressing the indices, we recover the known equation (2.34) (cf. [56])

$$\overline{\mathrm{W}} = (\Pi_\Sigma \mathrm{W})_\circ + (\mathring{\mathbf{L}} \owedge \mathring{\mathbf{L}})_\circ$$

Taking traces in the above equation, we get

$$0 = (\Pi_\Sigma \mathrm{W})_{ab}{}^{ad} + 2\,\mathring{\mathbf{L}}_{[a}{}^{a}\mathring{\mathbf{L}}_{b]}{}^{d} + 4\bar{\delta}_{[a}{}^{[a}\mathcal{F}_{b]}{}^{d]}$$

but

$$
\begin{aligned}
\bar{\mathbf{g}}^{ac}(\Pi_\Sigma \mathrm{W})_{abcd} &= \mathbf{g}^{ac}\Pi_{a}{}^{a'}\Pi_{b}{}^{b'}\Pi_{c}{}^{c'}\Pi_{d}{}^{d'}\mathrm{W}_{a'b'c'd'} \\
&= \mathbf{g}^{ac}\left(\delta_{a}{}^{a'} - \mathbf{N}_{a}\mathbf{N}^{a'}\right)\left(\delta_{c}{}^{c'} - \mathbf{N}_{c}\mathbf{N}^{c'}\right)\Pi_{b}{}^{b'}\Pi_{d}{}^{d'}\mathrm{W}_{a'b'c'd'} \\
&= \left(\mathbf{g}^{a'c'} - \mathbf{N}^{a'}\mathbf{N}^{c'} - \cancel{\mathbf{N}^{c'}\mathbf{N}^{a'}} + \cancel{\mathbf{N}^{a'}\mathbf{N}^{c'}}\right)\Pi_{b}{}^{b'}\Pi_{d}{}^{d'}\mathrm{W}_{a'b'c'd'}
\end{aligned}
$$

and therefore $\bar{\mathbf{g}}^{ac}(\Pi_\Sigma \mathrm{W})_{abcd} = -\overset{\mathrm{NN}}{\mathrm{W}}_{bd}$ where $\overset{\mathrm{NN}}{\mathrm{W}}_{bd} := \Pi_{b}{}^{b'}\Pi_{d}{}^{d'}\mathrm{W}_{a'b'c'd'}\mathbf{N}^{a'}\mathbf{N}^{c'}$.

Similarly,

$$2\,\mathring{\mathbf{L}}_{[a}{}^{a}\mathring{\mathbf{L}}_{b]}{}^{d} = \mathring{\mathbf{L}}_{a}{}^{a}\mathring{\mathbf{L}}_{b}{}^{d} - \mathring{\mathbf{L}}_{b}{}^{a}\mathring{\mathbf{L}}_{a}{}^{d} = -\mathring{\mathbf{L}}^{2d}_{b}$$

and

$$\begin{aligned} 4\bar{\delta}_{[a}{}^{[a}\mathcal{F}_{b]}{}^{d]} &= \bar{\delta}_a{}^a\mathcal{F}_b{}^d - \bar{\delta}_b{}^a\mathcal{F}_a{}^d + \bar{\delta}_b{}^d\mathcal{F}_a{}^a - \bar{\delta}_a{}^d\mathcal{F}_b{}^a \\ &= \bar{n}\mathcal{F}_b{}^d - \mathcal{F}_b{}^d + \bar{\delta}_b{}^d\mathcal{F}_a{}^a - \mathcal{F}_b{}^d \\ &= (\bar{n}-2)\mathcal{F}_b{}^d + \bar{\delta}_b{}^d\mathcal{F}_a{}^a \end{aligned}$$

We obtain that $0 = -\overset{\text{NN}}{\mathrm{W}}_{bd} - \mathring{\mathbf{L}}^2_{bd} + (\bar{n}-2)\mathcal{F}_{bd} + \bar{\mathbf{g}}_{bd}\mathcal{F}_a{}^a$

Taking traces again we get $0 = -|\mathring{\mathbf{L}}|^2 + (2\bar{n}-2)\mathcal{F}_a{}^a$ and therefore $\mathcal{F}_a{}^a = \frac{|\mathring{\mathbf{L}}|^2}{2(\bar{n}-1)}$

Thus we have $\mathcal{F}_{bd} = \frac{1}{\bar{n}-2}\left(\overset{\text{NN}}{\mathrm{W}}_{bd} + \mathring{\mathbf{L}}^2_{bd} - \frac{|\mathring{\mathbf{L}}|^2}{2(\bar{n}-1)}\bar{\mathbf{g}}_{bd}\right)$.

The conformal Codazzi equation revisited

Proposition 3.2.55. *The tractor projected part of the action of the ambient tractor curvature on the normal tractor is given explicitly in an intrinsic scale $\bar{g} \in \bar{c}$ by*

$$\Pi_\Sigma\left(\underline{\Omega}\cdot \mathrm{N}\right)_{ab}{}^C = \Pi^C{}_E\underline{\Omega}_{ab}{}^E{}_D\mathrm{N}^D \overset{\bar{g}}{=} \bar{Z}^C{}_c\underline{\mathrm{W}}_{ab}{}^{cd}\mathbf{N}_d - \bar{X}^C\overset{\text{N}}{\underline{\mathrm{Y}}}_{ab}$$

where

$$\overset{\text{N}}{\underline{\mathrm{Y}}}_{ab} := \underline{\mathrm{Y}}_{ab}{}^d\mathbf{N}_d = \Pi_a{}^{a'}\Pi_b{}^{b'}\mathrm{Y}_{a'b'}{}^d\mathbf{N}_d$$

Proposition 3.2.56. *The tractor exterior covariant derivative $\check{\nabla}\wedge\mathbb{L}$ of the tractor shape 1-form $\mathbb{L}_a{}^B$ with respect to the projected ambient tractor connection is given explicitly by*

$$\begin{aligned} 2\,\check{\nabla}_{[a}\mathbb{L}_{b]}{}^C &= \bar{Z}^C{}_c\left(2\,\overline{\nabla}_{[a}\mathring{\mathbf{L}}_{b]}{}^c - 2\,\bar{\delta}_{[a}{}^c\overset{\text{N}}{\underline{\mathsf{P}}}_{b]} - 2\,\bar{\delta}_{[a}{}^c\overline{\nabla}_b\mathbf{H})\right) \\ &\quad -\bar{X}^C\left(2\,\Pi_{[b}{}^c(\underline{\nabla}_{a]}\mathsf{P}_{cd})\mathbf{N}^d + 2\,(\Pi_\Sigma\mathsf{P})_{c[b}\mathbf{L}_{a]}{}^c + 2\,\mathbf{N}_{[a}\mathbf{L}_{b]}{}^c\overset{\text{N}}{\underline{\mathsf{P}}}_c + 2\,\check{\mathsf{P}}_{c[a}\mathring{\mathbf{L}}_{b]}{}^c\right) \end{aligned}$$

Comparing the Z-slots in the expressions for the terms of (3.78), we recover the conformal Codazzi equation (2.35)

$$2\,\overline{\nabla}_{[a}\mathring{\mathbf{L}}_{b]}{}^c = \underline{\mathrm{W}}_{ab}{}^{cd}\mathbf{N}_d + 2\,\bar{\delta}_{[a}{}^c\overset{\text{N}}{\underline{\mathsf{P}}}_{b]} + 2\,\bar{\delta}_{[a}{}^c\overline{\nabla}_b\mathbf{H})$$

Contracting the indices a and b, we obtain the contracted conformal Codazzi equation (2.36)

$$\overline{\nabla}^a\mathring{\mathbf{L}}_{ab} = (\bar{n}-1)\left[\overset{\text{N}}{\underline{\mathsf{P}}}_b + \overline{\nabla}_b\mathbf{H}\right]$$

Proposition 3.2.57. *The tractor shape form can be also presented as*

$$\mathbb{L}_{aB} \overset{\bar{g}}{=} \bar{Z}_B{}^b \mathring{\mathbf{L}}_{ab} - \tfrac{1}{\bar{n}-1} \bar{X}_B \overline{\nabla}^b \mathring{\mathbf{L}}_{ab} \qquad (3.83)$$

Proof. One needs to use the contracted conformal Codazzi equation to rewrite the last term of (3.72). □

3.3 Invariant tractor operators on hypersurfaces

Using the intrinsic tractor bundle, we can construct the tractor invariant operators acting on densities along the hypersurface, which we denote as $\overline{\mathbb{D}}_{AP}$ and $\overline{\mathrm{D}}_A$ and term the *intrinsic double-D* and the *intrinsic Thomas-D* operators , respectively.

As we have noticed in Section 3.1.4, the tractor operators can be twisted with any vector bundle equipped with an invariant connection, in particular, with a tractor bundle. We can use the projected ambient tractor connection $\check{\nabla}$ to twist the operators $\overline{\mathbb{D}}_{AP}$ and $\overline{\mathrm{D}}_A$ and obtain the operators $\check{\mathbb{D}}_{AP}$ and $\check{\mathrm{D}}_A$, which may be more convenient in some calculations, as the tractor Gauss and Codazzi equations allude.

The most crucial observation (which is due to A.R.Gover) developed in the present thesis is that this twisting can be done not only with the intrinsic tractor bundle, but also with the ambient tractor bundle of the hypersurface. The latter approach gives us a hope to escape certain restrictions, which are known from representation theory, and obtain a possibly larger class of invariants than it would be available from just the intrinsic conformal structure of the hypersurface.

Moreover, the operators should act tangentially (see e.g. in [11] or [29]) on the normal tractors to ensure that the results remain independent of the choice of an oriented defining function.

We introduce now the operators, which will be termed the twisted intrinsic tractor operators where the word *"twisted"* before "intrinsic" will always mean *"twisted with the ambient tractor bundle"* unless stated otherwise.

One can repeat the construction of the double-D and Thomas-D operator and obtain their twisted intrinsic versions by considering the following operator. The required properties of the twisted intrinsic operators also follow immediately from the general considerations, which we discuss in Sections 2.1.3 and 3.1.4.

Definition 3.3.1. The *twisted intrinsic pre-D operator* $\underline{\overline{\mathfrak{D}}}^A$ is the intrinsic pre-D operator twisted with the ambient tractor bundle. Explicitly, it can be given by the expression

$$\underline{\overline{\mathfrak{D}}}^A f := \bar{Y}^A w f + \bar{Z}^A{}_a \underline{\nabla}^a f \tag{3.84}$$

Remark 3.3.2. This is the first occurrence where our conventions to overline the intrinsic objects and to underline the operators twisted with the ambient tractor connection $\underline{\nabla}$ are combined in one symbol. This notation is proved to be convenient in the constructions of invariants, which we present in the next chapter.

The twisted intrinsic pre-D operator possesses the properties, which we need to construct the invariant tractor operators as we did in Section 3.1.4.

Proposition 3.3.3. *The twisted intrinsic pre-D operator acts on weighted ambient tractors and has the following properties:*

1. *it satisfies the Leibniz rule;*
2. *it acts tangentially;*
3. *it rescales according to the equation*

$$\widehat{\underline{\overline{\mathfrak{D}}}^A} f = \underline{\overline{\mathfrak{D}}}^A f + \bar{X}^A (\overline{\Upsilon}^a \underline{\nabla}_a f + \tfrac{|\Upsilon|^2}{2} w f) \tag{3.85}$$

The rescaling rule of the twisted pre-D operator allows us to introduce a version of the double-D operator

Definition 3.3.4. The twisted intrinsic double-D operator $\underline{\overline{\mathbb{D}}}_{AP}$ is

$$\underline{\overline{\mathbb{D}}}_{AP} f := 2\bar{X}_{[P} \underline{\overline{\mathfrak{D}}}_{A]} f \tag{3.86}$$

Proposition 3.3.5. *The twisted double-D operator $\underline{\overline{\mathbb{D}}}_{AP}$ acts on weighted ambient tractors and has the following properties*

1. *it satisfies the Leibniz rule;*
2. *it acts tangentially;*
3. *it is conformally invariant.*

In the usual tractor calculus we use the box operator $\Box f$ that can be obtained from the pre-D operator as $\mathfrak{D}^A\mathfrak{D}_A f = \Box f$. If we can compute the corresponding identity for the twisted pre-D operator. A short calculation shows that

$$\underline{\overline{\mathfrak{D}}}^A\underline{\overline{\mathfrak{D}}}_A f = \underline{\Delta} f + \mathsf{J}wf + \tfrac{\mathbf{H}^2}{2}\bar{n}wf = \underline{\Delta} f + \bar{\mathsf{J}}wf + \tfrac{\bar{n}^2-4\bar{n}+2}{(\bar{n}-2)(\bar{n}-1)}|\mathring{\mathrm{L}}|^2 wf$$

where we have used $\mathsf{J}+\frac{\mathbf{H}^2}{2}\bar{n} = \bar{\mathsf{J}}+\frac{\bar{n}^2-4\bar{n}+2}{(\bar{n}-2)(\bar{n}-1)}|\mathring{\mathrm{L}}|^2$ that follows from the contracted Gauss–Schouten equation (2.31).

The trailing term that we have obtained in the expression for $\underline{\overline{\mathfrak{D}}}^A\underline{\overline{\mathfrak{D}}}_A f$ is conformally invariant, so dropping it off will not hurt the conformal behavior of the remaining part. This way we arrive to the following object.

Definition 3.3.6. The twisted intrinsic box operator $\underline{\overline{\Box}}$ is

$$\underline{\overline{\Box}} f \coloneqq \bar{\mathbf{g}}^{ab}\Big(\underline{\nabla}_a\underline{\nabla}_b f + \overline{\mathsf{P}}_{ab}wf\Big) = \underline{\Delta} f + \bar{\mathsf{J}}wf \tag{3.87}$$

Proposition 3.3.7. *The twisted intrinsic box operator $\underline{\overline{\Box}}$ acts on weighted ambient tractors and has the following rescaling rule*

$$\widehat{\underline{\overline{\Box}}} f = \underline{\overline{\Box}} f + (\bar{n}+2w-2)\Big(\overline{\Upsilon}^a\underline{\nabla}_a f + \tfrac{|\overline{\Upsilon}|^2}{2}wf\Big) \tag{3.88}$$

Proof. This is a special case of Proposition 2.1.1. □

Hence this is the right candidate for our needs, and we give the following

Definition 3.3.8. The twisted Thomas-D operator $\underline{\overline{\mathrm{D}}}_A$ is

$$\underline{\overline{\mathrm{D}}}_A f \coloneqq (\bar{n}+2w-2)\underline{\overline{\mathfrak{D}}}_A f - \bar{X}_A\underline{\overline{\Box}} f \tag{3.89}$$

Proposition 3.3.9. *The twisted Thomas-D operator $\underline{\overline{\mathrm{D}}}_A$ acts on weighted ambient tractors and has the following properties*

- *it acts tangentially;*
- *it lowers the conformal weight by 1;*
- *it is conformally invariant.*

Remark 3.3.10. The action of the twisted intrinsic operators adds some intrinsic indices to the result, which may also have ambient indices, for instance $\mathrm{N}^A\underline{\overline{\mathrm{D}}}_A V^B = 0$, but in general $\mathrm{N}_B\underline{\overline{\mathrm{D}}}_A V^B \neq 0$. This requires a certain level of attention when performing actual calculations.

Another possible way to construct an invariant tractor operator, which is able to act on ambient tractors, is to consider the background pre-D operator $\mathfrak{D}^A$ restricted onto the hypersurface.

Definition 3.3.11. We denote this operator by $\underline{\mathfrak{D}}_A$ an refer to it as the *ambient pre-D operator*. Its action is given by the formula

$$\underline{\mathfrak{D}}^A f \stackrel{g}{=} Y^A w f + Z^A{}_a \underline{\nabla}^a f \tag{3.90}$$

The underline in the notation $\underline{\mathfrak{D}}^A$ is used to remind that the definition of this operator uses the ambient connection $\underline{\nabla}_a$.

This operator has a sightly different rescaling property.

Proposition 3.3.12. *The ambient pre-D operator is defined on weighted tractor densities along the hypersurface and has the following properties:*

1. *It satisfies the Leibniz rule* $\underline{\mathfrak{D}}_A(f_1 f_2) = f_1 \underline{\mathfrak{D}}_A(f_2) + f_2 \underline{\mathfrak{D}}_A(f_1)$*;*
2. *It acts tangentially;*
3. *it has the following rescaling rule:*

$$\widehat{\underline{\mathfrak{D}}^A} f = \underline{\mathfrak{D}}^A f - \mathrm{N}^A \mathrm{N} \cdot \Upsilon w f + X^A (\overline{\Upsilon}^a \underline{\nabla}_a f + \tfrac{|\Upsilon|^2}{2} w f - \mathbf{H} w f) \tag{3.91}$$

Proof. Everything is straightforward. The rescaling rule is found to be

$$\widehat{\underline{\mathfrak{D}}^A f} = \underline{\mathfrak{D}}^A f - Z^A{}_a \mathrm{N}^a \mathrm{N} \cdot \Upsilon w f + X^A (\overline{\Upsilon}^a \underline{\nabla}_a f + \tfrac{|\Upsilon|^2}{2} w f)$$

and then rewritten as in the statement. □

At the expense of adding three indices, we still are able to construct an invariant operator out of $\underline{\mathfrak{D}}_A$.

Definition 3.3.13. The triple-D operator $\underline{D}_{ABC}$ is given by

$$\underline{\mathrm{D}}_{ABC} f \coloneqq 6 \mathrm{N}_{[C} X_B \underline{\mathfrak{D}}_{A]} f \tag{3.92}$$

We keep the underlining in the notation for this operator to emphasize that it acts in tangential directions along the hypersurface.

Proposition 3.3.14. *The triple-D operator acts on weighted ambient tractors and has the following properties:*

1. *it satisfies the Leibniz rule* $\underline{\mathrm{D}}_{ABC}(f_1 f_2) = f_1 \underline{\mathrm{D}}_{ABC}(f_2) + f_2 \underline{\mathrm{D}}_{ABC}(f_1)$*;*
2. *it acts tangentially;*
3. *it is conformally invariant* $\widehat{\underline{\mathrm{D}}_{ABC}} f = \underline{\mathrm{D}}_{ABC} f$

3.3.1 The hypersurface Weyl tractors

Recall that the curvature of the ambient tractor connection can be viewed as an ambient-bundle-valued intrinsic 2-form $\underline{\Omega}_{ab}{}^{C}{}_{D}$ defined by $\underline{\Omega}_{ab}{}^{C}{}_{D} V^D = 2\underline{\nabla}_{[a}\underline{\nabla}_{b]} V^C$. Keeping this point of view, we define the ambient tractor expression of the ambient tractor connection as

$$\underline{\Omega}_{ABCD} \coloneqq Z_A{}^a Z_B{}^b \underline{\Omega}_{abCD}$$

and subsequently the corresponding W-tractor is defined by

$$\underline{\mathrm{W}}_{ABCD} \coloneqq \tfrac{3}{\bar{n}-2} \underline{\overline{\mathrm{D}}}^{A'} X_{[A'} \underline{\Omega}_{AB]CD} = \tfrac{3}{n-3} \underline{\overline{\mathrm{D}}}^{A'} X_{[A'} \underline{\Omega}_{AB]CD}$$

The purpose of introducing this version of the W-tractor is to be able to compare it with its intrinsic

$$\overline{\mathrm{W}}_{ABCD} \coloneqq \tfrac{3}{\bar{n}-2} \overline{\mathrm{D}}^{A'} \bar{X}_{[A'} \overline{\Omega}_{AB]CD} \overline{\mathrm{W}}_{ABCD} = \tfrac{3}{n-3} \overline{\mathrm{D}}^{A'} \bar{X}_{[A'} \overline{\Omega}_{AB]CD}$$

and the "projected ambient"

$$\check{\mathrm{W}}_{ABCD} \coloneqq \tfrac{3}{\bar{n}-2} \check{\mathrm{D}}^{A'} \bar{X}_{[A'} \check{\Omega}_{AB]CD} = \tfrac{3}{n-3} \check{\mathrm{D}}^{A'} \bar{X}_{[A'} \check{\Omega}_{AB]CD}$$

counterparts, where $\overline{\Omega}_{ABCD} \coloneqq \bar{Z}_A{}^a \bar{Z}_B{}^b \overline{\Omega}_{abCD}$ and $\check{\Omega}_{ABCD} \coloneqq \bar{Z}_A{}^a \bar{Z}_B{}^b \check{\Omega}_{abCD}$ are the corresponding tractor expressions of the curvatures of the intrinsic and the projected ambient tractor connections respectively.

Chapter 4

Construction of Hypersurface Conformal Invariants

In the previous chapter we have described the ingredients, which now can be used to make various families of conformal invariants of hypersurfaces. We modify the ideas of [26] and [27] and define some classes of such invariants.

Before we start talking about invariants of hypersurfaces, let us briefly recall the basic definitions and present a slightly simplified construction of conformal invariants following [27].

4.1 Elements of conformal invariant theory

Generally speaking, an invariant is a function, which does not change its values when its arguments are transformed in a certain way (cf. e.g. [54]). The value of the function may be an element of a certain class, a representative of which is obtained by evaluation of the function on a concrete choice of arguments, and then the whole class is regarded as the value of the invariant. This notion, of course, is very broad, since it has applications in many (if not every!) areas of mathematics.

Metric invariants

We begin with the following definition (cf.[5], [24], [27]).

Definition 4.1.1. Let (M, g) be a (pseudo)-Riemannian manifold. A *scalar metric invariant* is a function $P(g)$ that in any choice of local coordinates (x^i) around each point $p \in M$ is expressed as a universal polynomial in $(\det g)^{-1}$ and $\partial_{k_1} \dots \partial_{i_s} g_{ij}$, $s \geq 0$, such that for any local diffeomorphism φ of M

$$P(\varphi^* g) = \varphi^* P(g) \tag{4.1}$$

Remark 4.1.2. To be more precise, this is the definition of a local polynomial scalar-valued differential invariant of the Riemannian structure of manifold M. At least within this chapter we only speak about local differential invariants, therefore these adjectives can be safely dropped off. In the next chapter we will be dealing with global invariants too.

Remark 4.1.3. If we allow in the Definition 4.1.1 that P has values in a tensor or a vector bundle, we obtain tensor or vector-bundle valued invariants.

Metric Weyl invariants

Definition 4.1.1 gives no information about the existence of metric invariants, however examples can be easily given using the standard objects of Riemannian geometry. For instance, the scalar curvature is a scalar metric invariant: it is a complete contraction of the Riemannian curvature.

The Riemannian curvature R_{abcd} and Ricci curvature Ric_{ab} are examples of tensor valued invariants. There is also a metric-invariant first-order differential operator, the Levi-Civita connection ∇_a, associated with the given Riemannian metric. Using these objects as building blocks we can manufacture an infinite series of metric invariants.

Definition 4.1.4. A *scalar Weyl metric invariant* is a linear combination of complete contractions (denoted by contr) of the following two forms:
even invariants,

- $\operatorname{contr}\left(g^{-1}\cdot\dots\cdot g^{-1}\cdot \mathrm{R}^{(k_1)}\cdot\dots\cdot \mathrm{R}^{(k_r)}\right)$

and *odd* invariants, defined if the manifold M is oriented,

- $\operatorname{contr}\left(\epsilon\cdot g^{-1}\cdot\dots\cdot g^{-1}\cdot \mathrm{R}^{(k_1)}\cdot\dots\cdot \mathrm{R}^{(k_r)}\right)$

where in the above displays

$$\mathrm{R}^{(k)}_{abcd,p_1\dots p_k} \coloneqq \nabla_{p_1}\dots\nabla_{p_k}\mathrm{R}_{abcd}$$

and g^{-1} is the inverse Riemannian metric, that is g^{ab}, and ϵ is the Riemannian volume form.

Remark 4.1.5. The classical invariant theory ([73]) guarantees that all metric invariants can be obtained as Weyl invariants, so we have a complete set of invariants of Riemannian structure. See also [5].

Remark 4.1.6. Taking partial contractions of the monomials in the definition and tensor parts thereof we get tensor-valued metric invariants. Again, all tensor-valued metric invariants can be obtained through this procedure.

Remark 4.1.7. We can also consider invariant differential operators built from the iterated Levi-Civita connection and the Riemannian curvature, with some indices contracted, using g^{-1}. All metric invariant differential operators arise this way.

We deal with more sophisticated examples in this thesis, actually. The tractor curvature is a $\mathcal{E}_{[ab]} \otimes \mathcal{T}_{[CD]}$-valued metric invariant for each choice of a Riemannian metric from the conformal structure. Looking a little bit further, we can say that the normal tractor is an ambient-tractor valued metric invariant on a hypersurface, with a metric in the background manifold fixed. (this is just a part of the truth, however).

The tractor operators $\mathfrak{D}_A$, $\mathbb{D}_{AP}$ and D_A discussed in Chapter 3 are, of course, metric invariant differential operators when a metric on M is fixed.

In fact, all invariant tractors and invariant operators, acting on tractors, are examples of metric invariants for each choice of a metric on M.

Conformal invariants

Some metric invariants behave nicely when the metric is rescaled by a positive function. The best possible behavior is described in the following definition, however there are other types of deformation, which are encountered in interesting classes of objects (e.g. Q-like quantities) that we touch upon in the next chapter.

Definition 4.1.8. A scalar *conformal* invariant of *conformal weight* w is a metric invariant with values in the bundle $\mathcal{E}[w]$ of densities of conformal weight w, if in addition to the requirements of Definition 4.1.1 it also satisfies the following rescaling property:

$$P(\Omega^2 g) = P(g)$$

for any smooth nonvanishing positive function $\Omega\colon M \to \mathbb{R}$

We work with conformally weighted sections of vector bundles, or vector bundle valued densities, in order to simplify the notation and calculations. If we worked this with nonweighted scalars, we would have to write the condition in the above definition as $P(\Omega^2 g) = \Omega^w P(p)$ Working with densities absorbs the factors Ω^w in the definitions and calculations.

