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# Separability and complete reducibility of subgroups of the Weyl group of a simple algebraic group of type $E_7$

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**Abstract**

Let $G$ be a connected reductive algebraic group defined over an algebraically closed field $k$. The aim of this paper is to present a method to find triples $(G, M, H)$ with the following three properties. Property 1: $G$ is simple and $k$ has characteristic 2. Property 2: $H$ and $M$ are closed reductive subgroups of $G$ such that $H < M < G$, and $(G, M)$ is a reductive pair. Property 3: $H$ is $G$-completely reducible, but not $M$-completely reducible. We exhibit our method by presenting a new example of such a triple in $G = E_7$. Then we consider a rationality problem and a problem concerning conjugacy classes as important applications of our construction.

**Keywords:** algebraic groups, separable subgroups, complete reducibility

## 1 Introduction

Let $G$ be a connected reductive algebraic group defined over an algebraically closed field $k$ of characteristic $p$. In [15, Sec. 3], J.P. Serre defined that a closed subgroup $H$ of $G$ is *$G$-completely reducible* ($G$-cr for short) if whenever $H$ is contained in a parabolic subgroup $P$ of $G$, $H$ is contained in a Levi subgroup $L$ of $P$. This is a faithful generalization of the notion of semisimplicity in representation theory since if $G = GL_n(k)$, a subgroup $H$ of $G$ is $G$-cr if and only if $H$ acts complete reducibly on $k^n$ [15, Ex. 3.2.2(a)]. It is known that if a closed subgroup $H$ of $G$ is $G$-cr, then $H$ is reductive [15, Prop. 4.1]. Moreover, if $p = 0$, the converse holds [15, Prop. 4.2]. Therefore the notion of $G$-complete reducibility is not interesting if $p = 0$. In this paper, we assume that $p > 0$.

Completely reducible subgroups of connected reductive algebraic groups have been much studied [9], [10], [15]. Recently, studies of complete reducibility via Geometric Invariant Theory (GIT for short) have been fruitful [1], [2], [3]. In this paper, we see another application of GIT to complete reducibility (Proposition 3.6).

Here is the main problem we consider. Let $H$ and $M$ be closed reductive subgroups of $G$ such that $H \leq M \leq G$. It is natural to ask whether $H$ being $M$-cr implies that $H$ is $G$-cr and vice versa. It is not difficult to find a counterexample for the forward direction. For example, take $H = M = PGL_2(k)$ and $G = SL_3(k)$ where $p = 2$ and $H$ sits inside $G$ via the adjoint representation. Another such example is [1, Ex. 3.45]. For many examples where $H$ and $M$ are connected with $H$ being $M$-cr and $M$ being $G$-cr, but not $G$-cr, even when each group is simple, see [18]. However, it is hard to get a counterexample for the reverse direction, and it necessarily involves a small $p$. In [3, Sec. 7], Bate et al. presented the only known counterexample for the reverse direction where $p = 2$, $H \cong S_3$, $M \cong A_1A_1$, and $G = G_2$, which we call "the $G_2$ example". The aim of this paper is to prove the following.

**Theorem 1.1.** *Let $G$ be a simple algebraic group of type $E_7$ defined over $k$ of characteristic $p = 2$. Then there exists a connected reductive subgroup $M$ of type $A_7$ of $G$ and a reductive subgroup $H \cong D_{14}$ (the dihedral group of order 14) of $M$ such that $(G, M)$ is a reductive pair and $H$ is $G$-cr but not $M$-cr.*

Our work is motivated by [3]. We recall a few relevant definitions and results here. We denote the Lie algebra of $G$ by $\operatorname{Lie} \mathrm{G} = \mathfrak{g}$. From now on, by a subgroup of $G$, we always mean a closed subgroup of $G$.

**Definition 1.2.** Let $H$ be a subgroup of $G$ acting on $G$ by inner automorphisms. Let $H$ act on $\mathfrak{g}$ by the corresponding adjoint action. Then $H$ is called *separable* if $\operatorname{Lie} C_G(H) = \mathfrak{c}_{\mathfrak{g}}(H)$.

Recall that we always have $\operatorname{Lie} C_G(H) \subseteq c_{\mathfrak{g}}(H)$. In [3], Bate et al. investigated the relationship between $G$-complete reducibility and separability, and showed the following [3, Thm. 1.2, Thm. 1.4].

**Proposition 1.3.** *Suppose that $p$ is very good for $G$. Then any subgroup of $G$ is separable in $G$.*

**Proposition 1.4.** *Suppose that $(G, M)$ is a reductive pair. Let $H$ be a subgroup of $M$ such that $H$ is a separable subgroup of $G$. If $H$ is $G$-cr, then it is also $M$-cr.*

Recall that a pair of reductive groups $G$ and $M$ is called a *reductive pair* if $\operatorname{Lie} M$ is an $M$-module direct summand of $\mathfrak{g}$. This is automatically satisfied if $p = 0$. Propositions 1.3 and 1.4 imply that the subgroup $H$ in Theorem 1.1 must be non-separable, which is possible for small $p$ only.

Now, we introduce the key notion of *separable action*, which is a slight generalization of the notion of a separable subgroup.

**Definition 1.5.** Let $H$ and $N$ be subgroups of $G$ where $H$ acts on $N$ by group automorphisms. The action of $H$ is called *separable* in $N$ if $\operatorname{Lie} C_N(H) = \mathfrak{c}_{\operatorname{Lie} N}(H)$. Note that the condition means that the fixed points of $H$ acting on $N$, taken with their natural scheme structure, are smooth.

Here is a brief sketch of our method. *Note that in our construction, $p$ needs to be* 2.

1. Pick a parabolic subgroup $P$ of $G$ with a Levi subgroup $L$ of $P$. Find a subgroup $K$ of $L$ such that $K$ acts non-separably on the unipotent radical $R_u(P)$ of $P$. In our case, $K$ is generated by elements corresponding to certain reflections in the Weyl group of $G$.

2. Conjugate $K$ by a suitable element $v$ of $R_u(P)$, and set $H = vKv^{-1}$. Then choose a connected reductive subgroup $M$ of $G$ such that $H$ is not $M$-cr. Use a recent result from GIT (Proposition 2.4) to show that $H$ is not $M$-cr. Note that $K$ is $M$-cr in our case.

3. Prove that $H$ is $G$-cr.

*Remark* 1.6. It can be shown using [17, Thm. 13.4.2] that $K$ in Step 1 is a non-separable subgroup of $G$.

First of all, for Step 1, $p$ cannot be very good for $G$ by Proposition 1.3 and 1.4. It is known that 2 and 3 are bad for $E_7$. We explain the reason why we choose $p = 2$, not $p = 3$ (Remark 2.9). Remember that the non-separable action on $R_u(P)$ was the key ingredient for the $G_2$ example to work. Since $K$ is isomorphic to a subgroup of the Weyl group of $G$, we are able to turn a problem of non-separability into a purely combinatorial problem involving the root system of $G$ (Section 3.1). Regarding Step 2, we explain the reason of our choice of $v$ and $M$ explicitly (Remarks 3.4, 3.5). Our use of Proposition 2.4 gives an improved way for checking $G$-complete reducibility (Remark 3.7). Finally, Step 3 is easy.

In the $G_2$ and $E_7$ examples, the $G$-cr and non-$M$-cr subgroups $H$ are finite. The following is the only known example of a triple $(G, M, H)$ with positive dimensional $H$ such that $H$ is $G$-cr but not $M$-cr. It is obtained by modifying [1, Ex. 3.45].

