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Understanding How Undergraduate Students Formalise Their  
Written Argumentations in Mathematics

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## Abstract

All students who take an undergraduate university course in mathematics will confront the task of having to present a written argument to someone else. This could be one or more of their colleagues, or teaching staff who are also likely to be the markers of their course assignments, tutorials, tests and examinations. During lectures and tutorials, the lecturer and tutor may provide justifications for various aspects of mathematics presented – which might include elementary proofs, or give examples of how various theorems or known results contribute to solving a range of problems. Students will very likely be exposed to modes of argumentation that encourage certain habits of mind and speech, and at the same time, foster a disposition suited to engaging successfully with mathematics. These combine to shape the first experiences students have with argumentation in mathematics. Usually, by their second year, many students will have been introduced to formal mathematics through linear algebra, differential and integral calculus, or courses that are specifically designed to teach logic, proof and formal thinking in mathematics. Certainly, by their third year, reading and writing formal arguments of proof is an established part of the student's life-world. However, literature from mathematics education shows that proof is a difficult mathematical concept for students to master.

Underlying this research, is an assertion that argumentation plays an important role in developing the thinking, reasoning and communicative skills required to produce formal mathematical arguments. One way to investigate this is to see how students adapt their written argumentations from ones that give a personal conviction to ones aimed at convincing an other. This research applied Tall's model of Three Worlds of Mathematics, Toulmin's layout of an argument, and Habermas' three dimensions of rationality to create a framework that was used to examine the argumentations of students, investigate how they formalised them, and describe some obstacles that may have prevented them from doing so.

Students who were taking a first year mathematics course, and a smaller number taking a second year mathematics course participated in this research by completing tasks that were designed to promote the production of written argumentations. The argumentation of four students were tracked over three years in an attempt to capture, and explain, general changes that occurred over a longer term. These were analysed using the research framework developed in this study.

The results of this study suggest that the first year mathematics students had the readiness

to engage successfully with writing formal mathematical arguments. When they perceived it necessary to do so, they could select the necessary cognitive and mathematical tools that allowed them to reason algebraically and deductively. By the second year, students' mathematical reasoning were well established, and they could confidently use methods of argument and logic usually associated with proof. A significant inclusion in second year students' argumentations was the use of definitions. Over time, the students used increasingly sophisticated symbols, concepts and mathematical theory to produce more complex argumentations. Although the primary form of reasoning in formalising argumentations was deductive, other forms of reasoning that provided elaboration, understanding, and further support for personal conviction often supported it.

It is anticipated that this research might provide further support to the assertion that written argumentation has an important role in learning mathematics, particularly in developing an understanding of proof.

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# Chapter 1

## Introduction

Argumentation is an important component of mathematics education in the sense that it encourages students to learn how to do mathematics. It gives students an opportunity to interact with mathematical ideas through activities, such as exploration, conjecturing, justifying and proving. Argumentation also encourages them to communicate their ideas with others using the language of mathematics, which is seen as an important competency in mathematics (Durand-Guerrier et al., 2012). Thus it contributes to students' mathematical literacy by engaging them in reading, writing and talking mathematically (see Osborne & Chin, 2010, pp. 88–102). As students make the transition from mathematics at secondary school to mathematics at tertiary level, argumentation becomes increasingly associated with activities of proving – particularly in their second and third year of undergraduate study (University of Auckland, 2013). One change the students have to contend with is the systematisation of argumentational activities. Tall (1991) notes, that within this change, students shift from explaining to defining, and move from convincing to proving using mathematical definitions, theorems, and logic. Associated with these is the requirement for students to reason in a way that is consistent with the purpose of communicating mathematics in a formal scholarly manner (Hanna & Jahnke, 1993; Selden & Selden, 1995; Dreyfus, 1999). However, balancing formal aspects of mathematics with divergent and less formal exploratory activities is an importance noted by Durand-Guerrier et al. (2012, p. 359) who state:

In sum, students need to experience two main practices: a divergent exploratory one and a convergent validating one. They would then become familiar with the openness of exploration, that is, its “opportunistic” character and its flexible validation rules. They would also learn the rigorous rules

needed to write deductive text and the strict usage of words, symbols, and formulas when constructing or organising a theory... However, isolating one to the detriment of the other weakens both, because they are related dialectically.

The shift in argumentation outlined above forms the context for the research into undergraduate argumentation in the present study.

## 1.1 A Personal Perspective

Research has the potential to develop new understandings in mathematics education. However, there may be some ethical and judgement issues that relate to the researcher that might not be evidenced in the research. Hence, Becker (1998) argues that the researcher should make their perspective known so that the readers can judge the researcher's conclusions in that light. With this in mind, a brief background of the researcher is provided so that the reader can put the thesis into a context in which judgements can be made.

In my first year experience of studying mathematics at university, the courses I took were dominated by formal mathematics and learning proofs. The course textbooks, were to me at least, riddled with definitions, theorems, and proofs – something which I had not encountered during my final year of high school. From my standpoint, at the time, lectures and course books did not present proof in a way that linked it to certainty, or as means of persuading and convincing. Rather, the emphasis on proof seemed to suggest that proof defined mathematics. Mathematics was proof, and to know mathematics well, meant that one could prove results. A frustration with not having the time to understand fully the formal nature of definitions and theorems, and link them to lecture material, meant that I could not engage meaningfully with proof. Consequently I left university.

From mid 1984 to the early part of 1991, I was a primary school teacher, teaching at an Intermediate School. During that time I became particularly interested in finding ways that would make mathematics an exciting and challenging endeavour for adolescent children, and developed a mathematics programme that gave students opportunities to solve problems, do mathematical investigations, and model various situations mathematically. I became a Resource Teacher of Mathematics (RTM) and worked closely with the Northland Schools Mathematics Advisor and other RTMs in Auckland. As a group (RTMs and Northland and Auckland mathematics advisors), we recognised the need for students to understand mathematics, reason in a logical way, and to explain their mathematical ideas.

It was also during this time I began to realise the importance of algebraic reasoning to students in the early phase of secondary school, since algebra was by Year 10 (age approx. 15 years) an essential tool of mathematics. From 1991 to 2002, I was a lecturer in mathematics education at the Auckland College of Education. During this time my interest in conceptual understanding became more grounded, and I became more interested in pedagogies that encouraged mathematical understanding - both for primary school children, and for prospective primary school teachers. A positive outcome for me during this time, was that I re-familiarised myself with mathematics, went back to university, and completed an undergraduate degree in mathematics.

In 2002, I shifted to The University of Auckland, and worked with Maori and Pasifika students who were taking first, second and third year mathematics courses. During this time I embarked on a Master's degree, that focussed on the development of conceptual knowledge in mathematics at senior school and tertiary level. This culminated in a year long study that looked how a class of Year 13 (age approx. 18 years) mathematics (school) students conceptualised knowledge in their calculus course. In 2004, I also began to teach and tutor first year university mathematics courses.

My original interest in argumentation came about as a result of my work with students who were doing Maths 108, a course that covers linear algebra and calculus for students who would generally go on to study commerce or a social science. While lecturing and tutoring in this course, I realised that many students experienced difficulty in explaining the mathematical ideas behind a particular procedure (e.g. finding the vector equation of a line in  $\mathbb{R}^2$ , or a vector equation of a plane in  $\mathbb{R}^3$ ), or understanding the role a definition or theorem played in a solution to a problem. The conversations of groups of students, who were involved in collaborative class tutorials, seemed to centre around procedures and associated mathematical computations, rather than key mathematical ideas.

I also had the opportunity to work with a small group of students from Maths 150, a first year course designed for those who intend to major in mathematics. The students I worked with experienced problems interpreting statements in mathematics that were written in formal notation, particularly mathematical definitions and theorems. Their problems did not appear to be with the interpretation of quantifiers, and connectives, but rather with the logical structure of a statement. For example, one difficulty I recall came from interpreting the definition of 'onto' (i.e., A function  $f$  from  $A$  to  $B$  is called *onto* if for all  $b \in B$ , there is an  $a \in A$  such that  $f(a) = b$ ). Students found it difficult to understand the use of this definition because of the forward mapping of the function  $f$

with the apparent backward mapping in the second part of the definition.

Apart from tutoring, my interest in argumentation was also motivated by lecturing. As part of the teaching and tutoring team for Maths 102, I was able to orchestrate teaching and learning sessions that involved students in questioning, verifying, looking for errors and explaining to others. An observation from collaborative tutorials revealed that conversations of Maths 102 students appeared to focus more on mathematical ideas, mainly because the assignment and tutorial questions forced them to do so. However, the problem that these students experienced at the time, was how to write concise mathematical argument that justified their findings.

These, and other observations, eventually lead to me to consider argumentation as a pedagogical tool that might help students gain a better understanding of mathematics. The initial appeal of argumentation was that it involved all facets of mathematical literacy such as reading, writing, and talking mathematics, and the processes of thinking and reasoning. However, the scope covered by mathematical literacy was too wide, and it was decided to focus on one aspect – students’ written argumentation. The motivating factor behind this was that students’ written work forms a substantial part of the assessed coursework in mathematics – they need to be able to write mathematical arguments. An initial review of literature, suggested that argumentation was important to the way students develop their ability to prove (e.g. Inglis et al., 2007; Pedemonte, 2007a). Hence the title of this thesis.

## 1.2 Aim

The primary aim of this research was to investigate how students formalise their argumentation. This was explored by considering argumentation as a process of conviction and comparing argumentations that students wrote to convince themselves of whether a claim is true, sometimes true, or false, with those they wrote to convince an ‘other’ that they had made the right choice. The ‘other’ was characterised by a mathematics lecturer who we suggest, is a primary influence in shaping the life-world of a university mathematics student. The notion of a student’s life-world will be discussed in more detail in Chapter 5.

This research examined the cognitive processes and mathematical resources students call on to write a formal argumentation, and considered obstacles that prevented students

from doing so. As students progress through first, second and then third year courses, developments occur in their mathematical knowledge, skills and ability to communicate with, and through, mathematics. Thus this research tracking changes students made in their argumentations as they progressed through three years of study in mathematics.

To analyse the students' argumentations, a framework was constructed, which came to be called the Tertiary Argumentation Framework (TA Framework). This framework amalgamated Tall's Three Worlds of learning mathematics (Tall, 2008), Toulmin's layout of an argument (Toulmin, 1958), and Habermas' three types of rationality (Habermas, 1998). This gave a cognitive, structural, and rational dimension to the analyses of students' argumentations. In order to understand the ways students formalise their argumentations in mathematics, the following research question (RQ) was formed that provided the foundation for this study.

### 1.3 The Research Question

[RQ] How do tertiary students formalise their argumentations of conviction?

To help answer this question, two focus questions, RQ 1, and RQ 2 were developed. RQ 1 had three subsidiary questions that provided further focus in the analysis. These are presented below as RQ 1a, RQ 1b, and RQ 1c.

### 1.4 Specific Research Questions

[RQ 1] *What changes are evidenced in the structure and rationality of first and second year university mathematics students' argumentations when they change from giving argumentations that secure a personal conviction, to ones that convince someone else?*

[RQ 1a] *Do students recognise that argumentation aimed at establishing a personal conviction has a different epistemological basis to that directed at convincing an other?*

[RQ 1b] *What changes are evidenced in the shift from students' argumentation for self-conviction to one aimed at convincing an other?*

[RQ 1c] *What obstacles prevent students' writing a mathematically convincing argument?*

[RQ 2] *How do students' argumentations of conviction change as they progress through first, second, and third year courses in university mathematics?*

## 1.5 The Overview of the Thesis

The following is a brief overview of the following chapters of this thesis. The next chapter gives a brief overview of the history of argumentation and its development in mathematics. This flows into the third chapter which examines relevant literature and general theories of argumentation including theoretical perspectives from mathematics education. It also discusses a debate about the use of argumentation in mathematics. Details of the Tertiary Argumentation Framework (TA Framework), the main analytical tool of students' argumentation in this thesis, will be provided in Chapter 4. The research method underlying this research is presented in Chapter 5 and is followed by a detailed analysis of the data in Chapter 6. The results of a diachronic study of the evolution of mathematical argumentation of a selected group of students is presented in the next two chapters. A discussion of the analysis of the data and the diachronic study is given in Chapter 9 where conclusions are presented and linked to answering the research questions that have shaped this study. The last chapter (Chapter 10) considers contributions that this research might make to mathematics education, as well as its limitations and implications for further research.



## Chapter 2

# A Historical Overview

### 2.1 Introduction

This chapter presents historical account of argumentation. Its relevance to this thesis is that it backgrounds the relatively recent debate in mathematics education surrounding the relationship between mathematics and argumentation. On one side of this debate are those who see argumentation and mathematics as separate areas of study, and on the other side are those that view argumentation as an aspect of mathematics. This debate is made particularly manifest in literature surrounding mathematical proof. However, the seeds of the debate appear to be sown in logic, which has undergone numerous transformations from its beginnings in India, China and Greece. In this journey through history a consideration will only be given to the development of logic beginning in the Greek tradition of philosophical thought. The primary reason for this is that Greek logic became widely accepted in science and mathematics. We acknowledge that India and China formed their own systems of logic, but a choice has been made to base the discussion here on Aristotle's theorisation of argumentation.

From a reading of relevant literature, there appear to be three events in ancient Greek history that contributed significantly to the development of argumentation. These can be briefly summarised as, a search for an explanation of the world, developments in literacy, and the development of democracy. All three events occurred in the sixth century B.C. – a time period in which the Greeks developed a philosophical and scientific tradition in order to understand the world using the validity of logical arguments. The early part of the sixth century was a time when Ionian thinkers began to search for non-mythological explanations of the world. This prompted two metaphysical questions for the Greeks;

‘What is the world made of – what is its substance?’ and, ‘What is the ultimate nature of reality?’ To the Greeks, the word *world* was inclusive of the universe, so that many of their explanations of the world tried to explain the matter of the universe in material terms. Other questions also emerged, some related to epistemology, such as whether it is possible to know something is true or certain, and others related to ethics – how do we know what is right and wrong. How did this contribute to argumentation? If an assertion is made regarding whether something is true or certain (or false), then its status had to be justifiable. Argument as a way of reasoning, explaining and justifying came to be an integral aspect of Greek epistemology and ethics. A second event that would fertilise argumentation in Greek culture was literacy.

During the sixth century B.C. the Greeks began to develop as a literate society. Literacy increasingly became presumed as part of public life in Greece and Ionia (Goody & Watt, 1963). According to Goody and Watt, literacy highlighted logical thinking and analysis, and gave impetus to their development. In her survey of related literature Murray (2000, p. 45) supported this saying:

Although in post-Homeric Greece many people knew how to read and write the alphabetic script, communication was still mostly oral. Public oral discourse was the primary means for the development and distribution of knowledge. Such studies also show how the characteristics of logical thought which were claimed to be the result of literacy, were in fact precursors to its introduction. Writing merely amplified this already existing way of thinking.

The technology of literacy also allowed the Greeks to record the histories of other civilisations, including their religions, philosophies and sciences. Comparing different histories would likely have prompted the early Greeks to ask, ‘Which of the different world views is the correct one?’ This may have fuelled their desire to determine nature of truth. As with questions of metaphysics, epistemology and ethics, these were also matters of philosophy. However, a third significant development in early Greek culture in the 6th Century would see some of these same types of question occur, but this time, in a practical arena. This third key factor was the establishment of democracy.

The idealism of democracy, was first implemented in the sixth century B.C. in Athens. With its subsequent development (in various forms) in most Greek city states, a democracy meant that citizens (male only–not slaves, who were over 20 years of age) were obliged to

serve the state. They were given an opportunity through the Ekklesia (an assembly of citizens) to debate and influence political decisions. Thus the ability to influence and change the thinking of others through debate became an important skill to have. Citizens who wished to do so had to be able construct persuasive and convincing arguments. However, this evoked questions like; ‘What actually is a good opinion?’ – how was one to determine which argument among many was the best? and ‘When can we say that something is true?’ (see van Eemeren et al., 1996, p. 30). It is against this background that we begin our journey through history, beginning with a brief outline of the work of Aristotle. The aim here is not to present a complete review of his work but rather, to consider elements of his work that linked to argumentation in mathematics. A common thread in each of the three events described was the pursuit and determination of truth. In the next section we will see how this contributed to the theorising of argumentation.

## 2.2 Theorising Argumentation

From the sixth century B.C. on, intellectual debate and musings within the contexts of politics, law, and philosophy (which is also included science and mathematics), came to be a defining attribute of Greek culture (Gowers et al. 2008). The need to theorise argumentation was driven by two main factors. These are identified for us by Eemeren et al. (1996, p. 31) who writes:

First the comparison of the arguments for opposing views on all sorts of subject led to the general query of what is a good argument. Second, the practice of politics and the law led to the question of what is good, and above all effective, argumentation. This general reflection on argumentation took shape in classical logic, dialectic, and rhetoric.

Also contributing to the need for a theory of argument were the activities of Sophists, who were peddlers of rhetoric and oratory, teaching them to any who were willing to pay for their services. They worked on the premise that truth is an individual matter – it lies in the minds of those in an audience. Thus for the Sophist, a successful argument persuaded and convinced a particular audience of the truth of a viewpoint, regardless of whether in fact it was true or not. Because of the Sophists’ reduction of truth to personal opinion, little regard was given them as true philosophers, particularly by the likes of Socrates, Plato, and Aristotle, who saw them as being more interested in making money than in

promoting true wisdom. Thus a theory of argumentation that would also address the validity of an argument was needed. Aristotle (384–322 B.C.) is considered by some to be the first to give a theoretical treatment of argumentation.

Aristotle’s treatment of argumentation focussed on a triadic of *analytics* (logic), *dialectic* (theory or art of debate) and its counterpart, *rhetoric* (arguments aimed at convincing and persuading a particular audience of a certain standpoint) (van Eemeren et al., 1996). All together Aristotle produced six works on logic, *Categories*, *On Interpretation*, *Prior Analytics*, *Posterior Analytics*, *Topics*, and *On Sophistical Refutations*. In *Topics* and *Rhetoric* Aristotle establishes logical rules for arguing, debating, and oratory in the public arena. In *Prior Analytics*, Aristotle established the notion of deductive (or syllogistic) reasoning as a means to acquire new knowledge from old. In *Posterior Analytics* he investigated the starting points of knowledge and addressed the problem of the infinite regress of reasons. He did this by defining a starting point in an argument such as an axiom, hypothesis or definition as an object whose truth is self-evident. Aristotle claimed that self-evident truths came from instantiated properties that themselves were a result of inductive generalisation. Aristotle suggested that in the formation of self-evident truths, the intuitive leap to certainty is done in the mind – although he does not seem to be clear about how this is achieved.

Possibly the most important contribution of Aristotle was his analysis of syllogisms. He goes in to much detail to explain the grammatical–logical construction of sentences that make up the syllogism (see Smith, 2012 for an analysis of Aristotle’s syllogistic logic). His syllogisms dealt with 4 basic statements (or sentences) where  $a$  may be thought of as a predicate associated with the subject  $b$ . The following example illustrates the focus Aristotle had on this kind of logic structure:

1.  $Aab$ :  $a$  belongs to all  $b$  (All  $b$  is in  $a$ )
2.  $Eab$ :  $a$  belongs to no  $b$  (No  $b$  is an  $a$ )
3.  $Iab$ :  $a$  belongs to some  $b$  (Some  $b$  is an  $a$ )
4.  $Oab$ :  $a$  does not belong to all  $b$  (Some  $b$  is not an  $a$ )

Aristotle often expressed the same relationship between a subject and its predicate in different ways. For example for the statement, ‘every  $a$  is a  $b$ ’ he would often use the following alternatives, ‘ $a$  belongs to every  $b$ ’, or ‘ $a$  is predicated of every  $b$ ’. Anglin and Lambek

(1995) note that Aristotle was also aware that  $Eab$  and  $Eba$  were equivalent statements, and that  $Aab$  and  $Eba$  cannot together form a true statement. Anglin and Lambek (1995) state that Aristotle's syllogisms were created from these four statements – A, E, I, and O. For example, the valid syllogism:

*Aab:* All humans are mortal (This is the major premise since it contains the predicate of the conclusion)

*Aca:* All Greeks are humans (This is the minor premise as it contains the subject of the conclusion)

Therefore *Aab:* All Greeks are mortal.

In Aristotle's view a feature that distinguished analytic arguments from dialectic ones was that the former were based on assumptions or assertions whereas the latter are provisional statements that need to be agreed upon in the argumentation. An important point to make here is that in Aristotle's construction, logic, debate and rhetoric were all treated by him as facets of argumentation. Aristotle applied his ideas about logic to mathematics in order to show how mathematics can be true, and how one can come to have a knowledge of mathematical truths (particularly in arithmetic and geometry) (Lear, 1982). As stated by Gowers et al. (2008, p. 931), "... Aristotle created out of nothing the science of logic, whose aim is to distinguish valid from invalid inferences of conclusions from premises". There were other developments in logic in the third century B.C from the Stoic philosopher Chrysippus (c. 279–206 B.C.). These included the development of propositions types such as modus ponens, modus tollens and logical connectives (Gould, 1970; Bobzien, 1996), and the development of the truth tables (Anglin & Lambek, 1995). Chrysippus' logic was an attempt at better understanding the workings of the universe and role of humanity within it. Thus although his system of logic reflected aspects of logic we see in modern day mathematics, the driving force for its initial development was an understanding of elements that relate to everyday life. It would take until the nineteenth and early twentieth century for logicians to reinvent the work of Chrysippus, and develop a more rigorous form of logic that shifted it away from contemporary Aristotelian logic.

We now give an overview of some important moments in the development of mathematics. The purpose is to show that, as mathematics developed, a new tool would become available that would accomplish two things. The first would be to shift proof and justification away from a reliance on geometric methods (Kline, 1972), and the second was to treat logic as a formal abstraction – removing it from its original association with Aristotle's (and

Chrysippus') argumentation, and configuring it wholly with mathematics.

## 2.3 The Development of Argumentation in Mathematics

In this section the development of mathematics is considered, not in the way most texts on the matter do, but with specific reference to the development of argumentation. As we have already seen, logic was a facet of Aristotle's formulation of argumentation. However this type of logic was not present in early Mesopotamian and Egyptian mathematics. Although these cultures had developed a useful knowledge of arithmetic and geometry, justification was based on inductive and experiential methods (Gowers et al., 2008; Krantz, 2010). The Rhind Papyrus (c.1650 B.C.) indicates that much of Egyptian mathematics was built around arithmetic procedures. Its presumed writer, a scribe named A'h-mosé, gives no explanation of why various procedures worked but provides us with a practical handbook of arithmetic procedures that are useful to "employers of labour, granary overseers and brewers" (Calinger, 1999, p. 43). However, seeds of argumentation were sown in the late seventh century B.C. that would change the practice of mathematics. The traditional phenomenological and practical work of mathematics would come to include the pursuit of generalisations, and the development of abstraction. A brief overview of the contributions of some key mathematicians follows, beginning with Thales of Miletus.

Thales of Miletus, (c. 625–547 B.C.) proposed five geometric propositions, each of which formulated geometric properties as generalised statements (see Suzuki, 2009). The formation of Thales' propositions as generalisations was an important initial step in the transforming mathematics from a system employing inductive and empirical means of justification to the use of deductive reasoning since this invited justification and proof that could not rest on empirical evidence alone. That is, the proof had to be true for all cases (Gowers et al. 2008). This could only be accomplished by the use of representations that were not linked to a material and empirical world. Rather, there needed to be some other way that could be used to state a general case. It was left to another mathematician (and his followers), namely, Pythagoras (569–500 B.C.) and his school, the Pythagoreans, to take the initial step toward abstraction.

Pythagoras (569–500 B.C.) and the Pythagoreans viewed mathematics as being more closely related to wisdom than to the practical demands of life (Boyer, 1991). During his travels, Pythagoras observed how the Babylonians and Egyptians were able to make sophisticated and accurate calculations using various methods. However, he was more

interested in understanding various relationships amongst numbers themselves and how they interacted with geometry, music, and nature. The Pythagoreans found a formula that gave the perfect numbers (numbers that are equal to the sum of its positive divisors excluding the number itself). They also discovered that the square root of 2 is not commensurable with a rational number. Therefore the square root of 2 must be thought of in an abstract way. This result impacted negatively on Greek mathematics, for now measurement and geometry and any justification that appealed to proportion could not be accepted with certainty (Boyer, 1991). The Pythagoreans were able to move from ‘concrete’ representations to abstract ways of thinking in other areas as well. For example, Anglin and Lambek (1995, p. 35) provide the following example:

They found proofs for several algebraic relations by means of studying figurative numbers. For example, looking at the sequence of squares, expressed in terms of arrays of pebbles..., they noticed that  $n^2 + (2n + 1) = (n + 1)^2$  and hence  $1 + 3 + 5 + \dots + (2n - 1) = n^2$  (using modern day notation).

Although the contributions to mathematics of Pythagoras and the Pythagoreans continue to be debated (Boyer, 1991; Anglin & Lambek, 1995), it is their contribution to shifting mathematics from a practical phenomenological status to an intellectual activity of the mind that is that is noteworthy (Suzuki, 2009). However, mathematics needed a level of organisation that would bring rigour to deductive reasoning. This was given an initial nudge forward by Eudoxus of Cnidus (c. 408–355 B.C.), a Greek astronomer, mathematician, and pupil of Plato.

Eudoxus is credited with organising mathematics into theorems (Krantz, 2007). An example of one of these is, the area of two circles ‘are to each other’ as the squares on their diameters, that is, the ratio of the areas of two circles is the same as the corresponding ratio of their diameters. Eudoxus also developed a technique of proof that would later in the seventeenth century be called the *Method of Exhaustion* by Gregory of St. Vincent (1647) (Suzuki, 2002). To illustrate this technique, suppose one was required to show that for two quantities  $a, b$ ,  $a = b$ . First one would set up the trichotomy  $a < b$ ,  $a > b$  and  $a = b$ . All one needed to do was disprove  $a < b$  and  $a > b$ , and therefore make the conclusion that  $a = b$ . Eudoxus also provided a definition of equality between two ratios that would avert the potential crisis implied by the Pythagorean’s discovery of the existence of incommensurable numbers (Boyer 1991) (Definition 5, Euclid’s Book V):

Magnitudes are said to be in the same ratio, the first to the second and the

third to the fourth when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.

Although, as Krantz (2007) tells us, Eudoxus generally did not prove theorems, Euclid (c. 300 B.C.) did. Euclid developed a method of organising the way we think about mathematics that would last to our time.

In developing his method, Euclid defined terms such as the *axiom*; a principle that cannot be demonstrated for it is known or accepted as self evident; the *hypothesis*, something believed to be true, and the *postulate*; something whose truth is assumed to be part of the study of science. These ideas were used extensively by Euclid (c. 300 B.C.) in his treatise of geometry and number theory comprising thirteen books that make up his benchmark work, the *Elements*.

Euclid's contribution to mathematical argument lay in the way he organised and systematised mathematics into a deductive body of knowledge. He was aware of the need to avoid circularity and the potential of an infinitely regressing chain of reasoning. He was also aware that an argument must have clear and acceptable inference rules. For these reasons (the same reasons as Aristotle) Euclid provided starting points for mathematical reasoning that were stated as definitions. These were important because reasoning from these (and axioms) he was able to form statements called theorems. The Elements gave the first axiomatic and deductive approach to the provision of mathematical justifications and proof. Although through the Elements, Euclid had established an axiomatic approach to proof, logic as a subject (as developed by Aristotle for example) was not yet associated with mathematics (Mendell, 2008).

Euclid did not only study geometry. He also wrote on number theory. However, because he did not have the facility of algebraic symbolism, he wrote his arguments in a mixture of prose and representations that had physical meaning. For example, he used line segments of different lengths to represent numbers of different sizes, as well as products of numbers. Thus when he demonstrated that, 'prime numbers are more than any assigned multitude of primes', Euclid did so using line segments as representations of prime numbers (Calinger, 1999) (since the Greeks at that time did not have a notion of infinity). The main thinking tool used in the period of Euclid was geometry, and this would prove to be a robust tool even after the early establishment of algebra. A new thinking tool would however be



needed.

### 2.3.1 A New Thinking Tool for Proof

One outcome of Euclid's *Elements* was the expression of axiomatic proof. However, many abstractions were still linked to geometry through their explanations, and the presentation of arguments was still at a rhetorical stage. To free abstract thought from geometry, and argument from its rhetorical presentation, a new thinking tool was needed. That tool would turn out to be algebra. In this subsection we will briefly describe key moments in the development of algebra beginning with Diophantus of Alexandria (fl. 250 C.E.) and ending with Descartes (1596–1650), as they relate to the development of argumentation.

Diophantus of Alexandria (fl. 250 C.E.) wrote a text comprising 13 books entitled *Arithmetica*. In these he put forward problems and their solutions using a notation and argument that departed from the rhetorical style of previous writers and the geometric algebra used by Euclid (Gowers et al., 2008). With regard to his notation Suzuki (2009, p. 47) writes, “[he] created a type of algebraic notation called syncopated notation, which essentially replaces the words describing an algebraic operation with an abbreviation”. For example,  $\varsigma$ ,  $\Delta^\Upsilon$ ,  $K^\Upsilon$ ,  $\Delta^\Upsilon K$ ,  $\Delta K^\Upsilon$ , and  $K^\Upsilon K$  were used to represent  $x$ ,  $x^2$ ,  $x^3$ ,  $x^4$ ,  $x^5$ , and  $x^6$ , and  $\alpha = 1$ ,  $\beta = 2$ , and  $\gamma = 3$  were coefficients. All terms to be added were put first in a polynomial expression, all terms to be subtracted followed the symbol  $\psi$ , and the constant followed the symbol  $\overset{\circ}{M}$ . What Diophantus would write as,  $K^\Upsilon K \alpha K^\Upsilon \alpha \psi \Delta K^\Upsilon \beta \varsigma \beta \overset{\circ}{M} \gamma$  would be interpreted in modern day algebra as,  $x^6 - 2x^5 + x^3 - 2x - 3$  (Calinder, 1999). However, algebra was also developing in both Indian and Islamic culture and by the 7th Century A.D., Indian culture had developed an algebraic system that used symbols such as letters to represent unknown quantities and abbreviations and symbols to represent operations (Joseph, 1994)

The development of algebra in the Islamic world was inspired by the works of Euclid and Diophantus that had found their way into the Islamic world after the fall of the Roman Empire. A treatise of algebra was given by Abu Ja'far Muhammad ibn Mūsā al-Khwārizmī (c. 780–847) (Al-Khwārizmī) in his book *al-Kiāb al-mukhtasar fī hisāb al-jabr w'al-muqābala* (The compendious book on calculation by completion and balancing) written entirely in prose, which appeared about 830 C.E. (Calinger, 1999). In it al-Khwārizmī interprets geometrical arguments given by Euclid in his second book using

indigenous algorithmic methods. This book soon became the starting point of algebraic study for Islamic mathematicians (Gowers et al., 2008). An important contribution to algebra came from al-Khwārizmī's articulation of processes used to transform linear or quadratic equations into one of the six basic forms, which he identified as,  $ax^2 = bx$ ,  $ax^2 = b$ ,  $ax = b$ ,  $ax^2 + bx = c$ ,  $ax^2 + c = bx$ , and  $ax^2 = bx + c$  (Gowers et al., 2008). In addition, he arithmetised algebra by giving an explanation of how arithmetic techniques, such as multiplication, can be used with unknown quantities (Suzuki, 2002). Although his proofs were geometrically motivated, geometry only served as an auxiliary support for his (emerging) algebraic reasoning (Burton, 1999). While there is some debate regarding al-Khwārizmī's contribution to mathematics (for example see Anglin and Lambek, 1995), it is clear that his contribution to algebra was significant in pushing its development forward. After al-Khwārizmī two other middle eastern mathematicians played a large part in continuing to develop the generalisability of algebra. First, the work of Egyptian Islamic mathematician, Abū Kāmil (c. 850–930), suggested a shift in emphasis from geometric solutions to particular problems, to algebraic solutions of general equations (Suzuki, 2002, pp. 253–254). Second, the Persian mathematician al-Karajī (c. 953–1029) incorporated Diophantus' idea of notational representation of variables into his own work and synthesised then into more general terms (Gowers et al., 2008). Using his notation, he developed further the arithmetisation of algebra originally begun by al-Khwārizmī (for example, 'like terms can be added to each other') (Suzuki, 2002), and was seen by some to be the first person to free algebra from geometry.

However, it would not be until the late 1500s that Diophantus' work would be rediscovered and translated into Latin that western mathematicians were given access to his ideas. This led to a rapid advancement in algebra in the western world, first with Cardano (1501–1576) whose work *Ars Magna* (The Great Art) was foundational in the development of European algebra. Cardano did not have a concise and flexible written notation, and so reverted to proof by way of geometrical representations (Suzuki, 2002). However, he is said to have improved on the verbal stage of algebra (Calinger, 1999, p. 423). Other contributors to the development of algebra around the same time included Tartaglia (1449–1557), Bombelli (1526–c.1572) and Viète (1540–1603). Bombelli wrote the *Algebra* which was noted for its advances in mathematical notation, including for the first time, notation for writing exponents (Gowers et al., 2008) and Viète applied algebra to geometry. Viète saw that symbols could be used to represent unknown quantities as well as numbers (arbitrary quantities). Previously techniques for solving equations were gauged by the type

of equation, for example (using modern day format)  $3x+2 = 0$ , or  $x^2+5x+6 = 0$ . But now, it was possible to write an equation in a more general form such as  $ax^2 + bx + c = 0$ . The symbolic format we now use with  $x$ 's, powers of  $x$ , and alphabetic letters as representatives of constants, was introduced by Descartes (1596–1650) in his work entitled *la géométrie*. Another of his notable contributions to mathematics was his reduction of geometry to algebra by way of his suggestion that a point in the Euclidean plane could be represented by pairs of real numbers. Descartes viewed algebra as having a close association with human reasoning, and treated it as an extension of logic. Here Descartes gives an abstract form of algebra freeing it from geometry and from meaning in general. This would eventually lead to what we now know as symbolic logic (Kline, 1972).

## 2.4 The Meeting of Mathematics and Logic and the Demise of Argumentation

Until the late seventeenth century Aristotelian logic shaped the study of logic as a discipline and as a formal science. However when placed under scrutiny, Aristotle's syllogistic reasoning was found to be inadequate as a description of deductive logic, as there were logical arguments that could not be packaged into a syllogism (Devlin, 1997, p. 3). With the power of algebra to symbolise, Descartes (1596–1650) and Leibniz (1646–1716) were motivated to search for an abstract way of reasoning that would behave in the same way as algebra (Kline, 1972, p. 1187). They wanted to develop a universal algebraic logic, and Leibniz pursued this ambitious goal. As noted by Kline (1972, p. 1188), "Directly and indirectly Leibniz had in his algebra concepts we now describe as logical addition, multiplication, identity, negation, and the null class. He also called attention to the desirability of studying abstract relations such as inclusion, one-to-one correspondence, many-to-one correspondences, and equivalence relations". Unfortunately, Leibniz did not get to finish his work on logic, which remained unedited until the beginning of twentieth century.

Two reformers of Aristotelian logic reunited the theorisation of argumentation (as developed through the philosophy of Aristotle) with mathematics. In 1847 Augustus De Morgan published his work, *Formal Logic* and at the same time, George Boole (1815–1864) published *Mathematical Analysis of Logic*. De Morgan (1806–1871) worked on and extended Aristotelian logic. By observing that terms in a syllogism can be quantified he added to Aristotle's syllogistic forms and worked on eliminating some issues of existence in Aristotelian logic. With regard to symbolic logic, De Morgan developed laws that ex-

pressed the duality between the disjunction *or*, and the conjunction *and* – which came to be known as De Morgan’s Laws. Boole saw propositional logic as a branch of algebra and argued that logic belonged to mathematics rather than to philosophy (Anglin & Lambek, 1995. p. 267). He also argued that the essential characteristic of mathematics was not its content, but its form. Boyer (1989, p. 579) summarised this union saying, “If any topic is presented in such a way that it consists of symbols and precise rules of operation upon these symbols, subject only to the requirement of inner consistency, this topic is part of mathematics”. In this sense, Boole and De Morgan shared a common belief that specified symbols along with rules of operation and inference could be used effectively in mathematics.

Whereas Leibniz, De Morgan, and Boole were interested in mathematising logic, Gottlob Frege (1848–1925) worked on the assertion that mathematics was an expression of logic – that mathematics could be constructed from axioms of logic. Until Frege, it was thought that all reasoning in mathematics was of the syllogistic type. However, he showed that a more precise description of reasoning was needed and shifted the study of logic away from its Aristotelian roots. Among other things, Frege introduced universal and existential quantifiers (though not in symbolic form), introduced in symbolic form the ‘Megarian’ concept of *material implication* (which freed the antecedent and the conclusion of any relationship to each other), the logic of relations, and a set of axioms upon which he built up a set of logical premises (see Kline, 1972, p. 1191). His attempt to construct mathematics from logic implied that mathematics would be free of contradiction, inconsistencies, and paradoxes.

Others who made extensive contributions to the development of logic and its relationship to mathematics include Bertrand Russell (1872–1970) who believed that once all paradoxes are resolved mathematics would be seen as logic. Russell had himself found a contradiction in the set theory of Frege (which came to be known as Russell’s Paradox) which was devastating for him since it pointed to a possibility “that mathematics might be inherently contradictory” (Singh, 1998, p. 152). Russell, and colleague Alfred Whitehead (1861–1947) attempted to demonstrate the power of logic by proving that “all pure mathematics can be derived from a small number of fundamental logical principles” (Boyer, 1991, p. 611). This was the essence of the logicist thesis, that mathematics is part of logic (Hanna, 1983). Another contributor during this period of time was David Hilbert (1862–1943) whose goal it was to formally axiomatise mathematics and “... prove it free from contradictions ...” (Gowers et al., p. 789). Hilbert took a formalist approach

to mathematics viewing it as a science of formal systems – strings of meaningless symbols (Hanna, 1983). The price of introducing this level of rigour was to take the meaning out of mathematics. In 1932, Kurt Gödel (1906–1978) showed that it was impossible to give mathematics a secure foundation using logic or any other system and guarantee its freedom from contradiction (see Singh, 1998, pp. 149–161). The system would always be incomplete, that is, contain statements that could not be proved true or false.

As logic became more encased in the study of mathematics, its disassociation with argumentation became more marked. Logic, which began as a facet of argumentation in Aristotle’s triadic of logic, dialectic, and rhetoric was now stripped of its grounding in human reason, and rebuilt as a formal system composed of a set of mechanical rules and symbols exemplified in the work of Russell, Whitehead and Hilbert. Argumentation continued to be associated with arguments that occur in everyday discourse, except that a good logical argument was said to be sound. Logic – rebranded as formal logic, was shifted into mathematics, where a logical argument was said to be valid. The power of logic was asserted in the early twentieth century when it was thought that symbolic logic was the system upon which ordinary language was based. If it wasn’t, then it was thought that it should have been, and this needed to be shown. Frege and Russell both thought this way. For example Russell attempted to show that ordinary sentences can be formatted into logical symbolic structure (Russell, 1949).

In 1958, Stephen Toulmin challenged the use of formal logic as a tool that could explain argumentation and produced a what he called the *Layout of Argument* – a model that Toulmin argued could analyse and provide a rational basis to argumentation without the use of formal logic (Toulmin, 1958). Toulmin’s model is discussed in Chapter 3. Inadvertently, Toulmin’s model gave impetus to mathematics education to re-associate argumentation with mathematics, although there remained some who continued to defend the separation of argumentation from mathematics (e.g., Balacheff, 1999; Duval 1999).

## 2.5 Reunited: Argumentation and Mathematics

The layout of an argument as presented by Toulmin modelled argumentation and rationality in everyday discourse. In fact it really seemed like an attempt to recover the Aristotelian ideal. For example in their analysis of Toulmin’s work, Van Eemeren et al. (1996, p. 130) state:

Such a radical reorientation of logic would for Toulmin amount to going back to the Aristotelian roots of logic. He refers several times to the first sentence of the *Prior Analytics*, where Aristotle expresses the double aim of logic: Logic is concerned with *apodeixis* (i.e., the way in which conclusions are to be established), and it is also the formal deductive, and preferably axiomatic science (epistemic) of their establishment.

By the late twentieth century and early twenty-first century researchers in mathematics education had taken hold of Toulmin's layout, and developed it to model students' argumentation associated with learning mathematics, conviction, and proof (Krummheuer, 1995; Pedemonte, 2007a; Inglis, Mejia-Ramos & Simpson, 2007; Boero et al., 2010; Tall & Mejia-Ramos, 2010). Also, during this time, the purpose of proof was under examination. For example, Rav (1999) distinguished *conceptual proof* from *formal proof*, describing formal proofs as a finite string of formulae that are either axioms or derived from them, and to which no meaning needs to be assigned. On the other hand, he described a conceptual proof as one that presents a rigorous argument acceptable to mathematicians, but also makes reference to the meaning of concepts where appropriate. Tall et al. (2012) refer to these proofs as *ordinary proofs* – the type of proof seen in scholarly journals, and school and university level textbooks. Other writers distinguish proofs by whether they explain why a result is true, or whether they show that a result is true by giving evidential reasons only (Hanna, 1990). This notion of a proof having explanatory power is echoed in the writings of Hersh (1993, p. 398) who states, “In brief, the purpose of proof is understanding”, and Thurston (1995, p. 34) who writes, “However, we should recognise that humanly understandable and humanly checkable proofs that we actually do are what is most important to us, and they are quite different from formal proofs”. Thus although the primary role of a proof may be to establish the truth of a result or convince someone else of its truth, what seems to be more interesting to mathematicians is understanding why a result is true (Thurston, 1995).

Another force driving argumentation and mathematics together was the notion of process and product. For example, in mathematics proof was seen as the product of the process of proving. As a result researchers examined the process of proving in terms of argumentation (see Bills & Tall, 1998; Pedemonte, 2007a; Arzarello & Sabena, 2011, for examples). In a well documented study, Harel and Sowder (1998) defined proving as a process that removes or creates doubts about the truth of an assertion. They argued that proving includes

two subprocesses, ascertaining (removing or creating one's own doubts) and persuading (removing the doubts of others). In their view, proof was considered a kind of argument that need not necessarily be a deductive proof. For example a proof might be based on numerical or perceptual examples, or perhaps on a certain procedure (Harel & Sowder, 1998). Thus, according to Harel and Sowder, arguments of proof are products of a proving act, which in turn is characterised by a certain proof scheme. As argued by Harel (2008) the proof scheme used reflects a student's way of thinking. We will examine Harel and Sowder's categorisation of schemes in more detail in the literature review (Section 3.3.2). Since in argumentation theory, argumentation should be interpreted as either a process or a product (see van Eemeren, 2010, p. 26), Harel and Sowder's proof schemes along with their proof products can also be interpreted as typology of argumentation.

One result of these various theoretical advances in mathematics education was the promotion of the idea of a continuum of argumentation, with exploration at one end and proof at the other. For example, Boero (1999) gave six phases of proof production linked to the work of a mathematician, beginning with production of a conjecture and ending with the production of a formal proof using mathematical logic. Recently, Hsieh et al. (2012) characterise a spectrum of argumentational activities consisting of a continuum of exploration, conjecturing, informal explanation, justification argument, and proof. In a similar manner, Tall et al. (2012) give a development of proof that is composed of successive stages that are interrelated with each successive stage correlated to previous ones. Their model is based on maturation of mathematical thinking and begins with perceptual recognition leading to proof within a crystalline concept and then progressing to more generalised proofs with a deductive knowledge structure (see Tall et al., 2012, pp. 19–23). Thus in contemporary mathematics education, argumentation is considered a feature of mathematics.

## 2.6 Conclusion

One purpose of this discussion has been to show that argumentation started out as an aspect of discursive discourse. Against this background Aristotle developed a theory of argumentation grounded in logic, dialectic and rhetoric, which in his view formed the basis of discursive discourse. Although Aristotle was able to show how his logic and mathematics could be integrated to provide a theory of argumentation in mathematics, it was not developed any further during the early history of mathematics. Rather than relying on

the theoretical contributions of Aristotle to mathematical argumentation, mathematics developed its own form of argumentation comprising the elements of explanation, justification and proof. Prior to the 19th and twentieth century logic developed along a separate line to mathematics. Various developments in logic separated it from its place in Aristotle's concept of argumentation. In addition, with the development of algebra as a symbolic thinking tool, logic became a significant feature of mathematics and in particular, a means by which the validity of arguments could be checked. At this point argumentation in mathematics was displaced by formal logic – at least for the logicians such as Russell, Whitehead, and Frege, and the formalists such as Hilbert.

The power of logic to identify valid statements as well as contradictions was seen by some to be useful for analysing not only mathematical statements, but also everyday discursive discourse. The negative reaction to this from argumentation theorist Stephen Toulmin eventually energised researchers in mathematics education to establish a link between argumentation and mathematics, and to interpret their relationship (see Hanna et al., 2012). This was accomplished through research on proof – the very subject in which logic was seen as having its most powerful effect. Other researchers linked argumentation to mathematics by considering how the cognitive gap between the production of a conjecture and the reasoning requirements of proof might be overcome. The outcome for mathematics education was that argumentation and mathematics and even formal logic were united.



## Chapter 3

# Review of Literature

### 3.1 Introduction

Our ability to use language in all its different genres to communicate with others, is a feature of human beings that separates us from all other animals. Our facility with language helps us to explain, reason, justify our ideas, as well as listen to the explanations, reasoning and justifications of others. With language we can write, sing, tell stories, imagine, design and build. However, a particular function of language that this study is primarily interested in, is found in the study of argumentation. It has been suggested that the study of argumentation had its beginnings in ancient Greece (van Eemeren et al., 1996) with Aristotle (384–322 B.C.E.), who gave a theoretical treatment to analytics (logic), dialectic (theory, or art of debate) and its counterpart, rhetoric (arguments aimed at convincing and persuading a particular audience of a certain standpoint) (van Eemeren et al., 1996). Argumentation has continued to be a topic of theoretical interest down to our time, and elements of Aristotle’s analysis still continue to appear in contemporary perspectives of argumentation. The review that follows will consider various perspectives writers have taken of argumentation. It will look at how argumentation has become an integral aspect of mathematics, and why it continues to be part of on-going research in mathematics education. In this examination of argumentation, the reader will be asked to consider common features that different researchers have used to frame their viewpoint of argumentation. Toward the end of the review, some of the general features that have been gleaned from it will be provided.

This literature review will begin by presenting some contemporary views of argumentation taken from argumentation theory (Section 3.2). It should be noted that these do

not represent all the views there are of argumentation. Examples of argumentation have been chosen to show that each view of argumentation is guided by certain goals of the writer/researcher. Along the way, there will be times when an explanation of a theory, model or some specific language terms will be needed, and these will be signalled beforehand.

A consideration of argumentation in mathematics education will be provided in Section 3.3. This will begin with an explanation of what we have termed ‘the debate’, and include a description of some responses from members of the mathematics and mathematics education community. An examination of how some of these responses have led to the development of argumentation as a feature of mathematics will follow. Since argumentation in mathematics is a complex of interwoven theories, there is no one-sentence definition that satisfies everyone. Therefore, a review of various characterisations of argumentation in mathematics will be provided. The reader will note that each characterisation is given a label that explains its primary goal, or important aspects that contribute to the construction of its goal. The literature review is completed by looking at what research tells us about the relationship between argumentation and proof.

## **3.2 Different Perspectives on Argumentation**

This section will consider how argumentation has been interpreted by various researchers and writers. The purpose for doing this is to show that there is a range of interpretations that make finding a generic definition of argumentation in the literature difficult. As will be shown, each interpretation or definition, is phrased with certain epistemic and teleological goals in mind. Epistemic goals refer to those goals that relate to determining whether an argumentation is rational, whereas teleological goals refer to the intended outcomes of argumentation and the organisation of the means by which these outcomes are to be achieved. Thus, we will show that various views of argumentation are dependent on the goals of the researcher, or theorist. We will first present some examples from argumentation theorists, and then consider interpretations of argumentation from mathematics education. For each interpretation, a brief outline of key ideas of underlying each view of argumentation will be provided, and these will be linked to the goals of argumentation. The first viewpoint of argumentation that will be discussed, is that it is an activity directed at the goal of conflict resolution.

### 3.2.1 Resolution of Conflict

One well known definition of argumentation is provided by van Eemeren et al. (1996, p. 5). They view argumentation as “...a verbal and social activity of reason aimed at increasing (or decreasing) the acceptability of a controversial standpoint for the listener or reader, by putting forward a constellation of propositions intended to justify (or refute) the standpoint before a rational judge”. Behind this definition stands the interest of van Eemeren and his colleagues in theorising argumentation as a rational procedure for resolving situations of conflict. The view that argumentation involves the presentation of a constellation of propositions indicates the pervasiveness of reasoning in justifying one’s standpoint, or refuting someone else’s. In harmony with this, these writers see language as a crucial feature of argumentation because it is the vehicle by which reason is used to resolve a difference of opinion. Thus the Pragma-dialectical argumentation framework developed by van Eemeren along with Grootendorst (1992, 2004) and Houtlosser (2002) includes a discussion of certain principles that should govern argumentation, as well as a description of stages in argumentation (van Eemeren & Houtlosser, 2002, see also van Eemeren, 2010, pp. 43–45). The framework for Pragma-dialectics also details 10 rules for critical discussion that comprise a Pragma-dialectic code of conduct (van Eemeren & Grootendorst, 2004). The teleological goal of argumentation expressed in this view of argumentation is the settlement of a dispute or disagreement. Achieving this goal is a function of language discourse. Assuming the disagreement can be resolved, and the argumentation follows van Eemeren and Grootendorst’s rules for critical discussion, then the outcome reached is seen as both a rational and valid – otherwise the argumentation is said to be derailed. This view is certainly not universal and has been challenged by Siegel and Biro (2010) on its failure to satisfy epistemological norms of evaluation. A central feature of van Eemeren’s definition noted by Schwarz et al. (2010, p. 104) is that it, “positions argumentation in a social space and excludes monological discourse in favour of a collective one”. Thus a monological argument, such as one given by a speaker at a public meeting, is not considered an example of argumentation. Van Eemeren’s definition also places argumentation in a social sphere and therefore does not consider the possibility of intrapersonal argumentation. However, as will be shown in the following section, another viewpoint of argumentation – one that incorporates intrapersonal and traditional rhetoric into its definition, rests on quite a different goal to that of van Eemeren et al. (1996).

