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Topics in the General Topology of Non-metric Manifolds

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ABSTRACT

The main core of this thesis is the general topology of non-metrisable manifolds. Although our emphasis was studying non-metrisable topological manifolds, we did not hesitate to consider topological properties which are useful in studying manifolds in a more general context. This thesis consists of five chapters:

The first chapter consists of two parts and in the first part we provide a perspective of what has been done in the field of non-metrisable topological manifolds and motivation for doing research in this field from a set-theoretical point of view. In this overview our emphasis is on distinguishing some classical examples according to separation properties such as Hausdorffness, the Tychonoff property, normality and perfect normality. The second half of this introductory chapter is devoted to the classification of one dimensional topological manifolds.

The second chapter is mainly about topological manifolds which do not satisfy the Hausdorff property with the emphasis on the studying the neighbourhoods of singular points. In the third chapter we show that the Prüfer surface is realcompact.

Our emphasis in Chapter 4 is on the $\omega$-bounded property which lies between compactness and countable compactness. This property is equivalent to compactness in metric spaces. We also generalise the classical theorem which says every continuous function from the set of countable ordinals with order topology to the real line is constant on a tail of the set of countable ordinals. We study several generalisations of this classical theorem.

Chapter 5 is closely related to the Nyikos bagpipe theorem. The proof of Nyikos Bagpipe theorem can be divided into two parts. The first part has a more set-theoretic
nature and the second part has a more geometric topology nature. The first part of the proof of the Bagpipe theorem is closely related to exhausting the manifold with an increasing $\omega_1$-sequence of open subspaces of the manifold. We study some topological spaces which have a $\lambda$-exhaustion for some ordinal number $\lambda \geq \omega_1$. In this chapter we also generalise the definition of a long pipe in higher dimensions.
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Chapter 1

Introduction to topological manifolds

1.1 Motivation

“If geometry is the human body and coordinates are clothing, then the evolution of geometry has the following comparison:

Synthetic geometry  Naked man
Coordinate geometry  Primitive man
Manifolds  Modern Man”

– Shing-shen Chern –

Topological manifolds are special types of topological spaces. Topological manifolds can be covered by a family of open subspaces of the space such that any member of the family looks like the Euclidean space and members of this family are put together in a nice way. Almost
all topological spaces which are interesting from a geometric point of view are topological
manifolds. The concept of manifold arises in several different aspects of mathematics. In
differential geometry it is as a unification and generalisation of the notions of curve and
surface, and in algebraic geometry as algebraic manifolds or varieties. The relation between
manifolds and theoretical physics can traced be back to Riemann’s “In microstructure of
matter and its binding forces” and his famous Paris prize essay in which Riemann modeled a
3-dimensional heat flow problem in an ex ante in-homogenous matter region and translated
it into a differential geometric structure of a 3-dimensional Riemannian metric.
Although some mathematicians step towards a generalization language, the credit should
 go to Riemann who explained the concept which from an extensional point of view would
form a manifold in his famous Habilitationsvortag in 1854. Even Riemann’s definition was
not well defined and had a rather engineering approach, but the necessary ideas are all
present. Hermann Weyl gave the first rigorous and axiomatic description of the Riemann
surface which is a complex manifold of dimension one [65].
In this introduction we briefly give the set-theoretical and topological backgrounds that we
need in the other chapters.

1.2 Set theory

In this thesis we use the Zermelo-Frankel axioms for set theory with the axiom of choice
(ZFC) but we do not assume the Continuum Hypothesis (CH) unless otherwise stated.

1.2.1 Set of countable ordinals and Fodor’s lemma

The set of all countable ordinals $\omega_1$ is essential for constructing some non-metrisable mani-
ifolds which are called Type $I$. In this section we study some properties of $\omega_1$. 
**Definition 1.2.1.** A subset $C$ of $\omega_1$ is closed if and only if $C$ is closed subset of $\omega_1$ with the respect to the order topology of $\omega_1$.

**Lemma 1.2.2.** A subset $C$ of $\omega_1$ is closed if and only if for every nonempty set $A \subseteq C$ if $\sup A < \omega_1$, then $\sup A \in C$.

**Proof.** ($\Rightarrow$) Let $C$ be a closed subset of $\omega_1$, $A \subseteq C$ and $\sup A < \omega_1$. Let $\alpha = \sup A$. If $\alpha \notin C$ then there is an open neighbourhood $U$ of $\alpha$ such that $U \cap C = \emptyset$. So there is a countable ordinal $\beta < \alpha$ such that $(\beta, \alpha] \cap C = \emptyset$. We know that for every $\beta < \alpha$, $(\beta, \alpha] \cap A \neq \emptyset$. This contradicts the assumption $\alpha = \sup A$. Therefore $\sup A \in C$. ($\Leftarrow$) If $C$ is not a closed subset of $\omega_1$ then there is a limit point $\alpha$ of $C$ such that $\alpha \notin C$. So for every $\beta < \alpha$, $(\beta, \alpha) \cap C \neq \emptyset$. Let $A = \{ \beta \in C \mid \beta < \alpha \}$. $\sup A = \alpha$ so $\alpha \in C$ which contradicts our assumption $\alpha \notin C$. Therefore $C$ is closed.

**Notation 1.2.3.** Cub stands for closed and unbounded.

**Theorem 1.2.4.** [4] Suppose that for every $n \in \omega$, $C_n$ is a closed, unbounded subset of $\omega_1$, then $C = \cap_{n \in \omega} C_n$ is closed and unbounded.

**Definition 1.2.5.** [4] A subset $S$ of $\omega_1$ is called stationary provided that for every cub subset $C \subseteq \omega_1$, $C \cap S \neq \emptyset$.

It is easy to see from Theorem 1.2.4 that every cub subset of $\omega_1$ is a stationary subset. It is also straightforward to see from 1.2.4 that the intersection of any stationary subset of $\omega_1$ with a cub subset of $\omega_1$ is stationary.

The following is usually called Fodor’s lemma or the pressing down lemma.

**Theorem 1.2.6.** [4] Suppose $S$ is stationary and $f : \omega_1 \to \omega_1$ is a regressive function i.e. $f(\alpha) < \alpha$ for all $\alpha \in S$. Then there is a stationary subset $S'$ of $S$ on which $f$ is constant.

**Corollary 1.2.7.** [38] If $f : \omega_1 \to \omega_1$ is a regressive function, then there is $\delta \in \omega_1$ such that $|f^{-1}(\delta)| = \omega_1$. 

3
1.2.2 Martin and diamond axioms

For any cardinal number $\kappa$ the following statement is called the $\kappa$-Martin axiom.

**Definition 1.2.8. MA($\kappa$).** If $X$ is a compact Hausdorff topological space which satisfies the countable chain condition, ccc, and $\{U_\alpha | \alpha < \kappa\}$ is a family of dense open subsets of $X$, then $\bigcap\{U_\alpha | \alpha < \kappa\} \neq \emptyset$.

We use $MA$ instead of $MA(\omega_1)$ and $MA + \neg CH$ for $\omega_1$-Martin axiom and the negation of the Continuum Hypothesis. The following statement is called the diamond axiom ($\diamond$).

**Definition 1.2.9.** There exists a sequence $\{a_\alpha | \alpha < \omega_1\}$ so that each $a_\alpha \subseteq \alpha$ and, for each set $a \subseteq \omega_1$, the set $\{\alpha | a \cap \alpha = a_\alpha\}$ is stationary.

The relation between Gödel’s axiom of constructibility, $V = L$, the diamond axiom ($\diamond$), the Continuum Hypothesis (CH) and Martin axiom (MA) is as follows [59] [41]:

$V = L \implies \diamond \implies CH \implies MA$.

1.3 Topological Manifold

**Definition 1.3.1.** A topological manifold $M$ is a topological space in which each point has a neighborhood homeomorphic to the closed $n$-ball, $\mathbb{B}^n$, where $n$ is a fixed integer.

A point $p \in M$ is called a boundary point if $p$ does not have any neighbourhood homeomorphic to the open ball, $\mathbb{B}^n$. We call the manifold $M$, a manifold with boundary if it has some boundary points otherwise we call it a manifold without boundary or simply a manifold. If we do not specify the manifold is with boundary or without boundary we always mean a manifold without boundary. The integer $n$ in the definition of manifold is called
the dimension of manifold and it is unique except where $M = \emptyset$. If $M = \emptyset$ the dimension is not well defined; in fact the dimension of the empty manifold can be any integer $n$. If we need to emphasise the dimension of the manifold $M$ of dimension $n$ we write $M$ is an $n$-manifold, $M^n$ or $\dim(M) = n$. By a curve we mean a manifold of dimension 1 and surface is a 2-dimensional manifold. *Metric manifolds*, i.e. locally Euclidean metric spaces, have been amongst the most studied objects in the whole of mathematics. Despite this very few people have tried to do research on non-metrisable manifolds, (i.e. locally Euclidean topological spaces which do not carry a metric structure), and understand their similarity to and differences from metric manifolds. Mathematicians who mentioned non-metrisable manifolds in their research mostly mentioned them just as pathological examples. There are two different approaches to the study of non-metric manifolds. The first is to try to find topological properties equivalent to metrisability and the second is to try to construct some non-metrisable manifolds, categorizing them, studying their properties and structures. The question of finding topological properties equivalent to metrisability of manifolds is well studied by David Gauld and more than one hundred topological properties equivalent to metrisability of manifolds are available now [20].

In this introduction we will concentrate on the second approach and we will look at what has been done and the methodology used in studying non-metric manifolds. There is a familiar pattern in general topology of looking at spaces which do not have special topological properties and non-metrisability is no exception. To study non-metric manifolds, as a first step we can look at some topological properties which all metric spaces share and try to construct a topological space which violates one or more of those properties. By looking at the figure 1.1 we get some idea about the topological properties which are essential for metric spaces. We should mention that from now on all of our spaces are supposed to be connected unless otherwise stated. Figure 1.1 gives some idea about our approach to the theory of non-metrisable manifolds. The direction of an arrow shows what property implies
Figure 1.1: Relations between topological properties in topological spaces
what property in a general topological space and if we need an extra condition it is shown by the arrow.

The first step towards non-metric manifolds is to construct some topological spaces which share all local properties of Euclidean space but do not have one of the above properties. In our list obviously strong separation properties are the most interesting ones but axioms weaker than Hausdorffness such as $KC^*$, $US^\dagger$ and $T_1$ are not independent of local properties.

**Theorem 1.3.2.** [75] Let $X$ be a first countable topological space, then the following are equivalent:

1. $X$ is Hausdorff
2. $X$ is a $KC$ space
3. $X$ is a $US$ space.

Any locally Euclidean space is $T_1$ and first countable so the previous theorem shows that there is no difference between $T_2$ manifolds and $KC$ manifolds. Hence we cannot relax the $KC$ or the $US$ conditions without losing the Hausdorff property.

---

*A* topological space is called a $KC$ space if its compact subsets are closed.

†A topological space is called a $US$ space if every convergent sequence of the space converges to at most one point.
1.3.1 Metrisable manifolds

In the last century most of the research in the field of topological manifolds was about metrisable manifolds. Interesting topological spaces such as complex algebraic varieties, Riemann surfaces, Riemannian manifolds and PL-manifolds are examples of metrisable manifolds. In this section we mention some topological conditions which are equivalent to the metrisability for a topological manifold $M$.

**Theorem 1.3.3.** [17] Let $M$ be a manifold. Then the following are equivalent:

1. $M$ is $\sigma$-compact
2. $M$ is Lindelöf
3. $M$ is second countable;

and all of them imply that $M$ is separable.

**Theorem 1.3.4.** [50] Let $M$ be a connected Hausdorff manifold. The following are equivalent:

1. $M$ is metrisable.
2. $M$ is paracompact.
3. $M$ is Lindelöf.

For a more complete list of topological properties equivalent to metrisability see [20].

Figure 1.3 shows the relation between some topological properties for a metrisable manifold $M$.

A major objective of the theory of non-metrisable manifolds is to understand if the red arrows in Figure 1.3 which are labeled by numbers 1 to 4 are reversible or not and where they
are not to construct counterexamples and conditions under which they will be reversible. For example \( \mathbb{R} \) is an example of a manifold which is paracompact but it is not compact. In some cases the question of reversibility of the arrows can be complicated. In the next section we show that the long line is an example of a normal topological manifold which is not perfectly normal.

### 1.4 Non-metrisable manifolds

In this section we review the history of non-metrisable manifolds and study some known basic examples of non-metrisable manifolds which are essential for our studies in other chapters. We also give some examples that show the red arrows in Figure 1.3 in general
are not reversible even for topological manifolds.

1.4.1 Perfectly normal non-metrisable manifolds

Metrisation theorems are considered to be amongst the most important theorems in general topology. In this type of theorem we provide one or more topological properties which imply metrisability of the topological space. A normal topological space $X$ is called perfectly normal if every open subset of $X$ can be written as the union of countably many closed subsets of $X$ [49]. Every metric space is perfectly normal [49] [13]. So it is natural to ask whether perfect normality is equivalent to metrisability or not. The answer to this question is no, even for compact topological spaces [73]. The answer is yes for compact Hausdorff manifolds since compact connected manifolds are Linelöf, so by Theorem 1.3.4 they are metrisable. In 1975, Mary Ellen Rudin provided a consistent solution to a problem posed in 1949 by Wilder [77]. The problem was whether every perfectly normal generalised manifold is metrisable or not, and Mary Ellen Rudin showed that under the axiom $\Diamond$ (a consequence of $V = L$, Gödel’s axiom of constructibility) there is a perfectly normal, hereditarily separable, countably compact, non metrisable surface (2-manifold). By sacrificing countable compactness, she built an example using the Continuum Hypothesis (CH) alone, which P.L.Zenor helped to simplify [63]. In 1978 Mary Ellen Rudin showed that the existence of a perfectly normal nonmetrisable manifold is independent of the usual axioms of set theory [62]. Specifically, she showed $\text{MA+¬CH}$ implies all perfectly normal manifolds are metrisable [62]. These discoveries brought a new interest in the field of non-metrisable topological manifolds.
1.4.2 The long line, powers of the long line and the long cylinder

We mentioned that in [62] Mary Ellen Rudin shows that existence of perfectly normal non-metric manifolds is dependent on the axioms of set theory. Therefore if we assume the Continuum Hypothesis, then the first red arrow from the top on the Figure 1.3 is not reversible. If we assume Martin’s Axiom and the negation of the Continuum Hypothesis the answer to the question posed by Raymond Wilder in [77] is positive so the first red arrow on the top of the Table 1.3 which is labelled 1 is reversible.

In this section we show that the closed long ray introduced by Georg Cantor is an example of a one dimensional manifold which is a normal topological space but it is not a perfectly normal topological space.

**Definition 1.4.1.** Let $L_1$ and $L_2$ be two totally ordered sets. The lexicographic order on $L_1 \times L_2$ is defined as follows:

$$(x, y) \leq (z, w) \text{ if either } x < z \text{ or } x = z \text{ and } y \leq w.$$  

**Notation 1.4.2.** We write $L_1 \times_{lex} L_2$ for the lexicographically ordered product of $L_1$ and $L_2$.

**Definition 1.4.3.** The closed long ray $\mathbb{L}_+$ is the ordered set $\omega_1 \times_{lex} [0, 1)$ with the order topology induced by the lexicographic order on $\mathbb{L}_+$. The space $\mathbb{L}_o = \mathbb{L}_+ \setminus \{0\}$ is called the open long ray.

**Definition 1.4.4.** Let $\mathbb{L}_- = \{-x | x \in \mathbb{L}_+\}$ such that for every $-x, -y$ in $\mathbb{L}_-$, $-x \leq -y$ if and only if and only if $y \leq x$. Similar to $\mathbb{L}_+$, $\mathbb{L}_-$ has the natural topology induced by the order of $\mathbb{L}_-$.

**Definition 1.4.5.** Assume that 0 and $-0$ are the first and the last elements of $\mathbb{L}_+$ and $\mathbb{L}_-$ respectively. If we identify the two points 0 and $-0$ of $\mathbb{L}_+$ and $\mathbb{L}_-$, then the topological space $\mathbb{L} = (\mathbb{L}_+ \cup \mathbb{L}_-)/(0 \sim -0)$ with the quotient topology induced by the equivalence relation $\sim$ is called the long line.
Notation 1.4.6. If \((\alpha, t) \in L_+\) we write \(\alpha + t\) for \((\alpha, t)\).

The following lemmas are necessary to prove that the long line and the long ray are not perfectly normal.

Theorem 1.4.7. If \(A\) and \(B\) are two closed and unbounded subsets of \(L_+\), then \(A \cap B\) is a closed and unbounded subset of \(L_+\). Moreover \(A \cap B \cap \omega_1\) is a cofinal subset of \(\omega_1\).

Proof. \(A \cap B\) is closed because the intersection of any two closed subspaces of any topological space is closed. We need to show that \(A \cap B\) is unbounded. Let \(\lambda \in \omega_1\) be an arbitrary element of \(\omega_1\). We show that there is an element \(\delta \in A \cap B\) with \(\delta > \lambda\). \(A\) is an unbounded subset of \(L_+\) so there is an element \(a_1 = \alpha_1 + t_1 \in A\) such that \(\alpha_1 > \lambda\). Similarly there is an element \(b_1 = \beta_1 + s_1 \in B\) such that \(\beta_1 > \alpha_1\). If \(a_n = \alpha_n + t_n\) and \(b_n = \beta_n + s_n\) are chosen let \(a_{n+1} = \alpha_{n+1} + t_{n+1}\) be such that \(\alpha_{n+1} > b_n\). This choice of \(a_n\) is possible because \(A\) is an unbounded subset of \(L_+\). Similarly we can choose \(b_{n+1} = \beta_{n+1} + s_{n+1}\) be such that \(\beta_{n+1} > a_{n+1}\). By using a leapfrog argument it is easy to see that two sequences \(\{\alpha_n\}\) and \(\{\beta_n\}\) converge to the same point \(\delta \in \omega_1\). For every \(n \in \omega\), \(a_n < b_n < \alpha_{n+1} < \beta_{n+1} < \delta\). So \(\delta\) is an upper bound for these two sequences \(\{\alpha_n\}\) and \(\{b_n\}\). If \(\theta < \delta\) is a countable ordinal then there are some \(n \in \omega\) such that \(\alpha_n\) and \(\beta_n \in (\theta, \delta)\). So \(a_{n+1}, b_{n+1} \in (\theta, \delta) \subset L_+\). This shows that \(\delta \in \omega_1\) is the limit point of the two sequences \(\{\alpha_n\}\) and \(\{b_n\}\). Therefore \(\delta > \lambda\) is an element of \(A \cap B \cap \omega_1\).

Corollary 1.4.8. If \(\{A_n\}_{n \in \omega}\) is a sequence of cub subsets of \(L_+\), then \(\cap_{n \in \omega} A_n\) contains a cub subset of \(\omega_1\).

Proof. For every \(n \in \omega_1\), by Theorem 1.4.7 \(A_n \cap \omega_1\) is a cub subset of \(L_+\). Let \(B_n = \omega_1 \cap A_n\) for every \(n \in \omega\). Then for every \(n \in \omega, B_n\), is a closed and unbounded subset of \(\omega_1\). So using Theorem 1.2.4 we see that \(B = \cap_{n \in \omega} B_n\) is a cub subset of \(\omega_1\) and \(B \subseteq \cap_{n \in \omega} A_n\).
Corollary 1.4.9. If \( \{A_n\}_{n \in \omega} \) is a sequence of cub subsets of \( [\alpha, \omega_1) \subseteq \mathbb{L}_+ \), then \( \bigcap_{n \in \omega} A_n \) contains a cub subset of \( \omega_1 \).

Proof. The set \( [\alpha, \omega_1) \) is a closed subspace of \( \mathbb{L}_+ \) and \( A = \bigcap_{n \in \omega} A_n \) is a closed subspace of \( \mathbb{L}_+ \). By using Corollary 1.4.8 we see that \( A \) is a cub subspace of \( \mathbb{L}_+ \) so \( A \) is an unbounded subspace of \( [\alpha, \omega_1) \). On the other hand \( [\alpha, \omega_1) \) is a closed subspace of \( \mathbb{L}_+ \). So \( A \cap [\alpha, \omega_1) \) is a closed subspace of \( [\alpha, \omega_1) \). Therefore \( A \) is a cub subspace of \( [\alpha, \omega_1) \). \( \square \)

Definition 1.4.10. Let \( L \) be a totally ordered topological space with the order topology and \( X \) be a topological space. We say that the continuous function \( f : L \rightarrow X \) is eventually in \( S \subseteq X \) if and only if there is \( z \in L \) such that \( f(y) \in S \) for all \( y > z \). If \( S \) is a singleton we say \( f \) is eventually constant.

Lemma 1.4.11. Any continuous function \( f : \mathbb{L}_+ \rightarrow [a, b] \subseteq \mathbb{R} \) is constant on \( [\alpha, \omega_1) \subseteq \mathbb{L}_+ \) for some \( \alpha \in \omega_1 \).

Proof. To show that \( f \) is eventually constant we need to show that there is a unique \( r \in [a, b] \) such that \( f^{-1}(r) \) is a cofinal subset of \( \mathbb{L}_+ \).

Uniqueness. If there are two real numbers \( r_1 \) and \( r_2 \) such that \( r_1 \neq r_2 \) and \( f^{-1}(r_1) \) and \( f^{-1}(r_2) \) are cofinal subsets of the closed long ray, then choose two increasing interlacing sequences \( \{x_n\} \) and \( \{y_n\} \) of \( f^{-1}(r_1) \) and \( f^{-1}(r_2) \) respectively.

These two sequences converge to the same point \( z \). Hence \( f(x_n) \rightarrow f(z) \) and \( f(y_n) \rightarrow f(z) \).

So we have \( r_1 = r_2 = f(z) \) which contradicts the assumption \( r_1 \neq r_2 \).

Existence. \( f^{-1}([a, b]) = \mathbb{L}_+ \). Set \( J_0 = [a, b] \). Consider \( [a, \frac{a+b}{2}] = I'_1 \) and \( [\frac{a+b}{2}, b] = I''_1 \). Then at least one of \( f^{-1}(I'_1) \) or \( f^{-1}(I''_1) \) is cofinal in \( \mathbb{L}_+ \), otherwise \( f^{-1}[a, b]) \neq \mathbb{L}_+ \). Choose \( J_1 \) to be one of \( I'_1 \) and \( I''_1 \) so that \( f^{-1}(J_1) \) is a cofinal subset of \( \mathbb{L}_+ \).

Let \( J_n \) be chosen for the natural number \( n \) so that \( f^{-1}(J_n) \) is a cofinal subset of \( \mathbb{L}_+ \).

We write \( I'_{n+1} = [\min J_n, \max J_n, \max J_n] \) and \( I''_{n+1} = [\min J_n, \frac{\min J_n + \max J_n}{2}] \). Then at least one of \( f^{-1}(I'_{n+1}) \) or \( f^{-1}(I''_{n+1}) \) is a cofinal subset of \( \mathbb{L}_+ \) otherwise \( f^{-1}(J_n) \) is not a cofinal
subset of \( \mathbb{L}_+ \). Choose \( J_{n+1} \) to be one of \( I'_{n+1} \) or \( I''_{n+1} \) so that \( f^{-1}(J_{n+1}) \) is a cofinal subset of \( \mathbb{L}_+ \). The sequence \( \{J_n\}_{n\in\omega} \) is a nested sequence of closed subintervals of \([a,b]\). So \( J_1 \supset J_2 \supset \cdots \supset J_n \supset \cdots \) is a sequence of closed subintervals of \([a,b]\) such that \( f^{-1}(J_k) \) is cofinal for all \( k \in \omega \), then by using Theorem 1.2.4 \( f^{-1}(\bigcap_{k\in\omega} J_k) = \bigcap_{k\in\omega}(f^{-1}(J_k)) \) is a cofinal subset of \( \mathbb{L}_+ \). On the other hand \( \bigcap_{k\in\omega} J_k \) is a singleton \( r \in [a,b] \), so \( f^{-1}(r) \) is a cofinal subset of \( \mathbb{L}_+ \).

Eventually constant: Let \( \{x_n\}_{n\in\omega} \) and \( \{y_n\}_{n\in\omega} \) be two sequences approaching \( r \) for \( a < r < b \) from both sides. Then, \( f^{-1}([x_i, x_{i+1}]) \) is bounded for all \( i \in \omega \). \( f^{-1} \bigcup_{i\in\omega}[x_i, x_{i+1}] \) is bounded. Similarly \( f^{-1} \bigcup_{i\in\omega}[y_i, y_{i+1}] \) is bounded. Hence \( f^{-1}([a,b] - \{r\}) \) is bounded and we are done.

**Corollary 1.4.12.** Any continuous function \( f : [\beta, \omega_1) \subset \mathbb{L}_+ \to [a,b] \subseteq \mathbb{R} \) is constant on \([\alpha, \omega_1) \subseteq \mathbb{L}_+ \) for some \( \alpha \in \omega_1 \).

**Proof.** We extend the function \( f \) to the function \( g : \mathbb{L}_+ \to [a,b] \) as follows:

\[
g(x) = \begin{cases} 
  f(x) & : x \in [\beta, \omega_1) \\
  f(\beta) & : x \in [0, \beta]
\end{cases}
\]

The Lemma 1.4.11 shows that \( g \) is eventually constant. It is clear that for any \( \gamma > \beta \) we have \( f(x) = g(x) \) for all \( x > \gamma \). So \( f \) is eventually constant.

**Lemma 1.4.13.** Any continuous function \( f : \mathbb{L}_+ \to \mathbb{R} \) is eventually bounded.

**Proof.** Assume that for every \( x \in \mathbb{L}_+ \) and every \( n \in \mathbb{N} \) there is \( y \in \mathbb{L}_+ \) such that \( |f(y)| > n \). Let \( x_0 \) to be an arbitrary element of \( \mathbb{L}_+ \). If the image of \( f : \mathbb{L}_+ \to \mathbb{R} \) is eventually a subset of the interval \([-1,1] \), then we are done. If the image of the continuous function \( f : \mathbb{L}_+ \to \mathbb{R} \) is not eventually a subset of the interval \([-1,1] \), then there is \( x_1 \in \mathbb{L}_+ \) such that \( |f(x_1)| > 1 \) and \( x_1 > x_0 \). If \( x_n \) is chosen then we choose \( x_{n+1} > x_n \) such
that $|f(x_n)| > n + 1$. The sequence $\{x_n\}_{n \in \omega}$ is an increasing sequence. Then $\{x_n\}_{n \in \omega}$ converges to a point $x \in \mathbb{L}_+$. From the construction of the sequence $\{x_n\}_{n \in \omega}$ we see that $f$ is unbounded. Then the sequence $\{f(x_n)\}$ diverges. On the other hand $f$ is a continuous function so $f(x_n) \to f(x)$. This is a contradiction because the sequence $\{f(x_n)\}$ cannot be both convergent and divergent. Therefore there is $x_0 \in \mathbb{L}_+$ and $n_0 \in \mathbb{N}$ such that $f([x_0, \omega_1)) \subseteq [-n_0, n_0]$.

\[\]

Proposition 1.4.14. Any continuous function $f : \mathbb{L}_+ \to \mathbb{R}$ is constant on $[\alpha, \omega_1) \subseteq \mathbb{L}_+$ for some $\alpha \in \omega_1$.

Proof. By using the Lemma 1.4.13 it is easy to see that $f$ is bounded on $[\alpha, \omega_1)$ for some $\alpha \in \omega_1$. Then Proposition 1.4.12 shows that $f$ is eventually constant.

Before proving that any initial segment $[0, x]$ of the long line is homeomorphic to the unit interval $[0, 1]$ we need some background theorems and definitions from general topology.

Definition 1.4.15. [5] A partially ordered set $P$ is called conditionally complete if and only if every bounded subset of $P$ has a greatest lower bound and a least upper bound.

Definition 1.4.16. [5] A lattice $L$ is called complete if and only if every subset of $L$ has a greatest lower bound and a least upper bound.

Definition 1.4.17. A nonempty subset $S$ of a partially ordered set $(P, \leq)$ is called a chain if and only if every two elements $x$ and $y$ in $S$ are comparable that is, $x \leq y$ or $y \leq x$.

Theorem 1.4.18. [5] A chain $C$ is complete if and only if it is compact in its order topology.

Definition 1.4.19. A chain $L$ is dense in itself if and only if for every two distinct points $x$ and $y$ in $L$, there is another point $z \in L$ such that $x < z < y$. 

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Theorem 1.4.20. [5] A chain is connected with respect to the order topology if and only if it is conditionally complete and dense in itself.

Definition 1.4.21. [33] A point \( p \) of a connected topological space \( X \) is called a cut point if and only if \( X \setminus \{p\} \) is not connected. If \( p \) is not a cut point we call it non cut point.

Example 1.4.22. Let \( X = [0, 1] \). So \( X \setminus \{\frac{1}{2}\} = [0, 1] \setminus \{\frac{1}{2}\} = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1] \). \( X \setminus \{\frac{1}{2}\} \) is not connected so \( \frac{1}{2} \) is a cut point. The initial point 0 and the terminal point 1 are not cut points because \([0, 1) \) and \((0, 1] \) are connected. Similarly it is easy to see that if \( p \in (0, 1) \), then \( X \setminus \{p\} \) is not connected so any element of the open unit interval is a cut point.

Theorem 1.4.23. [49] (Urysohn Metrisation Theorem) Let \( X \) be a Hausdorff second countable topological space. If \( X \) is regular, then it is metrisable.

Definition 1.4.24. [33] A topological space \( X \) is called a continuum if and only if \( X \) is compact and connected.

Theorem 1.4.25. [33] If \( X \) is a metric continuum with just two non-cut points, then \( X \) is homeomorphic to the closed unit interval.

Definition 1.4.26. The subspace \( \mathbb{L}_\mathbb{Q} = \{\alpha + q \mid \alpha \in \omega_1 \text{ and } q \in [0, 1) \cap \mathbb{Q}\} \) of the closed long ray is called the closed rational long ray.

We use the following lemma later to give a characterisation of the closed long ray.

Lemma 1.4.27. For every \( x \in \mathbb{L}_+ \) the initial segment \([0, x] \subseteq \mathbb{L}_+ \) is a connected, compact metric space. Moreover \([0, x] \) has exactly two non cut points 0 and \( x \).

Proof. For every \( x \in \mathbb{L}_+ \) the interval \([0, x] \) is a chain with the initial point 0 and the terminal point \( x \). It is easy to see that \( \mathbb{L}_+ \) is a chain and dense in itself.

To see that \( \mathbb{L}_+ \) is regular let \( C \) be a closed subset of the closed long ray and \( p \notin C \). So \( p \in \mathbb{L}_+ \setminus C \) and clearly \( \mathbb{L}_+ \setminus C \) is open so there is an open neighbourhood \( U \) of \( p \)
which does not intersect \( C \). \( U \) contains an interval \((x, y)\) for some \( x \) and \( y \) in \( U \) such that \( x < p < y \). \( \mathbb{L}_+ \) is dense in itself so there are \( x' \in (x, p) \) and \( y' \in (p, y) \). Let \( W = (x', y') \) and \( V = [0, x) \cup (y, \omega_1) \subset \mathbb{L}_+ \). Clearly \( W \) and \( V \) are disjoint open subsets of \( \mathbb{L}_+ \). Moreover \( p \in W \) and \( C \subset V \). Therefore \( \mathbb{L}_+ \) is regular.

To see that \([0, x]\) is second countable we introduce a countable basis for \([0, x]\). In case one assume that \( x \in \omega_1 \) is a limit ordinal. It is easy to see that \( S = \{[0, a) \mid a \in \mathbb{L}_Q \text{ and } a < x\} \cup \{(a, b) \mid a, b \in \mathbb{L}_Q \text{ and } a, b < x\} \cup \{(b, x] \mid b \in \mathbb{L}_Q\} \) provides a countable basis for \([0, x]\).

Then \([0, x]\) is a metrisable totally ordered set with the maximum \( x \) and the minimum \( 0 \).

Theorem 1.4.18 implies that \([0, x]\) is compact. The Urysohn Metrisation Theorem implies that \([0, x]\) is a metric space. Theorem 1.4.20 implies that \([0, x]\) is connected.

Any bounded subset \( S \) of \((0, x)\) is clearly a bounded subset of \([0, x]\). Let \( S \) be bounded above by \( q \in (0, x) \) and bounded below by \( p \in (0, x) \). Therefore \( 0 \) and \( x \) can not be the greatest lower bound and least upper bound of \( S \) in \([0, x]\) respectively. The greatest upper bound of \( S \) as a subset of \([0, x]\) is the same as the greatest lower bound of \( S \) as a subspace of \((0, x)\).

Similarly the least upper bound of \( S \) in \((0, x)\) is the same as the least upper bound of \( S \) in \([0, x]\). So \( S \subset (0, x) \) is conditionally complete. On the other hand for every two elements \( p \) and \( q \) in \([0, x]\) there is an element \( y \in [0, x] \) such that \( p < y < q \).

This shows that \((0, x)\) is dense in itself. So Theorem 1.4.20 implies that \((0, x)\) is connected.

So \( 0 \) and \( x \) are the only non-cut points of \([0, x]\). Clearly for every \( p \in [0, x] \), the two disjoint open subsets \([0, p) \cup (p, x]\) separate \([0, x] \setminus \{p\} \). So every point of \([0, x]\) except \( 0 \) and \( x \) is a cut point. This completes the proof.

\textbf{Proposition 1.4.28.} For every \( x \in \mathbb{L}_+ \), the closed interval \([0, x]\) is homeomorphic to the unit interval.

The proof of 1.4.28 follows from Lemma 1.4.27 and Theorem 1.4.25.

The following lemma is immediate consequence of Proposition 1.4.28 and Lemma 1.4.13
Lemma 1.4.29. Every continuous function \( f : \mathbb{L}_+ \to \mathbb{R} \) is bounded.

The following theorem gives more general result than Proposition 1.4.28 and has a more constructive nature at a cost of a more effort.

Theorem 1.4.30. Let \( p \in \mathbb{L}_+ \). For every \( x \in \mathbb{L}_+ \setminus \{0\} \), such that \( x > p \), \((0,x)\) is an open neighbourhood of \( p \) homeomorphic to \( \mathbb{R} \).

Proof. By using induction on \( \omega_1 \) we show that for every \( x \in \mathbb{L}_+ \) such that \( x > 0 \), \([0,x)\) is homeomorphic to the half closed unit interval \([0,1)\).

First we show that for every limit ordinal \( \alpha \in \mathbb{L}_+ \), \([0,\alpha)\) is homeomorphic to \([0,1)\subset \mathbb{R} \). Let \( \alpha \in \omega_1 \) be a limit ordinal, then there is an increasing sequence \( \{\alpha_n\}_{n \in \mathbb{N}} \) which converges to \( \alpha \). Without loss of generality we assume \( \alpha_0 = 0 \) and \( \alpha_1 = 1 \), otherwise we can replace the first few terms of the sequence and it does not affect convergence of \( \{\alpha_n\} \). Clearly \([\alpha_0,\alpha_1)\) is order isomorphic to \([0,1)\). Assume for each \( \alpha_n \) we have an order isomorphism from \([\alpha_n,\alpha_{n+1})\) to \([0,1)\). Clearly \([0,1)\) can be written as \([0,1) = \bigcup_{n \in \mathbb{N}} \left(1 - \frac{1}{n}, 1 - \frac{1}{n+1}\right) \) and for every natural number \( n \), \([\frac{n-1}{n}, \frac{n}{n+1})\) is order isomorphic to \([0,1)\) so we have an order isomorphism \( f_n : [\alpha_n, \alpha_{n+1}) \to \left[\frac{n-1}{n}, \frac{n}{n+1}\right) \) for all \( n \in \mathbb{N} \).

Now define \( f : [0,\alpha) \to [0,1) \) by \( f(x) = f_n(x) \) for \( \alpha_n \leq x \leq \alpha_{n+1} \). We show that \( f \) is an order isomorphism.

Let \( 0 < x < y < \alpha \). If \( x \) and \( y \) both belong to \([\alpha_n,\alpha_{n+1})\) for some \( n \in \mathbb{N} \), then \( f_n \) is an order isomorphism. Therefore \( f_n(x) < f_n(y) \) so \( f(x) = f_n(x) < f_n(y) = f(y) \).

If \( \alpha_n \leq x < \alpha_{n+1} \) and \( \alpha_{n+k} \leq y < \alpha_{n+k} \) for some \( k \), then \( f(x) = f_n(x) < \frac{n}{n+1} < \frac{n+k}{n+k+1} \leq f_{n+1}(y) = f(y) \). So \( f : [0,\alpha) \to [0,1) \) is an isomorphism hence a homeomorphism from \([0,\alpha)\) onto \([0,1)\) where both ordered sets are equipped with the order topology.

Clearly \( f \) can be extended to a homeomorphism \( \overline{f} : [0,\alpha] \to [0,1] \) where \( \overline{f}(x) = f(x) \) for all \( x \in [0,\alpha) \) and \( \overline{f}(\alpha) = 1 \).

Assume for \( \alpha \in \omega_1 \) we have an order isomorphism \( \overline{f} : [0,\alpha) \to [0,1) \). We write \([0,\alpha+1) = \)
$[0, \alpha] \cup [\alpha, \alpha + 1)$ and $[0, 1) = [0, \frac{1}{2}] \cup \left(\frac{1}{2}, 1\right)$. $g : [0, 1] \to [0, \frac{1}{2}]$ defined by $g(x) = \frac{1}{2}x$ and $g' : [0, 1) \to \left(\frac{1}{2}, 1\right)$ defined by $g'(x) = \frac{x}{2} + \frac{1}{2}$ are order isomorphisms. Define $h : [0, \alpha + 1) \to [0, 1)$ by

\[
    h(x) = \begin{cases} 
    g \circ f(x) & (x \in [0, \alpha]) \\
    g'(t) & (x = \alpha + t \in [\alpha, \alpha + 1])
    \end{cases}
\]

$h$ is an order isomorphism so $h$ is a homeomorphism where both spaces carry the order topology. Thus for each $\alpha \in \omega_1$, $[0, \alpha)$ is homeomorphic to $[0, 1)$.

Similarly if $x \in \mathbb{L}_+$ we can write $x = \alpha + t$, then $[0, \alpha)$ is order isomorphic to $[0, \frac{1}{2})$ and if $t > 0$, then $[\alpha, \alpha + t)$ is also order isomorphic to $[\frac{1}{2}, 1)$ by $h : [0, t) \to [\frac{1}{2}, 1)$ where $h(x) = \frac{1}{2} + \frac{1}{2t}x$.

Finally we have proved that for each non zero $x \in \mathbb{L}_+$, $[0, x)$ is homeomorphic to $[0, 1)$ since $h$ is an order isomorphism. So $h$ is a homeomorphism. If $h : [0, 1) \to (0, x)$ is a homeomorphism which preserves the order, then $h(0) = 0$ so $h|_{(0, x)} : (0, x) \to (0, 1)$ is a homeomorphism.

\[\square\]

**Lemma 1.4.31.** Let $e : \omega_1 \to \mathbb{L}_+$ be an embedding. Then $e(\omega_1) \cap \omega_1$ is a closed unbounded subset of $\omega_1$.

**Proof.** Let $E = e(\omega_1) \subseteq \mathbb{L}_+$. First we show that $E$ is unbounded. If $E$ is bounded, then there is $\alpha_0 \in \omega_1$ such that $E \subseteq [0, \alpha_0]$. By Lemma 1.4.11 and Proposition 1.4.28 $e : \omega_1 \to [0, \alpha_0]$ is eventually constant which contradicts our assumption that $e$ is an embedding. To see that $E$ is a closed set we show that $\overline{E} \subseteq E$. Let $\{U_n \mid n \in \omega\}$ be a neighbourhood system of open sets of the point $p \in E'$, where $E'$ is the set of all limit points of $E$. Without loss of generality we assume $\{U_n \mid n \in \omega\}$ is nested. Otherwise by defining $V_k = \cap_{m \leq k} U_m$ where $k \in \omega$ we get a nested neighbourhood system of open sets around $p \in E'$. For every $k \in \omega$, $p \in U_k$, then there is $\lambda_k \in \omega_1$ such that $e(\lambda_k) \in U_k$ for $k \in \omega$ and define $\lambda_k = \min\{\lambda \mid e(\lambda) \in U_k\}$. So we have a non-decreasing sequence $\{\lambda_n\}_{n \in \omega}$ such that
for every \( k \in \omega, e(\lambda_k) \in U_k. \) This sequence converges to a limit ordinal \( \lambda' \) so \( e(\lambda_n) \to e(\lambda'). \) The long ray has the order topology so it is Hausdorff. Then \( e(\lambda') = p. \) This shows that \( E \) is a closed subset of \( \mathbb{L}_+. \) Then \( E \) is a closed and unbounded subset of \( \mathbb{L}_+. \) \( \omega_1 \) and \( E \) are closed subsets of \( \mathbb{L}_+ \), so \( E \cap \omega_1 \) is closed in \( \omega_1. \) \( E \) and \( \omega_1 \) are cub subspaces of \( \mathbb{L}_+ \) so by Theorem 1.4.7 \( E \cap \omega_1 \) is a cub subspace of \( \mathbb{L}_+. \)

Corollary 1.4.32. For any two points \( x, y \in \mathbb{L}_+, (x, y) \) is homeomorphic to \((0, 1)\).

The following lemma shows that the closed long ray is a manifold with boundary.

Corollary 1.4.33. The closed long ray is a manifold with boundary.

Lemma 1.4.34. The closed long ray is normal.

Proof. To show that \( \mathbb{L}_+ \) is normal we need to show that for every pair \( M, N \) of closed and disjoint subsets of \( \mathbb{L}_+ \) there is a pair \( U \) and \( V \) of open and disjoint subsets of \( \mathbb{L}_+ \) such that \( M \subset U \) and \( N \subset V. \) Either \( M \) and \( N \) both are bounded, both are unbounded or only one of them is unbounded so we need to discuss three different cases.

If \( M \) and \( N \) are two disjoint closed subsets of \( \mathbb{L} \) and both are bounded, then \( M, N \subseteq [a, b] \subseteq \mathbb{L} \) for some \( a \in \mathbb{L}_+ \) and some \( b \in \mathbb{L}_+. \) Proposition 1.4.28 and Theorem 1.4.25 imply that \([a, b] \) is homeomorphic to \([0, 1] \subseteq \mathbb{R}, \) so there are two open and disjoint subsets \( U \) and \( V \) of \([a, b] \) such that \( M \subset U \) and \( N \subset V \) respectively.

If \( M \) and \( N \) are both unbounded then Theorem 1.4.7 implies that \( M \cap N \neq \emptyset. \) So we are done.

If \( M \) is a bounded closed subset of \( \mathbb{L}_+ \) and \( N \) is an unbounded subset of \( \mathbb{L}_+, \) then let the point \( p \) be the least upper bound of \( M. \) It is clear that \( p \notin N \) because \( N \) and \( M \) are closed subspaces of \( \mathbb{L}_+. \) Let \( q > p \) be a point of \( \mathbb{L}_+. \) \( M \subset [0, q) \) and \( N \cap [0, q) \) are closed subspaces of \( [0, q). \) On the other hand it is easy to see that \( [0, q) \cong [0, +\infty) \) so \( [0, q) \) is normal. There are two open subspaces \( U \) and \( V' \) of \([0, q) \) such that \( M \subset U \) and \( N \cap [0, q) \subset V'. \) Now the
open subspaces $U$ and $V = V' \cup (p, \omega_1)$ are two open subspaces of $L_+$ such that $M \subset U$ and $N \subset V$. This completes the proof.

\[\square\]

**Lemma 1.4.35.** [33] The closed long ray is countably compact.

**Proof.** To show that $L_+$ is countably compact we show that every countable open cover $U$ of $L_+$ has a finite subcover.

Let $U = \{U_k | k \in \omega\}$ be a countable cover by open non-empty subsets of $L_+$. Then $V = \{V_k | k \in \omega\}$ is an open cover for $L_+$, where $V_n = \bigcup_{k=0}^{n} U_k$.

If for some natural number $m$, $V_m = V_k$ for all $k > m$, then we are done.

So without loss of generality we assume that for every natural number $m$ there is a natural number $k$ such that $k > m$ and $V_k \neq V_m$. Moreover we delete the consecutive terms of the sequence which are the same set. To do this define an increasing sequence $\{n_k\}$ as follows:

Let $n_1 = 1$. If $n_k$ is defined for the natural number $k$ we write $n_{k+1} = \min\{m \in \omega \mid V_m \neq V_{n_k}, m > n_k\}$. So $V_0 \subset V_{n_1} \subset V_{n_2} \subset V_{n_3} \subset V_{n_4} \subset \ldots$ and $V_{n_k} \neq V_{n_{k+1}}$ for all $k \in \mathbb{N}$. Note that $U_0 \neq \emptyset$ so $V_0 \neq \emptyset$.

Choose $p_0 \in V_0$ and for every natural number $k$ let $p_{k+1} \in V_{n_{k+1}} \setminus V_{n_k}$. Any countable subset of $L_+$ is bounded so $x = \sup\{p_k | k \in \omega\} \in L_+$ exists. Clearly $\{p_0, p_1, p_2, \ldots\} \subset [0, x]$.

Proposition 1.4.28 implies that $[0, x]$ is a compact subspace of $L_+$. The set $V = \{V_{n_k} | k \in \omega\}$ is a countable cover for $\{p_k | k \in \omega\}$. So finitely many of the elements of $V$ cover $\{p_k | k \in \omega\}$.

So for some $N \in \omega$ we have $\{p_k | k \in \omega\} \subseteq \bigcup_{k=1}^{N} V_{n_k}$. It is clear that $\bigcup_{k=1}^{N} V_{n_k} = V_{n_k}$.

Therefore $\{p_n | n \in \omega\} \subseteq V_{n_N}$ for some $N \in \mathbb{N}$. On the other hand $p_{N+1} \in V_{n_{N+1}} \setminus V_{n_N}$.

This is a contradiction. Therefore only finitely many of the elements of $V$ cover $L_+$. This shows that $L_+$ is countably compact.

\[\square\]

**Theorem 1.4.36.** [49] [13] A normal topological space $X$ is perfectly normal if and only if for every closed subset $C$ of $X$ there is a continuous function $f : X \to \mathbb{R}$ such that $C = \{x \in X \mid f(x) = 0\}$.
Lemma 1.4.37. [13] The closed long ray is not perfectly normal.