It is easy to see that the only *scalar* conformal invariants of zero weight are constants, so it is essential to consider arbitrary weights. Throughout this thesis we will only deal with integer weights.

It is well known that for manifolds of $\dim M = 2$ there are no local invariants of conformal structure, more precisely, any two metrics are locally conformally equivalent (cf. the uniformization theorem).

Starting from dimension 3, there are examples of conformally invariant quantities and maps. For instance, on 3-dimensional manifolds the Cotton tensor is invariant. In contrast with this, the Weyl tensor is invariant in any dimension $n \geq 3$, however in dimension $n = 3$ the Weyl tensor is identically zero (and it is not defined for dimension $n = 2$). Notice that we can form scalar invariants from these tensors taking their complete contractions, as discussed earlier. Thus, on any manifold of $\dim M \geq 3$ there is a conformally invariant scalar density $|\mathrm{W}|^2 = \mathrm{W}_{abcd}\mathrm{W}^{abcd}$. In dimension $n = 3$ a similar role is played by $|\mathrm{Y}|^2 = \mathrm{Y}_{abd}\mathrm{Y}^{abd}$.

The problem of finding all conformal and CR invariants was posed by Charles Fefferman in [23] as an attempt to find the coefficients of the asymptotic expansion of the Bergman kernel of a strictly pseudoconvex domain in $\mathbb{C}^n$. This stimulated an active research and generated rich literature.

To find a conformal invariant with certain properties one may take a Weyl invariant as above, and examine its conformal transformation behavior. Adding lower order terms may, in principle, cancel out the transformation, and an invariant will be constructed. This process is feasible for lower orders, and some examples have been found using it (e.g. the Yamabe and Paneitz operators, see e.g. [58]), but when the order grows, the number of terms increases exponentially, and the search of suitable cancellations becomes an extremely tedious endeavor.

Fortunately, there are ways to construct infinite series of conformal invariants. Fefferman and Graham ([24] and [25]) proposed a solution based on finding an ambient space $\widetilde{M}$ with the Riemannian structure such that the Weyl invariants of this ambient structure give rise to conformal invariants of the initial manifold M. This so-called *ambient construction* gives a complete solution if the dimension of M is odd, while for even dimensional manifolds it has an obstruction at a finite order.

Conformal Weyl invariants

Using methods of tractor calculus, Gover in [27] developed a method that works for both odd and even dimensional cases, and, save for some exceptional cases [7], yields a complete solution of the problem of generating all conformal invariants. This is the construction we describe in the nutshell here. It can be seen as a construction of conformal invariants that replaces the Weyl invariants of the Riemannian structure.

Definition 4.1.9. A scalar *conformal Weyl invariant* is a linear combination of complete contraction of the following two forms,
even invariants,

- $\operatorname{contr}\left(\mathrm{h}^{-1}\cdot\dots\cdot\mathrm{h}^{-1}\cdot\mathrm{W}^{(k_1)}\cdot\dots\cdot\mathrm{W}^{(k_r)}\right)$

and *odd* invariants, defined when M is oriented,

- $\operatorname{contr}\left(\eta\cdot\mathrm{h}^{-1}\cdot\dots\cdot\mathrm{h}^{-1}\cdot\mathrm{W}^{(k_1)}\cdot\dots\cdot\mathrm{W}^{(k_r)}\right)$

where in the above displays

$$\mathrm{W}^{(k)}_{ABCD,P_1\dots P_k} := \mathrm{D}_{P_1}\dots\mathrm{D}_{P_k}\mathrm{W}_{ABCD}$$

By construction, the Weyl invariants are conformally invariant densities that form a family with a countable list of generating elements.

Allowing partial contractions we obtain tractor valued Weyl invariants, that can be used further to extract tensor or scalar valued conformal invariants arising the the projected parts of tractors ([6]).

For instance, in dimension 3, the Weyl tractor W_{ABCD} has the projecting part containing the Cotton tensor Y_{abd}, that gives another proof that this tensor is conformally invariant on 3-dimensional manifolds. In dimension 4 the Weyl tractor W_{ABCD} is just $\mathbb{X}_{AB}{}^{b}\mathbb{X}^{d}_{CD}\mathrm{B}_{bd}$, and the projecting part is essentially the Bach tensor B_{bd}, so it is invariant in dimensions 4. In dimensions $n > 4$ the projecting part of the Weyl tractor contains the Weyl tensor W_{abcd}.

The Weyl tractor has conformal weight -2 and in dimension 6 we have a dimension-dependent conformal invariant $\mathrm{D}_P\mathrm{W}_{ABCD} = \mathrm{X}_P\Box\mathrm{W}_{ABCD}$, which can be used to form a density, e.g. $\mathrm{D}_P\mathrm{W}_{ABCD}\mathrm{D}_P\mathrm{W}^{ABCD}$.

One can also use other conformally invariant operators say, the double-D operator, to obtain invariants in the described way. We refer to them as the *generalized Weyl invariants*.

Conformal quasi-Weyl invariants

Not all invariants arise as conformal Weyl invariants described above. For instance $\mathrm{D}_A\mathrm{D}_B(\mathrm{W}^{CDAE}\mathrm{W}_{CD}{}^B{}_E)$ explicitly (see e.g. [26]) is of the form

$$\tfrac{1}{2}(n-4)^2(n-6)(n-8)\mathrm{FG} + \text{constant} \times \mathrm{W}^{ab}{}_{de}\mathrm{W}^{fg}{}_{ab}\mathrm{W}_{defg}$$

where FG is the invariant obtained by Fefferman and Graham in [24] using the ambient construction. This shows that in dimensions 4, 6 and 8 the tractor formula $\mathrm{D}_A\mathrm{D}_B(\mathrm{W}^{CDAE}\mathrm{W}_{CD}{}^B{}_E)$ gives a trivial invariant (a multiple of lower order invariants).

There is another construction of so called quasi-Weyl invariants, which gives a wider set of invariants. In [27] it is shown that almost all conformal invariants can be obtained as quasi-Weyl, the missed cases having orders that fit into finite segments with the determined limits.

We give here a simplified version of the quasi-Weyl construction which is equivalent to the original one, given in [27].

We present this construction in the three steps as follows.

Step 1. If $\dim M > 3$, then we take a juxtaposition of a finite number of monomials of the form

$$\mathbb{D}_{PP'} \ldots \mathbb{D}_{QQ'}\mathrm{C}_{A'ABCDD'}$$

and contract some indices with the tractor metric or the tractor volume form. Here $\mathrm{C}_{A'ABCDD'}$ is the lifted tractor expression of the Weyl tensor (3.47).

If $\dim M = 3$, then the construction is the same but instead of $\mathrm{C}_{A'ABCDD'}$ the lifted tractor expression of the Cotton tensor $\mathrm{C}_{A'ABDD'}$ is used, see (3.48).

Step 2. Let us for the simplicity assume that we have contracted all the unprimed indices in the previous step (this is not necessary, of course). Consider the symmetric trace-free part of the resulting tractor. In the considered case it will have only primed indices.

If it turns out that some of the X-s in the resulting expression can be moved to the leftmost position, that is we get an expression of the form

$$\mathrm{X}_{(I'} \ldots \mathrm{X}_{J'}J_{K'\ldots L')\circ}$$

then we have an object $J_{K'\ldots L'}$ which is invariant due to the injectivity of the map of taking the symmetric trace-free part (see [27, p. 230]).

Step 3. If the goal is to construct a scalar invariant density, we eliminate all the remaining tractor indices in $J_{K'\dots L'}$ by the Thomas-D operator we obtain an indexless density:

$$\mathrm{D}^{K'}\dots\mathrm{D}^{L'}J_{K'\dots L'}$$

Leaving out some indices not contracted will produce a tractor valued invariant.

Definition 4.1.10. Any linear combination of tractors obtained through the three step algorithm described above is called a *conformal quasi-Weyl invariant*.

Ultimately, we would like to be able to work with the tractor formulæ instead of using tensors. As with the Weyl invariants, we can extract invariant projecting parts from tractors when needed.

Example 4.1.11. Let us demonstrate this process. We can consider a coupled invariant

$$\mathrm{h}^{AB}\mathbb{D}_{AA'}\mathbb{D}_{BB'}f$$

where f is some weighted section with weight w. This is an invariant operator of zero weight. It can be thought as arising from the Laplacian $\nabla^a\nabla_a f$ by replacing the ∇-s with $\mathbb{D}$-s. Applying it to other Weyl or quasi-Weyl invariants we can get further examples.

Expanding the inner double-D operator by (3.24) and using the identities for the action of the double-D operator on adjoint tractor projectors and Table A.5, after some simplifications we obtain

$$\begin{aligned}\mathrm{h}^{AB}\mathbb{D}_{AA'}\mathbb{D}_{BB'}f =& - wnX_{A'}Y_{B'} - 2w^2\, X_{(A'}Y_{B')}f - wZ_{A'}{}^{a}Z_{B'}{}^{b}\mathbf{g}_{ab}f\\ &- (w-1)Z_{A'}{}^{a}X_{B'}\nabla_a f - wX_{A'}Z_{B'}{}^{b}\nabla_b f - (n-1)X_{A'}Z_{B'}{}^{b}\nabla_b f\\ &+ X_{A'}X_{B'}(\Delta f + w\mathsf{J}f)\end{aligned} \tag{4.2}$$

Symmetrizing with respect to indices A' and B' we get

$$\begin{aligned}\mathrm{h}^{AB}\mathbb{D}_{A(A'|}\mathbb{D}_{B|B')}f =& - X_{(A'}Y_{B')}\Big(w(n+2w-2)\Big)f\\ &- wZ_{(A'}{}^{a}Z_{B')}{}^{b}\mathbf{g}_{ab}f - 2wX_{(A'}Y_{B')}\\ &- X_{(A'}Z_{B')}{}^{b}\Big(n+2w-2\Big)\nabla_b f\\ &+ X_{(A'}X_{B')}(\Delta f + w\mathsf{J}f)\end{aligned} \tag{4.3}$$

The trace free part of this with respect to the tractor metric is now

$$\begin{aligned} \mathrm{h}^{AB}\mathbb{D}_{A(A'|}\mathbb{D}_{B|B')\circ}f = &- X_{(A'}Y_{B')\circ}\Big(w(n+2w-2)\Big)f \\ &- X_{(A'}Z_{B')\circ}{}^{b}\Big(n+2w-2\Big)\nabla_b f \\ &+ X_{(A'}X_{B')\circ}(\Delta f + w\mathsf{J}f) \end{aligned} \tag{4.4}$$

or,

$$\mathrm{h}^{AB}\mathbb{D}_{A(A'|}\mathbb{D}_{B|B')\circ}f = -X_{(A'}\mathrm{D}_{B')\circ}f \tag{4.5}$$

where $\mathrm{D}_A f$ is the Thomas-D operator (3.25).

This turns out to be not a good example since when we eliminate the tractor index we get $\mathrm{D}^A\mathrm{D}_A f = 0$. This problem can be circumvented by changing the weight of the original expression that we start with, e.g. for $\mathrm{h}^{AB}f^2\mathbb{D}_{AA'}\mathbb{D}_{BB'}f$ the above computations lead to $\mathrm{D}^A f^2\mathrm{D}_A f$ which is not zero in general

4.2 The notion of hypersurface invariants

4.2.1 Metric invariants of hypersurfaces

When we represent a hypersurface Σ as level set of smooth oriented function $s\colon M \to \mathbb{R}$, we want that geometric objects on the hypersurface, which we express using the defining functions, remain invariant when the defining function is changed.

Example 4.2.1. As we know from Proposition 1.2.9, the unit normal

$$\mathrm{N}_a = |\nabla s|^{-1}\nabla_a s$$

is independent of the choice of an oriented defining function. This is a fundamental example of hypersurface metric invariants, which we define now.

Let Σ be an oriented embedded hypersurface in an oriented Riemannian manifold (M, g), and let Σ be locally represented as the zero set of a smooth oriented defining function.

Definition 4.2.2. A scalar *hypersurface metric invariant* $P(s, g)$ is a universal polynomial expression in the variables from the list $\partial_{k_1}\dots\partial_{k_p}g_{ij}$, g^{ij}, $(\det g)^{-1}$, $\partial_{k_1}\dots\partial_{k_p}s$, $|\nabla s|^{-1}$ such that

1. $P(s, g)$ is natural, in the sense that for any diffeomorphism $\phi\colon M \to M$

we have

$$P(\phi^* s, \phi^* g) = \phi^* P(s, g)$$

2. $P(s,g)|_\Sigma$ is independent of the choice of oriented defining function, that is for any two oriented defining functions s' and s of Σ we have

$$P(s', g)|_\Sigma = P(s, g)|_\Sigma$$

Remark 4.2.3. Allowing the values of $P(s,g)$ to be in tensor (vector) bundles, we get tensor (vector-bundle) valued invariants.

Example 4.2.4. Background metric invariants[1] restricted to the hypersurface are obviously hypersurface metric invariants in the sense of the above definition (since they do not depend on the defining function at all).

This class of invariants is important for us because we want the Weyl invariants of the background structure to be included into the scope. For instance, the background Riemann and Weyl curvatures are in this class, as well as their covariant derivatives in any (not only tangential) directions.

Example 4.2.5. The projection operator $\Pi_a{}^b = \delta_a{}^b - \mathrm{N}_a \mathrm{N}^b$ is an ambient-valued hypersurface invariant since it is a linear combination of the invariants $\delta_a{}^b$ and N_a.

Example 4.2.6. Projected quantities $\Pi^{a'}{}_a \dots T_{a' \dots}$ (where $T_{a' \dots}$ is a background invariant) are hypersurface invariants.

Example 4.2.7. *Intrinsic metric invariants*, that is the metric invariants of the induced Riemannian structure[2] $\bar{g}$ on the hypersurface Σ, are hypersurface metric invariants.

Example 4.2.8. The invariants, which are not intrinsic, are termed *extrinsic*.

The background scalar curvature Scal, or any background scalar Weyl invariant, such as $|\mathrm{R}|^2$ or $|\mathrm{W}|^2$ restricted to the hypersurface, are extrinsic invariants.

We could say that the extrinsic invariants depend on the embedding $\Sigma \hookrightarrow M$, but the induced structure does depend on the embedding too: $\bar{g}_{ab} = \Pi^c{}_a \Pi^d{}_b g_{cd}$. This can be clarified if we regard the hypersurface as a Riemannian manifold *isometrically* embedded into another Riemannian manifold M. Then the intrinsic metric invariants are those that we have on Σ without any embedding.

[1] Invariants of the Riemannian structure g on the background manifold M

[2] That is, they arise from $\bar{g}$ by Definition 4.1.1

A classical striking example of an intrinsic invariant is the Gauss curvature (cf. *Theorema Egregium*).

The interaction between the intrinsic and extrinsic invariants is rather subtle, as the equations of Gauss and Codazzi allude.

4.2.2 Conformal invariants of hypersurfaces

Adding the appropriate modifications to the notion of hypersurface metric invariants, we obtain the following definition.

Definition 4.2.9. A scalar *hypersurface conformal invariant* of weight w is a $\mathcal{E}[w]$-valued metric invariant $P(g,s)$ of a hypersurface $\Sigma = \mathcal{Z}(s)$ in a conformal manifold (M,c) such that for any rescaling of the metric $\widehat{g} = \Omega^2 g$ the following holds:

$$P(\Omega^2 g, s) = P(g, s)$$

Remark 4.2.10. As before, we may allow $P(g,s)$ to have values in (weighted) tensor or vector bundles.

Remark 4.2.11. Again we can distinguish background, projected, ambient, extrinsic and intrinsic conformal invariants (We can safely drop off the adjective "hypersurface" from this terminology).

Example 4.2.12. The ambient conformal metric $\mathbf{g}_{ab}$ (the conformal metric in the ambient bundle $\mathcal{E}^a$ along Σ) is identified with the restriction onto Σ of the background conformal metric on (M,c). It is a background tensor-valued conformal invariant of conformal weight 2. It inverse $\mathbf{g}^{ab}$ has weight -2.

The intrinsic metric $\bar{\mathbf{g}}_{ab}$ is both projected and intrinsic conformal invariant of weight 2.

The projection operator $\Pi_a{}^b = \delta_a{}^b - \mathrm{N}_a \mathrm{N}^b$ is an ambient conformal invariant of weight 0.

Example 4.2.13. The unit conormal N_a and the unit normal N^a are ambient conformal invariants, that is a conformally invariant sections of the ambient bundles $\mathcal{E}_a[1]$ and $\mathcal{E}^a[-1]$.

Example 4.2.14. The curvature invariants W_{abcd} (the background Weyl tensor), $\Pi_\Sigma \mathrm{W}_{abcd}$ (the projected background Weyl tensor) and $\overline{\mathrm{W}}_{abcd}$ (the Weyl tensor of the intrinsic metric) fall under this definition too.

The square of the norm of the background Weyl tensor $|\mathrm{W}|^2 = \mathrm{W}_{abcd}\mathrm{W}^{abcd}$ is a background scalar conformal invariant of weight -4.

We can add similar constructions using the Cotton tensor Y_{abd} and the Bach tensor B_{bd} in the appropriate dimensions.

Example 4.2.15. The umbilicity tensor $\mathring{\mathbf{L}}_{ab}$ is an extrinsic hypersurface conformal tensor-valued invariant of weight 1.

Example 4.2.16. Another interesting example is the binormal part of the ambient Weyl tensor

$$\overset{\mathrm{NN}}{\mathrm{W}}_{bd} := \mathrm{W}_{abcd} N^a N^c \tag{4.6}$$

It is a symmetric trace-free conformally invariant tensor along hypersurface. It is an extrinsic invariant that can be seen as section of an intrinsic bundle $\overline{\mathcal{E}}_{(bd)\circ}$ because both its indices are tangential: $\overset{\mathrm{NN}}{\mathrm{W}}_{bd} N^b = \overset{\mathrm{NN}}{\mathrm{W}}_{bd} N^d = 0$.

Example 4.2.17. The Fialkow tensor (cf. (3.66))

$$\mathcal{F}_{ab} = \Pi_\Sigma \mathsf{P}_{ab} - \overline{\mathsf{P}}_{ab} + H\mathring{\mathbf{L}}_{ab} + \tfrac{H^2}{2}\bar{g}_{ab}$$

is an extrinsic conformal invariant on hypersurfaces of dimension $\bar{n} > 2$ (and it is defined to be 0 when $\bar{n} = 2$).

Example 4.2.18. According to the conformal Gauss equation (2.32)or (2.34), the intrinsic Weyl tensor is a combination of extrinsic conformal invariants.

The conformal Codazzi equation (2.35) expresses the intrinsic exterior covariant derivative of the umbilicity tensor as an extrinsic quantity.

Using the quantities described above, the list of examples of hypersurface invariants can be continued. Say, we can consider a quantity $|\mathrm{W}|^2\, \mathcal{F}_{ab} \overline{\mathrm{W}}^a{}_{cde} \overline{\mathrm{W}}^{bcde}$, and so on.

Generalizing the mentioned examples we can define a class of metric hypersurface invariants that we shall call the Weyl metric hypersurface invariants

Definition 4.2.19. A *scalar hypersurface metric Weyl invariant* is a linear combination of complete contractions of the following two forms:
even invariants

- $\operatorname{contr}\left(g \cdot g \cdot \dots \cdot g \cdot R^{(k_1)} \cdot \dots \cdot R^{(k_r)} \cdot \mathrm{N}^{(l_1)} \cdot \dots \cdot \mathrm{N}^{(l_s)}\right)$

and *odd* invariants, that require orientations on M and Σ,

- $\operatorname{contr}\left(\epsilon \cdot g \cdot g \cdot \dots \cdot g \cdot R^{(k_1)} \cdot \dots \cdot R^{(k_r)} \cdot \mathrm{N}^{(l_1)} \cdot \dots \cdot \mathrm{N}^{(l_s)}\right)$

In the above displays

$$R^{(k)}_{abcd,p_1\dots p_k} := \nabla_{p_1}\dots\nabla_{p_k}\mathrm{R}_{abcd}$$

and

$$\begin{cases} \mathrm{N}^{(0)}_a := \mathrm{N}_a \\ \mathrm{N}^{(l)}_{a,q_1\dots q_l} := \underline{\nabla}_{q_1}\dots\underline{\nabla}_{q_l}\mathrm{N}_a \text{ for } l \geq 1 \end{cases}$$

All these quantities are considered along the hypersurface Σ. For the $R^{(k)}$ this means that we restrict to Σ the result of the differentiation that takes place on the background manifold. The $\mathrm{N}^{(k)}$ may be seen as the sections of the ambient bundles along Σ. Here g, R and ϵ are the background tensors (defined on M).

Remark 4.2.20. If we allow partial contractions in the above definitions, we obtain tensor valued Weyl invariants. The remaining indices are in general ambient, so taking tangential projections on all indices may be desirable.

The definition of $\mathrm{N}^{(l)}$ is suggested by the fact that we only have the listed object being surely independent of extensions. The normal derivatives of the normal tensor do not enjoy this property.

To understand the set of all invariants of this sort one perhaps needs to call for the other methods of the invariant theory. An attempt to give a complete solution to the problems of classifying the invariants of immersions of manifolds with the metric tensors is given in [53].

4.3 Constructions of conformal invariants of hypersurfaces

We introduce the notions of conformal hypersurface Weyl and hypersurface quasi-Weyl invariants extending the corresponding constructions for the case of conformal structure on a manifold (Section 4.1) by adding iterated tangential invariant differential operators acting on the normal tractor.

4.3.1 Conformal Weyl invariants of hypersurfaces

We are now able to present the construction of hypersurface conformal invariants that mimics the corresponding definition of metric Weyl invariants with the appropriate modifications.

Definition 4.3.1. A scalar *hypersurface conformal Weyl invariant* is a linear combination of complete contractions (using the tractor metric h) of one of the following form
even invariants

- $\text{contr}\left(\mathrm{h}\cdot\cdots\cdot\mathrm{h}\cdot\mathrm{W}^{(k_1)}\cdot\cdots\cdot\mathrm{W}^{(k_r)}\cdot\mathrm{N}^{(l_1)}\cdot\cdots\cdot\mathrm{N}^{(l_s)}\right)$

odd invariants (require orientations)

- $\text{contr}\left(\eta\cdot\mathrm{h}\cdot\cdots\cdot\mathrm{h}\cdot\mathrm{W}^{(k_1)}\cdot\cdots\cdot\mathrm{W}^{(k_r)}\cdot\mathrm{N}^{(l_1)}\cdot\cdots\cdot\mathrm{N}^{(l_s)}\right)$

where

$$\mathrm{W}^{(k)}_{ABCD,P_1\dots P_k} := \mathrm{D}_{P_1}\dots\mathrm{D}_{P_k}\mathrm{W}_{ABCD}$$

and now the symbols $\mathrm{N}^{(l)}$ have tractor indices:

$$\begin{cases}\mathrm{N}^{(0)}_A := \mathrm{N}_A\\ \mathrm{N}^{(l)}_{A,Q_1\dots Q_l} := \underline{\overline{\mathrm{D}}}_{Q_1}\dots\underline{\overline{\mathrm{D}}}_{Q_l}\mathrm{N}_A \text{ for } l\geq 1\end{cases}$$

All these quantities are considered along the hypersurface Σ. For the $\mathrm{W}^{(k)}$ this means that we restrict to Σ the result of the differentiation that takes place on the background manifold. The $\mathrm{N}^{(k)}$ may be seen as the sections of the ambient tractor bundles along Σ. Here h, W and η are the background tractors (defined on M).

Remark 4.3.2. If we allow partial contractions in the above definition, we obtain tractor valued invariants. The tractor indices may be then projected into the intrinsic tractor bundles using the tractor projection operator $\Pi_A{}^B$. Now an index, which comes from the normal tractor in the construction, will not necessarily vanish in the projection (see the end of this chapter where we compute $\underline{\overline{\mathrm{D}}}_A\mathrm{N}_B$ explicitly).

Extracting projecting parts from the resulting tractors we obtain tensor-valued invariants.

Example 4.3.3. The normal tractor $\mathrm{N}_A = \mathrm{N}^{(0)}_A$ trivially falls under this definition. This means that the tractor projection operator $\Pi^A{}_B$ is a linear combination of tractor valued conformal hypersurface Weyl invariants, and therefore is a Weyl invariant too. In other words, taking tractor projections preserves the property of being conformal hypersurface Weyl invariant.

Example 4.3.4. The shape 2-tractor $\mathbf{L}_{AB}$ is symmetric 2-tractor defined as $\mathbf{L}_{AB} := \Pi^{B'}{}_B\underline{\overline{\mathrm{D}}}_A\mathrm{N}_{B'}$ is a conformal hypersurface Weyl invariant. It is examined in a higher detail in the last section of this chapter.

Example 4.3.5. The tractor-binormal part of the Weyl tractor $\overset{\mathrm{NN}}{\mathrm{W}}_{BD}$ is a symmetric 2-tractor, defined as

$$\overset{\mathrm{NN}}{\mathrm{W}}_{BD} \coloneqq \mathrm{W}_{ABCD}\mathrm{N}^A\mathrm{N}^C \tag{4.7}$$

which is another useful example of conformal hypersurface Weyl invariants.