### 3.2.2 Transmission of Knowledge

Philosopher Alvin Goldman develops a perspective of argumentation that includes an intrapersonal view of argumentation. Other aspects of his view of argumentation will be discussed in the following paragraphs. Goldman (1997) argues that justification – which he sees as a key feature of argumentation, is not just a social activity, suggesting that social interaction is not sufficient to guarantee genuine (personal) justification. He states, “personal justification is a personal matter, and that a cognitive agent can be personally justified in believing a proposition without having any relevant justificational relation to other people” (p. 156). Being convinced of an argument depends on the strength of belief a cognitive agent has in the truth of given premises, and whether they actually support the conclusion. This strength comes from practical interests that dictate the desirability of having epistemic states that are not merely reasonable degrees of belief, but reasonable high degrees of belief. Getting such epistemic states requires gathering enough suitable evidence (Goldman, 2009) from resources, which may include an individual’s own knowledge, perceptual experiences or beliefs, as well as, documents, conversation, or electronic media. Goldman’s intrapersonal view of argumentation appears to emphasise an epistemological analysis that considers psychological factors at work in an argumentation in its attempt to answer questions about knowledge and truth. However, a personal justification, and hence belief, may be wrong, or questioned by others. Thus Goldman’s apparent relativist position with regard to personal justification is tempered by interpersonal justification. In interpersonal justification all justification is sourced from a social or interpersonal arena. It is here that personal justifications can be examined, evaluated and readjusted if necessary. Goldman also considers monological discourse as an aspect of argumentation.

Monological discourse is one in which a speaker addresses a particular audience. This may take the form of a public address, a lecture, or perhaps a seminar. In defining his conception of argumentation Goldman (1999, p. 131) states, “If a speaker presents an argument to an audience, in which he asserts and defends the conclusion by appeal to the premises, I call this activity argumentation. More specifically, this counts as monological argumentation, a stretch of argumentation with a single speaker”. Goldman distinguishes this from dialogical argumentation which he describes as a situation in which the goal of two or more speakers is to resolve their differences of opinion with regard to the truth of a conclusion. Goldman’s writing suggests that this is best accomplished by transforming one’s personal conviction into an argument that would convince an other. This idea is

taken up further, later in this thesis.

As a research goal, Goldman is interested in developing a normative structure that includes deductive and inductive tests that can be used to evaluate a statement's truth (Goldman, 1999). The epistemology underscoring his view of argumentation is based on the reliability of the process that caused a belief to be justified (a theory he calls Reliabilism), and on what he refers to as the *truth-in-evidence* principle. The latter states, "a larger body of evidence is generally a better indicator of the truth-value of a hypothesis than a smaller, contained body of evidence, as long as all the evidence propositions are true and what they indicate is correctly interpreted" (Goldman, 1999, p. 145). Thus his goals of argumentation comprise three subgoals: to establish a personal conviction of the truth of an argument (Goldman, 1997); to persuade others of what is true; and to transmit knowledge (Goldman, 1994; Blair, 2005). This study will consider the argumentations students use to establish a personal conviction regarding a certain issue, since examining an individual epistemology appears to offer a method of gaining insight into how students ground and warrant their arguments.

Goldman (1994) makes reference to persuasion as a goal of argumentation. Persuasion requires that the arguers know something about the nature of their audience, such as what they already know, value and believe, and what things are of concern to them. It is from this basis that the next viewpoint of argumentation to be considered here is developed. Central to this view of argumentation is persuasion and conviction.

### 3.2.3 Persuasion and Conviction

Persuasion and conviction are treated differently by some argumentation theorists, although in some cases the difference is subtle. For example, Crucius and Channell (2003) see conviction as the securing of an agreement and persuasion as an argument that influences behaviour. Others associate conviction with a belief that a statement is true, or false, and persuasion with the giving of an argument with the objective of gaining a conviction in the (probable) truth or falsity of statement of conclusion (Harel & Sowder, 1998; Inglis & Mejia-Ramos, 2009). From linguistic theory we have a description of conviction as the modification of one's opinions and confidences as a result of appeals to rationality, and persuasion described in the same way except rationality is not necessarily used in the appeal (Pedemonte, 2007b).

A persuasive and conviction function of argumentation underlies the New Rhetoric model

of argumentation proposed by Chaim Perelman and Lucie Olbrechts-Tyteca (1969). Perelman and Olbrechts-Tyteca were influential in developing a contemporary role for Aristotelian rhetoric in argumentation. In the context of this discussion, rhetoric is taken to mean finding and using the most suitable means to persuade and convince an audience. For these theorists, argumentation is a matter of inducing or increasing the plausibility of one's proposition, with the objective of persuading and convincing an audience of its value or truth. In their opinion, the achievement of this overlies a successful argumentation. Perelman and Olbrechts-Tyteca differentiate persuasion from conviction on the basis of a theoretical and rather complex construct of an audience. Simply put, in their view argumentation that is persuasive claims validity for a specific (or 'ideal') audience only, or for a certain individual. These claims cannot be imposed on others because they reflect certain likes, tastes, inclinations, and beliefs, and so are subjective. Conviction however, means gaining the adherence of a wider group of people (a 'universal audience'), not just ourselves or a specific audience.

According to Perelman and Olbrechts-Tyteca (1969) the universal audience is a construct that lies in the mind of the speaker. It is a theoretical audience composed of individuals from diverse backgrounds who project the disposition of reasonableness. When a speaker configures his argumentation to such an audience his aim is to convince them that "the reasons adduced are of a such a compelling character, that they are self-evident, and possess an absolute and timeless validity, independent of local or historic contingencies" (Perelman & Olbrechts-Tyteca, 1969, p. 32). This viewpoint of persuasion and conviction highlights the New Rhetoric's focus on the nature of the audience, and the importance that having a knowledge of the audience – its realities, preferences and beliefs – plays in constructing a persuasive or convincing argument. The theory of the New Rhetoric, views argumentation as the justification of one's ideas to claims of rationality. Perelman and Olbrechts-Tyteca make a bold claim that rationality is a function of the audience, hence their attention on the composition of the ideal audience, the universal audience, and the determination of a departure point (a starting point for the engagement of argumentation with an audience). The concept of a universal audience is useful to this study which compares argumentations that convince the self, with those that are aimed at convincing an other. The idea of the universal audience conceptualises 'an other' as distinct from 'the self'. For example, in this research it is hypothesised that the universal audience will be conceptualised by the students as the audience of 'mathematicians' who teach courses similar to the ones they have taken. It is also hypothesised that the universal audience will

be sensitive to, and hold realistic expectations of these students' mathematical abilities. Perelman and Olbrechts-Tyteca's goal is to recast the Aristotelian notion of rhetoric – finding and using the most suitable means to persuade and convince an audience – as the basis for a theory of argumentation that includes dialectic interaction. Thus they place rhetoric in a web linking the psychology of arguers with the constitution of an audience, and with the validity of an argument. In the *New Rhetoric* they provide a typology of argumentation schemes that can be used to increase the likelihood of acceptance from the audience – many of these being based on the manipulation of the audiences' beliefs, and conceptions of rationality, reality, and sensibleness (van Eemeren et al., 1996). The reason we present this viewpoint of argumentation is because of its practical application. For example, when we reason with others about a certain matter, we often make choices regarding the language, illustrations, and other representations we use. In many instances, these choices are made on the basis of having some knowledge of the person(s) we are communicating with, and an idea of the representative group they might be categorised into (e.g., educators, or the 'business world').

Whereas the *New Rhetoric* is technique driven, other researchers have aligned argumentation with the goal of learning. The section that follows presents an example of argumentation theory that focusses on the relationship between argumentation and collaborative learning, along with an illustrative example from mathematics education. First however we will, very briefly provide a background for constructivist learning theory, since the epistemological status of both examples provided can be tracked back to constructivist roots. Another reason for presenting constructivism here is because it theorises branches of learning as both a social and individual activity.

### 3.2.4 Learning

From the mid 1980s, the theory of learning mathematics began to undergo a transformation. Constructivist learning theory was gaining popularity in the fields of research and teaching. In a 30 year reflection on the development of constructivism, Confrey and Kazak (2006) locate the roots of constructivist learning theory in problem solving: misconceptions; critical barriers and epistemological obstacles; and theories of cognitive development. They summarised constructivism as a theory that places the development of knowledge and understanding in the known world of the student (Confrey & Kazak, 2006). This perspective, suggested that students are able to build knowledge and understanding through

action – here we use the broad definition of action as proposed by Sinclair (1987) which is, “behaviour by which we bring about a change in the world around us or by which we change our own situation in relation to the world” (p. 28). Because of criticisms levelled at constructivism (see Confrey & Kazak, 2006, pp. 309–314 for a discussion of these), as well as the lack of precise educational implications of the theory, Confrey and Kazak (2006) set out 10 statements that, in their view, collectively articulated principles of constructivism. One of these is that constructivism should allow students to derive warrants for mathematics. The writers suggested that this is best achieved by allowing students to develop conjectures and create arguments and justifications. Confrey and Kazak (2006, p. 316) state, “We refer to these processes [those mentioned in the preceding sentence] as chains of reasoning which are the hallmark of mathematical thought, and they include intuition, visualization, generalization, problem solving, symbolizing, representing, demonstrating and proving , etc.” Although the primary unit of analysis of cognitive achievement is the individual, Confrey and Kazak (2006) argue that constructivism focusses on the development of knowledge, and therefore objectivity “must be redefined as the result of a consensus among a group of qualified individuals to authorize a particular description or explanation as viable and as shared among them” (p. 318). This harmonises with an earlier perspective of social constructivism as a theory that describes social genesis and warranting of both objective and subjective knowledge (Bartolini-Bussi, 1991; Lerman, 1996; Ernest, 1998). Central to the social constructivist view is language, negotiation and argumentation, as well as conversation.

Ernest (1998), traces the roots of social constructivism in mathematics back to the philosophies of Lakatos and Wittgenstein. For example, in his book *Proof and Refutations* (1976), Lakatos provides a pragmatic perspective regarding the development of mathematical knowledge and its warrant, suggesting that it occurs in a dialectical interchange involving a more knowledgeable other (such as a teacher) who can orchestrate the interchange. Ernest refers to Lakatos’ work as a “fallibilist critique of absolutist notions of formal proof in mathematics” that provides a “rhetorical and socially persuasive role of proof in mathematics” (Ernest, 1998, p. 135). On the other hand, Wittgenstein (1956) captures the epistemological backing for social constructivism by locating logical necessity and mathematical knowledge in linguistic rules and social practices. This supports the social constructivist view that “social conventions, norms, and lived patterns of behavior, including above all socially accepted patterns of language use, that provide the basis for notions of necessity of truth” (Ernest, 1998, p. 135) provide the epistemological foundation



for mathematics and logic.

Social constructivism is seen as a means through which intersubjectivity can be addressed (Lerman, 1996; Palinscar, 1998). Lerman (1996) backs this stance with four fundamental assumptions. The first is that the individual is part of the social world, and therefore his/her thinking is in a dialectical and reflexive relationship with that world. Second, the fundamental units of cognition are meanings, which are based on the social plane. Third, human function is the product of consciousness, which only takes place in social life. Against this background we introduce argumentation's relationship to collaborative learning.

Collaborative learning may be described as a situation where students from different performance levels work in small groups toward the achievement of a common goal. Michael Baker (2003) sets out to explain how argumentative activity can lead to collaborative learning. With this objective in mind he writes, "we see argumentative interaction fundamentally as a type of dialogical or dialectical game that is played upon and arises from the terrain of collaborative problem solving, and that is associated with collaborative meaning-making" (Baker, 2003, p. 48). Thus in Baker's view, the role of argumentation is to transform the epistemic statuses of competing problem solutions (or viewpoints) so that one of them appears more acceptable than the others to the group. An important aspect involved in achieving this is 'negotiation of meanings', which he refers to as redefining, or reformulating linguistic expressions. This is important to his view of argumentation because he argues that once meanings of statements shift, then so do their epistemic status. He further describes two other processes of argumentation, dissociation – the splitting of a concept into two differentiated parts in order to modify one's thinking, and compromise, where "new solutions are generated by complex combinations and elaborations of existing one's [solutions]" (Baker, 2003, p. 50). According to Baker, argumentation comprising negotiation, dissociation and knowledge elaboration (through compromise), is the fundamental means by which collaborative learning occurs. It is important to note that Baker argues that a teacher is a crucial part of the argumentation process. This is because the teacher is able to resolve certain questions that might prevent the argumentational process continuing. S/he is also able to moderate the processes of argumentation and negotiation, and assist in integrating what students have learned into their existing knowledge. Thus, teleologically, Baker's viewpoint of argumentation is oriented toward learning. Underlying this orientation is an epistemology that is rooted in the theory of social constructivism. A second example of a collaborative based argumentation theory was developed in 1995 by

Götz Krummheuer. This theory synthesised elements from argumentation theory, sociology, and education theory.

Krummheuer's study (see Krummheuer, 1995) analysed the argumentations of children in a primary school setting who were engaged in developing strategic thinking in number. Krummheuer argued that childrens' classroom argumentations could not be analysed using formal argumentation schemes of Pragma-dialectics or the New Rhetoric (Schwarz, 2009, p. 109). His objective was to create an argumentational scheme in such a way that it would theorise how children learn through argumentative discourse. At the basis of Krummheuer's scheme is a view that argumentation is primarily a social activity – a situation where co-operating individuals try to adjust their intentions and interpretations by orally presenting the rationale for their actions. He thus defines argumentation as, “interactions in the observed classroom that have to do with the intentional explication of the reasoning of a solution or after it” (Krummheuer, 1995, p. 231). Krummheuer saw argumentation as a source of ‘taken as shared’ understandings governed by sociomathematical norms. By sociomathematical norms we mean classroom norms that determine what counts as a mathematical discourse, what counts as being mathematically different, and what counts as an acceptable mathematical justification (See Yackel & Cobb, 1996 for a full discussion of sociomathematical norms). Thus Krummheuer's argumentation scheme is embedded in the practice of classroom communities, and is referred to as *collective argumentation*. To produce a theory of learning through argumentation, Krummheuer firstly employs Toulman's layout of an argument as a means to identify the roles that different aspects of interaction play in the construction of argumentational discourse. Toulmin's Layout is discussed more detail later in this section, however we give a very brief description here as it is used by Krummheuer. It should be noted that Krummheuer does not use all of Toulmin's layout – a point that is challenged in contemporary uses of Toulmin's layout in mathematics education (see Inglis, Mejia-Ramos, & Simpson, 2007).

We begin with a recapitulation of Toulmin's layout of an argument. Toulmin's layout begins with a claim that is made, or that is laid up for evaluation. Along with this claim the evidence or data that is given to ground it is provided. Connecting these two features (the claim and the grounding) is a step that authorises the use of the particular evidence or set of data. This link is referred to as the warrant. However, the warrant can also be justified if necessary. This justification comes by way of the backing – a theoretical construct that underlies the warrant. Krummheuer views Toulmin's components of, evidence – data, warrant and claim as the core of an argument. Upon Toulmin's

framework, Krummheuer places another theoretical layer that aims at explaining how intersubjective meaning emerges in an argumentation. Here he uses the sociological theory of ethnomethodology to propose that a classroom in which argumentation contributes to learning is based on framing difference – differences of opinions or viewpoints. The aim is to construct a working frame where these differences can be ‘resolved’ – at least enough to be able to come to a shared frame so that the argumentation can continue to move forward. He also argues that when one makes one’s argumentation known to others one gives at the same time, an accountability for his actions, and it is this ethnomethodological concept of accountability that, Krummheuer argues, gives collective argumentation its rationality. In essence, he gives a social constructivist account of the construction of knowledge.

The examples provided here indicate that various writers’ interpretations and definitions of argumentation are oriented toward specific goals that they may have in mind. Along with this, we have seen that it is also possible to uncover certain epistemological foundations or assumptions regarding the validity of the resulting argument. However, there are some underlying similarities among the different descriptions that contribute to a core meaning for argumentation. For example, the various viewpoints of argumentation presented here agree that reasoning and justification are key features of argumentation. Also, at the basis of the perspectives expressed so far is the belief that argumentation is a social activity, and one of its aims is to settle and resolve what Walton (1996) describes as a situation of unsettledness. These are situations of uncertainty, instability, lack of proof, or perhaps grievance. However, from the perspective of learning, argumentation is not only grounded in social activity, but also in the cognitive activity of an individual. This is underscored by constructivist views of learning, and is particularly inherent in Krummheuer’s elaboration of argumentation. It should be noted also, that each of the expressions of argumentation discussed can also be linked to a deliberate effort to shift the analysis of argumentation away from the using the formal notion of logic as a tool for assessing validity. So, how does argumentation fit in mathematics? At first glance, it might seem that argumentation fashioned on the basis of epistemologies that stem from a rejection of formal logic might not be compatible with mathematics – particularly mathematics at senior school and tertiary level. As we will see, argumentation has indeed found a niche in mathematics education. However, this has not occurred without debate (Duval, 1991; Balacheff, 1999; Douek, 2009; Durand-Guerrier, Boero, Douek, Epp & Tanguay, 2012), and this is discussed in the next section.

### 3.3 Argumentation in Mathematics Education

Over a period spanning at least two decades mathematics education has shown a sustained interest in the nature of argumentation, particularly with respect to proof and learning (Durand-Guerrier, Boero, Douek, Epp & Tanguay, 2012). Hence, proof, conviction and persuasion have continued to be major factors in studies of argumentation. In 1982 Nicholas Balacheff investigated mathematical proof in middle school. He proposed that one way to initiate students into proof is to give them opportunities to investigate mathematical problems through collective discourse. According to Duval (1991) this set the scene for research into students' argumentation in mathematics, which in turn led to investigations into the question of whether argumentation may be a route to proof. An important contribution to the development of a conception of argumentation in mathematics has been the debate over the relationship between argumentation and proof. This debate has prompted a theorisation of argumentation in mathematics, and in particular its contribution to proof. Although there is as yet no universally agreed-upon theory of argumentation in relation to mathematics, there is a general acceptance that it is multifaceted and that reasoning lies at its heart (Schwarz, Prusak, & Hershkowitz, 2010). We will now consider literature surrounding the debate around the role of argumentation in proof

#### 3.3.1 The Debate: Argumentation and Proof

As already noted above, a well subscribed view of argumentation is that it is rooted in everyday social situations and the usage of natural everyday language. With this understanding of argumentation one might conclude that the modes of reasoning used, as well as the methods used to evaluate an argument's validity, would differ from those practised in mathematics. This is the basis of Raymond Duval's (1991) argument that argumentation and proof in mathematics are different. Duval argues that proof in mathematics requires a specific mode of thinking – evidenced in deductive reasoning, and makes the following comment, "Deductive thinking does not work like argumentation. However these two kinds of reasoning use very similar linguistic forms and propositional connectives. This is one of the main reasons why most of the students do not understand the requirements of mathematical proofs" (Duval, 1991, p. 233). Here, Duval refers to the explicit way in which inference rules work in deduction in mathematics – that the conclusion of one step becomes the input for the next. He contrasts this with propositions in argumentation

which he argues are intrinsically connected by semantic content. According to Duval, unlike discursive activity in everyday discourse, epistemic values in mathematical proof are determined according to theoretical foundations that include axioms, definitions and logic. Thus he suggests that the gap between argumentation and proof is a function of the status (or value) of their statements. Pedemonte (2001, p. 35) summarises his thoughts this way, “[according to Duval] the distance between proof and argumentation is not only logic but is also cognitive: in a proof the epistemic value depends on the theoretical status whereas in argumentation it depends on the content”. Duval concludes that argumentation does not open the way to the construction of a valid mathematical proof; rather it inhibits it. Duval’s stance is supported by Nicholas Balacheff (1999). Balacheff focusses on analysing the hypothetical situation of mathematical argumentation. He begins by assuming the goal of mathematical argumentation is to produce a proof. He asserts that proof is a discourse that is devoid of contextual references and the human actor(s) – who is replaced by specific logic rules of an axiomatic mathematical system. However for Balacheff, argumentation is a social event that demands participatory action from actors. It also cannot be separated from context. In addition, Balacheff argues that argumentation is an open process that allows any means of persuasion including the use of natural (everyday) language, whereas the language of proof is precise and associated with deductive reasoning. From this basis he reasons that mathematical argumentation cannot exist. To Balacheff the purpose of a mathematical argument is to provide a mathematical proof or justification. Since argumentation necessarily involves the use of reasoning, lexicon and grammar that is non-mathematical according to Balacheff, a mathematical proof can never be obtained through argumentation. However, he concedes the existence of argumentation in mathematics, which he describes as “a context in which to develop argumentative practices using means which could be used elsewhere (metaphor, analogy, abduction, induction, etc)” (Balacheff, 1999, p. 4). Interestingly, Balacheff does not appear to connect argumentation in mathematics to mathematical proof, as noted by his conclusion, “argumentation constitutes an epistemological obstacle to the learning of mathematical proof, and more generally to proof in mathematics” (Balacheff, 1999, p. 5).

Both Duval and Balacheff appear to share a formal axiomatic view of mathematical proof that is limited to a structure. Thus in their view, formal criteria drive and control the proving process and hence its validation. However, this does not always appear to be the case in the proving and validation practices of professional mathematicians. For example, Gila Hanna (1995) concludes the following, ‘through an examination of mathematical

practice I came to the conclusion that in the eyes of practising mathematicians rigour is clearly of secondary importance to understanding and significance, and that proof is actually becomes legitimate and convincing to a mathematician only when it leads to real mathematical understanding” (Hanna, 1995, p. 42). Support for her conclusion comes from renowned mathematician William Thurston (1995) who argues that criteria based on content rather than formal rules of validity are more useful in proving and in the reading of proof. The term ‘useful’ is used in the context of understanding the ideas that lie behind the proof, that is in the argumentation on which it is based. In a study by Keith Weber and Lara Alcock (2004), which compared the proving strategies of doctoral mathematics students, undergraduate students, and algebraists, it was noted that mathematicians did not solely rely on formal criteria. They made use of intuition and instantiations when determining whether a statement may be true or not. Here we note that Weber and Alcock used the term ‘instantiation’ to refer to systematic and repeatable ways in which an individual thinks about and is able to represent a mathematical idea. They argued that instantiations were used to “make sense of a statement to be proven and to suggest formal inferences that can be drawn” (Weber & Alcock, 2004, p. 211) and concluded that instantiations, drawn from one’s conceptual understanding surrounding a problem at hand, can be a significant factor in one’s ability to produce a proof.

Another study by Keith Weber (2008) also showed that mathematicians were likely to use several modes of reasoning when validating a proof rather than just reasoning deductively with formal criteria. Some of the modes of reasoning he identified included “formal reasoning, informal deductive reasoning, and example based reasoning” (Weber, 2008, p. 431). As in his previous study with Lara Alcock (2004), Weber was also able to conclude that conceptual knowledge plays an important role in activities associated with mathematical validation. He makes it clear that the act of validating proofs involves reasoning that is situated in argumentation. These studies, as well as those by Hanna (1995), Thurston (1995), and Weber and Alcock (2004) link proof and argumentation through the process of understanding (and validation). These writers are consistent in their arguments that understanding (and validation) involve an activity of proving that includes the reconstruction of the ‘proving’ activity that has lead to a given proof. But what about Duval, and Balacheff’s claim that argumentation inhibits the production of proof – that argumentation and proof are two separate irreconcilable concepts?

Paolo Boero (1999) began the challenge to Balacheff by writing an article for *Proof: The Newsletter on the Teaching of Mathematical proof* (July/August 1999) entitled, *Argu-*

*mentation and proof: A complex, productive, unavoidable relationship in mathematics and mathematics education.* In this article Boero challenged Balacheff's assertion that argumentation might be an epistemological obstacle to the development of a mathematical proof. Boero (1999) argued for six phases in proof production relating to the work of a mathematician, beginning with production of a conjecture and ending with the production of a formal proof using mathematical logic. He suggested that in each phase, mathematicians use argumentation in a particular way. We will now outline his six phases and include some of Boero's (1999) explanation of them, along with his views on the action of argumentation in each phase.

Phase 1: Production of a conjecture (including: exploration of the problem situation; identification of "regularities"; identification of conditions under which such regularities take place; identification of arguments for the plausibility of the produced conjecture etc...

Phase 2: Formulation of the statement of conjecture according to shared textual conventions.

Boero (1999, p. 4) notes that, "During these first two phases, argumentation concerns inner [a mathematician's private] (and eventually public) analysis of the problem situation, questioning the validity and meaningfulness of the discovered regularity, refining hypotheses, discussing possible reformulations".

Phase 3: Exploration of the content of the conjecture (including: semantic elaborations; elaborating links in arguments; identifying appropriate arguments for validation; relating aspects of arguments to reference theory). Boero (1999) suggests that this phase usually belongs to the private side of a mathematician's work. During this phase, argumentation is focussed on organising content from Phases 1 and 2. This may be done by discussing the acceptability or relevance of certain arguments, or finding links between them.

Phase 4: Selection and enchaining of coherent, theoretical arguments into a deductive chain. Boero claims that this phase is where a mathematician would present his work to colleagues in an informal way. In this way, s/he is able to test the organisation and robustness of his argument and make necessary adjustments where required. According to Boero, argumenta-

tion in this phase is evidenced in the control of argument enchaining. It is also evidenced in the presentation choices made by the mathematician, and the responses he makes in defence of this arguments.

Phase 5: Organisation of the enchainned arguments into a proof that is acceptable according to current mathematical standards. According to Boero this is where a proof is worthy of being included in a publication. This is the most often reached form of proof. Argumentation may be expressed in this phase as an activity of comparison, where the proof produced is compared with current standards of rigour.

Phase 6: Reaching a formal logic proof in the Hilbert/Whitehead sense. Boero suggests that this stage may sometimes be reached if required.

Likening argumentation to a game, Boero (1999, p. 5) claims that mathematicians can play the game of using what he calls “rich and free argumentation” to investigate and explore (Phases 1 to 3). They can also switch to playing a new game that follows in which argumentation becomes increasingly constrained by the strict rules mathematical logic, reasoning and presentation (Phases 4 to 6). However, he suggests students in general cannot play these games. He reasons that this is not because argumentation is an epistemological obstacle, but because problem situations are not chosen well enough to invite students to engage with argumentation that would allow them to access the first 5 phases. Thus not only does Boero claim argumentation is not an epistemological obstacle to proof, but his phases indicate his rejection of Duval’s assertion that argumentation and mathematical proof must be separate. His views resonate with the research by Bettina Pedemonte (2007a, 2007b) who also argues that argumentation and proof are related.

Bettina Pedemonte, who has written extensively about argumentation, also rejects the claims made by Duval and Balacheff that argumentation and proof are irreconcilable concepts. While not denying the existence of a potential gap between argumentation and proof, Pedemonte sets out to explain its causes and how this gap might be avoided. This will be discussed in more detail in the next section. Before that however, her views of the relationship between argumentation and proof will be presented here. Pedemonte’s characterisation of argumentation is guided by her interest in contouring a relationship between argumentation and proof, in particular determining how conjecture can lead to proof. She claims that proof is an example of argumentation that is made valid using mathematical theory (Pedemonte, 2003). Her suggestion that proof can be conceived as a



particular example of argumentation is argued for from the basis that they share certain characteristics (Pedemonte, 2007a). For example, according to Pedemonte, argumentation and proof share teleological and epistemological commitments to rational justification and conviction. Proof and argumentation are both addressed to a ‘universal audience’, for which she gives the following examples, “the mathematical community, the classroom, the teacher, the interlocutor himself” (Pedemonte, 2007a, p. 27). Argumentation and proof also belong to a particular ‘field’. The use of the term ‘field’ is consistent with a broad interpretation of Toulmin’s (Toulmin, 1958) description, and refers to discourses in which claims are grounded. For example, according to Pedemonte, proof belongs to a theoretical field evidenced in the discourses of algebra, calculus, geometry, analysis, or topology. On the other hand, argumentation in mathematics belongs to a field that delimits validity criteria through discourses associated with representations, context and content. In addition Pedemonte argues that argumentation and proof can both be placed in Toulmin’s layout, and thus they share structural characteristics. In addition, Pedemonte (2007b) interprets argumentation as an aspect of problem solving. The rationale for this is that through the resolution of a problem a statement of conjecture is produced, which is followed by a proof phase and the result is a compression of argumentation into a text of proof.

A significant result of the responses made in the debate surrounding the claim that argumentation and proof should be separated, or that there is no mathematical argumentation, is that various researchers have developed different characterisations of argumentation in mathematics – much in the same way they were developed in argumentation theory. We have already discussed Krummheuer’s (1995) characterisation, and aspects of those drawn from the works of Boero (1999), and Pedemonte (2003, 2007a, 2007b). In the next section we will describe aspects of one of these in more detail, as well as other characterisations that have contributed to formulating a meaning of argumentation in mathematics that will be used in this thesis.

### 3.3.2 Characterisations of Argumentation in Mathematics

As we have seen, conceptions of argumentation are complex and involve the meshing together of various theoretical points of view. The characterisations of argumentation that follow are drawn from the field of mathematics education, in which argumentation is a relatively young research area compared to argumentation and communication theory. This

section will illustrate how argumentation has become part of the discourse of mathematics education, and will consider conviction, reasoning, the structure of argumentation, and rationality.

We will begin by discussing conviction, which is considered an important facet of mathematical activity (Mason, Burton, & Stacey, 1982). Conviction is seen to occur in several ways (Harel & Sowder, 1998; Inglis & Mejia-Ramos, 2008b), some of these will now be considered.

### **Argumentation and Conviction**

In Section 3.2.3 we noted that persuasion and conviction were central to the characterisation of argumentation given by the New Rhetoric of Perelman and Olbrechts-Tyteca (1969). They are also central to the characterisation of argumentation given in Harel and Sowder's (2007) framework of proof schemes (Fig. 3.1). Harel and Sowder considered proving in terms of either establishing a personal conviction (which they called *ascertaining*), or gaining the conviction of an other (which they referred to as *persuading*). Note that Harel and Sowder's association of the term 'persuading' with conviction of an other differs from that of Perelman and Olbrechts-Tyteca who use it as a means to securing a self-conviction. Harel and Sowder use their framework to explain the bases for students thinking and reasoning when producing arguments of proof (Harel, 2008).

Harel and Sowder (1998) found that students primarily construct three types of arguments when attempting to persuade an other. In the first type, their arguments rely on features that are external to their own ability to create mathematics. These features included the appearance and form of an argument, an authority figure such as the lecturer or a textbook, or meaningless symbolic manipulations. In the second type of argument, students rely on empirical arguments using different types of examples – perceptual (e.g., visuospatial images) and inductive (e.g., numerical substitutions, measurements). The third type of argument is characterised by the instantiation of deductive reasoning which ranges from reasoning with generic examples to reasoning with a given axiomatic system.

Another perspective considers conviction as a complex form of behaviour, whereby participants can request, interpret, and assess the persuasiveness of an argument. Using Toulmin's layout of an argument, Inglis and Mejia-Ramos (2008b) claim that when providing an argument or evaluating the persuasiveness of an argument, an individual will likely focus on one of the components of Toulmin's layout. Thus they may focus on the

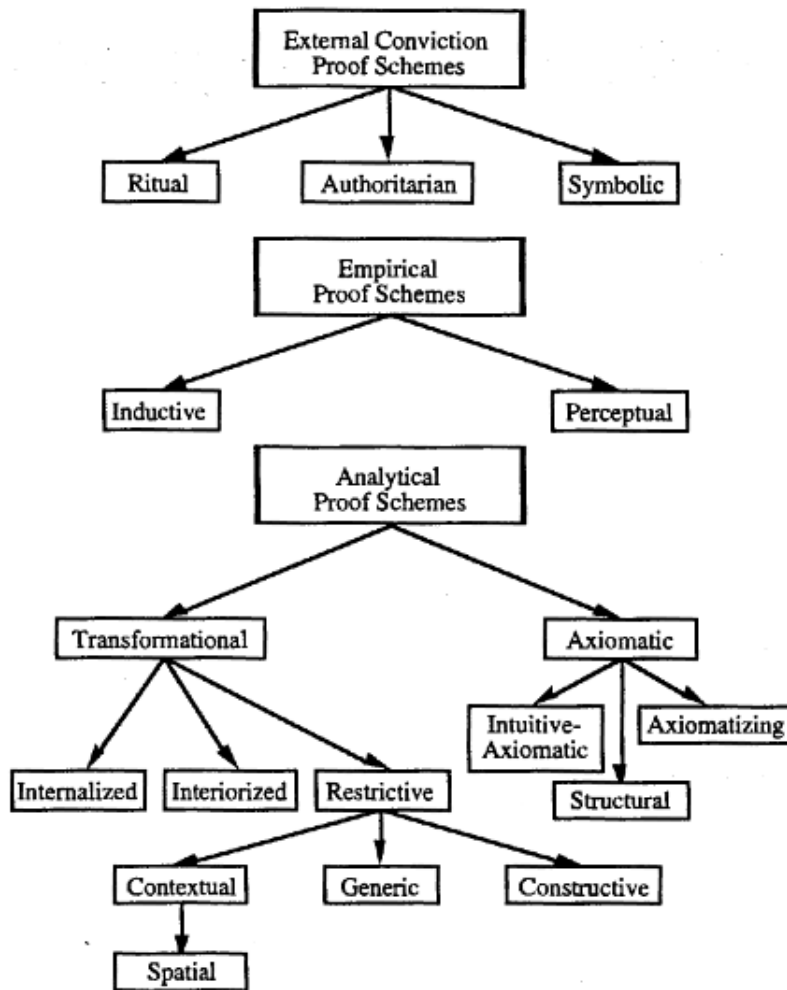


Figure 3.1: Harel and Sowder's proof schemes (Harel & Sowder, 1998, p. 245)

reliability and trustworthiness of the *data*, the likelihood of the conclusion, the appropriate use and strength of the *warrant* (and its associated *backing*), the appropriateness of a given *qualifier* (and its associated *rebuttal*) to the overall argument, or whether the argument is appropriate to the particular context in which it occurs. The context governs the type of data, warrants (and backings), qualifiers and rebuttals that may be used in an argument. An example of Inglis and Mejia-Ramos' evaluation of persuasiveness of an argument based on data is a scenario where one does not trust the data that is presented. Thus the doubt that arises is reflected on to his/her evaluation of the argument. On the other hand, someone who has trust in the warrant (and its backing), and sees how it links the data to the conclusion will evaluate the argument accordingly.

Conviction and its practical applications are explored by Mason, Burton, and Stacey (1982) who place it at the heart of mathematical activity. They distinguish three levels of conviction in increasing levels of sophistication, namely:

- Convince yourself;
- Convince a friend;
- Convince a sceptic.

Mason et al. (1982) also make reference to an ‘internal enemy’ – an internal sceptic. The role of the internal enemy is to develop a disposition that pushes aside complacency, and encourages probing and querying of assumptions; testing arguments with a view to defeating them, and modifying them until a convincing justification is found. The first level of conviction emphasises a private process; a psychological aspect of proof whereby personal doubts about the truth (or falsity) of mathematical claim are removed. However, the other two levels emphasise a social process associated with persuasion because they involve conviction based on the acceptance of shared criteria (see also Harel & Sowder, 1998, 2007). Traditionally, the difference between conviction and persuasion is seen in terms of communicative intention and communicative means, which are usually conjoined (O’Keefe, 2012). O’Keefe argues that conviction and persuasion are interpreted differently depending on how argumentation is defined. For example, if argumentation in mathematics is defined as a process (a means) of proving or disproving a proposition, then it is very likely that its communicative intentions will be to induce a new belief, establish the certainty or truth of a statement, or address an error in the mind of an other. Thus from O’Keefe’s point of view, argumentation, conviction, and persuasion are linked.

A similar viewpoint of argumentation appears in a study by Judith Segal (1999) that examined the relationship between first year university students’ mathematical self-conviction, and the bases on which they determined the persuasiveness certain arguments. In particular she explored the third level of conviction – convince a sceptic. In her study, Segal considered self-conviction as a private aspect of proof comprising criteria that is generally implicit and related to everyday reasoning experiences. On the other hand she viewed persuasion as a public aspect of proof arguing that criteria for determining whether an argument is judged acceptable or not are socially constructed. Hence Segal associates persuasion with validity. Defining exactly what these social criteria are is not easy. However what is true about them, according to research, is that they are not based on formal logic (Hanna, 1991; Sowder, 1997; Thurston, 1995). It should be noted here that Segal is not interested in formal logic proofs, but in ordinary proofs that can be found in a typical undergraduate textbook.

When asked to analyse a set of three empirical arguments for validity, Segal's study showed that first year university students' were able to distinguish between self-conviction and validity (persuasion). This got better as they gained more experience in mathematics. The differentiation in criteria used to secure a conviction and judge validity related to the distinction between conviction as an everyday personal attribute, and validity as dependent on the values of a potentially 'hostile' community of mathematicians. Hostile in this context, encouraged the students to look for any opportunity to criticise an argument. Thus the students examined the validity of each argument by checking to see whether potential opportunities for a counter-example were present. If they were, then they classified the argument as not valid. Because the students got better at using this strategy over time, Segal suggested that examining arguments in this way is a learned behaviour – part of students' mathematical experiences. This is in agreement with the *internal monitor* of Mason, Burton, and Stacey, (1982) who give practical steps to how this behaviour can be developed.

In the second part of her study, Segal gave students one of two deductive arguments to analyse – one valid and the other not. A totally different picture emerged, this time the students were unable to distinguish between criteria for self-conviction and validity. Rather, there was an increased tendency for students to call either argument valid whether or not they actually were. Segal argued that this was attributable to at least two things. The first was the probable inexperience, ability, time, and inclination of the students to translate the deductive argument into a more concrete register so as to make sense of it (which might include empirical examples, diagrams, or some algebraic manipulations). Rather, she suggested the students most likely took the easy option of equating self-conviction with validity (and thus persuasion). The second reason offered by Segal was that the students likely used spurious criteria associated with Harel and Sowder's (1998) ritualistic or symbolic proof schemes and were likely influenced by the 'look and feel' of an argument.

A question arising out of this brief discussion is; How do students' written arguments aimed at deriving a self-conviction impact on corresponding arguments aimed at persuading (or convincing) an other – if they do at all? This question is directly linked to the debate discussed in Section 3.3.1, and underlies the development of a characterisation of argumentation put forward by Bettina Pedemonte. Pedemonte (2007b) claimed that sometimes the reasoning students use in argumentation associated with conviction and persuasion does not make it possible for them to reformat it as a written proof. Pede-

monte's characterisation is a complex of interlaced processes aimed at convincing oneself or an audience about the truth of a matter. Framing her view of argumentation is a research goal of developing an understanding of how argumentation contributes to proving and proof. We now present more details of the characterisation of argumentation gleaned from the works of Bettina Pedemonte (2007a, 2007b, 2008).

### **Argumentation: Cognitive Continuity to a Theory of Argumentation**

Bettina Pedemonte has written extensively on the relationship between argumentation and proof (e.g., Pedemonte, 2007a, 2007b, 2008; Pedemonte & Reid, 2011). Her early work consisted of analysing the cognitive gap suggested by Duval (1999), which she refers to as a structural gap or distance. She uses the term 'structural' to refer to modes of reasoning (e.g., deductive, abductive), so the gap she refers to is one involving a discontinuity in reasoning. Pedemonte associates argumentation with problem solving, specifically solving problems that lead to the construction of a conjecture. Proof is seen as an inscription of this argumentation into a text that meets certain standards of deductive reasoning and communication. Pedemonte uses and extends the cognitive unity theorem of Boero, Garuti, Lemut and Marriotti (1996) as a tool to theorise the impact of students' conjecturing on subsequent proof construction. The original concept of cognitive unity put forward by Boero et al. (1996, pp. 119-120) states:

During the production of a conjecture, the student progressively works out his/her statement through an intensive argumentative activity functionally inter-mingled with the justification of the plausibility of his/her choices; during the subsequent statement proving stage, the student links up with this process in a coherent way, organizing some of the justifications ('arguments') produced during the construction of the statements according to a logical chain.

This conception only accounts for continuity in content and justifications between conjecture and proof. Pedemonte extends this beyond a consideration of content and justification to include reasoning strategies and knowledge. Thus she argues that there is continuity if elements of reasoning, knowledge and content used in conjecturing are also present in the proving phase – otherwise there is a structural distance between the two. Pedemonte used this idea to examine why some students were unable to construct a proof although they had a mathematical understanding of the concepts involved. In her work, Pedemonte

uses the term *cognitive continuity* to indicate her extended concept of cognitive unity (Pedemonte, 2007a, 2007b, 2008).

Pedemonte's examination lead to the identification of two systems through which continuity can be observed. The first is referred to as the *referential system*, and second as the *structural system* (Pedemonte, 2007b, 2008). The referential system is composed of two parts. The first part consists of representational, linguistic and symbolic knowledge, and the second consists of content and concept knowledge including knowledge of valid mathematical operations. Pedemonte suggests that a referential system supports the development of a statement of conjecture – a phase she refers to as 'constructive argumentation', while another referential system supports the development of a proof in the proof phase (Pedemonte, 2007b). She argues that there is continuity if some elements in the referential system supporting constructive argumentation are also present in the proving phase. Accompanying this is a structural system composed of modes of reasoning. These include the categories of abductive, inductive, and deductive reasoning. A structural gap occurs when there is a failure to convert a reasoning strategy in the construction phase into one required for the proving phase. According to Pedemonte (2007b), this transformative step does not always happen because the type of reasoning used in the construction phase (e.g., inductive, or abductive) works counter to the establishment of deductive process of reasoning required for proof. Pedemonte thus suggests another strand of argumentation, one that drives the transition to the proving phase. She refers to this form of argumentation as *structurant argumentation*. The goal of structurant argumentation is to bring about the justification of a conjecture after it has been 'stated' as a fact (e.g., after the student is personally convinced that the conjecture is true). Part of accomplishing this goal involves determining patterns and links that might lead to a justification or proof. Another important part, is marshalling aspects from the referential and structural systems to find an effective format for communicating the justification or proof to others. Thus Pedemonte (2007b) proposes that structurant argumentation frames the referential and structural systems supporting the proving phase.

Another important feature interlaced in Pedemonte's view of argumentation is Toulmin's layout – which she uses to analyse the structure of argumentation, and an adaptation of the Conception, Knowing, concept (CKc) model (Balacheff & Gaudin, 2002) to analyse the referential system. The CKc model is Balacheff and Gaudin's attempt to characterise a student's knowing of mathematics within a given situation (such as problem solving). From this model, Pedemonte (2007a) uses Balacheff and Gaudin's characterisation of a

conception, which can be written as  $C = (P, R, L, \Sigma)$  where:

$C$  is a conception;

$P$  represents a set of problems for which the conception is efficient;

$R$  represents a set of operators, i.e. physical and mental actions which an individual can perform to solve a problem;

$L$  is the representation system – where problems and the operators of  $R$  can be represented;

$\Sigma$  is the control structure that allows one to choose operators, decide their relevance, evaluate their efficiency and decide whether a problem is solved or not. This aspect describes the domain of validity of the conception.

The analysis of the referential system comes from observations of the content of the data, warrant, backing, modal qualifier and rebuttal, and their relationship to the claim.

The reason Pedemonte includes the CKc model for analysing students' conceptions is because in her view, argumentation is "constructed on the base of students' conceptions" (Pedemonte, 2008, p. 388). Thus using an adapted Toulmin–Ckc model, Pedemonte builds up a portrait of the conception a student is mobilising. However, rather than describe in detail how she does this and what information it yields (see Pedemonte, 2008, pp. 385–400), it is sufficient for our purpose, to state that the Toulmin layout of an argument and Balacheff and Gaudin's (2002) characterisation of a conception are integral aspects of Pedemonte's complex of argumentation. In this study, a view of continuity similar to that taken by Pedemonte will be used to compare the reasoning strategies of students when securing a self-conviction with those used when attempting to convince an other. This comparison will therefore refer to Pedemonte's concepts of referential and structural systems.

The use of various elements in one's argument of self-conviction to write a statement that would convince another is also examined by Hsieh, Horng, and Shy (2012). They assert that when students explore a mathematical statement their primary focus is to gain personal meaning. When they start to explain their ideas informally to others, Hsieh et al. claim that they reciprocally "associate their own explanations with their own actions; [which] requires the operation of both self-justification and social-justification" (Hsieh, Horng, & Shy, 2012, p. 298). These authors suggest that it is in this phase that



self-justification merges with social-justification through a filtering and adaptive process allowing an argumentation to pass the scrutiny of others, implying a reflexive relationship between a self-conviction, and the act of convincing an other. However, some have criticised the emphasis given to conviction in the construction of specialised arguments of proof. The root of this criticism comes from the question, ‘Why prove a result that is empirically or visually obvious?’

Michael de Villiers (1990), who is one such critic, addresses this question with an examination of the purpose of proof. He concludes with an assertion that proof cannot be reduced solely to the function of verification – which he elaborates as conviction and justification. He argues that there are at least six functions of proof:

- *verification*;
- *explanation*
- *systematisation*;
- *discovery*;
- *communication*; and
- *intellectual challenge*.

Rather than proof as a means of conviction, de Villiers (1990) claims that conviction is a prerequisite for proof. He also states that professional mathematicians use means other than an absolute rigorous proof to develop a conviction of a mathematical statement and describes the *verification* function of proof as the use of problem solving strategies to make sense of a mathematical statement. These strategies might include the construction of counter-examples or quasi-empirical testing. It is clear from his description that the purpose of the verification function is the development of one’s conviction and confidence in it, and the possible formation of a logical argument – not necessarily a rigorous proof. The *explanatory* function of proof addresses the reason(s) why the statement behaves the way(s) it does. In this aspect of proof, a mathematician (or student) focusses on searching for insight into, and an understanding of how, a certain mathematical statement is formed from other mathematical ideas. This function addresses the reason why a statement is true. De Villiers reports that a distinct aspect of proof is that it exposes logical relationships. The systematisation function involves placing a known result into a deductive system of axioms, definitions and theorems, and chaining together a deductive sequence resulting in

the result. His description provides a sense that this is an organisational function that involves verifying the truth of various statements, making and checking links to and between axioms, definitions, theorems, and relevant concepts, and then compressing what is formed into a coherent whole. The *discovery* function does not rely on verification. According to de Villiers, this function highlights the exploratory and analytic power of proof and includes problem posing, exploration, invention, and the production of new knowledge. The *communication* function of proof involves the transmission of mathematical knowledge to others. De Villiers argues that reporting one's proof to others is a social process involving the elements of subjectivity and the negotiation of meanings, concepts and criteria for an acceptable argument. Engaging with this function of proof subjects one's work to criticism (including identification of errors), refinement, and possible rejection. In a later paper (de Villiers, 1999), de Villiers adds *intellectual challenge* as another function of proof. Here, he argues that proof serves the function of self-realisation and intellectual gratification. Thus he refers to proof as a testing ground for intellectual stamina.

A key thought conveyed in his articulation of these functions is that they are interwoven. For example, in some cases a certain function(s) may dominate, whereas in others some functions may not feature at all. Based on his account of the various functions of proof, de Villiers argues that proof as a function of conviction should not be taught in isolation from the other functions of proof. In this thesis, conviction of an other is seen as potentially addressing functions other than verification. The key is the determination of the audience to which the students will direct their argument.

### **Argumentation: Modes of Reasoning**

Argumentation is discussed extensively in mathematics education with a view to its influence on proof, and its complex conceptions, as has been presented above (Pedemonte, 2003, 2007a, 2007b, 2008). However, there are other conceptions of argumentation with links to proof that are equally complex. For example, the classification of proof schemes by Harel and Sowder (1998), are linked to argumentation through various forms of reasoning strategies characterising students' different ways of thinking (see also Harel, 2008). Other conceptions, for example that of Boero (1999) discussed above, or that provided by Nadia Douek (2009, pp. 142–143) which will now be described, are less complex. Douek uses an adaptation of Lolli's four modes of reasoning (see Arzarello, 2007) to examine the impact of different types of reasoning on the production of a text of a proof. Douek finds that

although deductive reasoning is important to proof production, so too are other activities – almost all of which are related to semantic content. She was able to find these activities in Lolli’s modes of reasoning, which are:

Mode 1: Exploration, interpretation and production of reasons for a statement. This may include reasoning using metaphor or representations – the emphasis is not necessarily on using acceptable mathematical reasoning. This is an investigative phase;

Mode 2: The emergence of a teleological dimension leads to an organisation of reasoning into a cogent argumentation. This may involve searching for convincing coherent links (e.g., through the use of abductive reasoning). Deductive reasoning is not yet a priority;

Mode 3: Production of a deductive text according to specific cultural constraints concerning the nature of propositions and their enchaining;

Mode 4: Formal structuring of the text according to shared rules of communication.

In terms of Douek’s goal of linking argumentation with proof, these modes appear as a conception of argumentation cast as reasoning behaviour (as a process of proving) that is directed at acquiring a proof. The value of this model is that it firstly presents argumentation as consisting of modes of reasoning, and secondly as a collection of intentional goal-oriented reasoning activities. Third, the goal of argumentation as it is presented here is not necessarily the formation of a text of proof, but includes other texts in which reasoning (not necessarily deductive) is a key attribute. Thus a solution to a problem in which visual reasoning is used to justify certain components could be classified as an argumentation.

In many of the characterisations presented so far in this thesis, Toulmin’s layout of an argument appears as a common thread. In Section 3.2.4 we discussed how Krummheuer used Toulmin’s layout in conjunction with an ethnomethodological approach. In the next section, we will consider how Toulmin’s layout has been used as a model of argumentation.

### **A Structural Model of Argumentation: The Toulmin Layout of an Argument**

We have already considered in some detail the core of Toulmin’s layout (Section 3.3.2). We will take some time here to explain briefly the whole of his layout of an argument,

since a consideration of the core elements only, restricts the range of argumentation that can be analysed (Inglis, Mejia-Ramos, & Simpson, 2007). Recall that the conclusion ( $C$ ) is a statement or claim that is being scrutinised – the statement that an arguer is hoping to convince either himself or another person(s) of its truth. The data ( $D$ ) represents the evidence upon which the claim is founded. It may consist of facts, data and information that provide the reasons for the claim. The logical link from the data to the claim is provided by the warrant ( $W$ ). This might be likened to the reasoning processes used to arrive at the claim using the data provided. The warrant is seated in a ‘theoretical’ foundation called the backing and provides the epistemological basis for the argument. The degree of confidence one has in the claim is expressed in Toulmin’s layout as a modal qualifier ( $Q$ ). The modal qualifier limits the strength of an argument according to the conditions under which it is true. The rebuttal ( $R$ ), states the exception to the claim and indicates the circumstances under which the general authority of the warrant fails to hold. In some argumentations, either the modal qualifier, or a rebuttal may not be present. The structure of Toulmin’s layout can be seen in Figure 3.2.

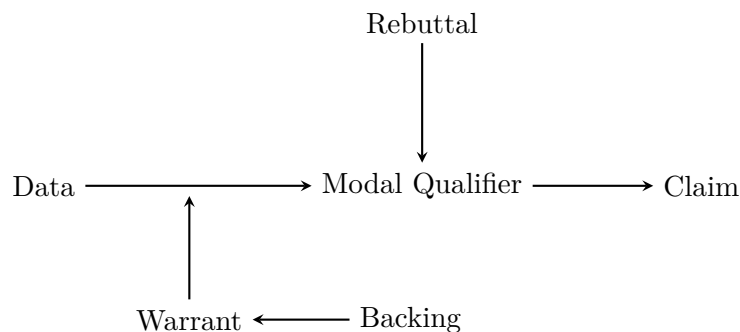


Figure 3.2: Toulmin’s Layout of an Argument

Toulmin’s framework for analysing an argument has been widely used. Two reasons for this have been proposed by Knipping (2008). The first is that it can be adapted to the analysis of formal deductive arguments and the second is that it allows for interpretation of students’ mathematical statements in terms of their function (e.g., whether a statement in an argument is used as a warrant, to give further data, or to provide a backing). As will be shown, Knipping’s comments are not coincidental.

