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Non-Markovian
Quantum Trajectories

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Abstract

The technique of quantum trajectories (stochastic Schrödinger equations or Monte Carlo wave functions) for open systems is generalized to the non-Markovian regime. I consider a microscopic model of an open system consisting of a boson field coupled linearly (with an excitation preserving coupling) to a localized system. The model allows for a field with an arbitrary dispersion relation and an arbitrary mode-dependent coupling to the system. The trajectories are formulated as continuous measurements of the output field from the system. For a general dispersive field these measurements must be distributed in space for this formulation to be possible. The result of this formulation is a non-Markovian equation for the system conditioned on the measurements. A method of numerically simulating this equation has been determined and implemented in some test cases. Numerical simulation is possible if one can introduce a finite memory time for the evolution of the reduced system.

As an illustration, the method is applied to the spectral detection of the emission from a driven two-level atom and also to an atom radiating into an electromagnetic field where the free space modes of the electromagnetic field are altered by the presence of a cavity. In both cases the non-Markovian behaviour arises from the uncertainty in the time of emission of a photon that is later detected (or reabsorbed), although, in the second case, the non-Markovian behaviour is intrinsic to the system environment coupling whereas, in the spectral detection case, it is a consequence of the choice of measurement process.

The generalization of the techniques of quantum trajectories to the non-Markovian regime promises to make a range of open system problems where the Born-Markov approximation is invalid tractable to numerical simulation.
Acknowledgments

I would like to dedicate this thesis to the memory of Prof. Dan Walls who died a few weeks before this thesis was submitted. I will remember Dan for his easygoing nature and his enthusiasm, which always managed to get the best out of those around him. Dan acted as my supervisor during the three years of my Ph.D. He provided me with support and guidance during this time and encouraged me to pursue my own interests. I'd like to thank Dan for the confidence he instilled in me to take part in the world of physics research.

Thanks go to my other supervisor, Dr. Matthew Collett. Without Matthew’s help I would never have completed a thesis on the topic of non-Markovian trajectories. Matthew originally encouraged me to consider the topic of this thesis and over the past three years he has provided me with endless assistance on the technical details of this topic. His intimate knowledge of the subtleties of open systems has proven invaluable.

The work on the Markov approximation for the output coupler was carried out in collaboration with Dr. Martin Naraschewski and I would like to thank him for his efforts.

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Finally, I would like to thank my partner xiao Xuan for her boundless patience and support during the past three years and my family for always being there when I need them.
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