STABILITY AND EFFICIENCY PROPERTIES
OF
IMPLICIT RUNGE-KUTTA METHODS

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NOTATION

Throughout this work vectors are denoted by bold face type and, unless otherwise stated, are understood to be column vectors. For each chapter there are three classes of numbering: definitions; examples; theorems, lemmas and corollaries. Each class is numbered in consecutive order. To avoid unsightly collection of brackets, double summations of the form $\sum_r \alpha_r (\sum_j B_j \delta_{j-r})$ will be written as $\sum_r \alpha_r \sum_j B_j \delta_{j-r}$.

TABLE OF CONTENTS

ABSTRACT

This thesis is divided into two sections. The first section examines certain stability properties of implicit Runge-Kutta methods. In particular, a new stability property is defined, which is a modification to non-autonomous problems of A-stability, and its relation to B-stability is considered.

A Runge-Kutta method is written as

$$\begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1s} \\ \vdots & \vdots & & \vdots \\ c_s & a_{s1} & \cdots & a_{ss} \\ \hline & b_1 & \cdots & b_s \end{array} = \begin{array}{c|c} c & A \\ \hline & b^T \end{array} ,$$

and classes of methods are constructed based on the property

$$\sum_{j=1}^{s} a_{ij} c_j^{k-1} = c_i^k / k \ , \quad i = 1,\dots,s \quad \text{and} \quad k = 1,\dots,s-1,$$

where $c_1,\dots,c_s$ are assumed to be distinct. Under this assumption a transformation is made, such that

$$A = V_s A_s V_s^{-1},$$

where V_s is the Vandermonde matrix whose (i,j) element is c_i^{j-1}, and A_s has a special structure. These methods are examined in the light of the various stability criteria. It is also shown that the growth of errors can be estimated by an extension of this new stability theory and a number of examples are given.

In the second section we consider the solution of stiff differential equations by implicit Runge-Kutta methods. In particular, we examine a procedure suggested by Butcher [6] which enables an efficient implementation

of Runge-Kutta methods. He has shown that the most efficient methods when using this implementation are those whose characteristic polynomial of the Runge-Kutta matrix has a single real s-fold zero. Based on this criterion a family of methods, called singly-implicit methods, is constructed, and results concerning their maximum attainable order and stability properties are given. Some consideration is also given to showing how local error estimates can be obtained, by the use of embedding techniques, for both singly-implicit methods and the more general family of implicit Runge-Kutta methods. Finally, an algorithm based on these singly-implicit methods is presented. It is tested on a number of stiff differential equations, and comparisons are made between this algorithm and others currently in use.