Early studies that have used Toulmin’s layout have tended to focus on the core, consisting of data – claim – warrant – backing (Krummheuer, 1995; Yackel, 2002; Pedemonte, 2007a, 2007b, 2008). However on the basis of their research, Inglis, Mejia-Ramos and Simpson (2007) stress the importance of using his complete scheme. In their 2007 study Inglis,

Mejia-Ramos and Simpson (2007) gave highly successful postgraduate mathematics students a set of conjectures and asked them to decide whether they were true or false, and then provide proofs for them. What the researchers found was that these mathematicians often used non-deductive warrants and paired these with appropriate model qualifiers and rebuttals. This approach served to reduce their doubts about a conjecture's truth value rather than remove them. To form an accurate picture of what processes were occurring the researchers had to use the whole of Toulmin's layout, since the core only allowed the analysis of arguments that had an absolute qualifier. Similarly, if one were to analyse the discursive modes of a proof, Toulmin's layout would also be helpful. For example, in a proof, one would expect to observe a deductive warrant (or as Rodd (2000) and Tall (2004) would put it, a warrant for truth) with a theoretical backing paired to an absolute qualifier (there would be no rebuttal).

But not all agree with the use of the Toulmin layout of argument as a model of argumentation. For example, Schwarz (2009) argues that, although a potentially useful pedagogical tool, Toulmin's model is inadequate for describing the mathematical activity of proving because it is too structural to "grasp the dynamic, dialectical nature of mathematical activity" (Schwarz, 2009, p. 108), particularly in proving. At the basis of Schwarz's comment is an ontological problem, namely that Toulmin's layout was developed to model argumentation using informal reasoning – and yet it is used to model mathematical reasoning, which is rooted in a system of formal logic. Using the 2007 study by Inglis, Mejia-Ramos and Simpson as an example, Schwarz argues that at a practical level the protocols of Toulmin's layout appear to be imposed rather than adapted to students' reasoning processes. As a result, Schwarz (2009, p. 109) suggests that the use of the Toulmin scheme is only of value if, "it is used as a tool for educational purposes rather than as a model to describe mathematical activity".

Although Toulmin's aim was to develop a framework for modelling informal argumentation, his model inherited attributes that could potentially be used for the analysis of formal reasoning. Perhaps this is not surprising since his undergraduate degree was in mathematics and physics. Toulmin's objection was with the view that formal logic (in the tradition of the formalist – see Hanna, 1983, pp. 47–51) could be used to analyse non-mathematical arguments. Neither should this come as a surprise, as his ideas about rationality were heavily influenced by Austrian philosopher Ludwig Wittgenstein, who was a close associate of his at Cambridge University. Toulmin did not object to mathematical reasoning when used within an appropriate field, such as science or mathematics. Thus

he noted two distinctive types of arguments, substantial (informal reasoning) and analytic (which includes technical, scientific and mathematical reasoning). Although his layout is structural in form, we have seen examples of gains in analytical strength when it is used in conjunction with other schemes that focus on reasoning activity and knowledge development. An example is the characterisation of argumentation given by Boero, Douek, Pedemonte and Morselli (2010), which will now be discussed.

### **Argumentation: An Alliance Between Toulmin's Structure and Habermas' Rationality**

An outcome of a collaboration between Boero, Douek, Pedemonte and Morselli (2010) has been the formulation of a framework of argumentation based on an integration of the works of Toulmin (1958), and Jürgen Habermas (2003) and including contributions from their own research. In this framework, Toulmin's layout of an argument (often referred to as Toulmin's model of argumentation) is combined with Habermas' construct of rationality to analyse the rational behaviour of a group of middle school students involved in proving and proof. As already noted, a strength of Toulmin's model is that it gives a structural view of an argument with components that are considered to be invariant (Toulmin, 1958). Another strength noted by Knipping (2008) is that it can be used to analyse both informal arguments, and formal arguments. Boero et al. (2010) used the Toulmin model in their framework for the purpose of analysing the structural characteristics students' argumentations. Since Toulmin's model has already been well covered, we will now consider the rest of the framework of argumentation put forward by Boero et al.

The second part of the framework provided by Boero et al. (2010) consists of Habermas' view of rationality. In his view, rationality comprises three components. A key feature of each of his components is the conscious efforts and choices students make when communicating with others.

The first component, *epistemic rationality*, surrounds the concept of knowledge with questions such as: On what basis does one believe the knowledge presented to be true? and, What reasons are given for its truth – how is it justified? Epistemic rationality reveals the validity of statements according to the propositional structure of knowledge, and ways of reasoning that are relevant to the context of justification (Habermas, 1998). Thus reasoning is framed against the socially accepted norms and knowledge that construct the discourse of 'the mathematics community' in which the argumentation is being developed.

Hence, validity comes from the correct application of inference rules (in particular deduction when a proof is the objective), premises, axioms, definitions and theorems. Also included is the correct use of syntactical rules, and logical and algebraic procedures. Boero et al. (2010) argue that expressions of validity reflect conscious behaviour that is directed toward controlling and managing argumentation. The epistemic component is seen in the justifications given in students' proving and proof and in their correctness of mathematical formalisations and the interpretation of resultant expressions (see Boero & Morselli, 2009).

The second component of Habermas' rationality is *teleological rationality*. The primary focus of teleological rationality is the setting and achievement of goals. Habermas (1998, p. 313) notes, "Action has a teleological structure, for every action-intention aims at the realization of a set goal". This involves intentionality of reasoning, along with the implementation of the strategies, and the organising of one's thoughts. These resources are marshalled in a manner that contributes to the achievement of the goals set. Habermas (1998) also explicitly links teleological rationality to knowledge, since the achievement of a goal requires that the right inputs of reliable information are selected. According to Boero et al. (2010) teleological rationality is reflected in the conscious choices students' make with regard to the presentation of their formalisations, transformations and interpretations that are useful to the aims of the argumentational activity.

The third aspect of rationality is *communicative rationality*. The focus of communicative rationality is on reaching an understanding about something with someone (Habermas, 1998). According to Habermas communicative rationality "...secures for participating speakers an intersubjectively shared life-world, thereby securing at the same time the horizon within which everyone can refer to one and the same objective world" (Habermas, 1998, p. 315). In their framework, Boero et al. tailor Habermas' concept of communicative rationality to fit with argumentation as it sits in their life-world of mathematics education. Thus they state that communicative rationality consists of, "the conscious adhering to rules that ensure both the possibility of communicating steps of reasoning and the conformity of the products (proofs) to standards in a given mathematical culture" (Boero et al., 2010). Evidence of this would be the conscious effort students make to demonstrate the socio-mathematical norms associated with argumentation, proving and proof, and their awareness of how elements of proof affect the ability to communicate with others.

The strength of combining Toulmin's layout and Habermas' rationality can be observed in the following statement:

Indeed, if considered separately, Toulmin's model is suitable to analyze the organization of arguments, not the subject's intentions nor the tensions between the different "components" of the proving process, conversely, Habermas' construct is suitable to take the subject's intentions and consciousness into account, but it does not offer the possibility of modelling and comparing different kinds of productions (in particular, the composition of arguments in the exploratory phases and in the final proof) and eliciting possible continuities and discontinuities (and related obstacles) between them. (Boero et al., 2010, p. 180).

Amongst all the different characterisations of argumentation discussed so far, an explicit cognitive perspective seems to be missing. The next section will consider the characterisation of argumentation by David Tall, who develops and extends Piaget's theory of psychological development to provide a cognitive perspective on the development of mathematical reasoning. We will briefly discuss some salient features of his observations, that will be utilised in this research.

### **Argumentation: A Cognitive View**

In a keynote address presented to the National Changhua University in Taiwan on May 2007, David Tall explained how his emerging cognitive framework of the three worlds of mathematical thinking could explain argumentation and proof for different individuals. Unlike other characterisations, Tall's description of argumentation is not centred on social situations (although its contribution is recognised), but rather on psychological changes in thinking and reasoning as an individual matures from a child into an adult. He argues that proof is led by argumentation, which develops as individuals build increasingly sophisticated experiences. A key to developing such increasingly sophisticated experiences is the development of language and its use in describing, defining and generalising properties.

Central to his characterisation of argumentation is Tall's construct of three overlapping worlds of mathematical thinking. He theorises three differentiated worlds in which individuals operate in mathematics, and come to an understanding of mathematical ideas. His theoretical worlds are partly based on Bruner's iconic, enactive and symbolic modes of representation. They also synthesise the cognitive theories of van Hiele (1986) (levels of geometric reasoning), Dubinsky (Cottrill et al., 1996) (APOS), Fauconnier and Turner



(2002) (compression and blending), and Pegg (Pegg & Tall, 2005) (SOLO Taxonomy). Tall gives the following description of his three worlds of mathematics (Tall, 2008, p. 7);

- The conceptual-embodied world, based on one's perception of and reflection on properties of objects, initially seen and sensed in the real world but then imagined in the mind;
- The proceptual-symbolic world that grows out of the embodied world through action (such as counting) and is symbolised as thinkable concepts (such as number) that function both as processes to do and concepts to think about (procepts);
- The axiomatic-formal world (based on formal definitions and proof), which reverses the sequence of construction of meaning from definitions based on known objects to formal concepts based on set theoretic definitions.

However, he also renames each world so that their interrelationship can be discussed in a more clear and concise manner. He refers to them simply as 'embodied', 'symbolic', and 'formal', so that he can then combine terms, for example, 'embodied formal', 'embodied symbolic', and 'symbolic formal' in order to describe certain modes of thinking (Fig. 3.3). The differentiation of Tall's three worlds of mathematics is driven by language development and cognitive growth (Tall, 2008).

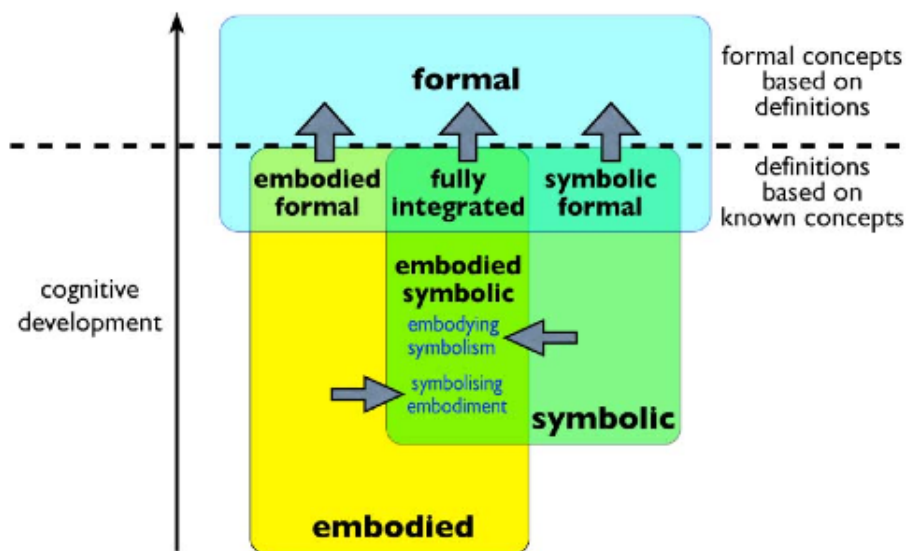


Figure 3.3: Tall's three worlds of mathematics (Tall, 2008, p.9)

Linguistic and representational resources used in the construction of an individual's argumentation can be related to a particular world of mathematics or an intersection of them (Tall, 2007, 2008; Tall et al., 2012), hence the differentiation of the three worlds also represents a differentiation of types of argumentation (Mejia-Ramos, 2006; Tall et al., 2012). Tall asserts that two processes, *compression* and *blending*, which he adapts from Fauconnier and Turner (2002), are crucial to individuals' cognitive development and progression through his worlds (Tall, 2007, 2008).

To process information more efficiently the brain connects ideas together compressing them into a single structure that Tall calls a *thinkable concept* (Tall 2008, p. 10). An example of the thinkable concept is the procept – a combination of *process* and *concept* represented by the same symbolism. For example, the symbolism in the expression  $\lim_{x \rightarrow -2} \frac{x^2 - 4}{x + 2}$  can be thought of as the process of approaching a limit and as the concept of the value of the limit. Compression is a key idea because it can be used to describe how an individual can shift from the embodied world, for example, to the symbolic world. Tall uses blending to refer to the bringing together of two (or more) met-befores – each consisting of an individual's thinkable concepts, embodiments and symbolisms. A met-before is described as “a current mental facility [of something] based on specific prior experiences of the individual” (Tall, 2008, p. 6). Thus we can talk about ‘multiplication makes bigger’, ‘division makes smaller’, and  $3 + 4 = 7$  as met-befores. The blending of met-befores may produce a misconception, for example, the cognitive blending ‘the fraction  $\frac{3}{4}$  is 3 out of 4’ with ‘addition of fractions’ may lead to the blend ‘when adding fractions, add the ‘tops’ and ‘bottoms’’. However, Tall suggests that this knowledge is tested when it is blended into a wider structure with properties that conflict with previous experience leading to the formation of new generalisations (or new potential met-befores) (Tall, 2007, see also Tall, 2013, pp. 81–83).

In Tall's writing about the three worlds of mathematics, the aim of argumentation is to lead to proof. Thus his view of proof is that it is a particular expression of argumentation. Proofs grow in increasing sophistication until they reach the formal axiomatic level where formal axioms are used to deduce relationships that may lead to proofs of theorems. An example drawn from second year mathematics is the introduction to students of the axioms defining a vector space (and vector subspace) from which theorems related to spanning sets, nullspace, row space, column space, linear independence (not an exhaustive list) can be developed.

The value of Tall's theory of Three Worlds of Mathematics to this research, is that it gives

a categorisation for the cognitive and mathematical resources that students select and use in their written argumentation. However as will be shown in the following chapter, it also makes an essential contribution to the analytical framework that will be used in this research.

### 3.4 Summary

So far in our review of literature we have discussed various ways in which argumentation has been viewed. As we have noted, each view is shaped by particular research goals. Our intention in presenting these various characterisations was twofold. As already noted, the first was to highlight how research goals influence various characterisations, and the second was to illustrate some important general features of argumentation. We summarise what we see as important features of argumentation for the purpose of establishing a research framework of our own – thus this is not an exhaustive list:

- Argumentation is a discursive activity that values reason, evidence, justification, and proof. Thus it should possess a communicative and epistemological dimension;
- Argumentation is goal oriented. The ‘audience’ to whom an argument is directed is important in its construction. Subgoals may be constructed in order to lead the audience through a reasoned journey to the conclusion. Thus argumentation should also possess a teleological dimension;
- Argumentation is dependent on language (written and oral), and individuals’ facility with representational features of language (everyday and mathematical). Argumentation in mathematics gets more sophisticated as an individual matures. Thus in a study of argumentation one should also be aware of the conceptual development of its participants.

All these will inform the construction of an analytical framework that will be used in this research. As shown in this literature review, searching for a commonly agreed meaning for argumentation is just as elusive as trying to secure a common meaning for proof (Durand-Guerrier et al. 2012). In this research, the view taken of argumentation is informed by the preceding summary and the assumptions given in introductory section of Chapter 4. Argumentation emerges from problem solving and involves the use of enactive, iconic, and symbolic forms of communication (sometimes all three simultaneously) to make a

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reasoned argument that can be ‘read’ by a reasonable person. It possesses an implicit structure connecting a conclusion (or claim) to justifiable evidence. It also reflects rational behaviour which is evidenced in a student’s knowledge and understanding of mathematics, and the manner in which s/he communicates these through written text. In this research, the view taken of proof is based on the those of Pedemonte (2007a, 2007b), Douek (1999a, 2009), and Tall et al. (2012) who assert that it is a specialised form of argumentation involving certain characteristics associated with symbolic inscription and syntax, logic, reasoning, and validity – which is dependent on satisfying standards of a representative community of mathematicians . These characteristics are augmented by Tall’s (2012) view that proof develops through each of ‘his’ Three Worlds of Mathematics, a view also endorsed by Mejia-Ramos (2006, pp. 173–180) who provides practical examples of these. It should be remembered that the primary aim of this research is to compare students’ mathematical argumentation with a view to determining the changes they make when writing to convince a representative of the community of mathematicians who is also a feature of their life-world. In Chapter 9 a model of argumentation that emerged during this research is given. In this model, the effects of a student’s life-world and the role of the teacher are included.

## Chapter 4

# A Framework for Analysing Argumentation

### 4.1 Introduction

In this chapter a framework will be developed that will be used to analyse the written argumentations of a first and second year group of tertiary mathematics students. It will also be used to provide a diachronic analysis of argumentations of a small number of students who were tracked from a first year course in mathematics to their third year of studying mathematics.

This discussion begins with a presupposition that, as students progress in their study of mathematics at tertiary level they mature in their mathematical scholarship. With respect to argumentation, this includes thinking mathematically and communicating their ideas with others in a discursive manner. The term *maturing scholarship* based on the work of Newmann, Marks, and Gamoran (1996, pp. 282–284) has been applied to this progress. According to these writers, authentic academic achievement represents accomplishments that are significant, worthwhile, and meaningful. It involves the possession of a disposition to engage in what they refer to as *disciplined inquiry*, a process whereby students participate in cognitive activity to gain knowledge, seek understanding, and communicate their ideas in increasingly sophisticated ways. Hence authentic academic achievement requires complex forms of communication which according to Newman, Marks, and Gamoran (1996) should be evidenced in the use of verbal, symbolic, and visual narratives that include details, explanations, and justifications. The term maturing scholarship indicates students' development toward authentic academic achievement. On the basis of our

presupposition, some preliminary assumptions will be stated.

The first assumption that will be made is that proving and proof are aspects of argumentation. The basis for this assumption is signalled in the works of David Tall, who used the Three Worlds of Mathematics framework to describe the maturation of argumentation in terms of cognitive growth (Tall, 2008), and then more recently, reinforced the role of argumentation in the development of proof (Tall et al., 2012). Further backing comes from the works of Pedemonte (2007a, 2007b) and Douek (1999b, 2009) who both argue that proof is a particular form of argumentation, and Schwarz et al. (2010) who summarise their analysis of the argumentational activities of mathematicians (or near to mathematicians) as enquiring, proving, and inscribing proofs. Traditionally however, the analysis of students' argumentations in mathematics has always had an association with spoken communication. Backing this is a view of argumentation as a verbal and social activity (Schwarz et al., 2010) situated in a discursive practice of communication (Habermas, 1998). The research described in this thesis differs from this position as it does not restrict argumentation to a verbal activity, but includes written forms of communication. Thus, this leads to our second assumption that will be made in this thesis which is that students' written mathematical statements reflect both their mathematical thinking and the communicative tools that they have on hand at a particular time.

Part of a student's maturing scholarship in mathematics is that they become adept at using symbolism in ways that reflect working in a mathematically logical manner, and as we have already seen, support for this comes from Newman, Marks, and Gamoran (1996). Drake and Amspbaugh (1994), and Moskal and Magone, (2000) highlight how students' written explanations and justifications are able to provide a robust account of their mathematical reasoning. As an example, Selden and Selden (1995) analysed the written work of 61 university undergraduate students who took part in a 'bridge' course designed to introduce them to proofs and mathematical reasoning. From the students' writings they were able to detail how students went about unpacking the logical structures in informal and formal statements of mathematics. Their research suggested that if students cannot unpack the logic of informal and formal mathematical statements, then they could not be expected to reliably validate the proofs of others, or construct proofs of their own. In later research, Selden and Selden (2009) acknowledged that a major factor in successful proof construction is the ability of students to bring to mind appropriate mathematical knowledge. This included algebraic and technical symbolic manipulations associated with quantification or logical implication (see also Fukawa-Connelly, 2012), as well as possessing behavioural

knowledge – that is “to know, and to act on, how parts of a statement relate to parts of its proof” (Selden & Selden, 2009, p. 348). These elements are expected to be evidenced in the texts of argumentation produced by the students in this research.

## 4.2 Constructing an Analytical Framework

This study presupposes three types of behaviours that should be ‘observable’ in students’ written argumentations. These are reasoning, structuring, and communicating. Thus, in this section, a framework is developed that will focus on accomplishing three goals. The first of these, is to identify reasoning strategies and mathematical tools and concepts, that students choose to use in their argumentations. The second, is to identify structural characteristics underlying students’ argumentations. Guiding this goal will be the following questions, “what is the claim underlying an argument, how is an argument grounded, and what gives an argument its force of conviction (or for some, persuasion)?” The third goal, is to articulate how rationality is expressed in students’ written communication. To accomplish these goals, the construction of the framework will draw on three different theoretical fields: the psychology of mathematics education (David Tall’s theory of three worlds of mathematics); argumentation theory (Stephen Toulmin’s theory of argumentation); and philosophy (Jürgen Habermas’ theory of communicative action). The use of Tall’s three worlds of mathematics, which we will from now on abbreviate to TWM is a critical feature of this framework because it draws the theories of Toulmin (1958) and Habermas (1998) into the field of mathematics. The TWM, also has two other strengths related to this thesis. First, is its description of strands of conceptual development (e.g., embodied, symbolic, and formal) and the second, is its inclusion of an embedded framework for the maturation of proof structures (see Tall, 2012, pp. 19–24).

A common feature linking Tall’s three worlds, Toulmin’s layout, and Habermas’ communicative action, is language. In Tall’s case, it is central to cognitive growth, and a key factor mediating the progression of an individual through each of his three theoretical worlds. It is also important in the transition from one world to the next. In Toulmin’s theory, the analysis of language used in an argumentation leads to the identification of certain invariant organisational features in the construction an argument and in Habermas’ theory of communicative action, discursive language is linked to the rationality of an argument. Tall’s theory of three worlds of mathematics brings together the theories of Toulmin and Habermas, by synthesising the cognitive tools and resources a student

uses, with their conscious intention to produce a convincing argument, and communicate it successfully to someone else.

The framework that will be used in this research is a synthesis of Tall's TWM, Toulmin's layout of an argument, and Habermas' forms of communicative rationality. This will be called the from now on, the Tertiary Argumentation Framework and will be referred to as the TA Framework. Table 6.10 gives a brief summary description of the TA Framework that will be used to analyse students' argumentations.

Table 4.1: Summary of the Components of the TA Framework

Three Worlds of Mathematics	Layout of Argumentation	Rationality
A World(s) (from TWM) is/are given along with characteristics of reasoning specific to it, as well as the nature of representations used to support it. A summary of the given categorisation will be provided	The structure of a student's argumentation is summarised using Toulmin's elements of data, warrant, backing, qualifier, rebuttal, and conclusion	The rationality underlying an argumentation is summarised in terms of the epistemic, communicative, and teleological components of Habermas' rationality

An analysis of an argumentation using the TA Framework will commence with its interpretation based on the TWM. This will identify the cognitive tools (e.g., symbols, other representations, modes of reasoning) that a student uses in his argumentation, and hence the world(s) from which his mathematical resources are being drawn. It will also provide a mathematical context for the rest of the analysis. Toulmin's layout will be used to analyse how the thinking tools and mathematical resources are used to structure an argument. Finally, Habermas' three dimensions of rationality will be used to examine how the elements from the TWM and structure from Toulmin's analysis combine to give a rational argument. Since the TA framework will be used to compare argumentations of self-conviction with those that are produced to convince an other, there are some matters contributing to the analysis that need further examination. One of these is the confidence a student has in their conviction.

An aspect of Toulmin's layout is that it allows us to identify the degree of confidence students have in their argumentation. In his layout, this is expressed as the modal qualifier. Inglis, Mejia-Ramos, and Simpson (2007) state that the modal qualifier is an important aspect of the Toulmin's layout that is often neglected in some analyses of students' argumentations. They argue that leaving it out constrains one's analysis of students' argumentations to only those that have an absolute conclusion. A modal qualifier might be paired with a rebuttal which describes the conditions under which a claim holds. This is



sometimes given as an ‘exception to the rule’. In particular we will consider how, if at all, students change their data, warrant, backing and modal qualifier when they switch from an activity of self-conviction, to that of convincing an other. Toulmin (1958) used the warrant to describe how the conclusion of an argument is linked to the data provided. The warrant may be a generalisation or rule, a definition, an analogy, or perhaps an appeal to authority. The force of the warrant – the modal qualifier and its rebuttal (if used) will also be examined because the qualifier denotes the degree of confidence that one has in the conclusion or equivalently, the level of conviction they have in their argument.

Toulmin’s use of the term warrant is situated in the context of his view of argumentation. Others however, have a tighter view of warrant situating it in the context of mathematical truth. For example, Melissa Rodd (2000) aligns knowledge with truth and defines warrant as, “that which secures knowledge” (p. 222), and as an “epistemic justification” (p. 226). Following on from her definition, Tall (2004) also uses warrant in terms of knowledge acquisition referring specifically to warrants for truth. As noted by Mejia-Ramos (2006, p. 174), Tall uses the notion of *warrant for truth* to denote a warrant that “secures someone’s personal belief and understanding in what is claimed to be known”. A warrant for truth implies that the conclusion necessarily follows the data – there is no rebuttal, because there is absolute confidence in a deductively reasoned conclusion. Tall’s TWM, privileges deductive reasoning which suggests that an analysis of an argumentation should focus on identifying the nature of the warrant. However, Mejia-Ramos (2006) provides a word of caution. He argues that analysing argumentation with respect to the warrant alone may obscure some subtleties in the categorisation of argumentation according to the TWM. He notes that when analysing an argumentation, attention should also be given to the backing which gives the warrant a principled or theoretical foundation. The backing may not be explicitly provided in students’ written argumentation, particularly with respect to self-conviction, but it is hypothesised that when writing an argument to convince others, elements of the backing will be observable and will become more so as students develop in their mathematics scholarship.

Converting a philosophical viewpoint into a practical application can be difficult. In this respect, we have been guided by Boero et al. (2010) in their treatment of epistemic, communicative and teleological rationality. They showed that a student’s argumentation can be analysed from an epistemological perspective by considering whether it is correct from a mathematical point of view. This involves an examination of how the student has reasoned to the conclusion and what mathematical theory or facts they have called

on in their argumentation. Meanwhile, communicative rationality can be examined by considering the way in which symbols and words combine together to create the text of the argument, and whether the text suggests that the student is communicating their understanding of the mathematics involved effectively. From a teleological perspective, an analysis of a student's argumentation will focus on intention and goal setting and the efficiency of the strategies and choices made in order to achieve them.

### 4.3 An Initial Trial of the TA Framework

In the next sections, the TA Framework will be used to see whether it is able to give a rich description of students' argumentations. An assortment of argumentations of self-conviction from a problem task given to a class of first year mathematics students will be examined. After completing the task, the students' argumentations of self-conviction were sorted, categorised and coded; a full coding is given in Chapter 6 (note also that argumentations to convince an other were also sorted, categorised and coded). The sample that is analysed here was categorised as *Arithmetic* because empirical and inductive reasoning were the primary strategies used to derive a conviction. These were subsequently sorted into subcategories and coded *Arithmetic Procedural* [AP] if the argumentation consisted of arithmetic examples only; *Arithmetic Structural* [AS] if the argumentation demonstrated a knowledge of the arithmetic structure underlying the solution; and *Arithmetic Generalising* [AG] if the solution indicated an algebraic generalisation. An extra example is provided that was categorised *Generalising with Algebra* and coded [GAlg]. In this case, algebra (in the absence of any empirical evidence) was used to draw a generalisation. The argumentations provided are typical of those in the categories given. The task given to the students was the following:

Task 1

The set of integers is the set  $\mathbb{Z}$  where  $\mathbb{Z} = \{\dots - 3, -2, -1, 0, 1, 2, 3, \dots\}$ .

Now, consider the following statement;

“The sum of any 3 consecutive integers is always a multiple of 3”.

- Investigate this statement, and convince yourself whether it is always true, sometimes true, or never true.
- Write an argument that you would use to convince another person such as a mathematics lecturer of your conclusion.

In the discussions that follow, the format that will be followed is that each category will be introduced with a general explanation. This will be followed by the analysis which will begin with a tabular summary of the three primary components of the framework and proceed to a more detailed written discussion.

### 4.3.1 Arithmetic–Procedural Mode of Argumentation

In this category of the framework, the use of numerical examples to support inductive reasoning is a central feature in students' strategies for gaining a conviction. Thus arithmetic procedures such as addition subtraction and division are used by the students. Tall identifies the use of symbols and arithmetic as characteristics of thinking in the Proceptual–Symbolic World (see Tall, 2008, p. 7; Tall et al., 2012, p. 29). Here, conclusions drawn often result from a generalisation from numerical examples. Thus from an epistemological point of view, truth comes from numerical data. In a structural analysis of argumentation, steps in a student's reasoning are linked to Toulmin's Layout, for example the warrant, backing, modal qualifier etc...The nature of reasoning a students might use may be evidenced in the words and phrases they use. For example, *let, suppose, if, then, therefore*. However, there may also be instances where a student might incorrectly generalise, because of an incorrect arithmetic concept (see Tall, 2013, in press). An example might be a statement they make like division leads to smaller numbers, whereas multiplication leads to large numbers.

Table 4.2 summarises some general characteristics of the examples provided of Arithmetic–Procedural modes of argumentation that are associated with the TA Framework.

Table 4.2: Arithmetic–Procedural

Three Worlds of Mathematics	Layout of Argumentation	Rationality
<p><b>Symbolic World:</b> Arithmetic reasoning using numerical symbols</p> <p>Specific arithmetic procedures are used (addition and multiplication)</p>	<p><b>Data:</b> Numeric</p> <p><b>Warrant:</b> Arithmetic calculations</p> <p><b>Backing:</b> Rules of arithmetic</p> <p><b>Qualifier:</b> (if provided): Strength is based on effectiveness of calculations</p> <p><b>Rebuttal:</b> (if provided): Exception to an arithmetic rule (based on a arithmetic misconception) or lack of sufficient evidence</p> <p><b>Conclusion:</b> Arithmetic generalisation</p>	<p><b>Epistemological:</b> Rules of arithmetic operations</p> <p>Specific arithmetic procedures are used</p> <p><b>Communicative:</b> Text uses a combination of ordinary language and standard arithmetic language, including symbols</p> <p><b>Teleological:</b> Example(s) lead to generalisations and a conclusion</p> <p>Text is illustrative of empirical and inductive reasoning</p>

Argumentation in this mode is characterised by warrants, backings, and rebuttals (when provided) that are numerical. Following the work of Boero et al. (2010), an argument in this category is said to be mathematically valid if it is warranted or backed by an arithmetic rule, and is said to be efficient if a sufficient number of numerical examples illustrating a consideration of the set of integers has been provided. If only one or two examples are given, a student's argumentation will still be classified as Arithmetic–Procedural because of the nature of its arithmetic approach. The following is an example of a student (AP1) who is in the Arithmetic–Procedural Category. An explanation of the student's argumentation in terms of the framework is provided below.

The argumentation securing a personal conviction in Figure 4.1 was student AP1's response to Task 1. AP1 uses arithmetic manipulation from the Symbolic World of the TWM. His conclusion is arrived at inductively – based on numerical examples. The warrant used is arithmetic and involves linking a sum to a product. Backing the warrant are rules of addition and multiplication from arithmetic. From the AP1's point of view, the warrant used acts as a warrant for truth because it is based on arithmetic rules, hence the absence of a modal qualifier and the concluding phrase "...the statement is true". Epistemic rationality is based on a knowledge of addition, multiplication, and equality. These are combined to give a set of examples that are used to reason a conclusion – reflecting communicative rationality of the student's argumentation. Teleological rationality is indicated by the achievement of the goal of showing that for each example, numerically, the sum could always be expressed as a multiple of three, and the generalisation expressed in the conclusion.

The next example in Figure 4.2 shows a similar argumentational strategy used by a student (AP2), but this time the AP2 uses a qualifier paired with a rebuttal. It should be noted however that their use is based on an arithmetic misconception that zero cannot be divided by a nonzero number.

Like the previous student (AP1) AP2 is also using arithmetic reasoning and manipulation consistent with working in the Symbolic World of the TWM. His conclusion is arrived at inductively through the use of numerical examples. In this example, the warrant is arithmetic (adding sets of consecutive numbers), and the backing is an arithmetic rule (multiples of 3 are divisible by 3). The modal qualifier is "sometimes", and the rebuttal used by the student is  $-1 + 0 + 1$  (which according to this student is not a multiple of 3). Epistemic rationality is derived from a rule of arithmetic but is compromised by a flaw in his mathematical knowledge. Communicative rationality is evident in the layout

$$\begin{aligned}
 & \cdot \underbrace{4 + 5 + 6}_{\text{positive integers}} = 15 = \underbrace{5 \cdot 3}_{\text{multiple of '3'}} \\
 & \cdot \underbrace{(-3) + (-2) + (-1)}_{\text{negative integers}} = (-6) = \underbrace{(-2) \cdot 3}_{\text{multiple of '3'}} \\
 & \cdot \underbrace{50 + 51 + 52}_{\text{positive integers}} = 153 = \underbrace{51 \cdot 3}_{\text{multiple of '3'}} \\
 & \cdot \underbrace{(-1)}_{\text{negative integers}} + \underbrace{0 + 1}_{\text{positive integers}} = 0 \rightarrow 0 \text{ is a multiple of 3, because } \underline{3 \cdot 0 = 0}
 \end{aligned}$$

• By choosing random consecutive integers, I convinced myself that the statement is true.

Figure 4.1: Student AP1's Arithmetic-Procedural argumentation.

of numeric examples followed by an explanation of the sums. This is also compromised by the assertion that 0 is not a multiple of 3. Teleological rationality is evident in the setup of two goals, the first of which is to produce an assortment of examples that include positive and negative integers. The second is to explain the results. These objectives are achieved. Some students reasoned on the structural properties of number and arithmetic operations. In this case, from an observer's point of view, the writer's conviction is based on a flawed argument. The second mode of argumentation, Arithmetic-Structural, describes the way in which these students framed their argumentation.

$$\begin{aligned} -3 + -2 + -1 &= -6 \\ -2 + -1 + 0 &= -3 \\ -1 + 0 + 1 &= 0 \\ 0 + 1 + 2 &= 3 \\ 1 + 2 + 3 &= 6 \\ 2 + 3 + 4 &= 9 \\ 3 + 4 + 5 &= 12 \end{aligned}$$

Question - Is the sum of any 3 consecutive integers always a multiple of 3?

In my investigation I decided to take a sample of 3 consecutive integers. Above in my calculations, it does show that the sum of 3 consecutive integers does equal a multiple of 3. But one sequence did not equal a multiple of 3. "-1 + 0 + 1" does not produce a valid answer. 0 is not a multiple of 3 because dividing 0 by 3 doesn't produce a multiple. Multiplying 0 by 3 also doesn't produce a multiple.

At first I thought the statement was completely true but after making some calculations I am convinced that the statement is sometimes true.

Figure 4.2: Student AP2's Arithmetic-Procedural argumentation

### 4.3.2 Arithmetic-Structural Mode of Argumentation

Characteristics of argumentation distinguishing Arithmetic-Structural mode from the previous one, are the references to the structure of arithmetic operations that are used, or the abstraction of a rule or method from the numeric and arithmetic structure underlying the examples given. These may be presented in an embodied form rather than simply numerical or a combination of the two. Properties and common features that are abstracted lead to the formation and implementation of certain thinking processes. In turn, this leads to an arithmetic generalisation that is used to provide a conclusion. Warrants used in this category are numerically based, but also reflect structural/operational abstractions used in students' argumentations. Thus justifications are based on the principles of arithmetic operations. Table 4.3 summarises some general characteristics of the examples provided of Arithmetic-Structural modes of argumentation that are associated with the TA Framework. An important feature of this table is the distinction between students whose argumentations are set in the Embodied and the Symbolic World of the TWM.

Table 4.3: Arithmetic–Structural

Three Worlds of Mathematics	Layout of Argumentation	Rationality
<p><b>Embodied World:</b> Visuospatial reasoning using pictures and symbols Visual embodiment of number, number operations, and relationships</p>	<p><b>Data:</b> Figurate/Numeric <b>Warrant:</b> Examples of visual representation of operations using an embodiment of number <b>Backing:</b> Arithmetic rule of division by measurement or sharing <b>Qualifier:</b> (if provided): Strength based on effectiveness of visual representations used <b>Rebuttal</b> (if provided): Exception based on limitations of embodiment used <b>Conclusion:</b> Visuospatial generalisation</p>	<p><b>Epistemological:</b> Rule of division (measurement, or sharing) Embodiment of repetitive addition or division <b>Communicative:</b> Text uses a combination of ordinary language and standard arithmetic language, including symbols and pictures, that lead to a conclusion <b>Teleological:</b> Pictures and accompanying numerical example(s) lead to generalisations and a conclusion Text is illustrative of visuospatial and inductive reasoning</p>
<p><b>Symbolic World:</b> Arithmetic reasoning using patterns in the data and/or arithmetic operations</p>	<p><b>Data:</b> Numeric <b>Warrant:</b> Either a pattern from the data or numerical calculations illustrates arithmetic structure <b>Backing:</b> Structure is based on rules of arithmetic <b>Qualifier:</b> (if provided): Strength is based on consistency of a pattern, or effectiveness of a rule/method (conclusion) <b>Rebuttal:</b> (if provided): Exception based on limitations in data and examples provided, or an exception to an arithmetic rule (stemming from an arithmetic misconception) <b>Conclusion:</b> A rule/method</p>	<p><b>Epistemological:</b> Structural properties of number operations can be validated by rules of arithmetic <b>Communicative:</b> Text uses a combination of ordinary language and standard arithmetic language, including symbols and pictures that leads to a conclusion <b>Teleological:</b> Numerical example(s) lead to generalisations and a conclusion Text is illustrative of inductive reasoning</p>

An example of a student’s argumentation reflecting structural embodied thinking is shown in Figure 4.3.

In this example, student AS1 appears to be reasoning in the intersection of the Embodied and Symbolic Worlds because pictures and numeric calculations have combined to contribute to her mathematical reasoning. Four numerical and two embodied examples are given. Addition and division are the primary arithmetic operations used. The figurative examples physically embody the grouping structure resulting from measuring out 3 items at a time. The warrant is arithmetic (numbers divisible by 3 are multiples of 3), and embodied (grouping in 3). The backing used by AS1 are arithmetic concept of a sum, and a rule that division is an action of measurement. There is a qualifier (‘if it is mathematically possible to be divisible’), and the rebuttal is  $-1 + 0 + 1$  (which according to AS1 is not divisible by 3). This misconception is possibly embodied in the figurate rep-

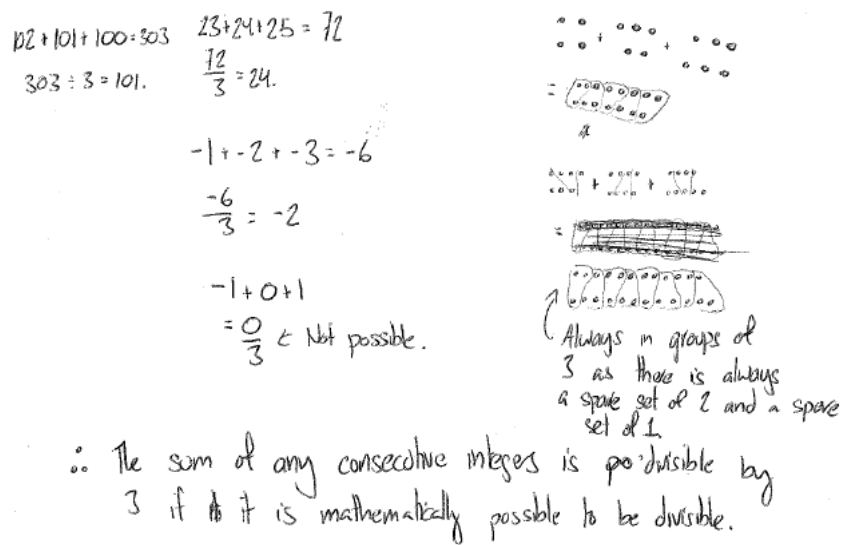


Figure 4.3: Student AS1's Arithmetic-Structural argumentation in the Embodied and Symbolic Worlds

resentation and the operational thinking used in the measurement process. For example, how do you measure out 0 things? Epistemologically, validation and truth are found in an embodiment of arithmetic rules (in this case division), and arithmetic calculation. However, there is a flaw in her knowledge of division. The text uses a combination of symbols, figurate diagrams, and mathematical language. Teleologically, AS1 uses numerical as well as pictorial evidence (an embodiment of division) to make a conclusion. It should be noted that AS1's argumentation to secure the conviction of an other also included the flawed reasoning regarding the division of zero, and hence was classified as a flawed argument.

The next example is of a student's argumentation whose reasoning is drawn from the Symbolic World of the TWM (Figure 4.4).

In this example, student AS2 identifies the middle integer of the 3 consecutive integers as important to transforming a summation to a multiplication. This is seen in Figure 4.4 where she underlines each of the middle integers in her examples. Reasoning on structure occurs when AS2 reasons that the two integers either side of the middle one can be made to equal it by adding 1 to the first, and subtracting 1 from the last. Thus AS2 concludes that the sum of 3 consecutive integers is the middle integer multiplied by 3. The warrant used is a multiplicative calculation and the backing is split into two parts. The first is the conservation of number under an additive rule, and the second is the equivalence relationship between repeated addition of integers and multiplication by 3. At this stage there is no attempt at using symbols and algebra to generalise the



Any given 3 consecutive integers will, in fact, be a multiple of 3.

I can prove this first by giving a few examples:

- 1)  $101 + \underline{102} + 103 = 306$   
 $(3 \times \underline{102} = 306)$
- 2)  $31 + \underline{32} + 33 = 96$   
 $(3 \times \underline{32} = 96)$
- 3)  $1 + \underline{2} + 3 = 6$   
 $(3 \times \underline{2} = 6)$
- 4)  $9 + \underline{10} + 11 = 30$   
 $(3 \times \underline{10} = 30)$

The reason why this is the case, is because if you ~~add~~ take 1 from your last number and add it to the first number, it will just be 3 equal numbers, as in above examples. Therefore, any given 3 consecutive numbers will ~~be~~ have a sum of number that is always 3 multiplied by the number in the middle. (underlined in above examples)

Figure 4.4: Student AS2's Arithmetic-Structural argumentation in the Symbolic World

result. Epistemic rationality is derived from an arithmetic property that equalises three integers and compresses the result into a multiplicative rule. Communicative rationality is demonstrated through the use of numerical examples, the use of a strategy to highlight a particular pattern, and the use of a generalisation to state a conclusion. However, from an observer's point of view this is compromised by the absence of negative integers in the array of examples. There is also a subtle (likely unintentional) shift from a discussion of integers to a discussion of numbers. Contributing to teleological rationality in AS2's argumentation is the goal of showing the relationship between an additive structure to a multiplicative one – which is accomplished in the explanation given. We now consider a category where algebraic thinking is involved. This is associated with the use of an Arithmetic-Generalising mode of argumentation. As we will see, students in this mode combined arithmetic and algebra in different ways depending on the goal of their argument.

#### Arithmetic-Generalising Mode of Argumentation

Argumentation in this mode is characterised by the combined use of numerical and alge-

braic reasoning in the construction of a conviction. In this mode, either numerical examples are used as a means of validating the truthfulness of an outcome of algebraic reasoning, or numerical examples are used as the basis for algebraic reasoning. Thus there are two distinct ways of deriving validation and truth, either through the use of empirical examples, or through algebra. In the former case, students attempt to communicate algebraically, but the normative requirements of algebraic reasoning are not evident. Table 4.4 summarises some general characteristics of the examples provided of Arithmetic–Generalising modes of argumentation that are associated with the TA Framework.

Table 4.4: Arithmetic–Generalising

Three Worlds of Mathematics	Layout of Argumentation	Rationality
<b>Symbolic World:</b>		
(i) Arithmetic reasoning is used to explain an algebraic formula Use of algebraic and arithmetic symbols	<b>Data:</b> An algebraic formula <b>Warrant:</b> Arithmetic calculation <b>Backing:</b> Rules of arithmetic <b>Qualifier:</b> (if provided): Strength based on the interpretation of the algebraic conclusion <b>Rebuttal:</b> (if provided): Exception (if used) is based on an arithmetic misconception <b>Conclusion:</b> An interpretation of an algebraic formula	<b>Epistemological:</b> Validity is based on arithmetic rules <b>Communicative:</b> Text uses a combination of ordinary language and standard arithmetic language, including algebraic and arithmetic symbols <b>Teleological:</b> Numerical example(s) confirm data, and conclusion Text is illustrative of inductive reasoning
<b>Symbolic World:</b>		
(ii) Arithmetic reasoning leads to algebraic reasoning and the development of an algebraic formula The conclusion is deductively reasoned	<b>Data:</b> Arithmetic calculations <b>Warrant:</b> Generalised arithmetic calculations <b>Backing:</b> Rules of arithmetic and algebra <b>Qualifier:</b> (if provided): Strength is based on interpretation of the last algebraic step (conclusion) <b>Rebuttal:</b> (if provided): Exception based on limitations in data <b>Conclusion:</b> An algebraic formula	<b>Epistemological:</b> Mathematical validity is by way of algebra <b>Communicative:</b> Text uses a combination of ordinary language and standard arithmetic language, including arithmetic and algebraic symbols <b>Teleological:</b> Numerical example(s) lead to an algebraic formula and conclusion Text is illustrative of deductive reasoning

Figure 4.5, illustrates an argumentation where arithmetic calculations are used as a means of establishing assurance and confidence in an algebraic result. In this example, validity and truth are seen in arithmetic procedures that involve ‘real’ numbers – not an ‘imaginary’ number represented as a variable. In this argumentation provided by a student who will be referred to as AG1, mathematical resources are drawn from the Symbolic World of the TWM. Arithmetic reasoning is used to support an algebraic construction. The data used

Sum of 3 consecutive integers always a multiple of 3.

$$+(x+1)+(x+2) = 3x+3.$$

say  $x$  was 4.

$$-3 + -2 + -1 = \frac{-6}{3} = 2$$

$$4 + (4+1) + (4+2) = 4 + 5 + 6 = 15$$

15 divisible by 3.

Always true. using 4 again.

$$3 \cdot 4 + 3 = 12 + 3 = 15.$$

Because the sum is multiplied by 3 it must be a multiple of 3. — multiply plus 3

Always true.

Figure 4.5: Student AG1's Arithmetic-Generalising argumentation

are the algebraic expressions  $x + (x + 1) + (x + 2)$  and  $3x + 3$  and the conclusion is a statement expressing the equality of these two expressions. The warrant being used by AG1 is arithmetic – the substitution of  $x = 4$  into  $x + (x + 1) + (x + 2)$  and  $3x + 3$  to show they are numerically equivalent. The warrant is backed by the rules of arithmetic with a particular emphasis on the rule of equality. Epistemic rationality is derived from the focus on arithmetic to establish a conviction of the result. From an observer's perspective, this might indicate that in this case, AG1's foundation for conviction lies in arithmetic examples rather than algebra. Thus, it appears that communicative rationality occurs through the use of arithmetic symbols and operations to support arithmetic reasoning. Teleologically, AG1 sets the goals of showing that 3 consecutive integers are always a multiple of 3, and that this is equal to 3 times the first of the 3 consecutive integers plus 3. This example is illustrative of a student whose conviction is grounded in arithmetic. The next example shown in Figure 4.6 gives a contrasting use of arithmetic thinking. Here student AG2 has used it as a springboard to thinking algebraically.

The example given in Figure 4.6 illustrates the second part of the Table 4.4 and provides a contrast to AG1's argumentation within the Arithmetic-Generalising category.

Student AG2's argumentation begins with arithmetic reasoning, verifying the sums are

Testing the statement by calculating the sum of random 3 consecutive integer sets:

$$\begin{array}{l} 1+2+3 = 6 \quad \checkmark \\ 7+8+9 = 24 \quad \checkmark \\ -3+-2+-1 = -6 \quad \checkmark \\ 99+100+101 = 300 \quad \checkmark \end{array} \left. \vphantom{\begin{array}{l} 1+2+3 \\ 7+8+9 \\ -3+-2+-1 \\ 99+100+101 \end{array}} \right\} \text{All multiples of 3}$$

$\swarrow$     $\swarrow$     $\swarrow$     $\swarrow$   
 $x, x+1, x+2$    (where  $x = \text{any integer}$ )

$$\begin{aligned} &x + (x+1) + (x+2) \\ &= 3x + 3 \\ &= 3(x+1) \end{aligned}$$

Therefore, 3 times any value of  $x$  is a multiple of 3.

Therefore, the statement "the sum of any 3 consecutive integers is always a multiple of 3" is always true.

Figure 4.6: Student AG2's Arithmetic-Generalising reasoning

multiples of 3 and structuring algebraic expressions for 3 general consecutive integers. From this point forward, reasoning involved is algebraic. As in the case of the previous student, AG2 is working with resources drawn from the Symbolic World of the TWM. The data used in this argumentation are sets of 3 consecutive integers, and the conclusion is an interpretation of an algebraic expression. Connecting the data to the conclusion is an algebraic warrant with a backing coming from the rules of algebra. Epistemic rationality is based on algebraic construction of the expression  $3(x+1)$ , and it seems that the absence of arithmetic verification indicates that AG2's conviction is founded on the interpretation of this algebraic expression – recorded at the end of his text. Communicative rationality is facilitated by the use arithmetic as well as algebraic symbols and processes to generalise an arithmetic argument. Teleological rationality is developed through the goals of demonstrating that the sum of 3 consecutive numbers is a multiple of 3 and constructing  $x$ ,  $x+1$  and  $x+2$  which are used to formulate a simplified algebraic expression. Thus in contrast to AS1, this student's conviction is derived from the validity of algebra.

In the next example, we will consider argumentation that is shaped solely by algebraic

reasoning. Table 4.5 summarises some general characteristics of examples of Arithmetic-Generalising modes of argumentation that are associated with the TA Framework.

Table 4.5: Generalising with Algebra

Three Worlds of Mathematics	Layout of Argumentation	Rationality
<b>Symbolic World</b>		
Algebraic reasoning using algebraic symbols	<p><b>Data:</b> Generalised numbers written using a variable</p> <p><b>Warrant:</b> Algebraic calculation</p> <p><b>Backing:</b> Rules of algebra</p> <p><b>Qualifier:</b> (if provided): Strength based on interpretation of algebraic formula</p> <p><b>Rebuttal:</b> None</p> <p><b>Conclusion:</b> An algebraic formula</p>	<p><b>Epistemological:</b> Mathematical validity is by way of algebra</p> <p><b>Communicative:</b> Text uses a combination of ordinary language and standard arithmetic language, including algebraic and arithmetic symbols</p> <p><b>Teleological:</b> Steps in algebra illustrate synthesis of data into an expression, simplification of an expression, and an interpretation of the result</p> <p>Text is illustrative of deductive reasoning</p>

The final mode of argumentation that will be described is Generalising Algebraically. In this mode students use algebraic and deductive reasoning to convince themselves of a result, or to convince someone else that the result they have found is the correct one.