Proof. The subset of all countable ordinals, $\omega_1$, of $\mathbb{L}_+$ is a closed subset of $\mathbb{L}_+$. Suppose that $f : \mathbb{L}_+ \to \mathbb{R}$ is a continuous function such that for every $\alpha \in \omega_1$, $f(\alpha) = 0$. Then Proposition 1.4.14 shows that there is $x_0 \in \mathbb{L}_+$ such that $f(x) = 0$ for all $x \geq x_0$. This shows that the zero set \{ $x \in \mathbb{L}_+ \mid f(x) = 0$ \} is not equal to $\omega_1$. Therefore by using Theorem 1.4.36 it is easy to see that $\mathbb{L}_+$ is not perfectly normal. This completes the proof. \qed

Lemmas 1.4.37, 1.4.14 and 1.4.33 show that the second red arrow from the top in the Figure 1.3 which is labelled 2 is not reversible for topological manifolds. Therefore the following proposition follows.

Proposition 1.4.38. L is normal, countably compact and locally Euclidean but it is not perfectly normal, hence it is an example of a normal manifold which is not perfectly normal.

In what follows we give a characterisation of the closed long ray but before that we need two theorems from the set theory.

Theorem 1.4.39. [37] Any countable linearly ordered topological space $X$ without first and last elements which is dense in itself is homeomorphic to $\mathbb{Q}$.

Theorem 1.4.40. [37] Any complete linearly ordered topological space $X$ without first and last elements which has a dense subspace homeomorphic to $\mathbb{Q}$ is homeomorphic to the real line.

Corollary 1.4.41. Any complete linearly ordered topological space $X$ with a first and a last element which has a dense subspace homeomorphic to $\mathbb{Q}$ is homeomorphic to the closed unit interval.

Lemma 1.4.42. Any totally ordered topological space $L$ with the order topology which satisfies the following properties is homeomorphic to the closed long ray.
1. Any bounded subset of $L$ is separable.

2. $L$ is connected.

3. Any countable subset of $L$ is bounded.

4. $L$ does not have a maximum point but it has a minimum point.

Proof. Suppose that $x_0 = \min(L)$ and $x_\alpha$ is defined for $\alpha \in \kappa$, where $\kappa$ is an ordinal number. We define $x_{\alpha+1}$ to be an arbitrary element of $L$ bigger than $x_\alpha$. If $\lambda < \omega_1$ is a limit ordinal, then define $x_\lambda = \sup\{x_\gamma | \gamma < \lambda\}$. If $\sup\{x_\gamma | \gamma < \lambda\}$ does not exist then $S = \{x \in L | x > x_\gamma \text{ for all } \gamma < \lambda\}$ and $S' = \{x \in L | x < x_\gamma \text{ for some } \gamma < \lambda\}$ are non-empty open subsets of $L$, since any countable subset of $L$ is bounded and $L = S \cup S'$ which contradicts the connectivity of $L$.

Now we show that $\kappa = \omega_1$. If $\kappa > \omega_1$, then for an ordinal number $\mu$ such that $\omega_1 < \mu < \kappa$ the set $A = \{x \in L | x < \mu\}$ is not separable. So $\kappa \leq \omega_1$.

If $\kappa < \omega_1$ it contradicts the condition (3). So $\kappa = \omega_1$. Now for every $\alpha < \omega_1$ consider $[x_\alpha, x_{\alpha+1}]$ which is a separable totally ordered space, with a maximum and a minimum hence is a metric continuum with two non cut-points and consequently is homeomorphic to unit interval $[0, 1]$ by Theorem 1.4.25.

Let $\phi_\alpha : [x_\alpha, x_{\alpha+1}] \to [\alpha, \alpha+1]$ be an oriented preserving homeomorphism. Then $\phi : L \to L_+$, where $\phi(x) = \phi_\alpha(x)$ for $x \in [x_\alpha, x_{\alpha+1}]$ is the desired homeomorphism. This completes the proof.

\textbf{Corollary 1.4.43.} Any totally ordered topological space $L'$ with the order topology which satisfies the following properties is homeomorphic to $L_-$.

1. Any bounded subset of $L'$ is separable.

2. $L'$ is connected.
3. Any countable subset of $L'$ is bounded.

4. $L'$ does not have a minimum point but it has a maximum point.

Proof. Let $(L', \leq')$ be a totally ordered set with all properties mentioned above and define $L = \{-x \mid x \in L'\}$. Now define $-x \leq -y$ if and only if $y \leq' x$ where $x, y \in L$. It is easy to see that $(L, \leq)$ satisfies all the conditions of Theorem 1.4.42. So $L \cong \mathbb{L}_+$. The function $f : L' \to L$ defined by $f(x) = -x$ is an order reversing function so it is a homeomorphism.

It is easy to see that most of the theorems and lemmas that we proved in this section are true for the long line $\mathbb{L}$ and the open long ray $\mathbb{L}_o$. We repeat these theorems and lemmas here for sake of completeness.

**Definition 1.4.44.** Let $M$ and $N$ be two manifolds with the same dimension. If $\partial M$ and $\partial N$ are boundaries of $M$ and $N$ respectively and $h : \partial M \to \partial N$ is a homeomorphism the connected sum $M \cup h N$ is the quotient space $\frac{M \cup N}{\sim}$ where any $x \in \partial M$ is identified with $h(x) \in \partial N$. We write $(M, \partial M)$ for a manifold $M$ with the manifold boundary $\partial M$. For a manifold $(M, \partial M)$, the connected sum $M \cup_{id} M$ is called the double of $M$.

**Lemma 1.4.45.** [44] Let $(M, \partial M)$ and $(N, \partial N)$ be two $n$-manifolds with boundary and $h : \partial M \to \partial N$ be a homeomorphism. Then $M \cup h N$ is an $n$-manifold without boundary.

**Lemma 1.4.46.** The open long ray and the long line are manifolds without boundary.

Proof. The closed long ray is a manifold with the boundary $0 \in \mathbb{L}_+$. By Lemma 1.4.45 it is easy to see that $\mathbb{L}$ is a manifold without boundary. It is clear from the definition of a manifold with boundary that any $n$-manifold $(M, \partial M)$, $M \setminus \partial M$ is a manifold without boundary. So $\mathbb{L}_o = \mathbb{L}_+ \setminus \{0\}$ is a manifold without boundary of dimension one. \qed
Lemma 1.4.47. The open long ray and the long line are normal spaces.

Proof. Let \( E \) and \( F \) be two disjoint closed subspaces of the long line. If for every open neighbourhood \( W \) of 0, \( W \cap E \neq \emptyset \), then \( 0 \in E \). Similarly if for every open neighbourhood \( W \) of 0, \( W \cap F \neq \emptyset \), then \( 0 \in F \). So either there is an open neighbourhood \( W \) of 0, such that \( W \cap F = \emptyset \) or there is a neighbourhood \( W' \) of 0 such that \( W' \cap E = \emptyset \).

Assume that there is an open neighbourhood \( W \) of 0 such that \( W \cap F = \emptyset \). \( W \) is open in \( L \). Then there is \( z \in L \) such that \( (-z, z) \subset W \). Choose an arbitrary \( \epsilon \in (0, z) \). Clearly \( [-\epsilon, \epsilon] \cap F = \emptyset \). Now let \( r \) be an element of \((0, \epsilon)\). So we have \(-\epsilon < -r < 0 < r < \epsilon\).

We have already proved in Lemma 1.4.34 that \( \mathbb{L}_+ \) is normal. So by splitting \( E \) and \( F \) into two pieces we will be able to use Lemma 1.4.34 and show that \( \mathbb{L} \) is normal. Details are as follows:

Let \( E_+ = E \cap \mathbb{L}_+ \) and \( E_- = E \cap \mathbb{L}_- \). Similarly let \( F_+ = F \cap \mathbb{L}_+ \) and \( F_- = F \cap \mathbb{L}_- \). \( E_+ \) and \( F_+ \) are closed and disjoint subspaces of \( \mathbb{L}_+ \) so by using Lemma 1.4.34 we can find two disjoint open subspaces \( U_+ \) and \( V_+ \) of \( \mathbb{L}_+ \) such that \( E_+ \subset U_+ \) and \( F_+ \subset V_+ \).

Similarly there are two open disjoint subspaces \( U_- \) and \( V_- \) of \( \mathbb{L}_- \) such that \( E_- \subset U_- \) and \( F_- \subset V_- \). \( V_+ \subset (\epsilon, \omega_1) \) since \( F \cap [-\epsilon, \epsilon] = \emptyset \). Clearly \( F_+ \subset V_+ \cap (r, \omega_1) \), \( E_+ \subset U_+ \cup (-r, r) \), \( F_- \subset V_- \cap (\omega_1, -r) \) and \( E_- \subset U_- \cup (-r, r) \).

Now let \( A = (V_+ \cap (r, \omega_1)) \cup ((-\omega_1, -r) \cap V_-) \) and \( B = (U_+ \cup (-r, r) \cup U_-) \). \( A \) and \( B \) are open disjoint subspaces of \( \mathbb{L} \), \( F \subset A \) and \( E \subset B \).

Lemma 1.4.48. The long line is countably compact.

Proof. Let \( \mathcal{U} = \{ U_n \mid n \in \omega \} \) be a countable open cover of \( \mathbb{L} \). \( \mathbb{L}_+ \) has the subspace topology so \( \{ U \cap \mathbb{L}_+ \mid U \in \mathcal{U} \} \) is an open cover of \( \mathbb{L}_+ \). Lemma 1.4.35 shows that \( \mathbb{L}_+ \) is a countably compact topological space. Then finitely many elements of \( \mathcal{U} \), say \( \{ U_{i_1}, U_{i_2}, U_{i_3}, \ldots, U_{i_m} \} \), cover \( \mathbb{L}_+ \). Similarly finitely many elements of \( \mathcal{U} \) cover \( \mathbb{L}_- \), say \( \{ U_{j_1}, U_{j_2}, U_{j_3}, \ldots, U_{j_m} \} \), so \( \{ U_{i_1}, U_{i_2}, U_{i_3}, \ldots, U_{i_n} \} \cup \{ U_{j_1}, U_{j_2}, U_{j_3}, \ldots, U_{j_m} \} \) is a finite subcover of \( \mathbb{L} \). This completes the
proof. □

**Lemma 1.4.49.** The long line is not perfectly normal.

**Proof.** The closed long ray $\mathbb{L}_+$ is a closed subspace of $\mathbb{L}$ and $\omega_1$ is a closed subspace of $\mathbb{L}_+$. So $\omega_1$ is a closed subset of $\mathbb{L}_+$. Let $f : \mathbb{L} \to \mathbb{R}$ be a continuous function. Then $f \mid \mathbb{L}_+ : \mathbb{L}_+ \to \mathbb{R}$ is constant on $[\alpha, \omega_1)$ for some $\alpha \in \omega_1$ by Proposition 1.4.14. Therefore $\{x \mid f(x) = 0\} \neq \omega_1$. Then Theorem 1.4.36 implies that $\mathbb{L}$ is not perfectly normal. □

**Lemma 1.4.50.** Any continuous function $f : \mathbb{L} \to \mathbb{R}$ is constant on $[\beta, \omega_1)$ for some $\beta \in \omega_1$.

**Proof.** We have already proved that any continuous function $f : \mathbb{L}_+ \to \mathbb{R}$ is eventually constant. So there is a $\beta \in \omega_1$ such that $(f \mid \mathbb{L}_+)(\alpha)$ is constant for all $\alpha > \beta$. Clearly $f(\alpha) = (f \mid \mathbb{L}_+)(\alpha)$ for all $\alpha > \beta$. So $f$ is eventually constant. □

**Corollary 1.4.51.** Any continuous function $f : \mathbb{L} \to \mathbb{R}$ is constant on $(-\omega_1, -\beta]$ for some $\beta \in \omega_1$.

**Example 1.4.52.** Let $f : \mathbb{L} \to \mathbb{R}$ be a continuous function defined by

$$f(x) = \begin{cases} 
1 & (x \geq 1) \\
x & (x \in [-1, 1]) \\
-1 & (x \leq -1). 
\end{cases}$$

$f$ is continuous and $f \mid [1, \omega_1) = 1$. Similarly $f \mid (-\omega_1, -1] = -1$. The function $f$ is eventually constant since $f \mid [1, \omega_1) = 1$. Moreover $f(\mathbb{L} \setminus (-1, 1)) \subset \{-1, 1\}$.

**Lemma 1.4.53.** Any totally ordered topological space $L$ with the order topology which satisfies the following properties is homeomorphic to the long line.
1. Any bounded subset of $L$ is separable.

2. $L$ is connected.

3. Any countable subset of $L$ is bounded.

4. $L$ does not have either a maximum point or a minimum point.

Proof. Pick an arbitrary element $x_0 \in L$. By using Lemma 1.4.42 we see that the set $S = \{x \in L \mid x \geq x_0\}$ is homeomorphic to the closed long ray. On the other hand by using Corollary 1.4.43 we see that $S' = \{x \in L \mid x \leq x_0\} \cong S$ is homeomorphic to the closed long ray. So by identifying the two spaces $S$ and $S'$ along $x_0$ it is easy to see that $L$ is homeomorphic to $L$.

The following examples provide normal topological manifolds which are not perfectly normal. In chapter four we study them in more detail.

Example 1.4.54. For any natural number $n \geq 1$, $\mathbb{L}^n$ is another example of a normal manifold which is not perfectly normal.

Example 1.4.55. If $M$ is a compact manifold, then $M \times \mathbb{L}_+$ is a normal manifold which is not perfectly normal. We are especially interested in $S^1 \times \mathbb{L}$ and $S^1 \times \mathbb{L}_+$.

1.4.3 Prüfer manifold

Any locally compact Hausdorff, topological space is completely regular [75]. So all Hausdorff manifolds are Tychonoff. The Prüfer manifold is an example of a Tychonoff topological manifold which is not normal. The main feature of the Prüfer manifold which makes it interesting is that it contains an uncountable discrete closed subspace.

Theorem 1.4.56. [75] Every locally compact Hausdorff space is a Tychonoff space.
Theorem 1.4.57. [54] The following properties are equivalent for a topological space $X$.

- $X$ is completely regular.
- $X$ is uniformisable.

Lemma 1.4.58 (Jones’ Lemma). [75] If $X$ is a separable and normal topological space, then for any closed and discrete subset $D \subseteq X$ we have $|D| < 2^\omega$.

Example 1.4.59. For every fixed real number $r \in \mathbb{R}$, let:

1. $P_r = \{(x, y, r) \in \mathbb{R}^3 | x < 0, y \in \mathbb{R}\}$.
2. $Q = \{(x, y, 0) \in \mathbb{R}^3 | x > 0, y \in \mathbb{R}\}$.
3. $L_r = \{(0, y, r) \in \mathbb{R}^3 | y \in \mathbb{R}\}$.

Consider $P = \bigsqcup_{r \in \mathbb{R}} (L_r \cup P_r) \cup Q$ as a subset of $\mathbb{R}^3$ without assuming any topology on $P$. Our goal is to introduce a topology $\tau$, different from the usual topology of $\mathbb{R}^3$, on $P$ which makes $P$ into a topological manifold. Moreover we need the space $(P, \tau)$ to be a Tychonoff space but not a normal topological space.

For every point $p = (x_0, y_0, 0) \in Q$ we define an open $\epsilon$ neighbourhood of $p$ to be the set

$$N_\epsilon(p) = \{(x, y, 0) \in Q | \sqrt{(x - x_0)^2 + (y - y_0)^2} < \min\{\epsilon, x_0\}\}$$

If $p = (x_0, y_0, r) \in P_r$ for some $r \in \mathbb{R}$, then we define

$$B_\epsilon(p) = \{(x, y, r) \in P_r | \sqrt{(x - x_0)^2 + (y - y_0)^2} < \min\{\epsilon, |x_0|\}\}$$

Open $\epsilon$-neighbourhoods of a point $p \in L_r$ are more complicated and consist of the union of
three different subspaces of \( P \).

To describe an open \( \epsilon \)-neighbourhood of a point \( p = (0, y_0, r) \in L_r \) for fixed \( r \in \mathbb{R} \) first we need to introduce three different subsets \( W_\epsilon(l) \), \( S_\epsilon(l) \) and \( I_\epsilon(l) \) of \( P \).

For \( \epsilon > 0 \) and \( r \) fixed in \( \mathbb{R} \) we define:

1. \( W_\epsilon(p) = \{(x, y, 0) \in Q | 0 < x < \epsilon, (y_0 - \epsilon)x + r < y < (y_0 + \epsilon)x + r\} \).

2. \( S_\epsilon(p) = \{(x, y, r) \in P_r | -\epsilon < x < 0, (y_0 - \epsilon) < y < (y_0 + \epsilon)\} \).

3. \( I_\epsilon(p) = \{(0, y, r) \in L_r | (y_0 - \epsilon) < y < (y_0 + \epsilon)\} \).

Then we let an open \( \epsilon \)-neighbourhood of \( p \in P_r \) be \( W_\epsilon(p) \cup I_\epsilon(p) \cup S_\epsilon(p) \).

The following convention helps us to unify our notation: For every \( p \in P \) we say \( V_\epsilon(p) \) is an open \( \epsilon \)-neighbourhood of \( p \) where \( V_\epsilon(p) \) is defined as follows:

\[
V_\epsilon(p) = \begin{cases} 
W_\epsilon(p) \cup I_\epsilon(p) \cup S_\epsilon(p) & : p \in L_r \\
N_\epsilon(p) & : p \in Q \\
B_\epsilon(p) & : p \in P_r 
\end{cases}
\]

Now we introduce a topology on \( P \) by declaring \( \{V_\epsilon(p) | \epsilon > 0\} \) to be a system of basic neighbourhoods of \( p \in P \).

**Lemma 1.4.60.** The Prüfer manifold is a locally Euclidean topological space.

**Proof.** Let \( p \) be a point in \( P \). In the first case we assume that \( p \in P_r \) for some \( r \in \mathbb{R} \).

If \( p = (x_0, y_0, r) \in P_r \) for a fixed real number \( r \in \mathbb{R} \), then choose a positive real number \( \epsilon < \frac{|x_0|}{2} \). So \( V_\epsilon(p) = B_\epsilon(p) \) is an open subset of \( P \) with respect to the topology \( \tau \). Clearly \( B_\epsilon(p) \) is homeomorphic to the open disc \( \{(x, y) \in \mathbb{R}^2 | \sqrt{(x-x_0)^2 + (y-y_0)^2} < \epsilon\} \) in \( \mathbb{R}^2 \).

The second case is similar to the first case. If \( p = (x_0, y_0, r) \in Q \), then we can choose a positive real number \( \epsilon < \frac{|x_0|}{2} \).

For the last case we assume that \( p = (0, y_0, r) \in L_r \) for some fixed real number \( r \in \mathbb{R} \).
Then $\mathcal{N}_p = \{V_\epsilon(p)|\epsilon > 0\} = \{W_\epsilon(p) \cup I_\epsilon(p) \cup S_\epsilon(p) | \epsilon > 0\}$ is an open neighbourhood system around $p$. Moreover the function

$$f(x, y, r) = \begin{cases} (x, \frac{y-r}{x} - y_0) : x > 0 \\ (x, y - y_0) : x \leq 0 \end{cases}$$

is a bijection between $V_1(p)$ in $P$ and $\{(x, y) \in \mathbb{R}^2 | -1 < x < 1, -1 < y < 1\} \subset \mathbb{R}^2$. Furthermore $f$ preserves the partially order relation $\subseteq$ between $\mathcal{N}$ and $\mathcal{N}'$ hence it is a homeomorphism. 

\[\square\]

**Lemma 1.4.61.** The Prüfer manifold is a Hausdorff topological space.

**Proof.** We realise nine different cases:

1. It is clear from the definition of the Prüfer manifold that $Q$ and $P_r$ for a fixed $r \in \mathbb{R}$ are open subspaces of the Prüfer manifold. So if $p \in Q$ and $q \in P_r$ for some fixed $r \in \mathbb{R}$, then we can simply separate $p$ and $q$ by $Q$ and $P_r$.

2. If $p = (x, y, r) \in P_r$ and $q = (x', y', r') \in P_{r'}$ and $r \neq r'$, then as in the previous case $P_r$ and $P_{r'}$ separates $p$ and $q$ since both $P_r$ and $P_{r'}$ are open and $P_r \cap P_{r'} = \emptyset$.

3. Let $p = (x_0, y_0, r) \in P_r$ and $q = (0, y_1, r) \in L_r$. For $\epsilon < \frac{|x_0|}{2}$, $V_\epsilon(p) \cap V_\epsilon(q) = \emptyset$.

4. If $p = (0, y', r) \in L_r$ and $q = (0, y'', r) \in L_r$, then we choose $\epsilon < \frac{|y' - y''|}{2}$.

5. If $p = (0, y', r') \in L_{r'}$ and $q = (0, y'', r'') \in L_{r''}$, then choose $\epsilon < \frac{|r'' - r'|}{2}$.

6. If $p = (0, y', r) \in L_r$ and $q = (x'', y'', 0) \in Q$, then choose $\epsilon < \frac{|x''|}{2}$.

7. If $p = (x', y', 0) \in Q$ and $q = (x'', y'', 0) \in Q$, then let $\epsilon < \min\left\{\frac{|x'|}{2}, \frac{|x''|}{2}, \frac{\sqrt{(x' - x'')^2 + (y' - y'')^2}}{2}\right\}$.
$S \in (p)$

$W \in (p)$

$I \in (p)$

$= p$

Figure 1: Prüfer manifold

Figure 1.4
8. If \( p = (x', y', r) \in P \) and \( q = (x'', y'', r) \in P \), then let \( \epsilon < \min\left\{ \frac{|x'|}{2}, \frac{|x''|}{2}, \frac{\sqrt{(x' - x'')^2 + (y' - y'')^2}}{2} \right\} \).

9. If \( p = (x', y', r) \in P \) and \( q = (0, y'', r') \in L' \), then \( P \) is an open neighbourhood of \( p \) and \( V_\epsilon(q) \) is an open neighbourhood of \( q \). Note that \( V_\epsilon(q) \subset P \cup L' \cup Q \) and \( (P \cup L' \cup Q) \cap P = \emptyset \) so \( V_\epsilon(q) \) and \( P \) separate \( p \) and \( q \).

In the all cases mentioned above \( V_\epsilon(p) \cap V_\epsilon(q) = \emptyset \), where \( V_\epsilon(p) \) and \( V_\epsilon(q) \) are chosen properly from the definition of the Prüfer manifold. Therefore the Prüfer manifold \( P \) is a Hausdorff topological space.

\[ \square \]

**Lemma 1.4.62.** The Prüfer manifold \( P \) is a Tychonoff topological space.

**Proof.** Lemma 1.4.61 states that the Prüfer manifold \( P \) is a Hausdorff space and Lemma 1.4.60 shows that the Prüfer manifold is a locally Euclidean topological space. So \( P \) is a locally compact Hausdorff topological space. Then by using Theorem 1.4.56 we see that \( P \) is a Tychonoff space.

\[ \square \]

**Lemma 1.4.63.** The Prüfer manifold is not normal.

**Proof.** The idea of the proof is similar to the proof of non-normality of the Moore tangent disc space and in both spaces we show that the space contains an uncountable closed discrete subspace. In fact \( Q \cup \{(0, 0, z)|z \in \mathbb{R}\} \subset P \) is homeomorphic to the Moore tangent disc space. So the Prüfer manifold \( P \) contains a subspace which is homeomorphic to the Moore tangent disc space. Now we describe it in more details.

Let \( L = \{(0, 0, r)|r \in \mathbb{R}\} \) be the \( z \)-axis of \( \mathbb{R}^3 \). \( Q \cup \{(0, 0, r)|r \in \mathbb{R}\} \) is separable as \( Q \) is dense in it. \( L = \{(0, 0, r)|r \in \mathbb{R}\} \) is discrete. Since \( |L| = 2^\omega \) by the Jones' Lemma \( Q \cup L \) is not normal. Since it is closed in \( P \) then nor is \( P \) normal. This completes the proof.

\[ \square \]

There are other topological spaces which have all the topological properties we mentioned above for the Prüfer manifold in common. We describe one of them in the next chapter.
Lemma 1.4.64. The Prüfer manifold is not separable.

Proof. The sets \( \{ P_r | r \in \mathbb{R} \} \) are open subsets of \( P \) and any dense subset of \( P \) must contain at least one point \( x_r \in P_r \). Since \( P_r \cap P_s = \emptyset \) when \( r \neq s \) we must have \( \{ x_r | r \in \mathbb{R} \} \) uncountable.

Lemma 1.4.65. The Suslin number of \( P \) is \( c \).

Proof. For every \( r \in \mathbb{R} \), let \( B_r = \{ (x, y, r) | \sqrt{(x+1)^2 + y^2} < \frac{1}{4} \} \subset P_r \). Then for distinct real numbers \( r_1 \) and \( r_2 \) we have \( B_{r_1} \cap B_{r_2} = \emptyset \). On the other hand we know that the cardinality of manifolds is \( | \mathbb{R} | \) so \( | \{ B_r | r \in \mathbb{R} \} | \leq | P | = c \). Then the Suslin number of \( P \) is the same as \( | \mathbb{R} | \).

The Prüfer manifold is an example which shows that the red arrow labelled 3 on Figure 1.3 is not reversible for manifolds.

1.4.4 Non-Hausdorff manifolds

Theorem 1.4.57 states that a topological space is completely regular if and only if it is uniformisable. So the Prüfer manifold is uniformisable. If we are looking for examples of topological manifolds which are not uniformisable the only choice is non-Hausdorff topological manifolds. The line with a double origin is an easy example of a non-Hausdorff manifold.

Example 1.4.66. (Real line with a double origin) We define a neighbourhood basis on the set \( X = (\mathbb{R}, 0) \cup \{ p_1, p_2 \} \cup (0, +\infty) \subseteq \mathbb{R}^2 \) which makes \( X \) into a locally Euclidean non-Hausdorff topological space, where \( p_1 = (0, 1) \) and \( p_2 = (0, -1) \) are points in \( \mathbb{R}^2 \). For any point \( p \) except \( p_1 \) and \( p_2 \) let \( p \) have the usual neighbourhood that it has in \( \mathbb{R}^* = \mathbb{R} \setminus \{ 0 \} \) but any neighbourhood of \( p_i, i = 1, 2 \) has the following form : \( N_\epsilon(p_i) = ( (\epsilon, \epsilon) - \{ 0 \}) \cup \{ p_i \} \)[68].
This example shows that the last red arrow in Figure 1.3 is not reversible. We study non-Hausdorff manifolds in more detail in the next chapter.

1.5 A quick review of the history of the classification of topological manifolds and dimension

The classification of closed manifolds has been a very active field of mathematics and mathematicians have introduced many different methods and techniques to classify manifolds of different dimension and different categories such as topological manifolds, differential manifolds and piecewise linear manifolds. In dimension one there are just four topologically different connected, Hausdorff manifolds without boundary which are the circle, the real line, the open long ray and the long line. There are three connected, Hausdorff manifolds with boundary as well: the closed unit interval, the half closed unit interval, and the closed long ray. Among the one dimensional manifolds the circle is the only closed, compact manifold with-
out boundary, of dimension one. The closed unit interval is a compact manifold with boundary and the half open interval is a manifold with boundary but it is not compact. The real line is the only paracompact and hence metrisable non-compact one dimensional manifold. The long line and the closed long ray are countably compact nonmetrisable manifolds and the open long ray is the only non metrisable non-countably compact one dimensional manifold. In dimension two we have the following theorem due to Brahana [6]. He was the first mathematician who gave a complete proof of the classification of closed surfaces i.e. compact two dimensional manifolds without boundary in 1921, and consequently in 1923 Kerckjártó successfully classified metrisable non compact surfaces without boundary in terms of their ideal boundaries [58] [23]. In 1979 Edward M. Brown and Robert Messer completed the classification of the metrisable 2-dimensional manifolds with boundary. Their result appeared in the paper titled “The Classification of Two Dimensional Manifolds” [7]. They developed some invariants that helped them to overcome the complication of the classification of metric 2-manifolds with boundary. Another type of surface, called $\omega$-bounded surface, an example being the long plane $L^2$, has been studied by Peter Nyikos [50]. He introduced his bagpipe theorem which reduces the question of classification of the $\omega$-bounded surfaces to the classification of some building blocks of manifolds which are called long pipes. Long pipes have a relatively simple structure. We will talk about $\omega$-bounded manifolds in different dimensions in Chapter 4.

**Theorem 1.5.1.** [6] (Brahana 1921) Any compact connected surface is homeomorphic to the connected sum of a sphere, finitely many tori and finitely many projective planes.

The recent book titled “A guide to the classification theorem for compact surfaces” by Jean Gallier and Dianna Xu is a good reference for the classification of compact surfaces [15].

**Theorem 1.5.2.** (Kerckjártó) Let $S$ and $S'$ be two metrisable surfaces without boundary of the same genus and orientability class. Then $S$ and $S'$ are homeomorphic if and only if their
ideal boundaries are topologically equivalent.

In dimension three, work on the classification of compact manifolds is still very active and compact 3-manifolds still are not classified.

### 1.6 The classification of one dimensional connected Hausdorff manifolds

Although the classification of all topological spaces is not feasible it is the ultimate aim of topology. Instead topologists try to classify some subclasses of topological spaces. Classification of manifolds has been one of the targets of topologists and the study of low dimensional manifolds is very active. Classification of 2-dimensional closed (compact, and without boundary ) manifolds is one of the important achievements of mathematics. Although the classification of one dimensional manifolds is easier than 2-manifolds, we could not find any complete proof in the literature.

On the page 643 Example 3.2 of [42] Peter Nyikos stated that there are only four connected one dimensional manifolds without boundary, namely the circle, the real line, the open long ray and the long line. In the same example he wrote that there are three one dimensional connected Hausdorff manifolds with the boundary namely, the closed long ray, the closed unit interval of the real line and the half closed interval of the real line. In this section we provide a complete proof for this classification. We should note that in Appendix 2 in [26], Victor Guillemin and Alan Pollack wrote a complete proof of the classification of the compact one dimensional connected Hausdorff differentiable manifolds. We should also mention the book titled “Beginner’s course in topology ” by D.B.Fucks and V.A.Rokhlin [14] from which we adopted some techniques to achieve our goal. In [14] the authors briefly discuss the classification of connected metrisable one dimensional manifolds. In this section we give a detailed proof of this classification. Our goal like any other classification problem
in mathematics is to consider a class of mathematical objects and provide a list of different mathematical objects in that class, here one dimensional manifolds, and then prove that any other objects of that class have to be the same as one element of our list of objects. Here we should be careful about the term the same because it all depends on the class that we are working on. In this section all the manifolds are assumed to be Hausdorff and connected unless otherwise mentioned.

**Definition 1.6.1.** By a chart \( c = (U, \phi) \) around a point \( p \) in an \( n \)-manifold \( M \) we mean an open subspace \( U \) of \( M \) and a homeomorphism \( \phi : U \to \mathbb{R}^n \). If \( M \) is a manifold with boundary \( \partial M \) then a chart around \( x \in \partial M \) is a homeomorphism \( \phi : U \to \{(x_1, x_2, x_3, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0 \} \).

In Definition 1.6.1 we can replace \( \mathbb{R}^n \) by \( D^n = \{x \in \mathbb{R}^n \mid \|x\| < 1 \} \) or any other subspace of \( \mathbb{R}^n \) which is homeomorphic to \( \mathbb{R}^n \).

**Lemma 1.6.2.** \( S^1 \) and the real line are one dimensional manifolds.

**Proof.** The identity function is a homeomorphism from \( \mathbb{R} \) to \( \mathbb{R} \).

We introduce two charts \( c_1 = (U, \phi) \) and \( c_2 = (V, \psi) \) such that \( U \cup V = S^1 \). Let \( U = S^1 \setminus \{(0, 1)\} \) and \( V = S^1 \setminus \{(0, -1)\} \). It is obvious that \( U \cup V = S^1 \). Now we introduce two homeomorphisms \( \phi : U \to \mathbb{R} \) and \( \psi : V \to \mathbb{R} \). Two functions \( \phi(x, y) = \frac{2x}{1 + y} \) and \( \psi(x, y) = \frac{2x}{1 + y} \) are desired homeomorphisms. So \( S^1 \) is a one dimensional manifold. The real line is a \( \sigma \)-compact topological space but it is not compact, while \( S^1 \) is compact so they are different topological manifolds.

**Lemma 1.6.3.** The closed interval and the half closed interval are manifolds with boundary.

**Proof.** We write \([0, 1] = [0, 1) \cup (0, 1) \cup (0, 1]. \) If \( 0 < x < 1 \), then \( ((0, 1) , \tan(\pi x - \frac{\pi}{2})) \) is a chart around \( x \). \( ([0, 1] , \tan(\frac{\pi}{2}) x) \) is a chart around \( 0 \) and \( ((0, 1] , \tan(\frac{\pi}{2})(1 - x)) \) is a chart around \( 1 \). This shows that \([0, 1] \) is a manifold with boundary. Hence so is \([0, 1). \)
Lemma 1.6.4. The closed long ray is a manifold with boundary. Moreover $\mathbb{L}_+^+$ has only one boundary component $0$.

Proof. Consider an arbitrary non-zero element $x$ of $\mathbb{L}_+^+$. By using Proposition 1.4.28 we see that $[0, \alpha]$ is homeomorphic to $[0, 1]$ where $\alpha = \min\{\beta \in \omega_1 | x < \beta\}$. So $(0, \alpha)$ is an open neighbourhood of $x$ and $(0, \alpha)$ is homeomorphic to $\mathbb{R}$.

The identity function provides a homeomorphism from $[0, 1) \subset \mathbb{L}_+^+$ onto $[0, 1) \subset \mathbb{R}$. Note that any arbitrary point $x$ of $\mathbb{L}_+^+$ except 0 has a neighbourhood homeomorphic to $\mathbb{R}$ so $x$ cannot be a boundary point. \hfill \qed

Lemma 1.6.5. The open long ray and the long line are one dimensional manifolds without boundary.

Proof. (long line)

$\{(-\alpha, \alpha) \mid \alpha \in \omega_1\}$ is an open cover for $\mathbb{L}$. We have already proved in Theorem 1.4.40 that for each $\alpha \in \omega_1$, $(-\alpha, \alpha)$ is homeomorphic to $(-1, 1)$. For each $x \in \mathbb{L}_+^+$ larger than or equal to 0 there is an $\alpha \in \omega_1$ larger than $x$ since $\omega_1$ is cofinal in $\mathbb{L}_+^+$ then $x \in (-\alpha, \alpha) \cong \mathbb{R}$.

Similarly if $x < 0$ there is an $\alpha \in \omega_1$ such that $-x < \alpha$, then $x \in (-\alpha, \alpha) \cong \mathbb{R}$.

(Open long ray )

Let $x$ be an element of the open long ray $\mathbb{L}_0$. Then $(0, x + 1)$ is an open neighbourhood of
Definition 1.6.6. An atlas $\mathcal{A}$ on an $n$-manifold $M$ is a family of charts $\{c_\alpha = (U_\alpha, h_\alpha) \mid \alpha \in I\}$ such that $\cup_{\alpha \in I} U_\alpha = M$. Given two charts $(U_\alpha, h_\alpha), (U_\beta, h_\beta)$, the transition map between them is a homeomorphism $h_{\alpha\beta} = h_\beta \circ h_\alpha^{-1} : h_\alpha(U_\alpha \cap U_\beta) \to h_\beta(U_\beta \cap U_\alpha)$.

Lemma 1.6.7. [67] Any open subspace of $\mathbb{R}$ can be written as a countable union of disjoint intervals, $\{I_n \mid n \in \mathbb{N}\}$.

Lemma 1.6.8. Any connected metrisable manifold without boundary $M$ of dimension one with an atlas $\mathcal{A}$, such that the atlas $\mathcal{A}$ consists of only one chart, is homeomorphic to $\mathbb{R}$.

Proof. The proof is obvious. Let $\mathcal{A} = \{(U, h)\}$ be an atlas on $M$. $\{U\}$ is a cover for $M$ so $M = U$. Then $h : M \to \mathbb{R}$ is the desired homeomorphism.

Notation 1.6.9. Let $(U, h)$ be a chart on a manifold $M$ of dimension one. We define a new chart $(U, \rho \circ h)$ on $M$ which has the same domain as $(U, h)$, where $\rho : \mathbb{R} \to \mathbb{R}$ defined by $\rho(x) = -x$ and call it the reversal of $(U, h)$.

The following lemma is briefly discussed in [14] and here we give a detailed proof with some changes in the proof due to some mistakes in [14].

Lemma 1.6.10. Any connected Hausdorff manifold $M$ of dimension one without boundary which has an atlas with two charts is homeomorphic to the circle or to the real line.

Proof. To prove the lemma first we consider an atlas $\mathcal{A} = \{(U, h_U), (V, h_V)\}$ on $M$ and we show that $h_U(U \cap V)$ and $h_V(U \cap V)$ can be either an initial segment of $\mathbb{R}$, a final segment of $\mathbb{R}$, or a union of an initial and a final segment of $\mathbb{R}$.

Then we realise ten different scenarios and show that each scenario either does not happen or it leads us to the desired result.
Let $M$ be a one dimensional connected metrisable manifold without boundary. If $U \subseteq V$ or $V \subseteq U$, then the problem reduces to an atlas with one chart so Lemma 1.6.8 implies that $M = \mathbb{R}$.

So let us assume that $U \setminus V \neq \emptyset$ and $V \setminus U \neq \emptyset$. Note that $U \cap V \neq \emptyset$ since $M$ is a connected manifold. Hence $h_U(U \cap V)$ and $h_V(U \cap V)$ are open and non-empty subspaces of $\mathbb{R}$.

Now we show that $h_U(U \cap V)$ and $h_V(U \cap V)$ cannot be bounded intervals of $\mathbb{R}$.

Lemma 1.6.7 shows that any open subspace of $\mathbb{R}$ can be written as a countable union of disjoint open intervals $I_m = \{I_n \mid n \in \mathbb{N}\}$.

Let $r_1$ and $r_2$ be two real numbers such that $I = (r_1, r_2) \in I$ is one of open intervals in $h_U(U \cap V)$. For every $m \in \mathbb{N}$, $r_1 \notin I_m$. Otherwise there are $r_3$ and $r_4$ such that $r_1 \in (r_3, r_4) \subset h_U(U)$ which contradicts disjointness of intervals. Similarly $r_2 \notin I_m$.

Recall that $h_{UV} = h_V \circ h_U^{-1} : h_U(U \cap V) \to h_V(U \cap V)$ is a homeomorphism so it takes the decomposition $I$ of $h_U(U \cap V)$ to the corresponding decomposition of $h_V(U \cap V)$.

Note that for a bounded element $I \in I$, $h_{UV}(I) \neq \mathbb{R}$, otherwise $h_U^{-1}(I) = h_V^{-1}(\mathbb{R})$, so $V = h_U^{-1}(I) \subseteq h_U^{-1}(\mathbb{R}) \subseteq U$ which contradicts the assumption $V \setminus U \neq \emptyset$. If the decomposition of $h_{UV}(U \cap V)$ to disjoint open intervals, $I$, contains a bounded interval $I$ we consider the two following cases and show that neither of them can happen.

I. The transition map $h_{UV}$ maps a bounded interval $I = (r_1, r_2) \in I$ onto an initial segment say $(-\infty, s_2)$ or a final segment say $(s_1, +\infty)$ of $\mathbb{R}$.

II. The transition map $h_{UV}$ maps a bounded interval $I = (r_1, r_2) \in I$ onto abounded interval $J = (s_1, s_2)$ of $\mathbb{R}$.

Case I.
Let $\rho : \mathbb{R} \to \mathbb{R}$ be defined by $\rho(x) = -x$. Replacing $h_V$ by $\rho \circ h_V$ if necessary we may assume that $h_{UV}(I) = (s_1, +\infty)$.

Since $(U, h_U)$ and $(V, h_V)$ are charts with $r_1 \in h_U(U)$ and $s_1 \in h_V(V)$, then for some $p \in U$ and $q \in V$ we have $h_U(p) = r_1$ and $h_V(q) = s_1$.

Note that $p \neq q$. For every neighbourhood $P$ of $p$ there is a positive real number $\delta$ such that $(r_1, r_1 + \delta) \subset h_U(P)$. Similarly for every neighbourhood $Q$ of $q$ there is a positive real number $\epsilon$ such that $(s_1, s_1 + \epsilon) \subset h_V(Q)$. Thus $h_U^{-1}(r_1, r_1 + \delta) \subset P$ and $h_V^{-1}(s_1, s_1 + \epsilon) \subset Q$.

Choose $x \in (r_1, r_1 + \delta)$ so that $h_U(x) \in (s_1, s_1 + \epsilon)$. Then $h_U^{-1}(x) \in P \cap Q$. So every neighbourhood of $p$ meets every neighbourhood of $q$. This contradicts $M$ being Hausdorff space.

Similarly $h_{UV}$ does not map a bounded element $I \in \mathcal{I}$ onto an initial segment $(-\infty, s_2)$ where $s_2 \in \mathbb{R}$.

Therefore $h_{UV}$ does not map bounded elements of $\mathcal{I}$ to the unbounded subsets of $\mathbb{R}$ so case I does not happen.

Case II.

Now we show that $h_{UV}$ does not map bounded elements of $\mathcal{I}$ to bounded subsets of $\mathbb{R}$.

Let $\mathcal{I}$ and $\mathcal{J}$ be decompositions of $h_U(U \cap V)$ and $h_V(U \cap V)$ into open intervals respectively. Assume that $h_{UV}(I) = J$, where $I = (r_1, r_2) \in \mathcal{I}$ and $J = (s_1, s_2) \in \mathcal{J}$.

Note that $h_{UV} : (r_1, r_2) \to (s_1, s_2)$ is a homeomorphism so it is a monotonic continuous function. We discuss the case where $h_{UV}$ is increasing. The other case is similar.

Assume that $h_{UV} : (r_1, r_2) \to (s_1, s_2)$ is an increasing continuous function. $h_{UV}(r_1) \neq s_1$ otherwise $h_U^{-1}(r_1) = h_V^{-1}(s_1) \in U \cap V$ but we know that $h_U^{-1}(r_1) \notin V \cap U$ so $h_U^{-1}(r_1) \notin V$.

Consider a decreasing sequence $\{x_n\}$ in $(r_1, r_2)$ such that $x_n \to r_1$. $h_{UV}$ is increasing and continuous so $y_n = h_{UV}(x_n)$ provides a decreasing sequence $\{y_n\}$ which converges to $s_1$. $h_U^{-1}$ and $h_V^{-1}$ are homeomorphism so $h_U^{-1}(x_n) \to h_U^{-1}(r_1)$ and similarly $h_V^{-1}(x_n) \to h_V^{-1}(s_1)$.

This shows that the sequence $\{t_n\}$ where $t_n = h_U^{-1}(x_n) = h_V^{-1}(y_n)$ for every $n \in \mathbb{N}$ converges
to two points. This contradicts the our assumption that $M$ is $T_2$.

This shows that the transition map $h_{UV}$ does not map bounded intervals of $I$ onto the bounded intervals of $J$. So case II does not happen.

*In view of the argument above it is not possible for $h_{UV}$ to map a finite end of its domain to a finite end of its range.* Note that $h_U(U \cap V)$ is an open subspace of $\mathbb{R}$ and it does not have any bounded interval in its decomposition into disjoint intervals. So $h_U(U \cap V)$ has one of the following forms:

1. $h_U(U \cap V) = (r, +\infty)$ for some $r \in \mathbb{R}$.
2. $h_U(U \cap V) = (-\infty, r)$ for some $r \in \mathbb{R}$.
3. $h_U(U \cap V) = (-\infty, r_1) \cup (r_2, +\infty)$ for some $r_1, r_2 \in \mathbb{R}$.

Similarly $h_V(U \cap V)$ has one of the following forms:

1. $h_V(U \cap V) = (r, +\infty)$ for some $r \in \mathbb{R}$.
2. $h_V(U \cap V) = (-\infty, r)$ for some $r \in \mathbb{R}$.
3. $h_V(U \cap V) = (-\infty, r_1) \cup (r_2, +\infty)$ for some $r_1, r_2 \in \mathbb{R}$.

Now we continue the proof for each combination of these cases for $h_U(U \cap V)$ and $h_V(U \cap V)$.

**Scenario one:**

\[
\begin{array}{l}
\begin{align*}
  h_U(U \cap V) &= (-\infty, r_1) \text{ and } h_V(U \cap V) = (r_2, +\infty) \text{ for some } \\
  r_1, r_2 &\in \mathbb{R}
\end{align*}
\end{array}
\]

Our goal in this scenario is to show that the manifold $M$ has to be homeomorphic to the real line.

We use the same method that we have used already to show that the image of $U \cap V$ under
the transition map $h_{UV}$ does not contain any bounded interval.

Consider the homeomorphism $h_{UV} = h_U \circ h_U^{-1} : h_U(U \cap V) \to h_V(U \cap V)$. We use the assumption $h_U(U \cap V) = (-\infty, r_1)$ and $h_V(U \cap V) = (r_2, +\infty)$ so $h_{UV} : (-\infty, r_1) \to (r_2, +\infty)$. Every homeomorphism between intervals $(-\infty, r_1)$ and $(r_2, +\infty)$ has to be monotonic; otherwise we can choose three points $a, b$ and $c$ in the initial interval $(-\infty, r_1)$ such that $h_{UV}(a) < h_{UV}(c) < h_{UV}(b)$ (or $h_{UV}(b) < h_{UV}(c) < h_{UV}(a)$) then by using intermediate value theorem there is a real number $r \in [h_{UV}(c), h_{UV}(b)]$ which will be acquired more than once by $h_{UV}$ which contradicts bijectivity of $h_{UV}$. So $h_{UV}$ is monotonic. Now we show that $h_{UV}$ is increasing.

Our method of showing that $h_{UV}$ is increasing is very similar to the argument that we used to prove that components of $h_{UV}(U \cap V)$ do not include any bounded intervals.

In fact we show that if $h_{UV}$ is decreasing, then it maps the finite end of $(-\infty, r_1)$ to the finite end of $(r_2, +\infty)$ which contradicts the Hausdorff property of $M$.

For completeness we give the details here.

Let $\{t_n\}_{n \in \mathbb{N}}$ be a sequence in $(-\infty, r_1)$ converging to $r_1$. The transition map $h_{UV}$ is decreasing so the sequence $\{h_{UV}(t_n)\}_{n \in \mathbb{N}}$ converges to $r_2$. We write $s_n = h_{UV}(t_n)$ for all $n \in \mathbb{N}$. So let us to write $p_n = h_U^{-1}(t_n) = h_V^{-1}(s_n)$ for all $n \in \mathbb{N}$. On the other hand $h_U^{-1}$ and $h_V^{-1}$ are continuous functions so $p_n = h_U^{-1}(t_n) \to h_U^{-1}(r_1) \in U \setminus V$ and $p_n = h_V^{-1}(s_n) \to h_V^{-1}(r_2) \in V \setminus U$. This contradicts the Hausdorff property of the manifold $M$ since in $T_2$ spaces every sequence converges at most to one point. So the transition map $h_{UV}$ is increasing.