Proposition 4.3.6. *The tractor $\overset{\mathrm{NN}}{\mathrm{W}}_{BD}$ has the following properties*

1. *it can be identified with a purely intrinsic section, because $\overset{\mathrm{NN}}{\mathrm{W}}_{BD}\mathrm{N}^B = 0$ and $\overset{\mathrm{NN}}{\mathrm{W}}_{BD}\mathrm{N}^C = 0$;*

2. *explicitly (in an intrinsic scale) it is given by*

$$\begin{aligned}\overset{\mathrm{NN}}{\mathrm{W}}_{BD} \overset{\bar{g}}{=} (n-4)\Big(&\bar{Z}_B{}^b\bar{Z}_D{}^d\mathrm{W}_{abcd}\mathbf{N}^a\mathbf{N}^c + 2Z_{(B}{}^b\bar{X}_{D)}\mathrm{Y}_{abc}\mathbf{N}^a\mathbf{N}^c\Big)\\ &+ \bar{X}_B\bar{X}_D\mathrm{B}_{cb}\mathbf{N}^b\mathbf{N}^c\end{aligned} \tag{4.8}$$

where W_{abcd}, Y_{abc} and B_{cb} are the background Weyl, Cotton and Bach tensors, respectively. Recall that $n = \dim M$.

We are not limited with the Thomas-D operator, of course, and using the (background or twisted intrinsic) double-D or triple-D operators is a viable way to expand the class of possible invariants, which we can construct.

Definition 4.3.7. If in Definition 4.3.1 we replace

- in the terms $\mathrm{W}^{(k)}_{ABCD,P_1\dots P_k}$ some (or all) of the Thomas-D operators D_P with the double-D operators $\mathbb{D}_{PP'}$,
- and\or in the terms $\mathrm{N}^{(l)}_{A,Q_1\dots Q_l}$ some (or all) of the twisted intrinsic Thomas-D operators $\underline{\overline{\mathrm{D}}}_P$ with the twisted intrinsic double-D operators $\underline{\overline{\mathbb{D}}}_{PP'}$ or with the triple-D operator $\underline{\mathrm{D}}_{ABC}$,

we obtain a potentially larger class of invariants, which we term *generalized hypersurface conformal Weyl invariants invariants.*

Observe that simple expressions like $\underline{\overline{\mathbb{D}}}_{AA'}\mathrm{N}_B$ often lead to trivial examples,

$$\begin{aligned}\underline{\overline{\mathbb{D}}}_{AA'}\mathrm{N}_B = -\overline{\mathbb{X}}_{AA'}{}^a\underline{\nabla}_a\mathrm{N}_B =& X_{A'}Z_A{}^a\Big(Z_B{}^b\mathring{\mathbf{L}}_{ab} - X_B(\dots)\Big)\\ &- X_A Z_{A'}{}^a\Big(Z_B{}^b\mathring{\mathbf{L}}_{ab} - X_B(\dots)\Big)\end{aligned}$$

so one may need to apply the double-D operator a number of times to recover an interesting invariant.

4.3.2 Conformal quasi-Weyl invariants of hypersurfaces

The construction of conformal Weyl invariants can produce expressions that "degenerate" in some dimensions. This is an interesting phenomenon that we do not discuss here. See e.g. [7], [26], [27]. The construction of quasi-Weyl invariants provides an approach to address this issue, but, more importantly, it yields a wider class of conformal invariants.

As before, we describe the construction of conformal hypersurface quasi-Weyl invariants in the following three steps.

Step 1. If $\dim M > 3$, then we take a juxtaposition of a finite number of monomials of the form

$$\mathbb{D}_{PP'} \dots \mathbb{D}_{QQ'} \mathrm{C}_{A'ABCDD'}$$

and of a finite number of juxtapositions of monomials of the form

$$\overline{\mathbb{D}}_{RR'} \dots \overline{\mathbb{D}}_{SS'} \mathbb{N}_{EE'}$$

and contract some indices with the tractor metric or the tractor volume form. Here $\mathrm{C}_{A'ABCDD'}$ is the lifted tractor expression (3.47) of the background Weyl tensor, and $\mathbb{N}_{AA'}$ is the *adjoint normal tractor* :

$$\mathbb{N}_{AA'} \stackrel{g}{=} \mathbb{X}_{AA'}{}^{a} \mathrm{N}_a \tag{4.9}$$

If $\dim M = 3$ the construction is the same but instead of $\mathrm{C}_{A'ABCDD'}$ the lifted tractor expression (3.48) of the Cotton tensor $\mathrm{C}_{A'ABDD'}$ is used, see equation.

The resulting expression can be contracted with a finite number of instances of the tractor metric and/or the tractor volume form. We also may take a tensor part of the result.

Step 2. Take the symmetric trace-free part of the resulting tractor.

Let us assume fro the simplicity that only primed indices remain in our expression after contracting them with the tractor metric and/or the tractor volume form.

If it turns out that some of the X-s in the resulting expression can be moved to the leftmost position, that is we get an expression of the form

$$\mathrm{X}_{(I'} \dots \mathrm{X}_{J'} J_{K' \dots L')\circ}$$

then the object $J_{K'\dots L'}$ is invariant due to the injectivity of the operation of taking the symmetric trace-free part.

Step 3. Eliminating the remaining tractor indices in $J_{K'\dots L'}$ by the twisted intrinsic Thomas-D operator we obtain an indexless density:

$$\underline{\overline{\mathrm{D}}}^{K'} \dots \underline{\overline{\mathrm{D}}}^{L'} J_{K'\dots L'}$$

Definition 4.3.8. Any linear combination of tractors obtained through the three step algorithm described above is called a *hypersurface conformal quasi-Weyl invariant*.

Contracting all the indices we obtain scalar valued hypersurface conformal quasi-Weyl invariants. If we leave some tractor indices free (not contracted with the tractor metric and/or the tractor volume form), the resulting tractor becomes a tractor-valued conformal quasi-Weyl hypersurface invariant.

Example 4.3.9. The restrictions onto the hypersurface of background conformal quasi-Weyl invariants are obviously covered with the above construction.

Example 4.3.10. The normal tractor can be recovered from the the tractor expression $\underline{\overline{\mathrm{D}}}^{A'}\mathbb{N}_{AA'}$.

4.4 Examples of hypersurface conformal invariants

The normal tractor has weight $w = 0$ and therefore we can write that

$$\underline{\overline{\mathrm{D}}}_A \mathrm{N}_B = (n-3)\bar{Z}_A{}^a \underline{\nabla}_a \mathrm{N}_B - \bar{X}_A \underline{\Delta} \mathrm{N}_B$$

and this implies that when the background dimension is $n = 3$, the tangential tractor Laplacian $\underline{\Delta}\mathrm{N}_B$ of the normal tractor is a conformally invariant section along the hypersurface (that has the dimension $\bar{n} = 2$ in this case). Its projecting part is then a conformally invariant (tensor)-density, which we are going to extract.

Let us compute the tangential tractor Laplacian explicitly:

$$\begin{aligned}
\underline{\Delta}\mathrm{N}_C &= \bar{\mathbf{g}}^{ab}\underline{\nabla}_a\underline{\nabla}_b\mathrm{N}_C \\
&= \bar{\mathbf{g}}^{ab}\underline{\nabla}_a\Big(Z_C{}^c\mathring{\mathbf{L}}_{bc} - \tfrac{1}{\bar{n}-1}X_C\overline{\nabla}^c\mathring{\mathbf{L}}_{bc}\Big) \\
&= \bar{\mathbf{g}}^{ab}\Big((\underline{\nabla}_a Z_C{}^c)\mathring{\mathbf{L}}_{bc} + Z_C{}^c\underline{\nabla}_a\mathring{\mathbf{L}}_{bc} - \tfrac{1}{\bar{n}-1}(\underline{\nabla}_a X_C)\overline{\nabla}^c\mathring{\mathbf{L}}_{bc} - \tfrac{1}{\bar{n}-1}X_C\underline{\nabla}_a\overline{\nabla}^c\mathring{\mathbf{L}}_{bc}\Big)
\end{aligned}$$

where in the first line we have used we have used that $\underline{\nabla}_b \mathrm{N}_C = Z_C{}^c \mathring{\mathbf{L}}_{bc} - \frac{1}{\bar{n}-1} X_C \overline{\nabla}^c \mathring{\mathbf{L}}_{bc}$ in the *ambient* splitting.

We need to recall the following rules for the ambient tractor covariant derivative: $\underline{\nabla}_a Z_C{}^c = -Y_C \underline{\delta}_a{}^c - X_C \underline{\mathsf{P}}_a{}^c$ and $\underline{\nabla}_a X_C = Z_C{}^c \bar{\mathbf{g}}_{ac}$. The ambient derivatives of the intrinsic tensors can be rewritten using the Gauss formula in terms of the intrinsic connection, the shape tensor and the normal vector, and we apply this to the terms $\underline{\nabla}_a \mathring{\mathbf{L}}_{bc}$ and $\underline{\nabla}_a \overline{\nabla}^c \mathring{\mathbf{L}}_{bc}$ to obtain

$$\underline{\nabla}_a \mathring{\mathbf{L}}_{bc} = \overline{\nabla}_a \mathring{\mathbf{L}}_{bc} - \mathrm{N}_b \mathbf{L}_a{}^f \mathring{\mathbf{L}}_{fc} - \mathrm{N}_c \mathbf{L}_a{}^f \mathring{\mathbf{L}}_{bf}$$

and

$$\underline{\nabla}_a \overline{\nabla}^c \mathring{\mathbf{L}}_{bc} = \overline{\nabla}_a \overline{\nabla}^c \mathring{\mathbf{L}}_{bc} - \mathrm{N}_b \mathbf{L}_a{}^f \overline{\nabla}^c \mathring{\mathbf{L}}_{fc}$$

Plugging all these identities into the expression for the tangential Laplacian obtained above, we continue:

$$\begin{aligned}\underline{\Delta} \mathrm{N}_C = \bar{\mathbf{g}}^{ab}\Big(&(-Y_C \underline{\delta}_a{}^c - X_C \underline{\mathsf{P}}_a{}^c)\mathring{\mathbf{L}}_{bc} + Z_C{}^c(\overline{\nabla}_a \mathring{\mathbf{L}}_{bc} - \mathrm{N}_b \mathbf{L}_a{}^f \mathring{\mathbf{L}}_{fc} - \mathrm{N}_c \mathbf{L}_a{}^f \mathring{\mathbf{L}}_{bf}) \\ &- \tfrac{1}{\bar{n}-1} Z_C{}^c \bar{\mathbf{g}}_{ac} \overline{\nabla}^e \mathring{\mathbf{L}}_{be} - \tfrac{1}{\bar{n}-1} X_C(\overline{\nabla}_a \overline{\nabla}^c \mathring{\mathbf{L}}_{bc} - \mathrm{N}_b \mathbf{L}_a{}^f \overline{\nabla}^c \mathring{\mathbf{L}}_{fc})\Big)\end{aligned}$$

Distributing the factors and collecting the terms, after some simplifications we get

$$\begin{aligned}\underline{\Delta} \mathrm{N}_C = \bar{\mathbf{g}}^{ab}\Big(&- Y_C \mathring{\mathbf{L}}_{ba} + Z_C{}^c(\overline{\nabla}_a \mathring{\mathbf{L}}_{bc} - \mathrm{N}_b \mathbf{L}_a{}^f \mathring{\mathbf{L}}_{fc} - \mathrm{N}_c \mathbf{L}_a{}^f \mathring{\mathbf{L}}_{bf} - \tfrac{1}{\bar{n}-1} \bar{\mathbf{g}}_{ac} \overline{\nabla}^e \mathring{\mathbf{L}}_{be}) \\ &- X_C(\underline{\mathsf{P}}_a{}^c \mathring{\mathbf{L}}_{bc} + \tfrac{1}{\bar{n}-1}(\overline{\nabla}_a \overline{\nabla}^c \mathring{\mathbf{L}}_{bc} - \mathrm{N}_b \mathbf{L}_a{}^f \overline{\nabla}^c \mathring{\mathbf{L}}_{fc})\Big)\end{aligned}$$

Contracting the indices a and b with the inverse intrinsic metric $\bar{\mathbf{g}}^{ab}$ and using that $\mathring{\mathbf{L}}_{ba}$ is trace free ($\bar{\mathbf{g}}^{ab} \mathring{\mathbf{L}}_{ba} = 0$) and that $\mathbf{L}_{ab}$ has no normal components, we further simplify the last display to the following form:

$$\underline{\Delta} \mathrm{N}_C = Z_C{}^c \Big(\overline{\nabla}^b \mathring{\mathbf{L}}_{bc} - \mathrm{N}_c \mathbf{L}^{bf} \mathring{\mathbf{L}}_{bf} - \tfrac{1}{\bar{n}-1} \overline{\nabla}^e \mathring{\mathbf{L}}_{ce}\Big) - X_C \Big(\underline{\mathsf{P}}^{bc} \mathring{\mathbf{L}}_{bc} + \tfrac{1}{\bar{n}-1}(\overline{\nabla}^b \overline{\nabla}^c \mathring{\mathbf{L}}_{bc})\Big)$$

After a few final simplifications we record this result as an expression for the tangential tractor Laplacian of the normal tractor in the ambient splitting:

$$\underline{\Delta} \mathrm{N}_C = Z_C{}^c \Big(\tfrac{\bar{n}-2}{\bar{n}-1} \overline{\nabla}^b \mathring{\mathbf{L}}_{bc} - \mathrm{N}_c |\mathring{\mathbf{L}}|^2\Big) - X_C \Big((\Pi_\Sigma \mathsf{P})^{bc} \mathring{\mathbf{L}}_{bc} + \tfrac{1}{\bar{n}-1}(\overline{\nabla}^b \overline{\nabla}^c \mathring{\mathbf{L}}_{bc})\Big) \quad (4.10)$$

where $\bar{n} = n - 1$ is the dimension of the hypersurface Σ, and $(\Pi_\Sigma \mathsf{P})_{ab}$ is the totally projected part of the ambient Schouten tensor.

It is easy to decompose this tractor into the intrinsic and normal parts just by inspection:

$$\begin{aligned}
\underline{\Delta}\mathrm{N}_C &= Z_C{}^c \tfrac{\bar n-2}{\bar n-1}\overline{\nabla}^b\mathring{\mathbf{L}}_{bc} - Z_C{}^c\mathrm{N}_c|\mathring{\mathbf{L}}|^2 - X_C\Big((\Pi_\Sigma \mathsf{P})^{bc}\mathring{\mathbf{L}}_{bc} + \tfrac{1}{\bar n-1}(\overline{\nabla}^b\overline{\nabla}^c\mathring{\mathbf{L}}_{bc})\Big)\\
&= Z_C{}^c \tfrac{\bar n-2}{\bar n-1}\overline{\nabla}^b\mathring{\mathbf{L}}_{bc} - \Big(\mathrm{N}_C + X_C\mathbf{H}\Big)|\mathring{\mathbf{L}}|^2 - X_C\Big((\Pi_\Sigma \mathsf{P})^{bc}\mathring{\mathbf{L}}_{bc} + \tfrac{1}{\bar n-1}(\overline{\nabla}^b\overline{\nabla}^c\mathring{\mathbf{L}}_{bc})\Big)\\
&= Z_C{}^c \tfrac{\bar n-2}{\bar n-1}\overline{\nabla}^b\mathring{\mathbf{L}}_{bc} - X_C\Big((\Pi_\Sigma \mathsf{P})^{bc}\mathring{\mathbf{L}}_{bc} + \tfrac{1}{\bar n-1}(\overline{\nabla}^b\overline{\nabla}^c\mathring{\mathbf{L}}_{bc}) + \mathbf{H}|\mathring{\mathbf{L}}|^2\Big) - \mathrm{N}_C|\mathring{\mathbf{L}}|^2
\end{aligned}$$

One can also apply the formulas for the action of the tractor projection operator (3.60) and get the same result. We can now write the component of $\underline{\Delta}\mathrm{N}_C$ in the intrinsic tractor bundle $\overline{\mathcal{T}}_C$ as follows:

$$\Pi^{C'}{}_C\Delta\mathrm{N}_{C'} = \bar Z_C{}^c \tfrac{\bar n-2}{\bar n-1}\overline{\nabla}^b\mathring{\mathbf{L}}_{bc} - \bar X_C\Big(\tfrac{1}{\bar n-1}(\overline{\nabla}^b\overline{\nabla}^c\mathring{\mathbf{L}}_{bc}) + (\Pi_\Sigma \mathsf{P})^{bc}\mathring{\mathbf{L}}_{bc} + \mathbf{H}|\mathring{\mathbf{L}}|^2\Big) \quad (4.11)$$

We also notice that the normal component is $\mathrm{N}^{C'}{}_C\Delta\mathrm{N}_{C'} = -\mathrm{N}_C|\mathring{\mathbf{L}}|^2$.

For the reasons that we explain in the next chapter, we introduce a special notation $\mathcal{W}$ for the coefficient at $\bar X_C$ in (4.11):

$$\mathcal{W} := \tfrac{1}{\bar n-1}\overline{\nabla}^b\overline{\nabla}^c\mathring{\mathbf{L}}_{bc} + (\Pi_\Sigma \mathsf{P})^{bc}\mathring{\mathbf{L}}_{bc} + \mathbf{H}|\mathring{\mathbf{L}}|^2 \quad (4.12)$$

Proposition 4.4.1. *In the ambient dimension $n = 3$, that is when the dimension of Σ is $\bar n = 2$, the density*

$$\mathcal{W}^{(2)} = (\overline{\nabla}^b\overline{\nabla}^c\mathring{\mathbf{L}}_{bc}) + (\Pi_\Sigma \mathsf{P})^{bc}\mathring{\mathbf{L}}_{bc} + \mathbf{H}|\mathring{\mathbf{L}}|^2 \quad (4.13)$$

is conformally invariant.

Proof. When $\bar n = 2$ the coefficient at $\bar Z_C{}^c$ in (4.11) vanishes. □

In the previous section we have mentioned the *shape 2-tractor* $\mathbf{L}_{AB}$ defined as

$$\mathbf{L}_{AB} := \Pi^{B'}{}_B\underline{\overline{\mathrm{D}}}_A\mathrm{N}_{B'} \quad (4.14)$$

as an example of hypersurface conformal Weyl invariants.

Proposition 4.4.2. *The shape 2-tractor $\mathbf{L}_{AB}$ is a symmetric 2-tractor (so the name is justified), more precisely, it can be identified with a section of the intrinsic tractor bundle $\overline{\mathcal{T}}_{(AB)}$, which explicitly it is given (in an intrinsic scale) by*

$$\mathbf{L}_{AB} = (\bar n-2)\bar{\mathsf{Z}}_{AB}{}^{ab}\mathring{\mathrm{L}}_{ab} - \tfrac{\bar n-2}{\bar n-1}\bar{\mathsf{W}}_{AB}{}^a\overline{\nabla}^b\mathring{\mathrm{L}}_{ab} + \bar{\mathsf{X}}_{AB}\mathcal{W} \quad (4.15)$$

where $\mathcal{W}$ is given by (4.12), and we use the intrinsic symmetric 2-tractor projectors $\bar{\mathsf{Z}}_{AB}{}^{ab}$, $\bar{\mathsf{W}}_{AB}{}^a$, and $\bar{\mathsf{X}}_{AB}$, see Appendix C.

In [34] it was proposed to call the quantity

$$\overline{\mathbf{L}}_{AB} = (\bar{n}-2)\bar{\mathsf{Z}}_{AB}{}^{ab}\mathrm{L}_{ab} - \tfrac{\bar{n}-2}{\bar{n}-1}\bar{\mathsf{W}}^{a}{}_{AB}\overline{\nabla}^{b}\mathrm{L}_{ab} + \bar{\mathsf{X}}_{AB}\big(\tfrac{1}{\bar{n}-1}\overline{\nabla}^{a}\overline{\nabla}^{b}\mathrm{L}_{ab} + \overline{\mathsf{P}}^{ab}\mathrm{L}_{ab}\big) \quad (4.16)$$

as the tractor second fundamental form[3] for $\bar{n} > 2$.

The tractor (4.16) is obtained by the application of the intrinsic middle operator (cf. [64]) to the tractor shape 1-form. In [34] and [68] some examples of hypersurface conformal invariants that arise from $\overline{\mathbf{L}}_{AB}$ are studied in detail.

The shape-2 tractor in either form, (4.16) or (4.14), can be used to make further conformal Weyl invariants, and it has properties similar to those of the umbilicity tensor (trace-free, conformally invariant).

The tractors $\mathbf{L}_{AB}$ and $\overline{\mathbf{L}}_{AB}$ have the conformal weight -1. The intrinsic box operator (2.12) becomes the conformal Laplacian on sections of this weight in dimension $\bar{n} = 4$.

Applying to $\mathbf{L}_{AB}$ the formula (C.17) for the intrinsic box operator on reduced symmetric 2-tractors (see Appendix C), we obtain the expression

$$\begin{aligned}
\overline{\Box}\mathbf{L}_{AB} = \bar{\mathsf{U}}_{AB}{}^{a}\Big(& -2(\bar{n}-2)\,\overline{\nabla}^{b}\mathring{\mathbf{L}}_{ab} + (\bar{n}+2)\tfrac{\bar{n}-2}{\bar{n}-1}\overline{\nabla}^{b}\mathring{\mathbf{L}}_{ab}\Big)\\
+ \bar{\mathsf{V}}_{AB}\Big(& 2(\bar{n}-2)\,\overline{\mathsf{P}}^{ab}\mathring{\mathbf{L}}_{ab} + 2\tfrac{\bar{n}-2}{\bar{n}-1}\overline{\nabla}^{a}\overline{\nabla}^{b}\mathring{\mathbf{L}}_{ab} - n\,\mathcal{W}\Big)\\
+ \bar{\mathsf{Z}}_{AB}{}^{ab}\Big(& -4(\bar{n}-2)\,\mathsf{P}^{c}{}_{a}\mathring{\mathbf{L}}_{bc} - 2(\bar{n}-2)\bar{\mathsf{J}}\mathring{\mathbf{L}}_{ab}\\
& + (\bar{n}-2)\overline{\Box}\mathring{\mathbf{L}}_{ab} - 4\tfrac{\bar{n}-2}{\bar{n}-1}\,\overline{\nabla}_{b}\overline{\nabla}^{c}\mathring{\mathbf{L}}_{ac} + 2\,\bar{\mathbf{g}}_{ab}\mathcal{W}\Big)\\
+ \dots &
\end{aligned}$$

(the lower slots shown as … are not essential here).

In dimension $\bar{n} = 4$ we see that one of the projecting parts of the above tractor appears at the term $\bar{\mathsf{Z}}_{AB}{}^{ab}$ (cf. the rescaling formula (C.11) for symmetric 2-tractors). This way we obtain a new example of hypersurface conformal invariants:

$$\overline{\Delta}\mathring{\mathbf{L}}_{ab} - \tfrac{4}{3}\overline{\nabla}_{(a}\overline{\nabla}^{c}\mathring{\mathbf{L}}_{b)c} + \tfrac{1}{3}\bar{\mathbf{g}}_{ab}\overline{\nabla}^{c}\overline{\nabla}^{d}\mathring{\mathbf{L}}_{cd} + \bar{\mathbf{g}}_{ab}(\Pi_{\Sigma}\mathsf{P})^{cd}\mathring{\mathbf{L}}_{cd} - 4\overline{\mathsf{P}}_{(a}{}^{c}\mathring{\mathbf{L}}_{b)c} - \bar{\mathsf{J}}\mathring{\mathbf{L}}_{ab} + \mathbf{H}|\mathring{\mathbf{L}}|^{2}\bar{\mathbf{g}}_{ab}$$

which can be further simplified, using the Simons and Weitzenböck identities.

[3] Our analysis in this thesis has shown that the tractor shape 1-form $\mathbb{L}_{aB}$ is a better candidate for this role. More precisely, by the tractor second fundamental form one should understand the operator $H_{a}{}^{B}{}_{C} = -\mathrm{N}^{B}\mathbb{L}_{aC}$, cf. (3.71).

Chapter 5

Towards higher dimensional analogues of the Willmore functional

5.1 The Willmore functional in 2 dimensions

5.1.1 Notes on the history of the Willmore functional

The bending energy of a an immersed surface Σ (with or without boundary) in the Euclidean space $\mathbb{R}^3$ is a classically known functional

$$\mathcal{U}(\Sigma) := \iint_{\Sigma} H^2 \mathrm{d}\Sigma \tag{5.1}$$

where H is the mean curvature and $\mathrm{d}\Sigma$ is the induced volume form. It has a long and fascinating history of study, with many mysteries and surprises. Already S.D. Poisson began generalizing the Euler's theory of elastica to surfaces that lead to this functional. It also appeared in the work of Sophie Germain who proposed it as a tool to explain the phenomenon of Chladni's vibrating plates. The functional 5.1 usually referred to as the *bending energy* in the context of the theory of elasticity and the mechanics of solids.

In 1923 G.Thomsen published the results of his PhD work in conformal geometry of surfaces [70] where he studied some variational problems related to the integral

$$\mathcal{C}(\Sigma) := \iint_{\Sigma} \left(\frac{1}{R_1} - \frac{1}{R_2}\right)^2 \mathrm{d}\Sigma \tag{5.2}$$

Here R_1 and R_2 are the principal radii of curvature of the surface Σ. Recall that the principal curvatures are the minimal and maximal values of the normal curvature at a given point p of the surface. The normal curvature is the signed curvature of the curve of intersection of the surface Σ and the plane Π passing through the point p on the surface in the direction of the unit normal N to the surface at that point (see Fig. 5.1). When the plane Π is rotated around the unit normal, the normal curvatures vary in general. Since we can parametrize the positions of the planes Π by the angles of rotation with respect to a fixed direction, which can be viewed as the points on the unit circle, the minimal $\kappa_{\min} =: \kappa_1$ and maximal $k\kappa_{\max} =: \kappa_2$ values of the normal curvature are achieved due to compactness of the unit circle. These values are called the *principal curvatures* of the surface at the given point. Their inverses $R_1 = \kappa_{\min}^{-1}$ and $R_2 = \kappa_{\max}^{-1}$ are the *principal radii of curvature* of the surface Σ. If the principal curvatures are equal, the point p, is called *umbilic*. If all the points of the surface are umbilic, the surface is called *totally umbilic*.