### Generalising with Algebra Mode of Argumentation

In this category, the warrants used are algebraic and are backed by rules of algebra. The strategy used in a student's argumentation is communicated through a combination of algebraic and ordinary language. Attention is given to correctly specifying variables, and there is evidence of deductive reasoning. Algebra is seen as the valid form of mathematical justification, with no need for numerical examples. The example of student A1 in Figure 4.7 illustrates an argumentation in this category.

Of particular note here is shift toward a more formal use of the language of mathematics. For example A1's argumentation begins with 'Let  $\{x, x + 1, x + 2\} \dots$ ', and the variable  $x$  is specified as 'any integer'. Algebraic reasoning and manipulation is used to simplify  $x + (x + 1) + (x + 2)$  to  $3(x + 1)$  and deduce the concluding statement. Part of A1's reasoning involves a proceptual view of  $3(x + 1)$ . The penultimate statement indicates that A1 reasons that  $3(x + 1)$  can be evaluated for an arbitrary  $x$ , and hence represents a process.

Let  $\{x, x+1, x+2\}$  represent 3 consecutive integers, where  $x = \text{any integer}$ .

The sum of the 3 consecutive integers:

$$\begin{aligned}x &+ (x+1) + (x+2) \\ &= 3x + 3 \\ &= 3(x+1)\end{aligned}$$

Therefore, 3 times any value of  $x$  is a multiple of 3.

Therefore, the sum of any 3 consecutive integers is always a multiple of 3.

Figure 4.7: Student A1's algebraic reasoning

Deductive reasoning is an integral part of A1's argumentation. There is no use of inductive reasoning, instead algebra is seen as the valid method of providing a mathematical argument.

The data used is algebraic,  $x, x+1, x+2$  (with  $x$  defined as an integer). The warrant used is also algebraic and involves manipulations according to algebraic rules – which provides the backing. The argumentation is deductively organised and begins with relevant 'definitions', followed by a stepwise development of the argument itself. The word 'therefore' is used to signify a deductive inference.

A preliminary analysis of students' argumentations using the TA Framework appears to give sufficient information about their thinking and reasoning, as well as the mathematical resources they are utilising. It also provides information about way in which students' argumentations are structured so as to make them appear rational. A strength of the framework is that it provides an effective focus for the researcher's observations and interpretations of students' argumentations. A question that will be addressed further on in this thesis is whether the TA Framework is robust enough to work in other areas of mathematics.

## Chapter 5

# Research Methodology

### 5.1 Introduction

This chapter outlines the research design used in this study including theoretical factors that contributed to the methodology used. It also includes a description of methods and procedures that were used to collect, analyse and interpret the data. Four factors were considered in the construction of a theoretical framework used in the design and methodology of this research. The first of these was the biases of the researcher which, as a natural part of research design and content, needed to be identified and exposed (Davis, 1996). The second factor considered was Romberg's research model (Romberg, 1992) comprising ten essential activities for research shown in Table 5.1. This model was used to shape the over-all research procedure used in this study. The notion of life-world, drawn from the work of Jürgen Habermas (1998) was the third factor considered as it suggested a theoretical lense through which the researcher could consider the context in which the phenomenon under investigation occurred. It also suggested the impact that various contextual elements might have on students' writing, and the researcher's observations and interpretations. The fourth factor considered, was determining a research paradigm upon which this research could be grounded (see Guba & Lincoln, 1985, pp. 105–117). Although given some individual attention in the following subsections, some explanation of each of these factors will be integrated throughout this thesis.

## 5.2 Toward a Research Process and Method

The purpose of constructing a theoretical framework for a research programme is to develop internal coherence, and provide clarity of objectives, design, and methodology. Zevenbergen and Begg (1999) suggest that a useful start point is to declare assumptions, and biases arguing that these effect a researcher's prejudices toward what is anticipated and what is attended to when information is collated and analysed. This is echoed by Davis (1996) who claims that the problematisation of distinctions and boundaries created by a researcher's assumptions leads to a richer understanding of the research situation at hand. The researcher is not only able to interpret the information through his researcher's lens, but also through the 'eyes of the students' (see Dowling 2001, p. 23). As noted by Hodder (1994) the data and the interpreter work to bring each other into existence in a dialectical and reflexive fashion. However this being said, the basis of a research design rests on the questions to be investigated (Romberg, 1992).

The formulation of the primary research question for investigation is important as it determines the choice of methods that will be used to gather and interpret information. A summary of a research procedure is given of Romberg's (1992) model of essential research activities (see Table 5.1). According to Romberg's (1992) model, the question to be

Table 5.1: Research Model (Romberg, 1992, p. 51)

- 
- 
- 1 Identify a phenomenon of interest
  - 2 Build a tentative model
  - 3 Relate the phenomenon and model to others' ideas
  - 4 Ask specific questions to make a reasoned conjecture
  - 5 Select a general research strategy for gathering evidence
  - 6 Select specific procedures
  - 7 Collect the information
  - 8 Interpret the information collected
  - 9 Transmit the result to others
  - 10 Anticipate the actions of others
- 

investigated is situated in the context of a certain world view wherein lies the phenomenon



of interest. The formulation of the question or conjecture is informed by two things, first a tentative model that provides a means of explaining the phenomenon of interest and secondly, the suppositions a researcher makes about the evidence needed to answer it. As noted by Peshkin (2000) the researcher is not indifferent to the subject of enquiry. Thus there is also a personal stake that the researcher has in it that influences the formulation of the question and its theorisation.

In the introductory chapter to this thesis, I gave a background to how I became interested in students' argumentation that progressed through career stages involving undergraduate study in mathematics, primary school teaching, and working in the education and mathematics tertiary sectors. These personal experiences contributed to the values and conceptions I presently hold with respect to mathematics and pedagogy and in particular, justification and proof. In this thesis, I propose that there are benefits to learning mathematics that ensue from engaging students in argumentational tasks based in a problem solving context. These values, conceptions and beliefs have influenced the framing of the research question, as well as the setting up of an analytical framework used in this study.

In Romberg's model (Table 5.1), activities one to four drive the selection of research methods in activities five to eight. Activities nine to ten are dependent on the identification a specific audience to whom the research and its outcomes will be addressed. The notion of *life-world* developed by Jürgen Habermas is an important facet affecting the implementation of this study and the interpretation of its results and will be discussed next.

### 5.2.1 The Life-world

The concept of life-world proposed by Habermas (1998) consists of three structural components, cultural, society, and personality. Habermas argues that these three components are maintained through linguistic processes of cultural reproduction (transmitting and developing cultural knowledge), social integration (of a life-world shared intersubjectively by its members), and socialisation (which serves to form and maintain personal identities) (Eriksen & Weigård, 2003; Segre, 2013; see also Habermas, 1998). Thus life-world is a means by which intersubjective communication is made possible, and through which knowledge grows, is reproduced, renewed, and warranted as rational (Outhwaite, 2009). Hence the term life-world refers to the background resources, contexts, and dimensions of social action that enable people to co-operate on the basis of mutual understanding. It also includes shared cultural systems that give rise to meaning, institutional orders that

stabilise patterns of action, and personality structures acquired in family, religion, neighbourhood, and educational institutions (Bohman & Rehg, 2011). Accordingly Habermas (1998, p. 246) notes:

The life-world as a whole comes in to view only when we, as it were, stand behind the back of the actor and view the communicative action as an element of a circular process in which the actor no longer appears as the initiator but rather as the product of the traditions within which she is situated, of solidarity groups to which she belongs, of socialization and learning process to which she is subjected.

Thus, in the context of a first year university student studying mathematics, their life-world comprises values, norms, attitudes, competencies and identities promoted by the institution constituting their mathematics programme. This includes their expectations and competencies with regard to teaching and learning, course content, assessment, interpersonal relationships, and communicative behaviour. Also contributing to a student's life-world are, knowledge (and understanding), experience and familiarity with the topic, and self-efficacy (with respect to mathematics, problem solving, and communicating mathematically). According to Habermas (1998) one's life-world is an implicit feature of one's written text (Habermas, 1998). This study uses ideas drawn from Habermas' theorisation of communicative action. In particular, it considers strategic actions students take when writing an argumentation.

This research studies the ways in which students shape their written argumentation according to two different audiences. It also examines the diachronic development of students' written argumentation. A key source of data for this research is students written argumentations. According to Dowling (2001), written texts can be interpreted as the construction of relations between an authorial and a reader voice. Dowling treats a text as an empirical object of analysis as if it were a bounded instance of social activity. He refers to a pedagogic text as one that constructs pedagogic relations between the voices of the author and the reader. In a pedagogic text, evaluation of potential performances is placed with the author's voice which the author knows but the reader doesn't – yet at least. Thus a student's argumentation facilitates the transmission of mathematical ideas in an attempt to construct the reader as a subject of mathematical practice (a conception formed in their life-world). Using Romberg's model, the reader (or researcher) then reflects on and interprets what has been transmitted through the eyes of both a 'constructed

mathematician' and a researcher.

### 5.2.2 Constructing a Research Paradigm

Interpreting a phenomenon is framed against a certain paradigm or interpretive framework that expresses the researcher's world view and belief system (Ernest, 1997). According to Guba and Lincoln (1994) the interpretive framework is informed by the researcher's response to three fundamental questions. The first of these is the ontological question, "What is the form and nature of reality, and therefore what can we know about it?" This question addresses the existence of a real objective world independent of our knowledge or the alternative that reality emerges as a result of human interaction. The second is an epistemological question, "What is the relationship between the researcher and the object of study?" Usually, but not always, this relationship is constrained by a researcher's ontological assumptions. The third question posed by Guba and Lincoln (1994) relates to methodology, "How can the researcher go about finding out about what he/she believes can be known?" Guba and Lincoln (*ibid*, p. 108) argue that a researcher's answers to these three questions will position their thesis within one of the following interpretive paradigms: positivism, post-positivism, critical theory, or constructivism. A response to these questions is now provided.

There are two ontological assumptions that are made in this thesis. The first is the assumption that an objective reality exists that is apprehended through the best experiences, knowledge and explanations of a phenomenon we have at a particular time. The second assumption is that apprehensions of reality emerge from mental (cognitive) constructions that become more or less informed and/or sophisticated over time. Thus conceptions of reality evolve and take time to crystallise into provisional structures of reality. The first assumption aligns itself with a post-positivist paradigm which is usually used to describe a paradigm that affirms an objective reality, but unlike positivism asserts that humans cannot know it for sure (Guba & Lincoln, 1994). This expresses my view of mathematics. The second is an eclectic mix of constructivist and critical theory. A constructivist paradigm ascribes to the notion that realities are individually and socially constructed whilst critical theory asserts that realities crystallise over time as more informed insights are developed through dialectical interaction. Subsumed in this is the interpretive paradigm which is labelled as a methodology. The constructivist paradigm aligns with my experience as a school teacher, teacher educator, and teacher of mathematics in a tertiary institution.

Underpinning the interpretation of students' argumentations is an epistemological assumption that some knowledge is constructed in the interaction between the researcher and the subject. In this study this occurred through the written medium and in other cases, through interviews and conversations. Another epistemological assumption that is made in this research is that knowledge evident in students' argumentations is mediated through the perspective of the researcher. The first assumption, is taken from the constructivist paradigm and links to Dowling's (2001) notion of the author's construction of the researcher as a mathematician, and Habermas' forms of rationality (Habermas, 1998). The second assumption, drawn from critical theory, reverses Dowling's (2001) pedagogic text. In this reversal, the researcher uses his voice and that of a theorised framework to evaluate the author's (student's) potential performance. The second assumption also means that the researcher's interpretations develop in a reflexive interplay between subjectivity and objectivity so as to give an authentic account of a situation (Banister et al., 1994). The epistemological positions just described indicate a correspondence of this research with an interpretive paradigm. Neuman (2000) notes that the interpretive approach to the examination of students' texts involves a systematic and detailed observation of them in order to discover embedded meanings, and develop an understanding of how its parts relate to the whole. Elements of this approach will be used in the investigation of students' argumentations.

A methodological assumption backing this thesis is that texts of written argumentational reflect the rational intentions of its writers as well as their knowledge of a mathematical situation, appropriate cognitive strategies, and a knowledge of the other to whom the argumentation is addressed. This assumption lines up with the description of a constructivist paradigm given by Guba and Lincoln (1994).

Assumptions underlying this thesis lead to a synthesis of an eclectic assortment of elements taken from different paradigms in Guba and Lincoln's (1994) classification (e.g., post-positivism, critical theory, and constructivism). Creswell (2009) describes a *pragmatic paradigm* through which an eclectic profile like this can be interpreted. He characterises pragmatism as follows (Creswell, 2009, p. 12):

- Pragmatism is not committed to any one system of philosophy and reality
- Individual researchers have freedom of choice. They are "free" to choose methods, techniques, and procedures of research that best meet their needs and purposes
- Pragmatists do not see the world as an absolute unity. In a similar way, researchers

look to many approaches to collecting and analyzing data rather than subscribing to only one way (e.g., multiple qualitative approaches)

- Truth is what works at the time; it is not based in a dualism between reality independent of the mind or with the mind
- Pragmatist researchers look to “what” and “how” of research based on its intended consequences – where they want to go with it
- Pragmatists agree that research always occurs in a social, historical, political, and other contexts
- Pragmatists have believed in an external world independent of the mind as well as those lodged in the mind. They believe (Cherryholmes, 1992) that we need to stop asking questions about reality and the laws of nature. “They just want to change the subject” (Rorty, 1983, p. xiv)

The paradigm in which a research project sits informs the selection of a research strategy (Denzin & Lincoln, 1994) such as a case study, ethnography, action research, or observational study (a list is given by Janesick (1994, p. 212)). In turn, the selected research strategy links the interpretive paradigm to methods that will be used to collect and analyse data (See Romberg’s activity 5, Table 5.1). The pragmatic paradigm appears to draw on elements of constructivism, critical theory, and post-positivism – at least enough to suggest an alignment with this study. Because of its eclectic nature and usefulness, this study will be adopt the pragmatic paradigm as its theoretical research basis. Backed by this theoretical discussion, the research strategy and methodology employed in this thesis followed a qualitative inquiry approach. This involved the use of both observational, and case study methods.

### 5.3 Methods

An observational methodology was used to interpret students’ written texts of argumentation. Of the various observational techniques available, the interpretive approach was used. This enabled the researcher to view artefacts of students’ thinking and actions (Patton, 2002) with an understanding that his experience and insight formed part of the data (Tuckman, 1999, p. 423). Hodder (1994) suggests that an important facet of an interpretive study is the parameters defining the context in which it takes place. In this

research the context in which the interpretation of students' texts took place was their tertiary mathematics course. This also provided a natural contextual boundary because of its content, coursework, and general lecturing pedagogy. Another boundary parameter was the cohort of the students who were taken from a particular semester of the university academic year. The assumption made here is that given a different semester cohort, different characteristics might have emerged. The boundary shifted with different types and levels of mathematics courses for example, the requirements and expectations of students taking a certain mathematics course such as Maths 102 were different to those for others such as Maths 150, or Maths 255. The final parameter defining the context was the distinction in the objectives of students' written argumentations – one focussed on deriving a self-conviction, and the other on convincing an other. Hodder (1994) also suggested attention be given to the identification of similarities and difference in students' argumentations. This study explored similarities and differences in first and second year students' written argumentations with a view to identifying possible categorisations and comparing how argumentations changed when there was a change in audience. The third suggestion from Hodder (1994) relates to the evaluation of the relevancy of theories used in the analysis to the collected data. In setting up the analytical framework for this research, a synthesis of three theoretical perspectives were used. Although each theory had been shown to be robust in previous research in mathematics education, this appears to be the first occasion that all three had been combined into an analytical framework.

This research also adopted a case study approach in order to present data that was illustrative of modes of argumentation in mathematics, and specific relationships involved in their production. Neuman (2000) explains that a case study is an in-depth examination of cases that should lead to the identification of generalities and the construction of theoretical representations that can be used to explain social structures and processes. According to Cohen and Manion (1994) a particular strength of the case study “lies in their attention to the subtlety and complexity of the case in its own right”. Like the interpretive approach, the case study takes place in a bounded system (Cohen, Manion & Morrison, 2007). However, the case study can be used as a research strategy in different ways, each providing a particular area of focus for the researcher. For example, Stake (1995) makes a distinction between an intrinsic, instrumental, and collective case study. In an intrinsic case study, the researcher's primary aim is to achieve a comprehensive understanding of a particular case – the focus is on the case itself whereas in an instrumental case study, the researcher attempts to get a better understanding of the phenomenon and relationships

within it (Merriam, 2002; Stake, 1995). The phenomenon of interest in this research was students' argumentations – those that focussed on self-conviction, and those that were written to convince an other. These were examined and compared using a theoretical framework. Finally, Stake's (1995) collective case study (also called multiple case studies) coordinates a set of case studies as a means to making various generalisations and or a proposed theories more convincing. According to Johnson and Christensen (2008) generalisability can be made more convincing by collecting and coordinating evidence from a number of individual case studies. Hence multiple cases were used in this study.

In this study there were two primary methods of collecting data, the interpretation of written texts including the observation of students writing their arguments using a 'think aloud' protocol, and structured interviews. Conversations were also included in some cases where some aspects of the particular case needed to be clarified.

### 5.3.1 The Participants

Data was gathered from a range of participants that included students who in 2011 were doing a first or second year university mathematics course, and a small group of participants who by 2012 were either completing or had completed a major in mathematics (or applied mathematics).

#### First Year Course in Mathematics

Participants were drawn from Maths 102 which is a course designed for those students who do not have a strong background in mathematics. At the time when the data was gathered, the researcher was also the lecturer of Maths 102 and had introduced to the course an element of justification through mathematical argumentation. This included examples that demonstrated the difference between inductive and deductive reasoning. Since this is a first year preparatory course, proof was not given a significant emphasis, but was used during lectures in situations where students had been given the necessary mathematical tools to handle their development. Proof and more generally argumentation was given to assist the understanding of certain mathematical ideas. Thus justification and elementary proving activities also became part of course assessment. The students were informed that as part of the ethics approval for this study, their participation or non-participation in this study would not affect their coursework, participation in the course, or relationship with their lecturer.

Those students who took Maths 102 in 2011 (Semester 1) were given two tasks written

specifically for this study as part of their coursework (see Appendix 1). Both tasks required that the students write an argumentation. The first task was handed out as part of the students' first assignment and they were given specific instructions that the task was to be completed as an individual without assistance from media or other students (at school or at any tertiary institution). If they did not agree to this, then they were not to hand in their script as part of this research, only as part of the coursework. Everyone, whether they participated in the study or not were given 5 marks for significant evidence of having engaged with the task – completed or not. After a week the students returned their task sheet to the lecturer. Task Two was handed out following the same protocol, as part of the students' second assignment. Eighty-three students participated in the first task by handing in their argumentation to the lecturer, and sixty-one students participated in the second task.

### **Second Year Students**

The recruitment of second year students took place in both the first and second semester of 2011. In Semester 1, students from Maths 250 a continuing mathematics stage two course were targeted, but none returned emails expressing interest in the study. In Semester 2, students from Maths 255 were invited to participate, and twelve students replied. Maths 255 is a course that introduces students to mathematical thinking and communication, and teaches them how to organise arguments logically and prove results. These students were provided with a specifically written task (Appendix 3), and given the same instructions as the first year participants. After a week they were asked to return their task sheet.

### **Students Tracked over Three Years**

Five students originally agreed to have their argumentations tracked from their first year to their third year. In nearly all cases, this involved interviews, and the presentation of a selection of work that demonstrated their progressive ability to write argumentations over the three year period. However, one student provided only one argumentation from his first year, and missed giving in any other.

## **5.4 Data Collection**

Data collected in this study included students' written argumentations, including some follow-up conversations with particular students for clarification of aspects in their argumentation, interviews, and observations. These are given further elaboration later in this



section.

#### 5.4.1 Collection of Students' Written Argumentation

Since the focus of this research was on how students formulate a coherent argumentation, students' written work comprised the majority of the data. As noted by Kelly et al. (2007), written argumentation poses unique possibilities and challenges in science education – and here we will also include mathematics education. First, writing an argumentation in mathematics requires that students draw on a range of skills and practices including mathematical knowledge, linguistic knowledge of specific lexicon and grammar (Halliday & Martin, 1993) as well as reasoning (Tall et al., 2012). Second, written argumentation involves justifying, persuading, or convincing a critical community of peers of one's standpoint. Thus, students' argumentations reflect the epistemic status of the claim (or conclusion) being made. Third, mathematical practices are specific to various levels of study, and particular courses of study (Kelly et al., 2007). Following on from the studies of Kelly et al. (2007), Kelly and Takao (2002), and Kelly and Bazerman (2003) of the analysis of written argumentation in science, this study will also examine these three general aspects through students' written argumentation.

In this study, students' argumentations were written on a researcher-designed task sheet that asked them to determine whether a proposition is true, sometimes true, or always true, and then provide an argument that would convince them of their choice (or give a counter-example). The next part of the task asked the students to provide an argument that would convince an other – who was characterised by their mathematics lecturer. These tasks were designed to compare the argumentations students wrote to convince themselves of a claim, with those they wrote to convince a more knowledgeable other. Each task was written so that in the analysis of the students' text the researcher would be able to locate cognitive and mathematical (Tall et al., 2012), structural (Toulmin, 1958) and rational (Habermas, 1998) aspects of their argumentation.

Once collected, data from the students was examined and sorted according to broad categories that were established from the literature review and evidenced in the framework developed for this study.

#### Observations

Although direct observation did not form the substantial part of data collection, it was used in some cases. These cases occurred during set office hours (of the researcher), or

during appointments made with individual students. The focus of these observations was to give the researcher an understanding and clarification of two things. First, how the student used mathematics in the construction of the argumentation and second, to confirm the backing that was being used in the argumentation and confirm warrant(s) being used. During some observations, casual conversation occurred that prompted some students to move forward in their construction of an argument (see the quote provided by Cohen et al., 2007, p. 361 below).

### **Interviews**

Interviews were introduced to the methodology so as to examine a particular group of students' experiences and conceptions of argumentation as they progressed through to studying third year courses in mathematics. A key aspect to be explored was how their perceptions of proof developed over their three years of study – note that here I am using the assumption that proof is a specific form of argumentation. The interview also aimed at getting these students' points of view on what elements of argumentation they considered to be important to engage successfully with proof. Where possible, their comments were linked to examples of their argumentations over a three year period. Two points of view shaped the way interview questions were developed and implemented. The first was Robson's suggestion (Robson, 2002) of using a semistructured, respondent style interview format with predetermined questions or themes that could be modified depending on individual circumstances. The second was the usefulness of prompts and probes in an interview. Cohen et al., (2007, p. 361) explains:

Prompts enable the interviewer to clarify topics or questions, while probes enable the interviewer to ask respondents to extend, elaborate, add to, provide detail for, clarify or qualify their response, thereby addressing richness, depth of response, comprehensiveness and honesty that are some of the hallmarks of successful interviewing.

Thus backed by these two compatible positions, questions were developed that addressed their experiences in and perceptions of proving and proof (see Appendix 4 for interview protocols). Students interviews were recorded, and transcribed.

## 5.5 Summary

This chapter has described the methodological framework and shown that it is rooted in the in the life-world of the researcher. It was developed through experience and beliefs and modified or reinforced through a consideration and review of literature. As noted, this study comprised a combination of post-positivist, constructivist, and critical theory paradigms that is best situated in a pragmatist paradigm. The study employed a case study with multiple cases that included interviews and some observations as means of gathering data. However, the primary method of data was students' written argumentations. With these data, an instrumental case study strategy was used to bring to the fore specific characteristics of students' argumentation that were then analysed with a particular interpretive framework.

# Chapter 6

## Analysis of the Data

### 6.1 Introduction

We will begin this section by presenting an overview of data that emerged from two tasks given to first year Maths 102 students. The first of these tasks was based on a problem from number theory, and the second from a problem in trigonometry. A discussion of the data will be provided through a detailed analysis of students' argumentations which will include a comparison of their modes of argumentation. Six cases will be considered with each case representing one of these modes. Within each case, a comparison will be made of the student's written argumentation of self-conviction, and that written to secure the conviction of an other. From the analyses, a general picture will be formed regarding how students approach each type of argumentation. An analysis of a task that were given to second year students will follow in the same way. The data and analysis of the first task will now be presented beginning with its description.

#### Task 1

The set of integers is the set  $\mathbb{Z}$  where  $\mathbb{Z} = \{\dots - 3, -2, -1, 0, 1, 2, 3, \dots\}$ .

Now, consider the following statement;

“The sum of any 3 consecutive integers is always a multiple of 3”.

- Investigate this statement, and convince yourself whether it is always true, sometimes true, or never true.
- Write an argument that you would use to convince another person such as a mathematics lecturer of your conclusion.

## 6.2 Organising the Data

After a review of the students' scripts, it was found that the modes of argumentation presented in Chapter 4 could be refined even further to give a clearer understanding of the ways in which they reasoned in their argumentations. Of the 128 students 83 of them returned their scripts and agreed to participate in the research. Students' modes of argumentations were categorised in the following way. If an argumentation's primary form of reasoning was inductive, sourced from numerical examples and backed by rules of arithmetic, then it was categorised *Arithmetic: Procedural* and coded [AP]. If an argumentation demonstrated reasoning on the numerical structure of three consecutive integers and/or the arithmetic involved, then it was categorised *Arithmetic: Structural* and coded [AS]. In the description of the initial trial given in Chapter 4 the categories, Arithmetic Generalising and Generalising with Algebra were used. However, an adjustment was made and all argumentations that used algebra were grouped together. Thus, these students were re-categorised as *Algebraic: Arithmetic Validation* (which in tables will be referred to as Arith-Valid) and coded [AlgAV], and *Algebraic: Construction* which was coded [AlgC]. Both employed arithmetic reasoning, but in two distinct ways. Algebraic: Arithmetic Validation argumentations developed an algebraic expression, but then used numerical examples at the conclusion to validate the algebra. On the other hand, Algebraic: Arithmetic Construction arguments used numerical examples and structural reasoning to formulate an algebraic expression. Arguments that used algebra only, without any numerical calculation, were classified *Algebra-Only* and coded [GAlg]. An overview of the data will now be given.

### 6.2.1 Presenting The Data

When asked to give an argument that would secure their personal conviction, the students began by using either numerical examples or algebra to construct a generalisation. They then made a decision about their conviction. Most stated they were convinced that the statement was always true, but others (18 of the 43 students) gave a conditional conviction citing a specific case for which it apparently failed. Table 6.1 indicates the different modes of argumentation that students used to secure a personal conviction, along with the number of students that used them. Although an explanation of each mode is given in the introduction to this chapter they will be elaborated further in the following sections. From Table 6.1, it is evident that arithmetic reasoning played a significant role in the

establishment of a personal conviction. For example, just under two-thirds of all students used numerical examples only to secure a personal conviction. Seven students (included in the table under the category Algebraic: Construction) used numerical examples as the basis for reasoning that eventually resulted in the construction of an algebraic expression. The strength of some students' perception of the power of numerical examples was exemplified by the arguments of 8 students, who used numerical examples to validate the algebraic expressions they had formed (see Cases 3 & 4). This group is classified in Table 6.1 as Algebraic: ArithValid. Only a small number of students constructed a generalised argument using algebra only as a means of securing a personal conviction.

Table 6.1: Modes of Argumentation used to Secure a Personal Conviction

<b>Arithmetic</b>		<b>Algebraic</b>		
Procedural	Structural	ArithValid	Constructive	Algebra-Only
43	12	8	7	13

When the students were asked to write an argument for the same problem, but this time with the goal of convincing an other, there was a significant shift in the data. The 'other' was purposefully identified for them as their mathematics lecturer – a relative expert and hence representative of the mathematics community. This targeted 'other' was chosen to see what changes, if any, they would make in their argumentation when shifting from writing a self-conviction to writing to convince an 'expert' in the field of mathematics. The data is given in Table 6.2 which shows a significant increase in the number of students who provided arguments based on algebraic reasoning – particularly those who used algebra only. There was a corresponding decrease in the number of students who continued to

Table 6.2: Modes of Reasoning used to Secure the Conviction of an Other

<b>Arithmetic</b>		<b>Algebraic</b>		
Procedural	Structural	ArithValid	Construction	Algebra-Only
11	9	13	1	49

reason arithmetically. Table 6.3 helps us to see the number of students who either changed, or kept to their initial mode of argumentation when writing to convince an other of their conclusion.

The left hand side of Table 6.3 gives the modes of argumentation students used to convince themselves while the columns represent the modes of argumentation they then used to convince an other.

Table 6.3: Transition from Self-conviction to Convincing an Other

		Arithmetic		Algebraic		
		Procedural	Structural	ArithValid	Construction	Algebra-Only
Arithmetic	Procedural	11	3	7	22	
	Structural		6		6	
Algebraic	ArithValid			5	0	3
	Construction			1	1	5
	Algebra-Only					13

For example, from the left side of the table, we see that of the 43 students whose arguments of self-conviction were classified as Arithmetic: Procedural, 11 produced a similar argument to convince an other while 3 gave a structural argument. However, 29 of the Arithmetic: Procedural students changed to algebraic reasoning with 7 giving a numerical validation of their algebra, and 22 using algebra only. Similarly, of the 12 students whose self-conviction was founded on arithmetic arguments focussed on arithmetic structure in the problem (categorised as Arithmetic: Structural), 6 gave a similar argument when convincing an other, whilst 6 switched to using algebra only. With this brief explanation of how Table 6.3 is to be read, we can see a significant shift of 35 students from arithmetic reasoning to algebraic reasoning. In the next section we will consider aspects that contributed to how the students argued to secure the conviction of an other. This will be done by considering various cases that highlight certain aspects of the modes of argumentation used by the students.

### 6.2.2 An Examination of Six Cases of Argumentation from First Year Mathematics Students

Six cases will now be presented to highlight aspects of the argumentations of first year Maths 102 students. Within five cases, a comparison will be made of the student's written argumentation of self-conviction, and that written to secure the conviction of an other.

The interpretation of their argumentations will include Pedemonte's (Pedemonte, 2007b, 2008) notion of continuity in the referential and structural systems (see Section 3.3.2). This will be used to examine links and breaks in students' argumentations of self-conviction and conviction of an other. Thus we will consider instances where students have implemented a new reasoning strategy in their conviction of an other. In all the cases presented, the primary tool of analysis will be the Tertiary Argumentation Framework (TA Framework). The sixth case highlights a particular observation that was made of some students' argumentations.

We begin with three cases that are drawn from the cohort of Maths 102 students who used one of the arithmetic modes of argumentation to develop a self-conviction, and then went on to use either the same, or a different arithmetic mode, to convince an other. Table 6.3 shows there were 20 students who did this. Case 1 typifies the way in which some of these students relied on the use of numeric examples only. Additionally this case will also illustrate a misconception that was a common occurrence in those arguments that were grounded with arithmetic. Case 2 will consider a situation where a structural consideration of the arithmetic involved in solving the problem led to deductive reasoning – and the resolution of a misconception. The third case will highlight the power of numerical examples. Pseudonyms will be used throughout this discussion. The cases that will be presented relate to Task 1: The Sum of 3 Consecutive Integers. The title given to each indicates the change in the mode of argumentation as students went from self-conviction to convincing an other.

### 6.2.3 Case 1: Arithmetic Procedural to Arithmetic Procedural

Maran is in his first of university and enrolled in a BSc. He is studying Maths 102 with a view to taking Maths 108 in his second semester. Maran's two argumentations are presented in Table 6.4.

We will first analyse the argumentation in the left hand column of Table 6.4 (Personal Conviction). This will provide a basis for determining whether there are any relevant links to his argumentation in the right hand column. Maran's argumentation is presented as a generalisation based on numerical examples. The data that he uses comes from the set of integers, and the warrant he uses is a division test to see whether the sums he has formed are multiples of 3. His backing comes from the rules of arithmetic (addition, and division). He has also used a modal qualifier to show that his conclusion is mostly true. The reason



Table 6.4: Case 1: Maran's Argumentation

Personal Conviction	Convincing an Other
<p>The statement is sometimes true. It is not always true.</p> <p>The set of integers consists of <math>t = \{\dots - 3, -2, -1, 0, 1, 2, 3, \dots\}</math></p> <p>It is true that for some of the consecutive sets of 3 numbers that if they are added together, the sum is divisible by 3. For example:</p> <p><math>1 + 2 + 3 = 6, 6/3 = 2</math>  <math>-3 + -2 + -1 = -6, -6/3 = -2</math>  <math>2 + 3 + 4 = 9, 9/3 = 3</math></p> <p>However I found one set of 3 integers that can't be divided by 3, <math>t = \{-1, 0, 1\}</math>. The sum of this set of integers is equal to 0, and 0 can't be divided by 3.</p> <p>So my conclusion is that although the sum of most of the sets of 3 consecutive integers can be divided by 3, <math>t = \{-1, 0, 1\}</math> can not.</p>	<p>The set of integers consists of <math>t = \{\dots - 3, -2, -1, 0, 1, 2, 3, \dots\}</math></p> <p>In the given set of integers, most of the time the sum of any 3 consecutive numbers can be divided by 3. This can be seen in the workings below:</p> <p><math>-3 + -2 + -1 = -6, -6/3 = -2</math>  <math>-2 + -1 + 0 = -3, -3/3 = -1</math>  <math>0 + 1 + 2 = 3, 3/3 = 1</math>  <math>1 + 2 + 3 = 6, 6/3 = 2</math>  <math>2 + 3 + 4 = 9, 9/3 = 3</math>  <math>3 + 4 + 5 = 12, 12/3 = 4</math></p> <p>There is only one set of 3 consecutive integers that can't be divided by 3, which is <math>t = \{-1, 0, 1\}</math>. This set will add up to 0, and 0 can't be divided by 3.</p>

for this is a misconception regarding zero, which he has included as his rebuttal. Thus his argumentation indicates that he is thinking and reasoning in the Symbolic World of the TWM. With regard to rationality, epistemic rationality comes from a backing based on rules of arithmetic. Communicative rationality consists of presenting a line by line set of numerical examples verifying that three consecutive integers are divisible by three, and hence multiples of three. Finally, teleological rationality is evidenced by the setting out of his personal statement at the beginning of the argumentation, followed by numerical examples leading to a possible conclusion, and an inductive step giving his conclusion.

His argumentation to convince an other (shown in the second column) is bolstered by extra examples, and thus extra data. This indicates a minor change to his warrant, which might be described as, 'the more examples – the more likely you are to become convinced'. This has the perceived affect of strengthening the qualifier 'mostly'. He has used the same empirical approach, reasoning, and layout as in his personal conviction. Hence there is continuity in his referential and structural systems, and no attempt is made to generalise using algebra. His attempt to convince an other has failed, since his argumentation does not adhere to the rules of arithmetic and is thus invalid. A common link in all argumentations that committed the same error with zero, was their reliance on division as a verification that an integer was a multiple of three. The example given in the trial in Chapter 4 was typical of those whose conception of division of zero appeared

to be linked to a physical embodiment of division. Perhaps an embodied partitive view of division may have led to Maran's misconception – but this is only speculation and would have needed further investigation. In the next case, a student (Freda) demonstrates a shift in thinking from reasoning with procedural examples, to reasoning on the procedure itself.

#### 6.2.4 Case 2: Arithmetic Structural to Arithmetic Structural

Freda is also a Maths 102 student who is in her first year of university. She is enrolled in a BCom, and is taking Maths 102 in order to enrol in Maths 108 in Semester 2. As a note for the reader, Maths 108 is a requirement for those who wish to complete a commerce degree. In this example, we see an important event unfold – the emergence of deductive reasoning. This appears to be a result of identifying an arithmetic property in the composition of 3 consecutive integers. Note that in her first argumentation, the problem with the division of zero is evident. Her argumentations are given in Table 6.5

Table 6.5: Case 2: Freda's Argumentation

Personal Conviction	Convincing an Other
$1 + 2 + 3 = 6, \frac{6}{3} = 2$ $52 + 53 + 54 = 159, \frac{159}{3} = 53$ $77 + 78 + 79 = 234, \frac{234}{3} = 78$ $96 + \underline{97} + 98 = 291, \frac{291}{3} = \underline{97}$	$96 + \underline{97} + 98 = 291, \frac{291}{3} = \underline{97}$ The multiple of 3 is always the middle number. $(96 + 1) + 97 + (98 - 1)$ $= 97 + 97 + 97$ $= 291$ $\frac{291}{3} = 97$
After testing numerous sets of 3 consecutive numbers, I have concluded that the sum of any 3 consecutive integers is usually always a multiple of 3 except when you have $-1 + 0 + 1 = 0$ which is not divisible by 0.	In any three consecutive integers, if you subtract 1 from the last number and add [it] to the first number you will always get the same number repeated 3 times. Adding the same number 3 times always gives a multiple of 3. This works for any three consecutive numbers. So it must work for 0, which it does because $3 \times 0 = 0$ .

Both of Freda's argumentations are backed by arithmetic procedures. However, her second one includes an additional backing of numerical conservation. The triplets  $\{1, 2, 3\}$ ,  $\{52, 53, 54\}$ ,  $\{77, 78, 79\}$ , and  $\{96, 97, 98\}$  are the data she uses in her personal conviction. Arithmetic examples illustrating that the derived sums are divisible by three secures her confidence in her conclusion, '...the sum of any 3 consecutive numbers is usually always a multiple of 3' – with a qualifier that this excludes the triple  $\{-1, 0, 1\}$ . The layout of her personal conviction

clearly indicates inductive reasoning based on four numerical examples – and one perceived non-example. Toward the end of her personal conviction, Freda indicates a connection regarding a pattern she has identified by underlining the middle integer, 97. This is used as the data for her convincing of an other.

In her argumentation to convince an other, the provision of numeric examples is no longer her focus. Rather,  $96 + \underline{97} + 98 = 291$ ,  $\frac{291}{3} = \underline{97}$  is used to construct an arithmetic generalisation. She does this by identifying that adding 1 to the preceding integer and subtracting 1 from the following one can replicate the central integer. Initially, this result is related to justifying the quotient. However, in her written summary she links this to repeated addition, from which she deduces a ‘multiplicative’ conclusion. The process leading to this conclusion also allows her to deduce that 0 is divisible by 3 (since it is the same as  $0 + 0 + 0 = 3 \times 0 = 0$  and hence a multiple of 3) which fixes her previous misconception.

In both argumentations, the cognitive and mathematical resources Freda uses to reason with place her in the Symbolic World of TWM. With respect to rationality, the epistemic status of her personal conviction is based on the rules of arithmetic, and inductive reasoning. However, in her second argumentation, epistemic rationality includes a generalisation based on conservation of number. Also, the division of zero is no longer a problem to the epistemological status of her argument. Communicative rationality is evident in her self-conviction by the provision of numerical examples, and the inclusion of a modal qualifier paired with a rebuttal. However, in her second argumentation communicative rationality is based on an explanation of a generalisation, first using symbols, and then using words. Her argumentation is laid out in a deductive manner. Teleological rationality is first evidenced by a focus in her first argumentation on the middle integer. This sets up three subgoals the first of which is to construct three equal numbers and link them to division, and the second, to link repetitive addition to multiplication. Her third subgoal is to show that the sum of 0 is a multiple of 3. This are achieved using deductive reasoning.

In this case study, there is continuity in the referential system used in each of the two argumentations, and a break in the structural system. This is seen in the format of the examples used in each argumentation, and in the identification of 97 in her self-conviction; which provides the basis for a generalisation in her second argumentation. The break in the structural system is seen in her switch from inductive to deductive reasoning.

Of the 43 students whose personal conviction was gained through the use of numerical

examples (arithmetic procedures), 29 of them changed to use algebraic reasoning in their argumentations to convince an other. A closer analysis reveals that these students used two different approaches in the presentation of their conclusions. In the first approach that will be examined in Case 3, students used arithmetic procedures to validate an algebraic expression. The second approach which will be examined in Case 4, was characterised by the construction of an algebraic expression and the absence of arithmetic validation. The last approach is the algebraic reasoning and symbols only. This will be examined in Case 5. Each of the three cases that will be presented typified the use of each strategy.

### 6.2.5 Case 3: Arithmetic Structural to Algebraic: Arithmetic Validation

Jacob is a first year university student taking Maths 102 as part of his BSc degree. His aim is to do Maths 150 so that he can get in to the first year of an engineering degree. Both of Jacob's argumentations are given in Table 6.6 below.

Table 6.6: Case 3: Jacob's Argumentation

Personal Conviction	Convincing an Other
$1 + 2 + 3 = 6 = 3 \times 2;$ $-4 + -3 + -2 = -9 = -3 \times 3;$ $10 + 11 + 12 = 33 = 11 \times 3;$ $-1 + 0 + 1 = 0 = 0 \times 3$ In all examples, the sum of any 3 consecutive integers is always a multiple of 3. Otherwise, in a set of 3 consecutive integers, one integer will be a multiple of 3, but the sum of the other 2 will not. But in any set 3 consecutive integers, one of the integers will be a multiple of 3, and the other 2 will always add to a multiple of 3. For example: $1, 2, 3; 1 + 2 = 3 = 3 \times 1$ $5, 6, 7; 5 + 7 = 12 = 3 \times 4$ $11, 12, 13; 11 + 13 = 24 = 3 \times 8$ Therefore the sum of any 3 consecutive integers is always a multiple of 3. This statement is true.	Let $n$ be an integer. 3 consecutive integers can be written as $n-1, n, n+1$ . Add these numbers together gives $n-1+n+n+1 = 3n$ . Therefore, no matter what integer $n$ is, $3n$ will always be a multiple of 3. For example if we substitute 5, 13, or 20 in for $n$ we get: $5 - 1 + 5 + 5 + 1$ $= 4 + 5 + 6 = 15$ $= 5 \times 3$ $13 - 1 + 13 + 13 + 1$ $= 12 + 13 + 14 = 39$ $= 13 \times 3$ $20 - 1 + 20 + 20 + 1$ $= 19 + 20 + 21 = 60$ $= 20 \times 3$ Therefore the statement is true. The sum of 3 consecutive integers is always a multiple of 3.

In both his argumentations, Jacob appears to be using reasoning and mathematical resources associated with the Symbolic World of the TWM. In his conviction of another, he selects a symbol ( $n$ ) as a variable which he uses to reason with algebraically. The argumentation for Jacob's personal conviction is grounded by numeric calculations which

are warranted by the equality of a sum with a corresponding multiple of 3. This argumentation can be divided into two sections – the first is numerical examples to derive a result, and the second is the use of a structure found in the construction of 3 consecutive integers to support a conjecture. This second section is warranted by a set of examples demonstrating his conjecture, and backed by the following rule:

... any set 3 consecutive integers, one of the integers will be a multiple of 3,  
and the other 2 will always add to a multiple of 3

In his self-conviction, epistemic rationality is demonstrated in his application of arithmetic rules, and his reference to a fact that at least one integer in a triplet of 3 consecutive integers must be a multiple of 3. Communicative rationality is achieved through the use of examples that result in a conclusion, and the use of examples to support a conjecture. Teleological rationality is seen in the achievement of his subgoal to show that at least one of the 3 consecutive integers is a multiple of 3, and the two other must always add to a multiple of 3.

In his argumentation to convince an other, Jacob uses resources from the Symbolic and Symbolic Formal Worlds of the TWM. He departs significantly from the empirical approach used in his personal conviction by beginning with algebraic data (i.e.,  $n - 1, n, n + 1$ ). He deduces from  $3n$  that  $n - 1 + n + n + 1$  will always be a multiple of 3. Up to this point, Jacob's argumentation is warranted and backed by rules of algebra along with deductive reasoning. However, something interesting now happens. He proceeds to validate the conclusion drawn from his algebraic work, by giving arithmetic examples.

Jacob's argumentation to secure the conviction of an other demonstrates a break in continuity of the referential and structural systems used in his self-conviction. The first break occurs when he replaces numerical data with algebraic data, and the second happens when he replaces inductive reasoning with algebraic and deductive reasoning. Consequently his warrant is backed by rules of algebra – until the last part of his second argumentation, which will now be discussed further.

In both Jacob's argumentations, numerical examples served similar epistemological purposes. In the first argument they served the purpose of providing the basis for his conclusion, however in the second, they served to validate his algebra. The use of arithmetic to validate algebra is the key characteristic identifying the Algebraic-Validation mode of argumentation. Using Toulmin's characterisation, Jacob's first argumentation begins with

arithmetic data, whilst the second begins with algebraic data. In his personal conviction, the arithmetic warrant (and hence backing) and qualifier are strengthened by an alternative argument based on an established pattern. However in convincing an other, the algebraic backing does not appear to be strong enough, rather, an arithmetic backing is selected. In his view, this shows that the algebra works.

From a wider analysis of students' argumentations, theoretical control appeared to be based on arithmetic, and numerical examples. For these students, algebra appeared to be built upon the 'rock-mass' of arithmetic, variables were seen as representatives of numbers, and algebraic operations linked to arithmetic. Therefore, in their view, algebra needed to be validated by showing that it worked arithmetically. For example, the following is an argumentation by a student who will be referred to by her pseudonym Polly (Table 6.7). This example illustrates a student who uses an Algebraic: Arithmetic Validation mode in both of her argumentations.

#### 6.2.6 Case 4: Algebraic: Arithmetic Validation to Algebraic: Arithmetic Validation

Polly is a first university student who is enrolled in a BSc and is hoping to major in chemistry. Polly's argumentation for a personal conviction, and conviction of an other are categorised as Algebraic: Arithmetic Validation because they both use arithmetic to validate an algebraic conclusion (Table 6.7). There is also evidence of Arithmetic Construction as she reasons toward  $3n + 3 = 3(n + 1)$ , but this appears to be a response to the subgoal of verifying that  $y = 3n + 3$  is of the form  $y = 3x$ . Her efforts in her self-conviction are directed at determining  $x$  for various input values  $n$ . A pattern emerges which provides a new subgoal – showing that for any integer  $n$  (in  $3n + 3$ ),  $x$  (in  $3x$ ) satisfies a certain structure namely  $n + 1$ , which is demonstrated arithmetically. Once the identity  $3n + 3 = 3(n + 1)$  is established (which now acts as her primary warrant), it is linked to the goal of the task itself, and she settles her original conjecture stated in the outset as a known fact. In her conviction of an other, Polly falls back on the surety of arithmetic to validate the identity she had previously established.

Thus in Polly's life-world, arithmetic is the actuating force behind algebra. Hence, although algebra is given, she provides an arithmetically validated argument to convince an other. From both Polly's arguments, it appears that she is yet unsure of the power of algebra as a method of generalisation. Her symbol sense seems to be still attached to

arithmetic. For example, she uses examples like  $3(-7) + 3 = 3 \times -6$ , and then generalises this to  $3n + 3 = 3x$ . Evidence from both argumentations suggest that she is in the Symbolic World, but also using some resources from the Formal World (e.g., using rules of algebra).

Table 6.7: Case 4: Polly's Argumentation

Personal Conviction	Convincing an Other
<p>Let <math>I = \dots - 2, -1, 0, 1, 2, 3, \dots</math>            The sum of 3 consecutive numbers is always a multiple of 3            (3,6,9,12,...)            3 consecutive numbers can be written:  <math>n + (n + 1) + (n + 2) = y</math>  <math>y = 3n + 3</math>            Substituting in some numbers for <math>n</math>: <math>n = 5</math>:  <math>3(5) + 3 = 18, y = 18, y = 3x</math>  <math>3(5) + 3 = 3 \times 6</math>  <math>n = 8</math>;  <math>3(8) + 3 = 3 \times 9</math>;  <math>n = -7</math>  <math>3(-7) + 3 = 3 \times -6</math>            Finding a connection  <math>3n + 3 = 3x</math>            When <math>n = 5, x = 6</math>            When <math>n = 8, x = 9</math>            When <math>n = -7, x = -6</math>            Hence; <math>3n + 3 = 3(n + 1)</math> Check: <math>n = 15</math>:  <math>3(15) + 3 = 48, 3(15 + 1) = 48</math>  <math>n = 100</math>;  <math>3(100) + 3 = 303, 3(100 + 1) = 303</math>  <math>n = 1</math>  <math>3(1) + 3 = 6, 3(1 + 1) = 6</math>            Therefore it is proven that for any values <math>n</math>, the equation <math>3n + 3 = 3(n + 1)</math> is satisfied. Hence it is proven that the sum of any 3 consecutive integers will always be a multiple of 3.</p>	<p>Given the formula:  <math>3n + 3 = 3(n + 1)</math> and <math>n</math> is any integer, <math>3n + 3</math> is derived from <math>n + (n + 1) + (n + 2)</math> because they are consecutive integers.            Let <math>n = -56</math>  <math>3(-56) + 3 = -165</math>  <math>3(-56 + 1) = -165</math>            Let <math>n = 28</math>  <math>3(28) + 3 = 87</math>  <math>3(28 + 1) = 87</math>            PROVEN</p>

There is clear evidence of continuity in the referential and structural systems used in both of Polly's argumentations. Her second argumentation is simply a contraction of the first, with all explanatory notes taken out. In both cases the data is algebraic and the backings are arithmetic. The difference between them lies in their conclusion and primary warrants. In the first argumentation (self-conviction), the conclusion is that any 3 consecutive integers is a multiple of 3. This is warranted by  $3n + 3 = 3(n + 1)$  and backed by arithmetic. In the second (to convince an other), her conclusion is that numerically  $3n + 3 = 3(n + 1)$ . This is warranted numerically and backed by arithmetic.

Both argumentations shared the same epistemic rationality since arithmetic validation was a common intention that was carried out. She also used her knowledge of patterning to link to her knowledge of elementary algebra. Communicative rationality differed, with the first argumentation providing more explanation to show that Polly had an understanding of the ideas she was trying to express. In the second argumentation however, these parts were erased. Teleological rationality was evidenced in the layout of both argumentations. For example, in the first one of these the layout consisted of; give the formula; show it works; then use it as the conclusion. The second argumentation had a different layout; give the conclusion; demonstrate that it works. However, Polly changed her goal in her second argumentation. In this argumentation, communicative rationality was evidenced in the goal of giving numerical evidence to support an algebraic simplification of  $3n + 3$ .

### 6.2.7 Case 5: Algebraic Construction to Algebra, Algebra–Only

For others, algebra was validated by its own rules and numerical calculations were unnecessary. A common feature of these argumentations was that when arithmetic and algebra were used, arithmetic reasoning led to an algebraic conclusion. The case that will now be presented is of a student (referred to as Jaxon) whose mode of argumentation for self-conviction was categorised as Algebraic: Construction, since he developed an algebraic approach from numerical examples. His argumentation to convince an other used algebra only, and was categorised Algebra–Only. Jaxon’s argumentation to convince an other is illustrative of those students’ whose conviction of an other were categorised as Algebra–Only. However, as we will soon see, other argumentations in this category were embellished with either an explanation of the algebra, or a consolidation of the warrant. Jaxon’s argumentations are outlined in Table 6.8.