The function $h_{UV} : (-\infty, r_1) \to (r_2, +\infty)$ is an increasing bijection so there are two points $p \in h_U(U \cap V) = (-\infty, r_1)$, and $q \in h_V(U \cap V) = (r_2, +\infty)$ such that $q$ is larger than $p$ and $h_{UV}(p) = q$. Hence $h_U^{-1}(p) = h_V^{-1}(q)$.

Note that $h_{UV} : (-\infty, r_1) \to (r_2, +\infty)$ is an increasing function so $h_{UV}((-\infty, p]) = ((r_2, q])$, $(q, +\infty)) = h_{UV}([p, r_1])$.

So $h_U^{-1}((-\infty, p]) = h_V^{-1}((r_2, q])$, $h_U^{-1}([q, +\infty)) = h_V^{-1}([p, r_1])$. 

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Figure 1.7: Decreasing transition map

Figure 1.8
Let $h^{-1}_U(p) = h^{-1}_V(q) = x^* \in U \cap V$. We claim that $M = h^{-1}_U([p, +\infty)) \cup h^{-1}_V((\infty, q])$.

Recall that $M = U \cup V$, $U = h^{-1}_U(\mathbb{R})$ and $V = h^{-1}_V(\mathbb{R})$ so $M = h^{-1}_U(\mathbb{R}) \cup h^{-1}_V(\mathbb{R}) = h^{-1}_U((-\infty, p]) \cup h^{-1}_U([p, +\infty)) \cup h^{-1}_V((\infty, q]) \cup h^{-1}_V([q, +\infty))$.

On the other hand we see that $h^{-1}_U(\mathbb{R}) = h^{-1}_U((-\infty, p]) \cup h^{-1}_U([p, +\infty))$ and $h^{-1}_V(\mathbb{R}) = h^{-1}_V((-\infty, q]) \cup h^{-1}_V([q, +\infty))$ so $h^{-1}_U((-\infty, p]) \subseteq h^{-1}_V([p, +\infty))$ and $h^{-1}_V([q, +\infty)) \subseteq h^{-1}_U([p, +\infty))$. Thus we have $M = h^{-1}_U([p, +\infty)) \cup h^{-1}_V((\infty, q])$. Consider two homeomorphisms

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{transition_map.pdf}
\caption{The transition map between two charts}
\end{figure}

$h_1 : (-\infty, q] \to (-\infty, 0]$ and $h_2 : [p, +\infty) \to [0, +\infty)$. Then $h^{-1}_U \circ h^{-1}_V([0, +\infty)) \cup h^{-1}_U \circ h^{-1}_V((-\infty, 0] = M$ and finally we can establish a homeomorphism $h : \mathbb{R} \to M$ defined by:

$$h(x) = \begin{cases} h^{-1}_U \circ h^{-1}_V(x) & (x \geq 0) \\ h^{-1}_V \circ h^{-1}_U(x) & (x \leq 0) \end{cases}$$

Scenario two:

$h_U(U \cap V) = (-\infty, r_1) \cup (r_2, +\infty)$, $h_V(U \cap V) = (-\infty, r'_1) \cup (r'_2, +\infty)$ and $h_{UV}(r_2, +\infty) = (-\infty, r'_1)$, $h_{UV}(-\infty, r_1) = (r'_2, +\infty)$
First we should note that $r_1 < r_2$, otherwise $h_U(U \cap V) = \mathbb{R}$ so $U \subset V$ and we showed that $U \subset V$ implies $M = \mathbb{R}$. So we assume that $r_1 < r_2$. Hence also $r'_1 < r'_2$.

The transition map $h_{UV} : h_U(U \cap V) \to h_V(U \cap V)$ is a homeomorphism, $h_{UV}(r_2, +\infty) = (\infty, r'_1)$, and $h_{UV}(\infty, r_1) = (r'_2, +\infty)$. Similar argument to that in the first scenario implies that $(h_{UV} \mid (-\infty, r_1)) : (-\infty, r_1) \to (r'_2, +\infty)$ is an increasing homeomorphism.

Similarly $(h_{UV} \mid (r_2, +\infty)) : (r_2, +\infty) \to (-\infty, r'_2)$ is increasing. Otherwise, by using the same method as we used in the scenario one, we end up with a contradiction. So $h_{UV}$ is an increasing homeomorphism.

If $h_{UV}((-\infty, r_1)) = ((-\infty, r'_1))$ and $h_{UV}((r_2, +\infty)) = (r'_2, +\infty)$ we follow $h_V : V \to \mathbb{R}$ by $\rho : \mathbb{R} \to \mathbb{R}$ defined by $\rho(x) = -x$ and the proof is similar to the previous situation so we do not discuss it.

Note that $(h_{UV} \mid (-\infty, r_1)) : (-\infty, r_1) \to (r'_2, +\infty)$ is an increasing homeomorphism. Then there are two points $p_1 \in (-\infty, r_1)$ and $q_1 \in (r'_2, +\infty)$ such that $h_{UV}(p_1) = q_1$. The homeomorphism $(h_{UV} \mid (r_2, +\infty)) : (r_2, +\infty) \to (-\infty, r'_2)$ is also increasing, so there are points $q_2 \in (-\infty, r'_1)$ and $p_2 \in (r_2, +\infty)$ such that $h_{UV}(p_2) = q_2$.

$h_{UV}$ is increasing so we get the following equalities:

$h_U^{-1}((-\infty, p_1]) = h_V^{-1}([r'_2, q_1])$
$h_U^{-1}([p_1, r_1]) = h_V^{-1}([q_1, +\infty))$
$h_U^{-1}([r_2, q_1]) = h_V^{-1}((-\infty, p_2])$
$h_U^{-1}([q_1, +\infty)) = h_V^{-1}([p_2, r'_1])$

We know that $M = U \cup V$ so $M = h_U^{-1}(\mathbb{R}) \cup h_V^{-1}(\mathbb{R})$. By using our equalities we can write $M = h_U^{-1}([p_1, p_2]) \cup h_V^{-1}([q_2, q_1])$.

Now we are ready to establish a homeomorphism $H : M \to S^1$. 

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Figure 1.10: The transition map of disconnected domain

\[ H(x) = \begin{cases} 
\exp(\pi i \frac{h_U(x)-p_1}{p_2-p_1}) & (x \in h_U^{-1}([p_1,p_2])) \\
-\exp(\pi i \frac{h_V(x)-p_2}{q_1-q_2}) & (x \in h_V^{-1}([q_2,q_1]))
\end{cases} \]

\[ M \] is compact because \( M \) has a cover with only two charts. By using the Pasting Lemma [47] we see that \( H \) is a continuous function. Moreover \( H \) is a bijection from a compact space \( M \) onto a Hausdorff space \( S^1 \) so \( H \) is a homeomorphism [47].

Scenario three:

Assume that \( A = \{(U,h_U),(V,h_V)\} \) is the atlas with two charts for \( M \). We consider the new atlas \( B = \{(U,h_U),(V,\rho \circ h_V)\} \) which consists of two charts \((U,h_U)\) and a reversal of
$(V,h_V)$. The new atlas reduces the question to the scenario one.

Scenario four:

\[
\begin{align*}
  h_U(U \cap V) &= (r_1, \infty) \\
  \text{and} \\
  h_V(U \cap V) &= (r_2, +\infty)
\end{align*}
\]

If we replace $(U,h_U)$ by its reversal this scenario will be similar to the scenario number one.

Scenario five:

\[
\begin{align*}
  h_U(U \cap V) &= (r_1, +\infty) \\
  \text{and} \\
  h_V(U \cap V) &= (-\infty, r'_1)
\end{align*}
\]

This scenario is similar to scenario one if we replace both charts by their reversals.

Scenario six:

\[
\begin{align*}
  h_U(U \cap V) &= (-\infty, r_1) \\
  \text{and} \\
  h_V(U \cap V) &= (-\infty, r'_1) \cup (r'_2, +\infty)
\end{align*}
\]

This scenario does not happen because $h_U(U \cap V) = (-\infty, r_1)$ implies that $U \cap V$ is connected but $h_V(U \cap V) = (-\infty, r'_1) \cup (r'_2, +\infty)$ implies that $U \cap V$ is not connected. Therefore this case does not happen.
Scenario seven:

\[
h_U(U \cap V) = (r_2, +\infty) \]
and
\[
h_V(U \cap V) = (-\infty, r'_1) \cup (r'_2, +\infty)
\]

This case is similar to the scenario six and never happens by replacing \( h_U \) by \( -h_U \).

Scenario eight:

\[
h_U(U \cap V) = (-\infty, r_1) \cup (r_2, +\infty)
\]
and
\[
h_V(U \cap V) = (r'_2, +\infty)
\]

This scenario is similar to the scenario number seven by replacing \( U \) by \( V \) and \( V \) by \( U \) so this scenario does not happen.

Scenario nine:

\[
h_U(U \cap V) = (-\infty, r_1) \cup (r_2, +\infty),
\]
and
\[
h_V(U \cap V) = (-\infty, r'_1)
\]

This scenario is similar to the scenario six and does not happen.
Scenario ten:

\[
h_U(U \cap V) = (-\infty, r_1) \cup (r_2, +\infty), \quad h_V(U \cap V) = (-\infty, r_1') \cup (r_2', +\infty)
\]

and

\[
h_{UV}(-\infty, r_1) = (-\infty, r_1'), \quad h_{UV}(r_2, +\infty) = (r_2', +\infty)
\]

If we replace \( h_V \) by \( \rho \circ h_V \), where \( \rho(x) = -x \) for all \( x \in h_V(U \cap V) \), then this scenario reduces to scenario two.

This completes the proof. \( \square \)

**Corollary 1.6.11.** If \( M \) is a connected one dimensional metrisable manifold without boundary and with an atlas consisting of two charts, then the topological boundary of the domain of every chart is either empty, has only one point or has two elements.

**Proof.** Any such one dimensional manifold is homeomorphic to the circle or the real line. So for any chart \( (U, h) \) the image \( h(U) \) is the image of an interval in the circle or the real line. If \( M \cong \mathbb{R} \) then \( h(U) \) has one of the following forms:

\( (-\infty, +\infty), (-\infty, r), (r, +\infty) \) or \( (r, r') \) for some real numbers \( r, r' \). Therefore it has at most two boundary points.

If \( M \cong S^1 \), then \( h(U) \) is a connected subset of \( S^1 \setminus \{p\} \) for some \( p \in S^1 \) so it has at most two boundary points. In both cases the boundary points of \( (U, h) \) are in \( M \setminus U \). So \( \partial_M U \subset V \) where \( (V, h_V) \) is the second chart of the atlas. \( \square \)

**Theorem 1.6.12.** Let \( M \) be a metrisable one dimensional connected manifold with a finite atlas \( \mathcal{A} \). Then \( M \cong S^1 \) or \( M \cong \mathbb{R} \).

**Proof.** We use strong induction to prove the theorem. If \( |\mathcal{A}| = 1 \) or \( |\mathcal{A}| = 2 \) the result is clear from Lemma 1.6.8 or Lemma 1.6.10 respectively. Assume that the result is true for
every manifold which has an atlas $\mathcal{A}$ with at most $n$ many charts.

Consider a manifold $M$ with the mentioned properties which has an atlas $\mathcal{A} = \{(U_k, \phi_k) \mid 1 \leq k \leq n + 1\}$ with $n + 1$ charts.

Assume that $M$ does not have any atlas of cardinality less than or equal to $n$ otherwise the result is clear from the assumption of the induction.

By rearranging the domain of charts in $\mathcal{A}$ we assume that $U_k \cap U_{k+1} \neq \emptyset$ for every $k \in \{1, \ldots, n\}$. Consider the subspace $V = \bigcup_{k=1}^{n} U_k$ of $M$. Note that $V$ is the union of open sets so $V$ is an open subspace of $M$.

Note that $V$ is a manifold of dimension one and $B = \{(U_k, \phi_k) \mid 1 \leq k \leq n\}$ is an atlas for $V$. So by our assumption $V \cong S^1$ or $V \cong \mathbb{R}$. If $V \cong S^1$, then $V$ is a compact subspace of $M$ so it is closed which contradicts the connectivity of $M$ so $V \cong \mathbb{R}$.

Now consider the atlas $\mathcal{A}'$ of $M$ with two charts $(V, \phi)$ and $(U_{n+1}, \phi_{n+1})$. By using Lemma 1.6.10 we see that $M \cong \mathbb{R}$ or $M \cong S^1$.

\textbf{Lemma 1.6.13.} Assume that $U$ and $V$ are two topological spaces homeomorphic to open intervals of $\mathbb{R}$. Let $h_U : U \to (a, b) \subset \mathbb{R}$ and $h_V : V \to (c, d) \subset \mathbb{R}$ be two homeomorphisms. If $U \subset V$, then there is a homeomorphism $h : V \to (\alpha, \beta) \subset \mathbb{R}$ such that $h \mid U = h_U$.

\textbf{Proof.} Following $h_V$ by an orientation-reversing homeomorphism if necessary, we assume that $h_U$ and $h_V$ introduce the same directions on $U$. More precisely for every pair $x, y \in U$ we have either $h_U(x) < h_U(y)$ and $h_V(x) < h_V(y)$ or $h_U(x) > h_U(y)$ and $h_V(x) > h_V(y)$.

$h_V(U) \subset (c, d)$ is connected so $h_V(U) = (s, t) \subset (c, d)$ for $c \leq s < t \leq d$. The transition map $h_{UV} = h_U \circ h_V^{-1}$ is a homeomorphism so $h_{UV}((s, t)) = (a, b)$. If $s \neq c$ and $t \neq d$, then let $f : (c, d) \to (\alpha, \beta)$, where $\alpha < a$ and $b < \beta$ be a function defined by

$$
f(x) = \begin{cases} 
\frac{a - \alpha}{s - c} (x - s) + a & (c < x \leq s), \\
h_U \circ h_V^{-1}(x) & (x \in (s, t)), \\
\frac{\beta - b}{d - t} (x - t) + b & (t \leq x < d), 
\end{cases}
$$

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If \( s = c \) but \( t \neq d \), then we define the function \( f : (c, d) \to (\alpha, \beta) \), where \( \alpha = a \) and \( b < \beta \) to be as follows:

\[
f(x) = \begin{cases} 
  h_U \circ h_V^{-1}(x) & (x \in (s, t)), \\
  \frac{\beta - b}{d - t}(x - t) + b & (t \leq x < d),
\end{cases}
\]

and if \( t = d \) but \( s \neq c \), then we let \( f : (c, d) \to (\alpha, \beta) \), where \( \alpha < a \) and \( \beta = b \) to be as follows:

\[
f(x) = \begin{cases} 
  \frac{a - \alpha}{s - c}(x - s) + a & (c < x \leq s), \\
  h_U \circ h_V^{-1}(x) & (x \in (s, t)).
\end{cases}
\]

Note that \( f \) is a homeomorphism from \((c, d)\) onto the interval \((\alpha, \beta)\). Now we define \( h = f \circ h_V : V \to (\alpha, \beta) \). Composition of two homeomorphisms is a homeomorphism and \( h \) is a composition of two homeomorphisms so it is a homeomorphism. Then \( h_U = (h_U \circ h_V^{-1}) \circ (h_V \mid U) = f \circ (h_V \mid U) \). This completes the proof. \( \square \)

**Corollary 1.6.14.** Let \( h_U : U \to (0, 1) \), \( h_V : V \to (0, 2) \) be two homeomorphisms and \( U \subset V \) and \( Fr_V(U) \) consists of exactly one point. Then there is a homeomorphism \( h : V \to (0, \beta) \) for some \( \beta \in \mathbb{R} \) such that \( h \mid U = h_U \).

**Definition 1.6.15.** [64] Let \( \{X_k \mid k \in I\} \) be a family of topological spaces indexed by a set \( I \) and \( X = \bigcup_{i \in I} X_i \). The weak topology on \( X \) with respect to \( \{X_k \mid k \in I\} \) is defined as follows:

\( U \subset X \) is open if and only if \( U \cap X_i \) is open in \( X_i \) for all \( i \in I \).

**Lemma 1.6.16.** [64] Let \( X = \bigcup_{k \in \mathbb{N}} X_k \) and \( Y = \bigcup_{k \in \mathbb{N}} Y_k \) such that for every \( k \), \( X_k \subset X_{k+1} \), and \( Y_k \subset Y_{k+1} \). Moreover assume that for every \( k \) there is \( g_k : Y_k \to X_{k+1} \) such that \( g_k \circ h_k = i_k : X_k \to X_{k+1} \), where \( h_k : X_k \cong Y_k \) is a homeomorphism for all \( k \in \mathbb{N} \). Then \( X \cong Y \).
Lemma 1.6.17. The only connected subspaces of $\mathbb{R}$ are intervals.

Proof. Let $C$ be a connected subspace of $\mathbb{R}$ and $x, y \in C$ such that $x < y$. If $z \notin C$ where $x < z < y$, then $C$ can be decomposed into two disjoint open subspaces $C \cap (-\infty, z)$ and $C \cap (z, \infty)$. This contradicts connectedness of $C$ so $z \in C$. Therefore $C$ is homeomorphic to $[0, 1]$, $(0, 1)$ or $(0, 1)$.

Lemma 1.6.18. Suppose that $X = \bigcup_{k \in \mathbb{N}} X_k$ where $\{X_k\}_{k \in \mathbb{N}}$ is an increasing sequence of open subsets of $X$ such that $X_k \cong \mathbb{R}$ for all $k \in \omega$, then $X \cong \mathbb{R}$.

Proof. We prove the Lemma by using induction and Lemma 1.6.16. It is clear from the assumption that $X_1 \cong \mathbb{R}$. So there is a homeomorphism $h_1 : X_1 \to (a_1, b_1)$ for some real numbers $a_1$ and $a_2$. Now assume that for every natural number $k \leq n$ there is a homeomorphism $h_k : X_k \to (a_k, b_k)$ for some real numbers $a_k, b_k$. By assumption we know that $X_{n+1} \cong \mathbb{R}$. So there are real numbers $c, d$ and a homeomorphism $f : X_{n+1} \to (c, d)$. The Lemma 1.6.13 provide us with a homeomorphism $h : X_{n+1} \to (\alpha, \beta)$ for some real numbers $\alpha$ and $\beta$ such that $h_{n+1} | X_n = h_n$, and $(c, d) \subset (\alpha, \beta)$. Now let $\alpha = a_{n+1}$ and $\beta = b_{n+1}$. So we have a sequence of homeomorphisms $\{h_k : X_k \to (a_n, b_n) | k \in \mathbb{N}\}$ such that $h_{k+1} | X_k = h_k$. Now letting $h : X \to \bigcup_{k \in \mathbb{N}} (a_k, b_k)$ be defined by $h | X_k = h_k$ we get a homeomorphism $h : X \to \bigcup_{k \in \mathbb{N}} (a_k, b_k)$. Note that $\bigcup_{k \in \mathbb{N}} (a_k, b_k)$ is an open interval of $\mathbb{R}$ by using Lemma 1.6.17. Hence $\bigcup_{k \in \mathbb{N}} (a_k, b_k)$ is homeomorphic to $\mathbb{R}$. So $X \cong \mathbb{R}$.

The following theorem by Morton Brown generalises the result of Lemma 1.6.18 to an arbitrary finite dimension but the proof is more complicated. For a short proof of the following theorem see [72].

Theorem 1.6.19 (Morton Brown). [8] Suppose that $X = \bigcup_{k \in \omega} X_k$ where $\{X_k\}_{k \in \omega}$ is an increasing sequence of open subsets of $X$ and $X_k \cong \mathbb{R}^n$ for all $k \in \omega$, then $X \cong \mathbb{R}^n$. 53
**Corollary 1.6.20.** Any non-compact, $\sigma$-compact one dimensional manifold $M$ without boundary is homeomorphic to the real line.

*Proof.* If $M$ is covered by finitely many charts, then the theorem is the result of Theorem 1.6.12. The manifold $M$ is $\sigma$-compact so it may be covered by countably many charts. Then by Corollary 1.6.18, $M$ is homeomorphic to the real line. \hfill $\square$

**Lemma 1.6.21.** Every connected proper subspace of $S^1$ is homeomorphic to an interval of $\mathbb{R}$.

*Proof.* Let $C$ be a proper connected subspace of $S^1$. Choose an arbitrary element $p \in S^1 \setminus C$. $S^1 \setminus \{p\}$ is homeomorphic to $\mathbb{R}$ so $C$ is homeomorphic to a connected subspace of $\mathbb{R}$. Using Lemma 1.6.17 we see that $C$ is homeomorphic to an interval. \hfill $\square$

**Theorem 1.6.22.** Any connected, metrisable manifold with boundary of dimension one is homeomorphic to either $[0,1]$ or $(0,1)$.

*Proof.* Let $M$ be a one dimensional manifold with boundary $\partial M$. Define the manifold $N = \frac{M \times \{1\} \cup M \times \{2\}}{\sim}$ where for all $x \in \partial M$ we have $(x,1) \sim (x,2)$. In Definition 1.4.44 this space is called the double of $M$, and Lemma 1.4.45 asserts that $N$ is a manifold without boundary with the same dimension as $M$.

So $N$ is homeomorphic to either $S^1$ or $\mathbb{R}$. $M$ is a connected subspace of $N$ and Lemmas 1.6.17 and 1.6.21 show that the only connected subspaces of $\mathbb{R}$ and $S^1$ are homeomorphic to $[0,1]$, $[0,1)$, $\mathbb{R}$ or $S^1$. But $S^1$ and $\mathbb{R}$ have no boundary so the only choices are $[0,1)$ and $[0,1]$. This completes the proof. \hfill $\square$

**Corollary 1.6.23.** Any compact, connected one dimensional manifold with boundary is homeomorphic to $[0,1]$. 

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Corollary 1.6.24. Any connected, metrisable non-compact one dimensional manifold with boundary is homeomorphic to $[0,1)$.

We have already proved that if a manifold $M$ has an atlas $\mathcal{A} = \{(U, h_U), (V, h_V)\}$, then the domain of each of those charts say $U$ has either empty boundary, a boundary consisting of only one point or a boundary consisting of two points. Now we show that this is the case for every connected Hausdorff one dimensional manifold.

Lemma 1.6.25. If $\mathcal{A}$ is an atlas for a topological manifold $M$, and $\mathcal{B} \subset \mathcal{A}$ with $\mathcal{B}$ non-empty, then either $\mathcal{B}$ is an atlas or there is $(U, \phi) \in \mathcal{A}$ such that $U$ meets both the subset which $\mathcal{B}$ covers and its complement.

Proof. Let $V = \bigcup_{U \in \mathcal{B}} U$. Then $V$ is open and non-empty. Suppose that there is no $(U, \phi) \in \mathcal{A}$ such that $U$ meets both $V$ and $M \setminus V$. Then for every $(U, \phi) \in \mathcal{A}$ either $U \subset V$ or $U \subset M \setminus V$. Since $\mathcal{A}$ is an atlas $M \setminus V$ must be covered by a subfamily of $\mathcal{A}$, and hence is open. So, since $M$ is connected, $M \setminus V$ is empty, i.e. $V = M$. It follows that $\mathcal{B}$ is an atlas. \hfill \Box

Lemma 1.6.26. If $\mathcal{A}$ is an atlas for a Hausdorff, connected one dimensional manifold $M$, then there is an atlas $\mathcal{B}$ for $M$ which satisfies one of the following:

1. There are two charts $(U, \phi)$ and $(V, \psi)$ in $\mathcal{B}$ such that $\psi(U \cap V)$ is an initial segment of $\mathbb{R}$, and $U \cap W = \emptyset$ for every chart $(W, \theta) \in \mathcal{B}$ where $W \neq V$.

2. There are two charts $(U, \phi)$ and $(V, \psi)$ in $\mathcal{B}$ such that $\psi(U \cap V)$ is a final segment of $\mathbb{R}$, and $U \cap W = \emptyset$ for every chart $(W, \theta) \in \mathcal{B}$ where $W \neq V$.

3. There are three charts $(U, \phi)$, $(V, \psi)$ and $(W, \theta)$ such that $W \cap V = \emptyset$, $U \cap W \neq \emptyset$, $U \cap V \neq \emptyset$, $\theta(U \cap W) = (-\infty, t)$ for some $t \in \mathbb{R}$ and $\psi(U \cap V) = (s, +\infty)$ for some $s \in \mathbb{R}$.

Proof. Let us agree to say that a chart $(U, \phi)$ is immediately to the left (right) side of a chart $(V, \psi)$ if $\psi(U \cap V)$ is an initial (final) segment of $\mathbb{R}$. Now consider the case that there
is a chart \((U_0, \phi_0) \in \mathcal{A}\) such that \((U_0, \phi_0)\) does not have any chart in its left side. Consider two points \(p, q \in U_0\) with \(\phi(p) < \phi(q)\). For every chart \((U, \phi) \in \mathcal{A}\) if \(U \cap U_0 \neq \emptyset\), then replace \(U\) by \(\phi^{-1}(\phi(p), +\infty)\) and let \(\mathcal{A}'\) be the new atlas. Note that \(\mathcal{A}'\) which consists of \((U_0, \phi_0)\), all modified charts \((\psi^{-1}(\phi(q), +\infty), \psi | \psi^{-1}(\psi(q), +\infty))\) and all charts \((W, \theta)\) such that \(W \cap U_0 = \emptyset\) is an atlas for \(M\). So far we have an atlas \(\mathcal{A}'\) which consists a chart \((U_0, \phi_0)\) such that no other chart of \(\mathcal{A}'\) is in the left side of \((U_0, \phi_0)\) and also there is an element \(p \in U_0\) which is not in the domain of any other chart in \(\mathcal{A}'\).

Now we introduce a new atlas \(\mathcal{B}\) for \(M\) which satisfies the conditions mentioned in 1.

Consider a chart \((U_1, \phi_1) \in \mathcal{A}'\) such that \(U_1 \cap U_0 \neq \emptyset\) and assume \(U_1\) is distinct and not a subset of \(U_0\). Consider \(x \in U_1 \setminus U_0\). Now consider an atlas \(\mathcal{B}\) consisting of \((U_0, \phi_0)\), \((U_1, \phi_1)\), all charts in \(\mathcal{A}'\) which have disjoint domain from \(U_1\) and \((\psi^{-1}(\psi(x), +\infty), \psi | \psi^{-1}(\psi(x), +\infty))\) where \((V, \psi) \in \mathcal{A}'\) and \(V \cap U_0 \neq \emptyset\). \(\square\)

So from now we always assume that every manifold \(M\) of dimension one has an uncountable atlas \(\mathcal{A}\) which satisfies conditions mentioned in the Lemma 1.6.26 and \(\mathcal{A}\) can not be reduced to a countable atlas.

Suppose \(\mathcal{A}\) is an atlas for \(M\) and there is a chart \((U_0, \phi_0) \in \mathcal{A}\) meeting only one other chart in \(\mathcal{A}\) such that \(U_0\) contains a point in no other chart of \(\mathcal{A}\). Set \(V_0 = U_0\). Following \(\phi_0\) by a homeomorphism \(\mathbb{R} \to (0, 1)\) gives us a homeomorphism \(h_0 : V_0 \to (0, 1)\). Let \((U_1, \phi_1)\) be the unique chart of \(\mathcal{A}\) meeting \((U_0, \phi_0)\). By the lemma \(U_1\) contains points of \(M \setminus U_0\).

Assume \(\alpha < \omega_1\) is such that \((V_\alpha, h_\alpha)\) has been chosen such that \(V_\alpha\) is a union of open sets obtained from charts of \(\mathcal{A}\) and \(h_\alpha : V_\alpha \to (0, \alpha + 1) \subset L_\alpha\) is a homeomorphism such that when \(\beta < \alpha\), then \(V_\beta \subset V_\alpha\) and \(h_\beta = h_\alpha\) restricted to \(V_\beta\). As \(V_\alpha\) is homeomorphic to a subset of \(\mathbb{R}\) it can not be all of \(M\) so by Lemma 1.6.25 there is \((U_\alpha+1, \phi_{\alpha+1}) \in \mathcal{A}\) with \(U_{\alpha+1}\) meeting both \(V_\alpha\) and \(M \setminus V_\alpha\).

Following \(\phi_{\alpha+1}\) by reflection of \(\mathbb{R}\) if necessary we may assume the transition \(\phi_{\alpha+1}h_\alpha\) preserves orientation. Let \(h_{\alpha+1}\) map \(\phi_{\alpha+1}^{-1}(\inf\{\phi_{\alpha+1}(t) \mid t \in U_{\alpha+1} \setminus V_\alpha\})\) to \(\alpha + 1\). Using sequential
continuity, for example, one can show $h_{\alpha+1}$ as defined so far is continuous.

Using $\phi_{\alpha+1}$ one can then extend $h_{\alpha+1}$ over all of $U_{\alpha+1} \setminus V_\alpha$ so as to obtain a homeomorphism $h_{\alpha+1} : V_{\alpha+1} \rightarrow (0, \alpha + 2)$.

Assume $\lambda$ is a limit ordinal and $(V_\alpha, h_\alpha)$ has been chosen for all $\alpha < \lambda$ such that $V_\alpha$ is a union of open sets obtained from charts of $A$, and $h_\alpha : V_\alpha \rightarrow (0, \alpha + 1) \subset L_\alpha$ is a homeomorphism such that when $\beta < \alpha$, then $V_\beta \subset V_\alpha$ and $h_\beta = h_\alpha$ restricted to $V_\beta$. Set $V_\lambda = \cup_{\alpha < \lambda} V_\alpha$.

Define $h_\lambda$ by $h_\lambda(x) = h_\alpha(x)$ when $x \in V_\alpha$.

So inductively we have defined $(V_\alpha, h_\alpha)$ for all $\alpha < \omega_1$ such that $V_\alpha$ is a union of open sets obtained from charts of $A$, and $h_\alpha : V_\alpha \rightarrow (0, \alpha + 1) \subset L_\alpha$ is a homeomorphism such that when $\beta < \alpha$, then $V_\beta \subset V_\alpha$ and $h_\beta = h_\alpha$ restricted to $V_\beta$. It is claimed that $\cup_{\alpha < \omega_1} V_\alpha = M$ and $h : \cup_{\alpha \in \omega_1} V_\alpha \rightarrow L_\alpha$ defined by $h \mid V_\alpha = h_\alpha$ for $x \in V_\alpha$ is a homeomorphism.

If $\cup_{\alpha < \omega_1} V_\alpha \neq M$ then the lemma 1.6.25 tells us that there is $(U, \phi) \in A$ such that $U$ meets both $\cup_{\alpha < \omega_1} V_\alpha$ and $M \setminus \cup_{\alpha < \omega_1} V_\alpha$. Much as above and replacing $\phi$ by $-\phi$ if necessary we consider $\phi^{-1}(\inf\{\phi(t) \mid t \in U \setminus \cup_{\alpha < \omega_1} V_\alpha\})$; call it $x$. Choose a sequence $\{x_n\}_{n \in \omega}$ in $\cup_{\alpha \in \omega_1} V_\alpha$ converging to $x$. Then $x_n \in \cup_{\alpha < \omega_1} V_\alpha$ so there is some $\alpha_n < \omega_1$ such that each $x_n$ is in $V_{\alpha_n}$.

Let $\alpha = \sup\{\alpha_n \mid n \in \omega\}$ and consider $h_{\alpha+1} : V_{\alpha+1} \rightarrow (0, \alpha + 1)$. Note that $h_{\alpha+1}(x_n) \in (0, \alpha)$ for all $n \in \omega$. Thus $h_{\alpha+1}(x) \in (0, \alpha] \subset (0, \alpha + 1)$ hence $x \in \cup_{\alpha \in \omega_1} V_\alpha$ a contradiction.

Note that we discussed the case when atlas $A$ of a one dimensional manifold $M$ satisfies the first condition in the Lemma 1.6.26. If $A$ satisfies the second or the third condition we can use similar argument and show that either $M \cong \mathbb{L}$ or $M \cong L_\alpha$.

So we have the following theorem:

**Theorem 1.6.27.** If $M$ is a Hausdorff, connected, one dimensional manifold without boundary with an uncountable atlas $A$ which is not reducible to a countable atlas, then $M$ is homeomorphic to either $\mathbb{L}$ or $L_\alpha$.
Theorem 1.6.28. Let $M$ be a connected, Hausdorff, one-dimensional manifold with boundary. If $M$ has an uncountable atlas $A$ which is not reducible to a countable atlas, then $M$ is homeomorphic to the closed long ray.

Proof. Let $M$ be a manifold with boundary $\partial M$ which satisfies the conditions of the theorem. The double of $M$, $D(M)$ is a manifold without boundary. Lemma 1.4.45 implies that $D(M)$ is either $\mathbb{L}$ or $\mathbb{L}_o$. On the other hand the only connected non metrisable subspaces of $\mathbb{L}_o$ or $\mathbb{L}$ are $(x, \omega_1)$, $\mathbb{L}$, $\mathbb{L}_o$, $(-\omega_1, x)$, $[x, \omega_1)$ and $(-\omega_1, x]$. The only manifolds with boundary among the mentioned connected subspaces of $\mathbb{L}$ and $\mathbb{L}_o$ are $[x, \omega_1)$ and $(-\omega_1, x]$ where $x \in \mathbb{L}$ or $x \in \mathbb{L}_o$. □

Lemma 1.6.29. If $x \in \mathbb{L}$, then $(x, \omega_1)$ and $(-\omega_1, x)$ are homeomorphic to $\mathbb{L}_o$. Similarly if $x \in \mathbb{L}_o$, then $(x, \omega_1)$ is homeomorphic to $\mathbb{L}_o$.

Proof. Let $x \in \mathbb{L}_o$. Consider $y \in \mathbb{L}_o$ such that $y > x$. Proposition 1.4.28 implies that there

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is an order preserving homeomorphism $h_1 : (0, y] \to (x, y]$. The identity map $id : [y, \omega_1) \to [y, \omega_1)$ is a homeomorphism. So by using the pasting lemma [47] it is easy to see that the function

$$h(x) = \begin{cases} h_1(x) & : x \in (x, y] \\ id(x) & : x \in [y, \omega_1) \end{cases}$$

is the desired homeomorphism.

If $x \in \mathbb{L}$ and $x < 0$, then $(-\omega_1, x) \cong (x, \omega_1) \subset \mathbb{L}_o$ by the homeomorphism $h(z) = -z$. So $(-\omega_1, x) \cong \mathbb{L}_o$. It is clear that for $x \in \mathbb{L}_o$, $-x < 0$, so $(-x, \omega_1) \cong (-\omega_1, x)$ by $h(z) = -z$ for every $z \in (-\omega_1, \omega_1)$. So as a final case let $-x \in \mathbb{L}$ and $-x < 0$. We can write $(-x, \omega_1) = (-x, 1] \cup [1, \omega_1)$. Moreover we have two homeomorphisms $(-x, 1] \cong (0, 1]$ and $[1, \omega_1) \cong [1, \omega_1)$ by using the pasting lemma we define the homeomorphism

$$h(x) = \begin{cases} h_1(x) & : x \in (-x, 1] \\ id(x) & : x \in [1, \omega_1) \end{cases}$$

$\square$

**Proposition 1.6.30.** Let $M$ be a nonmetrisable Hausdorff, connected manifold of dimension one. Then $M$ is homeomorphic to one of the following manifolds $\{\mathbb{L}_+, \mathbb{L}_o, \mathbb{L}\}$.

**Proof.** If $M$ has a countable atlas $\mathcal{A}$, then $M$ is metrisable. So assume $M$ has an uncountable atlas $\mathcal{A}$. We have already showed that if $| \mathcal{A} | = \omega_1$, then $M$ is either homeomorphic to $\mathbb{L}$, $\mathbb{L}_o$ or $\mathbb{L}_+$. Theorem 2.9 in [42] implies that the cardinality of a manifold is at most $\mathfrak{c}$. So the number of charts cannot exceed $\mathfrak{c}$. Therefore the only manifold with the mentioned properties in the theorem are $\mathbb{L}$, $\mathbb{L}_+$, and $\mathbb{L}_o$. $\square$

The table on the next page summarises some topological properties of one dimensional connected Hausdorff topological manifolds and the non-Hausdorff manifold line with two origins.
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**Table 1.1:** Topological properties of connected one manifolds
Chapter 2

Non-Hausdorff manifolds

“Non-Hausdorff spaces, often regarded as a technical nuisance, sometimes produce a global disaster.”
– Geometry of non-Hausdorff spaces and its significance for physics by M.Heller et al –

“Kelley’s book set the cat among the pigeons in 1955 by daring to omit the Hausdorff condition from many of its definitions.”
– On non-Hausdorff spaces by Ivan L. Reilly –

2.1 Motivation

Although there were some examples of non-Hausdorff topological spaces known in the first half of the 20th century such as Sierpinski space and Alexandroff’s Raüme, non-Hausdorff topological spaces attracted the interest of few topologists [57].
Ivan L.Reilly in [57] writes “Until about 1950 it seemed that, with few exceptions, topolo-
gists had a theorem which said all spaces are Hausdorff”.

Another example of a topology which is interesting is the Zariski topology of algebraic varieties. Surprisingly the Zariski topology was introduced shortly after Kelley’s book appeared in 1950. In general the Zariski topology is not a Hausdorff topology.

Only a few papers have been published about non-Hausdorff topological manifolds. In [28] Hájíček studied causality in non-Hausdorff space-times. Another two papers about non-Hausdorff manifolds that should be mentioned are [16] and [3]. In this chapter we study some examples of non-Hausdorff topological manifolds and describe three different kinds of points in the non-Hausdorff topological manifolds.

2.2 Filter and filterbase

Filters and filter bases are amongst the simplest structures on a set in set theory. In this section we study the system of filter bases on a set and the topological structure induced by such a system. Most of the results are well known and we just slightly modify some of them to study their relations to the topological manifolds.

**Definition 2.2.1.** A filter base $\mathcal{B}$ on a set $S$ is a non-empty family of subsets of $S$ such that:

1. $\emptyset \notin \mathcal{B}$.
2. If $B_1$ and $B_2$ are elements of $\mathcal{B}$, then there is an element $B \in \mathcal{B}$ such that $B \subset B_1 \cap B_2$.

If for each $B \in \mathcal{B}$, $x \in B$ and $\mathcal{B} \neq \emptyset$ we say $\mathcal{B}$ is based at $x$ and we write $\mathcal{B}_x$.

**Definition 2.2.2.** A filter $\mathcal{F}$ on the set $S$ is a non-empty family of subsets of $S$ such that:

1. $\emptyset \notin \mathcal{F}$.
2. If $F_1$ and $F_2$ are two members of $\mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$.
3. If $F_1 \in \mathcal{F}$ and $F_1 \subseteq F$, then $F \in \mathcal{F}$.
Definition 2.2.3. Let \( S \) be a set and \( \mathcal{B}_x \subseteq \mathcal{P}(S) \) a filter base based at \( x \). A family \( \mathcal{B} = \{ \mathcal{B}_x : x \in X \} \) is a system of neighbourhoods for \( S \) if it satisfies the following condition: if \( B \in \mathcal{B}_x \) and \( y \in B \), then there is \( B' \in \mathcal{B}_y \) such that \( B' \subseteq B \).

Definition 2.2.4. (Induced \( \mathcal{B} \)-topology) If \( \mathcal{B} \) is a system of neighbourhoods on the set \( S \) a subset \( U \) of \( S \) is \( \mathcal{B} \)-open if and only if \( U \) is the empty set or for every \( x \in U \) there is an element \( B \in \mathcal{B}_x \) such that \( B \subseteq U \). We denote all the \( \mathcal{B} \)-open subsets of \( S \) by \( \tau_\mathcal{B} \) or \( \tau_\mathcal{B}(S) \).

Lemma 2.2.5. \( \tau_\mathcal{B} \) is a topology.

Proof. Note that the empty-set is in \( \tau_\mathcal{B} \) by definition. To see that \( S \) belongs to \( \tau_\mathcal{B} \) for every \( x \in S \), \( \mathcal{B}_x \neq \emptyset \) choose an element \( B \in \mathcal{B}_x \). Then \( x \in B \subseteq S \) so \( S \) is open. To see that \( \tau_\mathcal{B} \) is closed under finite intersections let \( U \) and \( V \) be two elements of \( \tau_\mathcal{B} \), and \( W = U \cap V \). If \( x \in W \), then there are \( B_1 \subseteq U \) and \( B_2 \subseteq V \) such that \( x \in B_1 \in \mathcal{B}_x \) and \( x \in B_2 \in \mathcal{B}_x \). \( \mathcal{B}_x \) is a filter base so there is an element \( B_3 \in \mathcal{B}_x \) and \( B_3 \subseteq B_1 \cap B_2 \). Hence \( x \in B_3 \subseteq U \cap V \) and \( B_3 \in \mathcal{B}_x \). So \( W \) is open.

To see that \( \tau_\mathcal{B} \) is closed under arbitrary unions assume that \( \{ U_\alpha : \alpha \in I \} \) is a family of \( \mathcal{B} \)-open subsets of \( S \). For an arbitrary element \( x \in U = \cup_{\alpha \in I} U_\alpha \) there is \( \alpha_0 \in I \) such that \( x \in U_{\alpha_0} \) and \( U_{\alpha_0} \) is an open subset of \( S \) so there is \( B \in \mathcal{B}_x \) such that \( B \subseteq U_{\alpha_0} \subseteq U \). This completes the proof.

Definition 2.2.6. Let \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) be two filter bases on the set \( S \). We say \( \mathcal{B}_1 \) is finer than (or approximates) \( \mathcal{B}_2 \) if and only if for every \( B_2 \in \mathcal{B}_2 \) there is an element \( B_1 \in \mathcal{B}_1 \) such that \( B_1 \subseteq B_2 \) and we write \( \mathcal{B}_2 \preceq \mathcal{B}_1 \). If \( \mathcal{B}_2 \preceq \mathcal{B}_1 \) and \( \mathcal{B}_1 \preceq \mathcal{B}_2 \) we say the two filter bases are equivalent and we write \( \mathcal{B}_2 \cong \mathcal{B}_1 \).
2.3 Non-Hausdorff topological spaces

In this section we develop the terminology that we need to study non-Hausdorff manifolds. A topological space $X$ is non-Hausdorff if there is at least one pair $(x, y)$ of distinct elements of $X$ such that for any pair of neighborhoods $U$ and $V$ of $x$ and $y$ respectively, $U \cap V \neq \emptyset$.

Notation 2.3.1. Let $\mathcal{N}_x$ be open neighbourhood base at $x$ for a topological space $X$ as defined in [78].

Definition 2.3.2. Let $\mathcal{N}_x$ be open neighbourhood base at $x$ for a topological space $X$. For a cardinal number $\alpha$ we say $x$ is $\alpha$-adjacent to $y$ if each pair of open sets $(U, V) \in \mathcal{N}_x \times \mathcal{N}_y$ have $|U \cap V| \geq \alpha$. If $\alpha = 1$ we use the term adjacent instead of 1-adjacent.

$$adj(x) = \{y \in X : (\forall U \in \mathcal{N}_x)(\forall V \in \mathcal{N}_y)(U \cap V \neq \emptyset)\}$$

$$adj_\alpha(x) = \{y \in X : (\forall U \in \mathcal{N}_x)(\forall V \in \mathcal{N}_y)(|U \cap V| \geq \alpha)\}$$

Notation 2.3.3. If $x \in adj(y)$ or $y \in adj(x)$, then we write $x \equiv y$.

If $x \in adj_\alpha(y)$, equivalently $y \in adj_\alpha(x)$, then we write $x \equiv_\alpha y$.

Definition 2.3.4. If $X$ is a topological space and $y \in X \setminus adj(x)$ we say $x$ is apart from $y$.

$$apart(x) = X \setminus adj(x)$$

Definition 2.3.5. A point in a topological space $X$ is called regular if $adj(x) = \{x\}$, otherwise it is singular.

Definition 2.3.6. If $x$ is a singular point, then $|adj(x)|$ is called the singularity number of $x$. 

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Definition 2.3.7. If $Y \subseteq X$ is a topological space we define:

$$
Sing_X(Y) = \{ y \in Y : |\text{adj}(y)| > 1 \},
$$

$$
Reg_X(Y) = \{ y \in Y : |\text{adj}(y)| = 1 \}.
$$

We usually suppress the dependence on $X$ and write $Sing(Y)$ and $Reg(Y)$ instead of $Sing_X(Y)$ and $Reg_X(Y)$.

Definition 2.3.8. If $A, B$ are two subspaces of a topological space $X$ we define:

$$
\text{Adj}_A(B) = \{ a \in A | a \equiv b \text{ for some } b \in B, b \neq a \}.
$$

Definition 2.3.9. A subset $S$ of a topological space $X$ is called neat if and only if $\text{Adj}_S(S) = \emptyset$. If $\text{Adj}_X(S) = \emptyset$ we say $S$ is a clean subspace of $X$.

Definition 2.3.10. A topological space $X$ is locally Hausdorff if and only if each $p \in X$ has a Hausdorff neighborhood.

Lemma 2.3.11. A topological space $X$ is locally Hausdorff if and only if each $p \in X$ has a neat neighborhood.

Definition 2.3.12. A topological space $X$ is called US or semi-Hausdorff space if any sequence of elements of $X$ converges to at most one point.

Hausdorff topological spaces are semi-Hausdorff (see [78] page 86).

Definition 2.3.13. A topological space $X$ is called a KC space if compact subspaces of $X$ are closed.

Any KC topological space is US and in first countable topological spaces the KC and US properties are equivalent to the Hausdorff property (see [75] or [76]).
2.4 Non-Hausdorff manifolds

In this section we use the tools that we introduced in the previous section to explore neighbourhoods of singular points in non-Hausdorff manifolds. We realise two different adjacency types among singular points.

**Definition 2.4.1.** A non-Hausdorff manifold (or $T_1$-manifold) $M$ is a locally Euclidean, connected and second countable non-empty topological space such that $\text{Adj}_M(M) \neq \emptyset$.

The following pattern that we call sewing appears frequently in non-Hausdorff topological spaces.

**Example 2.4.2.** (Sewing a set to a topological space) Let $S \neq \emptyset$ be a set, $(X, \tau)$ a $T_1$ topological space and $f : S \to X$ a function. $B = \tau \cup \{(U \cup \{p\}) \setminus \{f(p)\} : p \in S$ and $U \in \tau$ contains $f(p)\}$ is a basis for a topology on the set $X \coprod S = Y$ (disjoint union of $S$ and $X$). Moreover elements of $f^{-1}(p)$ where $p \in X$ are adjacent to $p$.