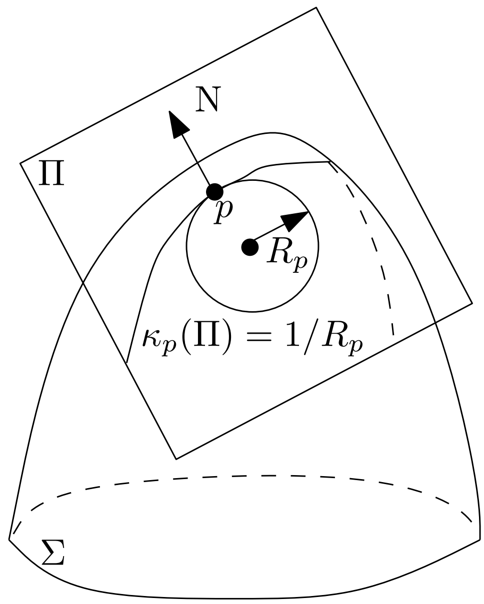

Figure 5.1: Normal curvature

The integrand in (5.2) can be rewritten as

$$(\kappa_1 - \kappa_2)^2 = (\kappa_1 + \kappa_2)^2 - 4\kappa_1\kappa_2 = 4H^2 - 4K$$

that shows the equivalence of (5.1) and (5.2) on closed[1] surfaces: $\mathcal{C} = 4\mathcal{U} - 4\chi$, where χ is the Euler characteristic of Σ, by the Gauss–Bonnet theorem.

Among other things, G.Thomsen obtained the Euler–Lagrange equation for

[1] The case when the surface Σ has boundary is qualitatively different form the point of view of conformal geometry, so we consider only closed surfaces. Another justification for this restriction is that we are interested in the applications where Σ is the boundary of some manifold.

the functional (5.2). In the final footnote to his paper he attributed this equation to an apparently unpublished work of W.Shadow. G.Thomsen was a student of W.Blaschke who published a three-volume treatise on geometry [9]. In the volume III of this work Blaschke discusses conformal properties of the integrand in 5.2 and mentions the results of Thomsen in connection with this.

More than 40 years later, T.J. Willmore proposed in 1965 [75] to study the critical surfaces of the functional (5.1) and obtained its Euler–Lagrange equation independently. He writes in [76] that he believed that this result was knew, and was surprised to learn that it had been already known since Thomsen and Schadow. After Willmore's work on the functional (5.1), it got into focus of active mathematical research. The famous Willmore's conjecture states that the only critical point of this functional on the class of surfaces of genus higher than one is the Hopf–Clifford torus up to conformal rescaling of the ambient space, which is either Euclidean 3-dimensional space or the 3-dimensional sphere. This conjecture stimulated a rich mathematical literature and was completely solved only recently by A.Neves and F.Marques [51].

The work of Blaschke [9] is only available in German, so the results on the functional (5.1) have not become widely known until White [74] published in 1973 a concise proof of conformal invariance of (5.1) for surfaces $\mathbb{R}^3$ using direct computations of the change of principal curvatures under inversion of $\mathbb{R}^3$ with respect to a sphere with center not lying on the surface.

Clearly, the integrand in (5.2) vanishes at the umbilic points of the surface. It can be shown, using the adapted coordinate system chosen at a point p on the surface, that the principal curvatures are the eigenvalues of the shape operator $\mathrm{L}_a{}^b$ of Σ at this point, which can be diagonalized by rotating the adapted system to the normal form, and the matrix of the shape operator will have the form $\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$. This shows that the integrand of (5.2) is the square of the length of the trace-free part of the second fundamental form, which we call the umbilicity tensor $\mathring{\mathrm{L}}_{ab}$.

This observation suggests the generalization of the functional (5.2) to the case of surfaces in arbitrary Riemannian spaces that we refer to as the Willmore functional.

Definition 5.1.1. The *Willmore functional* $\mathcal{W}(\Sigma)$ is defined on an embedded hypersurface Σ in a Riemannian manifold (M, g) of dimension $n = 3$ as

$$\mathcal{W}(\Sigma) = \int_\Sigma |\mathring{\mathbf{L}}|^2 \,\mathrm{d}\Sigma \tag{5.3}$$

5.1.2 The variation of the umbilicity tensor

A stationary point of functional (5.3) is called a Willmore surface. The necessary condition for for a surface to be a stationary point is known as the Euler–Lagrange equation for this functional. In order to find this equation it is convenient to have the identity, which we derive now.

Proposition 5.1.2. *Let f^t of $f\colon \Sigma^{n-1} \to M^n$ be a normal variation as in Chapter 1, and $V = \varphi N$ be its variational vector, where $\varphi\colon \Sigma \to \mathbb{R}$ is smooth compactly supported function on Σ. For each point $p \in \Sigma$ we have the umbilicity tensor $\mathring{\mathrm{L}}_{ab}$ defined along the curve p^t.*

The variation of the umbilicity tensor $\mathring{\mathrm{L}}_{ab}$ at $t = 0$ is given by

$$\delta_0 \mathring{\mathrm{L}}_{ab} = -\overline{\nabla}_{(a}\overline{\nabla}_{b)\circ}\varphi - \varphi\overset{\mathrm{NN}}{\mathrm{R}}_{(ab)\circ} + \varphi\mathring{\mathrm{L}}^2_{(ab)\circ} + \frac{2\varphi|\mathring{\mathrm{L}}|^2}{(n-1)}\bar{g}_{ab} \tag{5.4}$$

where $\overline{\nabla}_{(i}\overline{\nabla}_{j)\circ}\varphi = \overline{\nabla}_i\overline{\nabla}_j\varphi - \bar{n}^{-1}\bar{g}_{ij}\overline{\Delta}\varphi$ is the trace-free part of the intrinsic Hessian of the variational function φ, and $\overset{\mathrm{NN}}{\mathrm{R}}_{(ab)\circ} = \overset{\mathrm{NN}}{\mathrm{R}}_{acbd}\mathrm{N}^c\mathrm{N}^d - \bar{n}^{-1}\mathrm{Ric}_{cd}\mathrm{N}^c\mathrm{N}^d\bar{g}_{ab}$ is the trace-free part of the binormal part of the Riemannian curvature of the manifold M evaluated along the hypersurface $\Sigma = \Sigma^0$.

Proof. Since by definition $\mathring{\mathrm{L}}_{ab} = \mathrm{L}_{ab} - \mathrm{H}\bar{g}_{ab}$, using Theorem 1.3.3 we compute:

$$\begin{aligned}
\delta_0\mathring{\mathrm{L}}_{ab} &= \delta_0\mathrm{L}_{ab} - (\delta_0\mathrm{H})\bar{g}_{ab} - \mathrm{H}\delta_0\bar{g}_{ab}\\
&= -\overline{\nabla}_a\overline{\nabla}_b\varphi + \varphi\Big(\mathrm{L}^2_{ab} - \overset{\mathrm{NN}}{\mathrm{R}}_{ab}\Big) + \bar{n}^{-1}\bar{g}_{ab}\overline{\Delta}\varphi + \bar{n}^{-1}\varphi\Big(|\mathrm{L}|^2 + \overset{\mathrm{NN}}{\mathrm{Ric}}\Big)\bar{g}_{ab} - 2\varphi\mathrm{H}\mathrm{L}_{ab}\\
&= -\overline{\nabla}_{(a}\overline{\nabla}_{b)\circ}\varphi - \varphi\overset{\mathrm{NN}}{\mathrm{R}}_{(ab)\circ} + \varphi\mathrm{L}^2_{ab} + \bar{n}^{-1}\varphi|\mathrm{L}|^2\bar{g}_{ab} - 2\varphi\mathrm{H}\mathrm{L}_{ab}
\end{aligned}$$

Now we need to decompose the shape tensor in the last three terms of the above display using $\mathrm{L}_{ab} = \mathring{\mathrm{L}}_{ab} + \mathrm{H}\bar{g}_{ab}$.

Notice that $\mathrm{L}^2_{ab} = \mathring{\mathrm{L}}^2_{ab} + 2\,\mathrm{H}\mathring{\mathrm{L}}_{ab} + \mathrm{H}^2\,\bar{g}_{ab}$ and $|\mathrm{L}|^2 = |\mathring{\mathrm{L}}|^2 + \mathrm{H}^2\,\bar{n}$ so that we have

$$\begin{aligned}
\delta_0\mathring{\mathrm{L}}_{ab} = -\overline{\nabla}_{(a}\overline{\nabla}_{b)\circ}\varphi - \varphi\overset{\mathrm{NN}}{\mathrm{R}}_{(ab)\circ} + \varphi\left(\mathring{\mathrm{L}}^2_{ab} + 2\,\mathrm{H}\mathring{\mathrm{L}}_{ab} + \mathrm{H}^2\,\bar{g}_{ab}\right)&\\
+\,\bar{n}^{-1}\varphi\left(|\mathring{\mathrm{L}}|^2 + \mathrm{H}^2\,\bar{n}\right)\bar{g}_{ab} - 2\varphi\mathrm{H}\left(\mathring{\mathrm{L}}_{ab} + \mathrm{H}\bar{g}_{ab}\right)&\\
= -\overline{\nabla}_{(a}\overline{\nabla}_{b)\circ}\varphi - \varphi\overset{\mathrm{NN}}{\mathrm{R}}_{(ab)\circ} + \varphi\mathring{\mathrm{L}}^2_{ab} + \bar{n}^{-1}\varphi|\mathring{\mathrm{L}}|^2\bar{g}_{ab}&
\end{aligned}$$

We observe now that $\mathrm{tr}\,\mathring{\mathrm{L}}^2_{ab} = \bar{g}^{ab}\mathring{\mathrm{L}}_a{}^c\mathring{\mathrm{L}}_{cb} = |\mathring{\mathrm{L}}|^2$ and so the trace free part of $\mathring{\mathrm{L}}^2_{ab}$ is $\mathring{\mathrm{L}}^2_{(ab)\circ} = \mathring{\mathrm{L}}^2_{ab} - \bar{n}|\mathring{\mathrm{L}}|^2\bar{g}_{ab}$. This suffices to confirm the claim. □

5.1.3 The Euler–Lagrange equation of the Willmore functional

Now we compute the Euler–Lagrange equation of the Willmore functional (5.3) of a closed embedded oriented surface Σ in an oriented connected 3-dimensional Riemannian manifold (M, g).

Using a standard argument one can show that the tangential component of the variation vector field acts on Σ by diffeomorphism, and so amounts to a reparametrization of the surface and does not change the integral over the surface.

Proposition 5.1.3. *The necessary condition (the Euler–Lagrange equation) for an embedded 2-dimensional hypersurface Σ to be a critical point of the Willmore functional $\mathcal{W}(\Sigma) = \int_\Sigma |\mathring{\mathbf{L}}|^2 \mathrm{d}\Sigma$ is*

$$\overline{\nabla}^a \overline{\nabla}^b \mathring{\mathbf{L}}_{ab} + \mathring{\mathbf{L}}^{ab} \overset{\mathrm{NN}}{\mathrm{R}}_{(ab)\circ} + H|\mathring{\mathbf{L}}|^2 = 0 \tag{5.5}$$

Proof. Let us present the Willmore functional (5.3) as

$$\mathcal{W} = \int_\Sigma \bar{\mathbf{g}}^{ac} \bar{\mathbf{g}}^{bd} \mathring{\mathbf{L}}_{ab} \mathring{\mathbf{L}}_{cd} \mathrm{d}\Sigma$$

and apply the variational operator δ_0. We obtain

$$\begin{aligned}\delta_0 \mathcal{W} &= \delta_0 \int_\Sigma \bar{\mathbf{g}}^{ac} \bar{\mathbf{g}}^{bd} \mathring{\mathbf{L}}_{ab} \mathring{\mathbf{L}}_{cd} \mathrm{d}\Sigma \\ &= \int_\Sigma \Big[2\,(\delta_0\, \bar{\mathbf{g}}^{ac}) \bar{\mathbf{g}}^{bd} \mathring{\mathbf{L}}_{ab} \mathring{\mathbf{L}}_{cd} \mathrm{d}\Sigma + 2\, \bar{\mathbf{g}}^{ac} \bar{\mathbf{g}}^{bd} (\delta_0\, \mathring{\mathbf{L}}_{ab}) \mathring{\mathbf{L}}_{cd} \mathrm{d}\Sigma + \bar{\mathbf{g}}^{ac} \bar{\mathbf{g}}^{bd} \mathring{\mathbf{L}}_{ab} \mathring{\mathbf{L}}_{cd} (\delta_0\, \mathrm{d}\Sigma) \Big]\end{aligned}$$

Substituting the identities (1.68),(5.4) and (1.69) into the last line we get (recall that we work in the case $n = \dim M = 3$, so $\bar{n} = 2$)

$$\begin{aligned}\delta_0 \mathcal{W} = \int_\Sigma \Big[& 2\,(-2\,\varphi \mathbf{L}^{ac}) \bar{\mathbf{g}}^{bd} \mathring{\mathbf{L}}_{ab} \mathring{\mathbf{L}}_{cd} \mathrm{d}\Sigma \\ & + 2\, \bar{\mathbf{g}}^{ac} \bar{\mathbf{g}}^{bd} \Big(- \overline{\nabla}_{(a} \overline{\nabla}_{b)\circ} \varphi - \varphi \overset{\mathrm{NN}}{\mathrm{R}}_{(ab)\circ} + \varphi\, \mathring{\mathrm{L}}^2_{(ab)\circ} + \frac{2\varphi |\mathring{\mathrm{L}}|^2}{(n-1)} \bar{g}_{ab} \Big) \mathring{\mathbf{L}}_{cd} \mathrm{d}\Sigma \\ & + \bar{\mathbf{g}}^{ac} \bar{\mathbf{g}}^{bd} \mathring{\mathbf{L}}_{ab} \mathring{\mathbf{L}}_{cd}\, 2\varphi\, H\, \mathrm{d}\Sigma \Big]\end{aligned}$$

Notice that in the first term we have the shape tensor $\mathbf{L}^{ac} = \mathring{\mathbf{L}}^{ac} + \mathbf{H} \bar{\mathbf{g}}^{ac}$, and we can rewrite this term as

$$-4\varphi\,(\mathring{\mathbf{L}}^{ac} + \mathbf{H}\bar{\mathbf{g}}^{ac})\mathring{\mathbf{L}}^2_{ac}\mathrm{d}\Sigma = (-4\varphi\,\mathring{\mathbf{L}}^{ac}\mathring{\mathbf{L}}^2_{ac} - 4\,\varphi\mathbf{H}|\mathring{\mathbf{L}}|^2)\mathrm{d}\Sigma$$

using the fact that

$$\bar{\mathbf{g}}^{ac}\mathring{\mathbf{L}}^2_{ac} = |\mathring{\mathbf{L}}|^2$$

The second term of the expression for $\delta\mathcal{W}(f)$ above can be rewritten as

$$\begin{aligned}&2\,\mathring{\mathbf{L}}^{ab}\Big(-\overline{\nabla}_{(a}\overline{\nabla}_{b)\circ}\varphi - \varphi\,\overset{\mathrm{NN}}{\mathrm{R}}_{(ab)\circ} + \varphi\,\mathring{\mathrm{L}}^2_{(ab)\circ} + \frac{2\varphi|\mathring{\mathrm{L}}|^2}{(n-1)}\bar{g}_{ab}\Big)\mathrm{d}\Sigma\\ &\quad= (-2\,\mathring{\mathbf{L}}^{ab}\overline{\nabla}_{(a}\overline{\nabla}_{b)\circ}\varphi - 2\varphi\,\mathring{\mathbf{L}}^{ab}\,\overset{\mathrm{NN}}{\mathrm{R}}_{(ab)\circ} + 2\varphi\,\mathring{\mathbf{L}}^{ab}\mathring{\mathrm{L}}^2_{(ab)\circ})\mathrm{d}\Sigma\end{aligned}$$

The remaining term is easy to handle, it is just

$$2\varphi\,\mathbf{H}|\mathring{\mathbf{L}}|^2\,\mathrm{d}\Sigma$$

Adding up these intermediate results, we continue the calculation as

$$\begin{aligned}\delta_0\mathcal{W} = \int_\Sigma \Big[&-4\varphi\,\mathring{\mathbf{L}}^{ac}\mathring{\mathbf{L}}^2_{ac} - 4\varphi\,\mathbf{H}|\mathring{\mathbf{L}}|^2\\ &-2\,\mathring{\mathbf{L}}^{ab}\overline{\nabla}_{(a}\overline{\nabla}_{b)\circ}\varphi - 2\varphi\,\mathring{\mathbf{L}}^{ab}\,\overset{\mathrm{NN}}{\mathrm{R}}_{(ab)\circ} + 2\varphi\,\mathring{\mathbf{L}}^{ab}\mathring{\mathrm{L}}^2_{(ab)\circ}\\ &+2\varphi\,\mathbf{H}|\mathring{\mathbf{L}}|^2\Big]\,\mathrm{d}\Sigma\end{aligned}$$

Collecting the terms, we get

$$\delta_0\mathcal{W} = \int_\Sigma \Big[-2\,\mathring{\mathbf{L}}^{ab}\overline{\nabla}_{(a}\overline{\nabla}_{b)\circ}\varphi - 2\varphi\,\mathring{\mathbf{L}}^{ac}\mathring{\mathbf{L}}^2_{ac} - 2\,\varphi\mathbf{H}|\mathring{\mathbf{L}}|^2 - 2\varphi\,\mathring{\mathbf{L}}^{ab}\overset{\mathrm{NN}}{\mathrm{R}}_{(ab)\circ}\Big]\,\mathrm{d}\Sigma$$

Integrating the first term in the last display by parts (and suppressing the volume form $\mathrm{d}\Sigma$ for brevity)

$$\int_\Sigma \mathring{\mathbf{L}}^{ab}\overline{\nabla}_{(a}\overline{\nabla}_{b)\circ}\varphi = \int_\Sigma \mathring{\mathbf{L}}^{ab}\overline{\nabla}_a\overline{\nabla}_b\varphi = -\int_\Sigma (\overline{\nabla}_a\mathring{\mathbf{L}}^{ab})\overline{\nabla}_b\varphi = \int_\Sigma \varphi\,\overline{\nabla}_a\overline{\nabla}_b\mathring{\mathbf{L}}^{ab}$$

we arrive to the equation

$$\delta_0\mathcal{W} = -2\int_\Sigma \varphi\,\Big[\overline{\nabla}^a\overline{\nabla}^b\mathring{\mathbf{L}}_{ab} + \mathring{\mathbf{L}}^{ac}\mathring{\mathbf{L}}^2_{ac} + \mathbf{H}|\mathring{\mathbf{L}}|^2 + \mathring{\mathbf{L}}^{ab}\,\overset{\mathrm{NN}}{\mathrm{R}}_{(ab)\circ}\Big]\,\mathrm{d}\Sigma$$

One can regard the term $\mathring{\mathbf{L}}^{ac}\mathring{\mathbf{L}}^2_{ac}$ as the trace of the cube of a trace-free symmetric 2×2-matrix. More formally, the Cayley–Hamilton theorem for a 2×2-matrix A yields $A^2 + a_1 A + a_2 I = 0$ where $a_1 = -\operatorname{tr}(A)$, $a_2 = -\frac{1}{2}\left(\operatorname{tr}(A^2) - (\operatorname{tr}(A))^2\right)$. Thus, when $\operatorname{tr}(A) = 0$ we have $A^2 = \frac{\operatorname{tr}(A^2)}{2} I$ and $A^3 = \frac{\operatorname{tr}(A^2)}{2} A$, so $\operatorname{tr}(A^3) = \frac{\operatorname{tr}(A^2)}{2}\operatorname{tr}(A) = 0$. This means that $\mathring{\mathbf{L}}^{ac}\mathring{\mathbf{L}}^2_{ac}$ vanishes on a 2-dimensional hypersurface.

Using the fact that φ is an arbitrary smooth function on an open subset of hypersurface Σ, which is assumed to be closed (compact, without boundary), we deduce the claim. □

Remark 5.1.4. In case of the Euclidean background, equation (5.5) reduces to the known form ([76, 14]),

$$\overline{\Delta} H + 2(H^2 - K)H = 0$$

because in the adapted normal coordinates at a point of the surface the umbilicity tensor would look as

$$\mathring{\mathbf{L}}_a{}^b = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix} - \begin{pmatrix} \frac{\kappa_1+\kappa_2}{2} & 0 \\ 0 & \frac{\kappa_1+\kappa_2}{2} \end{pmatrix} = \begin{pmatrix} \frac{\kappa_1-\kappa_2}{2} & 0 \\ 0 & \frac{\kappa_2-\kappa_1}{2} \end{pmatrix}$$

and therefore $|\mathring{\mathbf{L}}|^2 = \operatorname{tr}_{\bar{g}} \mathring{\mathbf{L}}^2 = \frac{1}{2}(\kappa_1 - \kappa_2) = 2(H^2 - K)$. By the contracted conformal Codazzi equation (2.36) for $\dim \Sigma = \bar{n} = 2$ in the Euclidean background ($\mathsf{P}_{ab} = 0$) yields $\overline{\nabla}^a \mathring{\mathbf{L}}_{ab} = \overline{\nabla}_b \mathbf{H}$.

5.1.4 The Willmore invariant

The left hand side of the Euler–Lagrange equation (5.5) for the Willmore functional (5.3) is an important hypersurface conformal invariant. Clearly, it generalizes the Willmore invariant for surfaces in 3-dimensional Euclidean space to the case of arbitrary Riemannian background.

A.R.Gover conjectured [30] that in dimension $n = 3$ this quantity should coincide with the invariant (4.13) obtained from $\Pi^{A'}{}_A \Box N_{A'}$. As we can see now, this is indeed the case.

It was also pointed to the author by A.R.Gover [31], that this quantity was used in [1, pp.592,609] as a conformally invariant density of weight -3 on the boundary of a 3-dimensional manifold, where it was discovered by an asymptotic analysis of the Yamabe equation.

Proposition 5.1.5. *The left hand side of (5.5) coincides with the quantity*

$$\mathcal{W}^{(2)} = \overline{\nabla}^a \overline{\nabla}^b \mathring{\mathbf{L}}_{ab} + (\Pi_\Sigma \mathsf{P})^{ab} \mathring{\mathbf{L}}_{ab} + \mathbf{H}|\mathring{\mathbf{L}}|^2 \tag{5.6}$$

from Proposition 4.4.1. It is conformally invariant with respect to rescalings of the background metric.

Definition 5.1.6. The quantity $\mathcal{W}^{(2)}$ defined by (5.6) is called the *Willmore invariant* $\mathcal{W}^{(2)}$ of a hypersurface in a 3-dimensional conformal manifold.

Proof. Recall that $\mathrm{Ric}_{ab} = (n-2)\mathsf{P}_{ab} + \mathsf{J}g_{ab}$ and thus $\mathrm{Ric}_{(ab)\circ} = (n-2)\mathsf{P}_{(ab)\circ}$, which in dimension$n = 3$ becomes $\mathrm{Ric}_{(ab)\circ} = \mathsf{P}_{(ab)\circ}$. Notice also that $\overset{\mathrm{NN}}{\mathrm{R}}_{(ab)\circ} = \overset{\mathrm{NN}}{\mathrm{W}}_{(ab)\circ} + (\Pi_\Sigma \mathsf{P})_{(ab)\circ}$, but $W = 0$ in dimension $n = 3$, so the equality of $\mathcal{W}$ and the left hand side of (5.5) is clear.

The Willmore functional is invariant with respect to the rescaling of the background metric, and the condition to be equal to zero is also trivially conformally invariant. This implies the conformal invariance of $\mathcal{W}^{(2)}$.

Alternatively, we may verify the conformal invariance of $\mathcal{W}$ by using the conformal rescaling rules of its ingredients. We shall perform this verification as is quite instructive.

The following conformal rescaling identity

$$\widehat{\overline{\nabla}}^a \widehat{\overline{\nabla}}^b \mathring{\mathbf{L}}_{ab} = \overline{\nabla}^a \overline{\nabla}^b \mathring{\mathbf{L}}_{ab} + (n-2)\mathring{\mathbf{L}}_{ab} \overline{\nabla}^a \overline{\Upsilon}^b + (2n-6)\overline{\Upsilon}^a \overline{\nabla}^b \mathring{\mathbf{L}}_{ab} + (n-4)(n-2)\overline{\Upsilon}^a \overline{\Upsilon}^b \mathring{\mathbf{L}}_{ab}$$

is easy to obtain using the rules for the rescaling of the Levi-Civita connection (see e.g. Stafford, p. 36). Here n is the ambient dimension: $n = \dim M$.