Jaxon’s personal conviction involves the use of arithmetic and then a generalisation using algebraic symbols. These support algebraic reasoning which led to a series of deductive steps resulting in a valid conclusion. Thus, he appears to be using resources drawn from the intersection of the Symbolic and Formal Worlds of the TWM. Jaxon’s personal conviction is initially grounded by 4 sets of numerical data, which are arithmetically warranted and backed. However, he then includes a data set consisting of the variable  $x$  to represent the first integer in a consecutive set of 3 followed by  $x + 1$ , and  $x + 2$ . His treatment of  $x$  and  $x + 1$  as objects is seen in two instances. First in his statement, ‘ $x$  is any integer’, and second in his reference to  $x + 1$  as any value so that  $3(x + 1)$  is a multiple of 3. Jaxon’s



Table 6.8: Case 5: Jaxon's Argumentation

Personal Conviction	Convincing an Other
<p>I tested the statement by calculating the sum of random 3 consecutive integer sets:</p> <p><math>1 + 2 + 3 = 6</math>, a multiple of 3</p> <p><math>7 + 8 + 9 = 24</math>, a multiple of 3</p> <p><math>-3 + -2 + -1 = -6</math>, a multiple of 3</p> <p><math>99 + 100 + 101 = 300</math>, a multiple of 3</p> <p><math>x, x + 1, x + 2</math> (where <math>x</math> is any integer)</p> <p><math>x + (x + 1) + (x + 2)</math></p> <p><math>= 3x + 3</math></p> <p><math>= 3(x + 1)</math></p> <p>Therefore, 3 times any value of <math>x + 1</math> is a multiple of 3.</p> <p>Therefore, the statement "the sum of any 3 consecutive integers is always a multiple of 3" is always true.</p>	<p>Let <math>\{x, x + 1, x + 2\}</math> represent 3 consecutive integers, where <math>x</math> is any integer.</p> <p>The sum of the 3 consecutive integers is:</p> <p><math>x + (x + 1) + (x + 2)</math></p> <p><math>= 3x + 3</math></p> <p><math>= 3(x + 1)</math></p> <p>Therefore, 3 times any value of <math>x + 1</math> is a multiple of 3.</p> <p>Therefore, the statement "the sum of any 3 consecutive integers is always a multiple of 3" is always true.</p>

focus shifts from reasoning with arithmetic to reasoning with algebra, and his warrant is transformed to an algebraic process of addition and 'taking out the common factor'. Correspondingly, his backing shifts from rules of arithmetic to rules of algebra. There is no qualifier used, indicating his conviction is based on an algebraic backing.

In Jaxon's second argumentation all references to arithmetic procedures are deleted, and he uses algebra only. Additional adjustments to this argumentation include set notation to indicate a general set of 3 consecutive numbers, and an explicit statement that  $x$  is any integer. This signifies a break from arithmetic representation his referential system. However, there is continuity in algebraic representations used, and also continuity in the reasoning aspect of his structural system.

With regard to rationality, we will only consider Jaxon's argumentation to convince an other, as we have already analysed an example similar to Jaxon's first argumentation (e.g., Student AA2 in Chapter 4). In his second argumentation, epistemic rationality comes from a backing comprised of algebraic rules augmented with algebraic and deductive reasoning. Note also that the data,  $\{x, x + 1, x + 2\}$  indicates generality, and therefore provides a signpost to use algebra. Communicative rationality is revealed through logical lines of argument for example: *Let <set of integers>,... then <a sum>...therefore <conclusion>*. Teleological rationality is evidenced in the achievement of the goal of showing  $x + x + 1 + x + 2 = 3(x + 1)$ , and that  $3(x + 1)$  is a multiple of 3, therefore 'the sum of any three consecutive integers is always a multiple of 3 is true'.

Jaxon's argumentation to convince an other is typical of the 13 argumentations that used Algebra-Only in both self-conviction and conviction of an other (see Table 6.3). Thus, Jaxon's example serves to illustrate this category as well. In the next section we make a shift in our comparisons. The example of argumentation that has been chosen will highlight a feature that was included in some argumentations written to convince an other that were in the Algebraic: Algebra-Only category.

### 6.2.8 Case 6: A Need for Description and Explanation

Although not shown in the Table 6.3, 19 of the 49 students whose argumentations to convince an other were categorised as Algebraic: Algebra-Only, included unnecessary explanations. These students either described each algebraic step taken, or explained the validity of certain statements – often explaining the warrant they were using. It should be noted, that 14 of these students' personal convictions used an arithmetic approach (i.e., Arithmetic: Procedural, and Arithmetic: Structural) while 5 of them used an argumentation classified as Algebraic: Construction. The examples that are provided in Table 6.9 are from 2 of these 5 students. We shall refer to them as Kerry and Sadie. Both of these students are in their first year of university and are taking Maths 102. Kerry is a mature student who has been has not done mathematics for five years. Her secondary school mathematics finished at Year 12. Sadie on the other hand is an Arts student who did Year 13 Mathematics before coming to university.

Table 6.9: Case 6: A Comparison of Descriptions and Explanations

Kerry: Convincing an Other	Sadie: Convincing an Other
<p>Let <math>n</math> be an integer.            Then 3 consecutive numbers could be:  <math>n, n + 1, n + 2</math>            Adding these numbers would give the sum of <math>3n + 3</math>.            By factorising <math>3n + 3</math>, we get <math>3(n + 1)</math>.            There are two factors: 3 and <math>n + 1</math>:            Therefore the sum of any 3 consecutive integers is always a multiple of 3.            Another example:  <math>n - 4, n - 3, n - 2</math>  <math>\Rightarrow 3n - 9 = 3(n - 3)</math>  <math>\Rightarrow</math> We get the same conclusion that adding any 3 consecutive integers is always a multiple of 3.</p>	<p>Let <math>n</math> be an integer. Three consecutive integers could be <math>n - 1, n, n + 1</math>.            The sum of these integers is <math>3n</math>,  <math>(n - 1 + n + n + 1)</math>  <math>3n</math> is always divisible by 3 since <math>\frac{3n}{3} = n</math>, and <math>n</math> is an integer.            If the three integers were <math>n, n + 1, n + 2</math>, adding these will give a sum of <math>3n + 3 = 3(n + 1)</math>  <math>\frac{3n + 3}{3}</math> equals <math>n + 1</math>            Because <math>n + 1</math> is an integer, this sum will also always be divisible by 3.            Therefore every version of consecutive numbers will be divisible by 3 so that the sum of 3 consecutive integers is always a multiple of 3.</p>

Both of these students are operating in the intersection of the Symbolic and Formal Worlds of the TWM with both using concepts associated with rules of algebra (e.g., taking out a common factor) and arithmetic (e.g., divisibility). Kerry's argumentation includes a running commentary that describes what has been done to derive each algebraic step. The crux of her argumentation is to show that the sum of  $n$ ,  $n + 1$ , and  $n + 2$  can be decomposed into 2 factors, 3 and  $n + 1$ . This leads to her conclusion. However, she then gives a further algebraic example using  $n - 4$ ,  $n - 3$ ,  $n - 2$ . Sadie's argumentation is similar although her explanation focusses on divisibility of a multiple of 3, by 3. Sadie also gives an extra example to demonstrate that divisibility by 3 will work for another set of (generalised) numbers,  $n$ ,  $n + 1$ ,  $n + 2$ . It is clear from these two examples that both students are satisfied with their algebraic backings, but they do not appear to be satisfied with the strength of their warrants. The action that both took to strengthen their warrants indicate that they are possibly in the early stages of developing an understanding of the power of algebra, and so this extra step likely shored up their personal convictions in the ability of algebra to provide a valid conclusion. In both argumentations, rationality is attained through a display of algebraic and arithmetic knowledge (epistemic); the use of this knowledge to convey an understanding of a resulting generalisation (communicative); and fulfilling an intention to convince a reader that their conclusions are founded on deductive grounds.

### 6.2.9 Summary

This section began by showing that the use of numerical examples along with inductive reasoning (in response to Task 1) was a common strategy in students' argumentations of self-conviction. Nearly two-thirds of the 83 students used this approach, while the others used a strategy that incorporated algebra. When asked to convince an other however, there appeared a dramatic shift to the use of algebra in students' argumentations. About 76% of the students used an algebraic approach in their argumentations associated with convincing an other.

This section also analysed cases that typified certain shifts in modes of argumentation as students went from convincing oneself to convincing an other. The following features emerged:

- Students appeared to know the difference between the requirements for writing personal conviction and writing an argument to convince an other. However, there

were mixed views regarding the power of algebra to offer a valid conclusion. This is observed in examples where arithmetic was used to confirm an already valid algebraic conclusion, or extra algebraic examples were provided to show that a certain conclusion always followed;

- Students who used arithmetic backings in both their argumentations (AP to AP) appeared to be operating in the Symbolic World or intersection of the Embodied and Symbolic Worlds of the TWM. In these cases, both argumentations exhibited continuity in the arithmetic and linguistic representations used (the referential system), and the knowledge of valid arithmetic operations including inductive reasoning for numerical examples (the structural system). However, many students in this category demonstrated a misconception involving division of zero by 3;
- When students began to focus on arithmetic structure, and think about the arithmetic or numeric properties underlying it, they were more likely to find and use an arithmetic pattern, or use an identified property as a springboard to algebra and deductive reasoning;
- The 22 Students who went from Arithmetic: Procedural to Algebra-Only provided numerical examples in their self-conviction in same way Jacob did in the first half of his personal conviction (see Figure 6.6). They then gave an argument to convince an other in a way similar to that demonstrated by Jaxon (Figure 6.8). These students recognised a need for a break in referential and structural systems used in their self-conviction. It is likely that contributing to this were the focus in lectures, tutorials, and course related material on problem solving, reasoning, explaining and justifying;
- This analysis suggests that in tasks such as Task 1 that invite an algebraic interpretation, students appear to be more likely to use deductive reasoning; and
- Some students need guidance in writing concise argumentations.

The TA Framework was used to identify elements of students argumentations, and changes they made when they wrote an argumentation to convince an other. Once again it appears to offer an effective lense with which to examine students argumentation, first in terms of mathematics and mathematical processes students appear to possess, secondly how these are structured in terms of a theoretical layout, and finally, how the prose of an argument gives a rational statement.

## 6.3 The Analysis of Task 2: A Problem From Trigonometry

### 6.3.1 Introduction

The TA Framework has already been used to analyse students' argumentations using the context of elementary number theory. One of the questions that needed answering was whether it could be used to analyse other areas of mathematics as well. In this section the TA Framework will be used to analyse a second task that was presented to the Maths 102 students. This was chosen from the subject area of trigonometry and given to the students as part of an assignment that covered a trigonometry module in Maths 102 comprising 5 lectures. The second task will now be presented.

Task 2:

A Maths 102 student suggests that  $\cos^2 \theta + \sin^2 \theta = 1$  for all angles  $\theta$  from  $0^\circ$  to  $360^\circ$ .

- What do you think? Write an argument to convince yourself whether the student's suggestion is always true, sometimes true, or never true.
- Write an argument that you would use to convince a mathematics lecturer of your conclusion.

This task was set up to invite the students to form an opinion, construct a personal conviction, and then organise an argument that would convince an other. The students had a week in which to complete the task and hand it in. All students who participated in this research had to sign an ethics form indicating all work presented was their personal work and not copied from any form of media. This was also carefully explained in a lecture, and reiterated in the lecture in which the tasks were given out.

### 6.3.2 A Brief Review of the General TA Framework

A reminder of the framework that will be used in the analysis of students' argumentations is summarised by Table 6.10.

### 6.3.3 Categorising Students' Argumentations

Sixty-one students returned their scripts, and the argumentations that they used to convince themselves, analysed first. The first organisational task was to identify from the

Table 6.10: Summary of the Components of the TA Framework

Three Worlds of Mathematics	Layout of Argumentation	Rationality
A World(s) (from TWM) is/are given along with characteristics of reasoning specific to it, as well as the nature of representations used to support it. A summary of the given categorisation will be provided	The structure of a student's argumentation is summarised using Toulmin's elements of data, warrant, backing, qualifier, rebuttal, and conclusion	The rationality underlying an argumentation is summarised in terms of the epistemic, communicative, and teleological components of Habermas' rationality

scripts, common modes of argumentation. Experience from organising scripts for the analysis of Task 1 meant that these modes quickly appeared as *Procedural*, *Structural*, and *Generalising*. Within each of these, subcategories were found. The same procedure was carried out using students' argumentations aimed at convincing an other. The categories and subcategories found were consistent with those already identified. The categories of modes of argumentation used in this analysis are listed below (see also Table 6.11) along with the coding used in the sorting process.

- *Procedural: Numeric*, [PN] (Numerical calculation only is used as the basis for self-conviction)
- *Procedural: Visual*, [PV] (Numerical or algebraic reasoning is based on one visual case.)
- *Structural: Numerical*, [SN] (Trigonometric properties of angles [including special angles] around a circle are used to inform numeric examples)
- *Generalising: Visual*, [GV] (Mathematical property of a point on the unit circle is used to generalise the required result)
- *Generalising: Algebraic*, [GA] (Algebraic and deductive reasoning based on a trigonometric identity is used to generalise the required result)

Table 6.11 gives the totals of students' argumentations that were sorted into the categories described above.

The results and analysis that follow relate to Table 6.11 where we see that most students (85%) used a procedural approach in their argumentation, with 63% of these students using only numerical calculations. Visual representations were used by approximately 37% of the students. In each of the following subsections, the TA Framework will be used to

Table 6.11: Categorising Argumentation Used to Secure a Self-conviction

Modes of Argumentation	Numeric	Visual	Algebraic
Procedural	33	19	
Structural	2		
Generalising		4	3

characterise each mode of argumentation. We will begin by considering those argumentations that were classified as Procedural namely, Procedural: Numeric, and Procedural: Visual.

### 6.3.4 Procedural: Numeric

Argumentations classified as using a Procedural: Numeric mode were identified by their use of arithmetic procedures as their primary source of conviction, although there was some variation in method. In these argumentations, arithmetic data and warrants were used, which were usually backed by the computational technology of the calculator. In some cases, the backing came from a known result in trigonometry. The examples given in Figures 6.1, 6.2, and 6.3 illustrate argumentation strategies that were typical of those used by 33 of the students. The example given in Figure 6.1 will be now be analysed using the TA Framework.

$\theta$	$\cos^2 \theta$	$\sin^2 \theta$	Total	
0	1	0	1	✓
10	0.704	0.296	1	✓
-10	0.704	0.296	1	✓
180	0.358	0.6418	1	✓
360	0.0801	0.9195	1	✓

I think  $\cos^2 \theta + \sin^2 \theta = 1$  is always true

Figure 6.1: ‘Generalising’ from empirical evidence

The argumentation in Figure 6.1 illustrates a focus on gathering empirical evidence, and using inductive reasoning to lead to a conclusion. The writer of this argumentation is reasoning from the Symbolic World of the TWM.  $\theta$  is used to represent various angles,

and the symbols  $\cos^2 \theta$  and  $\sin^2 \theta$  are used to represent the squares of  $\sin \theta$  and  $\cos \theta$ . The data being used comprises different angular values for  $\theta$ , and the values for  $\cos^2 \theta$  and  $\sin^2 \theta$  calculated using a calculator (in this case set to radian mode). The conclusion is that the student thinks that  $\cos^2 \theta + \sin^2 \theta = 1$  is always true. The warrant being used here is a set of calculator computations. Thus the backing is the computational technology of the calculator. There is no evidence suggesting that the rules of trigonometry have been used to generalise any of the results provided.

Epistemic Rationality is derived from the computational technology/power of the calculator. The absolute qualifier (always true) indicates that this writer has absolute confidence in her calculator. In this example, communicative rationality comes from the tabular layout of the student's argument. There is little evidence in the argumentation that tells the reader whether the student understands the identity other than as a calculation. Thus communicative rationality also includes empirical evidence to show that the relationship works, therefore the conclusion must follow. The teleological element to rationality of this argumentation is demonstrated by the intention to show that equations in each horizontal line in the table adds to 1, which is verified by placing a tick next to it.

Other students used a different problem solving strategy. Hence their argumentations, although Procedural: Numeric, were also different. For example, in Figure 6.2 the strategy used by this student was to show that if  $\theta \neq \beta$  then  $\cos^2 \theta + \sin^2 \beta \neq 1$ . By specific choices of angles  $\theta$  and  $\beta$  the student demonstrated that the desired result can only be true when  $\theta = \beta$ .

A feature of this argumentation is inductive reasoning. The warrant used is the evaluation of  $\cos^2 \theta + \sin^2 \beta$  as  $\theta \rightarrow \beta$ , whilst the backing is the computational technology/power of the calculator. A qualifier is used to state the dependency of the conclusion on the rebuttal that the angles  $\theta$  and  $\beta$  must be the same. Thus this argumentation is differentiated from the one in Figure 6.1 by its goal. In the argument shown there, the goal was to demonstrate empirically that  $\cos^2 \theta + \sin^2 \theta = 1$ , whereas in this argumentation, the goal is to show that  $\cos^2 \theta + \sin^2 \beta = 1$  only when  $\theta = \beta$ . One other student presented his argumentation using this same strategy.

A third variation involved the use of the 'special angles' of  $30^\circ$ ,  $45^\circ$ ,  $60^\circ$ , and  $90^\circ$  to give exact values for cosine and sine. Two students used the approach shown in Figure 6.3. Their warrants did not differ from those of the other students in this category, but their backing did. Rather than relying on calculator technology, these students' arguments were backed



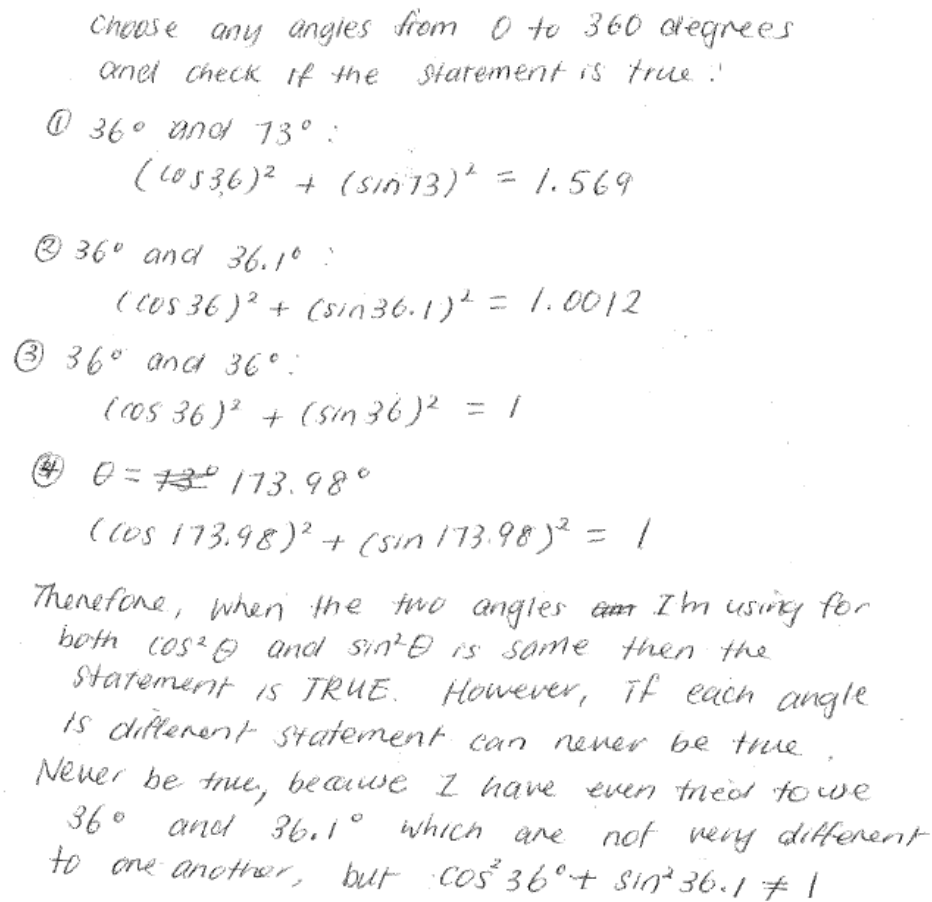


Figure 6.2: 'Generalising' using a problem solving strategy

by known trigonometric ratios for the 'special angles', and rules of arithmetic. Possession and use of this knowledge provided the epistemic rationality underlying argumentations of this type.

The Procedural: Numeric mode of argumentation was also evident in argumentations that were written to convince the lecturer. They also reflected characteristics described above. Thus in Table 6.12 we have summarised general features of argumentations in this category based on the TA Framework.

Students who used this argumentation strategy to convince their lecturer, and who only considered a few angles, or angles less than  $90^\circ$  were considered not to have answered the problem fully. Their argumentations would have failed to achieve epistemic rationality since the knowledge and understanding of the identity appears to be restricted to the interval  $0^\circ$  to  $90^\circ$ . They would also not have achieved communicative rationality because the problem is only partially addressed. Teleological rationality would likewise have failed because the goals did not include a range of angles in the interval  $90^\circ$  to  $360^\circ$ .

Using different angles:-

\* when  $\theta = 60^\circ$

$$\begin{aligned} & \sin^2(60^\circ) + \cos^2(60^\circ) \\ &= \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \\ &= \frac{3}{4} + \frac{1}{4} \\ &= 1 \quad \checkmark \end{aligned}$$

\* when  $\theta = 0^\circ$

$$\begin{aligned} & \sin^2 0^\circ + \cos^2 0^\circ \\ &= 0 + (1)^2 \\ &= 1 \quad \checkmark \end{aligned}$$

\* when  $\theta = 30^\circ$

$$\begin{aligned} & \sin^2(30^\circ) + \cos^2(30^\circ) \\ &= \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 \\ &= \frac{1}{4} + \frac{3}{4} \\ &= 1 \quad \checkmark \end{aligned}$$

\* when  $\theta = 90^\circ$

$$\begin{aligned} & \sin^2(90^\circ) + \cos^2(90^\circ) \\ &= (1)^2 + 0 \\ &= 1 \quad \checkmark \end{aligned}$$

\* when  $\theta = 45^\circ$

$$\begin{aligned} & \sin^2(45^\circ) + \cos^2(45^\circ) \\ &= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 \\ &= \frac{1}{2} + \frac{1}{2} \\ &= 1 \quad \checkmark \end{aligned}$$

$\therefore \sin^2 \theta + \cos^2 \theta = 1$  is always true.

Figure 6.3: Numerical calculations with exact sine and cosine values

The use of a procedural strategy in argumentations was not restricted to the context of numeric calculation. As shown in Table 6.11, 19 students used a visual representation in conjunction with a numeric, or algebraic procedure. These argumentations were categorised as Procedural: Visual. We will now analyse these further, and characterise them using the TA Framework.

### 6.3.5 Procedural: Visual

Two factors identified argumentations categorised as Procedural: Visual. The first of these was that data was derived from a diagram of a right angled triangle, and the second was the use of the Theorem of Pythagoras to show the conclusion that  $\cos^2 \theta + \sin^2 \theta = 1$ . The use of the term 'Procedural' in the description of this category indicates that these students used a trigonometric ratio that was followed by either an algebraic procedure or

Table 6.12: Procedural: Numeric

Three Worlds of Mathematics	Layout of Argumentation	Rationality
<p><b>Symbolic World:</b> Reasoning uses arithmetic and trigonometric symbols</p> <p>Trigonometric calculations require the use of a calculator</p> <p>Arithmetic calculations are used</p> <p>Trigonometric ratios from ‘special triangles’ are used</p>	<p><b>Data:</b> Numeric</p> <p><b>Warrant:</b> Arithmetic calculations Trigonometrical calculations using a calculator</p> <p><b>Backing:</b> Rules of arithmetic Calculator’s computational routines Properties of ‘special angles’</p> <p><b>Qualifier:</b> (if provided) Strength is based on correct calculation</p> <p><b>Rebuttal:</b> (if provided) A given range of <math>\theta</math>, or <math>\theta</math> must equal <math>\beta</math>.1cm</p> <p><b>Conclusion:</b> <math>\sin^2 \theta + \cos^2 \theta = 1</math></p>	<p><b>Epistemic Rationality:</b> Justification is grounded in rules of arithmetic operations, numerical rules of rounding, faith in the calculator to compute correctly, or the use of trigonometric ratios for ‘special angles’</p> <p><b>Communicative Rationality:</b> Text uses a combination of ordinary language and standard arithmetic and trigonometric language, including relevant symbols Layout of argumentation provides numerical cases that focus on the conclusion, for example the use of a tabular representation</p> <p><b>Teleological Rationality:</b> Empirical example(s) are constructed to achieve the intended goal Text is illustrative of inductive reasoning</p>

a numerical one. The first part of this analysis will consider argumentations that used a right angled triangle of the types shown in Figure 6.4 as their primary source of data. Of the 19 students whose argumentations were categorised Procedural: Visual, 14 used one of these triangles. An example is given in Figure 6.5.

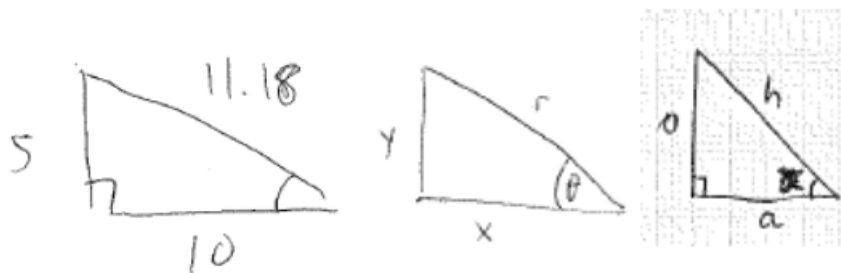


Figure 6.4: Examples of right angled triangles used in argumentations

Students who provided argumentations like this were thinking and reasoning in the intersection of the Embodied and Symbolic Worlds of the TWM. A diagram of a right angled triangle embodied sine and cosine as well as the Pythagorean Theorem. The data in this example of argumentation consists of trigonometric ratios associated with the right angled triangle, for example,  $\sin \theta = \frac{y}{r}$  and  $\cos \theta = \frac{x}{r}$  so that  $y = r \sin \theta$ , and  $x = r \cos \theta$ . The warrant comes from trigonometric substitution into the Pythagorean relationship  $r^2 = x^2 + y^2$ ,

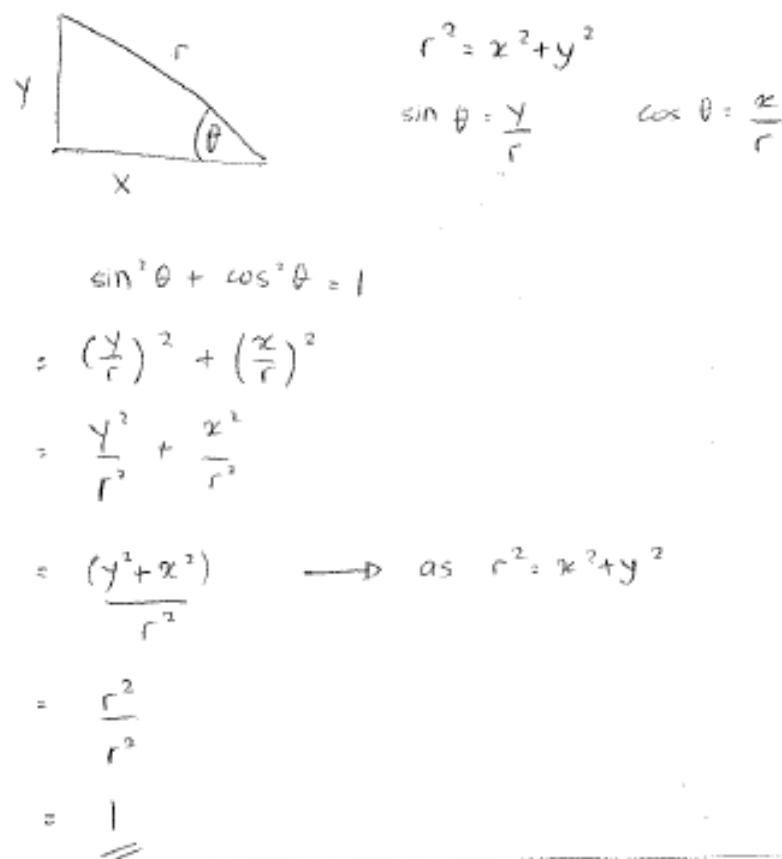


Figure 6.5: Using a right angled triangle to ‘generalise’  $\cos^2 \theta + \sin^2 \theta = 1$  for all  $\theta$ :  $90^\circ \leq \theta \leq 360^\circ$

and algebraic manipulation and simplification. The backing consists of known results in trigonometry and algebra, and the Theorem of Pythagoras.

Epistemic rationality comes from an application of trigonometric ratios and the Theorem of Pythagoras. Communicative rationality is established by showing a link between the Theorem of Pythagoras and  $\cos^2 \theta + \sin^2 \theta = 1$ , and teleological rationality comes from the step by step algebraic reasoning used to derive the result.

Five students in the Procedural: Visual category used a right angled triangle with either a circle of radius  $r$  or a unit circle ( $r = 1$ ). These students either labelled the two ‘legs’ of the triangle  $x$  and  $y$ , or  $r \cos \theta$  and  $r \sin \theta$  (for a base angle  $\theta$ ). In the latter case the students converted the trigonometric expressions to ratios. The same algebraic procedures as set out in the previous discussion were followed. None of these students considered a point on the unit circle, or explained how  $x^2 + y^2 = r^2$  ( $x^2 + y^2 = 1$ ) related to the circle. These examples also illustrated how the use of a right angled triangle inhibited a generalisation to all angles. One student stated as a rebuttal that her method ‘only works for right angled triangles’. Table 6.13 summarises general features of argumentations using

a Procedural: Visual used in students' self-convictions and conviction of an other based on the TA Framework.

Table 6.13: Procedural: Visual

Three Worlds of Mathematics	Layout of Argumentation	Rationality
<p><b>Intersection of the Embodied and Symbolic Worlds:</b> Reasoning uses a representation of a right angled triangle Arithmetic, or algebraic procedures are used</p>	<p><b>Data:</b> A right angled triangle from which numeric values, or trigonometric ratios for <math>\sin \theta</math> and <math>\cos \theta</math> are derived <b>Warrant:</b> Trig. ratios for sine and cosine; Theorem of Pythagoras <b>Backing:</b> Rules of arithmetic or... rules of trigonometry and algebra A theorem (Pythagoras) <b>Qualifier:</b> (If provided) Conclusion is sometimes true <b>Rebuttal:</b> (If provided) Result is true for angles from <math>0^\circ</math> to <math>90^\circ</math> <b>Conclusion:</b> <math>\sin^2 \theta + \cos^2 \theta = 1</math> always true</p>	<p><b>Epistemic Rationality:</b> Knowledge of arithmetic rules, algebra, trigonometry, and the Theorem of Pythagoras evident in text <b>Communicative Rationality:</b> Evidence suggest that knowledge of arithmetic, algebra, and trigonometry is applied to produce a reasoned argument using suitable diagram(s) and effective notation <b>Teleological Rationality:</b> Subsidiary goals (usually establishing trig. ratios and using Pythagoras' Theorem) are indicated and achieved. These are linked to the conclusion Deductive reasoning may be present</p>

In some cases, the use of a visual representation was implied. This was the case with two students who used numerical examples that included values of  $\theta$  that ranged through the interval  $0^\circ$  to  $360^\circ$ . An aspect of their argumentations was that they demonstrated a structural knowledge of the situation – that is, they had knowledge of representations and their trigonometric properties. They also demonstrated that they understood how to use the representations. Students who gave argumentations that reflected these attributes were categorised as Structural: Numeric.

### 6.3.6 Structural: Numeric

Students whose argumentation belonged to this category combined properties of the 'special right angled triangles' to compute  $\cos^2 \theta + \sin^2 \theta = 1$  for specific  $\theta$  in each of the four quadrants of the unit circle. This mode of argumentation featured numerical examples only. The two students in this category reasoned on the structure of angles measures, their sine and cosine and their relationship to the appropriate ratios developed from the special triangles. Angles were strategically chosen so that arithmetic ratios could be used to calculate their sine and cosine. An inductive generalisation of the conclusion for angles in the range from  $0^\circ$  to  $360^\circ$  followed. One of their argumentations is shown in Figure 6.6.

$$\begin{aligned}
 \text{when } \theta = 0: & \cos \theta = 1, \sin \theta = 0 \Rightarrow \sin^2 \theta + \cos^2 \theta = 1 \\
 \text{when } \theta = 45^\circ & \cos 45 = \frac{\sqrt{2}}{2}, \sin 45 = \frac{\sqrt{2}}{2} \Rightarrow \sin^2 \theta + \cos^2 \theta = 1 \\
 \text{when } \theta = 90^\circ & \sin 90 = 1, \cos 90 = 0 \Rightarrow \sin^2 \theta + \cos^2 \theta = 1 \\
 \text{when } \theta = 135^\circ & \sin 135^\circ = \sin (180 - 135) = \sin 45 = \frac{\sqrt{2}}{2} \\
 & \cos 135^\circ = -\cos (180 - 135) = -\cos 45 = -\frac{\sqrt{2}}{2} \\
 & \Rightarrow \sin^2 \theta + \cos^2 \theta = 1 \\
 \text{when } \theta = 180^\circ & \cos \theta = -1, \sin \theta = 0 \Rightarrow \sin^2 \theta + \cos^2 \theta = 1 \\
 \text{when } \theta = 225^\circ & \sin 225^\circ = \sin (225 - 180) = \sin 45 = \frac{\sqrt{2}}{2} \\
 & \cos 225^\circ = -\cos (225 - 180) = -\cos 45 = -\frac{\sqrt{2}}{2} \\
 & \Rightarrow \sin^2 \theta + \cos^2 \theta = 1 \\
 \text{when } \theta = 270^\circ & \sin \theta = -1, \cos \theta = 0 \Rightarrow \sin^2 \theta + \cos^2 \theta = 1 \\
 \text{when } \theta = 315^\circ & \sin 315 = -\sin (360 - 315) = -\sin 45 = -\frac{\sqrt{2}}{2} \\
 & \cos 315 = \cos (360 - 315) = \cos 45 = \frac{\sqrt{2}}{2} \\
 & \Rightarrow \sin^2 \theta + \cos^2 \theta = 1 \\
 \text{when } \theta = 360^\circ & \cos \theta = 1, \sin \theta = 0 \Rightarrow \sin^2 \theta + \cos^2 \theta = 1
 \end{aligned}$$

Figure 6.6: Using angle relationships in computation

A consideration of this argumentation suggests mathematical and reasoning resources are drawn from the Symbolic World of the TWM. The data used are specific choices of angles, and the conclusion provided is that on the basis of calculations,  $\sin^2 \theta + \cos^2 \theta = 1$  (for all  $0^\circ \leq \theta \leq 360^\circ$ ) is always true. The warrant is composed of the sine and cosine for special angles expressed as exact ratios, trigonometric properties of the unit circle, and computation using Pythagoras' Theorem. Backing the warrant is the Theorem of Pythagoras, and the angle properties of the 'special right angled triangles'.

Epistemic rationality is based on a knowledge of trigonometric ratios from the special triangles, and the relationship between sine and cosine in each of the four quadrants of their graphs (or unit circle).

Communicative rationality is evidenced by the correct use of trigonometric symbols, and correct calculations. The writer is clearly aware that he is writing to secure a conviction. The qualifier, ‘Therefore according to my calculations’, shows this. The argumentation is visually set out to show that calculations lead to the required generalisation in the original problem.

Teleological rationality is established through the intentional selection of data, and a series of successive numerical computations. There were no students who used this approach in writing an argumentation to convince a lecturer, hence a summary is not provided.

Some students’ argumentations considered an embodiment of a right angled triangle oriented in all quadrants of the unit circle. These visual representations supported trigonometric and algebraic reasoning. Students’ argumentation that reflected these attributes were classified as using a *Generalising: Visual* mode of argumentation, which will now be described.

### 6.3.7 Generalising: Visual

Four students considered the right angled triangle as it would appear in the four quadrants of the unit circle. In none of the argumentations, did the students use a point  $P(x, y)$  or  $P(\cos \theta, \sin \theta)$  on the unit circle. Rather, they used a blend of the Cartesian coordinate system and the unit circle, and considered the sign changes of  $x$ , and  $y$  ‘legs’ of a right angled triangle as it moved through each quadrant. A typical example of argumentation in this category is presented in Figure 6.7. An analysis using the TA Framework begins by noting that reasoning and mathematical resources used in this argumentation are consistent with those that come from an intersection of the Embodied and Symbolic Worlds of the TWM. The data is embedded in four right angled triangles in each of the quadrants of the unit circle, and consists of angles  $\theta$ , signed  $x$  and  $y$  ‘values’, and the trigonometric expressions  $x = \cos \theta$ , and  $y = \sin \theta$ . The warrant is the identity  $x^2 + y^2 = 1$ , which can be derived using the Theorem of Pythagoras, and a substitution of  $x$  and  $y$  for  $\cos \theta$  and  $\sin \theta$ . The backing includes properties of trigonometry and the unit circle, and the Theorem of Pythagoras.

Epistemic rationality is established by integrating the unit circle and the Theorem of Pythagoras to give rotations of  $\theta$  that give rise to reference angles in each of the four quadrants.

Communicative rationality is accomplished through the provision of diagrams that visually

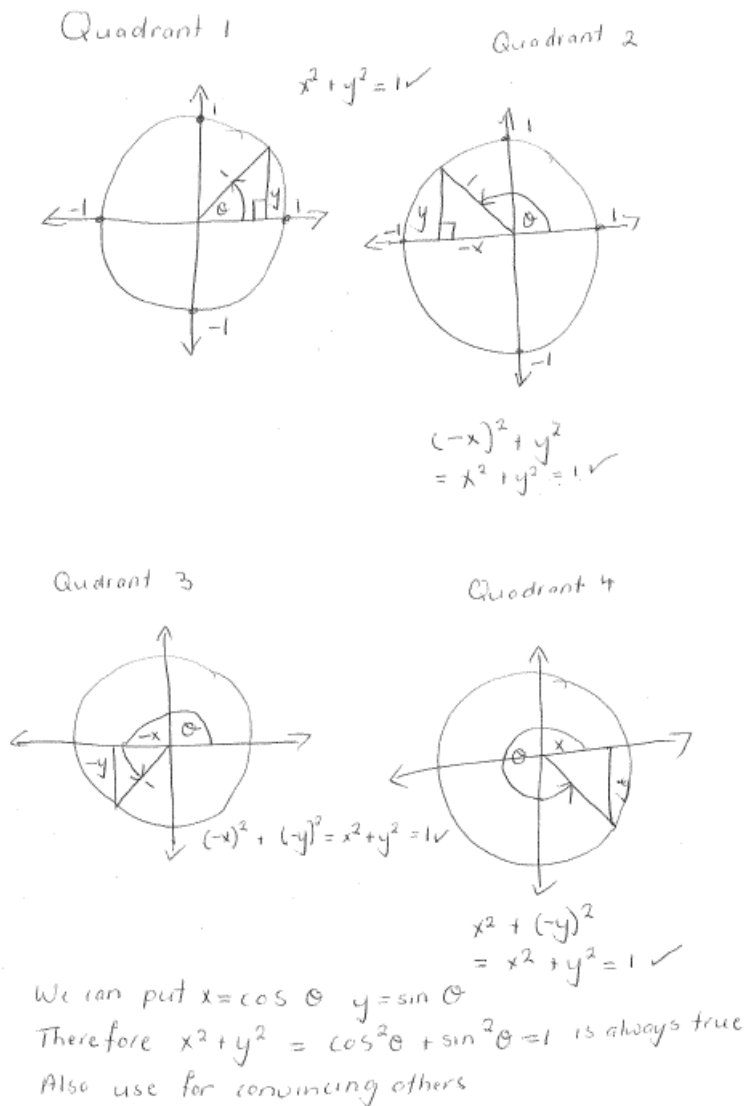


Figure 6.7: Generalising from four cases

support algebraic thinking. Alongside each diagram is an algebraic computation verifying the required identity. Teleological rationality is achieved by showing that  $x^2 + y^2 = 1$  is true in each quadrant, and therefore  $\cos^2 \theta + \sin^2 \theta = 1$  is true for all angles  $0^\circ \leq \theta \leq 360^\circ$ . Features of argumentations that were categorised as Generalising: Visual are provided in Table 6.14.

Another approach that was evidenced in students' argumentations relied on algebraic reasoning without the use of visual representations or numerical examples. These students' argumentations were classified as Generalising: Algebraic, and will be discussed in the following subsection.



Table 6.14: Generalising: Visual

Three Worlds of Mathematics	Layout of Argumentation	Rationality
<p><b>Intersection of the Embodied and Symbolic World:</b> Visual representations along with algebraic, and trigonometric symbols support trigonometric and algebraic reasoning</p> <p>Knowledge from mathematics is used such as <math>x^2 + y^2 = 1</math>, and <math>x = \cos \theta</math>, <math>y = \sin \theta</math></p>	<p><b>Data:</b> Data comes from a circle of radius <math>r</math> (or a unit circle when <math>r = 1</math>), and right angled triangles. The data includes; <math>\theta</math> (taken from the positive <math>x</math> axis), signed <math>x</math> and <math>y</math> 'values', and the trigonometric expressions <math>x = \cos \theta</math>, and <math>y = \sin \theta</math></p> <p><b>Warrant:</b> Formation of a reference angle and the Theorem of Pythagoras to give <math>x^2 + y^2 = 1</math> for each triangle</p> <p><b>Backing:</b> Theorem of Pythagoras Properties of the unit circle</p> <p><b>Conclusion:</b> <math>\sin^2 \theta + \cos^2 \theta = 1</math>, true for <math>0^\circ \leq \theta \leq 360^\circ</math></p>	<p><b>Epistemic Rationality:</b> Justifications are backed by mathematical theory. Generalisability (or potential generalisability) is linked to embodied, algebraic and trigonometric reasoning</p> <p><b>Communicative Rationality:</b> Text uses a combination of ordinary language, diagrams and algebraic and trigonometric symbols. These are managed in a way that leads the argumentation to its conclusion</p> <p><b>Teleological Rationality:</b> The intention is to show the identity is true in each quadrant of the unit circle Text is illustrative of deductive reasoning</p>

### 6.3.8 Generalising: Algebraic

Students whose argumentations were categorised Generalising: Algebraic used algebra to reason from a trigonometric identity to the conclusion that  $\sin^2 \theta + \cos^2 \theta = 1$  for all angles from  $0^\circ$  to  $360^\circ$ . Three students provided argumentations that used this approach – using them as both their self-conviction and conviction of an other. All three students used the identity,  $\sin \theta \sin \beta + \sin \beta \cos \theta = \cos(\theta - \beta)$ , with  $\beta = \theta$ . The argumentation provided in Figure 6.8 is representative of those used by these students. It is clear that the goal is to use algebra to get the '1' on right hand side of the equation  $\sin^2 \theta + \cos^2 \theta = 1$ .

The trigonometric identity that is chosen is used in the same manner as a definition. Hence, reasoning and mathematical resources used in this argumentation are drawn from an intersection of the Symbolic and Formal Worlds of TWM. The data grounding this student's argumentation is the statement,  $\sin^2 \theta + \cos^2 \theta = \sin \theta \sin \theta + \cos \theta \cos \theta$  and the conclusion is  $\sin \theta \sin \theta + \cos \theta \cos \theta = 1$ . The warrant used in this argumentation is  $\cos(\theta \pm \beta) = \cos \theta \cos \beta \mp \sin \theta \sin \beta$  with  $\theta = \beta$ . This is backed by the status of the compound angle formula.

Epistemic rationality is derived from a knowledge and application of the compound angle formula.

⊙ we can use compound angle <sup>formula</sup> to prove it.  
 Assume that  $\theta$  represent any angle from  $0^\circ$  to  $360^\circ$ .  
 The  $\sin^2 \theta + \cos^2 \theta = (\sin \theta)^2 + (\cos \theta)^2$   
 $= \sin \theta \cdot \sin \theta + \cos \theta \cdot \cos \theta$ .  
 According to cosine compound angle formula  $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$   
 $\therefore \sin \theta \cdot \sin \theta + \cos \theta \cdot \cos \theta$   
 $= \cos(\theta - \theta) = \cos 0 = 1$ .

Figure 6.8: Using a trigonometric identity

The deductive steps in the student's written argumentation in Figure 6.8 clearly show her reasoning, giving rise to communicative rationality.

Teleological rationality can be found in the strategy used by the student to link  $\sin^2 \theta + \cos^2 \theta$  to  $\cos(\theta - \theta)$ . Through this link, she achieves her intention of establishing a '1' on the right hand side of the equation,  $\sin^2 \theta + \cos^2 \theta = 1$ .

A summary of the general characteristics of a Generalising: Algebraic approach to argumentation as demonstrated by students in this research is given in Table 6.15.

Table 6.15: Generalising: Algebraic

Three Worlds of Mathematics	Layout of Argumentation	Rationality
<p><b>Intersection of the Symbolic and Formal Worlds:</b></p> <p>Trigonometric symbols used support algebraic and trigonometric reasoning</p> <p>The intention of the argumentation is to provide an algebraic proof</p> <p>A trigonometric identity is used</p>	<p><b>Data:</b> An angle <math>\theta</math> such that <math>0^\circ \leq \theta \leq 360^\circ</math>  <math>\sin^2 \theta + \cos^2 \theta = \sin \theta \sin \theta + \cos \theta \cos \theta</math></p> <p><b>Warrant:</b> Trigonometric identity</p> <p><b>Backing:</b> Theoretical status of the trigonometric identity            Rules of algebra</p> <p><b>Conclusion:</b> <math>\sin^2 \theta + \cos^2 \theta = 1</math>, true for <math>0^\circ \leq \theta \leq 360^\circ</math></p>	<p><b>Epistemic Rationality:</b> Truth and validity is grounded in the mathematical status of trigonometric identity and correct algebraic processes</p> <p><b>Communicative Rationality:</b> Reasoning is clearly communicated            Trigonometric symbols are correctly used to support reasoning</p> <p><b>Teleological Rationality:</b> Goals are established and achieved            Text is illustrative of deductive reasoning</p>

### 6.3.9 Summary

The analysis of Task 2 has shown that most of this cohort of students drawn from Maths 102 relied on procedural means to develop a self-conviction. Following a procedure using numerical examples meant putting various values for  $\theta$  into a calculator and letting the calculator confirm that  $\sin^2 \theta + \cos^2 \theta = 1$ . There were some variations that resulted from the use of a particular problem solving strategy. A Visual–Procedural approach involved a diagram (usually of a right angled triangle) from which trigonometric ratios, or lengths of sides were derived, and the procedure of the Theorem of Pythagoras used to show that  $\sin^2 \theta + \cos^2 \theta = 1$ . Variations in this category were due to some using their knowledge of ‘special triangles’. A key point to arise from this analysis was the basis on which students secured their conviction, with twenty (Procedural mode) students basing their conviction on angles less than  $90^\circ$ . The next section will consider how these first year students construction argumentations to convince an other (a lecturer of their conclusion).

## 6.4 How Students' in Task 2 Convinced An Other of Their Conclusion

From the analysis in the previous section, 52 of the 61 students who participated in this research used procedural methods involving numeric examples and algebra to convince themselves that  $\sin^2 \theta + \cos^2 \theta = 1$  was true for all  $\theta$  from  $0^\circ$  to  $360^\circ$ . The high number of students that used numerical examples as a significant part of their argumentations of self-conviction is consistent with studies by Coe and Ruthven (1994), Harel and Sowder (1998) and Healy and Hoyles (2000), that showed students preferred empirical arguments when convincing themselves of the truth of a statement.

An important point made by de Villiers (1990, 1999) is that students may not go beyond providing procedural arguments for a statement or point of view; they do not see the point of doing so because the result is already obvious. On the other hand, some students may not have a disposition for certainty, causality, algebraic computation, or logical structure (see also Zaslavsky et al., 2012). However, the Maths 102 students who participated in this research were involved in a course that emphasised justification and explanation using mathematical reasoning. A substantial part of each of their three assignments asked for mathematical justifications to be given, and lectures included a comparison of intuitive and empirical justifications with those that used deductive reasoning.

In the following subsections we will consider cases that illustrate how students changed their argumentations when asked to convince their lecturer. Please note that in most cases, a TA Framework analysis of the various modes of argumentation have already been done in the previous section. Table 6.16 shows a comparison of the modes of argumentation that the students used to convince themselves against those used to convince a lecturer. The rows represent the modes of argumentation used to convince themselves, and the columns give the modes used to convince a lecturer. Some interpretations of the table will now be presented.

Table 6.16: Transition from Self-conviction to Convincing an Other (a Lecturer)

		Procedural		Structural	Generalising	
		Numeric	Visual	Numeric	Visual	Algebraic
Procedural	Numeric	5	22		3	3
	Visual		15		3	1
Structural	Numeric				2	
Generalising	Visual				4	
	Algebraic					3

#### 6.4.1 From Procedural: Numeric

Of the 52 students who used a Procedural mode to convince themselves, 42 continued to use this same mode to convince a lecturer, with 10 changing to the mode of Generalising. Looking at this result more closely reveals a large shift in the 33 students who convinced themselves using a Procedural: Numeric mode in their argumentations. Only 5 continued to use this mode to convince a lecturer. Three of these students used the same argument they used to convince themselves, and 2 added more examples to verify the conclusion. The majority of students who used this mode in their self-conviction shifted to a Procedural: Visual mode. These students used the trigonometric properties of a right angled triangle in conjunction with the Theorem of Pythagoras in their arguments to convince a lecturer. Three students engaged with a Generalising mode and generalised the conclusion using a

trigonometric identity. However, their argumentations turned out to have a fundamental flaw. This will be discussed in detail in the next section.

### To Procedural: Visual

Twenty-two students changed from a Procedural: Numeric to Procedural: Visual mode of argumentation to convince a lecturer. Changes in these students' argumentations were initiated by the replacement of numerical with symbolic data, which came from a diagram. This signalled a change from using an empirical warrant to using a trigonometric and algebraic one. Consequently, the backing changed from rules of arithmetic, to rules of trigonometry, and the Theorem of Pythagoras.