**Proof.** We know that $\tau$ is a topology on $X$ so $\cup \tau = X$. If $p \in S$, then we consider an arbitrary element of $\tau$ which contains $f(p)$. So $p \in (U \cup \{p\}) \setminus \{f(p)\}$ where $f(p) \in U$ and $U \in \tau$. It shows that $B$ is a cover for $X \coprod S$. Let $B_1$ and $B_2$ be two elements of $B$. We realise the following cases:

If $B_1$ and $B_2$ are both elements of $\tau$ for $x \in B_1 \cap B_2$ there is a neighbourhood $B_3 \in \tau$ of $x$ and $B_3 \subseteq B_1 \cap B_2$. $B_3 \in B$ since $\tau \subseteq B$.

If $B_1 \notin \tau$ and $B_2 \notin \tau$, then there are $U_1 \in \tau$ and $U_2 \in \tau$ such that $B_1 = (U_1 \setminus \{x\}) \cup \{f(x)\}$ and $B_2 = (U_2 \setminus \{y\}) \cup \{f(y)\}$. Then $B_1 \cap B_2 \subseteq (U_1 \setminus \{x\}) \cap (U_2 \setminus \{y\})$ so there is $B_3 \subseteq (U_1 \setminus \{x\}) \cap (U_2 \setminus \{y\})$ such that $B_3 \in \tau$ since $\tau$ is a topology on $X$. If $B_1 \in \tau$ but $B_2 \notin \tau$, then there is $U \in \tau$, $x \in S$ such that $B_2 = U \setminus \{f(x)\} \cup \{x\}$ and $B_1 \cap U = B_1 \cap B_2$. So there is an element $B_3 \in \tau$ such that $B_3 \subseteq U \cap B_1 = B_2 \cap B_1$.

Finally we consider two elements $B_1$ and $B_2$ of $B$ which do not have empty intersection and
\[ B_1 = (U \setminus \{x\}) \cup \{f(x)\} \quad \text{and} \quad B_2 = (V \setminus \{y\}) \cup \{f(y)\}. \]

In this situation \( U \setminus \{x\} \in \tau \) and \( V \setminus \{y\} \in \tau \) so there is \( W \in \tau \) such that \( W \subset (U \cap V) \setminus \{x, y\} \). This completes the proof. \( \square \)

**Definition 2.4.3.** Let \( M \) be a non-Hausdorff topological manifold and \( x \cong y \). We say \( x \) and \( y \) are strongly adjacent if and only if any sequence \( \{x_n\} \) in \( M \setminus \{x, y\} \) which converges to \( x \), also converges to \( y \) and vice versa. If \( x \) and \( y \) are strongly adjacent we write \( x \cong^n y \) or \( x \in \text{sadj}(y) \). If \( x \) and \( y \) are two adjacent points but they are not strongly adjacent we call them weakly adjacent and we write \( x \in \text{wadj}(y) \).

**Definition 2.4.4.** Let \( X \) be a first countable, topological space and \( x \in X \). We say the sequence \( \{U_n\}_{n \in \omega} \) of open non-empty subsets of \( X \), which are not neighbourhoods of \( x \) tends to \( x \) if and only if every neighbourhood \( N \) of \( x \) contains a tail of the sequence of open sets.

**Lemma 2.4.5.** Let \( M \) be a non-Hausdorff topological manifold and \( x \cong y \). Then, there is a sequence \( \{U_n\}_{n \in \omega} \) which converges to both \( x \) and \( y \).

**Proof.** Every manifold is first countable so without loss of generality we assume that \( \{U_n|n \in \omega \text{ and } U_n \cong \mathbb{R}^n\} \) and \( \{V_n|n \in \omega \text{ and } V_n \cong \mathbb{R}^n\} \) are nested open base for \( x \) and \( y \) respectively, where \( x \cong y \).

Note that \( \{U_n|n \in \omega \text{ and } U_n \cong \mathbb{R}^n\} \) and \( \{V_n|n \in \omega \text{ and } V_n \cong \mathbb{R}^n\} \) exist because \( M \) is a locally Euclidean space. \( x \cong y \) so \( U_0 \cap V_0 \neq \emptyset \). Let \( p_0 \) be an arbitrary element of \( U_0 \cap V_0 \).

Consider an open neighbourhood \( W_0 \) of \( p_0 \) such that \( W_0 \subset U_0 \cap V_0 \). If \( p_n \) and \( W_n \) are chosen we define \( p_{n+1} \) to be an arbitrary element of \( U_n \cap V_n \cap W_n \) and \( W_{n+1} \) to be a Euclidean neighbourhood of \( p_{n+1} \) such that \( W_{n+1} \subset W_n \cap U_n \cap V_n \). The sequence \( \{W_n\} \) is the desired sequence. \( \square \)

Figures 2.1 and 2.2 illustrate two non-Hausdorff manifolds which have adjacent points which are not strongly adjacent. Figure 2.3 illustrates strongly adjacent points.
Figure 2.1: Splitting lines

Figure 2.2: Three splitting lines

Figure 2.3: Strongly adjacent points
We defined two points $p$ and $q$ of a non-Hausdorff space to be adjacent points if and only if every open neighbourhood $U$ of $p$ and every open neighbourhood $V$ of $q$ has non-empty intersection. The following lemma shows that for non-Hausdorff topological manifolds being adjacent is equivalent to being $c$-adjacent.

**Lemma 2.4.6.** If $M$ is an $n$-dimensional non-Hausdorff topological manifold, then any two adjacent points $x$ and $y$ of $M$ are $c$-adjacent, where $c$ is the cardinality of the continuum.

**Proof.** Let $U$ and $V$ be two open sets containing $x$ and $y$ respectively such that $U \cap V \neq \emptyset$. If $p \in U \cap V$, there is an open neighbourhood of $p$ contained in $U \cap V$ and homeomorphic to $\mathbb{R}^n$. To see this consider two charts $(U, \phi)$ and $(V, \psi)$. $p \in \phi(U \cap V) \subset \mathbb{R}^n$ so there is an open ball $D^n \subset \mathbb{R}^n$ which contains $\phi(p)$ which is contained $\phi(U \cap V)$. Then let $W = \phi^{-1}(D^n) \subset U \cap V$. $|W| = c$ since it is homeomorphic to $\mathbb{R}^n$ so $|U \cap V| \geq c$. \hfill $\square$

Note that in Lemma 2.4.6 if $U$ and $V$ are Euclidean neighbourhoods, then $|U \cap V| \leq |U|$ and hence by the Cantor-Schröder-Bernstein theorem $|U \cap V| = c$.

**Proposition 2.4.7.** If $M$ is a non-Hausdorff topological manifold, then $x \asymp y$ if and only if $x \asymp_c y$.

**Lemma 2.4.8.** Suppose $M$ is a non-Hausdorff topological manifold and $x \asymp y$, then there is no chart whose domain contains both $x$ and $y$.

**Proof.** Suppose $(U, \phi)$ is a chart and $x, y \in U$. Then $\phi(x)$ and $\phi(y)$ are two elements of $\mathbb{R}^n$ where $n$ is the dimension of the manifold so there are disjoint open balls $B_1$ and $B_2$ in $\mathbb{R}^n$ such that $\phi(x) \in B_1$ and $\phi(y) \in B_2$. On the other hand $\phi^{-1}(B_1) \cap \phi^{-1}(B_2) = \emptyset$, where $\phi^{-1}(B_1)$ and $\phi^{-1}(B_2)$ are open subsets of $M$ containing $x$ and $y$ respectively. This contradicts that $x$ and $y$ are adjacent. This completes the proof. \hfill $\square$

**Proposition 2.4.9.** Non-Hausdorff manifolds are $T_1$. 

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Proof. If \( x \) and \( y \) are two non-adjacent points of a non-Hausdorff manifold \( M \), then there are open neighbourhoods \( U \) and \( V \) of \( x \) and \( y \) respectively such that \( U \cap V = \emptyset \). So \( x \notin V \) and \( y \notin U \).

Let \( x \) and \( y \) be two adjacent points of the manifold \( M \). There is a chart with domain \( U \) which contains \( x \). Then \( y \) does not belong to \( U \). Similarly there is a chart with the domain \( V \) which contains \( y \). Then \( x \) does not belong to \( V \). \( \square \)

**Lemma 2.4.10.** The relation \( \simeq \) is symmetric and reflexive on a non-Hausdorff topological manifold \( M \).

**Proof.** For each \( x \in M \), \( x \simeq x \) because for every neighbourhood \( U \) of \( x \) clearly \( x \in U \). For every neighbourhood \( U \) of \( x \) and every neighbourhood \( V \) of \( y \), where \( U \cap V \neq \emptyset \), \( V \cap U \neq \emptyset \). So \( x \simeq y \) implies \( y \simeq x \). \( \square \)

Figure 2.4 illustrates that \( \simeq \) is not transitive.

**Example 2.4.11.** Consider the set \( X = \mathbb{R} \times \{-1, 0, 1\} \). Each line \( X_1 = \mathbb{R} \times \{-1\} \), \( X_2 = \mathbb{R} \times \{0\} \) and \( X_3 = \mathbb{R} \times \{1\} \) has a natural order defined by \( (r, i) \leq (r', i) \) if and only if \( r \leq r' \) for a fixed \( i \in \{-1, 0, 1\} \).

Assume that every line \( X_i \) for \( i = -1, 0, 1 \) has the order topology. Now we define an equivalence relation \( \sim \) on \( X \) as follows:

\((x, -1) \sim (x, 0)\) for \( x < 0 \) and \((x, 0) \sim (x, 1)\) for \( x > 0 \). The topological space \( X/\sim \) with the quotient topology is a non-Hausdorff topological manifold. Moreover \((0, -1) \simeq (0, 0)\) and \((0, 0) \simeq (0, 1)\) but \((0, -1)\) is not adjacent to \((0, 1)\).

**Lemma 2.4.12.** \( \simeq^n \) is an equivalence relation on a manifold \( M \).

**Proof.** If \( x \in M \) and \( p_n \to x \), then \( p_n \to x \). So \( \simeq^n \) is reflexive.

If \( p_n \to y \) and \( x \simeq^n y \), then it is clear from the definition that \( p_n \to x \). So \( y \simeq^n x \).
If $p \equiv^n q$, $q \equiv^n r$ and $x_n \to p$, then $x_n \to q$ and consequently $x_n \to r$. Similarly if $x_n \to r$, then $x_n \to p$. So $\equiv^n$ is an equivalence relation.

\begin{definition}
Let $X$ be a topological space. By a deleted neighbourhood $U$ of $x \in X$ we mean an open subspace $U$ of $M$ such that $U \cup \{x\}$ is an open neighbourhood of $x$ but $U$ is not a neighbourhood of $x$.
\end{definition}

\begin{definition}
Let $X$ be a non-Hausdorff topological space and $x \equiv y$. By a pitted neighbourhood $U$ of $x$ and $y$ in $X$ we mean an open subspace $U$ of $M$ such that $U \cup \{x\}$ is an open neighbourhood of $x$, $U \cup \{y\}$ is an open neighbourhood of $y$ and $x, y \notin U$. In this case $x$ and $y$ are called cores of $U$.
\end{definition}

In Figure 2.5 $U$ is a pitted neighbourhood of $x$ and $y$ since $U \cup \{x\}$ is a neighbourhood for $x$ and $U \cup \{y\}$ is a neighbourhood for $y$.

In Figure 2.6 two points $x$ and $y$ are adjacent and $U$ is a deleted neighbourhood of $x$ but it is not a pitted open neighbourhood of $x$ and $y$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.4.png}
\caption{Non-transivity of $\equiv$}
\end{figure}
Figure 2.5: Pitted neighbourhood of $x$ and $y$

Figure 2.6: Non-pitted neighbourhood of $x$ and $y$
Example 2.4.15. Consider the real line with two origins $x$ and $y$. It can be seen from the Figure 2.5 that $\mathbb{R} \setminus \{x, y\}$ is an example of a pitted neighbourhood for $x$ and $y$.

If $M \neq \emptyset$ is a manifold of dimension $n$, then for every point $p \in M$ there is a neighbourhood base $\mathcal{N}_p$ such that every element of $\mathcal{N}_p$ is homeomorphic to the Euclidean space $\mathbb{R}^n$.

To see this consider a homeomorphism $h : U \to \mathbb{R}^n$ where $U$ is an open neighbourhood of $p$. So $\{h^{-1}(B(p,n)) \mid n \in \mathbb{N}\}$ where $B(p, n) = \{x \in U \mid \|h(x) - 0\| < \frac{1}{n}\}$ is the desired neighbourhood base around $p$.

Every point $p$ of a locally Euclidean space $X$ has an open Euclidean neighbourhood so it is locally Hausdorff.

Theorem 2.4.16. Let $M \neq \emptyset$ be a non-Hausdorff topological manifold and $x \approx y$. Assume $\mathcal{N}_x$ and $\mathcal{N}_y$ are two open neighbourhood bases of Euclidean neighbourhoods of $x$ and $y$ respectively. Then $x \in \bigcap_{V \in \mathcal{N}_y} \partial M V$ and $y \in \bigcap_{U \in \mathcal{N}_x} \partial M U$, where $\partial M U$ is the topological boundary of $U$ in $M$.

Proof. For every $U \in \mathcal{N}_y$, $x \notin U$ since $U$ is locally Euclidean so Hausdorff. Similarly for every $V \in \mathcal{N}_x$, $y \notin V$. On the other hand $U \cap V \neq \emptyset$. Hence $U \cap V$ is a non-empty subset of $M$ and $x, y \notin U \cap V$ hence $x \in \partial U V$ and $y \in \partial M V$. So $x \in \bigcap_{V \in \mathcal{N}_y} V$ and $y \in \bigcap_{U \in \mathcal{N}_x} U$ since $U$ and $V$ are arbitrary elements of $\mathcal{N}_y$ and $\mathcal{N}_x$ respectively. \hfill \Box

Note that if $M$ is a manifold and $x \in M$, and $U$ a neighbourhood of $x$, then there is a sequence $\{B_n\}$ of Euclidean neighbourhoods of $x$ such that $B_n \subset U$ for all $n \in \omega$.

Lemma 2.4.17. Let $M$ be a non-Hausdorff manifold and $x \approx y$. Then $x \approx^n y$ if and only if every deleted neighbourhood of $x$ contains a deleted neighbourhood of $y$ and every deleted neighbourhood of $y$ contains a deleted neighbourhood of $x$.

Proof. ($\Rightarrow$) First we assume that $x \approx^n y$ and show that every deleted open neighbourhood $U$ of $x$ contains a deleted open neighbourhood $V$ of $y$. 
Assume to the contrary that there is an open deleted neighbourhood $U$ of $x$ such that for every deleted open neighbourhood $V$ of $y$, $U \cap V^c \neq \emptyset$. Consider a sequence $\{B_n\}$ of deleted nested open Euclidean neighbourhoods of $y$ such that $B_{n+1} \subset B_n$ for all $n \in \omega$. So for every $n \in \mathbb{N}$ we have $B_n \cap U^c \neq \emptyset$. For every $n \in \mathbb{N}$ we choose $p_n \in B_n \cap U^c$. Evidently $p_n \to y$ but $\{p_n\}$ does not converge to $x$ a contradiction.

$(\Leftarrow)$ Assume that every open deleted neighbourhood of $x$ contains an open deleted neighbourhood of $y$ and vice versa. Let $p_n \to x$ and $p_n \neq x$ for all $n \in \mathbb{N}$. We show that $p_n \to y$. Assume that $V$ is an open neighbourhood of $y$. So $V$ contains a deleted open neighbourhood $U$ of $y$ and $p_n \in U$ for all but finitely many $n \in \mathbb{N}$. Consequently $p_n \in V$ for all but finitely many $n \in \mathbb{N}$. So $p_n \to y$. So $x \equiv y$.

\begin{flushright}
\square
\end{flushright}

**Proposition 2.4.18.** Let $M$ be a non-Hausdorff manifold and $x \equiv y$. If $x \equiv y$, then every deleted neighbourhood of $x$ or $y$ is a pitted neighbourhood of $x$ and $y$.

**Proof.** Let $x \equiv y$. Consider an open deleted neighbourhood $U$ of $x$. By using the Lemma 2.4.17 there is a deleted neighbourhood $V$ of $y$ such that $V \subset U$. So $U \cup \{y\}$ is open. This shows that $U \cup \{x\}$ and $U \cup \{y\}$ are open neighbourhoods of $x$ and $y$ respectively. So $U$ is a pitted neighbourhood of $x$ and $y$.

\begin{flushright}
\square
\end{flushright}

**Theorem 2.4.19.** Let $M$ be a non-Hausdorff manifold and $x \equiv y$, then there is a path $\phi : I \to M$ such that $\phi(0) = x$ and $\phi(1) = y$, where $I$ is the closed unit interval.

**Proof.** Let $U$ be an open Euclidean neighborhood of $x$ and $V$ an open Euclidean neighbourhood of $y$. Pick an arbitrary point $z \in U \cap V$. Then there is a path $\alpha : I \to M$ such that $\alpha(0) = x$ and $\alpha(1) = z$. Similarly there is a path $\beta : I \to V$ such that $\beta(0) = z$ and
\( \beta(1) = y \). We define \( \phi : I \to M \) by

\[
\phi(t) = \begin{cases} 
\alpha(2t) & (t \in [0, \frac{1}{2}]) \\
\beta(2t - 1) & (t \in [\frac{1}{2}, 1])
\end{cases}
\]

We see that \( \alpha(1) = \beta(0) = z \) so \( \phi \) is a path from \( x \) to \( y \). □

**Definition 2.4.20.** (Hájíček) Let \( N \) be a connected non-Hausdorff manifold. An open subset \( M \subseteq N \) is called an \( H \)-submanifold of \( N \) if it is Hausdorff, connected and maximal.

We use \( \mathbb{H}_x(M) \) for the set of all \( H \)-submanifolds of \( M \) containing \( x \).

**Theorem 2.4.21.** (Hájíček) If \( M \) is a connected non-Hausdorff manifold, then for each point \( x \in M \), there is an \( H \)-submanifold of \( M \) containing \( x \).

### 2.4.1 Generalisation of the feather space

In [27] Haefliger and Reeb introduced an example of a one dimensional manifold which is called the feather space. For every point \( x \) in the feather space \( \text{adj}(x) \) consists of exactly two points.

Moreover every two adjacent points are weakly adjacent, i.e. for two adjacent points \( x \) and \( y \) of the feather space there is a sequence \( \{p_n\} \) such that \( p_n \) converges to only \( p \) or only to \( q \).

In this section we generalise the idea behind the feather and for every totally ordered space \( L \) we introduce the Haefliger-Reeb space of \( L \) or \( HR(L) \). Our definitions and terminology are the same as [3].

Let \( (L, \leq) \) be a linearly ordered topological space (LOTS). We define the relation \( < \) on the set \( HR(L) = \bigcup_{k \in \omega} L^{<k+1>} \), where \( L^{<k+1>} = \{(x_1, x_2, x_3, \ldots, x_k, x_{k+1}) | x_i \in L \text{ for } 1 \leq i \leq k \text{ and } x_1 < x_2 < x_3 < \cdots < x_k \leq x_{k+1}\} \) as follows:
Consider two elements $x$ and $y$ in $HR(L)$. We can write $x = (x_1, x_2, x_3, \ldots, x_{n-1}, x_n, x_{n+1})$ and $y = (y_1, y_2, y_3, \ldots, y_{n-1}, y_n, y_{n+1}, \ldots, y_{n+k}, y_{n+k+1})$, where $k \geq 0$.

**Definition 2.4.22.** Consider two points $x = (x_1, x_2, x_3, \ldots, x_n)$ and $y = (y_1, y_2, y_3, \ldots, y_m)$ of $HR(L)$. Then $x < y$ if and only if $n \leq m$ and either

- For every $k \in \{1, 2, 3, \ldots, n-1\}$ we have $x_k = y_k$, and $x_n < y_n$;

or

- For every $k \in \{1, 2, 3, \ldots, n\}$ we have $x_k = y_k$ and $x_n \leq y_{n+1}$.

**Lemma 2.4.23.** If $L$ is a linearly ordered topological space (LOTS), then the relation $\leq$ on $HR(L)$ is a partial order.

**Proof.** It is evidently clear from the definition 2.4.22 that $x < y$ implies $x \not< y$. So we prove the transitivity of $\prec$. Let $x = (x_1, x_2, x_3, \ldots, x_k)$, $y = (y_1, y_2, y_3, \ldots, y_m)$ and $z = (z_1, z_2, z_3, \ldots, z_n)$. Without loss of generality assume $k \leq m \leq n$.

Assuming $x < y$ and $y < z$ we show that $x < z$.

1. If $k = m = n$, then $x_i = y_i = z_i$ for $1 \leq i \leq n-1$ and $x_n < y_n < z_n$. So $x_n < z_n$ hence $x < z$.

2. If $k < m = n$, then $x_i = y_i = z_i$ for $1 \leq i \leq k-1$ and $x_k < y_{k+1} \leq z_{k+1}$. So $x_k < z_{k+1}$. Hence $x < z$.

3. If $k = m < n$, then $x_i = y_i = z_i$ for $1 \leq i \leq k-1$ and $x_k < y_k < z_{k+1}$. So $x_k < z_{k+1}$. Hence $x < z$.

4. If $k < m < n$, then we have $x_k \leq y_k \leq z_k$. Note that for every $i \in \{1, 2, 3, \ldots, k-1\}$ we have $x_i = y_i = z_i$ since $x < y < z$. We see that $x_k < y_k$ implies that $x < z$ since $y_k \leq z_k$.

Similarly $y_k < z_k$ implies $x < z$. So we consider the case where $x_k = y_k = z_k$.

We know that $x < y < z$ so $x_k \leq y_{k+1} \leq z_{k+1}$. Hence $x_k \leq z_{k+1}$ and consequently $x < z$.

This completes the proof. $\square$
One method of introducing a topology on a set is defining a neighbourhood system of open sets for every point of the set. Here we use the method described in [31]. We define a topology $\tau_{HR}$ on $HR(L)$ by defining the neighbourhood system for every point $p \in HR(L)$. We topologise the $HR(L)$ by defining an open neighbourhood system $N_x$ for every point $x = (x_i)_{i=1}^{i=n+1}$.

We distinguish two different type of points.

Consider $x = (x_1, x_2, x_3, \ldots, x_n, x_{n+1})$ and $y = (x_1, x_2, x_3, \ldots, x_n, x_n)$. Recall that $L$ is a linearly ordered topological space so there is a neighbourhood base $N_{x_{n+1}}$ of intervals at $x_{n+1}$ and neighbourhood base of intervals $N_{x_n}$ at $x_n$.

For every open interval $I \subset L$ such that $x_{n+1} \in I$ we define $I(x) = \{(x_1, x_2, x_3, \ldots, x_{n-1}, x_n, t)|t \in I\}$.

Similarly for every interval $I$ such that $x_n \in I$ we define $I(y) = \{(x_1, x_2, x_3, \ldots, x_{n-1}, x_n, t)|t \in I, t \geq x_n\} \cup \{(x_1, x_2, x_3, \ldots, x_{n-1}, t)|t \in I \text{ and } t < x_n\}$.

So we see from the definition of $I(p)$ that every point $p \in HR(L)$ has a neighbourhood $I(p)$. So for every $p \in HR(L)$, $N_p \neq \emptyset$. We also see from this definition that for every $p \in HR(L), p \in I(p)$.

Consider two open neighbourhoods $I_1(x)$ and $I_2(x)$ and let $p \in I_1(x) \cap I_2(x)$ where $I_1$ and $I_2$ are two open intervals of $L$ both containing $x_{n+1}$. Let $J = I_1 \cap I_2 \subset L$ so $x_{n+1} \in J$.

Now we see that $J(x) \subset I_1(x) \cap I_2(x)$. Similarly for all basic open neighbourhoods $I_1(y)$ and $I_2(y)$ we consider $J \subset I_1 \cap I_2$ so from the definition we see that $J(y) \subset I_1(y) \cap I_2(y)$.

Now if $p \in I(x)$, we can write $p = (x_1, x_2, x_3, \ldots, x_n, p_{n+1})$ where $p_{n+1} \in I$. If $p_{n+1} > x_{n+1}$, then we consider $J_1 = \{t \in I|t > x_{n+1}\}$ and if $p_{n+1} < x_{n+1}$, then we consider $J_2 = \{t \in I|t < x_{n+1}\}$. We see that $J_1(p) \subset I(x)$ and $J_2(p) \subset I(x)$. Similarly if $y \in I(y)$ and $p \in I(y)$ we can define $J_1(p)$ and $J_2(p)$ such that $J_1(p) \subset I(y)$ and $J_2(p) \subset I(y)$. So we have:

**Lemma 2.4.24.** If $L$ is a linearly ordered set, then $(HR(L), \tau_{HR})$ is a topological space.

If $L$ is a linearly ordered set, then $(HR(L), \tau_{HR})$ is not an order topology on $HR(L)$. 

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since every neighbourhood $U$ of $x = (x_1, x_2, x_3, \ldots, x_n, x_n)$ and every neighbourhood $V$ of $(x_1, x_2, x_3, \ldots, x_n)$ have nonempty intersection so $(HR(L), \tau_{HR})$ is not Hausdorff but every TOSET is Hausdorff.

**Lemma 2.4.25.** Let $L$ be a linearly ordered topological space, $p = (p_0, p_1, p_2, p_3, \ldots, p_{k-1})$ be a fixed element of $HR(L)$ and $I \subseteq L$ be an interval which contains $p_k$.

Then $S_{p,t} = \{(p_1, p_2, \ldots, p_{k-1}, x_k, x_k+1, \ldots, x_n) \in HR(L) | n \geq k, x_k \in I\}$ is an open subspace of $HR(L)$.

**Proof.** We show that every point of $S$ is an interior point.

Let $x = (p_1, p_2, p_3, \ldots, p_{k-1}, x_k, x_k+1, \ldots, x_n)$ from the definition of $HR(L)$ we see that $I(x)$ is an open subspace of $HR(L)$ and $I(x) \subseteq S$ so $x$ is an interior point. This shows that $S$ is an open subspace of $HR(L)$.

**Lemma 2.4.26.** Let $x$ and $y$ be two elements of $HR(L)$ and $L$ be a linear ordered set. Then $x \sim y$ if and only if $x = (x_1, x_2, x_3, \ldots, x_{n-1}, x_n)$ and $y = (x_1, x_2, x_3, \ldots, x_{n-1}, x_n, x_n)$ where $x_1 < x_2 < x_3 < \ldots < x_n$.

**Proof.** ($\Leftarrow$) First assume that $x = (x_1, x_2, x_3, \ldots, x_{n-1}, x_n)$ and $y = (x_1, x_2, x_3, \ldots, x_{n-1}, x_n, x_n)$ where $x_1 < x_2 < x_3 < \ldots < x_n$ we show that $x \sim y$.

Let $U$ be an open neighbourhood of $x$ and $V$ be an open neighbourhood of $y$. Every open neighbourhood of $x$ contains $\{(x_1, x_2, x_3, \ldots, x_{n-1}, t) | t \in I$ and $t \leq x_n\}$ for some open interval $I \subseteq L$ which contains $x_n$. On the other hand every open neighbourhood of $y$ contains $\{(x_1, x_2, x_3, \ldots, x_{n-1}, t) | t \in J$ and $t \leq x_n\}$ for some open interval $J \subseteq L$ which contains $x_n$. Consider $I \cap J$.

Now we see that $\{(x_1, x_2, x_3, \ldots, x_{n-1}, t) | t \in I \cap J$ and $t < x_n\}$ is a non-empty subspace of $U \cap V$. This shows that $x \sim y$.

($\Rightarrow$) Consider two points $x = (x_1, x_2, x_3, \ldots, x_m, x_{m+1})$ and $y = (y_1, y_2, y_3, \ldots, y_n, y_{n+1})$. Without loss of generality assume that $m \leq n$. We define $i = \min \{k \in \mathbb{N} | x_k \neq y_k\}$ and
\( p = (x_1, x_2, x_3, \ldots, x_{i-1}) \).

If \( i < m + 1 \) we consider two disjoint open intervals \( I, J \subset L \) such that \( x_i \in I \) and \( y_i \in J \).

We see that \( S_{p,I} \) and \( S_{p,J} \) are two disjoint open subspaces of \( HR(L) \). Hence \( x \) and \( y \) are not adjacent.

If \( i = m + 1 \), then we can write \( x = (x_1, x_2, x_3, \ldots, x_m, x_{m+1}) \) and \( y = (x_1, x_2, x_3, \ldots, x_m, x_{m+1}, x_{m+2}, \ldots, x_{n+1}) \). In this case we consider an arbitrary open interval \( I \) such that \( x_{m+1} \in I \) and \( U = \{ z \in HR(L) | z > (x_1, x_2, x_3, \ldots, x_{m+1}, x_{m+1}) \} \). It can be seen that \( I(x) \) and \( U \) are open disjoint subspaces of \( HR(L) \). So \( x \) and \( y \) are not adjacent unless \( y = (x_1, x_2, x_3, \ldots, x_{m+1}, x_{m+1}) \).

\[\Box\]

\textbf{Figure 2.7:} Feather space

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Definition 2.4.27. A partially ordered set $P$ is said to be dense in itself if and only if for every two elements $x$ and $y$ of $P$ where $x < y$ there is another element $z \in P$ such that $x < z < y$.

Lemma 2.4.28. If $L$ is a one dimensional manifold, then $HR(L)$ is a non-Hausdorff manifold.

Proof. Consider two points $x = (x_1, x_2, x_3, \ldots, x_n, x_{n+1})$ and $y = (x_1, x_2, x_3, \ldots, x_n, x_n)$ in $HR(L)$.

We know that $L$ is locally homeomorphic to $\mathbb{R}$ so assume $\phi : I \rightarrow \mathbb{R}$ is a homeomorphism where $I$ is an open interval of $L$ and $x_{n+1} \in I$. There is a natural homeomorphism $f$ from $I(x)$ onto $I$ defined by $(x_1, x_2, x_3, \ldots, x_n, t) \mapsto t$. We see that $\phi \circ f$ provides us with the desired homeomorphism.

Similarly if $y = (x_1, x_2, x_3, \ldots, x_n, x_n)$ in $HR(L)$ replacing $f$ by $g : I(y) \rightarrow I$ defined by $(x_1, x_2, x_3, \ldots, x_n, t) \mapsto t$ for $t > x_n$ and $(x_1, x_2, x_3, \ldots, x_{n-1}, t) \mapsto t$ for $t \leq x_n$ we get a homeomorphism from $I$ onto $I(y)$. This completes the proof.

Corollary 2.4.29. $HR(\mathbb{R})$, and $HR(\mathbb{L})$ are one dimensional non-Hausdorff manifolds.

The space $HR(\mathbb{R})$ is called the feather space. It is clear from the discussion above that for each point $p \in HR(L)$ we have $\overline{p} = \{x \in HR(\mathbb{R}) | x \equiv p\}$ consisting of exactly two points.

Lemma 2.4.30. $HR([0,1])$ is not compact.

Proof. Consider the countable nested open cover $\{\bigcup_{k \in \mathbb{N}} L^{<k} | n \in \omega\}$ of $HR(L)$. We see that for every $n \in \omega$, $\bigcup_{k=0}^n L^{<k}$ does not contain $(x_1, x_2, x_3, \ldots, x_{n+k}, x_{n+k+1})$ for $k \geq 2$. So there is no finite subcover for $\{\bigcup_{k \in \mathbb{N}} L^{<k} | n \in \omega\}$.

In the examples 2.4.32 and 2.4.33 we provide two examples of non-Hausdorff spaces which are not Haefliger-Reeb spaces but before that we introduce the Sorgenfrey circle which is helpful to understand 2.4.32 and 2.4.33 better.
Example 2.4.31. (Sorgenfrey circle) The Sorgenfrey circle is the unit circle with the topology which has \( \left\{ [e^{it}, e^{is}] \mid s, t \in \mathbb{R}, |s - t| < \pi, t < s \right\} \) as a basis for open sets, where \( [e^{it}, e^{is}) = \{ e^{ix} \mid x \leq s \} \).

Example 2.4.32. (Non-Hausdorff Double circle) Consider two concentric circles \( A \) and \( B \) of radius 1 and 2 respectively. We define a new topology \( \tau \) on \( X = A \cup B \) which makes \( X \) a non-Hausdorff manifold.

Let \( A = \{ e^{it} \mid 0 \leq t < 2\pi \} \) and \( B = \{ 2e^{it} \mid 0 \leq t < 2\pi \} \). Let \( B \) have the subspace topology of \( \mathbb{R}^2 \) and for every point \( x_t = e^{it} \) where \( t \in [0, 2\pi) \) define \( \varepsilon \)-basic open neighbourhood of \( e^{it_0} \) by \( U_\varepsilon(e^{it_0}) = \{ e^{it} \in A \mid t_0 \leq t < t_0 + \varepsilon \} \cup \{ e^{it} \in B \mid t_0 - \varepsilon < t < t_0 \} \) where \( 0 < \varepsilon < \pi \). In this example the outer circle is homeomorphic to the circle and the inner circle is homeomorphic to the Sorgenfrey circle. Note that in this example every point is singular. The outer circle and inner circle are neat subspaces. Moreover every two adjacent points are weakly adjacent i.e. there is a sequence which converges to a point \( p \) but it does not converge to another point \( e \) which is adjacent to \( p \).

Note that the only adjacent points are intersections of rays originated from circles with two circles and every two points on the same ray are adjacent.
The non-Hausdorff double-circle is not a manifold. To see this consider point $e^{it_0}$ with basic open neighbourhood $U_\epsilon(e^{it_0})$ as we defined in Example 2.4.32. $U_\epsilon(e^{it_0+\frac{\pi}{2}})$ is an open subset of $X$ so $U_\epsilon(e^{it_0}) \cap U_\epsilon(e^{it_0}) = [t_0 + \frac{\pi}{2}, t_0 + \epsilon)$ is not open in $(t_0 - \epsilon, t_0 + \epsilon)$.

**Example 2.4.33.** (Non-Hausdorff double circle with uncountable discrete subspace) Consider two concentric circles $A$ and $B$ of radius 1 and 2 respectively. We define a new topology $\tau$ on $X = A \cup B$ which makes $X$ a non-Hausdorff manifold.

Let $A = \{e^{it} | 0 \leq t < 2\pi\}$ and $B = \{2e^{it} | t \in [0, \pi)\}$. Let $B$ have the subspace topology of $\mathbb{R}^2$ and for every point $x_t = e^{it}$ where $t \in [0, 2\pi)$ define $\epsilon$-basic open neighbourhood of $p = e^{it_0}$ by $U_\epsilon(e^{it_0}) = \{2e^{it} \in B | t \in (t_0 - \epsilon, t_0 + \epsilon) \setminus \{t_0\}\} \cup \{e^{it_0}\}$ where $0 < \epsilon < \pi$. In this example the outer circle is homeomorphic to the circle and the inner circle is homeomorphic to a discrete space of cardinality $\mathfrak{c}$. Note that the inner circle is not a submanifold of $X$.

Note that in this example every point is singular. The outer circle and inner circle are neat.
subspaces. Moreover every two adjacent points are strongly adjacent i.e. every sequence which converges to a point \( p = e^{it_0} \) converges to \( q = 2e^{it_0} \). This space is a one dimensional non-Hausdorff manifold. Note that \( A \) is not a submanifold of dimension one.

Note that two points are adjacent if and only if they are on the intersection of a ray and two circles.

**Notation 2.4.34.** \( \mathcal{H}(X) = \{ h | h : X \rightarrow X \text{ is a homeomorphism } \} \), where \( X \) is a topological space.

**Definition 2.4.35.** A topological space \( X \) is homogeneous if and only if for each pair \( x, y \in X \) there is a homeomorphism \( h : X \rightarrow Y \) such that \( h(x) = y \).

**Lemma 2.4.36.** \( HR(L) \) is connected if and only if \( L \) is connected.

**Proof.** Assume that \( L \) is not connected. We show that \( HR(L) \) is not connected.

If \( L \) is not a connected ordered set, then there are two disjoint open subspaces \( L_1 \) and \( L_2 \) of \( L \) such that \( L = L_1 \cup L_2 \). Now we write \( HR(L) = \{ x \in HR(L) | x_1 \in L_1 \} \cup \{ x \in HR(L) | x_1 \in L_2 \} \). These two open subspaces of \( HR(L) \) are disjoint so \( HR(L) \) is not connected.

Now assume that \( L \) is connected we show that \( HR(L) \) is connected. Note that \( HR(L) = \bigcup_{k \in \mathbb{N}} L^{<k+1>} \) so we show that for every \( k \in \mathbb{N} \), \( \bigcup_{n<k} L^{<k+1>} \) for some \( n \in \mathbb{N} \) is a connected space.

Assume \( L \) is connected but \( HR(L)^n \) is not connected.

Consider \( n \in \omega \), and let \( HR(L)^n = \bigcup_{k<n+1} L^{<k+1>} \). If \( HR^n(L) = U \cup V \) where \( U \) and \( V \) are two open disjoint subspaces of \( HR(L)^n \) we see that every two adjacent points \( x \) and \( y \) either both belong to \( U \) or both belong to \( V \). On the other hand every subset \( X_k = \{ (x_1, x_2, x_3, \ldots, x_k, t) | x_1 < x_2 < \cdots < x_k, t \geq x_k \} \) is order isomorphic to \( \{ x \in L | x \geq x_k \} \) so connected. Hence \( X_k \subset U \) or \( X_k \subset V \). Moreover \( L \subset U \) or \( U \subset V \). Hence we have either \( HR(L)^n \subset U \) or \( HR(L)^n \subset V \). This shows that \( HR(L)^n \) is a connected subspace of \( HR(L) \).
Note that $HR(L) = \bigcup_{n \in \mathbb{N}} HR(L)^n$ and for every two natural numbers $m$ and $n$ $HR(L)^m \cap HR(L)^n \neq \emptyset$ so $HR(L)$ is connected.

**Lemma 2.4.37.** Let $X$ be a non-Hausdorff topological space and $h \in \mathcal{H}(X)$. Then $x \equiv y$ if and only if $h(x) \equiv h(y)$.

**Proof.** If $U$ and $V$ are two open neighbourhoods of $x$ and $y$ respectively, then $U \cap V \neq \emptyset$ if and only if $h(U) \cap h(V) = h(U \cap V) \neq \emptyset$. So self homeomorphisms preserve strong adjacency.

**Proposition 2.4.38.** If $X$ is a homogenous non-Hausdorff topological space and $h : X \rightarrow X$ is a homeomorphism, then $h(Sing(X)) = Sing(X)$.

**Proposition 2.4.39.** If $X$ is a homogenous non-Hausdorff topological manifold and $h : X \rightarrow X$ is a homeomorphism and $N$ is a neat (clean) subspace of $X$, then $h(N)$ is a neat (clean) subspace of $X$. 

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Chapter 3

Realcompactness and metrisability for a manifold

“One of the most impressive achievements in the mathematics of the past two centuries is the development of various remarkable techniques that can handle non flat spaces of various dimensions.”

— Roger Penrose —

3.1 Introduction

In differential topology, the celebrated Whitney embedding theorem states that any smooth $n$-dimensional metrisable manifold smoothly embeds in Euclidean space for some large enough integer $m$. Later Whitney improved his result and showed that $m = 2n$ was sufficient [36]. This number cannot be improved in general as the Klein bottle does not embed in $\mathbb{R}^3$. If we are just concerned about topological embeddings of metrisable topological manifolds into a finite power of the real line $\mathbb{R}$, then the question is more easier to answer.
The embedding Theorem of dimension theory states that every separable metric space of covering dimension at most \( n \) can be embedded in \( \mathbb{R}^{2n+1} \)[48]. The covering dimension of a metrisable topological manifold is equal to its dimension as a manifold [46]. So every metrisable topological manifold of dimension \( n \) embeds in \( \mathbb{R}^{2n+1} \).

Since every metrisable topological manifold \( M \) is paracompact and Hausdorff, for every indexed open cover of \( M \) there is a partition of unity subordinate to the open cover of \( M \) [44]. Consequently \( M \) admits a function \( f : M \to \mathbb{R} \) such that for every \( r \in \mathbb{R} \), \( f^{-1}((-\infty, r]) \) is a compact subset of \( M \) [44]. Now assuming that \( e_0 : M \to \mathbb{R}^{2n+1} \) is an embedding we establish an embedding \( e : M \to \mathbb{R}^{2n+2} \) defined by \( e(x) = (f(x), e_0(x)) \). The embedding \( e \) is a proper map so \( e(M) \) is closed [44].

This shows that every paracompact topological manifold embeds in \( \mathbb{R}^{2n+2} \) where \( e(M) \) is a closed subspace of \( \mathbb{R}^{2n+2} \). Paracompactness is equivalent to the metrisability for topological manifolds as we saw in the Theorem 1.3.4 so every metrisable topological manifold is homeomorphic to a closed subspace of \( \mathbb{R}^m \), where \( m \) is an integer greater or equal to \( 2\dim(M) + 2 \).

In this chapter we address the question: if a manifold embeds as a closed subset of some (possibly uncountable) power of \( \mathbb{R} \) must it be metrisable? A topological space which embeds as a closed subspace of some power of \( \mathbb{R} \) is called realcompact, so our question asks whether realcompactness of a manifold is equivalent to metrisability. We answer this question by introducing a realcompact non-metrisable manifold \( MNS(S) \) corresponding to each subset \( S \) of \( \mathbb{R} \) which has cardinality \( \omega_1 \). These manifolds are submanifolds of the Prüfer surface. While these manifolds are realcompact, they are not Lindelöf so are not metrisable. As a consequence of this result it follows that assuming the Continuum Hypothesis the Prüfer surface is realcompact.
3.2 Prüfer surface

In Chapter One we saw the classical Prüfer surface and studied some of its topological properties. In this section we study another non-metrizable topological manifold which has most of the properties in common with the classical Prüfer manifold but still there are some properties that one of these two spaces holds while the other one does not.

Example 3.2.1. (Prüfer surface)

Let $X = \{(x,y)|x,y \in \mathbb{R} \text{ and } x \neq 0\}$ and for each fixed real number $y$ define $B_y = \{(0,y,z)|z \in \mathbb{R}\}$. Write $y,z$ for a typical member $((0,y),z)$ of $B_y$. Set $M_P = X \cup (\cup_{y \in \mathbb{R}} B_y)$. To make $M_P$ into a topological manifold, we assume $X$ has the subspace topology of $\mathbb{R}^2$. So for every point $q = (x_0,y_0) \in X$ and $\epsilon < |x_0|$ we define the $\epsilon$-neighbourhood $N(q,\epsilon)$ of $q$ to be the set $\{(x,y) \in X | \sqrt{(x-x_0)^2+(y-y_0)^2} < \epsilon\}$.

If $p = < y_0, z_0 > = (0,y_0,z_0) \in B_{y_0}$, then an open basic $\epsilon$-neighbourhood of $p$ is the union of two subsets $U(p,\epsilon) = \{(x,y) \in X | y_0 + (z_0 - \epsilon)x < y < y_0 + (z_0 + \epsilon)x \text{ and } 0 < x < \epsilon\} \cup \{(x,y) \in X | y_0 - (z_0 - \epsilon)x < y < y_0 - (z_0 + \epsilon)x \text{ and } 0 < x < \epsilon\}$ and $V(p,\epsilon) = \{(0,y_0,z) | z_0 - \epsilon < z < z_0 + \epsilon\}$ of $M$ so we define $N(p,\epsilon) = V(p,\epsilon) \cup U(p,\epsilon)$.

The Prüfer surface is a connected Hausdorff topological manifold [20] [50].

Lemma 3.2.2. The Prüfer surface is separable.

Proof. Let $D = \mathbb{Q}^2 \cap X$ where $X = \{(x,y)|x,y \in \mathbb{R} \text{ and } x \neq 0\}$ is subset of the Prüfer surface $M$. We show that every basic open set of the topology defined in Example 3.2.1 contains some point of $\mathbb{Q}^2$. The subspace $X$ of $M$ has the subspace topology of $\mathbb{R}^2$ and $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$ so $\mathbb{Q}^2$ is dense in $X$. If $p \in P \setminus M$, then for every basic open neighbourhood $N(p,\epsilon) = U(p,\epsilon) \cup V(p,\epsilon)$ for some $\epsilon > 0$. Now consider a rational number $a \in (0,\epsilon)$. Let $b$ be a rational number in the open interval $(r_0 + (y_0 - \epsilon)a, r_0 + (y_0 + \epsilon)a)$. We see that $(a,b) \in U(p,\epsilon)$ so $(a,b) \in N(p,\epsilon)$. This completes the proof. 

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Corollary 3.2.3. The Prüfer surface $M_P$ is not homeomorphic to the Prüfer manifold in Example 1.4.59.

Proof. We see from Lemma 1.4.64 that the Prüfer manifold is not separable but Lemma 3.2.2 states that Prüfer surface is separable so they are not homeomorphic. □

3.3 Realcompactness and non-metrisable manifolds

Definition 3.3.1. [13] [49] A topological space $X$ is called pseudo-compact if and only if every continuous function $f : X \to \mathbb{R}$ is bounded.

Definition 3.3.2. [13] [49] A topological space $X$ is realcompact if and only if $X$ is homeomorphic to a closed subspace of $\mathbb{R}^\tau$ for some cardinal number $\tau$. 

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Theorem 3.3.3. [13] [49] A Hausdorff topological space $X$ is compact if and only if $X$ is pseudo-compact and realcompact.

Lemma 3.3.4. The closed long ray is pseudo-compact.

Proof. In Corollary 1.4.12 we proved that every continuous function $f : \mathbb{L}_+ \to \mathbb{R}$ is constant on $[\alpha_0, \omega_1)$ for some $\alpha_0 \in \mathbb{L}_+$ say $f((\alpha_0, \omega_1)) = x_0$. On the other hand $[0, \alpha_0]$ is homeomorphic to $[0, 1]$ so it is compact hence has a compact image under the continuous function $f$. We see that $f(\mathbb{L}_+) \subseteq [a, b]$ where $a = \min f([0, \alpha_0])$ and $b = \max f([0, \alpha_0])$.

Proposition 3.3.5. The closed long ray is not realcompact.

Proof. The closed long ray is pseudo-compact by Lemma 3.3 so if $\mathbb{L}_+$ is realcompact, then $\mathbb{L}_+$ is compact but we know that $\mathbb{L}_+$ is not compact since the open cover $U = \{[0, \alpha)|\alpha \in \omega_1\}$ does not have finite subcover. So $\mathbb{L}_+$ is not realcompact.

For every topological space $X$, the set of all continuous functions from $X$ in $\mathbb{R}$ with the pointwise addition and pointwise multiplication constitutes a ring which is called the ring of continuous functions and shown by $C(X)$. For realcompact topological spaces $X$ and $Y$, $X$ is homeomorphic to $Y$ if and only if $C(X)$ is isomorphic to $C(Y)$ [22].