We compute in dimension $n = 3$(or $\bar{n} = 2$):

$$\begin{aligned}
\widehat{\mathcal{W}^{(2)}} &= \widehat{\overline{\nabla}^a} \widehat{\overline{\nabla}^b} \mathring{\mathbf{L}}_{ab} + \widehat{(\Pi_\Sigma \mathsf{P})}^{ab} \mathring{\mathbf{L}}_{ab} + \widehat{\mathbf{H}}|\mathring{\mathbf{L}}|^2 \\
&= \overline{\nabla}^a \overline{\nabla}^b \mathring{\mathbf{L}}_{ab} + \mathring{\mathbf{L}}_{ab} \overline{\nabla}^a \overline{\Upsilon}^b - \cancel{\overline{\Upsilon}^a \overline{\Upsilon}^b \mathring{\mathbf{L}}_{ab}} \\
&\quad + \mathring{\mathbf{L}}_{ab} \Pi_{a'}{}^a \Pi_{b'}{}^b \mathsf{P}^{a'b'} - \mathring{\mathbf{L}}_{ab} \Pi_{a'}{}^a \Pi_{b'}{}^b \nabla^{a'} \Upsilon^{b'} + \cancel{\mathring{\mathbf{L}}_{ab} \Pi_{a'}{}^a \Pi_{b'}{}^b \Upsilon^{a'} \Upsilon^{b'}} \\
&\quad + \mathbf{H}|\mathring{\mathbf{L}}|^2 + \mathrm{N}_c \Upsilon^c |\mathring{\mathbf{L}}|^2
\end{aligned}$$

where we have canceled two terms using the chain rule $\Pi_a{}^b \Upsilon_b = \overline{\Upsilon}_a$.

Collecting and rearranging the remaining terms we get

$$\begin{aligned}
\widehat{\mathcal{W}^{(2)}} &= \overline{\nabla}^a \overline{\nabla}^b \mathring{\mathbf{L}}_{ab} + \mathring{\mathbf{L}}_{ab} \Pi_{a'}{}^a \Pi_{b'}{}^b \mathsf{P}^{a'b'} + \mathbf{H}|\mathring{\mathbf{L}}|^2 \\
&\quad + \mathring{\mathbf{L}}^{ab} \Big(\overline{\nabla}_a \overline{\Upsilon}_b - \Pi_a{}^{a'} \Pi_b{}^{b'} \nabla_{a'} \Upsilon_{b'} + \mathrm{N}_c \Upsilon^c \mathring{\mathbf{L}}_{ab} \Big)
\end{aligned}$$

where the last three terms (in the parentheses) annihilate due to the Gauss and Weingarten formulas:

$$\begin{aligned}\Pi_b{}^{b'}\underline{\nabla}_a\Upsilon_{b'} &= \Pi_b{}^{b'}\underline{\nabla}_a\Big(\overline{\Upsilon}_{b'} + (\mathrm{N}^c\Upsilon_c)\mathrm{N}_{b'}\Big)\\ &= \Pi_b{}^{b'}\Big(\overline{\nabla}_a\overline{\Upsilon}_{b'} - \mathrm{N}_{b'}\mathbf{L}_a{}^c\overline{\Upsilon}_c + \mathrm{N}_{b'}\underline{\nabla}_a(\mathrm{N}^c\Upsilon_c) + (\mathrm{N}^c\Upsilon_c)\mathbf{L}_{ab'}\Big)\\ &= \overline{\nabla}_a\overline{\Upsilon}_b + (\mathrm{N}^c\Upsilon_c)\mathbf{L}_{ab}\end{aligned}$$

Observing that $\mathring{\mathbf{L}}^{ab}\mathbf{L}_{ab} = \mathring{\mathbf{L}}^{ab}\mathring{\mathbf{L}}_{ab} = |\mathring{\mathbf{L}}|^2$ we confirm that $\widehat{\mathcal{W}^{(2)}} = \mathcal{W}^{(2)}$. □

5.2 A 4-dimensional analogue

5.2.1 The Branson–Gover operators

The Q-curvature originates from the work of T.P.Branson (1953 – 2006) in conformal geometry. It can be seen as a generalization of the Gaussian curvature on 2-dimensional surfaces to the Riemannian manifolds of higher dimensions.

In [12] T.P.Branson and A.R.Gover have proposed a new families of operators that shed more light on the nature of the Q-curvature and give ways to further generalizations.

We are interested in the Branson–Gover's Q_k-operators because of their strong invariance on the closed vector bundle valued forms, that has been proved by T.P.Branson and A.R.Gover in [13], Theorem 5.3. This allows to apply them to the tractor shape 1-form, and provided it is closed we obtain a hypersurface conformal invariant. It turns out that in conformally flat background of dimension $n > 3$ the shape tractor 1-form $\mathbb{L}_a{}^B$ is certainly closed. The precise condition is given in the Proposition below.

From the the tractor Codazzi equation (3.78) we can notice that $\mathbb{L}$ is $d^{\check{\nabla}}$-closed for $n \geq 3$ in the conformally flat background, but in the ambient dimension 3 the connection $\check{\nabla}$ is taken to be the intrinsic tractor connection $\overline{\nabla}$ by definition. We can improve this observation by directly computing the $\overline{\nabla}$-exterior derivative of $\mathbb{L}$.

Proposition 5.2.1. *Let (M, c) be a conformal manifold of dimension $n > 3$. The intrinsic covariant exterior derivative of the shape tractor 1-form $\mathbb{L}$ of a hypersurface Σ in M is given in a choice of conformal scale $\bar{g} \in \bar{c}$ explicitly by the formula*

$$2\overline{\nabla}_{[c}\mathbb{L}_{a]B} \stackrel{\bar{g}}{=} \bar{Z}_B{}^b\underline{\mathrm{W}}_{cabd}N^d - \tfrac{1}{n-3}\bar{X}_B\overline{\nabla}^b(\underline{\mathrm{W}}_{cabd}N^d) \tag{5.7}$$

where $\bar{c}$ is the induced conformal class on Σ.

Proof. Recall that $\mathbb{L}$ in a choice of scale is expressed as

$$\mathbb{L}_{aB} \stackrel{\bar{g}}{=} \bar{Z}_B{}^b \mathring{\mathbf{L}}_{ab} - \frac{1}{n-2} \bar{X}_B \overline{\nabla}^b \mathring{\mathbf{L}}_{ab}$$

Differentiating this with respect to $\overline{\nabla}$ and applying the Leibniz rule we get

$$\begin{aligned}
\overline{\nabla}_c \mathbb{L}_{aB} &= (\overline{\nabla}_c \bar{Z}_B{}^b) \mathring{\mathbf{L}}_{ab} + \bar{Z}_B{}^b \overline{\nabla}_c \mathring{\mathbf{L}}_{ab} - \tfrac{1}{n-2} (\overline{\nabla}_c \bar{X}_B) \overline{\nabla}^b \mathring{\mathbf{L}}_{ab} - \tfrac{1}{n-2} \bar{X}_B \overline{\nabla}_c \overline{\nabla}^b \mathring{\mathbf{L}}_{ab} \\
&= (-\bar{Y}_B \bar{\delta}_c{}^b - \bar{X}_B \overline{\mathsf{P}}_c{}^b) \mathring{\mathbf{L}}_{ab} + \bar{Z}_B{}^b \overline{\nabla}_c \mathring{\mathbf{L}}_{ab} - \tfrac{1}{n-2} \bar{Z}_B{}^b \bar{\mathbf{g}}_{bc} \overline{\nabla}^d \mathring{\mathbf{L}}_{ad} - \tfrac{1}{n-2} \bar{X}_B \overline{\nabla}_c \overline{\nabla}^b \mathring{\mathbf{L}}_{ab} \\
&= -\bar{Y}_B \mathring{\mathbf{L}}_{ac} + \bar{Z}_B{}^b \Big(\overline{\nabla}_c \mathring{\mathbf{L}}_{ab} - \tfrac{1}{n-2} \bar{\mathbf{g}}_{bc} \overline{\nabla}^d \mathring{\mathbf{L}}_{ad} \Big) - \bar{X}_B \Big(\overline{\mathsf{P}}_c{}^b \mathring{\mathbf{L}}_{ab} + \tfrac{1}{n-2} \overline{\nabla}_c \overline{\nabla}^b \mathring{\mathbf{L}}_{ab} \Big)
\end{aligned}$$

Skewing over the indices c and a we obtain

$$2\overline{\nabla}_{[c} \mathbb{L}_{a]B} = \bar{Z}_B{}^b \Big(2\overline{\nabla}_{[c} \mathring{\mathbf{L}}_{a]b} - \tfrac{2}{n-2} \bar{\mathbf{g}}_{b[c} \overline{\nabla}^d \mathring{\mathbf{L}}_{a]d} \Big) - \bar{X}_B \Big(2\, \overline{\mathsf{P}}_{[c}{}^b \mathring{\mathbf{L}}_{a]b} + \tfrac{2}{n-2} \overline{\nabla}_{[c} \overline{\nabla}^b \mathring{\mathbf{L}}_{a]b} \Big) \tag{5.8}$$

To handle the first term we rewrite the conformal Codazzi equation in the form

$$\underline{\mathrm{W}}_{ab}{}^{cd} N_d = 2\overline{\nabla}_{[a} \mathring{\mathbf{L}}_{b]}{}^c - 2\, \bar{\delta}_{[a}{}^c \underline{\mathsf{P}}_{b]}{}^d N_d - 2\, \bar{\delta}_{[a}{}^c \overline{\nabla}_{b]} \mathbf{H}$$

and using the contracted conformal Codazzi equation obtain a compact form of the (full) conformal Codazzi equation as

$$\underline{\mathrm{W}}_{ab}{}^{cd} N_d = 2\overline{\nabla}_{[a} \mathring{\mathbf{L}}_{b]}{}^c - \tfrac{2}{n-2}\, \bar{\delta}_{[a}{}^c \overline{\nabla}^b \mathring{\mathbf{L}}_{b]d}$$

The second term in (5.8) after substituting the contracted conformal Codazzi equation (2.36) and simplifying using the torsion-freeness of the Levi-Civita connection, i.e. $2\, \overline{\nabla}_{[c} \overline{\nabla}_{a]} \mathbf{H} = 0$, becomes $2\, \overline{\mathsf{P}}_{[c}{}^b \mathring{\mathrm{L}}_{a]b} + 2\overline{\nabla}_{[c} \overset{\mathrm{N}}{\underline{\mathsf{P}}}_{a]}$ where

$$\overset{\mathrm{N}}{\underline{\mathsf{P}}}_a = \Pi^{a'}{}_a \mathsf{P}_{a'b} \mathrm{N}^b$$

where P_{ab} is the ambient Schouten tensor restricted to the hypersurface Σ.

We claim now that

$$2\overline{\nabla}_{[c} \overset{\mathrm{N}}{\underline{\mathsf{P}}}_{a]} = \tfrac{1}{n-3} \overline{\nabla}^b (\underline{\mathrm{W}}_{cabd} N^d) - 2\, \overline{\mathsf{P}}_{[c}{}^b \mathring{\mathbf{L}} a]b \tag{5.9}$$

Indeed, the contracted Codazzi equation implies that

$$2\overline{\nabla}_{[c}\underline{\overset{\mathrm{N}}{\mathsf{P}}}_{a]} = \tfrac{2}{n-2}\overline{\nabla}_{[c}\overline{\nabla}^{b}\mathring{\mathbf{L}}a]b \tag{5.10}$$

We can swap the derivatives:

$$\begin{aligned}\bar{\mathbf{g}}^{bd}\overline{\nabla}_c\overline{\nabla}_d\mathring{\mathbf{L}}_{ab} &= \bar{\mathbf{g}}^{bd}\Big(\overline{\nabla}_d\overline{\nabla}_c\mathring{\mathbf{L}}_{ab} - \mathrm{R}_{cd}{}^{e}{}_{a}\mathring{\mathbf{L}}_{eb} - \mathrm{R}_{cd}{}^{e}{}_{b}\mathring{\mathbf{L}}_{ae}) \\ &= \overline{\nabla}^{b}\overline{\nabla}_c\mathring{\mathbf{L}}_{ab} + \overline{\mathrm{R}}^{b}{}_{c}{}^{e}{}_{a}\mathring{\mathbf{L}}_{eb} - \overline{\mathrm{Ric}}_{c}{}^{e}\mathring{\mathbf{L}}_{ae}\end{aligned}$$

After skewing the indices c and a the second term vanishes due to the swap symmetry of the Riemannian curvature, and the third one becomes

$$\overline{\mathrm{Ric}}_{c}{}^{e}\mathring{\mathrm{L}}_{ae} = (\bar{n}-2)\overline{\mathsf{P}}_{c}{}^{e}\mathring{\mathrm{L}}_{ae} + \bar{\mathsf{J}}\mathring{\mathrm{L}}_{ac} = (n-3)\overline{\mathsf{P}}_{c}{}^{e}\mathring{\mathrm{L}}_{ae} + \bar{\mathsf{J}}\mathring{\mathrm{L}}_{ac}$$

Thus we have

$$2\overline{\nabla}_{[c}\overline{\nabla}^{b}\mathring{\mathrm{L}}_{a]b} = 2\overline{\nabla}^{b}\overline{\nabla}_{[c}\mathring{\mathrm{L}}_{a]b} - 2(n-3)\overline{\mathsf{P}}_{[c}{}^{e}\mathring{\mathrm{L}}_{a]e}$$

□

5.2.2 Q_Σ-density

Coupled Q-operators

The operator $Q_1^D\colon \mathcal{E}^1\otimes\mathcal{F}\to\mathcal{E}_1\otimes\mathcal{F}$, introduced in [28, 13], is given explicitly by the formula

$$Q_1 := \mathrm{d}^D\delta^D - 4\mathsf{P}\# + 2\mathsf{J} \tag{5.11}$$

where $\mathcal{F}$ is a vector bundle with a connection D, d^D is the exterior covariant derivative associated to D, and δ^D is the formal adjoint of d^D. By $\mathcal{E}^1$ we denote the bundle of 1-forms.

Claim 5.2.2. *Operator Q_1 is Q-like on closed 1-forms in dimension 4.*

Proof. Write the action of Q-operator on a 1-form u_a explicitly

$$Qu_a = -\nabla_a\nabla^b u_b - 4\,\mathsf{P}_a{}^{b}u_b + 2\,\mathsf{J}u_a$$

Using (2.1), (2.4), (2.9) and (2.10) we compute in dimension $n=4$

$$\begin{aligned}\widehat{Q}u_a \quad = \quad & -(\nabla_a - 2\Upsilon_a)(\nabla^b u_b + 2\Upsilon^b u_b) - 4\mathsf{P}_a{}^{b}u_b + 4(\nabla_a\Upsilon^b)u_b - 4\Upsilon_a\Upsilon\cdot u \\ & + \cancel{2|\Upsilon|^2 u_a} + 2\mathsf{J}u_a - 2(\nabla\cdot\Upsilon)u_a - \cancel{2|\Upsilon|^2 u_a}\end{aligned}$$

Opening the parentheses we get some more cancellations

$$\begin{aligned}\widehat{Q}u_a = & -\nabla_a\nabla^b u_b - 2\nabla_a(\Upsilon^b u_b) + 2\Upsilon_a\nabla\cdot u + \cancel{4\Upsilon_a\Upsilon\cdot u} - 4\mathsf{P}_a{}^b u_b \\ & + 4(\nabla_a\Upsilon^b)u_b - \cancel{4\Upsilon_a\Upsilon\cdot u} + 2\mathsf{J}u_a - 2(\nabla\cdot\Upsilon)u_a\end{aligned}$$

Collecting the terms we arrive to the expression

$$\begin{aligned}\widehat{Q}u_a &= Qu_a - 2\nabla_a(\Upsilon^b u_b) + 2\Upsilon_a\nabla\cdot u + 4(\nabla_a\Upsilon^b)u_b - 2(\nabla\cdot\Upsilon)u_a \\ &= Qu_a - 2(\nabla_a\Upsilon^b)u_b - 2\Upsilon^b\nabla_a u_b + 2\Upsilon_a\nabla\cdot u + 4(\nabla_a\Upsilon^b)u_b - 2(\nabla\cdot\Upsilon)u_a \\ &= Qu_a + 2(\nabla_a\Upsilon^b)u_b - 2\Upsilon^b\nabla_a u_b + 2\Upsilon_a\nabla\cdot u - 2(\nabla\cdot\Upsilon)u_a.\end{aligned}$$

Note that up to this point we have not used that u is closed, or any other properties of u. Neither have any derivatives been commuted. The calculation so far is an expansion of the transformation formulas.

Now to identify the transformation terms we compute

$$\begin{aligned}\delta\mathrm{d}\,(\Upsilon u_a) &= -2\nabla^b\nabla_{[b}\Upsilon u_{a]} = 2\nabla^b\nabla_{[a}\Upsilon u_{b]} \\ &= \nabla^b\nabla_a(\Upsilon u_b) - \nabla^b\nabla_b(\Upsilon u_a) \\ &= \nabla^b\Big(\Upsilon_a u_b + \Upsilon\nabla_a u_b\Big) - \nabla^b\Big(\Upsilon_b u_a + \Upsilon\nabla_b u_a\Big) \\ &= (\nabla^b\Upsilon_a)u_b + \Upsilon_a\nabla\cdot u + \cancel{\Upsilon^b\nabla_a u_b} + \cancel{\Upsilon\nabla^b\nabla_a u_b} \\ &\quad -(\nabla\cdot\Upsilon)u_a - \Upsilon_b\nabla^b u_a - \cancel{\Upsilon^b\nabla_b u_a} - \cancel{\Upsilon\nabla^b\nabla_b u_a}\end{aligned}$$

where the cancellations occur because we assume that u_a is closed which is equivalent to

$$\nabla_a u_b = \nabla_b u_a.$$

Thus, using that $\nabla_a\Upsilon_b = \nabla_b\Upsilon_a$ and again that $\nabla_a u_b = \nabla_b u_a$, we get

$$\widehat{Q}u_a = Qu_a + 2\delta\mathrm{d}\,(\Upsilon u_a), \tag{5.12}$$

which is consistent with [13, Theorem 5.3]. □

Now observe that, at the level of a formal algebraic calculation, the entire calculation leading to (5.12) used only the identities

$$\nabla_a\Upsilon_b = \nabla_b\Upsilon_a \quad \text{and } \nabla_a u_b = \nabla_b u_a.$$

No other commutation of the derivatives was required. Therefore for the operator (5.11) we have the following result, which is just the specialization of [13, Theorem 5.3] to the case $n = 4$. Here $\mathcal{C}^1(\mathcal{F})$ denotes $\mathcal{F}$-valued 1-forms which are closed with respect to d^D.

Theorem 5.2.3. *For any vector bundle with connection $(\mathcal{F}, D)$*

$$Q_1^{D,\widehat{g}}\kappa = Q_1^{D,g}\kappa + 2\delta^D d^D(\Upsilon\kappa),$$

for $\kappa \in \mathcal{C}^1(\mathcal{F})$ and $\widehat{g} = e^{2\Upsilon}g$.

Now we want to use the fact that for a hypersurface Σ embedded in a conformally flat manifold the tractor shape 1-form $\mathrm{L}_{aB} := \underline{\nabla}_a N_B$ is closed. If the ambient manifold has dimension 5 then for each metric $\bar{g}$, from the conformal class on Σ, we can construct the quantity

$$Q_\Sigma^{\bar{g}} := \mathrm{L}^{aB} Q_1^{\bar{g}} \mathrm{L}_{aB}. \tag{5.13}$$

Theorem 5.2.4. *On a closed hypersurface Σ, embedded in conformally flat 5-manifold M, we have that*

$$\mathcal{Q}_\Sigma := \int_\Sigma Q_\Sigma^{\bar{g}},$$

is independent of the choice of metric $\bar{g}$ from the conformal class $\bar{c}_\Sigma$.

Proof. First observe that $Q_\Sigma^{\bar{g}}$ has conformal weight -4, and so it has the right weight to integrate against the conformal measure. Rescaling to $\widehat{\bar{g}} = e^{2\overline{\Upsilon}}\bar{g}$, we have

$$Q_\Sigma^{\widehat{\bar{g}}} = \mathbb{L}^{aB} Q_1 \mathbb{L}_{aB} + 2\,\mathbb{L}^{aB}\delta^{\overline{\nabla}}\mathrm{d}^{\overline{\nabla}}(\overline{\Upsilon}\,\mathbb{L}_{aB}),$$

where $\overline{\nabla}$ denotes the intrinsic tractor connection on Σ.

Since the tractor metric is preserved by $\overline{\nabla}$, integrating by parts the second term we get

$$\int_\Sigma \mathbb{L}^{aB}\delta^{\overline{\nabla}}\mathrm{d}^{\overline{\nabla}}\,(\overline{\Upsilon}\,\mathbb{L}_{aB}) = 0,$$

since $d^{\overline{\nabla}}\mathbb{L} = 0$. □

Remark 5.2.5. Note that there are other quantities with some properties similar to $Q_\Sigma^{\bar{g}}$. Most obviously there is the usual Q-curvature $Q^{\bar{g}}$ determined by the intrinsic structure $(\Sigma, \bar{g})$. This takes values in conformal weight $-n$ densities and and transforms under conformal changes $\bar{g} \mapsto \widehat{\bar{g}} = e^{2\Upsilon}\bar{g}$ according to

$$Q^{\widehat{\bar{g}}} = Q^{\widehat{\bar{g}}} + P_4\Upsilon$$

where P_4 is the Paneitz operator on functions. In a given scale $\bar{g}$ the latter takes the form $\delta Q_1 d$, with Q_1 the differential operator from above (see [28], and so

$$\int_\Sigma Q^{\hat{\bar{g}}}$$

is also a global invariant of even dimensional conformal hypersurfaces.

Moreover $\mathrm{tr}(\mathring{\mathrm{L}}^4)$ is also a conformally invariant density of weight -4. Thus we apparently have a family of global invariants

$$\int_\Sigma \left(\alpha Q^{\bar{g}}_\Sigma + \beta Q^{\bar{g}} + \gamma\,\mathrm{tr}(\mathring{\mathbf{L}}^4)\right)$$

depending on the 3 parameters $\alpha, \beta, \gamma \in \mathbb{R}$.

Variation of embedding

We shall see that $\mathcal{Q}_\Sigma$ may be distinguished from the total Q-curvature by considering total metric variations.

We consider a compactly supported variation of the embedding of the form $f^t = f + t\phi N$. Computing $\delta := \frac{\partial}{\partial t}\big|_t = 0$ we obtain the following result.

Theorem 5.2.6. *Let Σ be a 4 dimensional hypersurface in a conformal flat manifold (M, c). Then*

$$\delta \mathcal{Q}_\Sigma = \int_\Sigma \phi \mathcal{W}^{(4)} \tag{5.14}$$

where the coefficient of variation $\mathcal{W}^{(4)}$ is a natural weight -5 density-valued conformal invariant of the hypersurface. This takes the form

$$\mathcal{W}^{(4)} \stackrel{g}{=} a \cdot \bar{\Delta}^2 H^g + \text{lower order terms}, \tag{5.15}$$

where g is any metric in c and a is a non-zero constant. So the Euler–Lagrange equation of the action $\mathcal{Q}_\Sigma$, with respect to embedding variation, has linear leading term.

Proof. Everything is obvious by construction, except the claim concerning the leading term. Expanding the formula for Q_Σ we obtain

$$Q_\Sigma = \mathbb{L}^{aB} Q_1 \mathbb{L}_{aB} = -\tfrac{2}{3}\mathring{\mathbf{L}}^{ab}\overline{\nabla}_a\overline{\nabla}^e\mathring{\mathbf{L}}_{eb} - 4\,\mathring{\mathbf{L}}^{ab}\bar{\mathsf{P}}_a{}^e\mathring{\mathbf{L}}_{eb} + 2\,\bar{\mathsf{J}}\mathring{\mathbf{L}}^{ab}\mathring{\mathbf{L}}_{ab} - \tfrac{2}{9}(\overline{\nabla}^b\mathring{\mathbf{L}}^a{}_b)\overline{\nabla}^e\mathring{\mathbf{L}}_{ae}.$$

Thus

$$Q_\Sigma = \tfrac{4}{9}(\overline{\nabla}^b\mathring{\mathbf{L}}^a{}_b)\overline{\nabla}^e\mathring{\mathbf{L}}_{ae} + \div S + \text{lower order terms} = 4(\overline{\nabla}^a H)\overline{\nabla}_a H + \div S + \text{lower order terms},$$

where $\div S$ is some divergence. From this the formula for $\mathcal{W}^{(4)}$ follows easily as $\delta H = \frac{1}{2}\bar{\Delta}\phi +$ lower order terms. □

This shows that $\mathcal{W}^{(4)} = 0$ is an analogue of the Willmore equation and $\mathcal{Q}_\Sigma$ is a dimension 4 analogue of the rigid string action.

Lemma 5.2.7 (Expansion of Q_Σ)**.** *The quantity Q_Σ is given explicitly by*

$$\mathbb{L}^{aB}Q_1\mathbb{L}aB = -\tfrac{n-3}{n-2}\mathring{\mathbf{L}}^{ab}\overline{\nabla}_a\overline{\nabla}^e\mathring{\mathbf{L}}_{eb} - 4\,\mathring{\mathbf{L}}^{ab}\overline{\mathsf{P}}_a{}^e\mathring{\mathbf{L}}_{eb} + 2\,\bar{\mathsf{J}}\mathring{\mathbf{L}}^{ab}\mathring{\mathbf{L}}_{ab} - \tfrac{n-3}{(n-2)^2}(\overline{\nabla}^b\mathring{\mathbf{L}}^a{}_b)\overline{\nabla}^e\mathring{\mathbf{L}}_{ae}$$

where n is the dimension of the background manifold M.