*Example* 1.7. Let $p = 2$, $m \geq 4$ be even, and $(G, M) = (GL_{2m}(k), Sp_{2m}(k))$. Let $H$ be a copy of $Sp_m(k)$ diagonally embedded in $Sp_m(k) \times Sp_m(k)$. Then $H$ is not $M$-cr by the argument in [1, Ex. 3.45]. But $H$ is $G$-cr since $H$ is $GL_m(k) \times GL_m(k)$-cr by [1, Lem. 2.12]. Also note that any subgroup of $GL(k)$ is separable in $GL(k)$ (cf. [1, Ex. 3.28]), so $(G, M)$ is not a reductive pair by Proposition 1.4.

In view of this, it is natural to ask:

**Open Problem 1.8.** Is there a triple $H < M < G$ of connected reductive algebraic groups such that $(G, M)$ is a reductive pair, $H$ is non-separable in $G$, and $H$ is $G$-cr but not $M$-cr?

Beyond its intrinsic interest, our $E_7$ example has some important consequences and applications. For example, in Section 6, we consider a rationality problem concerning complete reducibility. We need a definition first to explain our result there.

**Definition 1.9.** Let $k_0$ be a subfield of an algebraically closed field $k$. Let $H$ be a $k_0$-defined closed subgroup of a $k_0$-defined reductive algebraic group $G$. Then $H$ is called *$G$-cr over $k_0$* if whenever $H$ is contained in a $k_0$-defined parabolic subgroup $P$ of $G$, it is contained in some $k_0$-defined Levi subgroup of $P$.

Note that if $k_0$ is algebraically closed then $G$-cr over $k_0$ means $G$-cr in the usual sense. Here is the main result of Section 6.

**Theorem 1.10.** *Let $k_0$ be a nonperfect field of charecteristic $p = 2$, and let $G$ be a $k_0$-defined split simple algebraic group of type $E_7$. Then there exists a $k_0$-defined subgroup $H$ of $G$ such that $H$ is $G$-cr over $k$, but not $G$-cr over $k_0$.*

As another application of the $E_7$ example, we consider a problem concerning conjugacy classes. Given $n \in \mathbb{N}$, we let $G$ act on $G^n$ by simultaneous conjugation:

$$g \cdot (g_1, g_2, \ldots, g_n) = (gg_1g^{-1}, gg_2g^{-1}, \ldots, gg_ng^{-1}).$$

In [16], Slodowy proved the following fundamental result applying Richardson's tangent space argument, [12, Sec. 3], [13, Lem. 3.1].

**Proposition 1.11.** *Let $M$ be a reductive subgroup of a reductive algebraic group $G$ defined over $k$. Let $n \in \mathbb{N}$, let $(m_1, \ldots, m_n) \in M^n$ and let $H$ be the subgroup of $M$ generated by $m_1, \ldots, m_n$. Suppose that $(G, M)$ is a reductive pair and that $H$ is separable in $G$. Then the intersection $G \cdot (m_1, \ldots, m_n) \cap M^n$ is a finite union of $M$-conjugacy classes.*

Proposition 1.11 has many consequences. See [1], [16], and [19, Sec. 3] for example. In [3, Ex. 7.15], Bate et al. found a counterexample for $G = G_2$ showing that Proposition 1.11 fails without the separability hypothesis. In Section 7, we present a new counterexample to Proposition 1.11 without the separability hypothesis. Here is the main result of Section 7.

**Theorem 1.12.** *Let $G$ be a simple algebraic group of type $E_7$ defined over an algebraically closed $k$ of characteristic $p = 2$. Let $M$ be the connected reductive subsystem subgroup of type $A_7$. Then there exists $n \in \mathbb{N}$ and a tuple $\mathbf{m} \in M^n$ such that $G \cdot \mathbf{m} \cap M^n$ is an infinite union of $M$-conjugacy classes. Note that $(G, M)$ is a reductive pair in this case.*

Now, we give an outline of the paper. In Section 2, we fix our notation which follows [4], [8], and [17]. Also, we recall some preliminary results, in particular, Proposition 2.4 from GIT. After that, in Section 3, we prove our main result, Theorem 1.1. Then in Section 4, we consider a rationality problem, and prove Theorem 1.10. Finally, in Section 5, we discuss a problem concerning conjugacy classes, and prove Theorem 1.12.

# 2 Preliminaries

## 2.1 Notation

Throughout the paper, we denote by $k$ an algebraically closed field of positive characteristic $p$. We denote the multiplicative group of $k$ by $k^*$. We use a capital roman letter, $G$, $H$, $K$, etc., to represent an algebraic group, and the corresponding lowercase gothic letter, $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{k}$, etc., to represent its Lie algebra. We sometimes use another notation for Lie algebras: $\operatorname{Lie} G$, $\operatorname{Lie} H$, and $\operatorname{Lie} K$ are the Lie algebras of $G$, $H$, and $K$ respectively.

We denote the identity component of $G$ by $G^\circ$. We write $[G, G]$ for the derived group of $G$. The *unipotent radical* of $G$ is denoted by $R_u(G)$. An algebraic group $G$ is *reductive* if $R_u(G) = \{1\}$. In particular, $G$ is *simple* as an algebraic group if $G$ is connected and all proper normal subgroups of $G$ are finite.

In this paper, when a subgroup $H$ of $G$ acts on $G$, $H$ always acts on $G$ by inner automorphisms. The adjoint representation of $G$ is denoted by $\mathrm{Ad}_\mathfrak{g}$ or just $\mathrm{Ad}$ if no confusion arises. We write $C_G(H)$ and $\mathfrak{c}_\mathfrak{g}(H)$ for the global and the infinitesimal centralizers of $H$ in $G$ and $\mathfrak{g}$ respectively. We write $X(G)$ and $Y(G)$ for the set of characters and cocharacters of $G$ respectively.

## 2.2 Complete reducibility and GIT

Let $G$ be a connected reductive algebraic group. We recall Richardson's formalism [14, Sec. 2.1–2.3] for the characterization of a parabolic subgroup $P$ of $G$, a Levi subgroup $L$ of $P$, and the unipotent radical $R_u(P)$ of $P$ in terms of a cocharacter of $G$ and state a result from GIT (Proposition 2.4).

**Definition 2.1.** Let $X$ be an affine variety. Let $\phi : k^* \to X$ be a morphism of algebraic varieties. We say that $\lim_{a\to 0} \phi(a)$ exists if there exists a morphism $\hat{\phi} : k \to X$ (necessarily unique) whose restriction to $k^*$ is $\phi$. If this limit exists, we set $\lim_{a\to 0} \phi(a) = \hat{\phi}(0)$.

**Definition 2.2.** Let $\lambda$ be a cocharacter of $G$. Define $P_\lambda := \{g \in G \mid \lim_{a\to 0} \lambda(a)g\lambda(a)^{-1} \text{ exists}\}$, $L_\lambda := \{g \in G \mid \lim_{a\to 0} \lambda(a)g\lambda(a)^{-1} = g\}$, $R_u(P_\lambda) := \{g \in G \mid \lim_{a\to 0} \lambda(a)g\lambda(a)^{-1} = 1\}$.

Note that $P_\lambda$ is a parabolic subgroup of $G$, $L_\lambda$ is a Levi subgroup of $P_\lambda$, and $R_u(P_\lambda)$ is a unipotent radical of $P_\lambda$ [14, Sec. 2.1-2.3]. By [17, Prop. 8.4.5], any parabolic subgroup $P$ of $G$, any Levi subgroup $L$ of $P$, and any unipotent radical $R_u(P)$ of $P$ can be expressed in this form. It is well known that $L_\lambda = C_G(\lambda(k^*))$.