Definition 3.3.6. A partially ordered set $D$ is a directed set if and only if for every pair of points $x$ and $y$ in $D$ there is a point $z \in D$ such that $z \geq x$ and $z \geq y$.

Definition 3.3.7. Let $D$ be a directed set and suppose that to each $\alpha \in D$ there corresponds a topological space $X_\alpha$ in such a way that whenever $\beta \geq \alpha$ there is a continuous map $\pi^\beta_\alpha : X_\beta \to X_\alpha$ such that:

1. $\pi^\alpha_\alpha = \text{id}$ for all $\alpha \in D$.
2. $\pi^\gamma_\beta \circ \pi^\beta_\alpha = \pi^\gamma_\alpha$ for $\alpha < \beta < \gamma$.

The maps $\pi^\beta_\alpha$ are called bonding maps and we call $\{(X_\alpha, \pi^\beta_\alpha) | \alpha, \beta \in D\}$ an inverse system.
Definition 3.3.8. The inverse limit of an inverse system \( \{(X_\alpha, \pi_\alpha^\beta) | \alpha \in D\} \) is the topological space \( X \subset \prod_{\alpha \in D} X_\alpha \) consisting of all elements \( x = (x_\alpha)_{\alpha \in D} \in \prod_{\alpha \in D} X_\alpha \) such that \( \pi_\alpha^\beta(x_\beta) = x_\alpha \) for all \( \alpha \leq \beta \). Elements of the inverse limit \( X \) are called threads.

Notation 3.3.9. Sometimes we write \( \{X_\alpha, \pi_\alpha^\beta, D\} \) instead of \( \{(X_\alpha, \pi_\alpha^\beta) | \alpha, \beta \in D\} \) and we also use \( \lim \limits_\leftarrow X_\alpha \) to denote the inverse limit of \( \{X_\alpha, \pi_\alpha^\beta, D\} \).

Lemma 3.3.10. [11] Let \( \{(X_\alpha, \pi_\alpha^\beta) | \alpha \in D\} \) be an inverse system of topological spaces and \( X \) the inverse limit of the inverse system \( \{(X_\alpha, \pi_\alpha^\beta) | \alpha \in D\} \). Then for every \( \alpha, \beta \in D \) such that \( \alpha \leq \beta \) the following diagram commutes:

\[
\begin{array}{ccc}
  X & \xrightarrow{\pi_\alpha} & X_
  \\
  \downarrow{\pi_\beta} & & \downarrow{\pi_\alpha^\beta} \\
  X_\beta & \xrightarrow{\pi_\alpha^\beta} & X_\alpha
\end{array}
\]

where \( \pi_\alpha^\beta \) for \( \alpha \leq \beta \) is bonding map and \( \pi_\alpha \) for \( \alpha \in D \) is a projection map.

Lemma 3.3.11. [55] Let \( \{(X_\alpha, \pi_\alpha^\beta) | \alpha \in D\} \) be an inverse system of topological spaces and for each \( \alpha \in D \) let \( B_\alpha \) be a basis for the topology of \( X_\alpha \). If \( X \) is the inverse limit of the system, then a basis for the topology of \( X \) consists of all the sets of the form \( (\pi_\alpha)^{-1}(U_\alpha) \), where \( U_\alpha \in B_\alpha \) and \( \alpha \in D \).

Lemma 3.3.11 uniquely defines the inverse limit of the inverse system \( \{(X_\alpha, \pi_\alpha^\beta) | \alpha \in D\} \). More precisely if \( \{(X_\alpha, \pi_\alpha^\beta) | \alpha \in D\} \) is an inverse system of topological spaces, \( Y \) is a topological space and for every \( \alpha \in D \), \( p_\alpha : Y \to X_\alpha \) is a continuous function such that the following diagram commutes for every \( \alpha \leq \beta \), then \( Y \) is homeomorphic to the inverse limit of the inverse system \( \{(X_\alpha, \pi_\alpha^\beta) | \alpha \in D\} \) [55].

\[
\begin{array}{ccc}
  Y & \xrightarrow{p_\alpha} & Y_
  \\
  \downarrow{p_\beta} & & \downarrow{p_\alpha} \\
  X_\beta & \xrightarrow{\pi_\alpha^\beta} & X_\alpha
\end{array}
\]
Lemma 3.3.12. [49] Every $T_1$, regular Lindelöf space is realcompact.

Corollary 3.3.13. Every subspace of $\mathbb{R}^n$ is realcompact.

Proof. Every subspace of a Tychonoff space is Tychonoff [78]. And every subspace of $\mathbb{R}^n$ is Lindelöf. So by using Lemma 3.3.12 we see that $X$ is realcompact. \qed


Example 3.3.15. (S-Double Moore-Nemitski plane) Let $S \subset \mathbb{R}$, $Y = \mathbb{R}^2 \setminus \{(0, y)\| y \in \mathbb{R} \setminus S\}$ and $X = \mathbb{R}^2 \setminus \{(0, y)\| y \in \mathbb{R}\}$. We assume that $X$ has the usual topology of $\mathbb{R}^2$ and for every point of $Y \setminus X$ we define an open neighbourhood base as follows:

For every $s \in S$ and $n \in \mathbb{N}$ define $p_{s,n} = \left(\frac{1}{n}, s\right)$ and $q_{s,n} = \left(-\frac{1}{n}, s\right)$. Now for every $n \in \mathbb{N}$ and $s \in S$ let

$$N_n(0, s) = \{p \in \mathbb{R}^2\| p - p_{s,n}\| < \frac{1}{n}\} \cup \\{p \in \mathbb{R}^2\| p - q_{s,n}\| < \frac{1}{n}\} \cup \{(0, s)\}.$$.

For every point $(0, s)$, the set $\{N_n(0, s)\| n \in \mathbb{N}\}$ is an open neighbourhood base at $(0, s)$. We are specially interested in the case where $|S| = \omega_1$. In this case we call the space $\omega_1$-Double the Moore-Nemitski plane.

In Example 3.3.15 if $S = \mathbb{R}$, then $\{(x, y) \in Y\| y \geq 0\}$ is called the Moore-Nemitski plane and we write $MN(S)$ (for details see [78]).

Notation 3.3.16. We write $MN$ instead of $MN(\mathbb{R})$.

Lemma 3.3.17. [49] The Moore-Nemitski plane is $T_{3^{\frac{1}{2}}}$.

We can use the same technique to prove that $S$-Moore-Nemitski plane is $T_{3^{\frac{1}{2}}}$.

Lemma 3.3.18. The $S$-Moore-Nemitski plane is $T_{3^{\frac{1}{2}}}$.
Lemma 3.3.19. [22] Let $X$ be a $T_{3\frac{1}{2}}$ space. If $f : X \to Y$ is a continuous injection and every subspace of $Y$ is realcompact, then every subspace of $X$ is realcompact.

Lemma 3.3.20. The $S$-Double Moore-Nemitski plane is realcompact.

Proof. Recall that every subspace of $\mathbb{R}^2$ is realcompact. Consider the identity function $id : MN(S) \to \mathbb{R}^2 \setminus \{(0, y) | y \in \mathbb{R} \setminus S\}$, where $MN(S)$ is the $S$-double Moore-Nemitski plane. By using Lemma 3.3.19 and Corollary 3.3.13 we see that $MN(S)$ is realcompact.

The following two examples are hybrids of the Prüfer surface and the Moore-Nemitski plane.

Example 3.3.21. (Euclidean plane with spikes) Let $X = \{(x, y) \in \mathbb{R}^2 | x \neq 0\}$, $L$ a subset of $\mathbb{R}$ of cardinality $\omega_1$ and $S \subset L$. For every $y \in S$ we define $B_y = \{(0, y, z) | z \in \mathbb{R}\}$, and usually we call $B_y$ a spike based at $(0, y)$.

The underlying set of the Euclidean plane with spikes is $M(S) = X \cup (\{0\} \times (L \setminus S)) \cup (\bigcup_{y \in S} B_y)$. If $S$ is a countable subset of $L$ we write $M(C)$ and if $S$ is finite we write $M(F)$ instead of $M(S)$. Now we are ready to introduce the topology on $M(S)$ by defining a neighbourhood base at each point of the set $M(S)$. For every $p = (0, y_0, z_0) \in B_{y_0}$ we define an $\epsilon$-neighbourhood of $p$ as follows and we call it an $\epsilon$-butterfly neighbourhood of $p$:

$N_{\epsilon}^{BF}(p) = \{(x, y) | y_0 + (z_0 - \epsilon) < y < y_0 + (z_0 + \epsilon) - \epsilon < x < \epsilon, x \neq 0\} \cup \{(0, y_0, z) | z_0 - \epsilon < z < z_0 + \epsilon\}$.

The rest of the space which consists of $X$ and $\{0\} \times L \setminus S$ has the usual topology of the Euclidean plane.

Example 3.3.22. (Moore-Nemitski plane with spikes) The underlying subset of the Moore-Nemitski plane with spikes $MNS(S)$ is the same as the Euclidean plane with spikes in Example 3.3.21. As in Example 3.3.21 we let $X$ have the usual topology of the Euclidean plane and for every point $p$ of the spike $B_y$ based at $(0, y) \in S$ we have a neighbourhood base
of $\epsilon$-butterfly neighbourhoods of $p$. If we let $\epsilon \in \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$ we see that $\frac{1}{n}$-butterfly neighbourhoods of $p$ provide us with a countable base at $p$.

Finally for every point $p = (0, y_0) \in \{0\} \times (L \setminus S)$ we define an $\epsilon$-neighbourhood of $p$ to be $N^B_\epsilon(p) = \{(x, y) \in \mathbb{R}^2 | (x - \epsilon)^2 + (y - y_0)^2 < \epsilon\} \cup \{(x, y) \in \mathbb{R}^2 | (x + \epsilon)^2 + (y - y_0)^2 < \epsilon\} \cup \{(0, y_0)\}$. We call $N^B_\epsilon(p)$, the $\epsilon$- bubble neighbourhood of $p$.

**Lemma 3.3.23.** The Euclidean plane with finitely many spikes, $M(F)$, is a metrisable manifold.

**Proof.** We show that every point $p \in M(F)$ has a Euclidean neighbourhood so it is a locally compact space. Moreover we need to show that $M(F)$ is a connected and Lindelöf topological space.

In [19] it is shown that $\frac{1}{n}$-butterfly neighbourhoods of every point $p$ on spike $B_y$ are locally Euclidean. So the same proof is applicable here as well. If $p = (x_0, y_0) \in X$, then we choose a natural number $n \in \mathbb{N}$ such that $\frac{1}{n} < |x_0|$. We see that $\{(x, y) | (x - x_0)^2 + (y - y_0)^2 < \frac{1}{n^2}\}$ is a neighbourhood of $(x_0, y_0)$ homeomorphic to $\mathbb{R}^2$ since $X$ has the subspace topology of $\mathbb{R}^2$.

If $p = (0, y_0) \in \{0\} \times L \setminus S$, then we consider $n \in \mathbb{N}$ such that $\frac{1}{n} < \min\{|(0, y_0) - (0, y)| | y \in F\}$. So every point of $M(F)$ has a locally Euclidean neighbourhood. The space $M(F)$ is connected since $\{(x, y) \in X | x > 0\}$ and $\{(x, y) \in X | x < 0\}$ are connected and every open neighbourhood of every point of $p \in \{0\} \times (L \setminus S)$ intersects both $\{(x, y) \in X | x > 0\}$ and $\{(x, y) \in X | x < 0\}$. So $M(F)$ is connected. To see that $M(F)$ is a Hausdorff space is similar to the Hausdorffness of the Prüfer surface [19].

So far we know that $M(F)$ is a connected Hausdorff locally Euclidean topological space. We need to show that $M(F)$ is a Lindelöf space and hence metrisable [20].

To see that $M(F)$ is Lindelöf we start with an arbitrary open cover $U$ of $M(F)$. Every subspace of $\mathbb{R}^2$ is Lindelöf so countably many elements of $U$ cover $X \cup \{0\} \times L \setminus S$ say $U_0$. Every spike $B_y$ is homeomorphic to $\mathbb{R}$ as a subspace of $M(F)$ so countably many elements of $U$ cover $B_y$. So countably many elements of $U$ cover $M(F)$ since we have only finitely
many spikes. By using Theorem 1.3.3 we see that Lindelöf manifolds are metrisable so $M(F)$ is a metrisable manifold.

Every locally compact Hausdorff space is Tychonoff [78] and every Lindelöf Tychonoff space is realcompact by Theorem 3.3.12. So $M(F)$ is a realcompact manifold. So we have:

**Lemma 3.3.24.** The Euclidean plane with finitely many spikes, $M(F)$, is realcompact.

**Lemma 3.3.25.** The Moore-Nemitski plane $MNS(F)$ with finitely many spikes is realcompact.

**Proof.** Consider $id : MNS(F) \to M(F)$. The topology of $MNS(F)$ is stronger than the topology of $M(F)$ so $id$ is a continuous function. The identity map is a bijection so Lemma 3.3.19 implies that $MNS(F)$ with finitely many spikes is realcompact.

---

**Figure 3.2:** Stereographic projection
Consider the line $L$ with the equation $x = r$ in the Euclidean plane and the circle $C(O, r)$ where $O = \left(\frac{r}{2}, y\right) \in \mathbb{R}^2$. Every ray $R$ originating from $a = (0, y)$ intersects $C(O, r) \setminus \{a\}$ in exactly one point. The ray $R$ intersects the line $L$ with the equation $x = r$ in exactly one point. So we have a bijection between the line $L$ and $C(O, r) \setminus \{a\}$. This correspondence provides a homeomorphism from $C(O, r) \setminus \{O\}$ onto $\mathbb{R}$ see Figure 3.2. By using a similar method we can find a homeomorphism $h : \mathbb{S}^n \setminus \{p\} \rightarrow \mathbb{R}^n$ which is called the stereographic projection [71]. Let $C(P, r)$ be a circle of the radius $r$ tangent to the $y$-axis at the point $a = (0, t)$ and let $P = \left(\frac{r}{2}, t\right)$ be the centre of the circle $C(P, r)$. Assume that $D$ is the open disc which has $C(P, r)$ as its boundary and $L$ be a line. Stereographic projection [71] provides a homeomorphism from $C(P, r) \setminus \{a\}$ onto $\mathbb{R}$. For every $\epsilon > 0$ and $z \in \mathbb{R}$ let $L_1$ and $L_2$ be two non-vertical lines through $a$ with slopes $z + \epsilon$ and $z - \epsilon$ and intersecting $C(P, r)$ in $p = (x_p, y_p)$ and $q = (x_q, y_q)$ respectively where $p \neq a \neq q$. If we choose a real
number $\delta < \min \{x_p, x_q\}$ the region which is bounded by $L_1$, $L_2$ and $x = \delta$ is a subset of $D$.

From what we discussed we conclude that the subspace $\{(x, y) \in \mathbb{R}^2 \mid (x - r)^2 + (y - y_0)^2 < r^2\} \cup \{(x, y) \in \mathbb{R}^2 \mid (x + r)^2 + (y - y_0)^2 < r^2\} \cup B_{y_0}$ is an open subspace of $M(C)$ where $y_0 \in C$.

**Lemma 3.3.26.** The Moore-Nemitski plane $MNS(C)$ with countably many spikes is real-compact.

*Proof.* The idea of proof is to show that for every countable set $C \subset \{0\} \times \mathbb{R}$, $MNS(C)$ can be written as an inverse sequence of a sequence of the Euclidean plane with finitely many spikes.

Consider $MNS(C)$ where $C = \{y_n\mid n \in \omega\}$. We also use $MNS(n)$ for $MNS(F)$ where $F = \{y_1, y_2, \ldots, y_n\}$. Consider the following diagram

$$
\cdots \xrightarrow{f_{n+1}} MNS(n) \xrightarrow{f_{k+1}} MNS(k) \xrightarrow{f_k} MNS(0)
$$

where $MNS(0)$ is the Moore-Nemitski double plane.

Let $f_{n+1}(x, y) = (x, y)$, $f_{n+1}(0, y_k, z) = (0, y_k, z)$ if $k < n + 1$ and $f_{n+1}(B_{y_{n+1}}) = (0, y_{n+1})$.

Consider an open neighbourhood $N^{B}_\epsilon(0, y_{n+1})$ of $(0, y_{n+1})$. We see that $f_{n+1}^{-1}(N^{B}_\epsilon(0, y_{n+1}) = N^{B}_\epsilon(0, y_{n+1}) \cup B_{y_{n+1}}$. 

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Note that

\[ f_{n+1}^{-1}(\{(x, y)|(x - \epsilon)^2 + (y - y_0)^2 < \epsilon^2, \ - \epsilon < x < \epsilon, \ x \neq 0\}) \]

\[ \cup \ \{(x, y)|(x + \epsilon)^2 + (y - y_0)^2 < \epsilon^2, \ - \epsilon < x < \epsilon, \ x \neq 0\} \]

\[ \cup \ \{(0, y_0)\} \]

\[ = \]

\[ \{(x, y)|(x - \epsilon)^2 + (y - y_0)^2 < \epsilon^2, \ - \epsilon < x < \epsilon, \ x \neq 0\} \]

\[ \cup \ \{(x, y)|(x + \epsilon)^2 + (y - y_0)^2 < \epsilon^2, \ - \epsilon < x < \epsilon, \ x \neq 0\} \]

\[ \cup \ B_{y_{n+1}} \]

and

\[ \{(x, y)|(x - \epsilon)^2 + (y - y_0)^2 < \epsilon^2, \ - \epsilon < x < \epsilon, \ x \neq 0\} \]

\[ \cup \ \{(x, y)|(x + \epsilon)^2 + (y - y_0)^2 < \epsilon^2, \ - \epsilon < x < \epsilon, \ x \neq 0\} \]

\[ \cup \ B_{y_{n+1}} \]

is an open subset of \( MNS(n + 1) \) (see Figure 3.4). So if we define \( f_m^n : MNS(n) \to MNS(m) \) by \( f_m^n = f_n \circ \cdots \circ f_1 \) we see that bonding maps are continuous so \( \{MNS(n), f_m^n, \omega\} \) is an inverse system.

For every \( n \in \omega \) we define the projection \( \pi_n : MNS(C) \to MNS(n) \) as follows:

If \((x, y) \in MNS(C)\), then we define \( \pi_n(x, y) = (x, y) \) and for \((0, y_m, z) \in B_{y_m}, \pi_n \) is defined by

\[ \pi_n(0, y_m, z) = \begin{cases} (0, y_m, z) : m \leq n \\ (0, y_m) : m > n \end{cases} \]

We see from the definitions of \( \pi_n \) and \( f_n \) that the following equalities hold:

If \((x, y) \in MNS(C)\), then \( \pi_n(x, y) = (x, y) = f_{n+1}(\pi_{n+1}(x, y)) \).
If \( m > n + 1 \), then \( \pi_n(0, y_m, z) = (0, y_m) = f_{n+1}\pi_{n+1}(0, y_m, z) \).

If \( m \leq n \), then \( f_{n+1}\pi_{n+1}(0, y_m, z) = (0, y_m, z) = \pi_n(0, y_m, z) \).

Finally if \( m = n + 1 \), then \( \pi_n(0, y_{n+1}, z) = (0, y_{n+1}) = f_{n+1}\pi_{n+1}(0, y_{n+1}, z) = f_{n+1}(0, y_{n+1}, z) \).

So \( \pi_n = \pi_{n+1}f_{n+1} \) for all \( n \in \mathbb{N} \). Then \( MNS(C) \) is the inverse limit. Hence by Theorem 3.3.14 \( MNS(C) \) is realcompact.

![Diagram](image)

**Figure 3.4:** Neighbourhood of a spike in \( MNS(C) \)

**Lemma 3.3.27.** [13] The limit of an inverse system of \( T_i \)-spaces is a \( T_i \)-space for \( i \leq 3\frac{1}{2} \).

If \( D \) is a directed set and \( D' \subset D \) such that for every \( x \in D \) there is \( y \in D' \) such that \( y \geq x \) we say \( D' \) is cofinal subset of \( D \).
Lemma 3.3.28. Let \( \{X_\alpha, \pi_\alpha^D, D\} \) be an inverse system and \( D' \) a cofinal subset of \( D \). Then the inverse limit of \( \{X_\alpha, \pi_\alpha^D, D\} \) and the inverse limit of \( \{X_\alpha, \pi_\alpha^D, D'\} \) are homeomorphic.

Consider \( L \subset \mathbb{R} \) such that there is a bijection \( f : L \to \omega_1 \). By \( L^{[\omega]} \) we mean all countable subsets of \( L \). \( (L^{[\omega]}, \subseteq) \) is a partially ordered set and the union of every two elements \( C_1 \) and \( C_2 \) of \( (L^{[\omega]}, \subseteq) \) belongs to \( (L^{[\omega]}, \subseteq) \). So \( (L^{[\omega]}, \subseteq) \) is a directed set. \( \{MNS(C), \pi_{C_1}^{C_2}, L^{[\omega]}\} \) is an inverse limit where \( \pi_{C_1}^{C_2} : MNS(C_2) \to MNS(C_1) \) and \( C_1 \subset C_2 \) is defined as follows:

For every \( (x, y) \in X \cup \{0\} \times L \setminus C_2 \) we have \( \pi(x, y) = (x, y) \) and for \( (0, y, z) \in B_y \) where \( y \in C_2 \setminus C_1 \) we have \( \pi_{C_1}^{C_2}(0, y, z) = (0, y) \).

Note that \( |L| = \omega_1 \) so there is a bijection from \( L \) onto \( \omega_1 \) so we write \( \{l_\alpha | \alpha \in \omega_1\} = L \).

We assume that \( \{(x, y) | x, y \in \mathbb{R}, x \neq 0\} \cup (\cup_{y \in L} B_y) \) has the subspace topology of \( \mathbb{R}^2 \) and every point \( p = (0, y_0, z_0) \in B_{y_0} \) has a neighbourhood base of \( \frac{1}{n} \)-butterfly neighbourhoods where \( n \in \mathbb{N} \). So a typical element of the neighbourhood base of \( p \) is

\[
N_{\frac{1}{n}}^{BF}(p) = \{(0, y_0, z) | z_0 - \epsilon < z < z_0 + \epsilon\}
\]

\[
\cup \{(x, y) \in \mathbb{R}^2 | y_0 + (z_0 - \epsilon) \leq y < y_0 + (z_0 + \epsilon) \mid |x| < y \} \setminus \{(x, y) | x \neq 0, -\epsilon < x < \epsilon\}
\]

Now consider a bijection \( f : L \to \omega_1 \) and let \( MNS(\alpha) = MNS(C) \), where \( C \subset L \) such that \( C = \{f^{-1}(\beta) | \beta \leq \alpha\} \).

We show that \( \{MNS(\alpha), \pi_\alpha^D, \omega_1\} \) is an inverse system in which \( MNS(\alpha) \) is realcompact by
Lemma 3.3.26 and by using Lemma 3.3.14 \( \lim_{\omega_1} MNS(\alpha) \) is realcompact. Consider the following diagram where \( f : L \to \omega_1 \) is a bijection and the projection \( \pi_\alpha \) is defined as follows:

\[
\pi_\alpha(p) = \begin{cases} 
  p & (p = (x, y), x \neq 0) \\
  (0, y, z) & (p = (0, y, z), \alpha < f(y)) \\
  (0, y) & (p = (0, y, z), f(y) \leq \alpha)
\end{cases}
\]

Continuity of bonding maps can be shown as in the proof of Lemma 3.3.26. For every \( \alpha \in \omega_1 \), \( \pi_\alpha((x, y)) = (x, y) \) if \( x \neq 0 \). For every \( \alpha \in \omega_1 \) the projection \( \pi_\alpha \) is continuous.

We see from the definition of the projection \( \pi_\alpha \) and the bonding map \( \pi_\beta^\alpha \) that the diagram commutes so \( MNS(\omega_1) \) is realcompact since \( MNS(\omega_1) \) is the inverse limit of an inverse system of realcompact spaces. So we have the following theorem:

**Theorem 3.3.29.** The topological space \( MNS(\omega_1) \) as described is realcompact.

Note that \( MNS(\omega_1) \) is a realcompact submanifold of the Prüfer surface with \( \omega_1 \) many spikes on it. If we assume the Continuum Hypothesis we can establish a bijection from \( \omega_1 \) onto \( \mathbb{R} \) and replace points with spikes all over the \( y \)-axis. So we have:

**Corollary 3.3.30.** Assuming the Continuum Hypothesis the Prüfer surface is realcompact.

**Proof.** If we assume the Continuum Hypothesis there is a bijection \( \mathbb{R} \to \omega_1 \). By letting \( L = \mathbb{R} \) in Theorem 3.3.29 we have \( M = M_P \). So the Prüfer surface is realcompact. \( \square \)
Figure 3.5: Inverse system in the Prüfer Surface

Figure 3.6: Neighbourhoods in the Prüfer surface and Moore-Nemitski plane
Lemma 3.3.31. [13] The inverse limit of an inverse system of compact Hausdorff topological spaces is a compact Hausdorff topological space.

Corollary 3.3.32. The long line is not the inverse limit of the following system:

\[ \ldots \to [0, \beta + 1] \xrightarrow{\pi_{\beta + 1}} [0, \beta] \xrightarrow{\pi_\beta} \ldots \to [0, \alpha] \xrightarrow{\pi_\alpha} \ldots \to [0, 1] \]

where projections and bonding maps are retractions. So $L$ is not homeomorphic to $\mathbb{L}_+$. 

Proof. The inverse limit of an inverse system of $T_2$ compact spaces is compact and $T_2$ by Lemma 3.3.31. $\mathbb{L}_+$ is not compact so $\mathbb{L}_+$ is not the inverse limit of the inverse system \{[0, \alpha], \pi_\alpha^\beta, \omega_1\} where $\pi_\beta^\alpha : [0, \beta] \to [0, \alpha]$ is retraction of $[0, \beta]$ into $[0, \alpha]$.

\[ \square \]
Chapter 4

$\omega$-bounded topological spaces and $D$-squat spaces

“One should always generalise.”

– Carl Gustav Jacobi –

It is well known that every continuous function $f$ from the set of all countable ordinals, $\omega_1$, with the order topology into the real line, $\mathbb{R}$, is constant on a tail of $\omega_1$, i.e. there is $\beta \in \omega_1$ such that $f(\alpha) = x$ for all $\beta < \alpha$. This result is a surprise to our intuition. In this chapter we generalise this result. We also study some properties of $\omega$-bounded topological spaces. Then we define the concept of $P$-squatness for a partially ordered topological space $P$ and generalise some of the results of Chapter One.
4.1 \(\omega\)-bounded topological spaces

**Definition 4.1.1.** A topological space \(X\) is an \(\omega\)-bounded space if and only if any countable subspace \(C\) of \(X\) is precompact, i.e. \(\overline{C}\) is a compact subspace of \(X\). If \(X\) is \(\omega\)-bounded but it is not compact we call it a strictly \(\omega\)-bounded space.

**Lemma 4.1.2.** A Hausdorff topological space \(X\) is \(\omega\)-bounded if and only if any countable subset \(C\) of \(X\) is contained in a compact subspace \(K\) of \(X\).

**Proof.** Let \(X\) be an \(\omega\)-bounded topological space and \(C\) be a countable subset of \(X\), then clearly \(C \subseteq \overline{C}\) and by definition \(\overline{C}\) is compact. So let \(K = \overline{C}\).

Now we prove the converse. Let \(C \subseteq K \subseteq X\), where \(C\) is countable, \(K\) is compact. Note that \(X\) is a Hausdorff space so \(K\) is a closed subspace of \(X\). Hence \(\overline{C} \subseteq K = \overline{K}\). Consequently \(\overline{C}\) is a closed subspace of the compact space \(K\) so it is compact. \(\square\)

A property \(p\) is called a topological property if and only if for any two spaces \(X\) and \(Y\) which are homeomorphic spaces with one of them satisfies \(p\), then the other space also satisfying \(p\). Compactness and connectedness are two examples of topological properties. If \(X\) and \(Y\) have a weaker relation, such as having the same homotopy type, and one of them is connected still the other one has to be connected but it is not the case for compactness.

As an example a singleton \(\{0\}\) and \(\mathbb{R}\) have the same homotopy type but \(\mathbb{R}\) is not compact while the singleton is compact. In this situation compactness is called a topological property but connectedness is called a homotopy property and connectedness seems to be preserved by a weaker relation between spaces, i.e. homotopy equivalence. We will show that \(\omega\)-boundedness and strict \(\omega\)-boundedness are both topological properties but they are not preserved by homotopy equivalence between spaces.

**Lemma 4.1.3.** The \(\omega\)-bounded property is a topological property.
Proof. To show that \( \omega \)-boundedness is a topological property we consider a homeomorphism \( f : X \to Y \) from an \( \omega \)-bounded space \( X \) onto \( Y \) and show that any countable subspace of \( Y \) has a compact closure.

Let \( f : X \to Y \) be a homeomorphism and \( C \subseteq Y \) be a countable subset of \( Y \). We show that \( \overline{C} \) is a compact subspace of \( Y \).

\( X \) is \( \omega \)-bounded and \( f^{-1}(C) \) is a countable subspace of \( X \) so \( \overline{f^{-1}(C)} \) is a compact subspace of \( X \). Note that \( f \) is a homeomorphism so \( f(\overline{f^{-1}(C)}) = \overline{f(f^{-1}(C))} = \overline{C} \) is a compact subspace of \( X \). \( \square \)

**Corollary 4.1.4.** Strict \( \omega \)-boundedness is a topological property.

**Proof.** We already know that the compactness is a topological property. So if \( X \) is not compact, then \( Y \) is not compact. Now we assume that \( X \) is \( \omega \)-bounded. So \( Y \) is \( \omega \)-bounded but \( Y \) is not compact. Therefore \( Y \) is strictly \( \omega \)-bounded. \( \square \)

**Definition 4.1.5.** A topological property \( p \) is a homotopy property if and only if whenever \( X \) and \( Y \) are spaces with the same homotopy type, then both have the property \( p \) or both do not satisfy \( p \).

**Example 4.1.6.** Let \( \tilde{\omega}_1 = \omega_1 \cup \{ -1 \} \) and assume \(-1 \leq \alpha \) for all \( \alpha \in \omega_1 \). Then \( f : \tilde{\omega}_1 \to \omega_1 \) defined by

\[
 f(\alpha) = \begin{cases} 
 \alpha + 1 & (\alpha < \omega_0) \\
 \alpha & (\alpha \geq \omega_0) 
\end{cases}
\]

is an increasing bijection. \( f \) induces a homeomorphism from \( \tilde{\mathbb{L}}_+ = \tilde{\omega}_1 \times [0,1] \) with the lexicographic order topology onto \( \mathbb{L}_+ \). Let \( F : \tilde{\mathbb{L}}_+ \times [0,1] \to \tilde{\mathbb{L}}_+ \) and \( G : (\tilde{\mathbb{L}}_+ - \{ -1 \}) \times [0,1] \to \tilde{\mathbb{L}}_+ - \{ -1 \} \) be defined by

\[
 F(x, t) = \begin{cases} 
 tx & (x \leq 0) \\
 x & (x \geq 0) 
\end{cases}
\]

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\[ F(\tilde{\mathbb{L}}_+ \times \{0\}) = \mathbb{L}_+ \text{ and } F(\tilde{\mathbb{L}}_+ \times \{1\}) = \tilde{\mathbb{L}}_+. \]

\[ G(x, t) = \begin{cases} 
  tx & (-1 < x \leq 0) \\
  x & (x \geq 0)
\end{cases} \]

\[ G(\tilde{\mathbb{L}}_+ - \{-1\}) \times \{0\} = \mathbb{L}_0 \text{ and } G(\tilde{\mathbb{L}}_+ - \{-1\}) \times \{1\} = \tilde{\mathbb{L}}_+ - \{-1\}. \]

\( F \) and \( G \) are deformation retractions of \( \tilde{\mathbb{L}}_+ \) and \( \tilde{\mathbb{L}}_+ - \{-1\} \) respectively to \( \mathbb{L}_+ \), so they have the same homotopy type but \( (\tilde{\mathbb{L}}_+) - \{-1\} \) is not \( \omega - \) bounded while \( \tilde{\mathbb{L}}_+ \) is strictly \( \omega - \) bounded.

**Example 4.1.7.** The long line, \( \mathbb{L} \), the long ray, \( \mathbb{L}_+ \) and all compact topological spaces are \( \omega\)-bounded spaces.

**Example 4.1.8.** The Euclidean space \( \mathbb{R}^n \) for all \( n \in \mathbb{N} \) is not an \( \omega\)-bounded space. Consider \( S = \{(k, 0, \ldots, 0) | k \in \mathbb{N}\} \subset \mathbb{R}^n \). \( S \) is not contained in any bounded subset of \( \mathbb{R}^n \) and compact subspaces of \( \mathbb{R}^n \) are bounded, so \( S \) is not \( \omega\)-bounded.

The following example shows that the Hausdorff property is essential in Lemma 4.1.2.

**Example 4.1.9.** (Line with \( \aleph_0 \) origins) Let \( X = \mathbb{R} \bigsqcup \{*_{1}, *_{2}, \ldots\} \) with the usual topology on \( \mathbb{R} \) and define \( \{U_{i}(\epsilon) | \epsilon > 0\} \) to be a basic system of open neighborhoods of \( *_{i} \) where \( U_{i}(\epsilon) = ((-\epsilon, \epsilon) \setminus \{0\} \cup \{*_{i}\}) \) and \( \epsilon \) is a positive real number. Then \( \mathcal{U} = \{(-1, 1) \setminus \{0\} \cup \{*_{i}\} | i \in \mathbb{N}\} \cup \{(-1, 1)\} \) is an open cover for \( \text{cl}(\{*_{i} | i \in \mathbb{N}\}) \) which does not have a finite subcover. So \( X \) is not \( \omega\)-bounded.

**Theorem 4.1.10.** Any \( \omega\)-bounded topological space is countably compact.

**Proof.** Suppose the topological space \( X \) is not countably compact, then there is a countable open cover \( \mathcal{U} = \{U_{n} | n \in \omega\} \) for \( X \) such that no finite subfamily of \( \mathcal{U} \) covers \( X \). Consider
the increasing open cover \( \{ V_n \mid n \in \omega \} \) defined by \( V_0 = U_0 \) and \( V_k = \bigcup_{n=0}^{k} U_n \). Define \( i_0 = 0 \) and

\[
i_{k+1} = \min\{ n \in \omega \mid n > i_k, V_n \cap (\bigcup_{m<n} V_m)^c \neq \emptyset \}.
\]

Then \( \mathcal{V} = \{ V_{i_0}, V_{i_1}, \ldots, V_{i_k}, \ldots \} \) is an open cover of \( X \) and any finite number of elements of \( \mathcal{V} \) does not cover \( X \). Choose an arbitrary point \( p_0 \) in \( V_0 \) and for any \( k \in \mathbb{N} \) let \( p_{k+1} \) be an arbitrary element of \( V_{i_{k+1}} - V_{i_k} \). Now \( C = \{ p_0, p_1, p_2, p_3, \ldots \} \) is a countable subset of \( X \) and \( \mathcal{V} \) is an open cover for \( C \). For any finite subset \( \mathcal{V}' \) of \( \mathcal{V} \) there is \( i_{k_0} \in \omega \) such that \( p_{i_{k_0}} \) does not belong to any element of \( \mathcal{V}' \). Then there is no finite subfamily of \( \mathcal{V} \) which covers \( C \). Then \( \overline{C} = \{ p_0, p_1, p_2, p_3, \ldots \} \) is not covered by finitely many elements of \( \mathcal{V} \). \( \mathcal{V} \) is a cover for \( X \), so it covers \( \overline{C} \). Then \( \overline{C} \) is not compact so \( X \) is not \( \omega \)-bounded. We conclude that \( X \) is countably compact.

The following Lemma is well known and easy to prove.

**Lemma 4.1.11.** Let \( X \) be a topological space and \( S' \) be a closed and discrete subspace of \( X \). Then every subspace of \( S' \) is closed and discrete.

**Lemma 4.1.12.** If \( X \) contains an infinite closed discrete subset, then \( X \) is not countably compact.

**Proof.** Let \( S' \subset X \) be an infinite closed discrete subset of \( X \). Let \( S \) be a countable subset of \( S' \). Note that \( S \) is a closed and discrete subspace of \( X \) by Lemma 4.1.11. So we can choose a family \( \mathcal{O} \) of open neighborhoods of elements of \( S \) such that for \( O_x \in \mathcal{O}, O_x \cap S = \{ x \} \). So \( \{ O_x \mid x \in S \} \cup \{ S^c \} \) is an open countable cover for \( X \). It is easy to see that there is no subfamily of \( \{ O_x \mid x \in S \} \cup \{ S^c \} \) which covers \( X \). Therefore \( X \) is not countably compact.

**Corollary 4.1.13.** If \( X \) contains a closed discrete infinite subset, then \( X \) is not \( \omega \)-bounded.
Example 4.1.14. \{(k,0,\ldots,0)|k \in \mathbb{N}\} is a closed discrete subset of $\mathbb{R}^n$. On the other hand by the Bolzano-Weierstrass property any uncountable subset of $\mathbb{R}^n$ has a limit point. Then $\mathbb{R}^n$ contains a countable discrete closed subset.

Theorem 4.1.15. Any closed subspace of an $\omega$-bounded space is $\omega$-bounded.

Proof. Let $X$ be an $\omega$-bounded topological space and $Y \subseteq X$ be a closed subspace of $X$. Let $C = \{y_1, y_2, y_3, \ldots\}$ be a countable subspace of $Y$. $C \subseteq Y$ and $Y$ is closed subset of $X$, so $\text{cl}_X(C) \subseteq Y$ so $\text{cl}_Y(C) = \text{cl}_X(C)$. Since $\text{cl}_X(C)$ is compact, then $\text{cl}_Y(C)$ is compact so $Y$ is $\omega$-bounded. \qed

The following example shows that the closedness of $Y$ can not be dropped from the previous theorem.

Example 4.1.16. The long line is $\omega$-bounded but $(0,1) \subset \mathbb{L}_+$ is not $\omega$-bounded, since $(0,1)$ is homeomorphic to $\mathbb{R}$, and $\mathbb{R}$ is not $\omega$-bounded.

Example 4.1.17. The long ray with $\aleph_0$ many origins contains a closed countable subspace which is not compact. So it is not $\omega$-bounded.

Theorem 4.1.18. Let $\{X_i | i \in I\}$ be a family of $\omega$-bounded Hausdorff topological spaces, then $X = \prod_{i \in I} X_i$ is an $\omega$-bounded topological space.

Proof. Let $S = \{p_k | k \in \omega\}$ be a countable subset of $X$. For any $i \in I$, $|\pi_i(S)| \leq |S| \leq \aleph_0$, where $\pi_i$ is the projection on the $i$'th coordinate. Then there is a compact subspace $K_i$ of $X_i$ such that $\pi_i(S) \subseteq K_i$. $K = \prod_{i \in I} K_i$ is a compact subspace of $X$ but $S \subseteq K$, and $X$ is a Hausdorff space, so $X$ is $\omega$-bounded by Lemma 4.1.2. \qed

Theorem 4.1.19. Let $D$ be a directed set, and $X = \{X_\lambda, \phi_{\lambda\lambda'}, D\}$ be an inverse system of $\omega$-bounded, Hausdorff topological spaces. Then $\lim_{\lambda \in D} X_\lambda$ is an $\omega$-bounded topological space.
Proof. For each pair of elements \( \lambda, \lambda' \in D \) such that \( \lambda \leq \lambda' \) define:

\[
X(\lambda, \lambda') = \{ p \in \prod_{\beta \in D} X_\beta \mid \phi_{\lambda \lambda'} \circ \phi_{\lambda'}(p) = \phi_\lambda(p) \}.
\]

Note that \( \prod_{\beta \in D} X_\beta \) is Hausdorff, \( \phi_{\lambda \lambda'} \circ \phi_{\lambda'} \) and \( \phi_\lambda \) are continuous, so \( X(\lambda, \lambda') \) is closed for all \( \lambda \) and \( \lambda' \in D \).

If \( I = \{ (\lambda, \lambda') \in D \times D \mid \lambda \leq \lambda' \} \), then \( \lim_{\lambda \in D} X_\lambda = \bigcap_{(\lambda, \lambda') \in I} X(\lambda, \lambda') \) is a closed subspace of \( \prod_{\beta \in D} X_\beta \).

Theorem 4.1.18 implies that \( \prod_{\beta \in D} X_\beta \) is \( \omega \)-bounded. Now by using Theorem 4.1.15 we see that \( \lim_{\lambda \in D} X_\lambda \) is \( \omega \)-bounded. \( \square \)

The following example shows that we cannot omit the Hausdorff condition in Theorem 4.1.18.

Example 4.1.20. For all \( k \in \mathbb{N} \), let \( X_k = L_+ \cup \{ \ast \} \), where \( \ast \notin X_k \) is a fixed point. \( L_+ \) has the order topology on it and basic neighbourhoods of \( \{ \ast \} \) have the following form. \( N_\epsilon(\ast_k) = \{ \ast_k \} \cup (0, \epsilon) \), where \( \epsilon \) is a positive real number.

Each \( X_k \) is \( \omega \)-bounded. On the other hand their product \( X = \prod_{k \in \mathbb{N}} X_k \) is not \( \omega \)-bounded as we now show.

For every \( n \in \mathbb{N} \) let \( s_n = (0, 0, 0, \ldots, 0, \ast, 0, \ldots, 0, \ldots) \in X \) be a sequence with all elements of the sequence zero except the \( n \)th element which is \( \ast \). Now assume that \( S = \{ s_n \mid n \in \mathbb{N} \} \subset X \) is the set of all such a sequences. If \( p = (0, 0, 0, \ldots, 0, \ldots) \) is the point with all coordinates 0 it is clear that \( s_n \rightarrow p \). On the other hand \( p \notin S \). Let \( V_k = \prod_{m \in \mathbb{N}} U_m \), where \( U_m = X_m \) for all \( m \neq k \) and \( U_k = \{ \ast \} \cup (0, \frac{1}{2}) \).

So \( \mathcal{U} = \{ V_n \mid n \in \mathbb{N} \} \cup \{ \prod_{m \in \mathbb{N}} L_+ \} \) is an open cover for \( S \) but there is no subfamily of \( \mathcal{U} \) which covers \( S \). Then \( \prod_{k \in \mathbb{N}} X_k \) is not \( \omega \)-bounded.
**Theorem 4.1.21.** Let $X$ be topological space and $Y$ a Hausdorff topological space. Let $f : X \rightarrow Y$ be a continuous function onto $Y$. If $X$ is an $\omega$-bounded space, then $Y$ is also $\omega$-bounded.

**Proof.** Let $C = \{y_k \mid k \in \omega\}$ be an arbitrary countable subset of $Y$. For any $y_k \in C$ let $x_k$ be an arbitrary element of $f^{-1}(y_k)$. We get a countable subset $\{x_k \mid k \in \omega\}$ and $\{x_k \mid k \in \omega\}$ is a compact subset of $X$. $f$ is a continuous function, so $f(\{x_k \mid k \in \omega\})$ is a compact subset of $Y$ which contains $f(\{x_k \mid k \in \omega\}) = \{y_k \mid k \in \omega\}$. Then by using Lemma 4.1.2 $Y$ is $\omega$-bounded. \hfill \Box

**Theorem 4.1.22.** If $M$ is a metric space and $N$ an $\omega$-bounded subspace of $M$, then $N$ is compact.

**Proof.** Let $N$ be an $\omega$-bounded subspace of $M$. Then $N$ is a countably compact metric space. Any countably compact metric space is compact. Then $N$ is compact. This completes the proof. \hfill \Box

We introduce the following definition.

**Definition 4.1.23.** A topological space $X$ is called strongly $\omega$-bounded provided $0 < |C'| \leq \aleph_0$ for any countably infinite subset $C$ of $X$.

**Example 4.1.24.** The space of all countable ordinals, $\omega_1$, is an example of a strongly $\omega$-bounded space.

**Lemma 4.1.25.** Any strongly $\omega$-bounded topological space is $\omega$-bounded.

**Proof.** Let $C$ be a countable subset of a strongly $\omega$-bounded topological space $X$. We show that $\overline{C}$ is compact.

Assume that $U$ is an open cover for $\overline{C}$. $X$ is strongly $\omega$-bounded so $\overline{C}$ is countable. Consider
a countable enumeration \( \{ p_n \mid n \in \omega \} \) of \( C \). For every \( p_n \in C \) there is an \( U_n \in \mathcal{U} \) such that \( p_n \in U_n \). So we can cover \( C \) by only countable many elements of \( \mathcal{U} \) say \( \{ U_n \mid n \in \omega_0 \} \). We show that \( C \) can be covered by only finitely many elements of \( \{ U_n \mid n \in \omega \} \). Assume to the contrary that \( C \) is not covered by finitely many elements of \( \{ U_n \mid n \in \omega \} \).

Choose \( x_0 \in C \) and let \( k_0 = \min \{ m \in \omega \mid x_0 \in U_m \} \). Assume that \( x_m \) and \( k_m \) are chosen for all \( m \leq n \), such that \( x_m \in U_{k_m} \) and \( x_m \not\in U_l \) for all \( l \leq k_m \). \( \{ U_m \mid m \leq k_n \} \) does not cover \( C \) otherwise we are done. So there is \( x_{n+1} \in C \) such that \( x_{n+1} \not\in \bigcup_{0 \leq m \leq k_n} U_m \). Let \( k_{n+1} = \min \{ m \in \omega \mid x_{n+1} \in U_m \} \). So we have two sequences \( \{ x_n \mid n \in \omega \} \) and \( \{ U_n \mid n \in \omega \} \) such that \( x_n \in U_{k_n} \) and \( x_n \not\in U_m \) for all \( m < k_n \).

Now for every \( n \in \omega \) define \( V_n = \bigcup_{m=0}^{k_n} U_m \). Note that \( \{ V_n \mid n \in \omega \} \) is an open cover for \( C \).

For \( m \neq n \) we have \( x_m \neq x_n \) so \( S = \{ x_n \mid n \in \omega \} \) is infinite. By assumption \( S \) has a limit point say \( p \). There is \( j \) such that \( p \in V_j \), and hence \( V_j \) contains infinitely many members of the sequence \( \{ x_n \} \). On the other hand the construction of the sequence \( \{ x_n \} \) implies that \( x_k \not\in V_j \) whenever \( k > j \). This is a contradiction. Hence finitely many members of \( \{ U_n \mid n \in \omega \} \) cover \( C \) and hence \( C \) is compact. Thus \( X \) is \( \omega \)-bounded.

\[ \square \]

**Example 4.1.26.** \( \mathbb{L}_+ \) is an example which shows the converse of Lemma 4.1.25 is not true. The subspace \( \omega \times (0,1) \cap \mathbb{Q} \) of \( \mathbb{L}_+ \) countable but it has uncountably many limit points.