Proof. Recall that

$$\mathbb{L}_{aB} \stackrel{\bar{g}}{=} \bar{Z}_B{}^b\mathring{\mathbf{L}}_{ab} - \tfrac{1}{n-2}\bar{X}_B\overline{\nabla}^b\mathring{\mathbf{L}}_{ab} \tag{5.16}$$

Differentiating with respect to the intrinsic tractor connection yields

$$\overline{\nabla}_c\mathbb{L}_{aB} = -\bar{Y}_B\mathring{\mathbf{L}}_{ac} + \bar{Z}_B{}^b\Big(\overline{\nabla}_c\mathring{\mathbf{L}}_{ab} - \tfrac{1}{n-2}\bar{\mathbf{g}}_{bc}\overline{\nabla}^d\mathring{\mathbf{L}}_{ad}\Big) - \bar{X}_B\Big(\overline{\mathsf{P}}_c{}^b\mathring{\mathbf{L}}_{ab} + \tfrac{1}{n-2}\overline{\nabla}_c\overline{\nabla}^b\mathring{\mathbf{L}}_{ab}\Big)$$

so the divergence becomes

$$\overline{\nabla}^e\mathbb{L}_{eB} = \bar{Z}_B{}^b\Big(\overline{\nabla}^e\mathring{\mathbf{L}}_{eb} - \tfrac{1}{n-2}\overline{\nabla}^d\mathring{\mathbf{L}}_{bd}\Big) - \bar{X}_B\Big(\overline{\mathsf{P}}^{ef}\mathring{\mathbf{L}}_{ef} + \tfrac{1}{n-2}\overline{\nabla}^e\overline{\nabla}^f\mathring{\mathbf{L}}_{ef}\Big)$$

or

$$\overline{\nabla}^e\mathbb{L}_{eB} = \tfrac{n-3}{n-2}\bar{Z}_B{}^b\overline{\nabla}^e\mathring{\mathbf{L}}_{eb} - \bar{X}_B\Big(\overline{\mathsf{P}}^{ef}\mathring{\mathbf{L}}_{ef} + \tfrac{1}{n-2}\overline{\nabla}^e\overline{\nabla}^f\mathring{\mathbf{L}}_{ef}\Big)$$

Differentiating this with respect to the intrinsic coupled tractor connection we see that

$$\begin{aligned}
\overline{\nabla}_a\overline{\nabla}^e\mathbb{L}_{eB} &= \tfrac{n-3}{n-2}(\overline{\nabla}_a\bar{Z}_B{}^b)\overline{\nabla}^e\mathring{\mathbf{L}}_{eb} + \tfrac{n-3}{n-2}\bar{Z}_B{}^b\overline{\nabla}_a\overline{\nabla}^e\mathring{\mathbf{L}}_{eb} \\
&\quad - (\overline{\nabla}_a\bar{X}_B)\Big(\overline{\mathsf{P}}^{ef}\mathring{\mathbf{L}}_{ef} + \tfrac{1}{n-2}\overline{\nabla}^e\overline{\nabla}^f\mathring{\mathbf{L}}_{ef}\Big) - \bar{X}_B\overline{\nabla}_a\Big(\overline{\mathsf{P}}^{ef}\mathring{\mathbf{L}}_{ef} + \tfrac{1}{n-2}\overline{\nabla}^e\overline{\nabla}^f\mathring{\mathbf{L}}_{ef}\Big) \\
&= \tfrac{n-3}{n-2}(-\bar{Y}_B\bar{\delta}_a{}^b - X_B\overline{\mathsf{P}}_a{}^b)\overline{\nabla}^e\mathring{\mathbf{L}}_{eb} + \tfrac{n-3}{n-2}\bar{Z}_B{}^b\overline{\nabla}_a\overline{\nabla}^e\mathring{\mathbf{L}}_{eb} \\
&\quad - \bar{Z}_B{}^b\bar{\mathbf{g}}_{ab}\Big(\overline{\mathsf{P}}^{ef}\mathring{\mathbf{L}}_{ef} + \tfrac{1}{n-2}\overline{\nabla}^e\overline{\nabla}^f\mathring{\mathbf{L}}_{ef}\Big) - \bar{X}_B\overline{\nabla}_a\Big(\overline{\mathsf{P}}^{ef}\mathring{\mathbf{L}}_{ef} + \tfrac{1}{n-2}\overline{\nabla}^e\overline{\nabla}^f\mathring{\mathbf{L}}_{ef}\Big)
\end{aligned}$$

so we get

$$\begin{aligned}-\overline{\nabla}_a\overline{\nabla}^e\mathbb{L}_{eB} &= \bar{Y}_B\tfrac{n-3}{n-2}\overline{\nabla}^e\mathring{\mathbf{L}}_{ae}\\ &\quad+\bar{Z}_B{}^b\Big(-\tfrac{n-3}{n-2}\overline{\nabla}_a\overline{\nabla}^e\mathring{\mathbf{L}}_{eb}+\bar{\mathbf{g}}_{ab}\overline{\mathsf{P}}^{ef}\mathring{\mathbf{L}}_{ef}+\bar{\mathbf{g}}_{ab}\tfrac{1}{n-2}\overline{\nabla}^e\overline{\nabla}^f\mathring{\mathbf{L}}_{ef}\Big)\\ &\quad+\bar{X}_B\Bigg(\overline{\nabla}_a\Big(\overline{\mathsf{P}}^{ef}\mathring{\mathbf{L}}_{ef}+\tfrac{1}{n-2}\overline{\nabla}^e\overline{\nabla}^f\mathring{\mathbf{L}}_{ef}\Big)+\tfrac{n-3}{n-2}\overline{\mathsf{P}}_a{}^b\overline{\nabla}^e\mathring{\mathbf{L}}_{eb}\Bigg)\end{aligned}$$

The next term to consider is

$$\begin{aligned}-4\,\overline{\mathsf{P}}_a{}^e\mathbb{L}_{eB} &= -4\,\overline{\mathsf{P}}_a{}^e\Big(\bar{Z}_B{}^b\mathring{\mathbf{L}}_{eb}-\tfrac{1}{n-2}\bar{X}_B\overline{\nabla}^b\mathring{\mathbf{L}}_{eb}\Big)\\ &= -4\,\bar{Z}_B{}^b\overline{\mathsf{P}}_a{}^e\mathring{\mathbf{L}}_{eb}+\tfrac{4}{n-2}\bar{X}_B\overline{\mathsf{P}}_a{}^e\overline{\nabla}^b\mathring{\mathbf{L}}_{eb}\end{aligned}$$

The last term in the expression (5.11) is

$$2\,\bar{\mathsf{J}}\mathbb{L}_{aB} = \bar{Z}_B{}^b 2\,\bar{\mathsf{J}}\mathring{\mathbf{L}}_{ab}-\tfrac{2}{n-2}\bar{X}_B\bar{\mathsf{J}}\overline{\nabla}^b\mathring{\mathbf{L}}_{ab}$$

Adding up we obtain

$$\begin{aligned}Q_1\mathbb{L}_{aB} &= -\overline{\nabla}_a\overline{\nabla}^e\mathrm{L}_{eB}-4\,\overline{\mathsf{P}}_a{}^e\mathrm{L}_{eB}+2\,\bar{\mathsf{J}}\mathrm{L}_{aB}\\ &= \bar{Y}_B\tfrac{n-3}{n-2}\overline{\nabla}^e\mathring{\mathbf{L}}_{ae}\\ &\quad+\bar{Z}_B{}^b\Big(-\tfrac{n-3}{n-2}\overline{\nabla}_a\overline{\nabla}^e\mathring{\mathbf{L}}_{eb}+\bar{\mathbf{g}}_{ab}\overline{\mathsf{P}}^{ef}\mathring{\mathbf{L}}_{ef}+\bar{\mathbf{g}}_{ab}\tfrac{1}{n-2}\overline{\nabla}^e\overline{\nabla}^f\mathring{\mathbf{L}}_{ef}-4\,\overline{\mathsf{P}}_a{}^e\mathring{\mathbf{L}}_{eb}+2\,\bar{\mathsf{J}}\mathring{\mathbf{L}}_{ab}\Big)\\ &\quad+\bar{X}_B\Bigg(\overline{\nabla}_a\Big(\overline{\mathsf{P}}^{ef}\mathring{\mathbf{L}}_{ef}+\tfrac{1}{n-2}\overline{\nabla}^e\overline{\nabla}^f\mathring{\mathbf{L}}_{ef}\Big)+\tfrac{n-3}{n-2}\overline{\mathsf{P}}_a{}^b\overline{\nabla}^e\mathring{\mathbf{L}}_{eb}+\tfrac{4}{n-2}\overline{\mathsf{P}}_a{}^e\overline{\nabla}^b\mathring{\mathbf{L}}_{eb}-\tfrac{2}{n-2}\bar{\mathsf{J}}\overline{\nabla}^b\mathring{\mathbf{L}}_{ab}\Bigg)\end{aligned}$$

Contracting this to (5.16) we arrive to

$$\begin{aligned}\mathbb{L}^{aB}Q_1\mathbb{L}_{aB} &= \mathring{\mathbf{L}}^{ab}\Big(-\tfrac{n-3}{n-2}\overline{\nabla}_a\overline{\nabla}^e\mathring{\mathbf{L}}_{eb}+\bar{\mathbf{g}}_{ab}\overline{\mathsf{P}}^{ef}\mathring{\mathbf{L}}_{ef}+\bar{\mathbf{g}}_{ab}\tfrac{1}{n-2}\overline{\nabla}^e\overline{\nabla}^f\mathring{\mathbf{L}}_{ef}\\ &\quad-4\,\mathsf{P}_a{}^e\mathring{\mathbf{L}}_{eb}+2\,\mathsf{J}\mathring{\mathbf{L}}_{ab}\Big)-\tfrac{n-3}{(n-2)^2}(\overline{\nabla}^b\mathring{\mathbf{L}}^a{}_b)\overline{\nabla}^e\mathring{\mathbf{L}}_{ae}\end{aligned}$$

which simplifies to

$$\mathbb{L}^{aB}Q_1\mathbb{L}_{aB} = -\tfrac{n-3}{n-2}\mathring{\mathbf{L}}^{ab}\overline{\nabla}_a\overline{\nabla}^e\mathring{\mathbf{L}}_{eb}-4\,\mathring{\mathbf{L}}^{ab}\overline{\mathsf{P}}_a{}^e\mathring{\mathbf{L}}_{eb}+2\,\bar{\mathsf{J}}\mathring{\mathbf{L}}^{ab}\mathring{\mathbf{L}}_{ab}-\tfrac{n-3}{(n-2)^2}(\overline{\nabla}^b\mathring{\mathbf{L}}^a{}_b)\overline{\nabla}^e\mathring{\mathbf{L}}_{ae}$$

□

In contrast the situation is rather different for the total Q-curvature.

Proposition 5.2.8. *Let Σ be a 4 dimensional hypersurface in any conformal*

5-manifold (M, c). Then

$$\delta \int_\Sigma Q = b \cdot \int_\Sigma \varphi B_{ab} \mathring{\mathrm{L}}^{ab}, \tag{5.17}$$

where b is a non-zero constant and B_{ab} is the Bach tensor of the hypersurface.

Proof. Under any compactly supported deformation of the metric the infinitesimal change of $\int Q$ (in dimension 4) is the Bach tensor integrated against the infinitesimal metric deformation [33]. On the other hand lower order terms with compactly supported normal variations of embedding of the form $f^t = f + t\varphi N$ and $\delta := \frac{\partial}{\partial t}|_{t=0}$ we have $\delta\bar{g} = -2\varphi L$. □

Remark 5.2.9. The expression 5.15 shows that for the Euler–Lagrange of the total Q-curvature the leading term is at least quadratic in the hypersurface curvature quantities. In particular this vanishes at level of linearisation.

Nevertheless the Branson Q-curvature of the first fundamental form is an interesting object on hypersurfaces. Observe that totally umbilic hypersurfaces are critical for the total Q-curvature functional.

Appendix A

Multiplication Tables

	$\mathbb{Y}^{AA'}{}_b$	$\mathbb{W}^{AA'}$	$\mathbb{Z}^{AA'}{}_{bb'}$	$\mathbb{X}^{AA'}{}_b$
$\mathbb{Y}_{AA'}{}^a$	0	0	0	$2\,\delta^a{}_b$
$\mathbb{W}_{AA'}$	0	-2	0	0
$\mathbb{Z}_{AA'}{}^{aa'}$	0	0	$4\,\delta^{[a}{}_{[b}\delta^{a']}{}_{b']}$	0
$\mathbb{X}_{AA'}{}^a$	$2\,\delta^a{}_b$	0	0	0

Table A.1: Complete contractions of the adjoint tractor projectors

	Y^{AB}	$\mathsf{U}^{AB}{}_e$	V^{AB}	$\mathsf{Z}^{AB}{}_{ef}$	$\mathsf{W}^{AB}{}_e$	X^{AB}
Y_{AB}	0	0	0	0	0	1
$\mathsf{U}_{AB}{}^c$	0	0	0	0	$2\delta_e{}^c$	0
V_{AB}	0	0	2	0	0	0
$\mathsf{Z}_{AB}{}^{cd}$	0	0	0	$\delta^{(c}{}_{(e}\delta^{d)}{}_{f)}$	0	0
$\mathsf{W}_{AB}{}^c$	0	$2\delta_e{}^c$	0	0	0	0
X_{AB}	1	0	0	0	0	0

Table A.2: Complete contractions of symmetric 2-tractor projectors

h^{CA}	Y_{AB}	$\mathsf{U}_{AB}{}^b$	V_{AB}	$\mathsf{Z}_{AB}{}^{ab}$	$\mathsf{W}_{AB}{}^b$	X_{AB}
Y_C	0	0	Y_B	0	$Z_B{}^b$	X_B
$Z_C{}^c$	0	$Y_B\mathbf{g}^{cb}$	0	$Z_B{}^{(b}\mathbf{g}^{a)c}$	$X_B\mathbf{g}^{cb}$	0
X_C	Y_B	$Z_B{}^b$	X_B	0	0	0

Table A.3: Mixed contractions of standard with symmetric 2-tractor projectors

h^{CA}	$\mathbb{Y}_{AB}{}^b$	$\mathbb{W}_{AB}$	$\mathbb{Z}_{AB}{}^{ab}$	$\mathbb{X}_{AB}{}^b$
Y_C	0	$-Y_B$	0	$Z_B{}^b$
$Z_C{}^c$	$-Y_B\mathbf{g}^{cb}$	0	$2\,Z_B{}^{[b}\mathbf{g}^{a]c}$	$-X_B\mathbf{g}^{cb}$
X_C	$Z_B{}^b$	X_B	0	0

Table A.4: Mixed contractions of standard with adjoint tractor projectors

$h^{A'B'}$	$\mathbb{Y}_{BB'}{}^{b'}$	$\mathbb{W}_{BB'}$	$\mathbb{Z}_{BB'}{}^{bb'}$	$\mathbb{X}_{BB'}{}^{b'}$
$\mathbb{Y}_{AA'}{}^{a'}$	$Y_AY_B\mathbf{g}^{a'b'}$	$-Z_A{}^{a'}Y_B$	$2\,Y_AZ_B{}^{[b}\mathbf{g}^{b']a'}$	$Y_AX_B\mathbf{g}^{a'b'}+Z_A{}^{a'}Z_B{}^{b'}$
$\mathbb{W}_{AA'}$	$-Y_AZ_B{}^{b'}$	$-2\,X_{(A}Y_{B)}$	0	$X_AZ_B{}^{b'}$
$\mathbb{Z}_{AA'}{}^{aa'}$	$2\,Z_A{}^{[a}Y_B\mathbf{g}^{a']b'}$	0	$2\,Z_A{}^{a}Z_B{}^{[b}\mathbf{g}^{b']a'}+2\,Z_A{}^{a'}Z_B{}^{[b'}\mathbf{g}^{b]a}$	$2\,Z_A{}^{[a}X_B\mathbf{g}^{a']b'}$
$\mathbb{X}_{AA'}{}^{a'}$	$X_AY_B\mathbf{g}^{a'b'}+Z_A{}^{a'}Z_B{}^{b'}$	$Z_A{}^{a'}X_B$	$2\,X_AZ_B{}^{[b}\mathbf{g}^{b']a'}$	$X_AX_B\mathbf{g}^{a'b'}$

Table A.5: Partial contractions of adjoint projectors

h^{CD}	Y_{DB}	$\mathsf{U}_{DB}{}^{b}$	V_{DB}	$\mathsf{Z}_{DB}{}^{db}$	$\mathsf{W}_{DB}{}^{b}$	X_{DB}
Y_{AC}	0	0	Y_AY_B	0	$Y_AZ_B{}^{b}$	Y_AX_B
$\mathsf{U}_{AC}{}^{a}$	0	$Y_AY_B\mathbf{g}^{ab}$	$Z_A{}^{a}Y_B$	$Y_AZ_B{}^{(d}\mathbf{g}^{b)a}$	$Y_AX_B\mathbf{g}^{ab}+Z_A{}^{a}Z_B{}^{b}$	$Z_A{}^{a}X_B$
V_{AB}	Y_AY_B	$Y_AZ_B{}^{b}$	$Y_AX_B+X_AY_B$	0	$X_AZ_B{}^{b}$	X_AX_B
$\mathsf{Z}_{AC}{}^{ac}$	0	$Z_A{}^{(a}Y_B\mathbf{g}^{c)b}$	0	$Z_A{}^{(a}\mathbf{g}^{c)(d}Z_B{}^{b)}$	$Z_A{}^{(a}X_B\mathbf{g}^{c)b}$	0
$\mathsf{W}_{AC}{}^{a}$	$Z_A{}^{a}Y_B$	$Z_A{}^{a}Z_B{}^{b}+X_AY_B\mathbf{g}^{ab}$	$Z_A{}^{a}X_B$	$X_AZ_B{}^{(b}\mathbf{g}^{d)a}$	$X_AX_B\mathbf{g}^{ab}$	0
X_{AC}	X_AY_B	$X_AZ_B{}^{b}$	X_AX_B	0	0	0

Table A.6: Partial contractions of symmetric 2-tractor projectors

h^{AC}	Y_{CD}	$\mathsf{U}_{CD}{}^{d}$	V_{CD}	$\mathsf{Z}_{CD}{}^{cd}$	$\mathsf{W}_{CD}{}^{d}$	X_{CD}
$\mathbb{Y}_{AB}{}^{b}$	0	$-Y_BY_D\mathbf{g}^{bd}$	$Z_B{}^{b}Y_D$	$Y_BZ_D{}^{(c}\mathbf{g}^{d)b}$	$Z_B{}^{b}Z_D{}^{d}-Y_BX_D\mathbf{g}^{bd}$	$Z_B{}^{b}X_D$
$\mathbb{W}_{AB}$	$-Y_BY_D$	$-Y_BZ_D{}^{d}$	$2\,X_{[B}Y_{D]}$	0	$X_BZ_D{}^{d}$	X_BX_D
$\mathbb{Z}_{AB}{}^{ab}$	0	$2\,Y_DZ_B{}^{[b}\mathbf{g}^{a]d}$	0	$2\,Z_B{}^{[b\vert}Z_D{}^{(c}\mathbf{g}^{d)\vert a]}$	$2\,X_DZ_B{}^{[b}\mathbf{g}^{a]d}$	0
$\mathbb{X}_{AB}{}^{b}$	$Z_B{}^{b}Y_D$	$Z_B{}^{b}Z_D{}^{d}-X_BY_D\mathbf{g}^{bd}$	$Z_B{}^{b}X_D$	$-X_BZ_D{}^{(c}\mathbf{g}^{d)b}$	$-X_BX_D\mathbf{g}^{bd}$	0

Table A.7: Mixed contractions of adjoint with symmetric 2-tractor projectors

Appendix B

Using Cadabra for tractor computations

In the course of our research we have encountered a great deal of tedious computations that would be desirable to perform with a computer algebra system.

One of such systems is **Ricci** for Mathematica [47], which has been used for tractor calculations for some time. For example, L.J. Peterson[1] has applied this package to obtain a number of important identities [32]. D.H.Grant [34] provides detailed listings with examples of computations, related to hypersurface conformal invariants. While the bundle *Mathematica+Ricci* is a convenient and powerful tool, it is not always accessible due to the license restrictions of Mathematica (Ricci is free). Another minor issue is that an effort is required to rewrite Ricci's output into a publishable form.

Cadabra ([59], [60] and [61]) is a open-source[2] standalone application, which is specifically dedicated for handling polynomial tensor expressions with multiple ranges of indices. It is very well suited for the abstract index notation conventions that we have adopted in this thesis. It accepts the input and produces the output in a subset of LaTeX that makes it very convenient for further reusing.

We have used Cadabra to make a lot of trial-and-error calculations and to double check the results obtained by hand. Many examples in this thesis, the multiplication tables from Appendix A, and the identities from Appendix C were obtained and verified with the help of Cadabra.

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[1] The author is grateful to Larry Peterson, who kindly explained to him how to get started with this package, during one of his visits to the University of Auckland.

[2] Available in the standard repository of Ubuntu.

The basics of Cadabra and a comprehensive documentation are available at its website [59].

In order to give an idea how Cadabra can be used for tractor calculations, we demonstrate the steps to compute the tractor Laplacian of the symmetric 2-tractor projector X_{AB}, see equation (C.14).

First of all, we need to specify the ranges for tensor and tractor indices.

```
#---Define bundles------------------------------------------------
{a,b,c,d,e,f,f#}::Indices(vector, position=fixed):
{A,B,C,D,E,F,F#}::Indices(tractor,position=fixed):
```

After that, Cadabra needs to be informed about the tensors, which serve as the metrics and the Kronecker symbols in the corresponding bundles. This is done by setting the properties of the tensors.

```
#---Define tensors------------------------------------------------
#------Metric tensors
g_{a b}::Metric:
g^{a b}::InverseMetric:
g^{a}_{b}::KroneckerDelta:
g_{a}^{b}::KroneckerDelta:
#---------The tractor metric
h_{A B}::Metric:
h^{A B}::InverseMetric:
h^{A}_{B}::KroneckerDelta:
h_{A}^{B}::KroneckerDelta:
```

There might be other tensors with the imposed symmetries:

```
#------Other tensors
{P_{a b}, P^{a b}}::Symmetric:
```

The symmetries are used by the algorithms of manipulating with the terms of expressions that we shall mention later.

The main feature of Cadabra for us is its ability to use the Leibniz rule, which is implemented by declaring the derivative operators:

```
#---Define derivatives------------------------------------------
# \nabla_{#}::Derivative:
# \nabla^{#}::Derivative:
```

The problem specific information is supplied by creating rules.

```
#---The rules defining the symmetric 2-tractor projectors
YY:=(YY_{A B}->Y_{A}Y_{B});
UU:=(UU_{A B}^{b}->Y_{A}Z_{B}^{b}+Y_{B}Z_{A}^{b});
VV:=(VV_{A B}->Y_{A}X_{B}+X_{A}Y_{B});
ZZ:=(ZZ_{A B}^{a b}->\frac{1}{2} (Z_{A}^{a}Z_{B}^{b}
     +Z_{B}^{a}Z_{A}^{b}));
WW:=(WW_{A B}^{b}->Z_{A}^{b}X_{B}+Z_{B}^{b}X_{A});
XX:=(XX_{A B}->X_{A}X_{B});
```

One of the efficient methods to implement covariant constants is to use **@unwrap** command. It requires that the user explicitly specifies which objects have nontrivial derivatives with respect to the derivation operator.

```
#---Inform that the following objects have \nabla derivatives
{P_{a b}, P^{a b},P_{a}^{b}, P^{a}_{b}}::Depends(\nabla):
{Y_{A},Z_{A}^{a},X_{A}}::Depends(\nabla):
{YY_{A B}, UU_{A B}^{a}, VV_{A B}, ZZ_{A B}^{a b},
         WW_{A B}^{a}, XX_{A B}}::Depends(\nabla):
```

We also need to inform Cadabra how to differentiate the quantities in the question, which is done by adding the rules for the action of the tractor connection to the script:

```
#---Define tractor derivative rules-----------------------------
#---For the standard 1-tractor projectors
DerY:=(\nabla_{b?}{Y^{A?}}->Z^{A?}_{c}P_{b?}^{c},
       \nabla_{b?}{Y_{A?}}->Z_{A?}^{c}P_{b?}_{c});
DerZ:=(\nabla_{b?}{Z^{A?}_{a?}}->-X^{A?}P_{a? b?}-Y^{A?}g_{a? b?},
  \nabla_{b?}{Z_{A?}^{a?}}->-X_{A?}P_{b?}^{a?}-Y_{A?}g_{b?}^{a?});
DerX:=(\nabla_{b?}{X^{A?}}->Z^{A?}_{b?},
      \nabla_{b?}{X_{A?}}->Z_{A?}^{c} g_{b? c});
#---For the standard symm. 2-tractor projectors
DerYY:=(\nabla_{c?}{YY_{A? B?}}->UU_{A? B?}^{a}P_{a c?});
DerUU:=(\nabla_{c?}{UU_{A? B?}^{a?}}->-2 YY_{A? B?}g^{a?}_{c?}
        -VV_{A? B?} P^{a?}_{c?}+2 ZZ_{A? B?}^{a? b}P_{b c?});
DerVV:=(\nabla_{c?}{VV_{A? B?}}->UU_{A? B?}^{a}g_{a c?}
                                 +WW_{A? B?}^{a}P_{a c?});
DerZZ:=(\nabla_{c?}{ZZ_{A? B?}^{a? b?}}->
```

```
      -\frac{1}{2}UU_{A? B?}^{a?} g^{b?}_{c?}
      -\frac{1}{2} UU_{A? B?}^{b?} g^{a?}_{c?}
      -\frac{1}{2} WW_{A? B?}^{a?} P^{b?}_{c?}
       -\frac{1}{2} WW_{A? B?}^{b?} P^{a?}_{c?});
DerWW:=(\nabla_{c?}{WW_{A? B?}^{a?}} ->
     -VV_{A? B?} g^{a?}_{c?} + 2 ZZ_{A? B?}^{a? b} g_{c? b}
     - 2 XX_{A? B?} P^{a?}_{c?});
DerXX:=(\nabla_{c?}{XX_{A? B?}} -> WW_{A? B?}^{a} g_{a c?});
```

The sequence of commands

```
XX:=XX_{A B}; # define the expression to work with
DXX:=\nabla_{c}{@(XX)}; #the first derivative
@substitute!(%)(@(DerXX)); # apply the rule of differentiation
DDXX:=\nabla_{d}{@(DXX)}; #the second derivative
@prodrule!(%); # distribute the derivatives using the Leibniz rule
@unwrap!(%); # drop the covariant constants
@substitute!(%)(@(DerWW)); # apply the rule of differentiation
@distribute!(%):@canonicalise!(%):@prodsort!(%):
@eliminate_kr!(%):@eliminate_metric!!(%);
LXX:=g^{c d} @(DDXX); # contract the derivatives, get the Laplacian
@distribute!(%): @canonicalise!(%): @prodsort!(%);
```

will produce the following output

$$LXX := -VV_{AB} g_{cd} g^{cd} + 2\, ZZ_{ABcd} g^{cd} - 2\, P_{cd} XX_{AB} g^{cd};$$

It is now easy to rewrite this expression as desired (C.14).