Let $M$ be a reductive subgroup of $G$. Then, there is a natural inclusion $Y(M) \subseteq Y(G)$ of cocharacter groups. Let $\lambda \in Y(M)$. We write $P_\lambda(G)$ or just $P_\lambda$ for the parabolic subgroup of $G$ corresponding to $\lambda$, and $P_\lambda(M)$ for the parabolic subgroup of $M$ corresponding to $\lambda$. It is obvious that $P_\lambda(M) = P_\lambda(G) \cap M$ and $R_u(P_\lambda(M)) = R_u(P_\lambda(G)) \cap M$.

**Definition 2.3.** Let $\lambda \in Y(G)$. Define a map $c_\lambda : P_\lambda \to L_\lambda$ by $c_\lambda(g) := \lim\limits_{a\to 0} \lambda(a) g \lambda(a)^{-1}$.

Note that the map $c_\lambda$ is the usual canonical projection from $P_\lambda$ to $L_\lambda \cong P_\lambda/R_u(P_\lambda)$. Now, we state a result from GIT (see [1, Lem. 2.17, Thm. 3.1], [2, Thm. 3.3]).

**Proposition 2.4.** *Let $H$ be a subgroup of $G$. Let $\lambda$ be a cocharacter of $G$ with $H \subseteq P_\lambda$. If $H$ is $G$-cr, there exists $v \in R_u(P_\lambda)$ such that $c_\lambda(h) = vhv^{-1}$ for every $h \in H$.*

## 2.3 Root subgroups and root subspaces

Let $G$ be a connected reductive algebraic group. Fix a maximal torus $T$ of $G$. Let $\Psi(G,T)$ denote the set of roots of $G$ with respect to $T$. We sometimes write $\Psi(G)$ for $\Psi(G,T)$. Fix a Borel subgroup $B$ containing $T$. Then $\Psi(B,T) = \Psi^+(G)$ is the set of positive roots of $G$ defined by $B$. Let $\Sigma(G,B) = \Sigma$ denote the set of simple roots of $G$ defined by $B$. Let $\zeta \in \Psi(G)$. We write $U_\zeta$ for the corresponding root subgroup of $G$ and $\mathfrak{u}_\zeta$ for the Lie algebra of $U_\zeta$. We define $G_\zeta := \langle U_\zeta, U_{-\zeta} \rangle$.

Let $H$ be a subgroup of $G$ normalized by some maximal torus $T$ of $G$. Consider the adjoint representation of $T$ on $\mathfrak{h}$. The root spaces of $\mathfrak{h}$ with respect to $T$ are also root spaces of $\mathfrak{g}$ with respect to $T$, and the set of roots of $H$ relative to $T$, $\Psi(H,T) = \Psi(H) = \{\zeta \in \Psi(G) \mid \mathfrak{g}_\zeta \subseteq \mathfrak{h}\}$, is a subset of $\Psi(G)$.

Let $\zeta, \xi \in \Psi(G)$. Let $\xi^\vee$ be the coroot corresponding to $\xi$. Then $\zeta \circ \xi^\vee : k^* \to k^*$ is a homomorphism such that $(\zeta \circ \xi^\vee)(a) = a^n$ for some $n \in \mathbb{Z}$. We define $\langle \zeta, \xi^\vee \rangle := n$. Let $s_\xi$ denote the reflection corresponding to $\xi$ in the Weyl group of $G$. Each $s_\xi$ acts on the set of roots $\Psi(G)$ by the following formula [17, Lem. 7.1.8]: $s_\xi \cdot \zeta = \zeta - \langle \zeta, \xi^\vee \rangle \xi$. By [5, Prop. 6.4.2, Lem. 7.2.1], we can choose homomorphisms $\epsilon_\zeta : k \to U_\zeta$ so that

$$n_\xi \epsilon_\zeta(a) n_\xi^{-1} = \epsilon_{s_\xi \cdot \zeta}(\pm a), \text{ where } n_\xi = \epsilon_\xi(1)\epsilon_{-\xi}(-1)\epsilon_\xi(1). \tag{2.1}$$

We define $e_\zeta := \epsilon'_\zeta(0)$. Then we have

$$\mathrm{Ad}(n_\xi) e_\zeta = \pm e_{s_\xi \cdot \zeta}. \tag{2.2}$$

Now, we list four lemmas which we need in our calculations. The first one is [17, Prop. 8.2.1].

**Lemma 2.5.** *Let $P$ be a parabolic subgroup of $G$. Any element $u$ in $R_u(P)$ can be expressed uniquely as*

$$u = \prod_{i \in \Psi(R_u(P))} \epsilon_i(a_i), \text{ for some } a_i \in k,$$

*where the product is taken with respect to a fixed ordering of $\Psi(R_u(P))$.*

The next two lemmas [8, Lem. 32.5 and Lem. 33.3] are used to calculate $C_{R_u(P)}(K)$.

**Lemma 2.6.** *Let $\xi, \zeta \in \Psi(G)$. If no positive integral linear combination of $\xi$ and $\zeta$ is a root of $G$, then*

$$\epsilon_\xi(a)\epsilon_\zeta(b) = \epsilon_\zeta(b)\epsilon_\xi(a).$$

**Lemma 2.7.** *Let $\Psi$ be the root system of type $A_2$ spanned by roots $\xi$ and $\zeta$. Then*

$$\epsilon_\xi(a)\epsilon_\zeta(b) = \epsilon_\zeta(b)\epsilon_\xi(a)\epsilon_{\xi+\zeta}(\pm ab).$$

The last result is used to calculate $\mathfrak{c}_{\mathrm{Lie}\,(R_u(P))}(K)$.

**Lemma 2.8.** *Suppose that $p = 2$. Let $W$ be a subgroup of $G$ generated by all the $n_\xi$ where $\xi \in \Psi(G)$ (the group $W$ is isomorphic to the Weyl group of $G$). Let $K$ be a subgroup of $W$. Let $\{O_i \mid i = 1 \cdots m\}$ be the set of orbits of the action of $K$ on $\Psi\left(R_u(P)\right)$. Then,*

$$\mathfrak{c}_{\mathrm{Lie}(R_u(P))}(K) = \left\{ \sum_{i=1}^{m} a_i \sum_{\zeta \in O_i} e_\zeta \,\middle|\, a_i \in k \right\}.$$

*Proof.* When $p = 2$, (2.2) yields $\mathrm{Ad}(n_\xi)e_\zeta = e_{n_\xi \cdot \zeta}$. Then an easy calculation gives the desired result. ☐

*Remark* 2.9. Lemma 2.8 holds in $p = 2$ but fails in $p = 3$.

# 3 The $E_7$ example

## 3.1 Step 1

Let $G$ be a simple algebraic group of type $E_7$ defined over $k$ of characteristic 2. Fix a maximal torus $T$ of $G$. Fix a Borel subgroup $B$ of $G$ containing $T$. Let $\Sigma = \{\alpha, \beta, \gamma, \delta, \epsilon, \eta, \sigma\}$ be the set of simple roots of $G$. Figure 1 defines how each simple root of $G$ corresponds to each node in the Dynkin diagram of $E_7$.

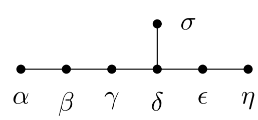


Figure 1: Dynkin diagram of $E_7$

From [6, Appendix, Table B], one knows the coefficients of all positive roots of $G$. We label all positive roots of $G$ in Table 1 in the Appendix. Our ordering of roots is different from [6, Appendix, Table B], which will be convenient later on.