**Definition 4.1.27.** A pair \( (S, \ast) \) consisting of a nonempty set \( S \) and an associative binary operation \( \ast \) on \( S \) is called a semigroup, and a semigroup \( (S, \ast) \) having an identity element \( 1 \) such that \( 1 \ast x = x \ast 1 = x \) for all \( x \in S \) is called a monoid.

**Definition 4.1.28.** A topological semigroup is an (algebraic) semigroup \( (S, \ast) \) such that \( S \) has a topology with respect to which the binary operation

\[
\begin{cases}
  \ast : S \times S \to S \\
  (x, y) \mapsto x \ast y
\end{cases}
\]

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is a continuous function, where $S \times S$ has the product topology.

**Definition 4.1.29.** A topological group is a group $(G, \star)$ such that $G$ has a topology with respect to which the binary operation

$$
\begin{align*}
\star : G \times G & \to G \\
(x, y) & \mapsto x \star y
\end{align*}
$$

and

$$
\begin{align*}
\text{inv} : G & \to G \\
x & \mapsto x^{-1}
\end{align*}
$$

are continuous functions, where $G \times G$ has the product topology.

**Theorem 4.1.30.** [78] If $F \subseteq Y^X$ has the pointwise topology, $(f_\lambda)$ converges to $f$ if and only if $(f_\lambda(x))$ converges to $f(x)$ for each $x \in X$.

**Theorem 4.1.31.** (Kakutani’s theorem) [49] A topological group is metrisable if and only if it is first countable.

Every \(\omega\)-bounded subset of metric space $X$ is compact. So if $M$ is an \(\omega\)-bounded manifold with a topological group structure, $M$ has to be compact since manifolds are first countable. In other words Kakutani’s theorem implies that strictly \(\omega\)-bounded manifolds do not carry a topological group structure.

The following example which is mentioned in [66] is an example of an \(\omega\)-bounded topological group which is not first countable.

**Example 4.1.32.** [66] Let $G = (2^{\omega_1}, +)$ be the set of all functions from $\omega_1$ in to $\{0, 1\}$, and $+$ defined by $(f + g)(\lambda) = f(\lambda) + g(\lambda)$ for all $\lambda \in \omega_1$, where $+$ is the mod2 addition of $\{0, 1\}$. 

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We show that $G$ is a topological group. By definition of $+ : 2^{\omega_1} \times 2^{\omega_1} \to 2^{\omega_1}$ for any two elements $f$ and $g$ of $2^{\omega_1}$, $f + g$ is a function from $2^{\omega_1} \times 2^{\omega_1}$ into $2^{\omega_1}$ which shows that the $+$ is a binary operation. The constant function $0 : 2^{\omega_1} \to \{0, 1\}$ defined by $0(\lambda) = 0$ for all $\lambda \in \omega_1$ is the identity of the group $G$ and each function is its own inverse.

Then $G$ is a group.

To see that $+ : 2^{\omega_1} \times 2^{\omega_1} \to 2^{\omega_1}$ is a continuous function consider the following diagram:

$$
\begin{array}{ccc}
2^{\omega_1} \times 2^{\omega_1} & \rightarrow & 2^{\omega_1} \\
\downarrow \pi \circ + & & \downarrow \pi \\
\{0, 1\} & \leftarrow & \{0, 1\}
\end{array}
$$

We show that $\pi_\alpha \circ + : 2^{\omega_1} \times 2^{\omega_1} \to \{0, 1\}$ is a continuous function for all $\alpha \in \omega_1$.

Note that $(\pi_\alpha \circ +)^{-1}(\{0\}) = 2^{\omega_1} \times 2^{\omega_1}, (\pi_\alpha \circ +)^{-1}(\{1\}) = \emptyset$. We need to show that $(\pi_\alpha \circ +)^{-1}(\{0\})$ and $(\pi_\alpha \circ +)^{-1}(\{1\})$ are open.

For every $\alpha \in \omega_1$ we have $(\pi_\alpha \circ +)^{-1}(\{0\}) = \{(f, g) \in 2^{\omega_1} \times 2^{\omega_1} \mid f(\alpha) = g(\alpha)\} = \{f \in 2^{\omega_1} \mid f(\alpha) = 0\} \times \{g \in 2^{\omega_1} \mid g(\alpha) = 0\} \cup \{f \in 2^{\omega_1} \mid f(\alpha) = 1\} \times \{g \in 2^{\omega_1} \mid g(\alpha) = 1\}$ which is an open subset of $G$ since $\{0\}$ and $\{1\}$ are open subsets of $\{0, 1\}$ and $2^{\omega_1}$ has the pointwise topology.

Similar argument shows that $\pi_\alpha \circ +$ is continuous for every $\alpha \in \omega_1$ so $+$ is continuous. The inverse operation on $G$ is equal to the identity $\text{id} : G \to G$ which is continuous. This shows that $G$ is a topological group.

Assume $H \leq G$ is the set of all functions $f : \omega_1 \to \{0, 1\}$ such that there is a $\lambda \in \omega_1$, and for any $\alpha \geq \lambda$, $f(\alpha) = 0$.

Consider $f, g \in H$. Note that $f$ and $g$ are eventually zero, i.e. there are $\alpha_1$ and $\alpha_2$ in $\omega_1$ such that $f(\alpha) = 0$ for every $\alpha > \alpha_1$ and $g(\alpha) = 0$ for every $\alpha > \alpha_2$. So if $\beta > \max\{\alpha_1, \alpha_2\}$ we have $f(\beta) = g(\beta) = 0$. On the other hand $(f - g)(\alpha) = (f + g)(\alpha) = f(\alpha) + g(\alpha)$. So assuming $f(\alpha) = 0$ for all $\alpha > \alpha_1$ and $g(\alpha) = 0$ for all $\alpha > \alpha_2$ we have $f(\alpha) = g(\alpha) = 0$.  

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for all $\alpha > \max\{\alpha_1, \alpha_2\}$ so $(f - g)(\alpha) = 0$ for all $\alpha > \max\{\alpha_1, \alpha_2\}$. This shows that $H$ is a subgroup of $G$. The restriction of $+ : G \times G \to G$ and $\text{inv} : G \to G$ to $H$ are continuous so $H$ is a topological group. Thus $H$ is a topological subgroup of $G$.

The topological subgroup $H$ is dense in $G$. To see that $\overline{H} = G$ we need to show that for every non-empty open set $U$ of $G$ there is an element $g \in H$ such that $g \in U$.

Note that a non-empty set $U \subset G$ has the form $\prod_{\alpha < \omega_1} U_\alpha$, where $U_\alpha = \{0, 1\}$ for all but finitely many elements of $\omega_1$ say $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ for some $n \in \omega$.

$U_{\alpha_1}, U_{\alpha_2}, U_{\alpha_3}, \ldots, U_{\alpha_n}$ are non-empty open proper subsets of $\{0, 1\}$ so for every $k$ such that $1 \leq k \leq n$ we have $U_k = \{0\}$ or $U_k = \{1\}$. Now consider $g \in H$ defined by $g(\alpha) = 0$ if $\alpha \notin \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ otherwise $g(\alpha_k) \in U_{\alpha_k}$. $g \in U$ so $H$ is a dense subspace of $G$. Thus $\overline{H} = G$.

Now we claim that $H$ is $\omega$-bounded so countably compact.

Consider $F = \{f_n \mid n \in \omega\} \subset H$ we need to show that $\text{cl}_H(F)$ is compact where the closure is taken in $H$. Let $g \in \text{cl}_G(F)$ so every non-empty open neighbourhood $U = \prod_{\alpha < \omega_1} U_\alpha$ where $U_\alpha \neq \{0, 1\}$ for only finitely many $\alpha$’s in $\omega_1$ contains $f_n$ for some $n \in \omega$. $f_n \in H$ so $f_n \in U \cap H$. So $\text{cl}_H(F) \subseteq \text{cl}_G(F)$. On the other hand $H$ has the subspace topology so $\text{cl}_H(F) \subseteq \text{cl}_G(F)$. Hence $\text{cl}_G(F) = \text{cl}_H(F)$. Note that $G$ is a compact space by using Tychonoff’s theorem so $\text{cl}_G(F)$ is compact and consequently $\text{cl}_H(F)$ is compact. Thus $G$ is $\omega$-bounded.

**Corollary 4.1.33.** The subgroup $H \leq G$ in Example 4.1.32 is countably compact.

**Corollary 4.1.34.** The subgroup $H \leq G$ in Example 4.1.32 is $\omega$-bounded.

**Corollary 4.1.35.** In Example 4.1.32, $G$ is not first countable.

**Proof.** If $G$ is first countable by using Kakutani’s theorem we see that $G$ is metrisable. We know that every $\omega$-bounded subspace of metric spaces are compact subspaces. So $H$ is compact which contradicts with $H \neq \overline{H} = G$. Thus $G$ is not first countable. \qed
Corollary 4.1.36. The subgroup $H \leq G$ in Example 4.1.32 is strictly $\omega$-bounded.

Proof. The topological subgroup $H \leq G$ is $\omega$-bounded but $H$ is not compact since $H$ is a dense proper subspace of $G$. Thus $H$ is strictly $\omega$-bounded. \qed

4.2 $D$-squat topological spaces

In Chapter One we proved that every continuous function from the set of all countable ordinals with the order topology onto the real line is eventually constant. In this section we generalise this result.

4.2.1 Ordered topological spaces

Order and topology are closely related. For example the usual topology on the real line has the open intervals as its basis. The open intervals on the real line are defined by using the ordered structure of the real line.

There are several ways to define a topology on partially ordered sets. In this section we mostly adopt the terminology used by Ward in [74], Birkhoff in [5] and specially the definitions used by Wolk in [79] and [80].

Definition 4.2.1. A preorder $(P, \leq)$ is a set $P$ and a binary operation $\leq$ which satisfies the following conditions:

1. For any $x \in P, x \leq x$.
2. For every $x, y, z \in P; x \leq y$ and $y \leq z$, implies $x \leq z$.
3. For any finite set $S \subseteq P$, there is an element $p_S \in P$ such that: for any $s \in S, s \leq p_S$.

Definition 4.2.2. If $(P, \leq)$ is a preordered set and $x_0 \in P$, we define the up set of $x_0 \uparrow$ to be $\{ x \in P \mid x_0 \leq x \}$ and the down set $x_0 \downarrow$ to be $\{ x \in P \mid x \leq x_0 \}$. We also define
$x_0 \uparrow = (x_0 \uparrow) \setminus \{x_0\}$ and $x_0 \downarrow = (x_0 \downarrow) \setminus \{x_0\}$.

**Definition 4.2.3.** If $(P, \leq)$ is a partially ordered set and $x_0 \in P$ and $S \subseteq P$, we define the up set of $S \uparrow$ to be $\{y \in P \mid$ there exists $x \in S$ such that $x \leq y\}$. Similarly the down set of $S \subseteq P$ to be $\{y \in P \mid$ there exists $x \in S$ such that $y \leq x\}$. We also write $E(S) = (S \uparrow) \cap (S \downarrow)$ where $S \subseteq P$.

**Definition 4.2.4.** A subspace $S$ of a partially ordered set $P$ is called monotone increasing (monotone decreasing) if and only if $S = S \downarrow$ ($S = S \uparrow$).

**Definition 4.2.5.** Suppose that $X$ is a topological space endowed with a partial order. The partial order is lower (upper) semicontinuous provided, whenever $a \not\leq b$ ($b \not\leq a$) in $X$, there is an open set $U$, with $a \in U$, such that if $x \in U$ then $x \not\leq b$ ($b \not\leq x$). The partial order is semicontinuous if it is both upper semicontinuous and lower semicontinuous. It is continuous provided that whenever $a \not\leq b$ in $X$, and there are open sets $U$ and $V$ containing $a$ and $b$ respectively such that if $x \in U$ and $y \in V$, then $x \not\leq y$. A partially ordered topological space (POTS) is a topological space with a semicontinuous partial order on $X$.

There are several different ways to define a topology on partially ordered sets to make them a POTS but the best known one is the interval topology.

**Definition 4.2.6.** If $(P, \leq)$ is a partially ordered set, the interval topology in $P$ is a topology with the set $\{(P \setminus x \uparrow) \mid x \in P\} \cup \{(P \setminus x \downarrow) \mid x \in P\}$ as a subbasis of open sets.

**Definition 4.2.7.** A subset $S$ of $P$ is called right Dedekind closed if and only if whenever $D$ is an up directed subset of $S$ and $p = \text{sup} \ D$, then $p \in S$.

**Definition 4.2.8.** A subset $S$ of $P$ is called left Dedekind closed if and only if whenever $D$ is a down directed subset of $S$ and $p = \text{inf} \ D$, then $p \in S$.

**Definition 4.2.9.** A subset $S$ of $P$ is called Dedekind closed if and only if $S$ is right Dedekind and left Dedekind.
Definition 4.2.10. A topology $\mathcal{D}$ on a partially ordered set $P$ is called Dedekind if the closed subsets of $P$ are exactly the Dedekind closed subsets of $P$.

Definition 4.2.11. A topology $\tau$ on a partially ordered set $P$ is called order compatible if and only if $\mathcal{I} \subset \tau \subset \mathcal{D}$ where $\mathcal{I}$ is the interval topology and $\mathcal{D}$ is the Dedekind topology on $P$.

Definition 4.2.12. Let $X$ be a topological space and $P$ a preorder equipped with a topology. We say $X$ is $P$-squat if and only if any continuous function $f : P \to X$ is constant on an up set $p_0 \uparrow$ for some $p_0 \in P$.

Definition 4.2.13. Let $P$ be a partially ordered set and $f : P \to X$ be a function. We say $f$ is eventually constant if and only if there is an element $\alpha_0 \in P$ and $x_0 \in X$ such that $f(\beta) = x_0$ for all $\beta > \alpha_0$.

Example 4.2.14. In Chapter One we showed that any continuous function from the set of all countable ordinals, $\omega_1$, with the order topology to the real line with the usual topology is eventually constant, so the real line $\mathbb{R}$ is $\omega_1$-squat.

Definition 4.2.15. A preordered topological space $P$ is eventually constant with respect to $X$ if and only if any continuous function $f : P \to X$ is eventually constant. If $P$ is eventually constant with respect to $\mathbb{R}$ we say that $P$ is eventually constant.

Definition 4.2.16. A topological space $X$ is locally $P$-squat if and only if every point $x \in X$ has a neighbourhood $U(x)$ such that every continuous function $f : P \to U(x)$ is eventually constant.

Definition 4.2.17. A subset $S$ of a totally ordered set $A$ is called cofinal (coinitial) if and only if for every $x \in A$, there exists $s \in S$ such that $s \geq x$ ($s \leq x$).

Definition 4.2.18. A subset $S$ of a partially ordered set $P$ is cofinal (coinitial) if and only if for each $p \in P$ there is $s \in S$, such that $s \geq p$ ($s \leq p$). If $S$ is not cofinal we call it final above.
Definition 4.2.19. A subset $S$ of a partially ordered set $P$ is bounded above (bounded below) if there is an element $p \in P$ such that $p \geq s$ ($p \leq s$) for all $s \in S$. If $S$ is not bounded above (below) we call it unbounded above (below). If $S$ is bounded below and bounded above we call it bounded.

Lemma 4.2.20. A bounded above subset of a partially ordered set $P$ is final.

Similarly we have:

Lemma 4.2.21. A bounded below subset of a partially ordered set $P$ is initial.

Definition 4.2.22. A preordered set $(P, \leq)$ is complete if and only if any subset of $P$ with an upper bound $p_0$ has a supremum.

4.2.2 Squatness in a more general context

This section contains the main theorem of this chapter which appears as Corollary 4.2.32.

Definition 4.2.23. [52] A topological space $X$ is called lob, linearly ordered base, space if every $x \in X$ has a linearly ordered base, i.e. every point $x \in X$ has an open neighbourhood base which is linearly ordered by the reverse set inclusion order.

Definition 4.2.24. For every regular ordinal $\lambda$ a topological space $X$ is called $\lambda$-small if and only if $|\chi(X)||L(X)| \leq \lambda$, where $L(X)$ is the Lindelöf number of $X$ and $\chi(X)$ is the character of $X$.

Definition 4.2.25. For every regular ordinal number $\lambda$ we say that a topological space $X$ is $\lambda$-lob space if and only if $X$ is a lob space and a $\lambda$-small topological space.

The cofinality of a lob space with respect to the reverse set inclusion relation is a regular cardinal, and equal to the character $\chi(x, X)$ of that space at $x$ [52]. Any other base at $x$ will
contain a subset order-isomorphic to $\chi(x, X)$ [52]. Whenever a point $x$ in any topological space $x$ has a lob, the cofinality of that base (with the respect to reverse inclusion) is a regular cardinal, and equal to the character $\chi(x, X)$ of that space at $x$. Any other base at $x$ will contain a subset order isomorphic to $\chi(x, X)$ [52].

**Theorem 4.2.26.** [40] A function $f : X \to Y$ is continuous at a point $x$ if and only if for every net $\{x_\alpha\}$, $x_\alpha \to x$ implies $f(x_\alpha) \to f(x)$.

**Theorem 4.2.27.** [40] Let $X$ be a topological space. $X$ is a Hausdorff topological space if and only if every net $\{x_\alpha\}$ converges to at most one point.

**Lemma 4.2.28.** Let $X$ be an $\alpha$-lob, Hausdorff topological space, $P$ a partially ordered topological space with the interval topology and $f : P \to X$ a continuous function subject to the following conditions:

1. Every subset of $P$ of cardinality at most $\alpha$ has a supremum in $P$.

2. There is an open neighbourhood basis $\mathcal{N}_x$ at $x$ such that for every $U \in \mathcal{N}_x$, $f^{-1}(U)$ is cofinal in $P$.

Then

$f^{-1}(\{x\})$ is cofinal in $P$.

**Proof.** Consider an arbitrary element $p \in P$. We show that there is an element $p_\theta \in P$ such that $p_\theta > p$ and $f(p_\theta) = x$.

$X$ is a lob space so without loss of generality we assume that $\mathcal{N}_x = \{U_\delta \mid \delta < \theta\}$ for some $\theta \leq \alpha$ such that for every $\beta, \gamma < \theta$, if $\beta < \gamma$, then $U_\gamma \subset U_\beta$.

Let $p_0$ be an element of $f^{-1}(U_0)$ such that $p_0 > p$. If $p_\beta$ is defined for ordinal number $\beta < \theta$ we define $p_{\beta+1}$ to be an element of the cofinal subset $f^{-1}(U_{\beta+1}) \subseteq P$ such that $p_{\beta+1} > p_\beta$.

For a limit ordinal $\lambda$ define $p_\lambda$ to be an element of the cofinal subset $f^{-1}(U_\lambda) \subseteq P$ such that $p_\lambda > p_\beta$ for all $\beta < \lambda$. Condition 1 implies that $\sup\{p_\beta \mid \beta < \theta\}$ exists, say $p_\theta = \sup\{p_\beta \mid \beta < \alpha\}$. 119
Now we show that \( f(p_\theta) = x \). The \( \theta \)-sequence \( \{p_\beta\}_{\beta < \theta} \) is an increasing sequence so for every \( \beta < \theta \), \( \{p_\delta \mid \delta > \beta\} \subset P \setminus (p_\beta \downarrow) \). So \( p_\beta \to p_\theta \).

Note that the \( \theta \)-sequence \( \{p_\beta\} \) is a net and \( f \) is a continuous function so Theorem 4.2.26 implies that \( f(p_\beta) \to f(p_\theta) \). For every \( \beta < \theta \), \( f(p_\beta) \in U_\beta \) so \( f(p_\beta) \to x \). The topological space \( X \) is Hausdorff so Theorem 4.2.27 implies that \( x = f(p_\theta) \). This completes the proof.

**Lemma 4.2.29.** Let \( X \) be an \( \alpha \)-small, Hausdorff topological space, \( P \) a partially ordered topological space and \( f : P \to X \) a continuous function such that:

1. The intersection of every two cofinal closed subsets of \( P \) with cardinality at most \( \alpha \) is not empty.
2. \( f^{-1}\{x\} \) and \( f^{-1}\{y\} \) are cofinal subspaces of \( P \).

Then

\( x = y \).

**Proof.** \( f^{-1}\{x\} \) and \( f^{-1}\{y\} \) are cofinal closed subspaces of \( P \) so \( f^{-1}\{x\} \cap f^{-1}\{y\} \) is a cofinal subset of \( P \). So \( f^{-1}\{x\} \cap f^{-1}\{y\} \neq \emptyset \). \( X \) is a Hausdorff space so \( x = y \). \( \square \)

**Theorem 4.2.30.** Let \( X \) be an \( \alpha \)-small, Hausdorff topological space, \( P \) a partially ordered topological space and \( f : P \to X \) a continuous function subject to the conditions 1-3 below:

1. Every subset of \( P \) of cardinality at most \( \alpha \) has a supremum in \( P \).
2. The union of every family of cardinality at most \( \alpha \) of final subsets of \( P \) is final.
3. The intersection of every family of cofinal subsets of \( P \) with the cardinality at most \( \alpha \) is cofinal.

Then

There is \( x \in X \) such that \( f^{-1}(U) \) is cofinal for every open neighbourhood \( U \) of \( x \).

**Proof.** Assume to the contrary that for every \( x \in X \) there is an open neighbourhood \( U \) of \( x \), \( f^{-1}(U) \) is a final subset of \( P \).
The subset $f^{-1}(U)$ of $P$ is final so there is $p_U \in P$ such that $f^{-1}(U) \cap (p \uparrow) = \emptyset$, so $f(p_U \uparrow) \cap U = \emptyset$.

For every $x \in X$ choose an open neighbourhood $U_x$ of $x$ such that $f^{-1}(U_x)$ is final. These open sets constitute an open cover for $X$ say $\mathcal{V}$. The space $X$ is $\alpha$-small so the Lindelöf number of $X$ is at most $\alpha$. Therefore we can cover $X$ by at most $\alpha$-many elements of $\mathcal{V}$ say $\{V_\beta \mid \beta < \alpha\}$. For every $\beta < \alpha$ let $p_\beta$ to be an element of $P$ such that $f^{-1}(V_\beta) \cap (p_\beta \uparrow) = \emptyset$.

The set of all such elements of $P$ say $S = \{p_\beta \mid \beta < \alpha\}$ has cardinality at most $\alpha$. For every $\beta < \alpha$, $f^{-1}(V_\beta) \cap (p_\beta \uparrow) = \emptyset$ so $f^{-1}(V_\beta) \cap (p_* \uparrow) = \emptyset$, where $p_* = \sup\{p_\beta \mid \beta < \alpha\}$.

Therefore $X \cap f(p_* \uparrow) = \emptyset$ since $X = \bigcup_{\beta < \alpha} V_\beta$. So $S$ is final. This contradicts that $f$ is a function from $P$ into $X$. So there is $x \in X$ such that for every open set $U$ of $x$ containing $x$, $f^{-1}(U)$ is cofinal.

Theorem 4.2.31. Let $X$ be an $\alpha$-small, Hausdorff topological space, $P$ a partially ordered topological space with the interval topology subject to the conditions 1-3 below and $f : P \to X$ a continuous function:

1. Every subset of $P$ of cardinality at most $\alpha$ has a supremum in $P$.
2. The union of every family of cardinality at most $\alpha$ of final subsets of $P$ is final.
3. Intersection of every family of cofinal subsets of $P$ with the cardinality at most $\alpha$ is cofinal.

Then there is $p_* \in P$ such that $f(p_* \uparrow) = x$.

Proof. Theorem 4.2.30 and Lemma 4.2.28 imply that there is $x \in X$ such that $f^{-1}(\{x\})$ is a cofinal subset of $P$. The topological space $X$ is a lob space so without loss of generality we assume that $|\mathcal{N}_x| = \chi(x, X)$ and $\mathcal{N}_x$ is totally ordered by reverse inclusion so we write $\mathcal{N}_x = \{U_\beta(x) \mid \beta < \alpha \text{ and } U_{\beta_2} \subset U_{\beta_1} \text{ for } \beta_1 < \beta_2 < \alpha\}$ [52].

We show that for every $\beta < \alpha$, $f^{-1}(X \setminus U_\beta(x))$ is final subset of $P$.

Assume to the contrary that for every $\beta < \alpha$, $f^{-1}(X \setminus U_\beta(x))$ is cofinal subset of $P$.

The function $f$ is a continuous function, $\{x\}$ and $f^{-1}(X \setminus U_\beta(x))$ are closed subspaces of $P$. So by using condition 3 we see that $\{x\} \cap f^{-1}(X \setminus U_\beta(x)) = \emptyset$ and this contradicts the
assumption that $f^{-1}(X \setminus U_\beta(x))$ is cofinal so $f^{-1}(X \setminus U_\beta(x))$ is a final subset of $P$ for every $\beta < \alpha$.

For every $\beta < \alpha$ there is $p_\beta \in P$ such that $f^{-1}(X \setminus U_\beta(x)) \cap p_\beta \uparrow = \emptyset$ since $f^{-1}(X \setminus U_\beta(x))$ is a final subset of $P$. Let $S$ be the set of such elements of $P$. $|S| = \alpha$ so $S$ has a supremum say $p_*$. For every $\beta < \alpha$, $p_* \subset p_\beta \uparrow$ so $(P \setminus f^{-1}\{x\}) \cap p_* \uparrow = \emptyset$. Therefore $p_* \uparrow \subset f^{-1}\{x\}$. So $f(p_* \uparrow) = x$.

**Corollary 4.2.32.** Let $X$ be an $\alpha$-small, Hausdorff topological space and $P$ a partially ordered topological space with the interval topology subject to the conditions 1-3 below and $f : P \to X$ is a continuous function:

1. Every subset of $P$ of cardinality at most $\alpha$ has a supremum in $P$.
2. The union of every family of cardinality at most $\alpha$ of final subsets of $P$ is final.
3. Intersection of every family of cofinal subsets of $P$ with the cardinality at most $\alpha$ is cofinal.

Then $X$ is $P$-squat.

**Proof.** The proof is the result of Theorem 4.2.31.

**Corollary 4.2.33.** Let $X$ be a Lindelöf Hausdorff first countable topological space. Then $X$ is $\omega_1$-squat.

**Example 4.2.34.** Let $\mathbb{R}_S$ be the Sorgenfrey line. the Sorgenfrey topology is finer than the interval topology on $\mathbb{R}$ so the Sorgenfrey line is aLOTS, linearly ordered topological space.

$\sup\{1 - \frac{1}{n} \mid n \in \mathbb{N}\} = 1$ but the sequence $\{\frac{n-1}{n}\}_{n \in \mathbb{N}}$ does not converge to 1 since for every $n \in \mathbb{N}$, $\frac{n-1}{n} \notin [1, +\infty)$.

Example 4.2.34 shows that if $P$ is a partially ordered topological space with the order topology, $S \subset P$ and $sup S = p$, then it is not necessarily true that $p$ is a limit point of $S$.

The following result is similar to Theorem 4.2.31.
Lemma 4.2.35. Let $X$ be an $\alpha$-small $T_3$ topological space, $P$ a partially ordered topological space with an order compatible topology and $f : P \to X$ a continuous function subject to the following conditions:

1. Every subset of $P$ of cardinality at most $\alpha$ has a supremum in $P$.
2. There is an open neighbourhood basis $N_x$ at $x$ such that for every $U \in N_x$, $f^{-1}(U)$ is a cofinal in $P$.

Then $f^{-1}(\{x\})$ is cofinal in $P$.

Proof. Consider an arbitrary element $p \in P$. We show that there is an element $p_\alpha \in f^{-1}\{x\}$ such that $p < p_\alpha$. Note that $X$ is a lob space so without loss of generality we assume that $N_x = \{U_\beta \mid \beta < \alpha\}$ such that for every two ordinals $\beta, \delta$ where $\beta < \delta < \alpha$, then $U_\delta \subset U_\beta$.

Let $p_0$ be an element of $f^{-1}(U_0)$ such that $p_0 > p$. If $p_\beta$ is chosen for some $\beta < \alpha$ we define $p_{\beta + 1}$ to be an element of the cofinal subset $f^{-1}(U_{\beta + 1})$ such that $p_{\beta + 1} > p_\beta$.

For a limit ordinal $\lambda < \alpha$ let $p_\lambda$ be an element of $f^{-1}(U_\lambda)$ such that $p_\lambda > p_\beta$ for all $\beta < \lambda$.

Note that $\{p_\beta \mid \beta < \lambda\}$ is not cofinal and $f^{-1}(U_\lambda)$ is cofinal in $P$ so the existence of $p_\lambda$ is the result of the condition 1. Now let $p_\alpha = \sup\{p_\delta \mid \delta < \alpha\}$. We claim that $p_\alpha \in f^{-1}(\{x\})$. For every $\beta < \alpha$, $p_\alpha \in \{p_\delta \mid \beta < \delta < \alpha\}$ so $f(p_\alpha) \in f(\{p_\delta \mid \beta < \delta < \alpha\}) \subseteq f(\{p_\delta \mid \beta < \delta < \alpha\}) \subseteq U_{\beta}$.

Note that $X$ is a $T_3$ topological space so for every $\delta < \alpha$, where $x \in U_\delta$ there is $\theta > \delta$ such that $U_\theta \subset U_{\delta}$. So $f(p_\alpha) \in \cap_{U \in N_x} U$. Therefore $f(p_\alpha) = x$. This completes the proof.

Note that Lemma 4.2.28 is not the same as Lemma 4.2.35. In Lemma 4.2.28 $P$ has the interval topology but in Lemma 4.2.35 $P$ has the order compatible topology. On the other hand in Lemma 4.2.28 $X$ is Hausdorff but in Lemma 4.2.35 $X$ is $T_3$. Let $\mathbb{L}_Q = \{(\lambda, q) \mid \lambda \in \omega_1 \text{ and } q \in [0,1) \cap \mathbb{Q}\}$. $\mathbb{L}_Q$ with the lexicographic order topology is called the long rationals.

Any continuous function from $\mathbb{L}_Q \to \mathbb{R}$ is eventually constant since all the conditions in the
previous theorem are satisfied.

**Definition 4.2.36.** A topological space $X$ is locally $L$-squat, where $L$ is a totally ordered set if and only if every point of $X$ has a neighbourhood which is $L$-squat.

**Example 4.2.37.** $\mathbb{L}_+$ is locally $\mathbb{L}_+$-squat but it is not $\mathbb{L}_+$-squat.

The following is an example of a space which is not locally $\omega_1$-squat.

**Example 4.2.38.** Let $id : \omega_1 \to (\omega_1, \tau)$ be the identity function, where the domain has the order topology and $\tau$ is the topology generated by $\{\alpha \uparrow | \alpha \in \omega_1\} \cup \{\omega - 1\} \cup \{\emptyset\}$. Define $\phi : \omega_1 \to \omega_1$ by
\[
\phi(x) = \begin{cases} 
1 + \alpha & (\delta \leq \alpha) \\
\delta & (\alpha < \delta) 
\end{cases}
\]
Then $\phi$ is continuous but it is not constant on $(\alpha, \omega_1)$.

**Lemma 4.2.39.** If $\{X_n | n \in \mathbb{N}\}$ is a family of $\omega_1$-squat spaces, then $\prod_{n=1}^{\infty} X_n$ is $\omega_1$-squat.

**Proof.** Consider the continuous function $f : \omega_1 \to X = \prod_{n=1}^{\infty} X_n$. For every $n \in \mathbb{N}$ the function $\pi \circ f : \omega_1 \to X_n$ is a continuous function so there are $x_n \in X_n$ and $\alpha_n \in \omega_1$ such that $f(\beta) = x_n$ for all $\beta > \alpha_n$. The countable set $\{\alpha_n | n \in \mathbb{N}\}$ is bounded above by some $\alpha \in \omega_1$. So $f(\beta) = x$ for all $\beta > \alpha$ where $x = (x_0, x_1, x_2, \ldots) \in X$. □

**Corollary 4.2.40.** $\prod_{n \in \mathbb{N}} \mathbb{R}$ is $\omega_1$-squat.

**Lemma 4.2.41.** (David Gauld) Let $U$ be an open subset of $\omega_1$ containing a Cub subset $C$ of $\omega_1$. Then $U$ contains a final segment (i.e. a subset of $\omega_1$ which has the form $[\lambda, \omega_1)$ for some $\lambda \in \omega_1$) of $\omega_1$.

**Proof.** Assume that $\lambda \in \omega_1 \cap C$ is a limit ordinal in $C$. Note that $U$ is open and $C \subset U$ so $\lambda \in U$. Then there is an ordinal number $p(\lambda) < \lambda$ such that $(p(\lambda), \lambda] \subset U$, since $\omega_1$ has the
order topology and $\lambda$ is a limit ordinal.

The function $p : C \cap \Lambda \to \omega_1$ is a regressive function so by the PDL, pressing down lemma, there is $\beta \in \omega_1$ such that $p^{-1}(\beta)$ is unbounded. So $(\beta, \omega_1) \subseteq \bigcup_{\alpha \in p^{-1}(\beta)} (\beta, \alpha] \subseteq U$. This completes the proof.

\[ \square \]

**Lemma 4.2.42.** Let $\Lambda$ be the set of all limit ordinals of $\omega_1$. Consider $\frac{\omega_1}{\Lambda}$ with the quotient topology on it, then $\frac{\omega_1}{\Lambda}$ is not first countable and it is not locally $\omega_1$-squat.

**Proof.** We show that $\frac{\omega_1}{\Lambda}$ is a Hausdorff and Lindelöf topological space but it is not $\omega_1$-squat. Let $\pi : \omega_1 \to \frac{\omega_1}{\Lambda}$ be the quotient map.

Consider two arbitrary elements $\delta_1$ and $\delta_2$ of $\frac{\omega_1}{\Lambda}$ such that $\pi^{-1}(\delta_1)$ and $\pi^{-1}(\delta_2)$ are successor ordinals. So $\{\pi^{-1}(\delta_1)\}$ and $\{\pi^{-1}(\delta_2)\}$ are singletons so both sets $\{\pi^{-1}(\delta_1)\}$ and $\{\pi^{-1}(\delta_2)\}$ are open. On the other hand $\pi$ is an open map so $\{\delta_1\}$ and $\{\delta_2\}$ are open and disjoint subsets of $\frac{\omega_1}{\Lambda}$. Now consider $\delta_1 \in \omega_1 \setminus \Lambda$ and $\Lambda$ where $\pi^{-1}(\delta_1)$ is a successor ordinal in $\omega_1$. We see that two open sets $\{\delta_1\}$ and $\omega_1 \setminus \{\delta_1\}$ separates $\delta_1$ and $\Lambda$. So $\frac{\omega_1}{\Lambda}$ is Hausdorff.

The image of $\omega$-bounded spaces under continuous functions is $\omega$-bounded and quotient map $\pi$ is continuous so $\frac{\omega_1}{\Lambda}$ is $\omega$-bounded so countably compact.

We claim that $\frac{\omega_1}{\Lambda}$ is compact. Already we showed that $\frac{\omega_1}{\Lambda}$ is countably compact so we need to show that $\frac{\omega_1}{\Lambda}$ is a Lindelöf space.

Consider an open neighbourhood $U$ of $\Lambda$ in $\frac{\omega_1}{\Lambda}$. Lemma 4.2.41 implies that for every neighbourhood $U$ of $\Lambda$, there is $\delta \in \omega_1$ such that $\pi((\delta, \omega_1)) \subseteq U$. For open neighbourhood $U$ of $\Lambda$ let $\lambda_U = \min \{ \delta \mid \pi((\delta, \omega_1)) \subseteq U \}$. To see that $\frac{\omega_1}{\Lambda}$ is Lindelöf we consider an arbitrary open cover $\mathcal{V}$ for $\frac{\omega_1}{\Lambda}$ and show that $\omega_1$ can be covered by countably many elements of $\{\pi^{-1}(V) \mid V \in \mathcal{V}\}$. Note that there is $\delta \in \omega_1$ and there is $V_{\Lambda} \in \mathcal{V}$ such that $(\delta, \omega_1) \subseteq \pi^{-1}(V_{\Lambda})$. The initial segment $[0, \delta + 1] \subseteq \omega_1$ is countable and $\{\pi^{-1}(V) \mid V \in \mathcal{V}\}$ is an open cover for $\omega_1$. So we can cover $[0, \delta + 1]$ by countably many elements of $\{\pi^{-1}(V) \mid V \in \mathcal{V}\}$.
so $\omega_1/\Lambda$ is a Lindelöf space. So $\omega_1/\Lambda$ is compact since $\omega_1/\Lambda$ is Lindelöf and countably compact. 

Now we show that $\omega_1/\Lambda$ is not locally $\omega_1$-squat. Consider the continuous function $\pi | (\delta U, \omega_1) : (\delta U, \omega_1) \to U$. The continuous function $\pi | (\delta U, \omega_1) : (\delta U, \omega_1) \to U$ is not eventually constant because $\Lambda \cap (\delta U, \omega_1)$ is unbounded in $(\lambda U, \omega_1)$ and for every $\lambda \in (\delta U, \omega_1) \cap \Lambda$, $\pi(\lambda + 1) \neq \Lambda$.

We can extend the continuous function $\pi | (\delta U, \omega_1) : (\delta U, \omega_1) \to U$ to a continuous function $f : \omega_1 \to U$ as follows

$$f(\delta) = \begin{cases} 
\pi(\delta U + 1) & (\delta \leq \delta U + 1) \\
\pi(\delta) & (\delta \geq \delta U + 1).
\end{cases}$$

The function $f$ is continuous but it is not eventually constant since $\pi | (\delta U, \omega_1)$ is not eventually constant. If $\omega_1/\Lambda$ is first countable, then Corollary 4.2.33 implies that $f$ is eventually constant which contradicts the fact that $f$ is not even locally eventually constant. So $\omega_1/\Lambda$ is not first countable.

\[ \square \]

**Proposition 4.2.43.** The topological space $\omega_1/\Lambda$ is not $\omega_1$-squat.

**Proof.** Any $\omega_1$-squat space is clearly locally $\omega_1$-squat, and $\omega_1/\Lambda$ is not locally $\omega_1$-squat. \[ \square \]

**Lemma 4.2.44.** Any countable $T_1$ topological space is $\omega_1$-squat.

**Proof.** Let $f : \omega_1 \to X$ be a continuous function and $X$ be a countable $T_1$ topological space. There is at least one element $x \in X$ such that $f^{-1}(x)$ is an uncountable subset of $\omega_1$ because $\omega_1$ cannot be written as a countable union of countable sets. If $x$ is not unique then we assume there is another element $y$ in $X$ such that $f^{-1}(y)$ is an uncountable subset of $\omega_1$. On the other hand $f^{-1}(x)$ and $f^{-1}(y)$ are closed unbounded subsets of $\omega_1$ so $f^{-1}(x) \cap f^{-1}(y)$ is an unbounded subset of $\omega_1$. Let $\alpha$ be an element of $f^{-1}(x) \cap f^{-1}(y)$, then $x = f(\alpha) = y$ which is not possible since $f$ is a function. Assume that for every $y \in X \setminus \{x\}$,
$f^{-1}\{y\}$ is a bounded subset of $\omega_1$. We show that $f^{-1}\{x\}$ is not bounded. Assume to the contrary that $f^{-1}\{x\}$ is bounded for every $x \in X$. The set $X$ is countable so we can write $X = \{x_n | n \in \omega\}$. Let $\alpha_n$ be an upper bound for $f^{-1}(\{x_n\})$ where $n \in \omega$. Any countable subset of $\omega_1$ is bounded so there is $\gamma \in \omega_1$ larger than all $\alpha_n$'s, so $f^{-1}(X)$ is a bounded subset of $\omega_1$. This contradicts the fact that $\omega_1$ is an unbounded subset of $\omega_1$. Then $X$ is $\omega_1$-squat.

Example 4.2.45. (Arens-Fort space) [69] Let $X = \mathbb{N}^2 \cup \{(0, 0)\} \subset \mathbb{R}^2$. The Arens-Fort topology on $X$ is defined as follows:

For every $p \in \mathbb{N}^2$, the singleton $\{p\}$ is open so the open neighbourhood system at $p$ is $\mathcal{N}_p = \{\{p\}\}$.

We need to define the neighbourhood system at $0 = (0, 0)$. Let $F \in \mathbb{N}^{<\omega}$ and for every $n \in \mathbb{N}$ let $C_n = \{(m, n) | m \in \mathbb{N}\}$ be the $n$th column and $F_n$ be a finite subset of $C_n$.

Now for every $F \in \mathbb{N}^{<\omega}$ and sequence $\{F_n\}_{n \in \mathbb{N} \setminus F}$, we define $\mathcal{N}(F, \{F_n\}_{n \in \mathbb{N} \setminus F}) = \bigcup_{n \in \mathbb{N} \setminus F} (C_n \setminus F_n) \cup \{0\}$. So $\mathcal{N}_0 = \{\mathcal{N}(F, \{F_n\}_{n \in \mathbb{N} \setminus F}) | F \in \mathbb{N}^{<\omega} \text{ and } F_n \subset C_n \text{ is finite for every } n \in \mathbb{N}\}$.

The Arens-Fort space is not first countable [69] but Lemma 4.2.44 implies that the Arens-Fort space is $\omega_1$-squat.

4.3 Realcompact spaces and squatness

In this section we study $\omega_1$-squatness of some topological spaces. We prove that every realcompact first countable topological space is $\omega_1$-squat but the converse is not true in general.
4.3.1 Some theorems and examples related to the squatness

In this section we show that $\Psi$-space which is also called a Mrowka space or an Isbell-Mrowka space introduced independently by Mrowka [45] and Isbell, is an $\omega_1$-squat space. This space is an example of an $\omega_1$-squat, pseudo-compact space which is not realcompact.

**Definition 4.3.1.** [59] (Hausdorff maximal principle) Any non-empty chain in a partially ordered set can be extended to a maximal chain.

**Definition 4.3.2.** Two infinite subsets $S_1$ and $S_2$ of $\mathbb{N}$ are called almost disjoint if and only if $|S_1 \cap S_2| < \omega$.

**Lemma 4.3.3.** [45] Let $\mathbb{N}$ be the set of all natural numbers, and $S$ the family of all its infinite subsets. There exists a family $\mathcal{E} \subset S$ such that:
1. The family $\mathcal{E}$ is infinite.
2. If $S_1, S_2 \in \mathcal{E}$, then $S_1 \cap S_2$ is finite.
3. For every $S \in S$, there is an element $S' \in \mathcal{E}$ such that $S \cap S'$ is infinite.

Now we are ready to define the topological space $\Psi$ by defining the open neighbourhood system for every element of $\mathbb{N} \cup \mathcal{E}$. More details about the $\Psi$-space can be found in [45] or [22].

**Example 4.3.4.** ($\Psi$-space) [45] Let $\Psi = \mathbb{N} \cup \mathcal{E}$. The open neighbourhood system $N_x$ at every point $x$ in $\Psi$ is defined as follows:
1. If $x \in \mathbb{N}$, then $N_x = \{x\}$.
2. If $x \in \mathcal{E}$, i.e. $x = S \in \mathbb{N}$, then $N_x = \{N(x, S') \mid S' \text{ is a finite subset of } S\}$ where $N(x, S') = (\{x\} \cup S) \setminus S'$ where $S'$ is an arbitrary finite subset of $S$.

**Theorem 4.3.5.** [45] [22] The $\Psi$-space has the following properties:
1. For every $x \in \Psi$, $N(x)$ is closed and open.
2. For every $E \subset \mathcal{E}$, $\overline{E} = E$.

3. $\Psi = \mathbb{N}$.

4. $\Psi$ is first countable.

5. $\Psi$ is completely regular.

6. $\Psi$ is not realcompact.

7. For every $x = E \in \mathcal{E}$, $\{x\} \cup E$ is a compact subspace of $\Psi$.

8. For every $x = E \in \mathcal{E}$, $\{x\} \cup E$ is one point compactification of $E$.

**Notation 4.3.6.** We usually use $x_E$ for subset $E$ of $\mathcal{E}$ to emphasis that we look at elements of $\mathcal{E}$ as points of topological space rather than as subsets of $\mathbb{N}$.

**Definition 4.3.7.** [34] A $\{0,1\}$-measure on a set $X$, where $|X| = \lambda$, is a function $\mu : 2^X \to \{0,1\}$ such that for every family $\{E_\delta \mid \delta \in \lambda\}$ of subsets of $X$, $\sum_{\delta \in \lambda} \mu(E_\delta) = \mu(\cup_{\delta \in \lambda} E_\delta)$ and $\mu(\emptyset) = 0$. A $\{0,1\}$-measure $\mu_0$ is called a zero or Dirac measure if and only if $\mu_0(E) = 0$ for all $E \subset X$. A $\{0,1\}$-measure $\mu_x$ is called pointed if and only if $\mu_x(E) = \chi_E(x)$ for every $E \subseteq X$. If the $\{0,1\}$-measure $\mu$ is not pointed or zero we call it proper.

**Definition 4.3.8.** [34] [49] A cardinal number $\lambda$ is called measurable if and only if there is a set $X$ and a proper $\{0,1\}$-measure $\mu : 2^X \to \{0,1\}$ such that $|X| = \lambda$.

All the cardinals produced by the cardinal arithmetic are non-measurable [34] so $\omega$, $\omega_1$ and $c$ are non-measurable cardinals.

**Lemma 4.3.9.** [22] [34] Every discrete space $X$ is realcompact if and only if $|X|$ is a non-measurable cardinal.

**Lemma 4.3.10.** [37] If $W_1$ and $W_2$ are two well-ordered sets, then exactly one of the following holds:

1. $W_1$ is isomorphic to $W_2$.
2. $W_1$ is isomorphic to an initial segment of $W_2$.
3. $W_2$ is isomorphic to an initial segment of $W_1$. 
Proposition 4.3.11. Every cub subset of $\omega_1$ is order isomorphic to $\omega_1$.

Proof. Let $S$ be a cub subset of $\omega_1$. $S$ is uncountable so it is not order isomorphic to any initial segment of $\omega_1$. On the other hand $\omega_1$ is not order isomorphic to any initial segment of $S$ otherwise $\omega_1 \subset S$. So by using Lemma 4.3.10 $S$ is order isomorphic to $\omega_1$. □

Lemma 4.3.12. If $X$ is a discrete topological space, then $X$ is $\omega_1$-squat.