Appendix C

Symmetric 2-tractors

Calculus for symmetric 2-tractors

As we have seen in Chapter 4, some of the interesting examples of hypersurface conformal invariants turn out to be symmetric 2-tractors, that is sections of $\mathcal{T}_{(AB)}$. In order to save space and simplify the notation, we introduce a modification of the tractor calculus for this specific case. This is no more than specialization of the technique of standard tractor projectors in view of the composition series for $\mathcal{T}_{(AB)}$ that we represent as

$$\mathcal{E}[2] \oplus \mathcal{E}_a[2] \oplus \begin{array}{c}\mathcal{E}[0] \\ \oplus \\ \mathcal{E}_{ab}[2]\end{array} \mathcal{E}_a[0] \oplus \mathcal{E}[-2] \tag{C.1}$$

This comes from considering a symmetrized tensor product of two tractors taken in forms

$$U_A = Y_A\sigma + Z_A{}^a\mu_a + X_A\rho \tag{C.2}$$

$$V_B = Y_B\sigma' + Z_B{}^b\mu'_b + X_A\rho' \tag{C.3}$$

and then organizing the terms according to their contribution into the other terms under conformal transformation.

A weightless tractor with two indices, i.e. a section V^{AB} of $\mathcal{T}^{AB} = \mathcal{T}^A \otimes \mathcal{T}^B$, can be seen as a finite sum of decomposable tractors V^AV^B representable in the form

$$\begin{aligned} V^AV^B = {} & Y^AY^BV^{++} + Y^AZ^B{}_bV^{+b} + Y^AX^BV^{+-} \\ & + Z^A{}_aY^BV^{a+} + Z^A{}_aZ^B{}_bV^{ab} + Z^A{}_aX^BV^{a-} \\ & + X^AY^BV^{-+} + X^AZ^B{}_bV^{-b} + X^AX^BV^{--} \end{aligned} \tag{C.4}$$

or as a matrix

$$V^A V^B = \begin{pmatrix} V^{++} & V^{+b} & V^{+-} \\ V^{a+} & V^{ab} & V^{a-} \\ V^{-+} & V^{-b} & V^{--} \end{pmatrix} \tag{C.5}$$

If we symmetrize (C.4) we will get

$$\begin{aligned} V^{(A}V^{B)} == {} & Y^{(A}Y^{B)}V^{[2]} + 2Y^{(A}Z^{B)}{}_b V^{[0]b} + 2Y^{(A}X^{B)}V^{[0]} \\ & + Z^{(A}{}_a Z^{B)}{}_b V^{[-2](ab)} + 2Z^{(A}{}_a X^{B)}V^{[-2]a} + X^{(B}X^{A)}V^{[-2]} \end{aligned} \tag{C.6}$$

where the numbers in brackets are the homogeneities of the terms.

We define the following sections Y_{AB}, $\mathsf{U}_{AB}{}^a$, V_{AB}, $\mathsf{Z}_{AB}{}^{ab}$, $\mathsf{W}_{AB}{}^a$, X_{AB} that we shall refer to as the *symmetric 2-tractor projectors*

$$\begin{aligned} \mathsf{Y}_{AB} &\coloneqq Y_{(A}Y_{B)} = Y_A Y_B \\ \mathsf{U}_{AB}{}^a &\coloneqq 2Y_{(A}Z_{B)}{}^a \\ \mathsf{V}_{AB} &\coloneqq 2Y_{(A}X_{B)} \\ \mathsf{Z}_{AB}{}^{ab} &\coloneqq Z_{(A}{}^a Z_{B)}{}^b = Z_A{}^{(a} Z_B{}^{b)} \\ \mathsf{W}_{AB}{}^a &\coloneqq 2Z_{(A}{}^a X_{B)} \\ \mathsf{X}_{AB} &\coloneqq X_{(A}X_{B)} = X_A X_B \end{aligned} \tag{C.7}$$

In terms of these projectors any symmetric 2-tractor $V_{AB} \in \mathcal{T}_{(AB)}$ can be written as

$$V_{AB} = \mathsf{Y}_{AB}y + \mathsf{U}_{AB}{}^a u_a + \mathsf{V}_{AB}v + \mathsf{Z}_{AB}{}^{ab} z_{ab} + \mathsf{W}_{AB}{}^a w_a + \mathsf{X}_{AB}x \tag{C.8}$$

which corresponds to the vector-like notation

$$\begin{pmatrix} & y & \\ & u_a & \\ v & | & z_{ab} \\ & w_a & \\ & x & \end{pmatrix} \tag{C.9}$$

where we think of this object as an element of (C.1)

Proposition C.0.10. *The conformal transformation rules for the symmetric 2-tractor projectors are:*

$$
\begin{aligned}
\widehat{\mathsf{Y}_{AB}} &= \mathsf{Y}_{AB} - \mathsf{U}_{AB}{}^{a}\Upsilon_a - \mathsf{V}_{AB}\tfrac{|\Upsilon|^2}{2} + \mathsf{Z}_{AB}{}^{ab}\Upsilon_a\Upsilon_b + \mathsf{W}_{AB}{}^{a}\tfrac{|\Upsilon|^2}{2}\Upsilon_a + \mathsf{X}_{AB}\tfrac{|\Upsilon|^4}{4}\\
\widehat{\mathsf{U}_{AB}{}^{a}} &= \mathsf{U}_{AB}{}^{a} + \mathsf{V}_{AB}\Upsilon^a - 2\mathsf{Z}_{AB}{}^{ab}\Upsilon_b - \mathsf{W}_{AB}{}^{b}(\Upsilon^a\Upsilon_b + \delta^a{}_b\tfrac{|\Upsilon|^2}{2}) - \mathsf{X}_{AB}|\Upsilon|^2\Upsilon^a\\
\widehat{\mathsf{V}_{AB}} &= \mathsf{V}_{AB} - \mathsf{W}_{AB}{}^{a}\Upsilon_a - \mathsf{X}_{AB}|\Upsilon|^2\\
\widehat{\mathsf{Z}_{AB}{}^{ab}} &= \mathsf{Z}_{AB}{}^{ab} + \mathsf{W}_{AB}{}^{(a}\Upsilon^{b)} + \mathsf{X}_{AB}\Upsilon^a\Upsilon^b\\
\widehat{\mathsf{W}_{AB}{}^{a}} &= \mathsf{W}_{AB}{}^{a} + 2\mathsf{X}_{AB}\Upsilon^a\\
\widehat{\mathsf{X}_{AB}} &= \mathsf{X}_{AB}
\end{aligned}
\tag{C.10}
$$

Proof. Straightforward calculations using (3.9) and (C.7). □

Remark C.0.11. It is instructive to write down the above rescaling rules in the "vector" notation , so we get the following transformation law

$$
\widehat{\begin{pmatrix} & y & \\ & u_a & \\ v & | & z_{ab} \\ & w_a & \\ & x & \end{pmatrix}} = \begin{pmatrix} & \widehat{y} & \\ & \widehat{u_a} & \\ \widehat{v} & | & \widehat{z_{ab}} \\ & \widehat{w_a} & \\ & \widehat{x} & \end{pmatrix}
\tag{C.11}
$$

where

$$
\begin{aligned}
\widehat{y} &= y\\
\widehat{u_a} &= u_a - \Upsilon_a y\\
\widehat{v} &= v - \tfrac{|\Upsilon|^2}{2}u + \Upsilon^a u_a\\
\widehat{z_{ab}} &= z_{ab} + \Upsilon_a\Upsilon_b y - 2\Upsilon_a u_b\\
\widehat{w_a} &= w_a + \tfrac{|\Upsilon|^2}{2}\Upsilon_a w - (\Upsilon^b\Upsilon_a + \delta^b{}_a\tfrac{|\Upsilon|^2}{2})u_b - \Upsilon_a v + \delta_a{}^c\Upsilon^b z_{cb}\\
\widehat{x} &= x + \tfrac{|\Upsilon|^4}{4}y - |\Upsilon|^2\Upsilon^a u_a - |\Upsilon|^2 v + \Upsilon^a\Upsilon^b z_{ab} + \Upsilon^a w_a
\end{aligned}
$$

In particular, these identities visualize the meaning of the composition series (C.1) for the symmetric 2-tractors, that is they show how the transformation propagates to the subspaces under conformal rescalings.

Proposition C.0.12. *The standard tractor connection acts on the symmetric 2-tractor projectors according to the following rules:*

$$\begin{aligned}
&\nabla_c \mathsf{Y}^{AB} = \mathsf{U}^{AB}{}_b \mathsf{P}_c{}^b \\
&\nabla_d \mathsf{U}^{AB}{}_c = 2\,\mathsf{Z}^{AB}{}_{ce}\mathsf{P}^e{}_d - 2\,\mathsf{Y}^{AB}\mathbf{g}_{dc} - \mathsf{V}^{AB}\mathsf{P}_{dc} \\
&\nabla_c \mathsf{V}^{AB} = \mathsf{W}^{AB}{}_d \mathsf{P}^d{}_c + \mathsf{U}^{AB}{}_c \\
&\nabla_c \mathsf{Z}^{AB}{}_{ab} = -\mathsf{U}^{AB}{}_{(a}\mathbf{g}_{b)c} - \mathsf{W}^{AB}{}_{(a}\mathsf{P}_{b)c} \\
&\nabla_d \mathsf{W}^{AB}{}_c = -\mathsf{V}^{AB}\mathbf{g}_{dc} - 2\mathsf{X}^{AB}\mathsf{P}_{dc} + 2\mathsf{Z}^{AB}{}_{cd} \\
&\nabla_c \mathsf{X}^{AB} = \mathsf{W}^{AB}{}_c
\end{aligned} \tag{C.12}$$

Proof. Straightforward calculations using (3.11) and (C.7). □

Using these identities and the Leibniz rule we can give an explicit expression for $\nabla_c V_{AB}$

Proposition C.0.13. *The tractor derivative of an arbitrary symmetric 2-tractor V_{AB} is given by the formula*

$$\begin{aligned}
\nabla_c V_{AB} = {} & \mathsf{Y}_{AB}\Big(\nabla_c y - 2\,u_c\Big) \\
&+ \mathsf{U}_{AB}{}^b\Big(\nabla_c u_b + \mathsf{P}_{bc} y + \mathbf{g}_{bc} v - z_{cb}\Big) \\
&+ \mathsf{V}_{AB}\Big(\nabla_c v - \mathsf{P}_c{}^b u_b - w_c\Big) \\
&+ \mathsf{Z}_{AB}{}^{ab}\Big(\nabla_c z_{ab} + 2\,\mathsf{P}_{ac} u_b + 2\mathbf{g}_{ac} w_b\Big) \\
&+ \mathsf{W}_{AB}{}^b\Big(\nabla_c w_b + \mathsf{P}_{bc} v - \mathsf{P}^a{}_c z_{ab} + \mathbf{g}_{bc} x\Big) \\
&+ \mathsf{X}_{AB}\Big(\nabla_c x - 2\mathsf{P}_c{}^b w_b\Big)
\end{aligned} \tag{C.13}$$

Next we record the formulas for the tractor Laplacian of V_{AB}.

Proposition C.0.14. *The action of the tractor Laplacian on the symmetric 2-tractor projectors is given by the following identities*

$$\begin{aligned}
&\Delta \mathsf{Y}_{AB} = -2\,\mathsf{Y}_{AB}\mathsf{J} + \mathsf{U}_{AB}{}^a\nabla_a \mathsf{J} - \mathsf{V}_{AB}|\mathsf{P}|^2 + 2\,\mathsf{Z}_{AB}{}^{ab}\mathsf{P}^2{}_{ab} \\
&\Delta \mathsf{U}_{AB}{}^a = -\mathsf{U}_{AB}{}^b\Big(4\,\mathsf{P}^a{}_b + \mathsf{J}\delta^a{}_b\Big) - \mathsf{V}_{AB}\nabla^a \mathsf{J} + 2\,\mathsf{Z}_{AB}{}^{ab}\nabla_b \mathsf{J} - \mathsf{W}_{AB}{}^b\Big(2\,\mathsf{P}^{2a}_b + |\mathsf{P}|^2\delta^a{}_b\Big) \\
&\Delta \mathsf{V}_{AB} = -2n\,\mathsf{Y}_{AB} - 2\,\mathsf{V}_{AB}\mathsf{J} + 4\,\mathsf{Z}_{AB}{}^{ab}\mathsf{P}_{ab} + \mathsf{W}_{AB}{}^a\nabla_a \mathsf{J} - 2\,\mathsf{X}_{AB}|\mathsf{P}|^2 \\
&\Delta \mathsf{Z}_{AB}{}^{ab} = 2\,\mathsf{Y}_{AB}\mathbf{g}^{ab} + 2\,\mathsf{V}_{AB}\mathsf{P}^{ab} - 4\,\mathsf{Z}_{AB}{}^{c(a}\mathsf{P}^{b)}{}_c - \mathsf{W}_{AB}{}^{(a}\nabla^{b)}\mathsf{J} + 2\,\mathsf{X}_{AB}\mathsf{P}^{2ba} \\
&\Delta \mathsf{W}_{AB}{}^a = -(n+2)\mathsf{U}_{AB}{}^a - \mathsf{W}_{AB}{}^b\Big(4\,\mathsf{P}^a{}_b + \mathsf{J}\delta^a{}_b\Big) - 2\,\mathsf{X}_{AB}\nabla^a \mathsf{J} \\
&\Delta \mathsf{X}_{AB} = -n\mathsf{V}_{AB} + 2\,\mathsf{Z}_{AB}{}^{ab}\mathbf{g}_{ab} - 2\,\mathsf{X}_{AB}\mathsf{J}
\end{aligned} \tag{C.14}$$

Proof. Using on the identities for the tractor connection acting on the symmetric 2-tractor projectors we compute:

$$
\begin{aligned}
\Delta \mathsf{Y}_{AB} &= \nabla^c \nabla_c \mathsf{Y}_{AB} \\
&= \nabla^c (\mathsf{U}_{AB}{}^b \mathsf{P}_{bc}) \\
&= (\nabla^c \mathsf{U}_{AB}{}^b) \mathsf{P}_{bc} + \mathsf{U}_{AB}{}^b \nabla^c \mathsf{P}_{bc} \\
&= (-2\,\mathsf{Y}_{AB} \mathbf{g}^{cb} - \mathsf{V}_{AB} \mathsf{P}^{cb} + 2\,\mathsf{Z}_{AB}{}^{dc} \mathsf{P}_d{}^b) \mathsf{P}_{bc} + \mathsf{U}_{AB}{}^b \nabla_b \mathsf{J} \\
&= -2\,\mathsf{Y}_{AB} \mathsf{J} + \mathsf{U}_{AB}{}^b \nabla_b \mathsf{J} - \mathsf{V}_{AB} |\mathsf{P}|^2 + 2\,\mathsf{Z}_{AB}{}^{ab} \mathsf{P}^2{}_{ab}
\end{aligned}
$$

$$
\begin{aligned}
\Delta \mathsf{U}_{AB}{}^a &= \mathbf{g}^{dc} \nabla_d \nabla_c \mathsf{U}_{AB}{}^a \\
&= \mathbf{g}^{dc} \nabla_d (-2\,\mathsf{Y}_{AB} \delta_c{}^a - \mathsf{V}_{AB} \mathsf{P}_c{}^a + 2\,\mathsf{Z}_{AB}{}^{ae} \mathsf{P}_{ec}) \\
&= \mathbf{g}^{dc} \Big(-2\,\nabla_d \mathsf{Y}_{AB} \delta_c{}^a - (\nabla_d \mathsf{V}_{AB}) \mathsf{P}_c{}^a - \mathsf{V}_{AB} \nabla_d \mathsf{P}_c{}^a \\
&\qquad + 2\,(\nabla_d \mathsf{Z}_{AB}{}^{ae}) \mathsf{P}_{ec} + 2\,\mathsf{Z}_{AB}{}^{ae} \nabla_d \mathsf{P}_{ec} \Big) \\
&= -2 \mathbf{g}^{da}\, \nabla_d \mathsf{Y}_{AB} - (\nabla_d \mathsf{V}_{AB}) \mathsf{P}^{da} - \mathsf{V}_{AB} \nabla^a \mathsf{J} \\
&\qquad + 2\,(\nabla_d \mathsf{Z}_{AB}{}^{ae}) \mathsf{P}_e{}^d + 2\,\mathsf{Z}_{AB}{}^{ae} \nabla_e \mathsf{J} \\
&= -2 \mathbf{g}^{da}\, (\mathsf{U}_{AB}{}^b \mathsf{P}_{bd}) - (\mathsf{U}_{AB}{}^b \mathbf{g}_{bd} + \mathsf{W}_{AB}{}^b \mathsf{P}_{bd}) \mathsf{P}^{da} - \mathsf{V}_{AB} \nabla^a \mathsf{J} \\
&\qquad + 2\,(-\mathsf{U}_{AB}{}^{(a} \delta^{e)}{}_d - \mathsf{W}_{AB}{}^{(a} \mathsf{P}^{e)}{}_d) \mathsf{P}_e{}^d + 2\,\mathsf{Z}_{AB}{}^{ae} \nabla_e \mathsf{J} \\
&= -2\,\mathsf{U}_{AB}{}^b \mathsf{P}_b{}^a - \mathsf{U}_{AB}{}^b \mathsf{P}_b{}^a - \mathsf{W}_{AB}{}^b \mathsf{P}^{2a}_b - \mathsf{V}_{AB} \nabla^a \mathsf{J} \\
&\qquad - \mathsf{U}_{AB}{}^a \delta^e{}_d \mathsf{P}_e{}^d - \mathsf{U}_{AB}{}^e \delta^a{}_d \mathsf{P}_e{}^d - \mathsf{W}_{AB}{}^a \mathsf{P}^e{}_d \mathsf{P}_e{}^d \\
&\qquad - \mathsf{W}_{AB}{}^e \mathsf{P}^a{}_d \mathsf{P}_e{}^d + 2\,\mathsf{Z}_{AB}{}^{ae} \nabla_e \mathsf{J} \\
&= -2\,\mathsf{U}_{AB}{}^b \mathsf{P}_b{}^a - \mathsf{U}_{AB}{}^b \mathsf{P}_b{}^a - \mathsf{W}_{AB}{}^b \mathsf{P}^{2a}_b - \mathsf{V}_{AB} \nabla^a \mathsf{J} \\
&\qquad - \mathsf{U}_{AB}{}^a \mathsf{J} - \mathsf{U}_{AB}{}^e \mathsf{P}_e{}^a - \mathsf{W}_{AB}{}^a |\mathsf{P}|^2 - \mathsf{W}_{AB}{}^e \mathsf{P}^{2a}_e + 2\,\mathsf{Z}_{AB}{}^{ae} \nabla_e \mathsf{J} \\
&= -\mathsf{U}_{AB}{}^b (4\,\mathsf{P}_b{}^a + \mathsf{J} \delta_b{}^a) - \mathsf{V}_{AB} \nabla^a \mathsf{J} + 2\,\mathsf{Z}_{AB}{}^{ae} \nabla_e \mathsf{J} - \mathsf{W}_{AB}{}^b (2\,\mathsf{P}^{2a}_b + \delta_b{}^a |\mathsf{P}|^2)
\end{aligned}
$$

$$
\begin{aligned}
\Delta \mathsf{V}_{AB} &= \nabla^c \nabla_c \mathsf{V}_{AB} \\
&= \nabla^c \Big(\mathsf{U}_{AB}{}^b \mathbf{g}_{bc} + \mathsf{W}_{AB}{}^b \mathsf{P}_{bc} \Big) \\
&= (\nabla^c \mathsf{U}_{AB}{}^b) \mathbf{g}_{bc} + (\nabla^c \mathsf{W}_{AB}{}^b) \mathsf{P}_{bc} + \mathsf{W}_{AB}{}^b \nabla^c \mathsf{P}_{bc} \\
&= (-2\,\mathsf{Y}_{AB} \mathbf{g}^{cb} - \mathsf{V}_{AB} \mathsf{P}^{cb} + 2\,\mathsf{Z}_{AB}{}^{dc} \mathsf{P}_d{}^b) \mathbf{g}_{bc} + \\
&\qquad (-\mathsf{V}_{AB} \mathbf{g}^{cb} + 2 \mathsf{Z}_{AB}{}^{dc} \delta_d{}^b - 2 \mathsf{X}_{AB} \mathsf{P}^{cb}) \mathsf{P}_{bc} + \mathsf{W}_{AB}{}^b \nabla_b \mathsf{J} \\
&= -2\,\mathsf{Y}_{AB} n - \mathsf{V}_{AB} \mathsf{J} + 2\,\mathsf{Z}_{AB}{}^{dc} \mathsf{P}_{dc} - \mathsf{V}_{AB} \mathsf{J} + \\
&\qquad 2 \mathsf{Z}_{AB}{}^{bc} \mathsf{P}_{bc} - 2 \mathsf{X}_{AB} |\mathsf{P}|^2 + \mathsf{W}_{AB}{}^b \nabla_b \mathsf{J}
\end{aligned}
$$

$$= -2n\,\mathsf{Y}_{AB} - 2\,\mathsf{V}_{AB}\mathsf{J} + 4\,\mathsf{Z}_{AB}{}^{bc}\mathsf{P}_{bc} + \mathsf{W}_{AB}{}^{b}\nabla_b\mathsf{J} - 2\mathsf{X}_{AB}|\mathsf{P}|^2$$

$$\begin{aligned}
\Delta\mathsf{Z}_{AB}{}^{ab} &= \nabla^c\nabla_c\mathsf{Z}_{AB}{}^{ab} \\
&= \nabla^c(-\mathsf{U}_{AB}{}^{(a}\delta^{b)}{}_c - \mathsf{W}_{AB}{}^{(a}\mathsf{P}^{b)}{}_c) \\
&= -\nabla^c\mathsf{U}_{AB}{}^{(a}\delta^{b)}{}_c - \nabla^c\mathsf{W}_{AB}{}^{(a}\mathsf{P}^{b)}{}_c - \mathsf{W}_{AB}{}^{(a}\nabla_c\mathsf{P}^{b)c} \\
&= \tfrac{1}{2}\Big(-(-2\,\mathsf{Y}_{AB}\mathbf{g}^{ca} - \mathsf{V}_{AB}\mathsf{P}^{ca} + 2\,\mathsf{Z}_{AB}{}^{dc}\mathsf{P}_d{}^a)\delta^b{}_c \\
&\quad - (-2\,\mathsf{Y}_{AB}\mathbf{g}^{cb} - \mathsf{V}_{AB}\mathsf{P}^{cb} + 2\,\mathsf{Z}_{AB}{}^{dc}\mathsf{P}_d{}^b)\delta^a{}_c \\
&\quad - (-\mathsf{V}_{AB}\mathbf{g}^{ca} + 2\mathsf{Z}_{AB}{}^{dc}\delta_d{}^a - 2\mathsf{X}_{AB}\mathsf{P}^{ca})\mathsf{P}^b{}_c \\
&\quad - (-\mathsf{V}_{AB}\mathbf{g}^{cb} + 2\mathsf{Z}_{AB}{}^{dc}\delta_d{}^b - 2\mathsf{X}_{AB}\mathsf{P}^{cb})\mathsf{P}^a{}_c\Big) - \mathsf{W}_{AB}{}^{(a}\nabla^{b)}\mathsf{J} \\
&= \tfrac{1}{2}\Big(2\,\mathsf{Y}_{AB}\mathbf{g}^{ab} + \mathsf{V}_{AB}\mathsf{P}^{ab} - 2\,\mathsf{Z}_{AB}{}^{db}\mathsf{P}_d{}^a + 2\,\mathsf{Y}_{AB}\mathbf{g}^{ab} + \mathsf{V}_{AB}\mathsf{P}^{ab} - 2\,\mathsf{Z}_{AB}{}^{da}\mathsf{P}_d{}^b \\
&\quad + \mathsf{V}_{AB}\mathsf{P}^{ab} - 2\mathsf{Z}_{AB}{}^{ac}\mathsf{P}^b{}_c + 2\mathsf{X}_{AB}\mathsf{P}^{2ba} + \mathsf{V}_{AB}\mathsf{P}^{ab} - 2\mathsf{Z}_{AB}{}^{bc}\mathsf{P}^a{}_c + 2\mathsf{X}_{AB}\mathsf{P}^{2ab}\Big) \\
&\quad - \mathsf{W}_{AB}{}^{(a}\nabla^{b)}\mathsf{J} \\
&= \tfrac{1}{2}\Big(4\,\mathsf{Y}_{AB}\mathbf{g}^{ab} + 2\mathsf{V}_{AB}\mathsf{P}^{ab} - 4\,\mathsf{Z}_{AB}{}^{db}\mathsf{P}_d{}^a - 4\,\mathsf{Z}_{AB}{}^{da}\mathsf{P}_d{}^b + 2\mathsf{V}_{AB}\mathsf{P}^{ab} + 4\mathsf{X}_{AB}\mathsf{P}^{2ba}\Big) \\
&\quad - \mathsf{W}_{AB}{}^{(a}\nabla^{b)}\mathsf{J} \\
&= 2\,\mathsf{Y}_{AB}\mathbf{g}^{ab} + \mathsf{V}_{AB}\mathsf{P}^{ab} - 2\,\mathsf{Z}_{AB}{}^{db}\mathsf{P}_d{}^a - 2\,\mathsf{Z}_{AB}{}^{da}\mathsf{P}_d{}^b + \mathsf{V}_{AB}\mathsf{P}^{ab} + 2\mathsf{X}_{AB}\mathsf{P}^{2ba} \\
&\quad - \mathsf{W}_{AB}{}^{(a}\nabla^{b)}\mathsf{J} \\
&= 2\,\mathsf{Y}_{AB}\mathbf{g}^{ab} + 2\,\mathsf{V}_{AB}\mathsf{P}^{ab} - 4\,\mathsf{Z}_{AB}{}^{d(a}\mathsf{P}_d{}^{b)} - \mathsf{W}_{AB}{}^{(a}\nabla^{b)}\mathsf{J} + 2\,\mathsf{X}_{AB}\mathsf{P}^{2ba}
\end{aligned}$$