The set of positive roots is $\Psi^+(G) = \{1, 2, \cdots, 63\}$. Note that $\{1, \cdots, 35\}$ and $\{36, \cdots, 42\}$ are precisely the roots of $G$ such that the coefficient of $\sigma$ is 1 and 2 respectively. We call the roots of the first type *weight-1 roots*, and the second type *weight-2 roots*. Define

$$L_{\alpha\beta\gamma\delta\epsilon\eta} := \langle T, G_{43}, \cdots, G_{63}\rangle,\ P_{\alpha\beta\gamma\delta\epsilon\eta} := \langle L_{\alpha\beta\gamma\delta\epsilon\eta}, U_1, \cdots, U_{42}\rangle.$$

Then $P_{\alpha\beta\gamma\delta\epsilon\eta}$ is a parabolic subgroup of $G$, and $L_{\alpha\beta\gamma\delta\epsilon\eta}$ is a Levi subgroup of $P_{\alpha\beta\gamma\delta\epsilon\eta}$. Note that $L_{\alpha\beta\gamma\delta\epsilon\eta}$ is of type $A_6$. We have $\Psi\left(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})\right) = \{1, \cdots, 42\}$. Define

$$q_1 := n_\epsilon n_\beta n_\gamma n_\alpha n_\beta,\ q_2 := n_\epsilon n_\beta n_\gamma n_\alpha n_\beta n_\eta n_\delta n_\beta,\ K := \langle q_1, q_2\rangle.$$

Let $\zeta_1, \zeta_2$ be simple roots of $G$. From the Cartan matrix of $E_7$ [7, Sec. 11.4] we have

$$\langle \zeta_1, \zeta_2\rangle = \begin{cases} 2, & \text{if } \zeta_1 = \zeta_2. \\ -1, & \text{if } \zeta_1 \text{ is adjacent to } \zeta_2 \text{ in the Dynkin diagram.} \\ 0, & \text{otherwise.} \end{cases}$$

From this, it is not difficult to calculate $\langle \xi, \zeta^\vee \rangle$ for all $\xi \in \Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))$ and for all $\zeta \in \Sigma$. These calculations show how $n_\alpha, n_\beta, n_\gamma, n_\delta, n_\epsilon$, and $n_\eta$ act on $\Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))$. Let $\pi : \langle n_\alpha, n_\beta, n_\gamma, n_\delta, n_\epsilon, n_\eta \rangle \to \mathrm{Sym}(\Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))) \cong S_{42}$ be the corresponding homomorphism. Then we have

$$\begin{aligned}\pi(q_1) =&(1\ 2)(3\ 6)(4\ 7)(9\ 10)(11\ 12)(13\ 14)(15\ 20)(16\ 17)(18\ 21)(19\ 23)(22\ 25)(24\ 26)\\ &(27\ 28)(29\ 32)(31\ 33)(34\ 35)(36\ 38)(37\ 39)(40\ 41),\\ \pi(q_2) =&(1\ 6\ 7\ 5\ 4\ 3\ 2)(8\ 10\ 12\ 14\ 13\ 11\ 9)(15\ 16\ 21\ 23\ 26\ 27\ 22)(17\ 20\ 25\ 28\ 24\ 19\ 18)\\ &(29\ 30\ 32\ 33\ 35\ 34\ 31)(36\ 38\ 39\ 41\ 42\ 40\ 37).\end{aligned}$$

It is easy to see that $K \cong D_{14}$. The orbits of $K$ in $\Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))$ are

$$\begin{aligned}O_1 =&\{1, \cdots, 7\}, O_8 = \{8, \cdots, 14\}, O_{15} = \{15, \cdots, 28\}, O_{29} =\{29, \cdots, 35\},\\ O_{36} =&\{36, \cdots, 42\}.\end{aligned}$$

Thus Lemma 2.8 yields

**Proposition 3.1.**

$$\begin{aligned}\mathfrak{c}_{\mathrm{Lie}(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))}(K) =&\left\{ a\left(\sum_{\lambda\in O_1} e_\lambda\right) + b\left(\sum_{\lambda\in O_8} e_\lambda\right) + c\left(\sum_{\lambda\in O_{15}} e_\lambda\right) + d\left(\sum_{\lambda\in O_{29}} e_\lambda\right)\right.\\ &\left. + m\left(\sum_{\lambda\in O_{36}} e_\lambda\right) \middle| a, b, c, d, m \in k\right\}.\end{aligned}$$

The following is the most important technical result in this paper.

**Proposition 3.2.** *Let $u \in C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})}(K)$. Then $u$ must have the form,*

$$u = \prod_{i=1}^{7} \epsilon_i(a) \prod_{i=8}^{14} \epsilon_i(b) \prod_{i=15}^{28} \epsilon_i(c) \prod_{i=29}^{35} \epsilon_i(a+b+c) \prod_{i=36}^{42} \epsilon_i(a_i) \text{ for some } a, b, c, a_i \in k.$$

*Proof.* By Lemma 2.5, $u$ can be expressed uniquely as $u = \prod_{i=1}^{42} \epsilon_i(b_i)$ for some $b_i \in k$. By (2.1), we have $n_\xi \epsilon_\zeta(a) n_\xi^{-1} = \epsilon_{s_\xi\cdot\zeta}(a)$ for any $a \in k$ and $\xi, \zeta \in \Psi(G)$. Thus we have

$$\begin{aligned}q_1 u q_1^{-1} =& q_1 \left(\prod_{i=1}^{42} \epsilon_i(b_i)\right) q_1^{-1}\\ =& \left(\prod_{i=1}^{7} \epsilon_{q_1\cdot i}(b_i)\right)\left(\prod_{i=8}^{14} \epsilon_{q_1\cdot i}(b_i)\right)\left(\prod_{i=15}^{28} \epsilon_{q_1\cdot i}(b_i)\right)\left(\prod_{i=29}^{35} \epsilon_{q_1\cdot i}(b_i)\right)\\ &\left(\prod_{i=36}^{42} \epsilon_{q_1\cdot i}(b_i)\right). \end{aligned}\tag{3.1}$$

A calculation using the commutator relations (Lemma 2.6 and Lemma 2.7) shows that

$$\begin{aligned}q_1 u q_1^{-1} =& \epsilon_1(b_2)\epsilon_2(b_1)\epsilon_3(b_6)\epsilon_4(b_7)\epsilon_5(b_5)\epsilon_6(b_3)\epsilon_7(b_4)\epsilon_8(b_8)\epsilon_9(b_{10})\epsilon_{10}(b_9)\epsilon_{11}(b_{12})\epsilon_{12}(b_{11})\epsilon_{13}(b_{14})\\ &\epsilon_{14}(b_{13})\epsilon_{15}(b_{20})\epsilon_{16}(b_{17})\epsilon_{17}(b_{16})\epsilon_{18}(b_{21})\epsilon_{19}(b_{23})\epsilon_{20}(b_{15})\epsilon_{21}(b_{18})\epsilon_{22}(b_{25})\epsilon_{23}(b_{19})\epsilon_{24}(b_{26})\\ &\epsilon_{25}(b_{22})\epsilon_{26}(b_{24})\epsilon_{27}(b_{28})\epsilon_{28}(b_{27})\epsilon_{29}(b_{32})\epsilon_{30}(b_{30})\epsilon_{31}(b_{33})\epsilon_{32}(b_{29})\epsilon_{33}(b_{31})\epsilon_{34}(b_{35})\epsilon_{35}(b_{34})\\ &\left(\prod_{i=36}^{41} \epsilon_i(a_i)\right) \epsilon_{42}(b_4 b_7 + b_{11} b_{12} + b_{22} b_{25} + b_{34} b_{35} + b_{42}) \text{ for some } a_i \in k. \end{aligned}\tag{3.2}$$