Four of the 22 students used a diagram of a circle in conjunction with a right angled triangle, with 3 of these students using a circle of radius  $r$ , and the other a unit circle. However, the primary focus of these three students was the geometric and trigonometric properties of the triangle. Consistent with the findings in Section 6.3.5 they did not make a link to either a point on the circle. A link to the equation of a circle was tentative, but not made explicit (Figure 6.9)

Keeping in mind that  $\sin \theta = y/r$  and  $\cos \theta = x/r$ , and

$$r^2 = x^2 + y^2.$$

By substitution,  $\sin^2 \theta = \frac{y^2}{r^2}$  and  $\cos^2 \theta = \frac{x^2}{r^2}$

So the left hand side of the identity becomes

$$\frac{y^2}{r^2} + \frac{x^2}{r^2}$$

Since these already have a common denominator, we can write them as one fraction:

$$\frac{(y^2 + x^2)}{r^2} = \frac{r^2}{r^2} = 1$$

$$\therefore \sin^2 \theta + \cos^2 \theta = 1.$$



Figure 6.9: Using the circle: Focus on the triangle

The other 18 students used a right angled triangle without a circle (or unit circle) and likewise primarily focussed on its trigonometric and geometric properties. All 22 students drew the same conclusion that  $\sin^2 \theta + \cos^2 \theta = 1$  was true for all  $\theta$  from  $0^\circ$  to  $360^\circ$ . Of note was that 19 of these students gave numerical examples that ranged throughout the interval from which  $\theta$  could be chosen in their self-conviction, but did not see the constraint that

the right angled triangle placed on the choices of angle. Examples of argumentations such as these have already been examined in Section 6.3.5.

### To Generalising: Visual

Three students changed from Procedural: Numeric mode to Generalising: Visual. These students used a visual representation that included the unit circle and a right angled triangle. An important feature of these argumentations that made them distinct from those already discussed in Section 6.3.7 was that students explicitly connected the equation  $x^2 + y^2 = r^2$  to the general equation of a circle before using the right angled triangle to derive the trigonometric ratios for sine and cosine. The lines of one student's written argumentation are shown in Table 6.17. These lines were similar to the other two students' argumentations with minor differences – mainly in descriptive language.

Table 6.17: Part of a Student's Argumentation: Convincing a Lecturer

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L1: We know that  $r^2 = y^2 + x^2$  is the equation of a circle. Each  $x$  and  $y$  lies on the circle

L2: Then  $\frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{r^2}{r^2} = 1$

L3: If we substitute into the equation  $\sin \theta = \frac{y}{r}$ , and  $\cos \theta = \frac{x}{r}$  we get  
 $\sin^2 \theta + \cos^2 \theta = 1$

---

This argumentation (and others that followed the same format) is warranted by the equation  $x^2 + y^2 = r^2$  and backed by the knowledge that  $x^2 + y^2 = r^2$  describes the equation of a circle of radius  $r$ .

The shift from Procedural: Numeric to Generalising: Visual involved the use of algebraic/geometric symbols/diagrams to support algebraic, visual, and deductive reasoning. This required a break in the continuity of students' referential and structural systems. We believe this was triggered by the change in audience, and the pedagogical style of using diagrams along with the algebraic techniques during the trigonometry module covered in lectures.

### To Generalising: Algebraic

Three students' argumentations aimed at convincing a lecturer were placed in this category.

All three students grounded their argumentations using the identity,

$$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

and were able to set up the required result of  $\sin^2\theta + \cos^2\theta = 1$ . From the student's point of view they had provided a valid and rational argument. But they were unsuccessful in providing an argument that would convince a lecturer. A key reason for this was the identity they used was based on the assumption that  $\sin^2\theta + \cos^2\theta = 1$ . Hence, a circular argument was given. An example is provided in Figure 6.11.

Assume that  $\theta$  represent any angle from  $0^\circ$  to  $360^\circ$ .

$$\therefore \cos 2A = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\therefore \cos^2\theta = \frac{\cos 2A + 1}{2} = \frac{\cos 2A}{2} + \frac{1}{2}$$

$$\sin^2\theta = \frac{1 - \cos 2A}{2} = \frac{1}{2} - \frac{\cos 2A}{2}$$

$$\therefore \cos^2\theta + \sin^2\theta = \left(\frac{\cos 2A}{2} + \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{\cos 2A}{2}\right)$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

Figure 6.10: Generalising – Algebra: An example of a circular argument

In these three cases, the students were unable to establish: epistemic rationality, since the choice of identity was flawed; communicative rationality, because of the circularity of the argument; and teleological rationality, because the goal of  $\sin^2\theta + \cos^2\theta = 1$  defaults the argument to a circular one.

#### 6.4.2 From Procedural: Visual

Nineteen students used a Procedural: Visual mode in writing a statement of self-conviction. Fifteen continued to use this mode to convince a lecturer while four switched to the Generalising mode. Three of these students provided a visually based generalisation of the required result, and one generalised the result using a trigonometric identity.

##### To Generalising: Visual

Three students were placed in this category. Two students generalised using a point  $(x, y) = (\cos\theta, \sin\theta)$  on the unit circle for a given angle  $\theta$ . The general formula for a circle was used (i.e.,  $x^2 + y^2 = 1$ ) and  $\sin\theta$  and  $\cos\theta$  substituted in for  $x$  and  $y$  to give

$\sin^2 \theta + \cos^2 \theta = 1$  for all angles from  $0^\circ$  to  $360^\circ$ . One student drew triangles in each quadrant with a common vertex on the origin, and used reference angles to confirm that for each triangle  $\sin^2 \theta + \cos^2 \theta = 1$ .

### **To Generalising: Algebraic**

The student in this category used the trigonometric identity,

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

for all angles from  $0^\circ$  to  $360^\circ$ , and then considered the situation when  $A = B$ . She correctly generalised the result  $\sin^2 \theta + \cos^2 \theta = 1$ . This student developed an approach that did not rely on a visual embodiment and the procedural use of the Pythagorean Theorem. There was no evidence of continuity in the referential and structural systems of her self-conviction and her argumentation to convince an other. She had gone from visual embodiment that did not generalise, to an algebraic expression that did. Validity was derived through reasoning deductively from a correctly chosen trigonometric identity, and a theoretically based backing (an identity from trigonometry).

### **6.4.3 From Structural: Numeric**

Argumentations that were categorised into the Structural: Numeric mode of argumentation were characterised by the use of reference angles to find the sine and cosine of angles greater than  $90^\circ$ .

Two students changed from a Structural: Numeric to Generalising: Visual. Both of these students used right angled triangles in each of the four quadrants of the Cartesian plane with a common vertex at the origin. Generalised reference angles were used to verify  $\sin^2 \theta + \cos^2 \theta = 1$  for all right angled triangles.

### **6.4.4 Summary**

Most of the students who were convinced that  $\sin^2 \theta + \cos^2 \theta = 1$  for all angles from  $0^\circ$  to  $360^\circ$  by checking numerical examples, changed their mode of argumentation when asked to convince a lecturer of the result. The new modes of argumentation involved data that consisted of symbols associated with trigonometry, along with an embodiment sine and cosine, and the use of the Theorem of Pythagoras. Algebra featured as the main method through which students attempted to convince the lecturer of their conclusion.



This result resonates with the research of Healy and Hoyles (2000) who also found that students in their study preferred empirical argumentations when focussed on securing a self-conviction, but preferred to give algebraic ones when attempting to convince their teacher.

However, in this study algebraic reasoning often consisted of following algebraic procedures. For example, those who used a Procedural: Visual mode in their argumentation to convince a lecturer did not show or give any explanation of how the algebra they derived from their representation, generalised to all angles  $\theta$  (The common representation used was a right angled triangle). Nor did those who included a circle of radius  $r$  or 1 relate  $x^2 + y^2 = r^2$ , or  $x^2 + y^2 = 1$  to the equation of a circle centred at the origin. It appeared the students followed a procedure using algebra without fully understanding the generalisations that emerged. A possible explanation may be that these students did not recognise that showing the generalisability of  $\theta$  was an important part of the problem. Another explanation may be that the students had only met trigonometry in terms of the right angle triangle in secondary school, and not had enough experience with the unit circle in Maths 102.

A similar situation arose with three students who used algebraic techniques correctly but gave a circular proof. Although this appeared to stem from the use of a ‘bad’ trigonometric identity, the root of the problem was that it was a faulty choice. A possible reason for this is discussed by Harel and Sowder (1998), whose conceptual framework of proof schemes suggest that these students’ argumentations may have been influenced by appearance and form, and not by ensuring the validity of the statements they were making.

## 6.5 Argumentations of Second Year Mathematics Students

In this section we will examine the argumentation of students who had studied mathematics at university for three semesters and who at the time of this research were studying Maths 255, a second year course in mathematics. This course introduces prospective mathematics majors to a formal study of mathematics in which proof is a significant feature. Students are exposed to techniques of proving which include, direct proof, proof by contradiction, and proof by contraposition. One reason for examining the argumentation of these students, is to find out how students who have had more experience in mathematics go about structuring their mathematical arguments. A class of Maths 255 students were invited to take part in this research of which twelve responded positively. These students

were given a task to complete over a period of a week. The task was taken from elementary number theory. An analysis of students' argumentations will be done using selected cases. The task that was given to the students will now be presented.

Task: Multiples of 2 and 3 in Three Consecutive Integers

A student makes a claim that in *any* set of three consecutive integers, either one of the integers is a multiple of 2 and another a multiple of 3, or one integer is a multiple of 2 and 3.

- What do you think? Write an argument to convince yourself whether this claim is always true, sometimes true, or never true.
- Write an argument that you would use to convince a lecturer of your conclusion.

The cases that provide the basis for our analysis will typify certain approaches taken by students in their argumentations. A consideration of how students secured a self-conviction will be considered, and compared with the argumentation they used to convince a lecturer. Unlike the analyses of the argumentations of first year mathematics students, it was expected that Maths 255 students' self-convictions might show more aspects of generalising and algebraic reasoning, and hence more of a deductive approach. Although this proved to be the case for some, others used numerically based evidence to convince themselves that the claim in the task was either true, false, or otherwise. Members of this small group of students will be referred to by their pseudonyms. This first case is of an argumentation belonging to a student called Tom.

### 6.5.1 Case 1: From Numerical Examples

In his first year, Tom took Maths 150, a first year course covering calculus and linear algebra. This course is designed for those who have very good mathematics skills and who wish to progress to second and third year courses in mathematics or go on to courses in engineering. In the second semester of his first year, Tom took Maths 250 and Maths 253 which are advancing mathematics courses that give a theoretical treatment of multivariable calculus and linear algebra. At the time of this study Tom was in his third semester taking Maths 255.

Tom's argumentation to secure a personal conviction is given in Table 6.18. We have inserted line identifiers to make it easier to reference parts of Tom's argument. You will see these as L1, L2 etc...We will refer to them in text as Line 1, and Line 2. We have also inserted in square brackets, words that make his argument more fluent for the reader.

Table 6.18: Tom's Argumentation

<b>Personal Conviction</b>
L1: First I consider simple cases to see if they follow the claim:
L2: 3, 4, 5 does as $3 = 3 \times 1$ , and $4 = 2 \times 2$
L3: 7, 8, 9 does as $8 = 4 \times 2$ and $9 = 3 \times 3$
L4: 17, 18, 19 does as $18 = 2 \times 9$ , and also $18 = 3 \times 6$
L5: So the claim seems true
L6: Also from the cases, it's obvious that one of the numbers must be even and thus a multiple of 2, and as it [the three integers] is [sic] 3 consecutive integers, one must also be a multiple of 3
L7: Thus the claim is true.

Tom's argumentation involves reasoning based on arithmetic symbols and arithmetic operations. This is consistent with thinking and reasoning in the Symbolic World of the TWM. In Line 1 Tom expresses the goal to use simple (numeric) cases to see whether the claim is true. The warrant being used is a numerical check that in every set of consecutive numbers, one them is a multiple of 2 and an other will be a multiple of 3 (or one is a multiple of both). Inglis and Mejia-Ramos (2008b) refer to a warrant like this that relies on numerical examples as an inductive warrant type. In Line 5, Tom pairs this inductive type warrant with an appropriate plausible modal qualifier, 'the claim *seems* true'. However in Line 6, Tom attempts to add another layer of conviction based on the observation that in each set of 3 consecutive integers one of the integers is always even, and, because it is three consecutive integers, one must be a multiple of 3. His comment, 'one must be a multiple of 3' in Line 6 is not adequately warranted and so his over-all warrant remains inductive. We will now consider the argumentation Tom used to convince a lecturer which is presented in Table 6.19 .

Tom adopts a new argumentation strategy from that which he used to gain a personal conviction. A feature that is immediately noticeable in this argumentation is the amount

of symbolism intermixed with written language.

Table 6.19: Tom's Argumentation

<b>Convincing an Other</b>
L1: 3 consecutive numbers can be represented by $n, n + 1, n + 2$ , where $n \in \mathbb{Z}$
L2: Consider when $n$ is even. Then $n = 2k$ , $k \in \mathbb{Z}$ , thus one of the integers is a multiple of 2.
L3: Consider when $n$ is odd. Then $n = 2k + 1$ and $n + 1 = 2k + 2 = 2(k + 1)$
L4: Thus one of the integers is a multiple of 2
L5: Consider the remainder theorem, $n = 3q + r$ , where $0 \leq r \leq 2$
L6: Then if $n = 3q + 0$ , $n$ is a multiple of 3.
L7: If $n = 3q + 1$ , then $n + 2 = 3q + 1 + 2 = 3(q + 1)$ which is a multiple of 3
L8: If $n = 3q + 2$ , then $n + 1 = 3q + 3 = 3(q + 1)$ which is a multiple of 3
L9: Thus one of $n, n + 1, n + 2$ must be a multiple of 3
L10: $\Rightarrow$ either one is a multiple of 2 or and an other a multiple of 3, or one is a multiple of 2 and 3.

Since he has directed his argument at an audience that has a measure of authority in mathematics, his goal is to generalise the claim made in the task, using algebraic symbols and rules. This is signalled in Line 1 where he begins by providing an algebraic construction of three consecutive integers and specifies necessary conditions on  $n$  ( $n \in \mathbb{Z}$ ). A logical flow is provided by a proving strategy that considers four separate cases, and then combines their outcomes to form a final conclusion. Tom's experience in mathematics is evidenced in Line 5 where he states the theoretical backing of the Remainder Theorem (also referred to as the Division Theorem) that he will use to consider one of the cases. This provides a measure of mathematical authority to his argument, a perspective that is further developed through his use of deductive reasoning. Logical expressions and deductive linking include:

- Consider...Then (Line 2, Line 3);
- If...Then (Lines 7 & 8); and
- deducing line 10 from Lines 4 and Line 9.

An analysis using the TA Framework begins by placing Tom's argumentation in the TWM. Since Tom supports his thinking and reasoning with symbols, algebra as well as a theorem

from number theory, his argumentation is most suitably placed in the intersection of the Symbolic and Formal Worlds. The data he uses to ground his argumentation consists of  $n, n + 1, n + 2$ , where  $n \in \mathbb{Z}$ . The conclusion that he arrives at, is that in three consecutive integers, either one of them is a multiple of 2 and another a multiple of 3, or one integer is a multiple of 2 and 3. Tom's argumentation is warranted by a consideration of four cases – when  $n$  is even (Line 2), when  $n$  is odd (Line 3), when  $n$  is a multiple of 3 (Line 6), and when  $n$  is not a multiple of 3 (Lines 7 & 8). The backing for his warrant consists of the nature of even and odd integers, the rules of algebra, and a mathematical theorem. An absolute modal qualifier is given in the final deductive step through the use of the logical implication symbol (Line 10).

Epistemic rationality comes from deductive reasoning within each line (e.g., Lines 3, 7, & 8) and from one line to another (e.g., lines 6, 7, & 8 to Line 9). Epistemic rationality also comes from the warrant and mathematical backing. Communicative rationality is evidenced by use of symbols and algebra to effectively communicate each step in his argumentation whilst at the same time ensuring the big picture (the main goal) was visible. Tom was able to convey his understanding of the claim made in the problem, and how it can be proved. Teleological rationality is shown in the setting up and achievement of four subgoals (i.e., the cases considered). The outcomes of these sub-goals were deductively linked to the conclusion. Both of Tom's argumentations are now compared using the TA Framework summary (Table 6.20).

Table 6.20: Tom's Argumentations and the TA Framework

Personal Conviction	Convincing an Other
<p><b>The TA Framework:</b> Numerical–Procedural: primarily because he uses arithmetic thinking – reasoning inductively on the multiplicative procedure he has used. Numerical examples used employ appropriate symbolisms and familiar arithmetic rules.</p> <p><b>The Three Worlds of Mathematics:</b> Symbolic World</p> <p><b>Layout of Argumentation:</b>  <i>Data:</i> Numeric  <i>Warrant:</i> Inductive (based on numerical evidence)  <i>Backing:</i> Arithmetic  <i>Modal Qualifier:</i> Plausible</p> <p><b>Rationality:</b>  <i>Epistemic Rationality:</i> Truth and validity of the conclusion is based on rules of arithmetic  <i>Communicative Rationality:</i> Numerical examples are used to reason inductively to the conclusion  <i>Teleological Rationality:</i> The goal is to determine whether the claim given in the problem could be true. A plausible conclusion is achieved. A case by case strategy to reaching the goal is set out</p>	<p><b>The TA Framework:</b> Generalising with Algebra: since algebraic reasoning is used to correctly deduce the claim</p> <p><b>The Three Worlds of Mathematics:</b> Intersection of the Symbolic and Formal Worlds</p> <p><b>Layout of Argumentation:</b>  <i>Data:</i> Algebraic  <i>Warrant:</i> 4 ‘cases’ considered algebraically  <i>Backing:</i> Rules of algebra and a mathematical theorem  <i>Modal Qualifier:</i> Absolute</p> <p><b>Rationality:</b>  <i>Epistemic Rationality:</i> Knowledge about the claim and its validity are derived from a combination of deductive and algebraic reasoning  <i>Communicative Rationality:</i> Layout of the argumentation is deductively sequenced. Symbols and algebra are used to support reasoning.  <i>Teleological Rationality:</i> Main goal is developed through 4 subgoals (or cases) each of which are achieved using deductive reasoning. An explicit link is made to the primary goal. Text is illustrative of deductive reasoning</p>

Another second year student, whom we will call Elise, convinced herself of the claim by reasoning mathematically without using symbols to write algebraic statements. Elise will be the second case that will be examined.

### 6.5.2 Case 2: From a Logical Narrative

In her first year of university Elise took Maths 108, a course with the same curriculum as Maths 150, but focusses more on developing skills and practical knowledge, rather than formal theory. After successfully completing Maths 108, Elise went on to take Maths 250 and Maths 253 in her second semester, and Maths 255 in her third semester.

Elise's argumentation of self-conviction given in Table 6.21 reads like a personal narrative of mathematical thinking. Her argumentation will now be examined in more detail.

Table 6.21: Elise's Argumentation: Self-conviction

<b>Self-conviction</b>
L1: In any set of 3 consecutive integers there is either 1 even number, or 2 even numbers.
L2: So one of the three integers must be a multiple of 2.
L3: If there is one even integer, then it must be in the middle.
L4: If the middle integer is a multiple of 3 the conjecture is proved. Assume it [the middle integer] is not [a multiple of 3].
L5: Then it is not divisible by 3, which means that it must be 1 or 2 bigger than a multiple of 3.
L6: If it [the middle integer] is 1 bigger, then the first integer must be a multiple of 3
L7: If it [the middle integer] is 2 bigger, then the next one [the 3rd integer] must be a multiple of 3.
L8: Suppose the even integer is in the first place and not divisible by 3.
L9: Using the same argument as above, either the second or the third integer is a multiple of 3.
L10: I am convinced that it [the conjecture] is true.

Elise's argumentation is noted for its logical language particularly its use of 'If – then' statements and deductive reasoning. Her argumentation demonstrates a structural knowledge of the composition of three consecutive integers (Lines 1 to 3), as well as the construction of an integer that is not divisible by three (Lines 5 to 7). This is not an example of an inductive argument, and neither does it use the symbolic tools of algebra. However, it is an example of algebraic reasoning as it reflects her ability to think about unknown quantities, reason on their relationships, and make generalisations from them. For example, the reference to 'even integer' in Line 3, and 'it' (referring to an even integer in the middle of the consecutive three integers with the property that it is not divisible by 3) in Line 4, are examples of unknown quantities. Lines 5 to 7 illustrate how Elise has reasoned on the unknown quantity (the middle integer that is a multiple of 2 and is not divisible by 3) in relation to the two integers either side to make a generalisation. In Line 8, Elise now places the integer with its special characteristics in the first position and repeats her reasoning strategy without restating it, to arrive at the conclusion in Line 10.

An analysis using the TA Framework will now be given, starting with a consideration of

the TWM. Elise's written argument does not show reasoning with mathematical symbols, rather her written text is set in a verbal narrative style. However, as already shown, there are many instances of abstract thought. In terms of the TWM Elise's text suggests that her argumentation is developed in the Embodied World. However, her use of algebraic reasoning suggest that this world does not give a sufficient description of her thinking and reasoning. Thomas and Stewart (2011) offer a possible answer to this dilemma. They connect APOS theory (Dubinsky 1991; Cottrill et al. 1996; Dubinsky & MacDonald 2001) with the TWM. In their resulting framework they consider a Process – Embodied World category in which a person demonstrates conscious control over their actions and can mentally examine their properties and steps, or examine their actions in terms of inputs and outputs. This seems to be a more appropriate description of Elise's thinking and reasoning which focussed on mental embodiments of a general number. Thus we will use Thomas and Stewart's (ibid) categorisation to place Elise's argumentation in the intersection of the Process World of APOS, and the Embodied World of the TWM.

Elise grounds her conclusion with the generalisation that in three consecutive integers there is at least one odd integer and one even integer of which one of these may or may not be a multiple of three (this is her data). Her conclusion is warranted by the placement of a general even integer (that is not a multiple of three) in the second or first position of the three consecutive integers, and then using algebraic reasoning to determine the nature of the other two integers. This warrant appears to be of a deductive type since Elise's argumentation indicates mental activity that is structured by deductive reasoning. The warrant is backed by a properties of number, and rules of counting. An absolute modal qualifier is given in Line 10 (i.e., 'is true').

Elise's argument is based on properties number (e.g., odd and even), counting (e.g., counting forward, counting backward), algebraic thinking, and logical implication. Thus, in the absence of mathematical symbolisms, Elise's argument appears to have a theoretical foundation. Its conclusion is brought about by appeal to these theoretical foundations so that her argumentation reflects the character of mathematical validity. Hence Elise argumentation reflects epistemic rationality.

Elise communicates in a manner that conveys her understanding of the mathematics involved in her argumentation. Logical thought is evidenced in the language she uses, particularly her extensive use of 'if-then' statements. Thus Elise appears to exhibit communicative rationality.



Elise's goal is to prove to herself that the claim is correct. This goal consists of three subgoals. The first of these is to place an even number with the property that is not a multiple of 3 in the middle of the set of integers, and consider its effect on the other two integers. The second and third subgoals are to consider the same, but this time with the even integer placed in the first and third positions. The achievement of each of these led to the statement of her conclusion. We now present, in Table 6.21, Elise's argumentation to convince an other.

Table 6.22: Elise's Argumentation: Convince an Other

<b>Convincing an Other</b>
L1: $2x - 1, 2x, 2x + 1$ are 3 consecutive integers where the middle integer is even.
L2: If $2x$ is not a multiple of 3 then $2x = 3y + 1$ , or $2x = 3y + 2$ .
L3: if $2x = 3y + 1$ , then $3y = 2x - 1$ is a multiple of 3.
L4: if $2x = 3y + 2$ , then $3y + 3 = 3(y + 1) = 2x + 1$ is a multiple of 3
L5: Suppose the 3 consecutive integers are $2x, 2x + 1, 2x + 2$
L6: If $2x$ is not divisible by 3, then $2x = 3y + 1$ , or $2x = 3y + 2$
L7: If $2x = 3y + 1$ then $3y + 3 = 3(y + 1) = 2x + 2$
L8: If $2x = 3y + 2$ then $3y + 3 = 3(y + 1) = 2x + 1$ .
L9: Suppose our even integer that is not divisible by 3 is in the first place.
L10: Using the same argument as above, either the second or the third integer will be a multiple of 3.
L11: In any set of three consecutive integers, either one of the integers is a multiple of 2 and another a multiple of 3, or one integer is a multiple of 2 and 3 is true.

As with our previous case, what is immediately striking here is that from Line 1 we are confronted with a symbolic presentation. Elise sets up three consecutive integers  $2x - 1, 2x, 2x + 1$ , with the constraint that the central integer is even. This is the same strategy she employed in her previous argumentation. The next constraint on  $2x$ , is that it is not a multiple of three (Line 2). Through the use of algebra in Lines 2 to 4, Elise deduces two possibilities, either  $2x - 1$ , or  $2x + 1$  is a multiple of 3. A key point worth mentioning here is that this argumentation is closely aligned with Elise's argumentation for self-conviction. This can be explained by Pedemonte's (2007b) referential and structural systems. There is continuity in Elise's referential system. Her content, concept,

and operational knowledge of number and algebra in the referential system used in her argumentation of self-conviction, is replicated in her argumentation to convince a lecturer. Likewise there is also continuity in the structural system, as deductive reasoning is also replicated. The transformation that Elise had to make was to map her narrative on to symbols, which did not seem to be an obstacle because her language was strongly influenced by logic, deduction, and algebraic thinking. Both of Elise's argumentations are now compared using the TA Framework (Table 6.23).

Table 6.23: Elise's Argumentations and the TA Framework

Personal Conviction	Convincing an Other
<p><b>The TA Framework:</b> Visual–Generalising: Visual reasoning along with algebraic reasoning is used to secure her conviction.</p> <p><b>The Three Worlds of Mathematics:</b> Process–Embodied World (see Thomas &amp; Stewart, 2011).</p> <p><b>Layout of Argumentation:</b>  <i>Data:</i> A set of 3 consecutive integers of which one of these may or may not be a multiple of 3  <i>Warrant:</i> An integer that is not a multiple of 3 must be 1 or 2 bigger than one that is  An even integer that is multiple of 2, but not a multiple of 3, is placed in the middle and then 1st position of the set of 3 consecutive numbers. The effect on the other two integers is given  <i>Backing:</i> Properties of number, and rules of counting  logical structure  <i>Modal Qualifier:</i> Absolute ('is true')</p> <p><b>Rationality:</b>  <i>Epistemic Rationality:</i> Justification is based on properties of numbers and rules of counting. Algebraic thinking along is used along with deductive reasoning  <i>Communicative Rationality:</i> Personal understanding of the problem and the mathematics involved is successfully communicated using a mixture of everyday and logical (if-then) language  <i>Teleological Rationality:</i> Two subgoals are set which involve a consideration of an even integer that is not a multiple of 3 being in the 1st, and 2nd positions in the set of 3 consecutive integers. A conclusion is derived for each case that contributes to the primary goal of her argumentation</p>	<p><b>The TA Framework:</b> Generalising with Algebra: Algebraic reasoning using symbols is used to correctly deduce the claim.</p> <p><b>The Three Worlds of Mathematics:</b> Intersection of the Symbolic and Formal Worlds.</p> <p><b>Layout of Argumentation:</b>  <i>Data:</i> Algebraic (<math>2x - 1</math>, <math>2x</math>, <math>2x + 1</math>)  <i>Warrant:</i> An integer that is not a multiple of 3 can be written as <math>3y + 1</math>, or <math>3y + 2</math>  Algebra is used to show that if any one of the three integers is not a multiple of 3, then one of the others must be  <i>Backing:</i> Rules of algebra and number. A mathematical theorem (Division Theorem implied in Line 2) is used  <i>Modal Qualifier:</i> Absolute.</p> <p><b>Rationality:</b>  <i>Epistemic Rationality:</i> Knowledge about the claim and its validity are derived from a combination of deductive reasoning and algebraic reasoning.  <i>Communicative Rationality:</i> Layout of the argumentation is deductively sequenced with 3 cases considered, that is when <math>2x</math> (which is not a multiple of 3) is placed in the 1st, 2nd, and 3rd positions in the consecutive set of integers. Symbols and algebra are used to support reasoning.  <i>Teleological Rationality:</i> Main goal is developed through 3 subgoals (or cases) each of which lead deductively to a conclusion. An explicit link is made to the primary goal.</p>

### 6.5.3 Case 3: Looking for a Proof

This case that is now presented, looks at an argumentation of a student who uses a ‘proof by contradiction’ approach to set up a self-conviction. We will refer to this student as Jerry. Jerry’s track through mathematics follows others in the group of participants. In her first year Jerry took Maths 150, Maths 250, and Maths 253. At the time of participating in this study, she was in her second year and taking Maths 255. Jerry’s argumentation is presented in Table 6.24.

Table 6.24: Jerry’s Argumentation: Self-conviction

<b>Self-conviction</b>
L1: Assume that in a set of 3 consecutive integers $\{n - 1, n, n + 1\}$ none of them is a multiple of 3.
L2: Then in the previous set, as well as the set after it, there’ll be no integer that is a multiple of 3.
L3: We can extend this to all consecutive sets of 3 consecutive integers on the number line.
L4: None of the sets would have a multiple of 3 in them.
L5: But we have a contradiction, because 3 is an integer on the number line.
L6: There is a set of 3 consecutive integers that will contain the integer 3.
L7: Therefore there’ll be at least 1 integer in any set of 3 consecutive integers that will be a multiple of 3.
L8: To show that one of the integers must be even, we know that if the multiple of 3 is even then we are satisfied. If it’s not, then the multiple [of 3] plus or minus 1 will give an even integer. Which means that there is an even integer before and after it.
L9: I conclude that in any set of 3 consecutive integers there must be a multiple of 2 and a multiple of 3.

Jerry’s strategy is to show why it is impossible to choose a set of three consecutive numbers that does not to contain a multiple of three. She begins by assuming that there is a set of three consecutive numbers that does not contain a multiple of three. She then argues that if this were the case, then a multiple of three will not be found in the next set of consecutive integers, or the next and so on (or any other preceding set). This assertion is based on a counting property of integers. She now uses the fact that 3 belongs to the set of integers, and so at least one set of three consecutive integers must contain the integer

3. Using the counting property, Jerry argues that every set of three consecutive integers will contain a multiple of 3.

An analysis of Jerry's argumentation using the TA Framework begins with placing her argumentation in the TWM. Jerry's argumentation indicated she was reasoning deductively with unknown quantities, and using a technique of proof. She also appeared to be mentally manipulating her thinking to provide logical structure to her written text. Jerry also embodied the set of integers using a number line. Like Elise in case 2, Jerry's argumentation is also situated in the intersection of the Process World of APOS, and the Embodied World of the TWM.

Jerry's primary goal is clear in Line 9, and so is her strategy for achieving it (Line 1). Consistent with the arguments of all the other participants, Jerry has two subgoals – showing that one of the integers is a multiple of three, and showing that one of them is also a multiple of two. The data used to ground her conclusion related to the first subgoal is a generalised set of three consecutive integers, and the assumption that none of them is a multiple of three (first part of Line 1). The conclusion is that the assumption she made is false. Her warrant consists of two parts. The first is a fact that three is an integer on the number line and is therefore in at least one consecutive set of three integers. The second is that given three is in one of the sets, then a multiple of three must occur in every other set of three consecutive numbers. Her warrant is backed by properties of integers on the number line, and rules of counting. The data that is used for the second conclusion is grounded by a selection of an odd multiple of three that is in any set of three consecutive integers. The conclusion is that in this set, one of the integers will be even. The warrant consists of two parts, the first being that if the chosen multiple of three happens to be even, then the conclusion is satisfied. Otherwise, an odd integer plus or minus one is even. This is backed by a property of integers.

Epistemic rationality comes from the use of an accepted mathematical technique for proving, combined with deductive reasoning (e.g., shifting from Line 1 to Line 2). Communicative rationality is evidenced in her intention to show that she has argued logically. She attempts to provide an understanding of both the mathematical content used in her argumentation and her proving strategy to the reader. However the links between Lines 6 and 7, and within Line 8, need to be clarified further. Teleological rationality comes from the achievement of reaching a contradiction and stating its consequence (Line 5 & 6). This has come about by an intentional implementation of a proving technique. The achievement of the second subgoal in Line 8 does not seem complete as it refers to the

central position only of a set of three consecutive integers.

We suggest these accounted for the validity Jerry perceived in her argumentation, and the processes involved accounted for Jerry's confidence (as indicated in Line 10). Jerry's argumentation for the conviction of another, followed a typical case by case approach shown by Tom. To describe it here would be a duplication.

#### 6.5.4 Summary

In the observations provided, all second year students were aware of the deductive nature of writing a formal statement of mathematics. They took care in their layout of argumentation, and relied heavily on algebra and logical connections. This was reflected in the attention they gave to the logical structure of their statements, and the deductive flow in their arguments. The students also reasoned on the generalised arithmetic structure of three consecutive integers, and in Elise's case (Case 2), this provided continuity in her referential and structural systems, between her self-conviction and conviction of another. The contradiction 'narrative' approach of Jerry (Case 3) highlighted her ability to implement an effective logical argument that did not involve intensive use of written algebra.

## Chapter 7

# Progressions in Argumentation

### 7.1 Introduction

In this chapter, the diachronic analysis of argumentation will be considered. This will be done by examining examples of argumentation from four students that are taken from each of their three years of study in mathematics. The samples of work were chosen by the student themselves to demonstrate the changing nature of their argumentations. However, there are examples of some of these students' work from a preliminary study of argumentation from their first year. These have been included by the researcher to highlight certain features associated with the changing nature of conviction and will be signalled when they are used. Each of the students' samples of work will be analysed against the TA Framework that was developed in Chapter 3. All care has been taken to ensure a faithful reproduction of the students' work. Where appropriate, diagrams and other relevant material have been included from the students' original work in order to illustrate their thinking and reasoning.

Each of the students had either completed a degree in mathematics, or were currently in a third year course of study. Three students participated in a preliminary study of argumentation. In this chapter, each student will be referred to by their research name, Lear, Cam, Able, and Tas.

It was hypothesised that in a diachronic analysis one would expect to see an increase in the sophistication of elements associated with mathematical language, structure and presentation of their arguments due to the skills, knowledge and understanding they would have developed over three years of studies in mathematics. Sophistication here refers to the complexity in cognitive and mathematical resources that they use to think, reason

and communicate with. It also includes the organised manner in which they manage and format their written argumentation. The long-term progress of each of the four students will now be presented beginning with Lear.

## 7.2 Long-term Progress in Argumentation: The Case of Lear

Lear has a BSc degree with a major in Applied Mathematics and at the time of this study was completing a postgraduate course in Applied Mathematics. In the three sections that follow, an analysis of three problems taken from each of the three years of Lear's study of mathematics will be presented. The following section begins with a problem presented in the Maths 102 course entitled 'Functioning in Mathematics' that she took in her first year.

### 7.2.1 A First Year Example of Argumentation

At the time when Lear did Maths 102, it was developing into a highly interactive course involving collaborative problem solving and assignment tutorials. It also included a re-developed module that focussed on reasoning in mathematics where examples of reasoning strategies such as inductive, deductive, and visual, were provided and discussed (including combinations of these). Questions requiring the production of mathematical arguments were also included in assignment material. The theme of reasoning mathematically was developed in each of the other modules that made up the content of the course. The problem Lear presented for analysis was as follows:

#### Problem 1

Are all integers that are multiples of 3 and divisible by 4, also multiples of 12?

- Investigate this statement, and convince yourself whether it is always true, sometimes true, or never true.
- Write an argument that you would use to convince another person such as a mathematics lecturer of your conclusion.

Table 7.1 shows the steps Lear wrote when attempting to solve the problem. Lear's initial approach is empirical, but in Line 2 she begins to use algebra to formulate a general

expression that will give multiples of 12. The empirical section (which acts as arithmetic data) confirms for Lear the construction of the multiples of 12 (multiples of 3 that are divisible by 4) and contributes to her algebraic representation of the problem. The additional data she uses is the algebraic expression  $3x$  ( $x \in \mathbb{N}$ ) which she supports with an algebraic warrant in Lines 3 and 4. This warrant appears to act as a warrant for truth (Rodd, 2000) and anchors her knowledge in the Symbolic World of the TWM (see Tall & Mejia-Ramos, 2010). Rodd’s use of the term ‘knowledge’ is useful to recall here as it is this usage that is applied to the context of argumentation being examined in this section. Rodd (2000, p. 222) refers to knowledge of an area of mathematics as involving an awareness of “connections, structures, meaning-carrying symbolism, and so forth, over a range of mathematical topics. Such mature understanding will include specific knowledge of the truth of certain mathematical propositions”. We also note that in Lear’s argumentation the warrant is appropriately paired with an absolute qualifier (recognised by the use of the words ‘must’ and ‘is’) and shored up in Line 3 by an algebraic backing namely that, if  $a$  and  $b$  are non-zero integers and  $k|ab$ , then either  $k|a$ , or  $k|b$ .

Table 7.1: Lear’s Example of a First Year Argumentation

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L1: 3, 6, 9, 12, 15, 21, 24, 27, 30, 33, 36

L2: A multiple of 3 has the form  $3x$  ( $x \in \mathbb{Z}$ )

L3: If  $3x$  is also divisible by 4 then  $x$  must be divisible by 4

L4: So,  $x$  is of the form  $4y$ ,  $y \in \mathbb{Z}$

L5: So we have  $3x \rightarrow 3(4y)$ , which is a multiple of 3 and 4, and therefore a multiple of 12.

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Rationality of Lear’s argumentation stems from the propositional nature of the warrant in Lines 3 and 4 and her backing. She also uses symbolic language to specify a list of multiples specifically identifying multiples of 3 and 4. Symbols  $x$  and  $y$  are used as variables, and the symbol for set membership used to signify that  $x$  comes from the set of Natural Numbers ( $x \in \mathbb{Z}$ ). Lear sets her argumentation out in a manner that clearly shows the use of deductive reasoning (e.g., the use of ‘If-then’, and ‘so...’). Table 7.2 summarises Lear’s argumentation using the TA Framework developed in this research.



Table 7.2: TA Framework Analysis of Lear's First Year Argumentation

Three Worlds of Mathematics	Toulmin Model	Rationality
<p><b>Symbolic World:</b> Reasoning uses numeric and algebraic symbols as well as a symbol for set membership</p> <p>L3 and L4 behave as a warrant for truth</p>	<p><b>Data:</b> (Arithmetic) A list of the 3 times tables where multiples of 3 and 4 are underlined. (Algebraic) <math>3x, x \in \mathbb{N}</math> (L2)</p> <p><b>Warrant:</b> (Algebraic) (L3 &amp; L4) If <math>3x</math> is also divisible by 4 then <math>x</math> must be divisible by 4 (acts as a theorem). So <math>x</math> is of the form <math>4y, y \in \mathbb{Z}</math></p> <p><b>Backing:</b> (Algebraic) A rule of divisibility</p> <p><b>Qualifier</b> Absolute qualifier (e.g. 'must' in L3, and 'is' for the conclusion in L5)</p> <p><b>Rebuttal</b> (None provided)</p> <p><b>Conclusion:</b> A multiple of three written as <math>3x</math> where <math>x \in \mathbb{N}</math> and <math>x</math> is of the form <math>4y</math> is also a multiple of 12</p>	<p><b>Epistemic:</b> An algebraic warrant is used to secure a truth (L3) and (L4)</p> <p>A theoretical backing is inferred from L3</p> <p><b>Communicative:</b> Text uses symbolic language, as well as logical language (e.g., If-then (L3)). Text is written to show a sequence of steps</p> <p><b>Teleological:</b> Goal is achieved by structuring <math>3x</math> in the form of <math>3(4y)</math> in L5</p> <p>Text is illustrative of deductive reasoning</p>

In the next subsection, we will consider a second year example of one of Lear's argumentations. The example is drawn from a problem presented in Maths 253 Advancing Mathematics 2, a course that covers linear algebra and multi-variable calculus.

### 7.2.2 A Second Year Example of Argumentation

A significant aspect of the study of mathematics in the second year at the University of Auckland is linear algebra. In Maths 253 this also includes links to multivariable calculus. However, the example chosen by Lear for this analysis is based on a problem involving an eigenvalue of a particular matrix.

#### Problem:

Prove that if  $\lambda$  is an eigenvalue of a matrix  $A$ , then  $\lambda - k$  is an eigenvalue of the matrix  $A - kI$ .

Lear's response to the problem is given in Table 7.3. One immediate contrast to her first year example, is that there appears to be an increased density of symbolism. Lear's argumentation hides many aspects of the concept image other students would need before being able to interpret her text with understanding. Primary amongst these are the role of the determinant in finding an eigenvalue, and valid arithmetic operations that can be done with matrices. Others might include for example, the dimensions of  $A$ , and the interpretation of solutions to  $(A - \lambda I)\mathbf{x} = 0$ . In Line 1 she recalls a definition of an eigenvalue  $\lambda$ . Line 2 is an application of the definition, and is an instrumental step towards

progressing her argumentation toward a proof. Lear's plan of attack is to transform  $|A - \lambda I|$  into  $|(A - kI) - (\lambda - k)I|$ , which she does in Line 3. She then applies the definition to show that  $\lambda - k$  is an eigenvalue of the matrix  $A - kI$ .

Rationality of Lear's argumentation is gained through the use of a definition in Line 1, and its extension in Line 2, appropriate use of matrix algebra (L3), and through the correct use of mathematical symbolism. The goal of the problem, is achieved in Line 3 which links the eigenvalue requirements in Lines 1 and 2. Rationality is also evidenced in the layout of five lines of working, which illustrates the deductive progression in her reasoning. An outline of how Lear's argumentation relates to the TA Framework is given in Table 7.4.

In the next subsection, an example from a mathematics course in Lear's third year

Table 7.3: Lear's Example of a Second Year Argumentation

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L1: $\lambda$ is an eigenvalue of matrix $A$ if and only if it is a solution to $ A - \lambda I  = 0$ .
L2: $\lambda - k$ is an eigenvalue of matrix $A - kI$ if and only if it is a solution to $ A - kI - (\lambda - k)I  = 0$ .
L3: $ A - \lambda I  = 0 \Rightarrow  A - kI + kI - \lambda I  =  (A - kI) - (\lambda - k)I  = 0$
L4: $ (A - kI) - (\lambda - k)I  = 0 \Rightarrow \lambda - k$ is an eigenvalue of $A - kI$

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Table 7.4: TA Framework Analysis of Lear's Second Year Argumentation

Three Worlds of Mathematics	Toulmin Model	Rationality
<p><b>Symbolic World:</b> Symbols are used from matrix algebra, comprising specific symbols for the determinant, eigenvalue, eigenvector, a matrix, and the identity matrix A definition of an eigenvalue is used in L1, and applied in L2</p>	<p><b>Data:</b> (Algebraic) <math> A - \lambda I  = 0</math> (L2) <b>Warrant:</b> (Algebraic) <math> A - \lambda I  =  (A - kI) - (\lambda - k)I </math> <b>Backing:</b> Definition of an eigenvalue (L1 and L2) Rules of matrix algebra used in L3 <b>Qualifier</b> Absolute qualifier (implied) <b>Rebuttal</b> (None provided) <b>Conclusion:</b> If <math>\lambda</math> is an eigenvalue of <math>A</math>, then <math>\lambda - k</math> is an eigenvalue of <math>A - kI</math></p>	<p><b>Epistemic:</b> A definition is given for an eigenvalue (L1 &amp; L2) and used to secure the conclusion (L4) Rules of matrix algebra are used correctly to transform <math> A - \lambda I </math> into <math> (A - kI) - (\lambda - k)I </math> <b>Communicative:</b> Text correctly uses symbolic language (e.g. <math> A - kI </math> for the determinant of <math>A - kI</math>). Implication symbols are also used to show a deductive sequence <b>Teleological:</b> Goal (conclusion) is achieved by interpreting <math> (A - kI) - (\lambda - k)I  = 0</math> using a generalised version of the definition of an eigenvalue in L2. Text is illustrative of deductive reasoning</p>

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will be examined and analysed. The work chosen by Lear comes Maths 361, an applied mathematics course that looks at partial differential equations and their applications to modelling real world phenomena.

### 7.2.3 A Third Year Example of Argumentation

The example of a third year argumentation was chosen by Lear to illustrate how she has come to use definitions, and how she has developed her understanding of mathematical concepts through the use of symbols and algebraic operations. The example comes from a problem Lear was given toward the beginning of her Maths 361 course, and revolves around showing whether or not an equation is linear. Lear's argumentation is presented in Table 7.5.

**Problem:**

Show whether Tricomi's equation  $u_{xx} + xu_{yy} = 0$  is linear or not linear.

Table 7.5: Lear's Example of a Third Year Argumentation

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L1:  $U_{xx} + xU_{yy} = 0.$

L2:  $L = \frac{\partial^2}{\partial x^2} + x \frac{\partial^2}{\partial y^2}$

L3:  $L(U) = \frac{\partial^2 U}{\partial x^2} + x \frac{\partial^2 U}{\partial y^2}, L(V) = \frac{\partial^2 V}{\partial x^2} + x \frac{\partial^2 V}{\partial y^2}.$  Let  $\alpha$  and  $\beta$  be constants then:

L4:  $L(\alpha U + \beta V) = \frac{\partial^2}{\partial x^2}(\alpha U + \beta V) + x \frac{\partial^2}{\partial y^2}(\alpha U + \beta V)$

L4(a):  $= \frac{\partial^2}{\partial x^2} \alpha U + \frac{\partial^2}{\partial x^2} \beta V + x \frac{\partial^2}{\partial y^2} \alpha U + x \frac{\partial^2}{\partial y^2} \beta V$

L4(b):  $= \alpha \frac{\partial^2 U}{\partial x^2} + \beta \frac{\partial^2 V}{\partial x^2} + x \alpha \frac{\partial^2 U}{\partial y^2} + x \beta \frac{\partial^2 V}{\partial y^2}$

L4(c):  $= \alpha \left( \frac{\partial^2 U}{\partial x^2} + x \frac{\partial^2 U}{\partial y^2} \right) + \beta \left( \frac{\partial^2 V}{\partial x^2} + x \frac{\partial^2 V}{\partial y^2} \right) = \alpha L(U) + \beta L(V)$

L5:  $\therefore L(\alpha U + \beta V) = \alpha L(U) + \beta L(V) \Rightarrow \underline{\underline{\text{Linear}}}$

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Lear begins her argumentation in Table 7.5 by restating Tricomi's Equation in Line 1, and rewrites it as a second order partial linear operator  $L$  in Line 2, using this form to solve the problem. In Line 4  $L(\alpha U + \beta V)$  is expanded, and from Lines 4(a) to 4(c) algebraic

reasoning is used to arrive at the required goal which is stated in Line 5.

Lines 1, 2 and 3 provide the grounding (data) for Lear's argumentation. Her warrant is the linearity of  $\frac{\partial^2}{\partial x^2}$  and  $\frac{\partial^2}{\partial y^2}$  provided in Line 4. This is crucial to the representation of  $L(\alpha U + \beta V)$  in Line 5 and the formation of the conclusion given in Line 5. The warrant provided can be backed by the definition of a linear operator  $L$  – which is implied Line 2 and its application in Line 3. The warrant is paired with an absolute qualifier (logical implication) in Line 5. Table 7.6 gives an analysis of Lear's argumentation using the TA Framework.

Table 7.6: TA Framework Analysis of Lear's Third Year Argumentation

Three Worlds of Mathematics	Toulmin Model	Rationality
<p><b>Intersection of the Symbolic and Formal Worlds:</b> Symbols used come from algebra and differential calculus</p> <p>Definition of a linear operator is assumed in Lines 2, 3, and 4</p> <p>The definition of linearity is also used</p>	<p><b>Data:</b> (Algebraic)  <math>U_{xx} + xU_{yy} = 0</math> (L1)</p> <p><math>L = \frac{\partial^2}{\partial x^2} + x\frac{\partial^2}{\partial y^2}</math> (L2)</p> <p><math>L(U)</math> and <math>L(V)</math> (L3)</p> <p><math>L(\alpha U + \beta V)</math>(L4)</p> <p><b>Warrant:</b> (Differential Calculus)  The linearity of <math>\frac{\partial^2}{\partial x^2}</math> and <math>\frac{\partial^2}{\partial y^2}</math>  (Lines 4 to 4(c))</p> <p><b>Backing:</b> Definition of a linear operator</p> <p><b>Qualifier</b> Absolute qualifier (L5)</p> <p><b>Conclusion</b> Tricomi's equation is linear</p>	<p><b>Epistemic:</b> Definitions are used in the argumentation (e.g. the linear operator-implied)  Rules of algebra are correctly applied.</p> <p><b>Communicative:</b> Text correctly uses symbolic language associated with partial derivatives.  Argumentation is set out so that the use of the linear operator is highlighted. The layout of argumentation beginning L4 (though Lines 4(a), 4(b), 4(c)) and ending L5 indicate purposeful deductive reasoning</p> <p><b>Teleological:</b> A subgoal to set up <math>L(U)</math> and <math>L(V)</math> (L(3)), and <math>L(\alpha U + \beta V)</math> (L(4)) is achieved.  The primary goal is achieved by observing the equivalence of the expansions of <math>L(\alpha U + \beta V)</math> and <math>\alpha L(U) + \beta L(V)</math></p>

An epistemological basis for the rationality of Lear's argumentation comes from her use of the definition of a known linear operator to confirm a linearity condition. Although it may be argued that these (the definitions of linearity and of a differential operator) are implicit, they are explicitly illustrated in her argumentation (for example Lines 4(a) to 4(c)). This was as an important aspect of her communicative rationality. Lear's step by step layout indicated how she used the definition along with algebraic operations to support her deductive reasoning. Teleologically, there appear to be three subgoals in her argumentation. The first was to set up  $L(U)$  and  $L(V)$  in Line 3 and then use Tricomi's equation to set up  $L(\alpha U + \beta V)$  in Line 4. The final subgoal was get an expression that could

be simplified using  $L(U)$  and  $L(V)$  ( e.g., the derivation of L4(b) and its simplification to the expression in L4(c)). From this, the final goal of her argumentation is reached.

#### 7.2.4 Summary of Lear's Progression in Argumentation

There is a series of observations we can make of the development of Lear's argumentation over the three years she has been studying mathematics. The first is that her warrants and backings became increasingly theoretical in nature. In her first year example of argumentation, the warrant was a combination of arithmetic and algebra, which could be backed by rules of algebra and an elementary result in number theory. In her second year example, her warrant was tied to a mathematical definition (of an eigenvalue), which meant her warrant could be backed theoretically. Lear's third year example illustrated a more complex situation that involved an increased knowledge of mathematical ideas, along with the use of more sophisticated symbols. Her warrant comprised mathematical statements that were theoretically situated in the definition of the linear operator and linearity. Thus her argumentation could be supported by a theoretical backing. Accordingly, Lear's progression showed an increased shift to the use of formally stated definitions and hence more theoretical warrants and backings in her argumentations.

A noticeable aspect of Lear's writing was the consistent way in which she laid out her argumentation line by line, ending with the desired conclusion. In all cases there was clear evidence of deductive reasoning. According to the TA Framework analysis, Lear reasoned from the Symbolic World of the TWM in her first year example of argumentation. As she progressed, reasoning strategies used in her second and third year argumentations, the use definitions, and an increased complexity in algebraic and deductive reasoning and symbolic representations appeared to situate her thinking in an intersection of the Symbolic and Formal Worlds of the TWM.