Proof. Let $f : \omega_1 \to X$ be a continuous function. We prove the Lemma in three steps. First we show that for some $x \in X$, $f^{-1}\{x\}$ is an unbounded subset of $\omega_1$. In the second step we show that there is at most one $x \in X$ which has an unbounded inverse image. Finally we show that if $f^{-1}\{x\}$ is unbounded, then $f^{-1}\{x\}$ contains a final segment $[\alpha, \omega_1) \subset \omega_1$ for some $\alpha \in \omega_1$.

Suppose on the contrary that for every $x \in X$, $f^{-1}\{x\}$ is a bounded subset of $\omega_1$. Choose an arbitrary element $x_0 \in X$, so $f^{-1}\{x_0\}$ is a bounded subset of $\omega_1$. Every bounded subset of $\omega_1$ has a supremum so let $\sup f^{-1}\{x_0\} = \alpha_0$. Assume that for $n \in \mathbb{N}$, $x_n$ and $\alpha_n$ are chosen. The union of finitely many bounded subsets of $\omega_1$ is bounded so $f^{-1}\{x_0, x_1, x_2, \ldots, x_n\}$ is a bounded subset of $\omega_1$. Pick an element $\beta > \max\{\alpha_k \mid 0 \leq k \leq n\}$ in $\omega_1$ and $x \in X \setminus \{x_0, x_1, x_3, \ldots, x_n\}$ such that $f(\beta) = x$. The existence of $\beta$ and $x$ is the result of our assumption that the inverse image of $f^{-1}\{x_k\}$ is bounded for $0 \leq k \leq n$. Let $x = x_{n+1}$ and $\alpha_{n+1} = \max\{\beta, \sup f^{-1}\{x_{n+1}\}\}$. Note that the sequence $\{\alpha_n\}_{n \in \omega}$ is an increasing sequence in $\omega_1$ so it converges to some $\alpha \in \omega_1$. The function $f$ is continuous so $f(\alpha_n) \to f(\alpha)$. The space $X$ is discrete so $\{f(\alpha)\}$ is an open subspace of $X$. Then for some $N \in \omega$, $f(\alpha_n) = f(\alpha_N)$ for all $n > N$. This is contradiction. Therefore there is at least one $x \in X$ such that $f^{-1}\{x\}$ is an unbounded subspace of $\omega_1$.

Now we show that there is only one $x \in X$ such that $f^{-1}\{x\}$ is an unbounded subset of $\omega_1$. On the contrary assume that there is more than one element of $X$ which has an unbounded inverse image. If $x_1$ and $x_2$ are two elements of $X$ with unbounded inverse images in $\omega_1$,
then $f^{-1}(x_1)$ and $f^{-1}(x_2)$ are Club subspaces of $\omega_1$ since $X$ is $T_1$ and $f$ is a continuous function. So $f^{-1}(x_1) \cap f^{-1}(x_2)$ is a Club subset of $\omega_1$. Hence $f^{-1}(x_1) \cap f^{-1}(x_2) \neq \emptyset$. This is impossible since $f$ is a function. So there is only one element of $X$ with unbounded inverse image.

Finally we need to show that if $x$ is a point in $X$ with an unbounded inverse image, then there is $\alpha_x \in \omega_1$ such that for every $\alpha > \alpha_x$, $f(\alpha) = x$.

Suppose the inverse image of $\{x\}$ and $X \setminus \{x\}$ are both unbounded. Two subspaces $\{x\}$ and $X \setminus \{x\}$ are closed subspaces of $X$ because $X$ is a discrete topological space so $f^{-1}\{x\}$ and $f^{-1}(X \setminus \{x\})$ have closed unbounded intersection which is a contradiction since $\{x\} \cap (X \setminus \{x\}) = \emptyset$. This completes the proof.

Lemma 4.3.12 and Lemma 4.3.9 imply that if there is a measurable cardinal $\lambda$, then every discrete space $X$ such that $|X| = \lambda$ is an example of $\omega_1$-squat topological space which is not realcompact. Now we are ready to give our example of $\omega_1$-squat space which is not realcompact.

**Lemma 4.3.13.** The $\Psi$ space is $\omega_1$-squat.

**Proof.** Let $f : \omega_1 \to \Psi$ be a continuous function. Suppose $f^{-1}(E)$ is unbounded. Since $\mathbb{N}$ is a union of isolated points it is open and so $E$ and $f^{-1}(E)$ are closed.

By using Proposition 4.3.11 every cub set in $\omega_1$ is a copy of $\omega_1$ i.e. there is a homeomorphism $e : \omega_1 \to f^{-1}(E)$. The function $f \circ e : \omega_1 \to E$ is a continuous function from $\omega_1$ to the discrete set $E$ so Lemma 4.3.12 implies that $f \circ e$ is eventually constant, say $E \in E$ is such that $f(\alpha) = E$ for all large enough $\alpha \in \omega_1$. For every $n \in \mathbb{N}$, $\{E\} \cup (E \setminus \{n\})$ is an open set containing $E$ so $f^{-1}(\{E\} \cup (E \setminus \{n\})$ is an open set containing a cub set in $\omega_1$.

By using Lemma 4.2.41 the complement of this open set is a bounded subset of $\omega_1$, so $f^{-1}(n) \subset \omega_1 \setminus f^{-1}(\{E\} \cup (E \setminus \{x\}))$ is bounded. So $f^{-1}(\mathbb{N}) = \cup_{n \in \mathbb{N}} f^{-1}(n)$ is bounded and
$f$ is eventually constant.

Now we consider the case that $f^{-1}(E)$ is a bounded subset of $\omega_1$. Let $\alpha_0 \in \omega_1$ be an upper bound for $f^{-1}(E)$. So for every $\beta > \alpha_0$, $f(\beta) \in \mathbb{N}$. $f : [\alpha_0, \omega_1) \to \mathbb{N}$ is a continuous function and $[\alpha_0, \omega_1)$ is a cub subset of $\omega_1$ so by using Lemma 4.3.11 and 4.3.12 we see that $f$ is eventually constant. This completes the proof.

The following lemma shows that $L$-squatness is a topological property.

**Lemma 4.3.14.** If $Y$ is an $L$-squat topological space where $L$ is a totally ordered topological space with the order topology and $h : Y \to X$ is a homeomorphism, then $X$ is an $L$-squat topological space.

**Proof.** Let $f : L \to X$ be a continuous function. Then $f \circ h : L \to Y$ is a continuous function so there is an $\alpha_0 \in L$ such that for every $\alpha > \alpha_0$, $(h \circ f)(\alpha) = y_0$ where $y_0$ is a fixed element of $Y$.

Then for every $\alpha \geq \alpha_0$, we have $f(\alpha) = h^{-1}(y_0)$. So $X$ is $L$-squat.

**Notation 4.3.15.** For every limit ordinal $\alpha$, $\text{Lim}(\alpha)$ is the set of all limit ordinals $\beta$ such that $\beta \subset \alpha$. We usually write $\text{Lim}(\omega_1) = \Lambda$.

**Lemma 4.3.16.** Suppose that $t \in (0, 1) \subset \mathbb{R}$. Then $L_t = \{\alpha + t \mid \alpha \in \omega_1\} \cup \Lambda \subset \mathbb{L}_+$ is homeomorphic to $\omega_1$.

**Proof.** We need to show that for every $t \in (0, 1)$, $L_t$ is order isomorphic to $\omega_1$. The following function is an order isomorphism, so it is a homeomorphism.

$$f(\alpha) = \begin{cases} 
  t + \alpha & (\alpha \notin \Lambda) \\
  \alpha & (\alpha \in \Lambda)
\end{cases}$$

To see that $f$ is an order isomorphism consider two arbitrary ordinals $\alpha$ and $\beta$ in $\omega_1$ such
that \( \alpha < \beta \). We have the following cases:

1. If \( \beta \) and \( \alpha \) are elements of \( \Lambda \), then \( f(\alpha) = \alpha < \beta = f(\beta) \).
2. If both \( \alpha \) and \( \beta \) are successor ordinals in \( \omega_1 \) such that \( \alpha < \beta \), then \( \alpha + t < \beta < \beta + t \). So \( f(\alpha) < f(\beta) \).
3. If \( \alpha \) is a successor ordinal and \( \beta \) is a limit ordinal such that \( \alpha < \beta \), then \( \alpha + t < \beta < \alpha + 1 < \beta = f(\beta) \).
4. If \( \alpha \) is a limit ordinal and \( \beta \) is a successor ordinal, then \( f(\alpha) = \alpha + t < \beta = f(\beta) \).

**Lemma 4.3.17.** Let \( t \in \mathbb{Q} \). Then \( \mathbb{L}_t \) is a cub subset of \( \mathbb{L}_\mathbb{Q} \).

**Proof.** For every \( \alpha + q \in \mathbb{L}_\mathbb{Q} \), \( \alpha + 1 + t \in \mathbb{L}_t \) so \( \mathbb{L}_t \) is an unbounded subset of \( \mathbb{L}_\mathbb{Q} \). Consider an arbitrary element \( \alpha + q \) where \( q \in \mathbb{Q} \) and \( q \neq t \) in \( \mathbb{L}_\mathbb{Q} \setminus \Lambda \). We show that \( \alpha + q \) is not a limit point of \( \mathbb{L}_t \). We consider an open neighbourhood \( I = (\alpha + (q - s), \alpha + (q + s)) \subset \mathbb{L}_\mathbb{Q} \), where \( 0 < s < \min\{|q|, |1 - q|, |\frac{|q|}{2}|\} \). We see that \( I \cap \mathbb{L}_t = \emptyset \). So \( \mathbb{L}_t \) is a closed and unbounded subspace of \( \mathbb{L}_\mathbb{Q} \).

**Lemma 4.3.18.** If \( X \) is \( \mathbb{L}_q \)-squat for all \( q \in (0, 1) \cap \mathbb{Q} \), then \( X \) is \( \mathbb{L}_\mathbb{Q} \)-squat.

**Proof.** Note that \( \bigcup_{n \in \omega} \mathbb{L}_{q_n} = \mathbb{L}_\mathbb{Q} \). Consider an enumeration \( \{q_n \mid n \in \omega\} \) of \( \mathbb{Q} \) for \( \mathbb{Q} \). For every \( n \in \omega \) there is an \( \alpha_n \in \Lambda \) and an element \( x_n \in X \) such that for every \( \alpha \in \mathbb{L}_{q_n} \), \( \alpha > \alpha_n \) implies \( f(\alpha) = x_n \).

Now consider the subset \( \{\alpha_n \mid n \in \omega\} \) of \( \omega_1 \). Every countable subset of \( \omega_1 \) is bounded above. Let \( \lambda \in \Lambda \) be an upper bound of \( \{\alpha_n \mid n \in \omega\} \). Note that \( \Lambda \subset \bigcap_{n \in \omega} \mathbb{L}_{q_n} \). So for every \( \alpha \in \mathbb{L}_\mathbb{Q} \), \( \alpha > \lambda \), implies \( f(\alpha) = f(\lambda) \). So for every \( m \in \omega \), every \( n \in \omega \), and every \( \alpha > \lambda \) in \( \mathbb{L}_\mathbb{Q} \) we have \( f(\alpha) = x_n = x_m = f(\lambda) \). This completes the proof.

**Corollary 4.3.19.** If \( X \) is \( \mathbb{L}_\mathbb{Q} \)-squat, then \( X \) is \( \mathbb{L}_+ \)-squat.
Proof. Let \( f \) be a continuous function from \( \mathbb{L}_+ \) into an \( \mathbb{L}_Q \)-squat topological space \( X \). The restriction of \( f \) to \( \mathbb{L}_Q \) is eventually constant. So there is a countable ordinal \( \lambda \in \Lambda \) and an element \( x_0 \in X \) such that for every \( \alpha \geq \lambda \), and every \( q \in \mathbb{Q} \cap [0,1) \) we have \( f(\alpha + q) = x_0 \). Now consider an arbitrary element \( \alpha + r \in \mathbb{L}_+ \) where \( \alpha > \lambda \). There is a sequence \( \{q_n\} \) of rational numbers in \( (0,1) \) which converges to \( r \).

Note that \( f \) is a continuous function and the sequence \( \{\alpha + q_n\}_{n \in \omega} \) converges to \( \alpha + r \) so \( f(\alpha + q_n) \) converges to \( f(\alpha + r) \). On the other hand for every \( n \in \omega \) we have \( f(\alpha + q_n) = x_0 \) so \( f(\alpha + r) = x_0 \). This completes the proof.

Corollary 4.3.20. If \( X \) is an \( \omega_1 \)-squat topological space, then \( X \) is \( \mathbb{L}_+ \)-squat.

Proof. Consider an arbitrary element \( t \in (0,1) \). Define \( f : \omega_1 \to \mathbb{L}_t \) by

\[
f(\alpha) = \begin{cases} 
\alpha + t & (\alpha \notin \Lambda) \\
\alpha & (\alpha \in \Lambda).
\end{cases}
\]

If \( \alpha < \beta \) and both are successor ordinals clearly \( \alpha + t < \alpha + 1 \leq \beta \). If \( \alpha < \beta \) and both are limit ordinals, then \( \alpha + t < \alpha + 1 < \beta \). Similarly if \( \alpha \) is a limit ordinal and \( \beta \) is a successor ordinal and \( \alpha < \beta \), then there is \( \delta \in \omega_1 \) such that \( \delta + 1 = \beta \) and we have the inequality \( \alpha < \delta < \beta \). Finally if \( \alpha < \beta \) for successor ordinal \( \alpha \) and limit ordinal \( \beta \) we have \( \alpha + t < \alpha + 1 < \beta \). Then \( f \) is an order isomorphism so it is a homeomorphism.

So far we have shown that a topological space \( X \) is \( \omega_1 \)-squat if and only if \( X \) is \( \mathbb{L}_t \)-squat.

We know from the assumption that \( X \) is \( \omega_1 \)-squat so \( X \) is \( \mathbb{L}_t \)-squat for all \( t \in (0,1) \). Lemma 4.3.18 and Corollary 4.3.19 imply that \( X \) is \( \mathbb{L}_+ \)-squat. This completes the proof.

The following example shows that in general \( \mathbb{L}_+ \)-squat spaces do not need to be \( \omega_1 \)-squat.

Example 4.3.21. The embedding \( e : \omega_1 \to \mathbb{L}_+ \) which takes \( \alpha \) into \( \alpha + 0 \) is not eventually constant. But if \( f : \mathbb{L}_+ \to \omega_1 \) is a continuous function, then the image of \( \mathbb{L}_+ \) is a connected subspace of \( \omega_1 \) so it has to be a singleton. Then \( \omega_1 \) is \( \mathbb{L}_+ \)-squat.
The following theorem is about another class of \(\omega_1\)-squat topological spaces.

**Theorem 4.3.22.** Assume that \(\{X_i \mid i \in I\}\) is a family of metric spaces and \(C\) is a closed and first countable subspace of \(\prod_{i \in I} X_i\). Then \(C\) is \(\omega_1\)-squat.

**Proof.** We show that every continuous function from \(\omega_1\) to \(C\) is eventually constant. Let \(f : \omega_1 \to C\) be a continuous function. Using Theorem 4.1.21, \(A = f(\omega_1)\) is an \(\omega\)-bounded subspace of \(C\). For every \(i \in I\), the projection \(\pi_i : \prod_{i \in I} X_i \to X_i\) is a continuous function so for every \(i \in I\), \((\pi_i \mid A) : A \to X_i\) is a continuous function. By using Theorem 4.1.22, \(\pi_i(A) = A_i\) is a compact subspace of \(X_i\) for every \(i \in I\). The Tychonoff theorem implies that \(B = \prod_{i \in I} A_i\) is a compact subspace of \(\prod_{i \in I} X_i\) so \(B\) is a Lindelöf space. Any subspace of a first countable space is a first countable space and any metric space is Hausdorff [13]. Moreover the product of Hausdorff spaces is Hausdorff so \(C \cap B\) satisfies the conditions of Corollary 4.2.33. Therefore \(C\) is \(\omega_1\)-squat. \(\square\)

**Proposition 4.3.23.** Any first countable realcompact topological space is \(\omega_1\)-squat.

**Proof.** Let \(X\) be a first countable realcompact topological space and \(f : \omega_1 \to X\) be a continuous function. Then by using Theorem 4.3.22 we see that \(f\) is eventually constant. \(\square\)

### 4.3.2 More examples about \(\omega_1\)-squatness

Our first two examples, the Sorgenfrey line and the Michael line, are well-known as counterexamples so it will be worthwhile to see if they are \(\omega_1\)-squat or not. Our third example is less known and has a heavily set-theoretic structure and is known as the Kunen line. All these three examples are built upon the real line and all of them have a finer topology than the usual one.

**Lemma 4.3.24.** If \((X, \tau_1)\) is an \(\omega_1\)-squat topological space and \(\tau_2\) is another topology on \(X\) and finer than \(\tau_1\), then \((X, \tau_2)\) is \(\omega_1\)-squat.
Proof. Let \( f : \omega_1 \to (X, \tau_2) \) be a continuous function. Then \( f^{-1}(U) \) is open subset of \( \omega_1 \) for every open subset \( U \in \tau_2 \). We know that \( \tau_1 \subseteq \tau_2 \), so \( f : \omega_1 \to (X, \tau_1) \) is continuous. Any continuous function from \( \omega_1 \) into \( (X, \tau_1) \) is eventually constant, so \( f \) is eventually constant. \( \square \)

Example 4.3.25. The Sorgenfrey line is the real line equipped with the topology with the base \( B = \{ [x, y) \mid x, y \in \mathbb{R} \text{ and } x < y \} \). The topology \( \tau \) induced by \( B \) is finer than the order topology, so the Sorgenfrey line is \( \omega_1 \)-squat by the previous lemma.

Example 4.3.26. If \( (\mathbb{R}, \tau) \) is the real line with the usual topology the Michael line, \( \mathcal{M} \), is the real line with the set \( \mathcal{B} = \{ U \cup F \mid U \in \tau \text{ and } F \subseteq \mathbb{R} - \mathbb{Q} \} \) as a basis of its topology. The topology of the Michael line is finer than the usual topology and the real line is \( \omega_1 \)-squat, so the Michael line is also \( \omega_1 \)-squat.

It is easy to see that the Michael line is not Lindelöf using some information from measure theory.

Definition 4.3.27. [24] Let \( E \) be a subset of \( \mathbb{R} \). The Lebesgue outer measure of \( E \) is

\[
\mu^*(E) = \inf \{ \sum_{k \in \omega} l(I_k) : \{ I_k \} \text{ is a sequence of open intervals such that } E \subseteq \bigcup_{k \in \omega} I_k \},
\]

where \( l(I_k) \) is the length of the open interval \( I_k \) for \( k \in \omega \).

Lemma 4.3.28. The Michael line, \( \mathcal{M} \), is not Lindelöf.

Proof. Let \( \{ q_n \mid n \in \omega \} \) be a numeration of all the rationals, and \( \mathcal{U} = \{ (q_n - \frac{1}{2^n + r}, q_n + \frac{1}{2^n + r}) \mid n \in \omega \} \cup \{ (r) \mid r \in \mathbb{R} - \mathbb{Q} \} \) be an open cover for \( \mathcal{M} \). \( \mu^*(\mathbb{R}) = \infty \) but \( \sum_{k \in \omega} \mu^*(I_k) \leq \sum_{n \in \omega} \frac{1}{2^n} < \infty \), so uncountably many irrationals are not in \( \bigcup_{k \in \omega} (q_k - \frac{1}{2^n + r}, q_k + \frac{1}{2^n + r}) \) and \( \mathcal{U} \) does not have a countable subcover. Hence \( \mathcal{M} \) is not Lindelöf. \( \square \)

Theorem 4.3.29. [10] The Michael line, \( \mathcal{M} \), is paracompact.
Lemma 4.3.30. The Michael line, $\mathbb{M}$, is not separable.

Proof. There are $c$ many open sets consisting of a singleton in $\mathbb{M}$, so for any countable subset $D$ of $\mathbb{M}$, there are some open sets disjoint from $D$. Then the Michael line is not separable. 

Our next example is called the Kunen line [39]. As with the Michael and Sorgenfrey lines the underlying set of the Kunen line is the real line $\mathbb{R}$.

The Kunen line $\mathbb{K}$ is the topological space consisting of $\mathbb{R}$ with a new topology $\tau_K$ on the reals that we will describe.

The new topology is finer than the usual topology so Lemma 4.3.24 shows that the Kunen line is $\omega_1$-squat.

To define the Kunen line we need some background information and historical motivation which makes the Kunen line interesting to study. A good reference for the definition of $\mathbb{K}$ is [60]. Another reference is the original paper by Juahász, Kunen and Rudin [39].

The topological properties “separability” and “Lindelöfness” are equivalent for metric spaces [10]. For general topological spaces these two properties are not equivalent. The lexicographic ordered square (see page 73 in [69]) or $\omega_1 + 1$ are examples of non-separable compact topological spaces. The cantor tree, the Moore-Nemytskii plane and the Prüfer surface are examples of separable spaces which are not Lindelöf [69] [10] [51].

A space $X$ is hereditarily separable if every subspace of $X$ has a countable dense subspace and $X$ is hereditarily Lindelöf if every subspace of $X$ is a Lindelöf space. In the late 1960’s a problem was posed by Hajnal and Juhász in [29] and independently by some other topologists and is called the S- and L- space problem. A $T_3$ topological space $X$ is called an L- space if it is hereditarily Lindelöf but is not separable, and a $T_3$ topological space $X$ is called an S-space if it is hereditarily separable but is not Lindelöf.

There are many spaces constructed from both types in different models of set theory. The
Gödel constructible universe contains both types [10]. Stevo Toderčević has constructed a model which does not have S-spaces. The existence of both types and the non-coexistence of both types does not lead to any contradiction. This shows that the S- and L-problems are independent of the axioms of set theory.

Similar constructions to the Kunen line $\mathbb{K}$ were introduced by Ostaszewski in [53] and Hajnal and Juhász in [29] and [30] under different set-theoretical assumptions to introduce S-spaces.

Ostaszewski in [53] assuming the $\diamond$ axiom introduced a construction which produces S-spaces which are countably compact but not realcompact.

A similar attempt was made by Hajnal and Juhász in [29] and [30] assuming the Continuum Hypothesis.

The construction that is closely related to the Kunen line and we are more interested in here first appeared in [39]. The authors assuming the Continuum Hypothesis as in [29] and [30] used a blend of methods used in [29], [30] and [53] to produce a $T_3$ S-space which is first countable but is not Lindelöf. Their construction is easier to understand and some properties such as local compactness and perfect normality are immediate.

Although the Kunen line is not a totally ordered topological space in fact the Kunen line is the result of introducing a new topology on the real line which is finer than the usual topology on $\mathbb{R}$.

The result of the construction used in [39] on the real line is called the Kunen line.

Let $K(X)$ be the output of this construction for the space $X$. The space $K(X)$ is not $T_3$ in general but if $X$ is a first countable Hausdorff topological space of the cardinality $\omega_1$, then $K(X)$ is first countable $T_3$ locally compact locally countable and not Lindelöf topological space.

We write $\mathbb{K} = K(\mathbb{R})$. The advantage of the Ostaszewski construction in [53] is that the resulting space is countably compact which is not the case in general for $K(X)$ even if $X$
is Hausdorff. There is another difference between these two constructions in that if \( X \) is a first countable \( T_2 \) topological space of cardinality \( \omega_1 \), then \( K(X) \) is realcompact while the resulting space of the Ostazewski construction is not in general realcompact.

**Definition 4.3.31.** A space \( X \) is called left separated if and only if \( X \) can be well ordered in such a way that every initial segment is closed.

**Definition 4.3.32.** A space \( X \) is called right separated if and only if \( X \) can be well ordered in such a way that every initial segment is open.

**Theorem 4.3.33.** [10] A space \( X \) is hereditarily separable if and only if \( X \) has no uncountable left separated subspace. A space \( X \) is hereditarily Lindelöf if and only if \( X \) has no uncountable right separated subspace.

**Definition 4.3.34.** [10] Let \( \tau \) be a topology on a set \( X \), and \( Y \subset X \). We write \( \tau \mid Y \) for the relative topology on \( Y \). If \( \tau \) is a topology on \( X \), \( \sigma \) a topology on \( Y \), and \( Y \subset X \), we say that \( \tau \) is a conservative extension of \( \sigma \) if and only if \( \sigma = \tau \mid Y \), i.e. if and only if the space \((Y, \sigma)\) is an open subspace of \((X, \tau)\).

**Definition 4.3.35.** [60] Suppose \( \{X_\alpha \mid \alpha < \kappa\} \) is an increasing collection of sets, \( \tau_\alpha \) is a topology on \( X_\alpha \) for each \( \alpha \), and \( \tau_\alpha \) is a conservative extension of \( \tau_\beta \) if \( \beta < \alpha < \kappa \). We define the simple limit topology \( \sum_{\alpha < \kappa} \tau_\alpha \) to be the unique conservative extension of \( \tau_\alpha \) on \( \cup_{\alpha < \kappa} X \), i.e. it is the topology on \( \cup_{\alpha < \kappa} X \) whose basis is \( \cup_{\alpha < \kappa} \tau_\alpha \).

**Lemma 4.3.36.** [60] Let \( \tau = \sum_{\alpha < \kappa} \tau_\alpha \). If every \( \tau_\alpha \) is Hausdorff, so is \( \tau \). If every \( \tau_\alpha \) is zero dimensional, so is \( \tau \).

**Definition 4.3.37.** (Kunen line) [60] [39] Assume \( CH \) and let \( \{x_\alpha \mid \alpha < \omega_1\} \) enumerate \( \mathbb{R} \). Let \( \{A_\alpha \mid \alpha < \omega_1\} \) enumerate all countably infinite subsets of \( \mathbb{R} \). Define \( X_\alpha = \{x_\beta \mid \beta < \alpha\} \) and let \( B_\alpha = \{A_\beta \mid \beta < \alpha, A_\beta \subset X_\alpha, x_\alpha \in \text{cl}_{\mathbb{R}}(A_\beta)\} \). Enumerate each \( B_\alpha \) as \( \{C_{\alpha, n} \mid n < \omega\} \) where each \( \tau_\alpha \) is a topology on \( X_\alpha \) and if \( \beta < \alpha \) then \( \tau_\alpha \) is a conservative extension of \( \tau_\beta \).
If $\tau = \sum_{\alpha < \omega_1} \tau_\alpha$ is the topology defined in Definition 4.3.35, then each $X_\alpha$ is open under $\tau$, so $(\mathbb{R}, \tau)$ is right separated in type $\omega_1$, and since each $\tau_\alpha$ is 0-dimensional so is $\tau$.

Now we construct $\tau_\alpha$ by induction for $\alpha < \omega_1$.

Define the topology $\tau_0 = \{\emptyset, \{x_0\}\}$ on $X$, where $X = \{x_0\}$.

Constructing $\tau_\alpha$ for $\alpha > 0$: The induction hypothesis at stage $\alpha < \omega_1$ is

(I) If $\beta < \alpha$, then $\tau_\beta$ is a 0-dimensional topology refining the usual topology (as a subspace of $\mathbb{R}$) on $X_\beta$.

(II) If $\gamma < \beta < \alpha$, then $\tau_\gamma = \tau_\beta \mid X_\gamma$.

(III) If $\beta + 1 < \alpha$ and $A \in B_\beta$, then $x_\beta \in cl_{\tau_{\beta+1}}(A)$.

If $\alpha$ is a limit ordinal, let $\tau_\alpha = \sum_{\beta < \alpha} \tau_\beta$. Then the induction hypothesis still holds.

Now suppose $\alpha = \beta + 1$. Pick a sequence $\{y_n \mid n \in \omega\}$ so each $y_n \in C_{\alpha,n}$ and the distances $\epsilon_n$ between $x_\beta$ and $y_n$ form a sequence decreasing to zero. Let $\{I_n \mid n \in \omega\}$ be a pairwise disjoint collection of intervals in $\mathbb{R}$ with $x_\beta \notin cl_{\mathbb{R}}(I_n)$ and $y_n \in I_n$.

For each $n$ pick $U_{n,\beta} \in \tau_\beta$, $U_{n,\beta}$ a clopen (in $\tau_\beta$) neighbourhood of $y_n$ contained in $I_n$.

Define

$$V_{n,\beta} = \{x_\beta\} \cup \bigcup_{k > n} U_{k,\beta}$$

and let $\tau_\alpha$ be the topology generated by $\tau_\beta \cup \{V_{n,\beta} \mid n < \omega\}$.

Checking the induction hypothesis we see that (II) is immediate, (III) follows from the fact that each $A \in B_\beta$ appears as infinitely many as $C_{\beta,n}$'s and the second half of (I) is immediate. We have to show that $\tau_\alpha$ is 0-dimensional.

Let $\mathcal{U}$ be a clopen basis for $\tau_\beta$ containing no set of the form $\bigcup_{n \in A} V_{n,\beta}$ where $A$ is infinite. $\mathcal{U}$ exists because $\tau_\beta$ refines the usual topology on $\mathbb{R}$.

Then $\mathcal{U}' = \mathcal{U} \cup \{U \cap V_{n,\beta} \mid n < \omega \text{ or } U = X_\alpha\}$ is a basis for $\tau_\alpha$. Suppose that $x \in X_\alpha$, $V \in \mathcal{U}'$, and $x \notin V$. We have to find a neighbourhood of $x$ avoiding $V$. We realise two cases:

1. $x \in X_\beta$ and $V \in \mathcal{U}$. Since $\tau_\beta$ is zero dimensional we are done.
2. $x \in X_\beta$ and $V = U \cap V_{n,\beta}$ for some $n$. Suppose $x$ lies in the left of $x_\beta$ in the usual ordering of $\mathbb{R}$. Pick $k$ so that if $s = \inf_{j > k} I_j$ then $s \neq x$. Since $\tau_\beta$ refines the usual topology on $\mathbb{R}$ we can find a neighbourhood $W$ of $x$ in $\tau_\beta$ so $W \cap (U \cap \bigcup_{n < j < k} U_{j,\beta}) = \emptyset$, $W \subset (-\infty, s)$, so we are done. If $x$ lies to the right of $x_\beta$ the proof is similar.

3. $x = x_\beta$, $V \in U$. By hypothesis, some $V_{n,\beta}$ misses $V$ and we are done.

By construction each $\tau_\beta$ is first countable and hence also $\tau$.

Note that by adding a further induction hypothesis (that each $\tau_\beta$ is locally compact) and requiring the $V_{n,\beta}$'s to be clopen compact, the final space is locally compact.

**Theorem 4.3.38.** [39] If $X$ is hereditarily separable, then

1. $K(X)$ is hereditarily separable.
2. If all closed sets of $X$ are $G_\delta$, then the same is true for $K(X)$.
3. If $X$ is $T_3$ and hereditarily Lindelöf, then $K(X)$ is normal.
4. If $X$ is $T_3$, and Lindelöf, then $K(X)$ is realcompact.

**Proposition 4.3.39.** The Kunen line is $\omega_1$-squat.

*Proof.* We already proved that first countable realcompact topological spaces are $\omega_1$-squat, so the Kunen line $\mathbb{K}$ is $\omega_1$-squat. 

It is clear from the construction of the Kunen line that it has $\mathbb{R}$ as its underlying set and has a finer topology than the usual topology so it is $\omega_1$-squat by 4.3.24.
Chapter 5

Long pipes and pseudo-products

“The most original contributions to the theory of infinity in ancient times were the theory of proportions and the method of exhaustion .”

–Mathematics and its history, John Stillwell –

Nyikos Bagpipe theorem introduced by Peter Nyikos in 1984 [50] shows that any $\omega$-bounded surface, (2-dimensional topological manifold), consists of a bag, a compact surface with finitely many open disks removed with a boundary homeomorphic to finitely many disjoint circles, and finitely many pipes. In [50] each pipe is a topological surface which can be represented as an increasing $\omega_1$-sequence of open cylinders such that each of them has a boundary homeomorphic to the circle inside the larger cylinders.

In [21] a modified definition of a long pipe is given which makes it easier to visualize and we adopt the later definition here. More precisely any $\omega$-bounded surface $S$ can be written as $S = K \cup (\cup_{i=1}^n P_i)$, where for each $i \in \{1,2,3,\ldots,n\}$, $P_i$ is a long pipe and $K$ is a compact surface with $n$-many open disks $D_1^2, \ldots, D_n^2$ of $S$ with disjoint boundaries removed.
In Nyikos Bagpipe decomposition theorem each $P_i$ can be written as $P_i = \bigcup_{\alpha<\omega_1} U_\alpha$, where $U_\alpha \cong S^1 \times [0,1)$ and $\partial U_\beta U_\alpha \cong S^1$. In [50] Peter Nyikos proves that there are uncountably many different, non-homeomorphic, long pipes, so there is no hope of classifying all $\omega$-bounded surfaces but still it is feasible to study intrinsic properties of long pipes. In this chapter we concentrate on long pipes and their generalisations to higher dimensions.

5.1 Examples of the long pipes

As we mentioned before, the concept of the long pipe first appeared in [50] as follows.

**Definition 5.1.1.** A space is a long pipe if it is the union of a chain $\{U_\alpha | \alpha < \omega_1\}$ of open subspaces $U_\alpha$ homeomorphic to $S^1 \times \mathbb{R}$, such that $U_\alpha \subset U_\beta$ and such that the boundary of $U_\alpha$ in $U_\beta$ is homeomorphic to the unit circle whenever $\alpha < \beta$.

**Example 5.1.2.** (Open long cylinder) Consider the space $S^1 \times L_o$. For every $\alpha \in \omega_1$ define $U_\alpha = S^1 \times (0,\alpha)$. So $\{S^1 \times (0,\alpha) | \alpha \in \omega_1\}$ is a decomposition for the long pipe $S^1 \times L_o$.

$S^1 \times L$ is not a long pipe. To see this assume that $\{U_\alpha | \alpha \in \omega_1\}$ is an increasing $\omega_1$-sequence of subspaces of $S^1 \times L$ such that $\overline{U_\alpha}$ is a Lindelöf space for all $\alpha \in \omega_1$, $S^1 \times \mathbb{R} \cong U_\alpha$ for all $\alpha \in \omega_1$ and $\overline{U_\alpha} \subset U_\beta$ where $\alpha < \beta$. So consider $U_0$. We see that the boundary of $U_0$ in $S^1 \times L$ consists of two disjoint copies of a circle, say $S_1$ and $S_2$. $\{U_\alpha | \alpha \in \omega_1\}$ is an increasing open cover for $S^1 \times L$, so there is $U_\alpha$ such that $S_1 \cup S_2 \subset U_\alpha$. We know that in the definition of the long pipe the boundary of $U_\alpha$ in $U_\beta$ is homeomorphic to the unit circle. This shows that $S^1 \times L$ is not a long pipe.

Note that in the definition of long pipe the boundary of $U_\alpha$ in $U_\beta$ is homeomorphic to the unit circle for all $\beta > \alpha$ and it is not enough if the boundary of $U_\alpha$ in $U_{\alpha+1}$ is homeomorphic to the unit circle.

We note that in the following decomposition of $S^1 \times L$, the topological boundary of $U_\alpha$ in
\( U_{\alpha+1} \) is homeomorphic to the unit circle but the topological boundary of \( U_\alpha \) in \( U_{\lambda+1} \) consist of two components each of which is homeomorphic to the unit circle, where \( \lambda \) is a limit ordinal and \( \lambda > \alpha \).

\[
(0, 1) \times S^1 \subset (0, 2) \times S^1 \subset \cdots \subset (0, \omega) \times S^1 \subset \cdots \subset (-\omega, \omega) \times S^1 \subset \cdots \subset (-\omega, \omega+1) \times S^1 \subset \cdots \subset (-\omega, \omega+2) \times S^1 \subset \cdots \subset (-\omega, 2, \omega.2) \subset \cdots
\]

Note that \( \partial(U_{\omega+1}) \) is disjoint union of two copies of \( S^1 \) where \( U_0 = S^1 \times (0, 1) \) and \( U_{\omega+1} = S^1 \times (-1, \omega) \).

**Example 5.1.3.** *(Closed long cylinder)* The topological space \( \mathbb{L}_+ \times S^1 \) is called the closed long cylinder.

In [21] the long pipes of [50] are called open long pipe and the closed long pipe is defined as follows:

**Definition 5.1.4.** A closed long pipe \( P \) is a closed 2-manifold with a boundary homeomorphic to the unit circle such that \( P = \cup_{\alpha \in \omega} U_\alpha \), where each \( U_\alpha \) is an open submanifold of \( P \) with the same boundary homeomorphic to \( S^1 \times [0, \infty) \) such that the topological boundary of \( U_\alpha \) in \( U_\beta \) is homeomorphic to the unit circle and \( \overline{U_\alpha} \subset U_\beta \) whenever \( \alpha < \beta \).

Following [21] by long pipe we usually mean the closed long pipe and where we mean open long pipe we mention it.

**Example 5.1.5.** \( \mathbb{L}_+ \times S^1 \) is a closed long pipe. To see the decomposition we simply need to define \( U_\alpha = S^1 \times [0, \alpha) \) for each \( \alpha \in \omega_1 \). In this example \( \partial U_\alpha U_\alpha = S^1 \times \{\alpha\} \) which is homeomorphic to the unit circle.

**Definition 5.1.6.** An open long pipe \( P_o \) is \( P \setminus \partial P \), where \( P \) is a closed long pipe and \( \partial P \) is the boundary of \( P \) as a manifold (all points of \( P \) which have a neighborhood homeomorphic to the half closed plane).
Example 5.1.7. The closed long cylinder is a manifold with boundary. \( \partial(\mathbb{L}_+ \times S^1) = \{0\} \times S^1 \). We see that \((\mathbb{L}_+ \times S^1) \setminus (\{0\} \times S^1) = \mathbb{L}_o \times S^1\).

It is worth mentioning that in [50] Nyikos shows that there are \(2^{\aleph_1}\) non-homeomorphic long pipes.

5.2 Long pipes and their generalisations

In [50] page 668 Nyikos asks how the bagpipe theorem can be generalised to higher dimensions. More precisely he asks: Is it true that an \(\omega\)-bounded \(n\)-manifold has a compact submanifold with boundary \(K\) such that \(M \setminus K\) is the union of finitely many subspaces, each of which is the union of a chain of subsets homeomorphic to \(N \times \mathbb{R}\) for the same compact \(n-1\) manifold \(N\)?

In this section we imitate the construction of the long pipes in more a general context, and discuss alternative definitions in higher dimensions. In general the long pipe is not canonical and this makes it complicated to study. The situation is even more complicated in higher dimensions. The following definition has the advantage that it helps us to unify the results about subsets of the product of metric spaces and the the long lines, and the product of the long line and a metric space.

Definition 5.2.1. A partial function \(f\) from \(X\) to \(Y\) is a subset of \(X \times Y\) such that for two pairs \((x, y), (x, y') \in f\) we have \(y = y'\). We usually write \(f : X \hookrightarrow Y\) to show \(f\) is a function from a subset of \(X\) to \(Y\).

Definition 5.2.2. A pseudo-product \(P\) is a triple \((P, \{\phi_\alpha\}_{\alpha \in \omega_1}, B)\), where \(P\) is an \(n\)-dimensional topological manifold, \(\{\phi_\alpha\}_{\alpha \in \omega_1}\) is a family of partial homeomorphisms from \(P\) onto \(B \times [0,1)\) and \(B\) is an \((n-1)\)-dimensional closed manifold such that:

1. \(\{\text{dom}(\phi_\alpha)\}_{\alpha \in \omega_1}\) is an open cover for \(P\).
2. If $\alpha < \beta$, then $dom(\phi_\alpha) \subset dom(\phi_\beta)$.

3. $\phi_\alpha : dom(\phi_\alpha) \to B \times [0,1)$ is a homeomorphism and it is possible to extend $\phi_\alpha$ to $\bar{\phi}_\alpha : cl_{dom(\phi_\beta)}(dom(\phi_\alpha)) \to B \times [0,1)$ for all $\beta > \alpha$.

Our definition of the pseudo-product is the same as the following definition which is easier to visualize.

**Definition 5.2.3.** (David Gauld) An $n$-manifold $P$ with boundary is a pseudo-product if and only if there is an $(n - 1)$-manifold $B$ which is called the base and an $\omega_1$-sequence $\{P_\alpha | \alpha \in \omega_1\}$ such that for every $\alpha < \beta$, $\overline{U_\alpha} \subset U_\beta$, $\partial U_\beta(U_\alpha) \cong B$, and $U_\alpha \cong B \times [0,1)$.

**Example 5.2.4.** The closed long ray is a pseudo-product. For each $\alpha \in \omega_1$ let $\phi_\alpha$ be a homeomorphism from $[0, \alpha)$ to $[0,1)$. In this case $B$ is the zero dimensional connected manifold which is a singleton.

**Example 5.2.5.** If $B = S^1$ in Definition 5.2.2, then we see that every long pipe is a pseudo-product.

The following is an example of a pseudo-product of dimension higher than 2 that is not homeomorphic to a long pipe or the long ray.

**Example 5.2.6.** The product of the 2-sphere and the closed long line, $S^2 \times L_+$, is a pseudo-product of dimension 3.

### 5.3 Nonmetrisable subsets of the product of the pseudo-products and a metric space

In [50] closed non-metrisable subspaces of $L_+ \times \mathbb{R}$ contain a closed copy of $\omega_1$. In this section we generalise some of these results for subspaces of the product of a metrisable manifold $M$ and a long pipe $P$. 

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Lemma 5.3.1. [50] Every closed, non-Lindelöf subspace of $\mathbb{L}^+$ contains a closed unbounded subset of $\omega_1$.

Theorem 5.3.2. [50] In the 2-manifold $\mathbb{L}^+ \times \mathbb{R}$:
1. Every closed non-metrisable subspace of $\mathbb{L}^+ \times \mathbb{R}$ contains a closed copy of $\omega_1$.
2. Every copy of $\omega_1$ is, with the exception of at most countably many points, a subset of $\mathbb{L}^+ \times \{r\}$ for some $r \in \mathbb{R}$.
3. For every copy $M$ of $\mathbb{L}^+$ in $\mathbb{L}^+ \times \mathbb{R}$, there exists $\alpha$ such that $M \cap [\alpha, \omega_1) \times \mathbb{R} = [\alpha, \omega_1) \times \{r\}$ for some $r \in \mathbb{R}$.

We introduce the concept of the length of a subset $Y \subset X$ in a closed subset $C$ of $X \times P$ to study non-metrisable subsets of $X \times P$ where $X$ is a topological space and $P$ is a pseudo-product.

Definition 5.3.3. Let $X$ be a topological space, $Y \subset X$ and $P = \cup_{\alpha \in \omega_1} P_\alpha$ be a pseudo-product as defined in Definition 5.2.2. We define the length of $Y$ in a closed subset $C$ of $X \times P$ to be the minimum of

$$C' = \{ \lambda \in \omega_1 | (Y \times (P \setminus P_{\lambda+1})) \cap C = \emptyset \}$$

if the minimum exists. Otherwise we say the length of $Y$ is $C$ is $\omega_1$. We denote the length of $Y$ in $C$ by $l_C(Y)$. For a singleton $Y = \{y\}$ we write $l_C(y)$.

Example 5.3.4. The closed long ray, $\mathbb{L}^+$, is an example of a pseudo-product. To see this let $B$ be a singleton, and for every $\alpha \in \omega_1$ let $P_\alpha$ be $[0, \alpha) \subset \mathbb{L}^+$. So $\mathbb{L}^+$ is a pseudo-product. Theorem 5.3.2 implies that for every closed non-metrisable subspace $C$ of $\mathbb{L}^+ \times \mathbb{R}$ there is a point $r \in \mathbb{R}$ such that $l_C(r) = \omega_1$.
Figure 5.1: Length of points in subspace of the product of circle and the long line

Lemma 5.3.5. [78] Every second-countable space is hereditarily Lindelöf.

Lemma 5.3.6. [78] If $X$ is a hereditarily Lindelöf topological space and $E \subset X$, then the set $E^* = \{ x \in E \mid x$ is not an accumulation point of $E \}$ is countable.

Lemma 5.3.7. Let $P = \bigcup_{\alpha \in \omega_1} P_\alpha$ be a pseudo-product and $M$ a metrizable manifold. If $C$ is a closed non-metrisable subspace of $M \times P$, $S_n \subset M$ for $n = 1, 2$ and $S_1 \subseteq S_2$, then $l_C(S_1) \leq l_C(S_2)$.

Proof. Note that $S_1 \subseteq S_2$ implies that $\{ \lambda \in \omega_1 \mid (S_1 \times (P \setminus P_{\lambda+1})) \cap C = \emptyset \} \subseteq \{ \lambda \in \omega_1 \mid (S_2 \times (P \setminus P_{\lambda+1})) \cap C = \emptyset \}$ so $\min \{ \lambda \in \omega_1 \mid (S_1 \times (P \setminus P_{\lambda+1})) \cap C = \emptyset \} \leq \min \{ \lambda \in \omega_1 \mid (S_2 \times (P \setminus P_{\lambda+1})) \cap C = \emptyset \}$. This completes the proof.

Corollary 5.3.8. Let $P = \bigcup_{\alpha \in \omega_1} P_\alpha$ be a pseudo-product and $M$ a metrizable manifold. If $C$ is a closed non-metrisable subspace of $M \times P$, $S_n \subset M$ and $l_C(S_n) \neq \omega_1$ for all $n \in \mathbb{N}$, then $l_C(\bigcup_{n \in \mathbb{N}} S_n) \neq \omega_1$.

Proof. Let $\alpha_n = l_C(S_n)$ for every $n \in \omega$. The sequence $\{ \alpha_n \}_{n \in \omega}$ is bounded above by some countable ordinal $\alpha$.

Note that for every $n \in \omega$, $S_n \times (P \setminus P_{\alpha_n}) \cap C = \emptyset$ so $S_n \times (P \setminus P_{\alpha}) \cap C = \emptyset$ since $P \setminus P_{\alpha} \subseteq P \setminus P_{\alpha_n}$. Then $(\bigcup_{n \in \omega} S_n) \times (P \setminus P_{\alpha}) \cap C = \emptyset$. Hence $l_C(\bigcup_{n \in \omega} S_n) \neq \omega_1$. 

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Theorem 5.3.9. Let $M$ be a metrisable topological manifold and $C$ be a closed, non-metrisable subset of $M \times P$, where $P$ is a pseudo-product. Then there is a countable ordinal number $\lambda_C \in \omega_1$ such that for all $x \in M$, $l_C(x) < \lambda_C$ or $l_C(x) = \omega_1$.

Proof. Assume to the contrary that for every $\lambda \in \omega_1$, there is an $x \in M$ such that $l_C(x) \neq \omega_1$ and $\lambda < l_C(x)$.

Let $N = \{x \in M \mid l_C(x) \neq \omega_1\}$. We need to show that for some $\lambda_0 \in \omega_1$, we have $l_C(N) \leq \lambda_0$. Note that $M$ is a metric manifold so by Lemma 5.3.5, $M$ is hereditarily Lindelöf.