Since $q_1$ and $q_2$ centralize $u$, we have $b_1 = \cdots = b_7,\ b_8 = \cdots = b_{14},\ b_{15} = \cdots = b_{28},\ b_{29} = \cdots = b_{35}$. Set $b_1 = a,\ b_8 = b,\ b_{15} = c,\ b_{29} = d$. Then (3.2) simplifies to

$$q_1 u q_1^{-1} = \prod_{i=1}^{7} \epsilon_i(a) \prod_{i=8}^{14} \epsilon_i(b) \prod_{i=15}^{28} \epsilon_i(c) \prod_{i=29}^{35} \epsilon_i(d) \left( \prod_{i=36}^{41} \epsilon_i(a_i) \right) \epsilon_{42}(a^2 + b^2 + c^2 + d^2 + b_{42}).$$

Since $q_1$ centralizes $u$, comparing the arguments of the $\epsilon_{42}$ term on both sides, we must have

$$b_{42} = a^2 + b^2 + c^2 + d^2 + b_{42},$$

which is equivalent to $a + b + c + d = 0$. Then we obtain the desired result. $\square$

**Proposition 3.3.** *$K$ acts non-separably on $R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})$.*

*Proof.* In view of Proposition 3.1, it suffices to show that $e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 \notin \operatorname{Lie} C_{R_u(P_\lambda)}(K)$. Suppose the contrary. Since by [17, Cor. 14.2.7] $C_{R_u(P_\lambda)}(K)^\circ$ is isomorphic as a variety to $k^n$ for some $n \in \mathbb{N}$, there exists a morphism of varieties $v : k \to C_{R_u(P_\lambda)}(K)^\circ$ such that $v(0) = 1$ and $v'(0) = e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7$. By Lemma 2.5, $v(a)$ can be expressed uniquely as $v(a) = \prod_{i=1}^{42} \epsilon_i(f_i(a))$ for some $f_i \in k[X]$. Differentiating the last equation, and evaluating at $a = 0$, we obtain $v'(0) = \sum_{i \in \{1,\cdots,42\}} (f_i)'(0) e_i$. Since $v'(0) = \sum_{i \in O_1} e_i$, we have

$$(f_i)'(0) = \begin{cases} 1 & \text{if } i \in O_1, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$f_i(a) = \begin{cases} a + g_i(a) & \text{if } i \in O_1, \\ g_i(a) & \text{otherwise,} \end{cases}$$

where $g_i \in k[X]$ has no constant or linear term.

Then from Proposition 3.2, we obtain $(a + g_1(a)) + g_8(a) + g_{15}(a) = g_{29}(a)$. This is a contradiction. $\square$

## 3.2 Step 2

Let $C_1 := \left\{ \prod_{i=1}^{7} \epsilon_i(a) \mid a \in k \right\}$, pick any $a \in k^*$, and let $v(a) := \prod_{i=1}^{7} \epsilon_i(a)$. Now, set

$$\begin{aligned} H :=& v(a) K v(a)^{-1} = \langle q_1 \epsilon_{40}(a^2) \epsilon_{41}(a^2) \epsilon_{42}(a^2), q_2 \epsilon_{36}(a^2) \epsilon_{39}(a^2) \rangle, \\ M :=& \langle L_{\alpha\beta\gamma\delta\epsilon\eta}, G_{36}, \cdots, G_{42} \rangle. \end{aligned}$$

*Remark* 3.4. By Proposition 3.1 and Proposition 3.2, the tangent space of $C_1$ at the identity, $T_1(C_1)$, is contained in $\mathfrak{c}_{\operatorname{Lie}(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))}(K)$ but not contained in $\operatorname{Lie}(C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})}(K))$. The element $v(a)$ can be any non-trivial element in $C_1$.

*Remark* 3.5. In this case $\sigma$ is the unique simple root not contained in $\Psi(L_{\alpha\beta\gamma\delta\epsilon\eta})$. $M$ was chosen so that $M$ is generated by a Levi subgroup $L_{\alpha\beta\gamma\delta\epsilon\eta}$ containing $K$ and all root subgroups of $\sigma$-weight 2.

We have $H \subset M, H \not\subset L_{\alpha\beta\gamma\delta\epsilon\eta}$. Note that $\Psi(M) = \{\pm 36, \cdots, \pm 63\}$. Since $M$ is generated by all root subgroups of even $\sigma$-weight, it is easy to see that $\Psi(M)$ is a closed subsystem of $\Psi(G)$, thus $M$ is reductive by [3, Lem. 3.9]. Note that $M$ is of type $A_7$.

**Proposition 3.6.** *$H$ is not $M$-cr.*

*Proof.* Let $\lambda = 3\alpha^\vee + 6\beta^\vee + 9\gamma^\vee + 12\delta^\vee + 8\epsilon^\vee + 4\eta^\vee + 7\sigma^\vee$. We have

$$\langle\alpha,\lambda\rangle = 0, \langle\beta,\lambda\rangle = 0, \langle\gamma,\lambda\rangle = 0, \langle\delta,\lambda\rangle = 0,$$
$$\langle\epsilon,\lambda\rangle = 0, \langle\eta,\lambda\rangle = 0, \langle\sigma,\lambda\rangle = 2.$$

So $L_{\alpha\beta\gamma\delta\epsilon\eta} = L_\lambda$, $P_{\alpha\beta\gamma\delta\epsilon\eta} = P_\lambda$.

It is easy to see that $L_\lambda$ is of type $A_6$, so $[L_\lambda, L_\lambda]$ is isomorphic to either $SL_7$ or $PGL_7$. We rule out the latter. Pick $x \in k^*$ such that $x \neq 1, x^7 = 1$. Then $\lambda(x) \neq 1$ since $\sigma(\lambda(x)) = x^2 \neq 1$. Also, we have $\lambda(x) \in Z([L_\lambda, L_\lambda])$. Therefore $[L_\lambda, L_\lambda] \cong SL_7$. It is easy to check that the map $k^* \times [L_\lambda, L_\lambda] \to L_\lambda$ is separable, so we have $L_\lambda \cong GL_7$.

Let $c_\lambda : P_\lambda \to L_\lambda$ be the homomorphism as in Definition 2.3. In order to prove that $H$ is not $M$-cr, by Theorem 2.4 it suffices to find a tuple $(h_1, h_2) \in H^2$ which is not $R_u(P_\lambda(M))$-conjugate to $c_\lambda((h_1, h_2))$. Set $h_1 := v(a)q_1v(a)^{-1}$, $h_2 := v(a)q_2v(a)^{-1}$. Then

$$\begin{aligned} c_\lambda((h_1, h_2)) &= \lim_{x\to 0} \left(\lambda(x)q_1\epsilon_{40}(a^2)\epsilon_{41}(a^2)\epsilon_{42}(a^2)\lambda(x)^{-1},\ (\lambda(x)q_2\epsilon_{36}(a^2)\epsilon_{39}(a^2)\lambda(x)^{-1}\right) \\ &= (q_1, q_2). \end{aligned}$$

Now suppose that $(h_1, h_2)$ is $R_u(P_\lambda(M))$-conjugate to $c_\lambda((h_1, h_2))$. Then there exists $m \in R_u(P_\lambda(M))$ such that

$$mv(a)q_1v(a)^{-1}m^{-1} = q_1,\ mv(a)q_2v(a)^{-1}m^{-1} = q_2.$$

Thus we have $mv(a) \in C_{R_u(P_\lambda)}(K)$. Note that $\Psi(R_u(P_\lambda(M))) = \{36, \cdots, 42\}$. So, by Lemma 2.5, $m$ can be expressed uniquely as $m := \prod_{i=36}^{42} \epsilon_i(a_i)$ for some $a_i \in k$. Then we have

$$mv(a) = \epsilon_1(a)\epsilon_2(a)\epsilon_3(a)\epsilon_4(a)\epsilon_5(a)\epsilon_6(a)\epsilon_7(a)\left(\prod_{i=36}^{42}\epsilon_i(a_i)\right) \in C_{R_u(P_\lambda)}(K).$$

This contradicts Proposition 3.2. ∎

*Remark* 3.7. In [3, Sec. 7, Prop .7.17], Bate et al. used [1, Lem. 2.17, Thm. 3.1] to turn a problem on $M$-complete reducibility into a problem involving $M$-conjugacy. We have used Proposition 2.4 to turn the same problem into a problem involving $R_u(P \cap M)$-conjugacy, which is easier.