### 7.3 Long-term Progress in Argumentation: The Case of Cam

Cam began his study of mathematics at university by taking Maths 102. As already noted in case studies elsewhere, this course is designed for those who have not had a good preparation for mathematics in their Year 13 secondary mathematics courses. Maths 102 sets students up for Maths 108 or Maths 150 (which requires at least an A grade pass

in Maths 102). Maths 108 and Maths 150 allow students to continue in their study of mathematics into the second year and beyond. Since Maths 102, Cam has gone on to take courses in mathematics that have led him into a BSc Honours degree with a major in mathematics. Along with Able and Tas, Cam also participated in an early preliminary study that considered how collective argumentation might lead to proof. We now present examples that represent Cam's progression in argumentation during his studies of mathematics over three years. His first example comes from a problem presented during the preliminary study, and a second example from Maths 150 'Advancing Mathematics 1' – a course he took following Maths 102. Cam's second and third year examples come from a set of assignment problems he presented as illustrations of his progress in advanced undergraduate mathematics courses.

### 7.3.1 A First Year Example of Argumentation:

#### Example 1

##### Problem 2:

If a square number  $n^2$  is a multiple of 3 then  $n$  must be a multiple of 3.

- Investigate this problem, and convince yourself whether it is always true, sometimes true, or never true.
- Write an argument that you would use to convince another person such as a mathematics lecturer of your conclusion.

In a discussion associated with defining a departure point for this argument, he chose to use the set of integers from which to draw  $n$  from. Cam's argumentation is shown in Table 7.7

Cam had two attempts at solving the problem presented. They have been referred to as Attempt A, and Attempt B. Attempt A begins with a literal translation of the problem into algebraic text. Of particular note is that the required conclusion is assumed true in the initial set-up of his argumentation. Thus the conclusion reached in Line 7 is the result of a circular argument and is an example of a logical fallacy. A second attempt (Attempt B) resulted from his suggestion that the problem provided could be solved using another method, which he referred to as 'contradiction'. There are elements such as symbol selection and usage present from his first attempt. Cam begins his second attempt in Line 8 by writing  $n^2 = 3k$  to define  $n^2$  as a multiple of 3. In the following line he assumes  $n$

Table 7.7: Cam's Example of a First Year Argumentation: Example 1

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Attempt A:

L1:  $n^2 = 3k \Rightarrow n = 3l$

L2:  $(3l)^2 = 9l^2$ , which a multiple of 3

L3:  $n^2 = 3k$ , and  $3k = 9l^2 \Rightarrow 3k = 3(3l^2)$ .

L4: Therefore  $k = 3l^2$ , and so  $k$  is a multiple of 3.

L5: Choose  $k = 3l^2$

L6:  $n^2 = 3(3l^2) = 9l^2 = (3l)^2$ .

L7: Then  $n$  is a multiple of 3.

Cam now changes to an alternative approach (Attempt **B**) which he termed “contradiction”.

Attempt B:

L8:  $n^2 = 3k$

L9: consider  $n \neq 3l$

L10:  $n^2 = 3k$

L11:  $n = \sqrt{3}\sqrt{k}$

L12: But  $n \in \mathbb{Z}$ , so  $\sqrt{k} = \sqrt{3}l$

L13: Therefore  $k = 3l^2$

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is not a multiple of 3. As with his first attempt  $k$  and  $l$  are not defined. He now searches for a ‘contradiction’, which arises in Line 12. However, Cam’s argumentation appears lack detail, and structure in its presentation. For example,  $n \in \mathbb{Z}$  should have been declared as the start, and Lines 8 and 9 should have included specifications for  $k$  and  $l$  respectively. Also,  $n = -\sqrt{3}\sqrt{k}$  should have been identified as an additional consideration in Line 11. However, the idea behind his argumentation appears sound.

So far, the TA Framework has been used to analyse valid argumentations. However, this time, we will consider how the TA Framework can be used to analyse a fallacious argumentation as shown in Attempt A. Beginning with the TWM, it appears that Cam is reasoning in the Symbolic World since symbolic notation, along with inferential language and algebra, are used to support what appears to be deductive reasoning. The data used consists of  $n^2 = 3k$  and what is required to be shown, namely  $n = 3l$ . Circularity is

created by the use of the statement  $n = 3l$  in Lines 2, 3, 4 and 6. Thus, although he uses an algebraic warrant backed by rules of algebra along with deductive reasoning, his argumentation is invalid. Cam's argumentation has failed to secure epistemic rationality since his search for a way of communicating has resulted in an invalid starting point expressed in Line 2. A goal of communicative rationality (Habermas, 1998) is that a shared understanding of a problem and its resolution is achieved between the writer and the reader – who in this case is representative of the community of mathematicians. From the reader's point of view a shared understanding has not been achieved since the reader identified a key fault in his argument. Cam set out with the goal of showing that  $n$  is a multiple of 3. However, using this as a built-in assumption has resulted in the setting of faulty goals (e.g., to show that  $k$  is a multiple of 3 in Line 4, and to show that  $n^2 = (3l)^2$  in Line 6). Therefore, neither does his argumentation reflect teleological rationality. A summary is provided using the TA Framework in Table 7.8.

Table 7.8: TA Framework Analysis of Cam's First Year Argumentation: Example 1

Three Worlds of Mathematics	Toulmin Model	Rationality
<p><b>Symbolic World:</b> Numeric and Algebraic symbols are used along with algebraic operations and inferential language to support deductive reasoning</p>	<p><b>Data:</b> (Algebraic) <math>n^2 = 3k</math>, and <math>n = 3l</math> (L1)  <b>Warrant:</b> (Algebraic) Algebraic operations (squaring and substitution).  <b>Backing:</b> (Algebraic) Rules of algebra  <b>Qualifier</b> Absolute qualifier implied (L13)  <b>Rebuttal</b> (None provided)  <b>Conclusion:</b> <math>n</math> is a multiple of 3 (L7) from <math>n = 3l</math> in Line 6</p>	<p><b>Epistemic:</b> Circular argumentation used, hence epistemic rationality is not established  <b>Communicative:</b> The argumentation assumes the conclusion and relies on it to lead to the goal of showing <math>n</math> is a multiple of 3. It fails to convince an authoritative reader of its validity. Thus communicative rationality is not established  <b>Teleological:</b> Goals and subgoals are set and achieved (e.g., L4), but these are based on a faulty assumption. Teleological rationality has failed to be established  Text is illustrative of deductive reasoning</p>

The next example of Cam's argumentation is based on a problem given in Maths 150 Advancing Mathematics 1; a course he took immediately after Maths 102. The argumentation for this problem is a contrast to the one just given (Table 7.7) because of its use of formal mathematics. The Maths 150 problem along with Cam's argumentation is now presented.



### 7.3.2 A First Year Example of Argumentation:

#### Example 2

##### Problem:

Show that 4 is the Least Upper Bound of the set  $(-\infty, 4)$ .

Cam's argumentation is shown in Table 7.9

Table 7.9: Cam's Example of a First Year Argumentation: Example 2

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- L1: We know that 4 is an Upper Bound because for all  $s \in (-\infty, 4)$ ,  $4 > s$ .
- L2: Now consider if 4 is not the Least Upper Bound for the set  $(-\infty, 4)$ , then there will exist an  $\epsilon > 0$ , such that:  
 $4 - \epsilon > 4 \Rightarrow \epsilon < 0$ . However  $\epsilon > 0$ .
- L3:  $\therefore$  4 is the Least Upper Bound for the set  $(-\infty, 4)$ .
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An analysis using the TA Framework indicates that Cam is thinking, reasoning, and using mathematical resources in the intersection of the Symbolic and Formal Worlds. He uses notational and logical symbols as well as the definition of an Upper Bound, and implies the definition of a Least Upper Bound. Cam also uses argument by contradiction to reason deductively to the conclusion stated in Line 3. The data that is used is that 4 is an Upper Bound but not a Least Upper Bound of the set  $(-\infty, 4)$ . A logical warrant is being used here, namely the existence of  $\epsilon > 0$ , such that  $4 - \epsilon > 4$ . This leads directly to the contradiction. The warrant is backed by definitions of an Upper Bound and Least Upper Bound, and the laws of logic.

Epistemic rationality is gained through the use of mathematical definitions, as well as the use of contradiction as a recognised (logical) model of argumentation in mathematics. Thus the basis for truth in Cam's argument is grounded in mathematical theory.

Communicative rationality is observable in the manner in which Cam carries out his intention to show that 4 is a Least Upper Bound for  $(-\infty, 4)$ . This intention is realised by first showing the upper boundedness of  $(-\infty, 4)$ . A negation is then set up with the goal of determining a contradiction (L2). The lines of Cam's argumentation all show the use of deductive inference.

Teleological rationality is evidenced in the setting and achievement of two distinct goals. The first of these was to show that 4 is a Upper Bound, and the second was to show

through a contradiction, that there could not exist a number smaller than 4 that would also be an upper bound.

The second year example of argumentation chosen by Cam comes from a problem given in Maths 260 Differential Equations, a course in applied mathematics. This was chosen to show argumentation involved in response to an open ended question.

### 7.3.3 A Second Year Example of Argumentation

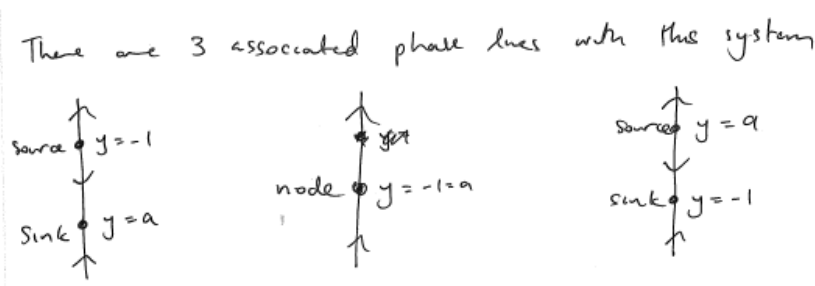
#### Problem:

Given the differential equation  $\frac{dy}{dt} = (y + 1)(y - a) = 0$  comment on its equilibrium conditions.

Cam's argumentation is shown in Table 7.10

Table 7.10: Cam's Example of a Second Year Argumentation

L1: Suppose  $\frac{dy}{dt} = (y + 1)(y - a) = 0$ . Then  $y = -1$  or  $y = a$  are the equilibria for this system.



L2: There is, I claim, a unique bifurcation at  $a = -1$

L3: Let us consider  $\frac{\partial f_a}{\partial y} = 0$

L4: Then  $\frac{\partial}{\partial y}(y + 1)(y - a) = y + 1 + y - a = 0$

$$y = \frac{a}{2} - \frac{1}{2}$$

Substitute this into  $\frac{dy}{dt}$

L5:  $\frac{dy}{dt} = \left(\frac{a}{2} - \frac{1}{2} + 1\right) \left(\frac{a}{2} - \frac{1}{2} - a\right)$   
 $= \left(\frac{a}{2} + \frac{1}{2}\right) \left(-\frac{a}{2} - \frac{1}{2}\right) = -\left(\frac{a}{2} + \frac{1}{2}\right)^2 = 0 \Rightarrow a = -1$

Using the TA Framework to analyse Cam's argumentation, we begin by suggesting that he is reasoning in the Symbolic World of the TWM with elements from the Embodied World. He states a condition that must hold for an equilibrium point and then uses the operation of differentiation along with algebraic operations to identify a bifurcation point. In his exploration, Cam also uses an embodiment of a system with equilibria points (phase diagrams) to compare behaviour around equilibrium points, including the bifurcation point.

Underlying Cam's argumentation is his claim that there is a unique bifurcation at the point  $y = a = -1$ . This is suggested by an examination of phase line diagrams representing three cases regarding the position of the equilibrium points  $y = -1$ , and  $y = a$ . These cases are  $a < -1$ ,  $a = -1$ , and  $a > -1$ . In each case he tests the sign of  $\frac{dy}{dt}$  above and below each equilibrium point to determine whether the solutions are increasing or decreasing. These are shown as arrows on his phase line diagrams (arrows down represents a decreasing solution, while arrows upward represent a solution that is increasing). From here, the role of each equilibrium point is identified as either a source, sink or node. The middle diagram shown in Line 1 giving only one equilibrium point (identified as a node), guides the direction (structure) of Cam's argument.

The data that is used consists of an optimisation condition expressed in Line 3 and its evaluation in Line 4. The warrant is the substitution of the critical point  $y = \frac{a}{2} - \frac{1}{2}$  in Line 4 into  $\frac{dy}{dt}$  in Line 5. The backing is a theory of optimisation drawn from differential calculus along with rules of algebra.

Epistemic rationality is evidenced by the use of a known method of optimisation in calculus to derive a critical point  $y$  which is then used to find the value of the node  $a$ . His text indicates that he has used algebraic reasoning to establish the behaviour of the possible solutions to the differential equation, and to arrive at his conclusion in a deductive manner. Algebra is the main tool that is used to implement the calculus ideas supporting his argumentation.

Communicative rationality is established through a sequential development of algebraic reasoning resulting in the conclusion that  $a = -1$ . This is supported by the use of an embodiment and mathematical symbolisms drawn from differential calculus. Thus Cam's written argumentation is an illustration of his mathematical knowledge and his understanding of how it can be used.

Teleological rationality is noticeable by the achievement of certain goals. The first of these

involved the analysis of equilibria points which then lead to the identification of a bifurcation point. The next goal that was achieved was the implementation of a strategy that would show that  $a = -1$  is the bifurcation point. However, the reason for its uniqueness remained implicit in the calculations and there was no reference to this aspect of his claim in the conclusion. Rather it appeared that Cam may have thought this was obvious as it was the only value  $y$  could take given the optimisation procedure followed. However a fully justified claim would have included some reference to the uniqueness of  $a = -1$ .

The third year example of argumentation chosen by Cam comes from a problem given in Maths 333 Analysis in Higher Dimensions. This has been chosen by Cam because it reflects a connected understanding of a complex concepts, and demonstrates a reliance on definitional and theorem knowledge. Another aspect that Cam's argumentation highlights is an assumption about the mathematical knowledge of the reader. Cam refers to this example of his argumentation as a proof. The problem will now be given, and his argumentation presented in Table 7.11.

### 7.3.4 A Third Year Example of Argumentation

**Problem:**

Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfies the following:

- (i)  $f(s + t) = f(s) + f(t)$
- (ii)  $f$  is continuous

Claim:  $f(t) = tf(1)$  for all  $t \in \mathbb{R}$

Cam begins by giving preliminary information that he will use in his argumentation. This includes a generalisation of  $f(s + t) = f(s) + f(t)$  to a summation, and a short argument to show that the function  $f$  defined in the problem must be an odd function. Both of these results are integral to his proof of the proposition (problem) for  $t \in \mathbb{N}$ .

Only Cam's argumentation related to proof of the proposition  $f(t) = tf(1)$  for all  $t \in \mathbb{Z}$  and for all  $t \in \mathbb{Q}$  (given the specified conditions (i) and (ii) in the problem) will be presented for analysis. We have omitted an analysis of his proof that  $f(t) = tf(1)$  for all  $t \in \mathbb{Q}$ . The preamble to Cam's proof now follows.

1. Note that property (i) generalises to the following for finite sums,

$$f\left(\sum_{n=1}^N a_n\right) = \sum_{n=1}^N f(a_n).$$

2. Also, by the following argument,  $f$  is odd:

$$\begin{aligned} f(-x) &= f(x - 2x) = f(x) + f(-2x) \\ &= f(x) + f((-x) + (-x)) = f(x) + f(-x) + f(-x) \end{aligned}$$

$$0 = f(x) + f(-x) \Rightarrow f(x) = -f(-x)$$

So  $f$  is odd and we have  $f(0) = 0 = 0f(1)$

3. Since  $f(-t) = -f(t)$ , it is sufficient to prove that  $f(t) = tf(1)$  for the positive integers, and we can immediately extend the result so it holds in  $\mathbb{Z}$

Table 7.11: Cam's Example of a Third Year Argumentation

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L1 We will prove the proposition successively for  $t$  in  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$  and then  $\mathbb{R}$

L2: Proof for  $t \in \mathbb{N}$  follows by induction.

L3: Base case:  $f(1) = 1f(1)$  holds

L4: Inductive case: Suppose  $f(k) = kf(1) \Rightarrow f(k+1) = f(k)+f(1) = kf(1)+f(1) = (k+1)f(1)$

L5: So  $f(t) = tf(1)$  holds for all  $t \in \mathbb{N}$

L6: Since  $f$  is odd,  $f(-t) = -f(t) = -tf(1)$ . So  $f(t) = tf(1)$  holds for all  $t \in \mathbb{Z}$ .

From this point forward Cam now shows that the proposition is true for  $t \in \mathbb{R} \setminus \mathbb{Q}$

L6: Finally, let  $t \in \mathbb{R} \setminus \mathbb{Q}$ .

L7: Since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ ,  $t \in \overline{\mathbb{Q}}$

L8: So there exists a sequence  $\{q_1, q_2, q_3, \dots\} \subset \mathbb{Q}$  s.t.  $\lim q_n = t$

L9: By the continuity of  $f$ ,  $\lim f(q_n) = f(t)$ , and since each  $q_n \in \mathbb{Q}$

L10:  $\Rightarrow \lim q_n f(1) = tf(1)$

L11:  $\therefore f(t) = tf(1)$  for all  $t \in \mathbb{R}$ .

---

Cam's argumentation can be split into two parts. The first is his use of induction to show that  $t \in \mathbb{N}$ , and since  $f$  is odd then  $t \in \mathbb{Z}$ . In the second part, he uses the idea that  $t$  as a limit point and along with continuity of  $f$  to show that  $f(t) = tf(1)$  for all  $t \in \mathbb{R}$ .

The data used in the first part of his argumentation is  $f(t) = tf(1)$ . The warrant Cam uses is mathematical induction on  $f(t)$ . This is theoretically backed by mathematical induction as a valid tool of proof. However, he also uses the result that  $f(t)$  is an odd function to warrant the step from  $t \in \mathbb{N}$  to  $t \in \mathbb{Z}$ . This warrant is backed by the definition of an odd function. The conclusion is in Line 5:  $f(t) = tf(1)$  holds for all  $t \in \mathbb{Z}$ .

The second aspect of his argumentation is more complex, since some of the definitions he uses are implied. For example in Line 7 he has used a definition for ‘ $\mathbb{Q}$  to be dense in  $\mathbb{R}$ ’ to justify the selection of  $t$  ( $t \in \overline{\mathbb{Q}}$ ) and the existence of a sequence  $q_n, n \in \mathbb{N}$  that converges to a limit point  $t$  (written as  $\lim f(q_n) = f(t)$ ). He has also employed a theorem that says that a sequence  $q_n, n \in \mathbb{I}$  has an image  $f(q_n), n \in \mathbb{N}$  which is also a sequence. In Line 9 he has used Line 8 and the definition of continuity (though not formally stated) on an arbitrary sequence. Loosely speaking, two conditions that define continuity are satisfied. The first is all sequences  $q$  in  $Q$  converge to  $t$ , and second is that the images of these sequences  $f(q)$ , all converge to  $f(t)$ . In this second part of Cam’s argumentation the data is the selection of  $t \in \mathbb{R} \setminus \mathbb{Q}$ . The warrant used consists of the continuity of  $f$  on a sequence  $q_n, n \in \mathbb{I}$  that converges to  $t$ . This is supported with a theoretical backing comprising the definition for  $\mathbb{Q}$  to be dense in  $\mathbb{R}$ , and a theorem of continuity. The conclusion that is achieved is  $f(t) = tf(1)$  for all  $t \in \mathbb{R}$  (Line 11).

Using the TA Framework to analyse Cam’s argumentation, it is clear from the descriptions of his argumentation in the preceding two paragraphs that he is reasoning in the intersection of the Symbolic and Formal Worlds of the TWM.

Epistemic rationality is achieved through the use of a theoretical backing and warrant. Cam’s argumentation also reflects communicative rationality. For example, the layout of the argumentation begins with a preamble that states certain features of a function  $f$  defined in the problem. He has employed this knowledge to accomplish his goal of showing that  $f(t) = tf(1)$  holds for all  $t \in \mathbb{Z}$ , and elsewhere in his argumentation to show that  $f(t) = tf(1)$  holds for all  $t \in \mathbb{Q}$  (which has not been recorded here in Table 7.11). Cam’s written text also shows that he understands the concepts that he is using and is able to connect them (e.g., connecting the concept of ‘ $\mathbb{Q}$  is dense in  $\mathbb{R}$ ’ in Line 7, with ‘continuity’ in Line 9). In addition, deductive reasoning is supported by the use of sophisticated symbolism. In the full version of his proof, each goal that he has set is developed separately. It is evident from his writing, that Cam has written for a specific audience – a mathematical analyst. Thus, he has not dwelled on concepts that he assumes would be taken for granted by the reader. From a teleological viewpoint, rationality is

established by the justification of his claims that  $f(t) = tf(1)$  holds for all  $t \in \mathbb{Z}$ ,  $t \in \mathbb{R}$  and  $t \in \mathbb{Q}$  (not shown in this analysis). Each claim formed the basis of three distinct argumentations.

### 7.3.5 Summary of Cam's Progression in Argumentation

The examples of Cam's argumentation begin with an Example 1 in his first year. In this example he made a real attempt to make sense of the problem, and create an argumentation that might lead to a solution. However, an analysis of Attempt A indicated that a valid and useful starting point was not found (recall, he began with  $n^2 = 3k \Rightarrow n = 3l$ ). His argumentation suggested that he was guided by the language of the problem which, in his case, was not helpful to determining a starting point. His mathematical text in Attempt B indicated a new and valid departure point ( $n^2 = 3k$ ). However, Cam failed to consider a case that would cover all values of  $n$  (note  $n \in \mathbb{Z}$ ). Although there were issues of validity in Attempt A, and a fault in algebraic reasoning in Attempt B, a positive aspect emerged. Cam demonstrated the ability to use algebra to reason with, and the ability to reason in a deductive manner. In neither attempt did he begin with inductive reasoning. At this beginning stage of his university study of mathematics, it appears that Cam needed more exposure to problems that focus on the production of argumentation, and strategies for determining a useful starting point. These facets were developed further in his Maths 102 course, and concurrently in a first year philosophy course that he also took focussing on logic.

By semester two of his first year (Maths 150), Cam's argumentation had developed markedly. The written text of his argumentation suggested that he had not only developed his knowledge of mathematics, but also an awareness of linguistic norms required in writing a formal mathematical argument. Thus there is a beginning sense of formality in notation and logical reasoning. This is particularly noted by the implicit support of his warrant by a formal backing. In this example, Cam appears to be operating in the intersection of the Symbolic and Formal Worlds of the TWM. He has at least begun to realise the role of definitions in writing mathematical and hence the rationality of his argumentation is determined by reference to mathematical theory through definitions, as well as the use of a recognised method of mathematical argument, namely contradiction (epistemic rationality). The mathematical language used involved the formal use of symbols as well as phrases that are indicative of deductive reasoning. There is an alignment of

intention and outcome with respect to the goals set in his argumentation (communicative rationality). These (goals) are achieved through a series of deductive steps (teleological rationality).

By his second year Cam's argumentation reflects the use of algebra as a tool for reasoning. His writing continues to reflect his knowledge of definitions and theorems (albeit implicitly), and their use in reasoning deductively to the conclusion. Iconic and symbolic representations are also used. These facets of his argumentation indicate that his cognitive resources are drawn from the Symbolic World of the TWM. This is also reflected in a structural analysis of his argumentation. Both the data and warrant are backed by theory, and the warrant (along with algebraic processes) is used to secure the truthfulness of his claim. His knowledge of differential calculus supported the epistemic rationality of his argumentation. Implicit references are made to definitions and theorems (e.g., a single parameter differential equation, equilibrium points, bifurcation) and he relied on algebraic processes to establish certain results. However, in the example of argumentation provided, the conclusion needed to be linked to all elements of the claim.

In his third year example of argumentation, there is a leap in complexity of his knowledge of mathematics and proof. For example, in his argumentation, he combines specific proof strategies of direct proof, mathematical induction, and 'proof by cases'. Symbolisms used are dense with meaning, and support the logical coherence of his arguments. His arguments are presented in a more formal style of mathematics using specific phrases to deductively link lines of his argumentation (and also ideas within lines of argument). Definitions and theorems are implied in lines of work and not specified, however, they are visible to a reader who has a knowledge of analysis. He appears to have a concept knowledge surrounding the problem that includes number systems, even and odd functions, continuity, and limits of sequences. It is therefore clear that he is operating in the intersection of the Symbolic and Formal Worlds of the TWM. In the structure of his argumentation, multiple warrants are used with each having a theoretical backing. The data is also theoretically based (e.g., a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  that satisfies two certain conditions). Rationality comes from the backing of mathematical theory (epistemic), the logical development of argumentation leading to set goals (communicative), and the partitioning of his argumentation into goals that contribute to the final conclusion (teleological).



## 7.4 Long-term Progress in Argumentation: The Case of Able

In this section we consider the changing argumentation of Able, who showed some exceptional ability in mathematics in his second and third year undergraduate studies. He is an above average student who has received student scholarships to work with research mathematics educators and mathematicians, and is currently completing a BSc in applied mathematics. Usually, students who have done well in Year 13 mathematics at secondary school would have taken Maths 150 Advancing Mathematics 1 or Maths 108. However, Able began his studies of mathematics by taking Maths 102 Functioning in Mathematics, a course that would normally be taken by students needing to get their mathematics up to an acceptable level so that they could engage with the mathematics of Maths 150 or Maths 108. The following section begins with a problem presented in a preliminary study of argumentation that was given in his second semester of mathematics. It is presented here because it illustrates the role that personal conviction plays in his work.

### 7.4.1 A First Year Example of Argumentation

#### **Problem 2:**

If a square number  $n^2$  is a multiple of 3 then  $n$  must be a multiple of 3.

- Investigate this problem, and convince yourself whether it is always true, sometimes true, or never true.
- Write an argument that you would use to convince another person such as a mathematics lecturer of your conclusion.

Table 7.12 shows Able's argumentation. It is split into two parts, Attempt A and Attempt B to highlight the difference in the use of logical reasoning. In Attempt A Able uses an inductive approach that leads him to make an algebraic formulation. A closer look reveals that his argumentation strategy is based on a sequential interpretation of the *If-then*, namely that, if it is true that a square number  $n^2$  is a multiple of 3, then the consequence that  $n$  must be multiple of 3 necessarily follows. Thus Able follows a direct proof model. This may be because he has not yet seen proof by contradiction or contraposition.

The grounding of Able's argumentation in Attempt A are the numerical examples in Line 2 (which uses the square numbers generated in Line 1). The conclusion is that  $n^2 = 3a$ ,

for  $a = 0, 2, 12, 27, 48, 75$  in Line 5. The warrant being used is the factorisation of square numbers into a multiple of 3 and another factor. Able's backing is a rule of arithmetic (numerical factorisation).

In Attempt B Able begins with the same factorisation strategy used in his first attempt and identifies a result in mathematics stated in Line 7. He applies this result to a particular case of square numbers, those that are multiples of 3 and deduces their factors are multiples of 3 also (Line 9). Only Able's second attempt will be analysed using the TA Framework, as it is the argumentation that best illustrates an example of logical reasoning in his first year.

Table 7.12: Able's Example of a First Year Argumentation

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Attempt **A**:

L1: Check some square numbers that are multiples of 3. We have 0, 9, 36, 81, 144, 225, 324. Square factors are 0, 3, 6, 9, 12, 15, 18. All factors are multiples of 3

L2:  $0 = 3 \times 0$ ,  $9 = 3 \times 3$ ,  $36 = 3 \times 12$ ,  $81 = 3 \times 27$ ,  $144 = 3 \times 48$ ,  $225 = 3 \times 75$

L3:  $n^2 = 3a$ ,  $a \in \mathbb{Z}$

L4:  $\frac{n^2}{3} = a$

L5:  $n^2 = 3a$ ,  $a = 0, 3, 12, 27, 48, 75, \dots$

Attempt **B**:

L6: If 12 is a multiple of 3, then one of its factors must be a multiple of 3. 189 is a multiple of 3 then one, maybe both, of its factors will be a multiple of 3.

L7: If an integer is a multiple of 3, then one of its factors must be a multiple of 3.

L8: Let  $n^2$  be a square number that is a multiple of 3. Then one of its factors must be a multiple of 3. The factors are the same, and must both be multiples of 3.

L9: Then yes I agree. If  $n^2$  is a square number that is a multiple of 3, then  $n$  must be a multiple of 3.

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An analysis of Attempt B using the TA Framework, shows that Able is reasoning with mental a construction of factors and multiples of 3. Thus according to the framework proposed by Stewart and Thomas (2011), Able is reasoning deductively in the Process-Embodied World.

A structural analysis begins with the data which consists of integers of the form  $n^2$  that are also multiples of 3. The conclusion is precisely stated in Line 9, ‘If  $n^2$  is a square number that is a multiple of 3, then  $n$  must be a multiple of 3’. The warrant that Able is using is the result expressed in Line 7, which also serves as his backing.

Epistemic rationality is achieved through Able’s use of a result in mathematics to both warrant and back his argumentation. The language used in his argumentation includes logical ‘If-then’ statements that are accompanied by the related pairing, ‘let-then’. These contribute to the logical deductive coherence of his argumentation, both within his lines of statements, as well as from one statement to the next. Communicative rationality is expressed in the way in which he logically flows from one statement to the next. His argumentation is presented in a way that allows the reader to understand why his conclusion makes sense. One of the outcomes of his argumentation is the development of a personal conviction regarding the result. The deductive development of his argumentation indicates that certain conditions have been met that allow an inference to take place. For example the first inference in Line 8 is based on the assurance that the conditions in Line 7 have been met. The second inference in Line 8 satisfies the condition that  $n$  is a factor of  $n^2$ . Finally, Able’s argumentation uses a warrant that gives mathematical credence to his conclusion, one that if necessary, can be further validated using a theoretical backing. Teleological rationality is expressed in Able’s statement of self-conviction in Line 9, as this indicates that he believes that he has arrived at the conclusion on the basis of a sound reasoning. Able appears to understand why he was successful in securing the desired conclusion both in terms of the mathematical ideas involved in his argumentation, and the deductive enchaining of his statements. A summary of this analysis is presented in Table 7.13.

#### 7.4.2 A Second Year Example of Argumentation

This second example of Able’s argumentation is taken from a problem given to him in his second year. It comes from a set of problems that were given to some second year students who were part of a preliminary study of argumentation. This particular problem was fashioned from one used by Mejia-Ramos and Tall (2005) involving the odd nature of a derivative given that the function under consideration was even. Able had access to a copy of the definition of an even and odd function taken from his course textbook (Anton, Bivens, & Davis, 2005 p. 35) as well as a whiteboard to record his argumentation. One

Table 7.13: TA Framework Analysis of Able's First Year Argumentation

Three Worlds of Mathematics	Toulmin Model	Rationality
<p><b>Process–Embodied World:</b> Reasons on mental construction of multiples and factors of 3</p> <p>A known result from mathematics is used (L7)</p> <p>The conclusion is reasoned deductively</p>	<p><b>Data:</b> (Algebraic) <math>n^2</math> that are multiples of 3 (L8)</p> <p><b>Warrant:</b> (Algebraic) If an integer is a multiple of 3, then one of its factors must be a multiple of 3 (L7).</p> <p><b>Backing:</b> If an integer is a multiple of 3, then one of its factors must be a multiple of 3 (L7)</p> <p><b>Qualifier</b> Absolute qualifier (e.g. 'must' in L9)</p> <p><b>Rebuttal</b> (None provided)</p> <p><b>Conclusion:</b> If <math>n^2</math> is a square number that is a multiple of 3, then <math>n</math> must be a multiple of 3</p>	<p><b>Epistemic:</b> A result from mathematics regarding warrant is used to secure a statement of truth. The warrant is paired with an absolute qualifier</p> <p>Deductive reasoning is used to construct conclusion</p> <p><b>Communicative:</b> Text uses logical language (e.g., If-then (L3)). Text is written to show a sequence of steps of reason</p> <p><b>Teleological:</b> Lines of argument are intentionally set out to show logically coherent development of thinking</p> <p>Text is illustrative of deductive reasoning</p>

reason this example has been chosen is because it includes a problem that arose during the production of his argumentation in which the researcher intervened through a brief conversation.

**Problem:**

The derivative of an even function is an odd function.

- Is this statement always true, sometimes true, never true?
- Write an argument that you would use to convince another person such as a mathematics lecturer of your conclusion.

Able's response is given in Table 7.14. The Lines of his argumentation are given as usual (e.g., L1 means Line 1), but also included are lines of conversation between Able and the researcher. Thus A1 refers to Able's 1st spoken dialogue and R1 to the researcher's 1st spoken comment.

Able begins by writing down the definition of an even function, then uses the Chain Rule to differentiate both sides of the equality. After (correctly) completing this, he was unsure whether he had proved the result.

A key finding in this second year example of Able's argumentation was that, although he possessed the necessary mathematical tools to solve the problem (A2), he was unable to interpret the result he established in Line 5 without assistance from the interviewer

Table 7.14: Able's Example of a Second Year Argumentation

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Able refers to copy of the definition of an even function taken from the second year calculus textbook Anton, Bivens, and Davis (2005). This also has an accompanying set of diagrams of examples of even and odd functions.

L1:  $f(-x) = f(x)$  then  $f(x)$  is even

L2: Put  $-x = g(x)$  so we have  $f(-x) = f(g(x))$ . This is a composite function.

L3: The Chain Rule says that  $\frac{d}{dx}f(g(x)) = f'(g(x)) \times g'(x)$

L4:  $\therefore \frac{d}{dx}f(g(x)) = f'(-x) \times -1 = -f'(-x)$

L5: On this side [points to the right hand side of the equation in L1] you have  $f'(x)$ . Actually we can write  $-f'(-x) = f'(x)$

R1: What does the result you have here in this line [points to Line 5] tell you?

A1: No response [At this point Able did not appear to know how to interpret the information in Line 5.]

R2: Is this line [Line 5] incorrect?

A2: No I used the Chain Rule, it's right. I have the definition [of an even function] and it's right.

R3: Take another look at the definition of an odd function. [Able refers again to definition of an odd function.]

L6: If  $f(-x) = -f(x)$  then  $f(x)$  is odd [writes this statement from the copy of the definition taken from (Anton, Bivens, and Davis, 2005)]

L7:  $-f(-x) = f(x)$

A3: Yes I have it. We have this [pointing to Line 1:  $f(-x) = f(x)$  then  $f(x)$  is even], and its derivative here [points to Line 4:  $-f'(-x) = f'(x)$ ] is odd ... by definition.

L8: If  $f(x)$  is an even differentiable function then  $f'(x)$  is odd

R4: Does your conclusion hold for constant functions like  $f(x) = 4$  say?

A4: If  $f(x) = 4$  its derivative is zero ... the derivative of all constant functions is zero isn't it. Doesn't work for any constant function. Yes it must be an exception because  $f(x) = 0$  is an even function, see here its graph. Then we should really write this at the end [of Line 8].

L9: [If  $f(x)$  is an even differentiable function then  $f'(x)$  is odd] except when  $f(x)$  is a constant function.

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(R3). His derived statement in Line 5 did not appear to trigger a corresponding statement regarding an odd function. To move forward in his argumentation the researcher asked Able to look at the definition of an odd function (R3). In Line 7 Able adjusted the algebraic presentation of the definition of an odd function (in Line 6) to look like his statement in Line 5. In A3, Able sees that the statement  $-f'(-x) = f'(x)$  is a statement about the derivative of  $f(x) = f(-x)$ , and  $-f'(-x) = f'(x)$  is actually a statement expressing an odd function. The prompt by the researcher in R4 was given to ask Able to think about the constant function and whether or not it fitted in with his conclusion in Line 8. Interestingly, he used the two diagrams presented in Anton, Bivens, and Davis (2005, p. 35) that establish the symmetrical property of the graph of even and odd functions. Hence the reference he makes to the graph of an even function in A4.

We now analyse Able's argumentation including of the dialogical component using the TA Framework. The data used by Able is that  $f(x)$  is an even function and his conclusion is that the derivative of  $f'(x)$  is an odd function, except when  $f(x)$  is a constant function. The warrants used are the derivative of  $f(-x)$  which is backed by the Chain Rule from differential calculus, and an odd function  $f(x)$  which is by also backed by a definition. There is a modal qualifier indicated in Line 9 by the word 'except' which is paired with a statement of rebuttal specifying the conditions under which the assertion that the derivative is an odd function is true. The rebuttal (in italics) is the exception *except when  $f(x)$  is a constant function*.

Epistemic rationality in Able's argumentation comes from his reference to, and use of, definitions for odd and even functions function and his use of the Chain Rule which are both seated in mathematical theory. There is evidence of deductive reasoning along with reasoning from a diagram

If we consider just Able's written argumentation, it would appear that communicative rationality may not have been established if he had finished on Line 5 where he experienced difficulty with understanding the result ( $-f'(-x) = f'(x)$ ). However the moment of understanding is evidenced in A3 which is not part of his text, but is part of the conversation with the researcher. This leads to an argumentation that possesses communicative rationality on the basis that it demonstrate an understanding of the mathematics involved in the problem, and presents a clear account of his thinking.

Teleological rationality is expressed in Lines 1 to 5 where Able uses the Chain Rule to produce a mathematical statement. However, he needs to be able to interpret Line 5. This

goal is achieved when he reorganises his knowledge to see that  $-f'(x) = f'(x)$  involves a derivative satisfying the definition of an odd function. Teleological rationality is also expressed by his achievement of a further goal of determining whether there are any functions that are exceptions to the statement in Line 8, 'If  $f(x)$  is an even differentiable function then  $f'(x)$  is odd'. We summarise Able's argumentation using the TA Framework in Table 7.15.

Table 7.15: TA Framework Analysis of Able's Second Year Argumentation

Three Worlds of Mathematics	Toulmin Model	Rationality
<p><b>Intersection of the Embodied, Symbolic and Formal Worlds:</b> Algebraic symbols in the context of differential calculus are used as well as Definitions of even and odd functions used (L1–4, 6, &amp; 7) The conclusion is reasoned deductively (A3, A4) supported by reasoning from diagrams</p>	<p><b>Data:</b> (Algebraic) <math>f(x)</math> is an even function (L1)  <b>Warrant:</b> (Algebraic) differentiation using the Chain Rule            The representations of an even and odd function  <b>Backing:</b> A rule of differentiation from differential calculus            Definition of an odd and even function  <b>Qualifier</b> 'except' in L9 implying almost always true  <b>Rebuttal</b> Exception to the rule, 'when <math>f(x)</math> is a constant function' in L9  <b>Conclusion:</b> If <math>f(x)</math> is an even differentiable function then <math>f'(x)</math> is odd except when <math>f(x)</math> is a constant function.</p>	<p><b>Epistemic:</b> A theoretical backing is used to secure a statement of truth            Deductive reasoning (some of which is included in the provided conversation) is used to construct a conclusion in L9  <b>Communicative:</b> Intentions are developed (e.g. to understand a result in L4, and to determine whether the result is true for all (differentiable) functions)            Communicates an understanding of the conclusion (L8 &amp; 9)  <b>Teleological:</b> Achieves intended goals using an understanding of a definition, and deductive reasoning</p>

A feature to emerge out of Able's first and second year argumentations can be inferred. The first is a need to make sense of his thinking, and understand his argumentation. This relates to his desire for personal conviction of his solution, and his secondary aim of producing an argumentation that a reader would follow and understand as well. The second feature is an attitude of persistence in finding a solution. In the next subsection, we will consider two examples of Able's argumentations taken from his third year. Both examples are drawn from problems set in Maths 340, Real and Complex Calculus. One example illustrates the use of definitions in constructing an elementary proof, while the other illustrates how Able uses definitions in what is essentially a computation problem.

### 7.4.3 A Third Year Argumentation: Example 1

#### Problem:

Show that the function  $u(x, y) = e^y \cos x$  is harmonic on  $\mathbb{R}^2$ .

Able prepares for his argumentation by setting up the partial derivatives of  $u(x, y)$  as shown below in Table 7.16.

Table 7.16: Able's Example of a Third Year Argumentation: Example 1

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$u_x = -e^y \sin x$	$u_y = e^y \cos x$
$u_{xx} = -e^y \cos x$	$u_{yy} = e^y \cos x$
L1: $u_{xx} + u_{yy} = -e^y \cos x + e^y \cos x = 0$ i.e. $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy} = 0 \dots$ (A)	
L2: Also, functions are $C^2$ i.e. $u$ and its first and second partial derivatives w.r.t. $x$ and $y$ are continuous functions $\dots$ (B)	
L3: Since A and B are true, then $u(x, y)$ is harmonic on $\mathbb{R}^2$ .	

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The first thing to note about Able's argumentation in Table 7.16 is the implied definition of a harmonic function in Lines 1 and 2. From his two explanatory notes it can be seen that the definition he is using involves two aspects, the first being that  $\nabla^2 u = 0$ , and the second that  $u(x, y)$  is  $C^2$  (see Greenberg, 1998, p. 1145). Able's strategy is to show that the second partials of  $u(x, y)$  satisfy both these requirements, and hence the definition.

In mapping Able's argumentation to the TA Framework, we first note that he is reasoning in the intersection of the Symbolic and Formal Worlds of the TWM. He is using symbols that are particular to differential calculus as tools to justify with (e.g.,  $u_{xx} + u_{yy} = 0$ , and the continuity of  $u$ ,  $u_x$ ,  $u_{xx}$ ,  $u_y$ , and  $u_{yy}$ ). He reasons on a given definition in mathematics. Able does not appear to think it is necessary to provide a formal argument showing why the first and second partials are continuous. Regarding continuity Able made the following comment when asked to clarify why he did not provide reasons for L2:

"It's obvious because  $e^y$  is continuous, therefore  $-e^y$  is continuous. Cosine and sine are both continuous. The product of continuous functions are continuous, therefore, these [the first and second partials of  $u(x, y) = e^y \cos x$ ] functions are continuous." [Able]

The data Able uses are the function  $u(x, y) = e^y \cos x$  and its 1st and 2nd partial derivatives which he sets up at the beginning of his argumentation. The conclusion that  $u(x, y)$



is harmonic on  $\mathbb{R}^2$  is given in Line 3. The warrant is satisfaction of the Laplace equation using the second partials provided (Line 1), and the continuity of the first and second partials of  $u(x, y)$  in Line 2. The warrants are backed by rules from differential calculus, and two facet comprising a definition. The warrant is given an absolute qualifier represented by the word ‘is’ in the phrase ‘is harmonic’. An indication that Able’s argumentation includes an element of self-conviction comes from Lines 1 and 2, where he seems to state, and then check off each condition of the definition. In the next example we will see this occur more explicitly.

Able’s argumentation can be said to be epistemologically rational because his warrant is backed theoretically by differential calculus, and he uses deductive reasoning to arrive at his conclusion. Communicative rationality is observed in his provision of additional notes that indicate that he understands aspects of the definition that he is employing to show the conclusion. Teleological rationality is evidenced by the achievement of his goals using the definition. It is also achieved because through he has presented his argumentation in such a way that reader can understand that his reasoning strategy can only lead to the stated conclusion. His argumentation is summarised using the TA Framework in Table 7.17.

Table 7.17: TA Framework Analysis of Able’s Third Year Argumentation:  
Example 1

Three Worlds of Mathematics	Toulmin Model	Rationality
<p><b>Intersection of the Symbolic and Formal Worlds:</b> Symbols associated with differential calculus are used. A given definition of a harmonic function is used (L1 &amp; L2). The conclusion is reasoned deductively with assumptions regarding continuity (L3).</p>	<p><b>Data:</b> (Algebraic) A function <math>u(x, y)</math> and its 1st and 2nd partials (see argumentation preamble)</p> <p><b>Warrant:</b> (Differential Calculus) The Laplace equation is satisfied (L1) and, the continuity <math>u(x, y)</math> and its 1st and 2nd partials (expressed as <math>C^2</math>) (L2).</p> <p><b>Backing:</b> A definition from Mathematics (Differential Calculus)</p> <p><b>Qualifier:</b> Absolute qualifier (in L3)</p> <p><b>Rebuttal:</b> (None provided)</p> <p><b>Conclusion:</b> <math>u(x, y)</math> is harmonic on <math>\mathbb{R}^2</math> (L3)</p>	<p><b>Epistemic:</b> Warrant is backed by a formal definition in mathematics. Deductive reasoning from L1 and 2 is used to construct conclusion in L3 (with assumptions)</p> <p><b>Communicative:</b> Appropriate symbolic language is used to communicate understanding and self-conviction. Goals are established that are related to the definition. An explanatory note is included in L2</p> <p><b>Teleological:</b> Goals are achieved by showing each elements of the definition are satisfied</p>

The next example of argumentation is also taken from Maths 340 and set in the context of solving what is essentially a computation problem. This argumentation indicates the emphasis Able continues to place on the satisfaction of definitional requirements, and on ensuring personal understanding.

#### 7.4.4 A Third Year Argumentation: Example 2

At first glance, this problem seems to be procedural. However, it is also an example of argumentation because of Able's goal to convince himself that he understands the concepts involved, as well as convincing an other that he understands the process involved.

**Problem:**

Compute the integral  $\oint_C \frac{1}{z^3 - z^2 + z - 1} dz$  when:

$C$  is a circle of radius 1 centred at  $z = 1$ , oriented clockwise

Able's argumentation is laid out in Table 7.18. Of particular interest is the justification of his use of the Cauchy Integral Formula (see Greenberg, 1998, p. 1200). In Line 3 Able uses a diagram to show that his choice of  $f(z)$  is analytic in a simply connected domain  $D$  which he has drawn to avoid the two singular points at  $\pm i$ . He has also shown a simple closed curve (a circle of radius 1) centred at the pole  $z = 1$  oriented anticlockwise and in the domain  $D$ . These conditions are checked off in Line 4. Thus Able considers that he is justified in using the Cauchy Integral Formula. We see from Lines 1 to 3 that Able uses algebra and the Factor Theorem to simplify the function  $\oint_C \frac{1}{z^3 - z^2 + z - 1} dz$  to  $\oint_C \frac{1}{(z - 1)(z - i)(z + i)} dz$ . Using further algebra, he is able to rewrite this into a form suitable for the use of the Cauchy Integral Formula  $\left( \oint_C \frac{f(z)}{z - a} dz \right)$ . Once this expression is found, the Cauchy Integral Formula becomes a simple computation of  $2\pi i f(a)$  with  $a$  determined from the integral. This example highlights the importance of Able places on understanding the problem (in this instance this meant understanding a definition), being personally convinced of elements within his argumentation and thus the argumentation itself.

Table 7.18: Able's Example of a Third Year Argumentation: Example 2

L1: First it is a good idea to factorise  $g(z) = z^3 - z^2 + z - 1$   
 $g(1) = 1 - 1 + 1 - 1 = 0$ ,  $\therefore (z - 1)$  is a factor of  $g(z)$

L2:  $(z^3 - z^2 + z - 1) = (z - 1)(z^2 - 1) = (z - 1)(z - i)(z + i)$

L3:  $\therefore \oint_C \frac{1}{g(z)} dz = \oint_C \frac{1}{(z - 1)(z - i)(z + i)} dz$

x - singularity points.

**Cauchy Integral formula..**  
 let  $f(z)$  be analytic in a simply connected ~~region~~ domain  $D$ , let  $C$  be a piecewise smooth, simple, closed curve in  $D$  oriented counterclockwise, and let  $a$  be any point within  $C$   
 Then  

$$\oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

To use Cauchy integral formula,  
 by choosing  $f(z) = \frac{1}{z^2+1}$   
 $a = 1$

L4: Checking the conditions:

- ✓  $C$  is a piecewise smooth, simple, closed curve oriented counterclockwise in  $D$
- ✓  $f(z) = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}$  is analytic in a simply connected domain  $D$
- ✓  $a = 1$  is a point within  $C$

L5: Then  $\oint_C \frac{1}{(z^2+1)(z-1)} dz = \oint_C \frac{\left(\frac{1}{z^2+1}\right)}{(z-1)} dz, \left( = \oint_C \frac{f(z)}{z-a} dz \right)$

$$\oint_C \frac{\left(\frac{1}{z^2+1}\right)}{(z-1)} dz = 2\pi i f(a) = 2\pi i f(1) = \pi i$$

L6:  $\therefore \oint_C \frac{1}{(z^2+1)(z-1)} dz = \pi i.$

An analysis of Able's argumentation using the TA Framework begins with the suggestion that he is reasoning in the intersection of the Symbolic and Formal Worlds of the TWM. He has referred to a given definition in mathematics, and uses symbols from integral calculus as well as algebraic operations to support his reasoning. The data that Able uses is the integrand  $\frac{1}{(z-1)(z-i)(z+i)}$ . From this data, Able constructs the diagram in Line 3 that corresponds to the 'checked conditions' in Line 4 thus justifying his use of the Cauchy Integral Formula. This provides the warrant for his argument which is backed by the definition of the Cauchy Integral Formula. The conclusion is simplified in Line 6 to  $\oint_c \frac{1}{(z^2+1)(z-1)} dz = \pi i$ .

Epistemic rationality is attained because gives a visual account of the theoretical backing he is using and then uses algebra to reformulate the date so that the theorem can be used. Also of note is the development of the singular points and the pole at  $z = 1$  resulting from an application of the Factor Theorem.

Communicative rationality comes from the diagram in Line 3, where he demonstrates his understanding of the theorem and its conditions of use in (given in Line 4). Able's first goal is to determine whether the use of the Cauchy Integral Formula is justified. His next goal is to form an integrand so that the theorem can be applied, and the third is to compute the integral using the Formula.

Teleological rationality is embedded in the achievement of his two goals. The first was to show that the conditions for the use of the Cauchy Integral Formula are were satisfied, and the second was to apply the formula. Able's argumentation is summarised using the TA Framework in Table 7.19

#### 7.4.5 Summary of Able's Progression in Argumentation

From a diachronic analysis of Able's argumentation spanning three years of study in mathematics, it is possible to see elements that have remained stable in his progression as well as changes that have emerged. One element that has remained stable has been Able's focus on understanding the mathematics that he is engaged with. This has resulted in an explanatory style of argumentation that satisfies both his need for a personal conviction, and his goal of communicating his understanding and solution to a reader. An aspect of his progression has been an increasing focus on providing theoretical backings for his convictions. Since his first year, he has gone from using elementary results in arithmetic and mathematics to using complex definitions and theorems so that by his third year,

Table 7.19: TA Framework Analysis of Able's Third Year Argumentation: Example 2

Three Worlds of Mathematics	Toulmin Model	Rationality
<p><b>The intersection of the Symbolic and Formal Worlds:</b> Symbols from integral calculus, and algebraic operations are used to support reasoning A given theorem from mathematics is used(L3)</p>	<p><b>Data:</b> (Algebraic) An integrand <b>Warrant:</b> (Theorem) All statements in a theorem in must be satisfied (L4) <b>Backing:</b> A Theorem from Mathematics <b>Qualifier</b> Implied absolute qualifier (in L5) <b>Rebuttal</b> (None provided) <b>Conclusion:</b> The Cauchy Integral Formula can be used (L5)</p>	<p><b>Epistemic:</b> Warrant used is theoretically backed Deductive reasoning from L3 to L4 along with algebraic operations in L5 are used to construct a valid conclusion in L6 <b>Communicative:</b> Argumentation develops understanding of the solution through analysis of a theorem, and the support of a diagram Subsidiary goals are set based on conditions of the theorem, and the construction of an appropriate integrand. The main goal is to compute the integral <b>Teleological:</b> All goals are achieved by showing that all conditions of the theorem were satisfied and using algebra to make a suitable integrand form. The original integral was correctly computed using the Formula Text is illustrative of deductive reasoning</p>

Able was setting his argumentations in a formal structure involving the use of sophisticated symbolic representations, accompanied by an understanding of important features of complex definitions.