Note that $N \subseteq M$ and so $N$ is a hereditarily Lindelöf space. By Lemma 5.3.6, all elements of $N$ except countably many of them are limit points of $N$. Hence $N \setminus N'$ is a countable subset of $N$, where $N' = \{x \in N \mid x$ is a limit point of $N\}$. Note that $N'$ is Lindelöf.

We realise two cases:

Case 1. There is a point $p \in N'$ such that for every neighbourhood $U$ of $p$ in $M$ we have $l_C(U \cap N') = \omega_1$.

Case 2. For every point $p \in N'$ there is an open neighbourhood $U$ of $p$ in $M$ such that $l_C(U \cap N') < \omega_1$.

Now we discuss these two cases:

Case 1. Assume that there is a point $p \in N'$ such that for every neighbourhood $U$ of $p$ in $M$ we have $l_C(U \cap N') = \omega_1$ and $l_C(p) = \lambda_0 < \omega_1$.

Note that $M$ is a manifold so first countable so there is a countable nested decreasing neighbourhood base of open sets of $p$ say $\{U_n \mid n \in \mathbb{N}\}$ such that $U_{n+1} \subset U_n$ for all $n \in \mathbb{N}$.

Assume $p \in N'$ has length $\alpha_0$ in $C$ and let $p_1 \in U_1 \cap N'$ be an arbitrary element of length $\alpha_1 > \alpha_0$. So there is $\lambda_1 \in \omega_1$ such that $\lambda_1 = l_C(p_1)$.

For every $n \in \mathbb{N}$ choose $p_{n+1} \in U_{n+1} \cap N' \setminus U_n \cap N'$ such that $\alpha_n = l_C(p_n) < l_C(p_{n+1}) = \alpha_{n+1} < \omega_1$ and choose $x_n \in \overline{(P_{\alpha_n} \setminus P_{\alpha_{n-1}})} \cap C$ such that $(p_n, x_n) \in (U_n \cap N') \times (\overline{P_{\alpha_n}} \setminus P_{\alpha_{n-1}}) \cap C$. Note that $\alpha_n \to \alpha$ for some $\alpha \in \omega_1$ and $x_n$ is an element of the compact set.
\( P_{\alpha+1} \cap C \) for all \( n \in \mathbb{N} \). So \( \{x_n\} \) has an accumulation point \( x \) in \( P_{\alpha} \cap C \).

\( P_n \to p \) so \( (p, x) \in \{p\} \times P_{\alpha} \setminus \bigcup_{n \in \mathbb{N}} P_{\alpha_n} \) which contradicts \( p \in N' \) and \( l_C(p) = \alpha_0 < \alpha_1 \). Hence if \( p \in N' \), then there is a neighbourhood \( U \) of \( p \) such that \( l_C(N' \cap U) < \omega_1 \). So case 1 does not happen. Now we discuss the second case.

Case 2. Assume that for every point \( p \in N' \) there is an open neighbourhood \( U \) of \( p \) in \( M \) such that \( l_C(U \cap N') < \omega_1 \).

For every \( p \in N' \) choose an open neighbourhood \( U(p) \) of \( p \) in \( M \) such that \( l_C(V(p)) < \omega_1 \) where \( U(p) \cap N' = V(p) \). Consider the set \( V = \{V(p) \mid p \in N'\} \). For every \( p \in N' \), \( V(p) \) is open subset of \( N' \). Note that \( V \) is an open cover of \( N' \) and \( N' \) is a Lindelöf space so \( N' \) can be covered by countably many elements of \( V \) say \( \{V_n \mid n \in \mathbb{N}\} \).

By using Lemma 5.3.9 we see that \( l_C(\bigcup_{n \in \mathbb{N}} V_n) < \omega_1 \) so \( l_C(N') < \omega_1 \). So \( \lambda_C = l_C(N') \) is the desired ordinal number.

\[ \square \]

**Corollary 5.3.10.** Let \( M \) be a metrisable topological manifold and \( C \) be a closed, non-metrisable subset of \( M \times P \), where \( P \) is a long pipe. Then there is a countable ordinal number \( \lambda_C \in \omega_1 \) such that for all \( x \in M \), \( l_C(x) < \lambda_C \) or \( l_C(x) = \omega_1 \).

**Corollary 5.3.11.** Let \( M \) be a metrisable topological manifold and \( C \) be a closed, non-metrisable subset of \( M \times \mathbb{L}_+ \). Then there is a ordinal number \( \lambda_C \in \omega_1 \) such that for all \( x \in M \), \( l_C(x) < \lambda_C \) or \( l_C(x) = \omega_1 \).

**Theorem 5.3.12.** [10] If \( M \) is a metric space, then the following are equivalent:

1. \( M \) is Lindelöf.
2. \( M \) is separable.
3. \( M \) is second countable.

**Corollary 5.3.13.** A metric space \( M \) is hereditarily Lindelöf if and only if it is hereditarily separable.
Definition 5.3.14. Let $M$ be a metrisable manifold and $C$ a closed non-metrisable subspace of $M \times \mathbb{L}_+$. We define the $x$-section of $C$ by $(M \times \{x\}) \cap C$ and denote it by $\text{sec}_C(x)$.

Theorem 5.3.15. Let $M$ be a metrisable manifold and $C$ be a closed unbounded subspace of $M \times \mathbb{L}_+$, then there is a subspace $N$ of $M$ and a closed unbounded subset $C'$ of $\omega_1$ such that for any $\lambda \in C'$, $\pi_M[\text{sec}_C(\alpha)] = N$.

Proof. Note that $M$ is a connected metrisable manifold so by Corollary 5.3.13 and Lemma 5.3.5, $M$ is hereditarily separable.

Let $N = \{x \in M \mid l_C(x) = \omega_1\}$ and let $D$ be a countably dense subspace of $N$. For every $d \in D$ define $S_d = (\pi_{\mathbb{L}_+}\{d\} \times \mathbb{L}_+) \cap C$, where $\pi_{\mathbb{L}_+}$ is the projection on $\mathbb{L}_+$.

For every $d \in D$, $S_d$ is a closed and unbounded subspace of $\mathbb{L}_+$ so is $S_D = \cap_{d \in D} S_d$ closed and unbounded. The subspaces $M \times S_D$ and $C$ are closed subspaces of $M \times \mathbb{L}_+$ so is their intersection. We see that for every $x \in S_D$, $\pi_M(\text{sec}_C(x)) = D$ so for every $x \in S_D$ we have $\text{sec}_X(x) = N$. This completes the proof. \hfill \Box

5.4 Contractibility of pseudo-products

In the following theorem we show that pseudo-products are not contractible. The proof is similar to the proof of non-contractibility of the long ray given in [18].

Definition 5.4.1. Let $X$ be a topological space. A subspace $U$ of $X$ is called relatively Lindelöf if and only if $\overline{U}$ is a Lindelöf subspace of $X$.

Lemma 5.4.2. [13] The cartesian product $X \times Y$ of a countably compact topological space $X$ and a compact topological space $Y$ is countably compact.

Lemma 5.4.3. [13] Every closed subspace $C$ of a countably compact topological space $X$ is countably compact.
Lemma 5.4.4. [13] The cartesian product of a Lindelöf topological space $X$ and a compact topological space $Y$ is a Lindelöf topological space.

Lemma 5.4.5. [13] Every closed subspace $C$ of a Lindelöf topological space $X$ is a Lindelöf topological space.

Lemma 5.4.6. Let $\{U_n \mid n \in \omega\}$ be a countable family of relatively Lindelöf subspaces of a pseudo-product $P = \bigcup_{\alpha \in \omega_1} P_\alpha$ where for every $\alpha \in \omega_1$, $P_\alpha \cong B \times [0,1)$. Then $\bigcup_{n \in \omega} U_n$ is relatively Lindelöf.

Proof. First we show that for every Lindelöf subspace $L \subset P$, there is $\alpha \in \omega_1$ such that $L \subset P_\alpha$. Assume to the contrary that for every $\alpha \in \omega_1$, $L \cap (P \setminus P_\alpha) \neq \emptyset$. So $\{P_\alpha \mid \alpha \in \omega_1\}$ is an open cover for $L$ which does not have a countable subcover. This contradicts the Lindelöfness of $L$. So every Lindelöf subspace of $P$ is contained in $P_\alpha$ for some $\alpha \in \omega_1$.

For every $n \in \omega$, $\overline{U}_n$ is a Lindelöf space so there is a countable ordinal $\alpha_n \in \omega_1$ such that $\overline{U}_n \subset P_{\alpha_n}$. Consider the subset $S = \{\alpha_n \mid n \in \omega\}$ of $\omega_1$. The set $S \subset \omega_1$ is countable so $S$ is bounded above. Let $\alpha$ be an upper bound for $S$. For every $n \in \omega$, $\overline{P_{\alpha_n}} \subset P_\alpha$ so $U = \bigcup_{n \in \omega} U_n \subset P_\alpha$. Hence $\overline{U} \subset \overline{P_\alpha} \subset P_{\alpha+1}$. On the other hand $P_{\alpha+1} \cong B \times [0,1)$ which is a Lindelöf space since it is a product of a compact space and a Lindelöf space. So the closed subspace $\overline{U}$ of $P_{\alpha+1}$ is a Lindelöf space. This completes the proof. $\square$

Consider a pseudo-product $P = \bigcup_{\alpha \in \omega_1} P_\alpha$, where $P_\alpha \cong B \times [0,1)$ for a compact manifold $B$ and $\alpha \in \omega_1$. For every $\alpha \in \omega_1$, there is $\beta \in \omega_1$ such that $\beta > \alpha$ and $\overline{P_\alpha} \subset P_\beta$. Note that $P_\beta$ is homeomorphic to $B \times [0,1)$ so it is countably compact by Lemma 5.4.2. The set $\overline{P_\alpha}$ is a closed subspace of $P_\beta$ so by Lemma 5.4.3 it is countably compact.

On the other hand by Lemma 5.4.4 $P_\beta$ is a Lindelöf space. The subset $\overline{P_\alpha}$ of $P_\beta$ is closed so by Lemma 5.4.4 it is Lindelöf. So for every $\alpha \in \omega_1$, $\overline{P_\alpha}$ is Lindelöf and countably compact space so it is compact.

Consider a countable subset $\{x_n \mid n \in \omega\}$ of a pseudo-product $P$. For every $n \in \omega$ there is
a countable ordinal $\alpha_n$ such that $x_n \in P_{\alpha_n}$. Every countable subset of $\omega_1$ is bounded above by a countable ordinal. So $\{\alpha_n\}_{n \in \omega}$ is bounded above by a countable ordinal say $\alpha$. So $\{x_n \mid x \in \omega\} \subset \bigcup_{n \in \omega} P_{\alpha_n} \subset P_\alpha \subset \overline{P_\alpha}$. So we have the following lemma:

**Lemma 5.4.7.** Every Pseudo-product is $\omega$-bounded.

**Corollary 5.4.8.** Every pseudo-product is countably compact.

**Definition 5.4.9.** A subset $S$ of a topological space $X$ is sequentially closed if and only if for every point $p \in X$, and every sequence $\{s_n\}$ in $S$ if $\{s_n\}$ converges to $p$, then $p \in S$.

**Definition 5.4.10.** A topological space $X$ is is called sequential space if and only if every sequentially closed subspace of $X$ is closed subspace of $X$.

**Theorem 5.4.11.** [13] Every first countable topological space is sequential space.

**Theorem 5.4.12.** Pseudo-products are not contractible.

*Proof.* Assume there is a pseudo-product $P$ which is contractible. Let $P = \bigcup_{\alpha < \omega_1} P_\alpha$, where $P_\alpha \cong B \times [0,1)$ for all $\alpha < \omega_1$ and $B$ be a compact manifold.

We assumed that $P$ is contractible, so there is a contraction $F : P \times [0,1] \to P$ such that $F(P \times \{0\}) = \{p_0\}$, and $F \mid P \times \{1\} = id$.

Define the set $S = \{t \in [0,1] \mid \overline{F(P \times [0,t])}$ is compact $\}$. Note that $\overline{F(P \times \{0\})} = \{p_0\}$ is compact, so $0 \in S$. If $t \in S$ and $s < t$, then $\overline{F(P \times [0,s])}$ is a closed subset of compact set $\overline{F(P \times [0,t])}$, so $\overline{F(P \times [0,s])}$ is compact, and $s \in S$.

The set $S$ is a bounded subset of $\mathbb{R}$. Then $S$ has a least upper bound say $sup S = r$. We show that $S$ is sequentially closed and first countable, so it is closed.

To show that $S$ is sequentially closed let $\{r_n\}_{n \in \omega}$ be a sequence converging to $r$. Without loss of generality we assume $\{r_n\}_{n \in \omega}$ is increasing, otherwise we consider an increasing subsequence of $\{r_n\}_{n \in \omega}$. 
The function $F$ is continuous, and Lemma 5.4.6 implies that a countable union of relatively Lindelöf subspaces of $P$ is relatively Lindelöf so $F(P \times [0, r]) = \overline{F(P \times [0, r])} \subseteq F(P \times \bigcup_{n \in \omega} [0, r]) = \bigcup_{n \in \omega} F(P \times [0, r_n])$. On the other hand for some $\alpha_n \in \omega_1$ we have $F(P \times [0, r_n]) \subseteq P_{\alpha_n}$, so $\bigcup_{n \in \omega} F(P \times [0, r]) \subseteq \bigcup_{n \in \omega} P_{\alpha_n} \subset P_{\alpha + 1}$, where $\alpha$ is the limit of the increasing sequence $\{\alpha_n \mid n \in \omega\}$.

This shows that $F(P \times [0, r])$ is relatively compact, and $r \in S$. To complete the proof we need to show that $r = 1$.

Assume to the contrary that $r \neq 1$. We show that this assumption contradicts the maximality of $r$.

Let $\beta \in \omega_1$ be a countable ordinal greater than $\alpha + 1$. Note that $F(P \times [0, r]) \subset \overline{F(P \times [0, r])} \subset P_\beta$. So $F^{-1}(P_\beta)$ is an open subset of $P \times [0, 1]$ containing $P_\lambda \times [0, r]$, for an arbitrary $\lambda \in \omega_1$.

We have $\pi_2(P_\lambda \times [0, r]) = [0, r] \subset \pi_2(F^{-1}(P_\beta))$ and $\pi_2$ is an open map so $\pi_2(F^{-1}(P_\beta))$ is an open interval of $[0, 1]$ containing 0.

This shows that there is $n_\lambda \in \omega$ such that $P_\lambda \times [0, r + \frac{1}{n_\lambda}] \subset \pi_2(F^{-1}(P_\beta))$. Define the function $f : \omega_1 \to \omega$ such that $f(\lambda) = n_\lambda$. The function $f$ is eventually constant by Lemma 4.2.44.

Let say $f(\lambda) = n_0$ for all $\lambda > \delta_0$ for some $\delta_0 \in \omega_1$.

Therefore $P \times [0, r + \frac{1}{n_0}] \subset F^{-1}(P_\beta)$ and consequently $F(P \times [0, r + \frac{1}{n_0}]) \subset \overline{P_\beta} \subset P_{\beta + 1}$.

This contradicts the maximality of $r$ and this completes the proof.

\textbf{Corollary 5.4.13.} The closed long pipes are not contractible.

\textbf{Corollary 5.4.14.} [18] The closed long ray is not contractible.

\section{5.5 Boundedness and $\omega$-boundedness}

Richard Dedekind formulated the theory of ideals in algebra [2] when he was dealing with the ring of integers. The concept of ideal is the result of an evolution of ideal numbers
introduced by E. E. Kummer [2]. In 1936 M. H. Stone in the paper “Boolean algebras and their applications to topology” used ideals in the context of Boolean algebra [70]. Shortly after Stone’s paper appeared J. W. Alexander published a paper titled “On the concept of topological space” [1]. In his paper Alexander modifies the definition of a topological space in such a way that it is possible to distinguish between bounded and unbounded parts of the space [1]. He defines the closed bounded subsets of a set $X$ to be a subfamily $B$ of $X$ which has the empty set as its member; it is closed under finite union and arbitrary intersection. Then he introduced closed subsets of $X$ to be subsets of $X$ which have closed bounded intersection with closed bounded subsets of $X$. Finally he defined a subset of $X$ to be bounded if it had a closed bounded closure, where the closure of a set $S$ is defined to be the intersection of all closed sets containing $S$ [1]. In 1949 Sze-Tsen-Hu published a paper “Boundedness in a topological space” and in that paper he defines a boundedness on a topological space $(X, \tau)$ to be a family $B$ of subsets of $X$ which is closed under finite union and any subset $C$ of an element $B \in B$ is an element of $B$ [35]. It is clear that a boundedness on a space $X$ is nothing new but an ideal on $X$. We choose Hu’s approach since his terminology is more compatible with geometric intuition. On the other hand Henri Cartan developed the theory of filters in 1937 which proved to be very useful in studying topological properties of a topological space. We use a boundedness universe to generalize the concept of $\omega$-boundedness of topological spaces.

**Definition 5.5.1.** A boundedness in a topological space $X$ is a non-empty family $B$ of subsets of $X$ such that:

1. If $B \in B$ and $S \subseteq B$, then $S \in B$.
2. The union of a finite number of elements of $B$ is an element of $B$.

**Definition 5.5.2.** A universe $(X, \tau, B)$ is a topological space $(X, \tau)$ with a given boundedness
\( B \) on \( X \). Elements of \( B \) are called bounded subsets of \( X \).

**Example 5.5.3.** Let \( X \) be a topological space. The family of all relatively compact subspaces of \( X \) is a boundedness.

Note that the family of compact subsets of a topological space \( X \) may not constitute a boundedness since an arbitrary subset of a compact space does not need to be compact.

**Example 5.5.4.** Let \( X \) be a topological space. The family of all relatively Lindelöf subspaces of \( X \) is a boundedness.

**Example 5.5.5.** If \( X \) is a topological space all countable subspaces of \( X \) constitute a boundedness.

Note that a topological space is \( \omega \)-bounded if and only if every element of boundedness \( B \) is an element of the boundedness \( C \), where \( B \) is the family of all countable subsets of \( X \) and \( C \) is the family of all relatively compact subsets of \( X \).

**Definition 5.5.6.** Let \( \mathcal{U} = (X, \mathcal{B}, \tau) \) be a universe. A \( \lambda \)-exhaustion of \( \mathcal{U} \) is an increasing family of bounded open subsets \( U_\alpha \) of \( X \) indexed by \( \lambda \) which covers \( X \), and for every \( \alpha < \beta < \lambda \), we have \( U_\alpha \subseteq U_\beta \). Moreover if for every limit ordinal \( \alpha < \lambda \), \( U_\beta = \bigcup_\alpha U_\beta \) we say \( \mathcal{U} \) is a canonical exhaustion, \( \mathcal{B} \)-canonical or simply canonical.

**Definition 5.5.7.** A universe \( \mathcal{U} = (X, \mathcal{B}, \tau) \) is called proper if there is a \( \mathcal{B} \)-canonical exhaustion \( \Sigma = \{B_\alpha \mid \alpha < \lambda_0\} \) which satisfies the following conditions, where \( \lambda_0 \) is a regular cardinal number:

1. \( X = \bigcup_{\alpha<\lambda_0} B_\alpha \).

2. \( X \neq \bigcup_{\alpha<\lambda_1} B_\alpha \), for \( \lambda_1 < \lambda_0 \).

3. \( B_\alpha \) is open and connected for \( \alpha < \lambda_0 \).
Our universes are usually proper except when we mention specifically otherwise.

**Definition 5.5.8.** Let $\mathcal{U} = (X, \mathcal{B}, \tau)$ be a proper universe where $X$ is locally connected with a $\mathcal{B}$-canonical exhaustion $\Sigma$. We define the small upsilon tree $\upsilon_\mathcal{B} (\Sigma)$ to be a rooted tree with an imaginary node $x_U$ for every unbounded component $U$ of $X \setminus \overline{B_\alpha}$ as a node in level $\alpha$, where $\alpha \geq 0$ and an imaginary node $x$ as the root with the following order: $x_U \leq x_V$ if and only if $U = V$ or $V \subset U$.

**Definition 5.5.9.** Let $\mathcal{U} = (X, \mathcal{B}, \tau)$ be a proper universe where $X$ is locally connected with a $\mathcal{B}$-canonical exhaustion $\Sigma$. We define the upsilon tree $\Upsilon_\mathcal{B} (\Sigma)$ to be a rooted tree with an imaginary node $x_U$ for every boundary of unbounded components of $X \setminus \overline{B_\alpha}$ as nodes in level $\alpha$, where $\alpha \geq 0$ and an imaginary node $x$ as the root with the following order: $x_U \leq x_V$ if and only if $U = V$ or $\partial X V \subset U$.

**Definition 5.5.10.** Let $\mathcal{U} = (X, \mathcal{B}, \tau)$ be a universe with a canonical exhaustion $\Sigma$. Then $X$ is called $\mathcal{B}$-trunklike if given any closed bounded subset $C$, $X \setminus C$ has at most one unbounded component.

**Lemma 5.5.11.** Let $\mathcal{U} = (X, \mathcal{B}, \tau)$ be a universe and $X$ be a locally connected topological space. If $\upsilon_\mathcal{B} (\Sigma)$ is a chain, then $X$ is $\mathcal{B}$-trunklike.

**Proof.** If $X$ is not $\mathcal{B}$-trunklike $X \setminus \overline{B_\alpha}$ has at least two unbounded components $U$ and $V$. Consider $x_U$ and $x_V$ of $\upsilon_\mathcal{B} (\Sigma)$. $U \cap V = \emptyset$, so $x_U \neq x_V$. $U \cap V = \emptyset$, so $\overline{V} \not\subseteq U$ and $\overline{U} \not\subseteq V$. Then $x_U \nleq x_V$ and $x_V \nleq x_U$. \hfill $\square$

The following lemma suggests that we can use either $\Upsilon$-tree or $\upsilon$-tree to find out if a space $X$ is trunklike or not but in general these two trees do not need to be the same.

**Lemma 5.5.12.** Let $\mathcal{U} = (X, \mathcal{B}, \tau)$ be a proper universe and $X$ a locally connected topological space, then the following statements are equivalent:
1. \( \nu_B(\Sigma) \) is a chain.

2. \( \Upsilon_B(\Sigma) \) is a chain.

Proof. (1) \( \Rightarrow \) (2) Assume \( \Upsilon_B(\Sigma) \) is not a chain, so \( \Upsilon_B(\Sigma) \) has two non-comparable nodes \( \partial_X(U) \) and \( \partial_X(V) \), then two nodes \( x_U \) and \( x_V \) are not comparable in \( \nu_B(\Sigma) \), then \( \nu_B(\Sigma) \) is not a chain.

(2) \( \Rightarrow \) (1) If \( \nu_B(\Sigma) \) is not a chain, then let \( x_U \) and \( x_V \) be two non comparable nodes of \( \nu_B(\Sigma) \), corresponding to two open components \( U \) and \( V \) of \( X \setminus B_\alpha \) for some \( \alpha \in \lambda_0 \) respectively. \( (X \setminus \overline{B_\alpha+1}) \cap U \) and \( (X \setminus \overline{B_\alpha+1}) \cap V \) have two unbounded components \( U' \) and \( V' \) respectively. \( \partial_X U' \subset U \) and \( \partial_X V' \subset V \), so \( \partial_X U' \) and \( \partial_X V' \) are not comparable in upsilon tree \( \Upsilon_B(\Sigma) \). Then \( \Upsilon_B(\Sigma) \) is not a chain.

Lemma 5.5.13. If \( \Sigma_1 \) and \( \Sigma_2 \) are two canonicals for the boundedness \( B \) of the proper universe \( U = (X, B, \tau) \), then \( \nu_B(\Sigma_1) \) is a chain if and only if \( \nu_B(\Sigma_2) \) is a chain.

Proof. Assume that \( \nu_B(\Sigma_1) \) is not a chain, then there are at least two uncomparable nodes \( x_U \) and \( x_V \) in the same level say level \( \alpha \). So \( X \setminus \overline{B_\alpha} \) has two unbounded components \( U \) and \( V \). Let \( \overline{B_\beta} \) be a bounded basic set of \( \Sigma_2 \), then \( (X \setminus \overline{B_\beta}) \cap U \) has at least one unbounded component \( U' \). The set \( (X \setminus \overline{B_\beta}) \cap V \) has at least one unbounded component \( V' \). \( U \cap V = \emptyset \), so \( U' \cap V' = \emptyset \), then \( x'_U \) and \( x'_V \) are not comparable in \( \nu_B(\Sigma_2) \). Then \( \nu_B(\Sigma_2) \) is not a chain.

Lemma 5.5.14. Assume \( \alpha_0, \beta_0 \) and \( \lambda_0 \) are regular cardinals and \( \alpha_0^+ \leq \beta_0 < \lambda_0 \). Let \( T \) be a tree with the following properties:

1. \( \text{ht}(T) = \lambda_0 \).

2. \( |T(\alpha)| \leq \alpha_0 \) for all \( \alpha < \lambda_0 \).

3. All branches are level cofinal, \( b \cap T(\alpha) \neq \emptyset \) for every branch \( b \) of \( T \) and \( \alpha < \lambda_0 \).

Then

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\[ T = S \cup (\cup_{i<\alpha_0} b_i), \] where the \( b_i \)'s are cofinal branches and \( S = \{ x \in T \mid ht(x) \leq \alpha_0 \} \).

**Proof.** Let \( T^* \) be the subset of \( T \) consisting of all nodes \( x \in T \) which have at least two branching nodes, nodes with at least immediate successors. If \( |T^*| < \alpha_0 \), we write \( T^* = \{ x_\alpha \mid \alpha < \alpha_0 \} \). Choose an increasing sequence \( \beta_n \) such that \( ht(x_n) < \beta_n \) for \( n < \alpha_0 \) so \( \beta_n \to \beta' \) for ordinal number \( \beta' < \alpha_0 \). Then \( ht(x) < \alpha_0 + 1 \) for all \( x \in T^* \). If \( |T^*| \geq \alpha_0 \), then let \( T^* = \{ x_0, x_1, x_2, \ldots, x_{\alpha_0}, \ldots \} \) and show that for some \( \alpha' < \lambda_0 \), \( |T^*(\alpha')| > \alpha_0 \) which contradicts the second assumption. For each \( x \in T^* \) we have at least two branches and all branches are level cofinal so any branch \( b \) intersects which have an element \( x \in T^* \) as its element intersect \( T(\alpha_0) \), \( |T(\alpha') \cap b| > 1 \). Then \( |T'(\alpha')| > |T^*| = |\alpha'| \) which contradicts the second assumption. \( T(\alpha_0) \) for \( \beta > \alpha_0 + 1 \) does not contain any branching node and its cardinality is \( |\alpha_0| \) so \( T = S \cup (\cup_{i<\alpha_0} b_i) \), where \( S = \cup_{\alpha<\alpha_0+1} T(\alpha) \).

**Theorem 5.5.15.** Let \( U = (X, B, \tau) \) be a universe such that:

1. \( X \) is locally connected and connected.
2. Elements of \( B \) are connected open and locally connected.
3. There is a canonical exhaustion for \( U \) of length \( \lambda \) of connected open bounded sets.
4. The Suslin number of \( X \), \( c(X) = \beta \) is less than the length of the exhaustion, \( \lambda_0 \).
5. Any open cover of \( X \) of cardinality \( \beta \) has an open subcover of cardinality \( \alpha_0 \) where \( \alpha_0 < \beta_0 < \lambda_0 \).

Then,

\[ X = B \cup (\cup_{i<\beta} T_i), \] where \( T_i \)'s are trunklike.

**Proof.** Let \( B_\alpha \) be a bounded set in the exhaustion \( \{ B_\alpha \mid \alpha \in \lambda \} \). Any component of \( X \setminus \overline{B_\alpha} \) is open because \( X \) is locally connected. If \( C \) is a component of \( X \setminus \overline{B_\alpha} \), \( C \cap \overline{B_\alpha} \neq \emptyset \).
Let $p \in C \cap \overline{B_\alpha}$, and let $U(p)$ be an open neighbourhood of $p$ in $B_{\alpha+1}$. The closure of $B_\alpha$ is a subset of open bounded set $B_{\alpha+1}$ so $C \cap B_{\alpha+1}$ is an open subset of $B_{\alpha+1}$. The Suslin number of $B_{\alpha+1}$, $s(\beta_{\alpha+1}) = \beta$, is absolutely less than $\lambda$, the length of exhaustion. So $\beta$, the number of components of $X \setminus \overline{B_\alpha}$, is absolutely less than $\lambda$. Now $\{B_{\alpha+1}\} \cup \{C \mid C$ is a component of $X \setminus \overline{B_\alpha}\}$ is an open cover for $X$, so it has a subcover of the cardinality $\alpha_0$, and we have at most $\alpha_0$ many unbounded components. Assume the number of bounded components of $X \setminus \overline{B_\alpha}$ is $\alpha_1$. For every bounded component $C$ of $X \setminus \overline{B_\alpha}$ there is an ordinal number $\delta < \alpha_1$ such that $C \subset B_\delta$. So there is a bounded set $B_{\alpha_1}$ such that all components of $X \setminus \overline{B_\alpha}$ are unbounded. The number of components cannot exceed the Suslin number of the $B_\alpha$ so only $\alpha$ many of them cover the space $X$. For any $\alpha$ such that $\alpha_1 < \alpha < \lambda$, $X \setminus \overline{B_\alpha}$ has at most $\alpha$ many unbounded components and this shows that any node of the $\nu$-tree of $X$ has at most $\alpha$ many successors. Hence $|T(\alpha)| \leq \alpha$. Let $\phi : \lambda \to \alpha$ be a function that assigns the cardinality of the $T(\delta)$ to $\delta$. This function has to be constant on a tail $[\beta_0, \lambda)$ so we can find a bounded connected open set $B_{\beta_0}$ such that any node of $\Upsilon_B(X)$ has only one successor. We write $X = B_{\beta_0} \cup (\cup_{i<\alpha} B_i)$.

Corollary 5.5.16. *(Bagpipe lemma [42]*) If $M$ is an $\omega$-bounded manifold, then $M = K \cup (\cup_{i=1}^n U_i)$, where the $U_i$’s are mutually disjoint non-Lindelöf connected open submanifolds of $M$.

*Proof.* Theorem 5.5.15 guarantees that any $\omega$-bounded manifold has a canonical of open Lindelöf connected subspaces which is the same as an exhaustion of the length $\omega_1$. This manifold is countably compact because it is $\omega$-bounded. So by Theorem 5.5.15 we can write $M = K \cup (\cup_{i=1}^n U_i)$, where $K$ is a compact manifold and the $U_i$’s are non-Lindelöf mutually disjoint submanifolds of $M$.

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Appendices
Appendix A

Topology and order

A.1 Some definitions

Definition A.1.1. A partially ordered set (Poset) is a set $P$ in which a binary relation $\leq$ is defined, which satisfies for all $x, y, z$ in $P$ the following conditions:

- For all $x \in P$, $x \leq x$. (Reflexive)

- If $x \leq y$ and $y \leq z$, then $x \leq z$. (Transitivity)

- If $x \leq y$ and $y \leq x$, then $x = y$. (Antisymmetry)

Definition A.1.2. A poset $P$ which satisfies the following condition is called simply ordered, totally ordered or a chain.

- Given any $x$ and $y$, $x \leq y$ or $y \leq x$.

Definition A.1.3. A function $f : P \rightarrow Q$ from a poset $P$ into a poset $Q$ is called isotonic or order preserving if and only $x \leq y$ implies $f(x) \leq f(y)$. An isotonic bijection with an isotonic inverse is called an isomorphism.
Definition A.1.4. An upper bound (lower bound) for subset $X$ of a poset $P$ is an element greater (smaller) than or equal to all elements of $X$. If $U(X)(L(X))$ is the set of all upper bounds (lower bounds) of $X$, then $p \in U(X)(q \in L(X))$ which is smaller (larger) than all elements of $X$ is called, l.u.b, least upper bound (g.l.b, greatest lower bound) of $X$ and we write $p = l.u.b(X) = \sup(X)(q = g.l.b(X) = \inf(X))$.

Definition A.1.5. A poset $L$ is called a lattice if and only if any pair $\{x, y\}$ have l.u.b and g.l.b and we show them by $x \lor y$ and $x \land y$ respectively. If any subset of the lattice $L$ has g.l.b and l.u.b the lattice is called complete.

Definition A.1.6. A poset $D$ is called a directed set if and only if any pair $\{x, y\}$ of elements of $D$ has an upper bound.

Definition A.1.7. A subset $Q$ of a poset $P$ is cofinal if and only if for any $p \in P$ there is $q \in Q$ such that $p \leq q$.

Definition A.1.8. A topology $\tau$ on a set $X$ is the set $\{X - C|C \in FIX(cl)\}$ where $cl : (2^X, \subseteq) \rightarrow (2^X, \subseteq)$ is an order preserving function which satisfies the following conditions:

- $cl(A \cup B) = cl(A) \cup cl(B)$ or equivalently $cl(A \lor B) = cl(A) \lor cl(B)$.
- $\emptyset, X \in FIX(cl)$
- $cl^2 = cl$

Definition A.1.9. A topological space $X$ is connected if and only if $FIX(cl) \bigwedge \tau = \{\emptyset, X\}$.

Definition A.1.10. Let $L$ be a poset $x \in L$ is minimum (maximum) if and only $x$ is smaller (greater) than or equal to all elements of $L$.

Definition A.1.11. A closed basis for topological space $(X, cl)$ is a set $B \subseteq FIX$ such that any element of $FIX$ is the greatest lower bound of a subset of $B$.
Definition A.1.12. A line is a totally linear set $(L, \leq)$ with a topology $\tau$ such that:

- $L$ does not have maximum and minimum elements.
- $\{U(x) \cup L(y) | x, y \in L\} \cup \{U(x) | x \in L\} \cup \{L(y) | y \in L\}$ is a closed basis for $(L, \tau)$.
- $L$ is connected.

Definition A.1.13. A topological space $(X, \tau)$ is called separable if and only if there is a countable subset $D \subseteq X$ such that any $U \in \tau$ has a nonempty intersection with $D$.

Theorem A.1.14. [61] The only separable line is $\mathbb{R}$.

Definition A.1.15. [61] A topological space $(X, \tau)$ has the Suslin property if and only if $\tau$ contains a countable family of mutually disjoint sets but any uncountable subfamily of $\tau$ contains at least one pair which has nonempty intersection.

Theorem A.1.16. [61] $\mathbb{R}$ has the Suslin property.

A.2 Topologies on the posets defined by a basis

Definition A.2.1. Let $(X, \leq, \tau)$ be an ordered set with a topology on it. We say the topology is order compatible if and only if for each pair $(x, y)$ of points of $X$ that $x \not\leq y$ there are open neighbourhoods $U, V$ of $x$ and $y$ respectively such that $z \not\leq y$ for all $z \in U$ and $x \not\leq z$ for all $z \in V$.

Definition A.2.2. Coarsest topology on a poset $(P, \leq)$ which is compatible with the order $\leq$ is called order topology.

The set of all subsets of $X$ which has one of the following forms is a closed basis for the order topology on $X$. $L(S), U(S)$ and $E(S)$ where $L(S) = \{x | x \leq y$ for some $y \in S\}$, $U(S) = \{x | x \geq y$ for some $y \in S\}$, and $E(S) = L(S) \cap U(S)$. 
A.2.1 Topology and Convergence

Definition A.2.3. A net is a function from a directed set $D$ into a set $X$, i.e., $n : D \rightarrow X$.

Definition A.2.4. Let $X$ be a topological space. A net $n : D \rightarrow X$ converges to $x \in X$ if and only if for any neighbourhood $U$ of $x$ there is $d_U \in D$ such that $d' \geq d_U$ implies $n(d') \in U$.

Definition A.2.5. Consider a set $X$ and let $\mathcal{N}(D, X) = \{h : D' \rightarrow X | h$ is a function and $D' \subseteq D$ is a directed subset of $D\}$. An $L$ function is defined by $L : \mathcal{N} \rightarrow X$ such that:

- $L$ of constant functions of $\mathcal{N}$ are constant, i.e., $L(n) = x$ where $n(d) = x$ for all $d \in D$.
- If $f : D \rightarrow D$ is a function such that $f(x) \geq x$ for all $x \in D$, then $L(nof) = L(n)$
- If for cofinal $f : D \rightarrow D$ there is a $g : D \rightarrow D$ such that $L(nogof) = x$, then $L(n) = x$
- If $\hat{n} \in \mathcal{N}(D, \mathcal{N}(D, X))$, then there are $f, g : D \rightarrow D$ such that $L(nof) = L^2(\hat{n})$, where $n \in \mathcal{N}(g(D), X)$.

Definition A.2.6. For $S \subseteq X$ define $cl_L(S) = \{x \in X |$ there is $n \in \mathcal{N}(D, S)$ and $f : D \rightarrow D$ such that $L(nof) = x\}$.
Appendix B

Ideals and Filters

B.1 Dual and De Morgan’s laws

Definition B.1.1. If \( \{ S_i \mid i \in I \} \) is a family of sets the following are called De Morgan’s laws:

1. \( (\bigcup_{i \in I} S_i)^c = (\bigcap_{i \in I} S_i^c) \).
2. \( (\bigcap_{i \in I} S_i)^c = (\bigcup_{i \in I} S_i^c) \).

Note that if \( A \subset B \), then \( A \cup B = B \). So by using De Morgan’s laws we have \( A^c \cap B^c = B^c \) so \( B^c \subset A^c \).

Definition B.1.2. Let \( S \) be a family of subsets of a set \( S \). The dual family \( S^* \) associated to \( F \) consists of complements of members of \( S \) and any condition placed on the members of \( S \) can be transformed, according to the De Morgan’s laws.
B.2 Ideals and Filters

Definition B.2.1. An ideal $\mathcal{I}$ on a set $X \neq \emptyset$ is a family of subsets of $X$ such that:

1. If $A \in \mathcal{I}$ and $B \subseteq A$ then $B \in \mathcal{I}$.
2. If $S \subseteq \mathcal{I}$ and $|S| < \omega$, then $\cup S \in \mathcal{I}$.

If $X \notin \mathcal{I}$ then $\mathcal{I}$ is called proper ideal.

Definition B.2.2. A filter $\mathcal{F}$ on a set $X \neq \emptyset$ is a family of subsets of $X$ such that:

1. If $A \in \mathcal{I}$ and $A \subseteq B$ then $B \in \mathcal{F}$.
2. If $S \subseteq \mathcal{F}$ and $|S| < \omega$, then $\cap S \in \mathcal{F}$.

If $\emptyset \notin \mathcal{F}$ then $\mathcal{I}$ is called proper filter.

Definition B.2.3. A filter $\mathcal{F}$ on a set $X$ is called an ultrafilter if and only if for every $S \subset X$ either $S \in \mathcal{F}$ or $X \setminus S \in \mathcal{F}$.

Definition B.2.4. A filter $\mathcal{F}$ on a set $X$ is called uniform if and only if for every two arbitrary elements $F_1, F_2 \in \mathcal{F}$, $|F_1| = |F_2|$.

Definition B.2.5. If $\mathcal{F}$ is a filter on a set $X$, then $\mathcal{F}^* = \{X \setminus I \mid I \in \mathcal{F}\}$ is called the dual ideal.

Definition B.2.6. If $\mathcal{I}$ is an ideal on a set $X$, then $\mathcal{I}^* = \{X \setminus I \mid I \in \mathcal{I}\}$ is called the dual filter.

Definition B.2.7. [41] For ideal $\mathcal{I}$ on a set $X \neq \emptyset$ we define:

1. $\text{add}(\mathcal{I}) = \min\{\alpha \mid \alpha \text{ is an initial ordinal and } \cup S \notin \mathcal{I}, \text{ for some } S \in [\mathcal{I}]^\alpha\}$,
2. $\text{cov}(\mathcal{I}) = \min\{\alpha \mid \alpha \text{ is an initial ordinal and } X = \cup \mathcal{E}, \text{ for some } \mathcal{E} \in [\mathcal{I}]^\alpha\}$,
3. $\text{non}(\mathcal{I}) = \min\{\alpha \mid \alpha \text{ is an initial ordinal}, A \notin \mathcal{I}, \text{ for some } A \in [X]^\alpha\}$. 

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B.3 Trees

**Definition B.3.1.** A tree is a partially ordered set $(T, \leq)$ such that for every $t \in T$, the set $\hat{t} = \{s \in T \mid s < t\}$ is well-ordered.

**Notation B.3.2.** If the order on the tree $(T, \leq)$ is known from the context, then often we write $T$ instead of $(T, \leq)$.

If $(T, \leq)$ is a tree and $t \in T$, then $t$ is called a node of $T$. If there is an element $t_0 \in T$ such that $t_0 \leq t$ for all $t \in T$, then $t_0$ is called the root of the tree and the tree is called rooted.

**Definition B.3.3.** The height of a node $t$ in a tree $T$ is the order type of $\hat{t} = \{s \in T \mid s < t\}$ and denoted by $ht(t)$.

**Definition B.3.4.** The height of a tree $T$ is defined by $\sup\{ht(t) + 1 \mid t \in T\}$ and denoted by $ht(T)$.

**Definition B.3.5.** For an ordinal number $\alpha$, the $\alpha$th level of a tree $T$ is $\{t \in T \mid ht(t) = \alpha\}$ and denoted by $l(\alpha)$.

**Definition B.3.6.** Let $T$ be a tree. Then we define $T_\alpha = \cup_{\beta < \alpha} l(\beta)$

**Definition B.3.7.** Let $T$ be a tree. We say that a subset $P$ of $T$ is a path in $T$ if $P$ is a chain, i.e., a linearly ordered set, by $\leq$, such that $\hat{t} \subseteq P$ for all $t \in P$.

**Definition B.3.8.** A branch in a tree $T$ is a maximal path.

**Definition B.3.9.** A branch $b$ of a tree $T$ is called cofinal if for every ordinal number $\alpha$, if $l(\alpha) \neq \emptyset$, then $b \cap l(\alpha) \neq \emptyset$.

**Definition B.3.10.** An antichain in a tree $T$ is a subset $A$ of $T$ of pairwise incomparable elements of $T$. 
Every tree $T$ of height $\omega_1$ which does not have uncountable chain or anti chain is called the Suslin tree. The assertion that there is no Suslin tree is called the Suslin Hypothesis (SH).
Appendix C

Direct limit of direct system of topological spaces and groups

C.1 Direct limit and long pipes

Consider a family \( \{X_\alpha \mid \alpha \in D\} \) of topological spaces indexed by a directed set \( D \). We denote the disjoint union of \( \{X_\alpha \mid \alpha \in D\} \) by \( \prod_{\alpha \in D} X_\alpha \).

**Definition C.1.1.** A direct system \( \{X_\alpha, \pi_\beta^\alpha, D\} \) of topological spaces consists of a family \( \{X_\alpha \mid \alpha \in D\} \) of topological spaces, a directed set \( D \) and a family \( \{\pi_\beta^\alpha : X_\alpha \to X_\beta \mid \alpha, \beta \in D \text{ and } \alpha \leq \beta\} \) of continuous functions such that \( \pi_\beta^\gamma \circ \pi_\beta^\alpha = \pi_\alpha^\gamma \) for \( \alpha \leq \beta \leq \gamma \) and \( \pi_\alpha^\alpha = \text{id} \) for all \( \alpha \in D \).

**Definition C.1.2.** Let \( X = \prod_{\alpha \in D} X_\alpha \). If \( x \in X_\alpha \) and \( y \in X_\beta \), then declare \( x \sim y \) if and only if there is a \( \gamma \in D \) such that \( \alpha \leq \gamma, \beta \leq \gamma \), and \( \pi_\beta^\gamma(x) = \pi_\beta^\gamma(y) \). The quotient space \( \prod_{\alpha \in D} X_\alpha \sim \) is called the direct limit of the direct system \( \{X_\alpha, \pi_\beta^\alpha, D\} \).

**Definition C.1.3.** Let \( X \) be a topological space and \( X = \bigcup_{\alpha \in I} X_\alpha \). We say that \( X \) has the
weak topology if $U$ is open (closed) in $X$ if and only if $U \cap X_\alpha$ is open (closed) in $X_\alpha$ for all $\alpha \in I$.

If $D$ in Definition C.1.1 is an ordinal $\Gamma$ and $\pi^\beta_\alpha$ is inclusion for every $\alpha, \beta \in \Gamma$, then the inverse system is called a tower. We are specifically interested in the case when $\Gamma = \omega_1$.

Note that a function $f : \prod_{\alpha \in D} P_\alpha \rightarrow P$ is continuous if and only if $f \circ P$ is continuous, where $P : \prod_{\alpha \in D} P_\alpha \rightarrow \prod_{\alpha \in D} P_\alpha$ is the quotient map. Note that $P \circ f = id$, so $f$ is continuous. To see that the inverse of $f$ is continuous consider an open set $U \subset P$, $f^{-1}(U) \subset \prod_{\alpha \in \omega_1} P_\alpha$ is open since $f^{-1}(U) \cap P_\alpha$ is open for all $\alpha \in \omega_1$. So every long pipe is a tower and consequently a direct limit [50]. For more details about direct limits and direct systems see [11], [64], [9] or [25]. So we have:

**Lemma C.1.4.** Let $P = \cup_{\alpha < \omega_1} P_\alpha$ be a pseudo-product, then $\{P, id^\beta_\alpha, \omega_1\}$ is a direct system and $P = \lim_{\rightarrow} P_\alpha$.

### C.2 Direct limit of groups

**Definition C.2.1.** A direct system $\{G_\alpha, \pi^\beta_\alpha, D\}$ of abelian groups consists of a family $\{G_\alpha \mid \alpha \in D\}$ of abelian groups, a directed set $D$ and a family $\{f^\beta_\alpha : G_\alpha \rightarrow G_\beta \mid \alpha, \beta \in D$ and $\alpha \leq \beta\}$, of group homomorphisms such that $f^\gamma_\beta \circ f^\beta_\alpha = f^\gamma_\alpha$ for $\alpha \leq \beta \leq \gamma$ and $f^\alpha_\alpha = id$ for all $\alpha \in D$.

**Definition C.2.2.** The direct limit of a direct system $\{G_\alpha, \pi^\beta_\alpha, D\}$ of abelian groups is the quotient group $\frac{\oplus_{\alpha \in D} G_\alpha}{S}$, where $S = \{x_\alpha - f^\beta_\alpha(x_\alpha) \mid \alpha, \beta \in D, \alpha \leq \beta$ and $x_\alpha \in G_\alpha\}$.

**Theorem C.2.3.** [43] The direct limit of a direct system of groups exists.

For more details about direct limit see [32], [12] or [56].
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