*Remark* 3.8. Instead of using $C_1$ to define $v(a)$, we can take $C_8 := \left\{\prod_{i=8}^{14}\epsilon_i(a) \mid a \in k\right\}$, $C_{15} := \left\{\prod_{i=15}^{28}\epsilon_i(a) \mid a \in k\right\}$, or $C_{29} := \left\{\prod_{i=29}^{35}\epsilon_i(a) \mid a \in k\right\}$. In each case, a similar argument goes through and gives rise to a different example with the desired property.

## 3.3 Step 3

**Proposition 3.9.** *$H$ is $G$-cr.*

*Proof.* First note that $H$ is conjugate to $K$, so $H$ is $G$-cr if and only if $K$ is $G$-cr. Then, by [1, Lem. 2.12, Cor. 3.22], it suffices to show that $K$ is $[L_\lambda, L_\lambda]$-cr. We can identify $K$ with the image of the corresponding subgroup of $S_7$ under the permutation representation $\pi_1 : S_7 \to SL_7(k)$. It is easy to see that $K \cong D_{14}$. A quick calculation shows that this representation of $D_{14}$ is a direct sum of a trivial 1-dimensional and 3 irreducible 2-dimensional subrepresentations. Therefore $K$ is $[L_\lambda, L_\lambda]$-cr. ∎

# 4 A rationality problem

We prove Theorem 1.10. The key here is again the existence of a 1-dimensional curve $C_1$ such that $T_1(C_1)$ is contained in $\mathfrak{c}_{\mathrm{Lie}\,(R_u(P_\lambda))}(K)$ but not contained in $\mathrm{Lie}(C_{R_u(P_\lambda)}(K))$. The same phenomenon was seen in the $G_2$ example.

*Proof of Theorem 1.10.* Let $k_0$, $k$, and $G$ be as in the hypothesis. We choose a $k_0$-defined $k_0$-split maximal torus $T$ such that for each $\zeta \in \Psi(G)$ the corresponding root $\zeta$, coroot $\zeta^\vee$, and homomorphism $\epsilon_\zeta$ are defined over $k_0$. Since $k_0$ is not perfect, there exists $\tilde{a} \in k \backslash k_0$ such that $\tilde{a}^2 \in k_0$. We keep the notation $q_1, q_2, v, K, P_\lambda, L_\lambda$ of Section 3. Let

$$\begin{aligned} H &= \langle v(\tilde{a}) q_1 v(\tilde{a})^{-1}, v(\tilde{a}) q_2 v(\tilde{a})^{-1} \rangle \\ &= \langle q_1 \epsilon_{40}(\tilde{a}^2)\epsilon_{41}(\tilde{a}^2)\epsilon_{42}(\tilde{a}^2), q_2\epsilon_{36}(\tilde{a}^2)\epsilon_{39}(\tilde{a}^2)\rangle. \end{aligned}$$

Now it is obvious that $H$ is $k_0$-defined. We already know that $H$ is $G$-cr by Proposition 3.9. Since $G$ and $T$ are $k_0$-split, $P_\lambda$ and $L_\lambda$ are $k_0$-defined by [4, V.20.4, V.20.5]. Suppose that there exists a $k_0$-Levi subgroup $L'$ of $P_\lambda$ such that $L'$ contains $H$. Then there exists $w \in R_u(P_\lambda)(k_0)$ such that $L' = wL_\lambda w^{-1}$ by [4, V.20.5]. Then $w^{-1}Hw \subseteq L_\lambda$ and $v(\tilde{a})^{-1}Hv(\tilde{a}) \subseteq L_\lambda$. So we have $c_\lambda(w^{-1}hw) = w^{-1}hw$ and $c_\lambda\left(v(\tilde{a})^{-1}hv(\tilde{a})\right) = v(\tilde{a})^{-1}hv(\tilde{a})$ for any $h \in H$. We also have $c_\lambda(w) = c_\lambda\left(v(\tilde{a})\right) = 1$ since $w, v(\tilde{a}) \in R_u(P_\lambda)(k)$. Therefore we obtain $w^{-1}hw = c_\lambda(w^{-1}hw) = c_\lambda(h) = c_\lambda\left(v(\tilde{a})^{-1}hv(\tilde{a})\right) = v(\tilde{a})^{-1}hv(\tilde{a})$ for any $h \in H$. So we have $w = v(\tilde{a})z$ for some $z \in C_{R_u(P_\lambda)}(K)(k)$. By Proposition 3.2, $z$ must have the form

$$z = \prod_{i=1}^{7}\epsilon_i(a)\prod_{i=8}^{14}\epsilon_i(b)\prod_{i=15}^{28}\epsilon_i(c)\prod_{i=29}^{35}\epsilon_i(a+b+c)\prod_{i=36}^{42}\epsilon_i(a_i) \text{ for some } a, b, c, a_i \in k.$$

Then

$$\begin{aligned} w &= \left(\prod_{i=1}^{7}\epsilon_i(\tilde{a})\right)\prod_{i=1}^{7}\epsilon_i(a)\prod_{i=8}^{14}\epsilon_i(b)\prod_{i=15}^{28}\epsilon_i(c)\prod_{i=29}^{35}\epsilon_i(a+b+c)\prod_{i=36}^{42}\epsilon_i(a_i) \\ &= \prod_{i=1}^{7}\epsilon_i(\tilde{a}+a)\prod_{i=8}^{14}\epsilon_i(b)\prod_{i=15}^{28}\epsilon_i(c)\prod_{i=29}^{35}\epsilon_i(a+b+c)\prod_{i=36}^{42}\epsilon_i(b_i) \text{ for some } b_i \in k. \end{aligned}$$

Since $w$ is a $k_0$-point, $b$, $c$, and $a+b+c$ all belong to $k_0$, so $a \in k_0$. But $a + \tilde{a}$ belongs to $k_0$ as well, so $\tilde{a} \in k_0$. This is a contradiction. $\square$

*Remark* 4.1. As in Section 3, we can take $v(\tilde{a})$ from $C_8$, $C_{15}$, or $C_{29}$. In each case, a similar argument goes through, and gives rise to a different example.

*Remark* 4.2. [1, Ex. 5.11] shows that there is a $k_0$-defined subgroup of $G$ of type $A_n$ which is not $G$-cr over $k$ even though it is $G$-cr over $k_0$. Note that this example works for any $p > 0$.

# 5 A problem of conjugacy classes

We prove Theorem 1.11. Here, the key is again the existence of a 1-dimensional curve $C_1$ such that $T_1(C_1)$ is contained in $\mathfrak{c}_{\mathrm{Lie}\,(R_u(P_\lambda))}(K)$ but not contained in $\mathrm{Lie}(C_{R_u(P_\lambda)}(K))$ as in the $G_2$ example. Let $G$, $M$, $k$ be as in the hypotheses of the theorem. We keep the notation $q_1, q_2, v, K, P_\lambda, L_\lambda$ of Section 3. A calculation using the commutator relations (Lemma 2.6) shows that

$$Z(R_u(P_\lambda)) = \langle U_{36}, U_{37}, U_{38}, U_{39}, U_{40}, U_{41}, U_{42}\rangle.$$

Let $K_0 := \langle K, Z(R_u(P_\lambda))\rangle$. It is standard that there exists a finite subset $F = \{z_1, z_2, \cdots, z_{n'}\}$ of $Z(R_u(P))$ such that $C_{P_\lambda}(\langle K, F\rangle) = C_{P_\lambda}(K_0)$. Let $\mathbf{m} := (q_1, q_2, z_1, \cdots, z_{n'})$. Let $n := n' + 2$. For every $x \in k^*$, define $\mathbf{m}(x) := v(x) \cdot \mathbf{m} \in P_\lambda(M)^n$.