$$\begin{aligned}
\Delta\mathsf{W}_{AB}{}^{a} &= \nabla^c\nabla_c\mathsf{W}_{AB}{}^a \\
&= \nabla^c(-\mathsf{V}_{AB}\delta_c{}^a + 2\mathsf{Z}_{AB}{}^{ab}\mathbf{g}_{bc} - 2\mathsf{X}_{AB}\mathsf{P}_c{}^a) \\
&= -(\nabla^c\mathsf{V}_{AB})\delta_c{}^a + 2(\nabla^c\mathsf{Z}_{AB}{}^{ab})\mathbf{g}_{bc} - 2(\nabla^c\mathsf{X}_{AB})\mathsf{P}_c{}^a - 2\mathsf{X}_{AB}\nabla^c\mathsf{P}_c{}^a \\
&= -(\mathsf{U}_{AB}{}^b\delta_b{}^c + \mathsf{W}_{AB}{}^b\mathsf{P}_b{}^c)\delta_c{}^a \\
&\qquad + 2(-\mathsf{U}_{AB}{}^{(a}\mathbf{g}^{b)c} - \mathsf{W}_{AB}{}^{(a}\mathsf{P}^{b)c})\mathbf{g}_{bc} - 2\mathsf{W}_{AB}{}^b\delta_b{}^c\mathsf{P}_c{}^a - 2\mathsf{X}_{AB}\nabla^a\mathsf{J} \\
&= -\mathsf{U}_{AB}{}^a - \mathsf{W}_{AB}{}^b\mathsf{P}_b{}^a - \mathsf{U}_{AB}{}^a n - \mathsf{U}_{AB}{}^a \\
&\qquad - \mathsf{W}_{AB}{}^a\mathsf{J} - \mathsf{W}_{AB}{}^b\mathsf{P}_b{}^a - 2\mathsf{W}_{AB}{}^b\mathsf{P}_b{}^a - 2\mathsf{X}_{AB}\nabla^a\mathsf{J} \\
&= -(n+2)\mathsf{U}_{AB}{}^a - \mathsf{W}_{AB}{}^b(4\mathsf{P}_b{}^a + \mathsf{J}\delta_b{}^a) - 2\mathsf{X}_{AB}\nabla^a\mathsf{J}
\end{aligned}$$

$$
\begin{aligned}
\Delta \mathsf{X}_{AB} &= \nabla^c \nabla_c \mathsf{X}_{AB} \\
&= \nabla^c \mathsf{W}_{AB}{}^b \mathbf{g}_{bc} \\
&= (-\mathsf{V}_{AB}\mathbf{g}^{cb} + 2\mathsf{Z}_{AB}{}^{cb} - 2\mathsf{X}_{AB}\mathsf{P}^{cb})\mathbf{g}_{bc} \\
&= -n\mathsf{V}_{AB} + 2\mathsf{Z}_{AB}{}^{cb}\mathbf{g}_{bc} - 2\mathsf{X}_{AB}\mathsf{J} \qquad \square
\end{aligned}
$$

It is now straightforward to derive the action of the box operator $\square f = \Delta f + \mathsf{J} w f$ on the symmetric 2-tractor projectors, so we just give the statement and omit the proof.

Proposition C.0.15. *The box operator $\square$ acts on the symmetric 2-tractor projectors as follows*

$$
\begin{aligned}
\square \mathsf{Y}_{AB} &= -4\,\mathsf{Y}_{AB}\mathsf{J} + \mathsf{U}_{AB}{}^a \nabla_a \mathsf{J} - \mathsf{V}_{AB}|\mathsf{P}|^2 + 2\,\mathsf{Z}_{AB}{}^{ab}\mathsf{P}^2{}_{ab} \\
\square \mathsf{U}_{AB}{}^a &= -\mathsf{U}_{AB}{}^b\Big(4\,\mathsf{P}^a{}_b + 3\mathsf{J}\delta^a{}_b\Big) - \mathsf{V}_{AB}\nabla^a \mathsf{J} + 2\,\mathsf{Z}_{AB}{}^{ab}\nabla_b \mathsf{J} \\
&\quad - \mathsf{W}_{AB}{}^b\Big(2\,\mathsf{P}^{2a}_b + |\mathsf{P}|^2\delta^a{}_b\Big) \\
\square \mathsf{V}_{AB} &= -2n\,\mathsf{Y}_{AB} - 2\,\mathsf{V}_{AB}\mathsf{J} + 4\,\mathsf{Z}_{AB}{}^{ab}\mathsf{P}_{ab} + \mathsf{W}_{AB}{}^a\nabla_a \mathsf{J} - 2\,\mathsf{X}_{AB}|\mathsf{P}|^2 \\
\square \mathsf{Z}_{AB}{}^{ab} &= 2\,\mathsf{Y}_{AB}\mathbf{g}^{ab} + 2\,\mathsf{V}_{AB}\mathsf{P}^{ab} - 4\,\mathsf{Z}_{AB}{}^{c(a}\mathsf{P}^{b)}{}_c - 2\mathsf{Z}_{AB}{}^{ab}\mathsf{J} \\
&\quad - \mathsf{W}_{AB}{}^{(a}\nabla^{b)}\mathsf{J} + 2\,\mathsf{X}_{AB}\mathsf{P}^{2ba} \\
\square \mathsf{W}_{AB}{}^a &= -(n+2)\mathsf{U}_{AB}{}^a - \mathsf{W}_{AB}{}^b\Big(4\,\mathsf{P}^a{}_b + \mathsf{J}\delta^a{}_b\Big) - 2\,\mathsf{X}_{AB}\nabla^a \mathsf{J} \\
\square \mathsf{X}_{AB} &= -n\mathsf{V}_{AB} + 2\,\mathsf{Z}_{AB}{}^{ab}\mathbf{g}_{ab}
\end{aligned}
\tag{C.15}
$$

Applying this to a symmetric 2-tractor we can obtain the full expression.

Proposition C.0.16. *The box operator of a symmetric 2-tractor of weight w is given explicitly by*

$$
\begin{aligned}
\square V_{AB} &= \mathsf{Y}_{AB}\Big(-(4\,\mathsf{J}y - \square y) - 4\,\nabla^b u_b - 2n\,v + 2\,\mathbf{g}^{ab}z_{ab}\Big) \\
&\quad + \mathsf{U}_{AB}{}^a\Big(y\nabla_a\mathsf{J} + 2\mathsf{P}_{ac}\nabla^c y - (4\,\mathsf{P}^b{}_a u_b + 3\mathsf{J}u_a - \square u_a) \\
&\qquad + 2\,\nabla_a v - 2\,\nabla^b z_{(ab)} - (n+2)w_a\Big) \\
&\quad + \mathsf{V}_{AB}\Big(-|\mathsf{P}|^2 y - \Big(u_b\nabla^b\mathsf{J} + 2\,\mathsf{P}_c{}^b\nabla^c u_b\Big) \\
&\qquad - \Big(2\,\mathsf{J}v - \square v\Big) + 2\,\mathsf{P}^{ab}z_{ab} - 2\nabla^a w_a - n\,x\Big) \\
&\quad + \mathsf{Z}_{AB}{}^{ab}\Big(2\,\mathsf{P}^2{}_{ab}y + \Big(2\,u_b\nabla_a\mathsf{J} + 4\,\mathsf{P}_{ac}\nabla^c u_b\Big) + 4\,\mathsf{P}_{ab}v \\
&\qquad - 4\,\mathsf{P}^c{}_a z_{(bc)} - 2\mathsf{J}z_{ab} + \square z_{ab} + 4\,\nabla_b w_a + 2\,\mathbf{g}_{ab}x\Big) \\
&\quad + \mathsf{W}_{AB}{}^a\Big(-\Big(2\,\mathsf{P}^{2b}_a + |\mathsf{P}|^2\delta^b{}_a\Big)u_b + \Big(v\nabla_a\mathsf{J} + 2\,\mathsf{P}_{ac}\nabla^c v\Big)
\end{aligned}
\tag{C.16}
$$

$$- z_{(ab)}\nabla^b \mathsf{J} - 2\,\mathsf{P}^b{}_c \nabla^c z_{(ab)} - \Big(4\,\mathsf{P}^c{}_a w_c + \mathsf{J} w_a - \Box w_a\Big) + 2\nabla_a x\Big)$$
$$+ \mathsf{X}_{AB}\Big(- 2\,|\mathsf{P}|^2 v + 2\,\mathsf{P}^{2ba} z_{ab} - \Big(4\,\mathsf{P}_c{}^a \nabla^c w_a + 2\,w_a \nabla^a \mathsf{J}\Big) + \Box x\Big)$$

This expression looks rather intimidating and hard to use in practice. Fortunately, many examples employ the reduced symmetric 2-tractors on which the expression is much simpler.

Definition C.0.17. The tractors of the form

$$V_{AB} = \mathsf{Z}_{AB}{}^{ab} z_{ab} + \mathsf{W}_{AB}{}^a w_a + \mathsf{X}_{AB} x$$

are termed the *reduced symmetric 2-tractors*.

The reduced symmetric 2-tractors form an invariant linear subbundle of the bundle of all symmetric 2-tractors.

Restricting to the reduced symmetric 2-tractors we obtain more manageable formulas, yet the we can produce interesting examples from this class. The shape 2-tractor $\mathbf{L}_{AB}$, the tractor $\overset{\mathrm{NN}}{\mathrm{W}}_{AB}$ and their powers, such as $\mathbf{L}^{(k)}_{AB} := \mathbf{L}_A{}^{C_1} \dots \mathbf{L}_{C_{k-1}B}$, and the symmetric parts of their partial contractions are of this form (see Chapter 4).

Proposition C.0.18. *On a reduced symmetric 2-tractor V_{AB} the action of the box operator is given by*

$$\begin{aligned}
\Box V_{AB} = {} & \mathsf{Y}_{AB}\Big(2\,\mathbf{g}^{ab} z_{ab}\Big) \\
& + \mathsf{U}_{AB}{}^a\Big(- 2\,\nabla^b z_{(ab)} - (n+2) w_a\Big) \\
& + \mathsf{V}_{AB}\Big(2\,\mathsf{P}^{ab} z_{ab} - 2\nabla^a w_a - n\,x\Big) \\
& + \mathsf{Z}_{AB}{}^{ab}\Big(- 4\,\mathsf{P}^c{}_a z_{(bc)} - 2\mathsf{J} z_{ab} + \Box z_{ab} + 4\,\nabla_b w_a + 2\,\mathbf{g}_{ab} x\Big) \\
& + \mathsf{W}_{AB}{}^a\Big(- z_{(ab)}\nabla^b \mathsf{J} - 2\,\mathsf{P}^b{}_c \nabla^c z_{(ab)} - \Big(4\,\mathsf{P}^c{}_a w_c + \mathsf{J} w_a - \Box w_a\Big) + 2\nabla_a x\Big) \\
& + \mathsf{X}_{AB}\Big(2\,\mathsf{P}^{2ba} z_{ab} - \Big(4\,\mathsf{P}_c{}^a \nabla^c w_a + 2\,w_a \nabla^a \mathsf{J}\Big) + \Box x\Big)
\end{aligned} \tag{C.17}$$

The Thomas-D operator on symmetric 2-tractors

Explicit calculations with the tractor-D operator can be very tedious as the following results demonstrate.

The reason for deriving these identities was to find some interesting examples out of the shape 2-tractor $\mathbf{L}_{AB}$ and the binormal part of the Weyl tractor $\overset{\mathrm{NN}}{\mathrm{W}}_{AB}$.

Proposition C.0.19. *The action of the tractor-D operator on a symmetric 2-tractor $V_{AB} \in \mathcal{T}_{AB}[w]$ is given explicitly by the following expression*

$$\begin{aligned}
\mathrm{D}_C V_{AB} = \mathsf{Y}_{AB}\Big[&(n+2\mathbf{w}-2)\mathbf{w}Y_C y + (n+2\mathbf{w}-2)Z_C{}^c\Big(\nabla_c y - 2\,u_c\Big) \\
&+ X_C\Big((4\,\mathsf{J}y - \Box y) + 4\,\nabla^b u_b + 2n\,v - 2\,\mathbf{g}^{ab}z_{ab}\Big)\Big] \\
+ \mathsf{U}_{AB}{}^a\Big[&(n+2\mathbf{w}-2)\mathbf{w}Y_C u_a \\
&+ (n+2\mathbf{w}-2)Z_C{}^c\Big(\nabla_c u_a + \mathsf{P}_{ac}y + \mathbf{g}_{ac}v - z_{ac}\Big) \\
&+ X_C\Big(-y\nabla_a\mathsf{J} - 2\mathsf{P}_{ac}\nabla^c y \\
&\quad + (4\,\mathsf{P}^b{}_a u_b + 3\mathsf{J}u_a - \Box u_a) - 2\,\nabla_a v + 2\,\nabla^b z_{ab} + (n+2)w_a\Big)\Big] \\
+ \mathsf{V}_{AB}\Big[&(n+2\mathbf{w}-2)\mathbf{w}Y_C v + (n+2\mathbf{w}-2)Z_C{}^c\Big(\nabla_c v - \mathsf{P}_c{}^b u_b - w_c\Big) \\
&+ X_C\Big(|\mathsf{P}|^2 y + \Big(u_b\nabla^b\mathsf{J} + 2\,\mathsf{P}_c{}^b\nabla^c u_b\Big) \\
&\quad + \Big(2\,\mathsf{J}v - \Box v\Big) - 2\,\mathsf{P}^{ab}z_{ab} + 2\nabla^a w_a + n\,x\Big)\Big] \\
+ \mathsf{Z}_{AB}{}^{ab}\Big[&(n+2\mathbf{w}-2)\mathbf{w}Y_C z_{ab} + (n+2\mathbf{w}-2)Z_C{}^c\Big(\nabla_c z_{ab} + 2\,\mathsf{P}_{ac}u_b + 2\mathbf{g}_{ac}w_b\Big) \\
&+ X_C\Big(-2\,\mathsf{P}^2{}_{ab}y - \Big(2\,u_b\nabla_a\mathsf{J} + 4\,\mathsf{P}_{ac}\nabla^c u_b\Big) \\
&\quad - 4\,\mathsf{P}_{ab}v + 4\,\mathsf{P}^c{}_a z_{bc} + 2\mathsf{J}z_{ab} - \Box z_{ab} - 4\,\nabla_b w_a - 2\,\mathbf{g}_{ab}x\Big)\Big] \\
+ \mathsf{W}_{AB}{}^a\Big[&(n+2\mathbf{w}-2)\mathbf{w}Y_C w_a \\
&+ (n+2\mathbf{w}-2)Z_C{}^c\Big(\nabla_c w_a + \mathsf{P}_{ac}v - \mathsf{P}^b{}_c z_{ab} + \mathbf{g}_{ac}x\Big) \\
&+ X_C\Big(\Big(2\,\mathsf{P}^{2b}_a + |\mathsf{P}|^2\delta^b{}_a\Big)u_b - \Big(v\nabla_a\mathsf{J} + 2\,\mathsf{P}_{ac}\nabla^c v\Big) \\
&\quad + z_{ab}\nabla^b\mathsf{J} + 2\,\mathsf{P}^b{}_c\nabla^c z_{ab} + \Big(4\,\mathsf{P}^c{}_a w_c + \mathsf{J}w_a - \Box w_a\Big) - 2\nabla_a x\Big)\Big] \\
+ \mathsf{X}_{AB}\Big[&(n+2\mathbf{w}-2)\mathbf{w}Y_C x + (n+2\mathbf{w}-2)Z_C{}^c\Big(\nabla_c x - 2\mathsf{P}_c{}^b w_b\Big) \\
&+ X_C\Big(2\,|\mathsf{P}|^2 v - 2\,\mathsf{P}^{2ba}z_{ab} + \Big(4\,\mathsf{P}_c{}^a\nabla^c w_a + 2\,w_a\nabla^a\mathsf{J}\Big) - \Box x\Big)\Big]
\end{aligned} \tag{C.18}$$

Proposition C.0.20. *The action of the tractor-D operator on a reduced symmetric 2-tractor V_{AB} is given by the expression*

$$
\begin{aligned}
\mathrm{D}_C V_{AB} = {} & \mathsf{Y}_{AB}\Big[X_C\Big(-2\,\mathbf{g}^{ab}z_{ab}\Big)\Big] \\
& + \mathsf{U}_{AB}{}^{a}\Big[(n+2\mathbf{w}-2)Z_C{}^{c}\Big(-z_{ac}\Big) + X_C\Big(2\,\nabla^b z_{ab} + (n+2)w_a\Big)\Big] \\
& + \mathsf{V}_{AB}\Big[(n+2\mathbf{w}-2)Z_C{}^{c}\Big(-w_c\Big) + X_C\Big(-2\,\mathsf{P}^{ab}z_{ab} + 2\nabla^a w_a + n\,x\Big)\Big] \\
& + \mathsf{Z}_{AB}{}^{ab}\Big[(n+2\mathbf{w}-2)\mathbf{w}Y_C z_{ab} + (n+2\mathbf{w}-2)Z_C{}^{c}\Big(\nabla_c z_{ab} + 2\mathbf{g}_{ac}w_b\Big) \\
& \qquad + X_C\Big(4\,\mathsf{P}^c{}_a z_{bc} + 2\mathsf{J}z_{ab} - \Box z_{ab} - 4\,\nabla_b w_a - 2\,\mathbf{g}_{ab}x\Big)\Big] \\
& + \mathsf{W}_{AB}{}^{a}\Big[(n+2\mathbf{w}-2)\mathbf{w}Y_C w_a + \\
& \qquad (n+2\mathbf{w}-2)Z_C{}^{c}\Big(\nabla_c w_a - \mathsf{P}^b{}_c z_{ab} + \mathbf{g}_{ac}x\Big) \\
& \qquad + X_C\Big(z_{ab}\nabla^b\mathsf{J} + 2\,\mathsf{P}^b{}_c\nabla^c z_{ab} + \Big(4\,\mathsf{P}^c{}_a w_c + \mathsf{J}w_a - \Box w_a\Big) - 2\nabla_a x\Big)\Big] \\
& + \mathsf{X}_{AB}\Big[(n+2\mathbf{w}-2)\mathbf{w}Y_C x + (n+2\mathbf{w}-2)Z_C{}^{c}\Big(\nabla_c x - 2\mathsf{P}_c{}^b w_b\Big) \\
& \qquad + X_C\Big(-2\,\mathsf{P}^{2ba}z_{ab} + \Big(4\,\mathsf{P}_c{}^a\nabla^c w_a + 2\,w_a\nabla^a\mathsf{J}\Big) - \Box x\Big)\Big]
\end{aligned}
\tag{C.19}
$$

In the process of making invariants we shall frequently need to eliminate an index with the tractor-D operator, so explicit identities for this will be useful.

Proposition C.0.21. *The tractor-contracted action of the tractor-D operator on a symmetric 2-tractor V_{AB} is given by*

$$
\begin{aligned}
\mathrm{D}^A V_{AB} = {} & Y_B\Big[(n+2\mathbf{w}+2)\mathsf{J}y - \Box y + (n+2\mathbf{w}+2)\nabla^a u_a \\
& + (n^2+3n\mathbf{w}+2\mathbf{w}^2-2\mathbf{w})v - (n+2\mathbf{w})\,\mathbf{g}^{ab}z_{ab}\Big] \\
& + Z_B{}^{b}\Big[-y\nabla_b\mathsf{J} - 2\mathsf{P}_{bc}\nabla^c y + (n+2\mathbf{w}+2)\mathsf{P}^a{}_b u_a + \\
& \qquad (n+2\mathbf{w}+1)\mathsf{J}u_b - \Box u_b \\
& \qquad - 2\,\nabla_b v + (n+2\mathbf{w})\,\nabla^a z_{ab} + (n+2\mathbf{w})(n+\mathbf{w})w_b\Big] \\
& + X_B\Big[|\mathsf{P}|^2 y + u_b\nabla^b\mathsf{J} + 2\,\mathsf{P}_c{}^b\nabla^c u_b + (n+2\mathbf{w})\mathsf{J}v - \Box v - (n+2\mathbf{w})\,\mathsf{P}^{ab}z_{ab} \\
& \qquad + (n+2\mathbf{w})\nabla^a w_a + (n+\mathbf{w}-1)(n+2\mathbf{w})\,x\Big]
\end{aligned}
\tag{C.20}
$$

And again, this expression simplifies on the reduced tractors.

Proposition C.0.22. *On a reduced symmetric 2-tractor V_{AB} the tractor-*

contracted action of the tractor-D operator is given by

$$\begin{aligned}\mathrm{D}^A V_{AB} = Y_B\Big(-(n+2\mathbf{w})\, z^a{}_a\Big)\\ + Z_B{}^b\Big((n+2\mathbf{w})\nabla^a z_{ab} + (n+2\mathbf{w})(n+\mathbf{w})\, w_b\Big)\\ + X_B\Big(-(n+2\mathbf{w})\,\mathsf{P}^{ab} z_{ab} + (n+2\mathbf{w})\nabla^a w_a + (n+\mathbf{w}-1)(n+2\mathbf{w})\, x\Big)\end{aligned} \quad (C.21)$$

To complete the picture let us also give the formulas for the action of the double-D operator on the symmetric 2-tractor projectors (C.7), which may be useful too:

$$\begin{aligned}\mathbb{D}_{AA'}\mathsf{Y}_{BC} &= -2\,\mathbb{W}_{AA'}\mathsf{Y}_{BC} - \mathbb{X}_{AA'}{}^a\mathsf{U}_{BC}{}^c\mathsf{P}_{ca}\\ \mathbb{D}_{AA'}\mathsf{U}_{BC}{}^c &= -2\,\mathbb{W}_{AA'}\mathsf{U}_{BC}{}^c + 2\,\mathbb{X}_{AA'}{}^a\mathsf{Y}_{BC}\delta_a{}^c + \mathbb{X}_{AA'}{}^a\mathsf{V}_{BC}\mathsf{P}_a{}^c - 2\,\mathbb{X}_{AA'}{}^a\mathsf{Z}_{BC}{}^{bc}\mathsf{P}_{ba}\\ \mathbb{D}_{AA'}\mathsf{V}_{BC} &= -\mathbb{X}_{AA'}{}^a\mathsf{U}_{BC}{}^c\mathbf{g}_{ac} - \mathbb{X}_{AA'}{}^a\mathsf{W}_{BC}{}^c\mathsf{P}_{ac}\\ \mathbb{D}_{AA'}\mathsf{Z}_{BC}{}^{bc} &= -2\,\mathbb{W}_{AA'}\mathsf{Z}_{BC}{}^{bc} + \mathbb{X}_{AA'}{}^a\mathsf{U}_{BC}{}^{(b}\delta_a{}^{c)} + \mathbb{X}_{AA'}{}^a\mathsf{W}_{BC}{}^{(b}\mathsf{P}_a{}^{c)}\\ \mathbb{D}_{AA'}\mathsf{W}_{BC}{}^c &= \mathbb{X}_{AA'}{}^a\mathsf{V}_{BC}\delta_a{}^c - 2\,\mathbb{X}_{AA'}{}^a\mathsf{Z}_{BC}{}^{bc}\mathbf{g}_{ab} + 2\,\mathbb{X}_{AA'}{}^a\mathsf{X}_{BC}\mathsf{P}_a{}^c\\ \mathbb{D}_{AA'}\mathsf{X}_{BC} &= 2\,\mathbb{W}_{AA'}\mathsf{X}_{BC} - \mathbb{X}_{AA'}{}^a\mathsf{W}_{BC}{}^c\mathbf{g}_{ac}\end{aligned} \quad (C.22)$$

Many of the formulas in this section have been obtained or verified using Cadabra, see Appendix B.

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