**Lemma 5.1.** *$C_{P_\lambda}(K_0) = C_{R_u(P_\lambda)}(K_0)$.*

*Proof.* It is obvious that $C_{R_u(P_\lambda)}(K_0) \subseteq C_{P_\lambda}(K_0)$. We prove the converse. Let $lu \in C_{P_\lambda}(K_0)$ for some $l \in L_\lambda$ and $u \in R_u(P_\lambda)$. Then $lu$ centralizes $Z(R_u(P_\lambda))$, so $l$ centralizes $Z(R_u(P_\lambda))$, since $u$ does. It suffices to show that $l = 1$. Let $l = t\tilde{l}$ where $t \in Z(L_\lambda)^\circ = \lambda(k^*)$ and $\tilde{l} \in [L_\lambda, L_\lambda]$. We have

$$\langle i, \lambda\rangle = 4 \text{ for any } i \in \{36, \cdots, 42\}. \tag{5.1}$$

So for any $z \in Z(R_u(P_\lambda))$, there exists $\alpha \in k^*$ such that $t \cdot z = \alpha z$. Then we have $\tilde{l} \cdot z = \alpha^{-1} z$. Now define $A := \{\tilde{l} \in [L_\lambda, L_\lambda] \mid \tilde{l} \text{ acts on } Z(R_u(P_\lambda)) \text{ by multiplication by a scalar}\}$. Then it is easy to see that $A \trianglelefteq [L_\lambda, L_\lambda]$. Since $[L_\lambda, L_\lambda] \cong SL_7$ and $L_\lambda \cong GL_7$, we have $A = Z([L_\lambda, L_\lambda])$. Therefore we obtain $\tilde{l} \in A = Z([L_\lambda, L_\lambda]) \subseteq \lambda(k^*)$. So we have $l = c\tilde{l} \in \lambda(k^*)$. Then we obtain $l \in C_{\lambda(k^*)}(Z(R_u(P_\lambda)))$. By (5.1) this implies $l = 1$. □

**Lemma 5.2.** *$G \cdot \mathbf{m} \cap P_\lambda(M)^n$ is an infinite union of $P_\lambda(M)$-conjugacy classes.*

*Proof.* Fix $a' \in k^*$. By Lemma 5.1, we have $C_{P_\lambda}(K_0) = C_{R_u(P_\lambda)}(K_0) \subseteq C_{R_u(P_\lambda)}(K)$. Then we obtain

$$C_{P_\lambda}(v(a')K_0 v(a')^{-1}) = v(a')C_{P_\lambda}(K_0)v(a')^{-1} \subseteq v(a')C_{R_u(P_\lambda)}(K)v(a')^{-1}. \tag{5.2}$$

Choose $b' \in k^*$ such that $\mathbf{m}(a')$ is $P_\lambda(M)$-conjugate to $\mathbf{m}(b')$. Then there exists $m \in P_\lambda(M)$ such that $m \cdot \mathbf{m}(b') = \mathbf{m}(a')$. By (5.2), we have

$$mv(b')v(a')^{-1} \in C_{P_\lambda}(v(a')K_0 v(a')^{-1}) \subseteq v(a')C_{R_u(P_\lambda)}(K)v(a')^{-1}.$$

By Proposition 3.2, we have

$$v(a')^{-1}mv(b') = \prod_{i=1}^{7}\epsilon_i(a)\prod_{i=8}^{14}\epsilon_i(b)\prod_{i=15}^{28}\epsilon_i(c)\prod_{i=29}^{35}\epsilon_i(a+b+c)\prod_{i=36}^{42}\epsilon_i(a_i), \text{ for some } a, b, c, a_i \in k.$$

This yields

$$m = \prod_{i=1}^{7}\epsilon_i(a+a'+b')\prod_{i=8}^{14}\epsilon_i(b)\prod_{i=15}^{28}\epsilon_i(c)\prod_{i=29}^{35}\epsilon_i(a+b+c)\prod_{i=36}^{42}\epsilon_i(b_i), \text{ for some } a, b, c, b_i \in k.$$

But $m \in P_\lambda(M)$, so $a + a' + b' = 0, b = 0, c = 0, a + b + c = 0$. Hence we have $a' = b'$. Thus we have shown that if $a' \neq b'$, then $\mathbf{m}(a')$ is not $P_\lambda(M)$-conjugate to $\mathbf{m}(b')$. So, in particular, $G \cdot \mathbf{m} \cap P_\lambda(M)^n$ is an infinite union of $P_\lambda(M)$-conjugacy classes. □

We need the next result [11, Lem. 4.4]. We include the proof to make this paper self-contained.

**Lemma 5.3.** *$G \cdot \mathbf{m} \cap P_\lambda(M)^n$ is a finite union of $M$-conjugacy classes if and only if it is a finite union of $P_\lambda(M)$-conjugacy classes.*

*Proof.* Pick $\mathbf{m_1}, \mathbf{m_2} \in G \cdot \mathbf{m} \cap P_\lambda(M)^n$ such that $\mathbf{m_1}$ and $\mathbf{m_2}$ are in the same $M$-conjugacy class of $G \cdot \mathbf{m} \cap P_\lambda(M)^n$. Then there exists $m \in M$ such that $m \cdot \mathbf{m_1} = \mathbf{m_2}$. Let $Q = m^{-1}P_\lambda(M)m$. Then we have $\mathbf{m_1} \in (P_\lambda(M) \cap Q)^n$. Now let $S$ be a maximal torus of $M$ contained in $P_\lambda(M) \cap Q$.

Since $S$ and $m^{-1}Sm$ are maximal tori of $Q$, they must be $Q$-conjugate. So there exists $q \in Q$ such that

$$qSq^{-1} = m^{-1}Sm. \tag{5.3}$$

Since $Q = m^{-1}P_\lambda(M)m$, there exists $p \in P_\lambda(M)$ such that $q = m^{-1}pm$. Then from (5.3), we obtain $pmSm^{-1}p^{-1} = S$. This implies $m^{-1}p^{-1} \in N_M(S)$. Fix a finite set $N \subseteq N_M(S)$ of coset representatives for the Weyl group $W = N_M(S)/S$. Then we have

$$m^{-1}p^{-1} = ns \text{ for some } n \in N, s \in S.$$

So we obtain $\mathbf{m_1} = m^{-1} \cdot \mathbf{m_2} = (nsp) \cdot \mathbf{m_2} \in (nP_\lambda(M)) \cdot \mathbf{m_2}$. Since $N$ is a finite set, this shows that a $M$-conjugacy class in $G \cdot \mathbf{m} \cap P_\lambda(M)^n$ is a finite union of $P_\lambda(M)$-conjugacy classes. The converse is obvious. □

*Proof of Theorem 1.12.* By Lemma 5.2 and Lemma 5.3, we conclude that $G \cdot \mathbf{m} \cap P_\lambda(M)^n$ is an infinite union of $M$-conjugacy classes. Now it is evident that $G \cdot \mathbf{m} \cap M^n$ is an infinite union of $M$-conjugacy classes. □


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## Appendix

| | | | |
|---|---|---|---|
| (1) $\begin{smallmatrix} &&&1&& \\ 0&0&1&1&1&0 \end{smallmatrix}$ | (2) $\begin{smallmatrix} &&&1&& \\ 1&1&1&1&0&0 \end{smallmatrix}$ | (3) $\begin{smallmatrix} &&&1&& \\ 0&1&1&2&1&1 \end{smallmatrix}$ | (4) $\begin{smallmatrix} &&&1&& \\ 0&0&1&2&2&1 \end{smallmatrix}$ |
| (5) $\begin{smallmatrix} &&&1&& \\ 1&1&2&2&1&0 \end{smallmatrix}$ | (6) $\begin{smallmatrix} &&&1&& \\ 0&1&1&2&2&1 \end{smallmatrix}$ | (7) $\begin{smallmatrix} &&&1&& \\ 1&2&2&2&1&1 \end{smallmatrix}$ | (8) $\begin{smallmatrix} &&&1&& \\ 0&0&0&0&0&0 \end{smallmatrix}$ |
| (9) $\begin{smallmatrix} &&&1&& \\ 0&0&0&1&0&0 \end{smallmatrix}$ | (10) $\begin{smallmatrix} &&&1&& \\ 0&1&1&1&1&0 \end{smallmatrix}$ | (11) $\begin{smallmatrix} &&&1&& \\ 0&0&1&2&1&1 \end{smallmatrix}$ | (12) $\begin{smallmatrix} &&&1&& \\ 1&2&2&2&2&1 \end{smallmatrix}$ |
| (13) $\begin{smallmatrix} &&&1&& \\ 1&1&2&3&2&1 \end{smallmatrix}$ | (14) $\begin{smallmatrix} &&&1&& \\ 1&2&3&3&2&1 \end{smallmatrix}$ | (15) $\begin{smallmatrix} &&&1&& \\ 0&0&1&1&0&0 \end{smallmatrix}$ | (16) $\begin{smallmatrix} &&&1&& \\ 0&0&0&1&1&0 \end{smallmatrix}$ |
| (17) $\begin{smallmatrix} &&&1&& \\ 0&1&1&1&0&0 \end{smallmatrix}$ | (18) $\begin{smallmatrix} &&&1&& \\ 0&0&0&1&1&1 \end{smallmatrix}$ | (19) $\begin{smallmatrix} &&&1&& \\ 0&0&1&2&1&0 \end{smallmatrix}$ | (20) $\begin{smallmatrix} &&&1&& \\ 1&1&1&1&1&0 \end{smallmatrix}$ |
| (21) $\begin{smallmatrix} &&&1&& \\ 0&1&1&1&1&1 \end{smallmatrix}$ | (22) $\begin{smallmatrix} &&&1&& \\ 1&1&1&2&1&1 \end{smallmatrix}$ | (23) $\begin{smallmatrix} &&&1&& \\ 1&2&2&2&1&0 \end{smallmatrix}$ | (24) $\begin{smallmatrix} &&&1&& \\ 1&1&2&2&1&1 \end{smallmatrix}$ |
| (25) $\begin{smallmatrix} &&&1&& \\ 0&1&2&2&2&1 \end{smallmatrix}$ | (26) $\begin{smallmatrix} &&&1&& \\ 1&1&2&2&2&1 \end{smallmatrix}$ | (27) $\begin{smallmatrix} &&&1&& \\ 0&1&2&3&2&1 \end{smallmatrix}$ | (28) $\begin{smallmatrix} &&&1&& \\ 1&2&2&3&2&1 \end{smallmatrix}$ |
| (29) $\begin{smallmatrix} &&&1&& \\ 0&0&1&1&1&1 \end{smallmatrix}$ | (30) $\begin{smallmatrix} &&&1&& \\ 0&1&1&2&1&0 \end{smallmatrix}$ | (31) $\begin{smallmatrix} &&&1&& \\ 1&1&1&2&1&0 \end{smallmatrix}$ | (32) $\begin{smallmatrix} &&&1&& \\ 1&1&1&1&1&1 \end{smallmatrix}$ |
| (33) $\begin{smallmatrix} &&&1&& \\ 0&1&2&2&1&0 \end{smallmatrix}$ | (34) $\begin{smallmatrix} &&&1&& \\ 0&1&2&2&1&1 \end{smallmatrix}$ | (35) $\begin{smallmatrix} &&&1&& \\ 1&1&1&2&2&1 \end{smallmatrix}$ | (36) $\begin{smallmatrix} &&&2&& \\ 0&1&2&3&2&1 \end{smallmatrix}$ |
| (37) $\begin{smallmatrix} &&&2&& \\ 1&1&2&3&2&1 \end{smallmatrix}$ | (38) $\begin{smallmatrix} &&&2&& \\ 1&2&2&3&2&1 \end{smallmatrix}$ | (39) $\begin{smallmatrix} &&&2&& \\ 1&2&3&3&2&1 \end{smallmatrix}$ | (40) $\begin{smallmatrix} &&&2&& \\ 1&2&3&4&2&1 \end{smallmatrix}$ |

| | | | |
|---|---|---|---|
| (41) $\begin{smallmatrix} &&&2&& \\ 1&2&3&4&3&1 \end{smallmatrix}$ | (42) $\begin{smallmatrix} &&&2&& \\ 1&2&3&4&3&2 \end{smallmatrix}$ | (43) $\begin{smallmatrix} &&&0&& \\ 1&0&0&0&0&0 \end{smallmatrix}$ | (44) $\begin{smallmatrix} &&&0&& \\ 0&1&0&0&0&0 \end{smallmatrix}$ |
| (45) $\begin{smallmatrix} &&&0&& \\ 0&0&1&0&0&0 \end{smallmatrix}$ | (46) $\begin{smallmatrix} &&&0&& \\ 0&0&0&1&0&0 \end{smallmatrix}$ | (47) $\begin{smallmatrix} &&&0&& \\ 0&0&0&0&1&0 \end{smallmatrix}$ | (48) $\begin{smallmatrix} &&&0&& \\ 0&0&0&0&0&1 \end{smallmatrix}$ |
| (49) $\begin{smallmatrix} &&&0&& \\ 1&1&0&0&0&0 \end{smallmatrix}$ | (50) $\begin{smallmatrix} &&&0&& \\ 0&1&1&0&0&0 \end{smallmatrix}$ | (51) $\begin{smallmatrix} &&&0&& \\ 0&0&1&1&0&0 \end{smallmatrix}$ | (52) $\begin{smallmatrix} &&&0&& \\ 0&0&0&1&1&0 \end{smallmatrix}$ |
| (53) $\begin{smallmatrix} &&&0&& \\ 0&0&0&0&1&1 \end{smallmatrix}$ | (54) $\begin{smallmatrix} &&&0&& \\ 1&1&1&0&0&0 \end{smallmatrix}$ | (55) $\begin{smallmatrix} &&&0&& \\ 0&1&1&1&0&0 \end{smallmatrix}$ | (56) $\begin{smallmatrix} &&&0&& \\ 0&0&1&1&1&0 \end{smallmatrix}$ |
| (57) $\begin{smallmatrix} &&&0&& \\ 0&0&0&1&1&1 \end{smallmatrix}$ | (58) $\begin{smallmatrix} &&&0&& \\ 1&1&1&1&0&0 \end{smallmatrix}$ | (59) $\begin{smallmatrix} &&&0&& \\ 0&1&1&1&1&0 \end{smallmatrix}$ | (60) $\begin{smallmatrix} &&&0&& \\ 0&0&1&1&1&1 \end{smallmatrix}$ |
| (61) $\begin{smallmatrix} &&&0&& \\ 1&1&1&1&1&0 \end{smallmatrix}$ | (62) $\begin{smallmatrix} &&&0&& \\ 0&1&1&1&1&1 \end{smallmatrix}$ | (63) $\begin{smallmatrix} &&&0&& \\ 1&1&1&1&1&1 \end{smallmatrix}$ | |

Table 1: The set of positive roots of $G = E_7$

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