Given that the examples chosen here to portray his progression in argumentation during his third year appear to be standard questions and responses for a complex analysis course, some may argue that Able may have followed a ritualistic proof scheme (see Harel & Sowder, 1998), possibly basing his solutions on examples from the board or from the textbook. However, this is not consistent with the evidence showing that Able clearly understood the conditions that allowed definitions and theorems to be used. For example, Able had a concept image of the Cauchy Integral given by  $\oint_c \frac{f(z)}{(z-a)} dz = 2\pi i f(a)$  that included, analytic functions, the line integral, a simply connected path, singular points, and deformation of a path to a circle. Note, these are just some aspects in his concept image. As shown in his third year examples, Able could link a definition or theorem to a problem and develop argumentation leading to an appropriate problem solving strategy that could be used to use them.

The Habermasian attributes of rationality in Able's argumentation have also developed over three years. In particular, from his second year, epistemological rationality has emerged from warrants that have become increasingly theoretical in their backing. These backings have taken the form of either a definition or theorem. Along with this is the use of increasingly sophisticated symbols (e.g.,  $\oint_C$ ,  $C^2$ ,  $u_{xx}$ , and  $\nabla^2$ ) and a thickening of his symbol sense. From the content of his argumentations it is apparent that Able has developed an elaborate concept image for these. Communicative rationality has continued to develop by ensuring that his reasoning strategies are packaged in a way that ensures self-conviction as well as the attainment of the conviction of the reader in the solution. Since his second year, he has continued to do this by constructing subsidiary objectives in his argumentation that are linked to the theoretical backing of his warrant. Teleological rationality has developed from achieving goals through the inclusion of inductive means, to reaching these using theoretical ideas drawn from mathematics. This includes the use of reasoning deductively although other modes of reasoning may also be present.

## 7.5 Long-term Progress in Argumentation: The Case of Tas

Tas has already completed a BSc degree in mathematics. He is currently working as a private tutor of mathematics. Tas is a talented mathematician who in his second year received a Summer Scholarship to work with a research mathematician over the summer vacation. Unlike most students, Tas undertook third year courses in group theory and analysis in his second year at university. In his first year, he was also a member of a pilot study that looked at how students construct collective argumentation in mathematics. The following subsection presents his selection of problems and their solutions chosen to represent his progress in argumentation. The first is taken from a problem that was given him as part of the pilot study.

### 7.5.1 A First Year Example of Argumentation

#### **Problem 3:**

A postgraduate mathematics student suggests there may be a number that is not a multiple of 3 that is a factor of square number that is a multiple of 3. Is he correct, or incorrect? Tas' argumentation is presented in Table 7.20.

This example illustrates how Tas uses a theoretical backing very early in his experience

Table 7.20: Tas' Example of a First Year Argumentation

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L1: If  $a \in \mathbb{Z}$  then  $3|a$ , or  $3 \nmid a$

L2: If  $3 \nmid a$ , then  $a = 3k + r, k \in \mathbb{Z}, (r = 1, 2)$

L3:  $\Rightarrow a^2 = (3k + r)^2 = 9k^2 + 6kr + r^2$

L4: Case (1)  $r = 1 \Rightarrow a^2 = 9k^2 + 6k + 1$ , which is not divisible by 3

L5: Case (2)  $r = 2 \Rightarrow a^2 = 9k^2 + 12k + 4$ , which is not divisible by 3

L6: Therefore he is incorrect

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of university mathematics. He uses a method of direct proof involving a consideration of cases to show that the postgraduate student's statement is incorrect.

An analysis of his argumentation using the TA Framework reveals that Tas is reasoning in the intersection of the Symbolic and Formal Worlds of the TWM. Tas reasons with algebraic symbols, and uses logical statements such as, 'if-then', ' $\Rightarrow$ ' for implication, and 'therefore' (as an implication). Other symbols from mathematics are also employed in his argumentation for example, ' $\in$ ' representing membership and ' $|$ ' and ' $\nmid$ ' representing 'divides' and 'does not divide'. His recording in Line 2 ( $3 \nmid a$ , then  $a = 3k + r, k \in \mathbb{Z}, (r = 1, 2)$ ) indicates that he is aware of the Division Theorem.

The data grounding Tas' argumentation is the statement  $3 \nmid a$  found in Line 2. The conclusion is that if  $a$  is not a multiple of 3, then neither is its square, which is summarised in Line 6 by the conclusion, 'therefore the postgraduate student is incorrect'. His conclusion is warranted by the constructions of  $a^2$ ,  $a^2 = 9k^2 + 6k + 1$  and  $a^2 = 9k^2 + 12k + 4$  in Lines 3 to 5. These are backed by the Division Theorem from mathematics, and the rules of algebra. His warrant is paired with an absolute qualifier in Line 6 (the word 'is', and the underlining of 'incorrect').

Tas' argumentation has epistemological rationality on account of the theoretical backing that is used (the Division Theorem). From this epistemic core, Tas is able to justify his construction of  $a$ ,  $a^2$ , and his conclusion. It is clear that the approach taken by Tas could only result in the conclusion in Line 6, given the correct interpretation of the divisibility of  $a^2$  by 3 in Lines 4 and 5.

Communicative rationality in Tas' argumentation is evidenced in two ways. Firstly, Tas provides an understanding of what it means for a number not to be divisible by another

Table 7.21: TA Framework Analysis of Tas' First Year Argumentation

Three Worlds of Mathematics	Toulmin Model	Rationality
<p><b>Intersection of the Symbolic and Formal World:</b> Reasoning is done with algebraic symbols, and uses logical statements such as, 'if-then', '<math>\Rightarrow</math>', to refer to as implication. Other symbols are correctly used from mathematics to describe ideas (e.g. the idea of '3 does not divide <math>a</math>')</p>	<p><b>Data:</b> (Arithmetic/Algebraic) <math>a</math> is not divisible by 3, represented as <math>3 \nmid a</math> in Line 2.</p> <p><b>Warrant:</b> (Algebraic) Presented in Line 2 as <math>a = 3k + r, k \in \mathbb{Z}, (r = 1, 2)</math></p> <p><b>Backing:</b> (Algebraic/Theory or Theorem) Rules of algebra, and the Division Theorem</p> <p><b>Qualifier</b> Absolute qualifier implied by the word 'is', and the underlining of 'incorrect' in Line 6</p> <p><b>Rebuttal</b> (None provided)</p> <p><b>Conclusion:</b> He (the postgraduate student) is incorrect.</p>	<p><b>Epistemic:</b> Warrant is backed by a theoretical backing (Division Theorem &amp; rules of algebra) so that his argumentation is mathematically justified.</p> <p><b>Communicative:</b> Text conveys an understanding of what it means if a number is divisible by 3, and provides a way in which divisibility can be checked. Reasoning is communicated through sequences of deductive steps</p> <p><b>Teleological:</b> Goals and sub-goals are set, and achieved using a case by case examination of <math>a^2 = 9k^2 + 6k + r^2</math>, for when <math>r = 1</math> and when <math>r = 2</math>.</p>

and provides a strategy for determining divisibility.

Tas sets up a subgoal that is directly related to the accomplishment of his claim that the postgraduate student is wrong. The subgoal is to show that for  $a = 3k + r$  and  $r = 1$  or  $2$ ,  $a^2$  cannot be divisible by 3. Teleological rationality is evidenced in the achievement of this subgoal, which is accomplished through a case by case consideration of the algebraic expressions  $a^2 = 9k^2 + 6k + 1$  and  $a^2 = 9k^2 + 6k + 4$ . In both cases he shows that they are not divisible by 3. The fact that they are both of the form  $3z + 1$  is 'obvious' to him. This in turn leads him to deduce that the postgraduate student is incorrect. Tas' argumentation is summarised using the TA Framework in Table 7.21.

The next problem provides an example of Tas' argumentation in his second year. The problem was given to him to complete during a problem solving session related to this study. This interaction occurred in an office, and Tas recorded his argumentation on a whiteboard. Tas referred to his argumentation as an example of a proof.



### 7.5.2 A Second Year Example of Argumentation

**Problem:**

Show that if  $f(x) = x + 1$ , and  $g(x) = (x - 1)^2$ , then  $f \geq g$  whenever  $x$  is taken from the set  $S = \{x \in \mathbb{R}, 0 \leq x \leq 3\}$

Tas' argumentation is given in Table 7.22. It is based on what he refers to as 'working back' and 'working forward'. In this strategy, Tas starts with the conclusion and works his way back in steps to a suitable starting point for a proof. After formalising his start point in Line 5, Tas constructs his proof by 'working forward' to a conclusion by reversing the order steps.

Table 7.22: Tas' Example of a Second Year Argumentation

<b>Argumentation Associated with Proof</b>
L1: $f(x) = x + 1, g(x) = (x - 1)^2$ [Tas] I'll work backwards to find a starting point.
L2: $\forall x_0 \in S$ assume that $x_0 + 1 \geq (x_0 - 1)^2$
L3: $\Leftrightarrow x_0 + 1 \geq x_0^2 - 2x_0 + 1$
L4: $\Leftrightarrow x_0(3 - x_0) \geq 0$
L5: Proof: $\forall x_0 \in S, x_0(3 - x_0) \geq 0$
L6: $\Leftrightarrow 3x_0 - x_0^2 \geq 0$
L7: $\Leftrightarrow 3x_0 \geq x_0^2$
L8: $\Leftrightarrow x_0 + 1 \geq x_0^2 - 2x_0 + 1$
L9: $\Leftrightarrow x_0 + 1 \geq (x_0 - 1)^2$
L10: $\Leftrightarrow f(x_0) \geq g(x_0)$ .

There are some aspects worth noting in Tas' argumentation. The first is that he uses an arbitrary choice of  $x$  to generalise the conclusion that  $f(x) \geq g(x)$  for all  $x$  chosen from  $S$ . The second point to note occurs in Line 2 where Tas appears to treat the set  $S = \{x \in \mathbb{R}, 0 \leq x \leq 3\}$  as part of a definition (this will be discussed in more detail below). In Line 5 he asserts that for an arbitrary  $x$  chosen from set  $S$ , the relationship  $x_0(3 - x_0) \geq 0$  holds. This is justified by Lines 1 to 4. Had Tas not provided the 'working backwards' section to derive a start point, it is likely that he would have had to find a start point using the fact that  $S = \{x \in \mathbb{R}, 0 \leq x \leq 3\}$  implies that  $x - 3 \leq 0$ , and  $x \geq 0$ .

Using the TA Framework to analyse Tas' argumentation, we see that he reasoning in the Symbolic World of the TWM with some formal world concepts. Support for this comes from the deductive reasoning supported by algebraic operations that has led to a start point for his proof in Line 5, and which subsequently leads to the conclusion in Line 10. Symbols of formal mathematical logic are also used such as  $\forall$  for 'for all', and  $\Leftrightarrow$  for 'if and only if'. The first occurrence of the if and only if symbol is in Line 3. Assuming this was done purposefully, then Tas would have taken as true that the statement that if  $f \geq g$  whenever  $x \in S$ , then whenever  $x \in S$ ,  $f \geq g$ . This leads to the suggestion that he is treating the statement in the problem that  $f \geq g$  whenever  $x$  is taken from the set  $S = \{x \in \mathbb{R}, 0 \leq x \leq 3\}$  as a definition. This is important because Tas does not have to justify his choice of  $x_0$  at Line 5 of his argumentation, the choice is treated as being a self evident generalisation.

The data consists of two functions  $f(x)$  and  $g(x)$  in Line 1 that are related to each-other by an inequality (in Line 2), and the arbitrary choice of an  $x_0$  from the set  $S$ . There are two conclusions that make up Tas' argumentation. Both of these appear to share the same warrant and backing. The first is the conclusion that if  $x_0$  is chosen from set  $S$ , then relationship  $x_0(3 - x_0) \geq 0$  holds (Lines 4 & 5). The warrant being used is algebraic operations on the inequality  $x_0 + 1 \geq (x_0 - 1)^2$  in Line 2 to yield  $x_0(3 - x_0) \geq 0$  in Line 4. The backing consists of the definition of  $S$  ( $S = \{x \in \mathbb{R}, 0 \leq x \leq 3\}$ ), and the rules of algebra. The second and main conclusion of Tas' argumentation is in Line 13 which shows that given  $x_0 \in S$ ,  $f(x_0) \geq g(x_0)$ . The warrant here is that  $f(x) \geq g(x)$  if and only if  $\forall x_0 \in S$  assume that  $x_0 + 1 \geq (x_0 - 1)^2$ , because it means that the deductive steps can be reversed. The backing comes from mathematical logic regarding the logical construction of 'if and only if' (or as a symbol,  $\Leftrightarrow$ ).

Epistemic rationality is attained from the deduction that  $\forall x_0 \in S$ ,  $x_0(3 - x_0) \geq 0$  if and only if  $f(x) \geq g(x)$  because this means that the deductive steps from Line 5 on are valid, and so the conclusion is valid.

Communicative rationality is evident in the way in which Tas shows how successive steps of reasoning are the result of an algebraic process. The algebra helps the reader to understand the construction of his argumentation. There is some unnecessary duplication in Line 2, but this does not appear to affect the logical progression set out in his argumentation.

Tas sets a subgoal to determine a start point in Line 1 with the intention of using it to derive the necessary and sufficient conclusion in Line 10. Teleological rationality is derived

Table 7.23: TA Framework Analysis of Tas' Second Year Example of Argumentation

Three Worlds of Mathematics	Toulmin Model	Rationality
<p><b>Intersection Embodied, Symbolic, Formal Worlds:</b> Reasons with algebraic symbols</p> <p>Uses a definition to construct a bi-conditional relationship</p> <p>The start point in Line 5 is reasoned deductively, and so is the conclusion in Line 10</p>	<p><b>Data:</b> (Algebraic) Functions <math>f(x)</math> and <math>g(x)</math> in line 1</p> <p>The inequality: <math>f(x) \geq g(x)</math></p> <p>A choice of <math>x_0 \in S</math></p> <p><b>Warrant:</b> (Algebraic) Algebraic operations that take <math>x_0 + 1 \geq (x_0 - 1)^2</math> to <math>x_0(3 - x_0) \geq 0</math></p> <p><b>Backing:</b> (Algebraic) Rules of algebra and set theory</p> <p><b>Qualifier</b> Absolute qualifier implied by deductions from Lines 5 to 10</p> <p><b>Rebuttal</b> (None provided)</p> <p><b>Conclusion:</b> <math>\forall x_0 \in S, f(x_0) \geq g(x_0)</math> which is found partially in Line 5, and in Line 10.</p>	<p><b>Epistemic:</b> Deductive reasoning, supported by algebraic operations, is used to construct a step by step argumentation where the conclusion is necessary and sufficient.</p> <p>The backing is theoretical, consisting of the rules of algebra and elementary set theory</p> <p><b>Communicative:</b> Text focusses on establishing a conclusion that is mathematically correct. Therefore each step is derived algebraically from the one before it.</p> <p>Tas' use of 'if and only if' indicates that each step can be reversed and that his argumentation can be bi-conditionally justified.</p> <p><b>Teleological:</b> Goals and subgoals are established. These are achieved through deductive reasoning.</p>

from the successful achievement of his goals in this argumentation. The first of these is the reduction of  $x_0 + 1 \geq (x_0 - 1)^2$  to its simplest form,  $x_0(3 - x_0) \geq 0$  to formulate the starting point for his proof. It turns out that this form is explicitly linked to the definition of the set S. His second goal (actually his primary goal) was to show that  $f(x) \geq g(x)$ . The argumentation accomplished this by a reversal of steps used to secure his first goal. Tas' argumentation is summarised using the TA Framework in Table 7.23.

The next problem is an example chosen by Tas to represent an example of argumentation in his third year. Tas' rationale for his choice was that a proof should stem from problem solving, which this proof illustrates. This example comes from a problem that was posed to him during one of his online tutoring sessions.

### 7.5.3 A Third Year Example of Argumentation

Prove that  $\pi(e - 1) < \int_0^\pi e^{|\cos 4x|} dx < 2 \left( e^{\frac{\pi}{2}} - 1 \right)$

Tas' argumentation is laid out in Tables 7.24 and 7.25.

Tas breaks the problem into two natural parts based on the inequalities,  $\pi(e - 1) < \int_0^\pi e^{|\cos 4x|} dx$  and  $\int_0^\pi e^{|\cos 4x|} dx < 2 \left( e^{\frac{\pi}{2}} - 1 \right)$ . He begins with a consideration of  $\pi(e - 1) < \int_0^\pi e^{|\cos 4x|} dx$ .

The difficulty in this problem appears to be the absolute value in the integrand. Tas overcomes this by rewriting the integral  $\int_0^\pi e^{|\cos 4x|} dx$  as  $8 \int_0^{\frac{\pi}{8}} e^{\cos 4x} dx$  in Line 1. To do this, he considered the symmetry of the graph of  $f(x) = |\cos 4x|$  over the interval  $[0, \pi]$ . In discussion with Tas, he said that it took him approximately 4 hours to conceive the next step, and he referred to it as the critical step.

The critical step for Tas was to use this second integral to find a way of showing that  $\pi(e-1) < \int_0^\pi e^{|\cos 4x|} dx$ . The strategy he devised was based on reducing the domain to the interval  $\left[0, \frac{\pi}{8}\right]$ , making the sketch of the graph of  $\cos 4x$ , with  $x \in \left[0, \frac{\pi}{8}\right]$  and then finding a function that was less than or equal to it. The function he used was the simplest possible – a linear function. The graph is also shown in the diagram on Line 1. He determined that this line segment comes from a line whose equation is  $y = -\frac{8}{\pi}x + 1$ .

The next part of his strategy was to take the inequality constructed in Line 3 (i.e.,  $e^{\cos 4x} \geq e^{-\frac{8}{\pi}x+1}$  for  $x \in [0, \frac{\pi}{8}]$ ) and transform it into a statement expressing the strict inequality of two integrals (Line 4). The argument underlying Line 4 is based on the representation of the integral as an area. Finally Line 5 extends the integral inequality to a consideration of  $\int_0^\pi e^{|\cos 4x|} dx = 8 \int_0^{\frac{\pi}{8}} e^{\cos 4x} dx > 8 \int_0^{\frac{\pi}{8}} e^{-\frac{8}{\pi}x+1} dx$ . An evaluation of the right hand integral leads to the desired conclusion in Line 6. The same strategy is used to establish the result that  $\int_0^\pi e^{|\cos 4x|} dx < 2(e^{\frac{\pi}{2}} - 1)$  in Line 10. However this time the calculation of the equation of the line in the diagram preceding Line 7 requires a knowledge of the gradient at the inflection point  $(\frac{\pi}{8}, 0)$  of  $f(x) = \cos 4x$  at  $(\frac{\pi}{8}, 0)$ .

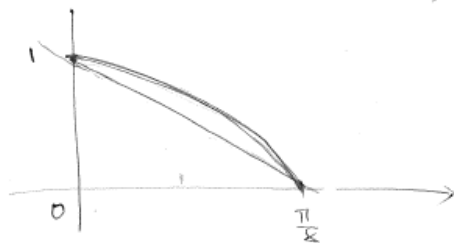
An analysis of Tas' argumentation begins with the suggestion that he is working in the intersection of the Symbolic, Embodied, and Symbolic Worlds of the TWM. There are at least three aspects of his argumentation that suggest this. The first is that he reasons from embodiments of symbols, for example interpreting  $\int_0^\pi e^{|\cos 4x|} dx = 8 \int_0^{\frac{\pi}{8}} e^{\cos 4x} dx$  as a comparison of areas under the graph of each integrand. Secondly, he uses two diagrams to progressively unpack his knowledge and understanding of the mathematical ideas contained in them. Devolving of his proof is done through a deductive process of reasoning supplemented with computational processes. Tas also reasons in an embodied way – using area an area comparison.

Tas' argumentation is grounded by the inequalities  $\cos 4x \geq -\frac{8}{\pi}x + 1$  for  $x \in [0, \frac{\pi}{8}]$  and  $\cos 4x \leq -4(x - \frac{\pi}{2})$  for  $x \in [0, \frac{\pi}{8}]$ . The conclusion involves two strict integral inequalities,  $\pi(e-1) < \int_0^\pi e^{|\cos 4x|} dx$ , and  $\int_0^\pi e^{|\cos 4x|} dx < 2(e^{\frac{\pi}{2}} - 1)$ , which when combined give the original claim that  $\pi(e-1) < \int_0^\pi e^{|\cos 4x|} dx < 2(e^{\frac{\pi}{2}} - 1)$ .

Table 7.24: Tas' Example of a Third Year Argumentation: Continued Over-page

$$\text{L1: } \int_0^\pi e^{|\cos 4x|} dx = 8 \int_0^{\frac{\pi}{8}} e^{\cos 4x} dx$$

We need to find a function that is less than  $\cos 4x$ . Consider the following diagram:



$$\text{L2: } \text{The equation of the line is } y - 1 = \frac{-1}{\frac{\pi}{8}}(x - 0) \Rightarrow y = -\frac{8}{\pi}x + 1$$

$$\text{L3: } \cos 4x \geq -\frac{8}{\pi}x + 1 \text{ for } x \in [0, \frac{\pi}{8}]$$

Taking the natural exponent of both sides gives,  $e^{\cos 4x} \geq e^{-\frac{8}{\pi}x+1}$  for  $x \in [0, \frac{\pi}{8}]$

L4: The area under the function on LHS of this inequality will be  $>$  then the area under the function on the RHS over the interval  $[0, \frac{\pi}{8}]$  i.e.

$$\int_0^{\frac{\pi}{8}} e^{\cos 4x} dx > \int_0^{\frac{\pi}{8}} e^{-\frac{8}{\pi}x+1} dx$$

$$\text{L5: } \text{Therefore, } 8 \int_0^{\frac{\pi}{8}} e^{\cos 4x} dx > 8 \int_0^{\frac{\pi}{8}} e^{-\frac{8}{\pi}x+1} dx = 8 \left[ -\frac{\pi}{8} e^{-\frac{8}{\pi}x+1} \right]_0^{\frac{\pi}{8}}$$

$$8 \left[ -\frac{\pi}{8} e^{-\frac{8}{\pi}x+1} \right]_0^{\frac{\pi}{8}} = 8 \left( -\frac{\pi}{8} - \left( -\frac{\pi}{8} e \right) \right) = 8 \left( \frac{\pi}{8} e - \frac{\pi}{8} \right) = \pi(e - 1)$$

$$\text{L6: } \text{We have shown } \pi(e - 1) < \int_0^\pi e^{|\cos 4x|} dx$$

To prove the right hand inequality, we consider the following diagram.

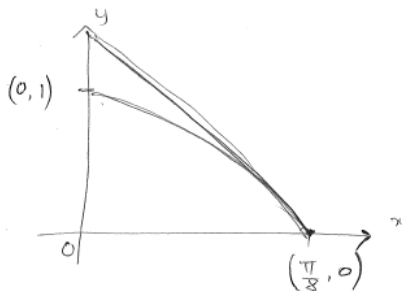


Table 7.25: Tas' Example of a Third Year Argumentation: Continued

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- L7: The equation of the line has gradient  $-4$  since at the inflection point of  $\cos 4x$ , the gradient is  $4(-1)$ .  $y - 0 = -4(x - \frac{\pi}{8}) \Rightarrow y = -4x + \frac{\pi}{2}$ , for all  $x \in [0, \frac{\pi}{8}]$
- L8: So,  $\cos 4x \leq -4(x - \frac{\pi}{2})$
- L9: Therefore  $8 \int_0^{\frac{\pi}{8}} e^{\cos 4x} dx < 8 \int_0^{\frac{\pi}{8}} e^{-4x + \frac{\pi}{2}} dx = 8 \left[ -\frac{1}{4} e^{-4x + \frac{\pi}{2}} \right]_0^{\frac{\pi}{8}}$   
 $8 \left[ -\frac{1}{4} e^{-4x + \frac{\pi}{2}} \right]_0^{\frac{\pi}{8}} = 8 \left( -\frac{1}{4} - \left( -\frac{1}{4} e^{\frac{\pi}{2}} \right) \right) = 2 \left( e^{\frac{\pi}{2}} - 1 \right)$
- L10: Thus,  $8 \int_0^{\frac{\pi}{8}} e^{\cos 4x} dx < 2 \left( e^{\frac{\pi}{2}} - 1 \right)$  and therefore,  $\int_0^{\pi} e^{|\cos 4x|} dx < 2 \left( e^{\frac{\pi}{2}} - 1 \right)$
- 

The warrant Tas uses is the Riemann integral of a function as a representation of the area under its graph. Standing alongside this warrant are trigonometric relationships, algebraic operations and integral calculus. The warrant facilitates the construction of algebraic inequalities (Lines 3 & 8) and their transformation into the strict inequalities of integrals (Lines 4 & 9). Tas' argumentation thus has a theoretical backing comprising the rules of algebra, and a knowledge of trigonometry and integral calculus.

Epistemic rationality is attained from a theoretical backing consisting of a knowledge from trigonometry, and integral calculus, and also a knowledge of the rules of algebra. Steps of working are deductively enchaind and so the shift from one line to the next is mathematically justified. Justification also occurs within lines. For example in Line 7, Tas justifies the gradient of his linear function by referring to the gradient of the cosine function at its inflection point. He knows that for  $\cos x$  the gradients are 1 and  $-1$  at inflection points, so for  $\cos 4x$  these gradients are scaled by a factor of 4.

Communicative rationality comes from his apparent focus on conveying an understanding of what he is doing in his argumentation to the reader (assuming his reader is proficient with the warrant and backing he is using). Two subgoals are identified that relate to establishing the claim in the given problem. The first of these goals is establishing the inequality  $\pi(e - 1) < \int_0^{\pi} e^{|\cos 4x|} dx$ , and the second is for  $\int_0^{\pi} e^{|\cos 4x|} dx < 2 \left( e^{\frac{\pi}{2}} - 1 \right)$ .

Teleological rationality is evidenced in the achievement of the two subgoals thus proving the claim that  $\pi(e - 1) < \int_0^{\pi} e^{|\cos 4x|} dx < 2 \left( e^{\frac{\pi}{2}} - 1 \right)$ . The subgoals are met through the development of a trigonometric and integral relationship evidenced in  $\int_0^{\pi} e^{|\cos 4x|} dx = 8 \int_0^{\frac{\pi}{8}} e^{\cos 4x} dx$  and the two diagrams. Lines of working indicate that Tas understands both the argumen-

Table 7.26: TA Framework Analysis of Tas' Third Year Argumentation

Three Worlds of Mathematics	Toulmin Model	Rationality
<p><b>Intersection of Embodied, Symbolic, Formal Worlds:</b> Reasons with numeric and algebraic symbols</p> <p>Reasons through trigonometry, and calculus</p> <p>Embodiments are used as an essential aspect of Tas' argumentation</p> <p>Diagrams are used to prompt and explain mathematical ideas</p> <p>The truth of the claim is reasoned deductively</p>	<p><b>Data:</b> (Trigonometry, Algebraic &amp; Calculus) The inequalities:  <math>\cos 4x \geq -\frac{8}{\pi}x + 1</math> for <math>x \in [0, \frac{\pi}{8}]</math> and,  <math>\cos 4x \leq -4(x - \frac{\pi}{2})</math> for <math>x \in [0, \frac{\pi}{8}]</math></p> <p><b>Warrant:</b> (Trigonometry, Algebraic &amp; Calculus) The embodiment of integration as area under a graph of a function supported by trigonometric relationships, algebraic operations and integral calculus</p> <p><b>Backing:</b> (Trigonometry, algebra &amp; calculus) Rules of algebra and arithmetic, knowledge of trigonometry and integral calculus</p> <p><b>Qualifier</b> Absolute qualifier implied 'we have shown' Line 6 and a deductive 'therefore' in Line 10</p> <p><b>Rebuttal</b> (None provided)</p> <p><b>Conclusion:</b> Comes in 2 parts;  <math>\pi(e - 1) &lt; \int_0^{\pi} e^{ \cos 4x } dx</math> and,  <math>\int_0^{\pi} e^{ \cos 4x } dx &lt; 2(e^{\frac{\pi}{2}} - 1)</math>, which are then combined</p>	<p><b>Epistemic:</b> A theoretical backing is used consisting of a knowledge of trigonometry, integral calculus, and algebra</p> <p>The use of the words and phrases such as 'therefore', 'we have shown', 'so', and 'thus' indicate a logical construction of statements</p> <p>Reasoning used to construct the conclusion is rooted in deductive mathematical logic</p> <p><b>Communicative:</b> Text in the argumentation focusses on establishing a shared understanding of mathematical ideas used to construct the proof for the reader</p> <p>The argumentation is presented in a way that shows that a theoretical backing is being used</p> <p><b>Teleological:</b> Two subgoals are established that relate directly to the proof of the claim</p> <p>Both subgoals are met and the claim is proved. These achievement of these goals is accomplished through deductive reasoning, an appropriate argumentational strategy that includes the implementation of selected mathematical ideas</p>

tational strategy, and the mathematical content of his presented work. The analysis of Tas' argumentation is summarised in via the TA Framework in Table 7.26.

#### 7.5.4 Summary of Tas' Progression in Argumentation

An examination of Tas' argumentations over each of the three years reveals that he consistently resorts to algebra as a tool for thinking. He has also consistently used language or symbols that imply a logical relationship between mathematical ideas, hence deductive reasoning has continued to be a feature of his argumentations. In all the argumentations presented, it is noticeable that he does not explicitly state the definitions or theorems that he uses. However, evidence in his argumentations strongly suggests that he has a knowledge and understanding of these. Thus all his argumentations presented reflect formal proof.

One element to emerge from his third year example of argumentation was the complexity of his warrant and backing. These appeared to involve a connected knowledge of mathematics that spans across content areas, for example, trigonometry, calculus, algebra, and the real number system. Although predominantly a user of algebra, Tas appears to be aware that visual methods can be a helpful a problem solving strategy, that they can also provide an understanding of mathematical ideas. Whereas the problems solved in his first and second year provided argumentations were traditionally focussed on mathematical content being considered at the time, his third year example demonstrated an ability to handle many areas of mathematics. This was reflected in the development of the three facets of rationality as well as his ability to move from using only one or two worlds of Tall's TWM, to using all three.

In his first and second year argumentation, epistemological rationality was evidenced in algebraic warrants that were backed by rules of algebra and a definition drawn from a particular theory in mathematics. In his third year example however, the warrant provided had embedded supporting warrants, and so the backing covered a range of mathematical fields.

Communicative rationality across all three years appears to be consistent in the achievement of producing a script reflecting personal understanding of the problem at hand, and of the mathematics involved. The script also reflects an intent to secure the reader's understanding and conviction of the argumentation process, and hence the result. In his first and second year, Tas' argumentations centred around one area of mathematics such as number theory or algebra. However, in his third year argumentation, Tas has drawn together different fields of mathematics using a range of techniques including diagrams, calculus, algebra, trigonometry, and deduction. Also consistent over the three years is Tas' focus on the use of logic and the phrases and symbols that go with it. It is noticeable that in Tas' first and second year argumentation, all steps of 'working' were recorded. However, in his third year example Tas omitted certain justifications for example,  $\int_0^\pi e^{|\cos 4x|} dx = 8 \int_0^{\frac{\pi}{8}} e^{\cos 4x} dx$ , and  $\cos 4x \geq -\frac{8}{\pi}x + 1$  for  $x \in [0, \frac{\pi}{8}]$  therefore  $e^{\cos 4x} \geq e^{-\frac{8}{\pi}x+1}$  for  $x \in [0, \frac{\pi}{8}]$ . Although it appears that Tas does not see that their justification is essential to his argument, if asked he would need to demonstrate their validity – this requirement is also an aspect of communicative rationality.

Teleological rationality refers to the realisation of one's intentions. This means the accomplishment of subgoals, and goals using a purposeful reasoning strategy. The writer also



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should reflect in the text of his argumentation that he knows and understands what he has done. In all of his argumentations Tas was able to set subgoals that would lead to the establishment of his claim or proof and achieve them. Deductive reasoning featured significantly in the argumentational strategies Tas used to achieve these subgoals. In Tas' first and second year, the subgoals were achieved using one field of mathematics. However, his third year argumentation involved reference to multiple fields of mathematics.

## Chapter 8

# Interviews: Third Year Students Conceptions of Proof

### 8.1 Introduction

In this chapter the assumption will be made that proving and proof are aspects of argumentation. This chapter will highlight some important observations that emerged from interviews with students who had either completed a degree in mathematics or were in their third year of mathematics study. We will present conceptions of proof these students developed in their first and second year mathematics courses, and compare them with how they viewed proof in their final year of studying mathematics. We will also consider significant moments they identified in their experiences of mathematics that contributed to their ability to read and write proofs. Finally, we will consider suggestions these students have for the way proof might be taught in the first year of studying mathematics.

Four students were interviewed for this part of the study. At the time of their interviews two were in their third year of studying mathematics, one was in a postgraduate year studying computer science, and the other had completed his degree in mathematics and was working as an educator. All of the students had taken Maths 150, 250, 253, and either Maths 255 or CompSci 225 (Discrete Structures in Mathematics and Computer Science), which was taught conjointly by members of the Departments of Mathematics and Computer Science. The focus of this computer science course was to consider mathematical logic and proofs (including methods of proof), developing these in the context of discrete mathematics and problems skewed to computer science. Maths 255 on the other hand, considered the rigorous development and organisation of mathematical arguments into

proofs. Here students were introduced to formal mathematical language and associated notations, with the ideas developed through mathematical contexts.

We will refer to our students by their pseudonyms, Able, Aaron, Tas, and Cam. Aaron and Tas had both recently completed a BSc in mathematics, while Able and Cam still had two semesters to go before completing their degrees in applied mathematics and pure mathematics respectively. It should be noted that Able, Aaron and Tas had participated in a pilot argumentation programme three years earlier (while in their first year).

Proof is not introduced as a study topic in the first year mathematics courses at Auckland, although it is often touched upon in Maths 102, 190 and 150. All students were unanimous in noting that they first encountered proving in their first year, but this was usually not with the rigour they encountered in their second and third year courses. Neither was their introduction to proof in their first year extensive – not covering techniques of proving. According to these students, their first year mathematics learning was spent on developing thinking, knowledge and understanding and language associated with definitions and theorems.

For two of the students, mathematical language appeared to be an aspect of learning they had to struggle with in some way. Their problems related to the logical structure of a statement or the logical equivalence of two statements. In the next section we will explore these aspects in more detail.

## 8.2 Coping with the Language

In their first year of mathematics courses students are usually introduced to mathematical language that has a different register from pragmatic everyday language. Students are confronted with mathematical terms such as *axiom*, *definition*, *theorem*, and *lemma*. Along with these come words and phrases that are given precise meanings. For example, ‘let  $\mathbb{R}$  be the set of real numbers’. Now compare this with the statement ‘let  $P$  be a set of real numbers’. The replacement of ‘the’ in the first statement with ‘a’ changes its meaning. Students are also introduced to particular logic forms, in particular *if – then*, and meet up with the existential quantifier *there exists* and the universal quantifier *for all*, even if the symbols for these are not at first introduced. Understanding these was not easy for some. Able noted:

*I first came across the words ‘axiom’, ‘definition’ and ‘theorem’ in Maths 150. These were new to me and I had to spend a lot of time sorting out for myself what the differences between them were, especially between axioms and definitions. (Able)*

*It took me a long time to interpret ‘for all – there exists’ and ‘there exists – such that for all’, they seemed so similar to me at the time. Also, what does ‘there exists’ mean? Does it mean that there exists only one specific value, or that there exists at least one value. The word ‘some’ was another difficult word for me. For example ‘some  $K$ ’. What does ‘some  $K$ ’ mean? Does it mean ‘at least one  $K$ ’, or ‘exactly one  $K$ ’? I spent quite a bit of time trying to understand what these words meant and how to use them ... frustrating sometimes. (Able)*

The need to be wary of quantifiers was also mentioned by Tas, who also said that he worked hard at understanding what the mean (which he later clarified as meaning their logical affect in a definition or theorem)

*Quantifiers can be tricky, because just a small misinterpretation of any one of them or sequence of them can lead to very massive consequences. I worked hard at understanding what they mean – absolutely. (Tas)*

The logical nature of mathematical language is different from that of ordinary language. Mathematical language attempts to exclude ambiguity so that there is only one meaning, whereas in ordinary language, this potential for ambiguity still exists. Meaning is dependent on context and situational knowledge. However, ordinary language is often used to try and make sense of mathematical language. One suggestion, proposed by Lee and Smith (2009), is that students need to separate the language of mathematics from the register of ordinary language in order to avoid confusion when interpreting mathematical statements. A related study (Epp, 2003) indicated that it was not unusual for students to have difficulty accepting the logical equivalence of *if  $p$  then  $q$*  and  *$p$  only if  $q$* , and to confuse these with the *if and only if* statement. She suggested that this would be one of the consequences of students trying to make sense of the *if  $p$  then  $q$*  and  *$p$  only if  $q$*  equivalence using ordinary language. For example, it took Cam until the end of his second year to understand the phrase *if and only if* fully.

*When I first came in contact with proof in Maths 150, the language used was ordinary language so you actually wrote ‘for all’ or ‘for every’ – it was quite natural to use these words. We also wrote ‘if and only if’. You know in 150 [Maths 150] I got a very good final grade, but I still wasn’t really sure what ‘if and only if’ was. I didn’t really know when it*

*was applicable and when it wasn't. 'If and only if' made sense to me when I did Maths 250 and Phil 101. In Phil 101 I could see the logic structure, and I came to see it as a matter of logical equivalency. Cam*

It will be useful to clarify the course Phil 101. It is a first year philosophy course that provides an introduction to logic. It is taught in the Faculty of Arts by members of the Department of Philosophy at the University of Auckland. The course studies arguments and evaluates them using a logical systems, examining propositional as well as predicate logic, and employs truth tables as an important feature of the course.

In Phil 101 Cam saw the equivalence through an analysis of *A if and only if B* in a truth table, while in Maths 250 he saw the structure of proofs involving *if and only if*. When pressed to explain how understanding the logical equivalence of two statements affected his understanding of proof, Cam said:

*I understood why you needed to prove both ways, because if you show just one way only that way is true, you haven't shown equivalence. Sometimes though, even I still forget to prove both ways. (Cam)*

Able's comments regarding quantifiers is important because of their importance in mathematical proof. Quantifiers have a direct impact on the logical construction of mathematical statements – particularly with respect to order. Combinations of the existential and universal quantifiers used in a statement may be a natural part of mathematical logic but they are not a common feature of everyday language. Students' difficulties in making sense of statements that make use of the existential and universal qualifiers together has been documented by Dubinsky and Yiparaki (2000) and Piatek-Jimenez (2010). Dubinsky and Yiparaki (2000) for example found that the university students in their study did not have a strong understanding of quantifiers phrased in ordinary language contexts – in particular the *there exists – for all* combination. They also found that what understanding students did have, did not transfer across to mathematics. Piatek-Jimenez (2010) showed that in the context of mathematics, undergraduate mathematics students experience difficulty in interpreting statements that contain a *there exists – for all* combination. She further noted that students' difficulties appear to be associated with the order in which the quantifier pairing occurs and that they frequently changed the statement to a form that they were more familiar with. Thus understanding a statement in which a qualifier combination occurs is not trivial matter.

Justifying, reading and writing proofs occurred at different stages for our students. How-

ever, all of them noted the significant contribution made by Maths 255 (Principles of Mathematics) or CompSci 225, both of which have a focus on proof, to their ability to write proofs. In the next section we will discuss the students' first introduction to proving and proof and then their introduction to formal proof. Although some of the students used rigour in their conversations, an overview of their interviews indicates that they held a multifaceted view of this word. One facet was that rigour referred to the use of abstract notations usually described using symbols, and another was that it referred to the use of axioms, definitions and theorems. A third facet was that rigour was reflected in certain methods of proof.

### 8.3 Introduction to Formal Proof

Tas, Cam and Able had experienced justifying and elementary proving in Maths 102. Further, Tas had studied mathematical proof as part of his personal study and had engaged with it at school (not in New Zealand) before coming to university. As part of his leisure time he studied and answered questions from previous international mathematics competitions. Cam also began to develop his proving ability through personal study, and Able through assignment work.

*I first started proving statements in Maths 102. They didn't ask me to, but I would often go and prove stuff that was in the course to challenge myself. (Cam)*

*Maths 102 was the first time I started having to prove statements ... in assignments. I enjoyed that more because we were free to think the way we want, rather this being restricted to using certain methods and pushing us to think in one way. (Able)*

Tas, Cam, and Able all went on to identify Maths 150 as their first real introduction to structured proof as part of a mathematics course. In this course students need to be able to read proofs and interpret them, as well as do some of their own. Tas spoke about how in Maths 150 a theorem was given and ideas essential to its proof provided. From these ideas, they (the students) had to construct a proof for the theorem. However, he noted that the proofs at this level were not as formal as the ones he encountered in his third year analysis course. It was at this stage that these students also came to realise the importance of axioms, definitions and theorems to the construction of a proof, as indicated by the following statements relating to Maths 150.

*When looking for a way in to proving a statement, I would look through my axioms defini-*

*tions and theorems and sort out which ones applied to the situation, and then by process of elimination determine which ones were useful, and then which ones were absolutely essential [to making a proof of a statement]. (Cam)*

*Proof is based on the definitions really, because once you understand the definition you can prove. (Tas)*

*For me in Maths 150, it was really important to understand the definitions and axioms because a time would come when I might need to use them in a proof. You can't use them if you don't know what they able to let you do. (Able)*

However, although they all had some experience of proof in their first year, all four students pointed to either Maths 255, or CompSci 225 as the courses that introduced them to the idea of formal proof. In particular, each student identified methods of proof as a significant feature of these courses that contributed to the development of their proving strategies. They noted that these methods were important tools for the rest of their studies in mathematics. Each student could recount the various methods of proof, which they identified as proof by induction, proof by contradiction and by contraposition, direct and bi-directional proofs, and proof by counterexample. These methods were added to in some of these students' third year courses. For example, Tas spoke of the *proof by pigeon hole principle* and Aaron noted *proof by smallest counterexample*. About his first encounter with proving in Maths 225 Aaron said:

*To be fair, I think it was the first time [CompSci 225] that I fully understood what I was doing when I was writing out lines of equations and then finishing with  $x$  equals  $y$  say. It was really like going this therefore this, therefore this, then that, therefore the conclusion,  $x$  equals  $y$ . Each step necessarily came from the one before it, except the first [step] of course. (Aaron)*

Aaron's original conception of proving was that it was the manipulation of algebra until you got the result you were after. However, his statement here indicates that a breakthrough for him was understanding the deductive process in creating a string of logical consequences. However Able, who also did the same course, focussed on his introduction to logic.

*[CompSci.]225 was the first time I had to do proof according to certain frames of thinking. I found that at the beginning we used truth tables and had to follow certain rules to prove different statements, which was against my natural thinking. I really wanted to know how the truth tables worked rather than just being showed them and [told] that's it, that's the way it is. (Able)*

The frame of thinking Able was referring to was procedural reasoning – following rules. As with other aspects of his interview, Able clearly indicated that he was a relational learner and his primary aim in learning was understanding. However, in his first introduction to formal proof he found himself working in a rule based environment. One issue concerned the use of truth tables, which occurred in both Maths 255 and CompSci 225. Epp (2003) suggests that truth tables should not be interpreted mechanically, but should always be accompanied by an explanation of what the truth table shows, along with particular reference to the hypothesis, conclusion or statement forms, depending on what is being highlighted in the truth table. Using logical equivalence as an example, Epp makes the following statement about the importance of explanation, “In the absence of an explanation, the process of constructing and interpreting a truth table may do little to advance their [students’] understanding of the concept of logical equivalence” (Epp, 2003, p. 897). Able’s understanding of truth tables was clarified when he used actual examples of proof to link propositional logic to different methods of proof.

*After a time we used the [truth] tables to justify the different methods you could use for proving. That’s when the tables made a lot more sense to me. (Able)*

The students made direct reference to methods of proving when asked what things they did when they first began to prove a statement. For example the following comment by Aaron common in the comments made by the other students.

*I would clarify the problem, identify or highlight key information including relevant definitions, axioms, and theorems, and then determine which method of proof to use – like contradiction, counterexample, direct proof, or induction. If my initial method didn’t work, I would either recheck the problem and the information, or change my method. (Aaron)*

*At first, proof is usually hard I think, but once you learn a framework for proof like contradiction, contrapositive – that kinda stuff, then you follow the thread of what’s going on in a proof a lot easier. Sometimes I’ll make a conjecture, and by a conjecture I mean, do I think that it [the statement] is true or false, and then I’d try and select a tool [a method of proving] that I could use and then I’d take it from there. Usually which one I choose would depend on the way that the problem is phrased. (Cam)*

*Before studying them in [Maths]225 I never thought about proof strategies. I just thought about how I could get to the conclusion. I suppose, now when I think about it, I used direct proof all the time without really knowing that’s what it was. So knowing different ways to prove gives me more ways to think about a problem – can I do it this way, or can I do it*



that way. (Able)

Tas, had met most of the methods of proving before entering university study. He clarified one further method he had developed, which he named ‘Roll Back’.

*Roll Back is where you look at the conclusion and then work back to the statement you have to start with. Then when you write your proof you just work in the other direction [from the statement to the conclusion]* (Tas)

During the interview, Tas was shown one of his earlier proofs as an example of Roll Back. However, he proceeded to make amendments to the original proof, which is shown in Table 8.1.

The original problem was:

**Problem:** Show that if  $f(x) = x + 1$ , and  $g(x) = (x - 1)^2$ , then  $f \geq g$  whenever  $x$  is taken from the Set  $S = \{x \in \mathbb{R}, 0 \leq x \leq 3\}$

Table 8.1: Tas’ Approach to Proof: Roll Back

Original Proof	Revised Proof
L1: $f(x) = x + 1$ , $g(x) = (x - 1)^2$	L1: Given
L2: I’ll work backwards to find a starting point.	$f(x) = x + 1$ , $g(x) = (x - 1)^2$
L3: $\forall x_0 \in S$ , assume that $x_0 + 1 \geq (x_0 - 1)^2$ .	L2: Assume that $x + 1 \geq (x - 1)^2$
L4: $\Rightarrow x_0 + 1 \geq (x_0 - 1)^2$	L3: $\Leftrightarrow x + 1 \geq x^2 - 2x + 1$
L5: $\Leftrightarrow x_0 + 1 \geq x_0^2 - 2x_0 + 1$	L4: $\Leftrightarrow x(3 - x) \geq 0$
L6: $\Leftrightarrow x_0(3 - x_0) \geq 0$	L5: $\Leftrightarrow x(3 - x) \geq 0$
L7: Proof	L6: $\Rightarrow 0 \leq x \leq 3$ for all $x \in \mathbb{R}$
L8: $\forall x_0 \in S$ , $x_0(3 - x_0) \geq 0$	L7: Let $S = \{x \in \mathbb{R}, 0 \leq x \leq 3\}$
L9: $\Leftrightarrow 3x_0 - x_0^2 \geq 0$	L8: $\forall x_0 \in S$
L10: $\Leftrightarrow 3x_0 \geq x_0^2$	L9: $\Leftrightarrow 3x_0 - x_0^2 \geq 0$
L11: $\Leftrightarrow x_0 + 1 \geq x_0^2 - 2x_0 + 1$	L10: $\Leftrightarrow 3x_0 \geq x_0^2$
L12: $\Leftrightarrow x_0 + 1 \geq (x_0 - 1)^2$	L11: $\Leftrightarrow x_0 + 1 \geq x_0^2 - 2x_0 + 1$
L13: $\Leftrightarrow f(x_0) \geq g(x_0)$ .	L12: $\Leftrightarrow x_0 + 1 \geq (x_0 - 1)^2$
	L13: $\Leftrightarrow f(x_0) \geq g(x_0)$
	L14: Therefore: $x + 1 \geq (x - 1)^2$ , $\forall x \in \mathbb{R}$ .

Lines 1 to 8 in his revised proof was the example he chose to represent ‘Roll Back’. This set up his actual proof in Lines 9 to 15. Tas found this method useful when dealing with direct proofs. We note that Tas revised his earlier proof because when he looked at the problem again he said that he really needed to construct the set  $S$  using the inequality,

and then take a general  $x_0$  out of it.

Also, when considering another problem during his interview Tas referred to the usefulness of intuition in his way of approaching problems:

*I have a problem here. Show that there is no integer solution to the equation  $x^3 + y^3 = 7z^3 + 3$ . Well you can look at the right hand side and see that there is a remainder of 3 on division by 7. But because the question says there are no integer solutions, there cannot be a remainder of 3 when you divide the left hand side by 7. So I sort of convince my self that the statement is true, that's the easy part. The hard part will be to show  $x^3 + y^3$  divided by 7 doesn't have a remainder of 3. (Tas)*

Another aspect to emerge concerning some students thinking about proof is that the proofs often hinge on a 'trick', although a better word for this would be a 'technique'. Finding such a technique was recognised as a difficult aspect of proving, and is illustrated by the following two comments.

*Something I found hard was in the problem itself ... looking for that one trick to make it [the proof] tick over. (Aaron)*

*Well a lot of proofs come down to little neat tricks, that kinda make things go klonk – like in the proof of the product rule [for single variable differentiation]. You add a bit and take it away at the same time just so you can get a form that will eventually give the rule you're after ... like, you just add zero. (Cam)*

Cam also noted that reading and understanding a proof can be difficult.

*Mathematicians are good at hiding their tracks. They get all the messy stuff and just sweep it away ... like why they do little tricks, why they are a good idea, and why they did a certain step. That's the messy stuff that doesn't get explained in the proof. That's why people find them [proofs] hard to follow. (Cam)*

The comments made by the students highlight two main factors in their learning experiences. The first is that axioms, definitions and theorems became increasingly important to them in their proving, and the second is the usefulness they found in knowing fundamental proving strategies. One student (Able) queried the effectiveness of introducing the logic of proof through the use of proof tables, and for the same reason queried the way in which methods of proof were introduced. He suggested that it would have been better to have the students come up with the truth or falsity of statements for themselves, rather than having the logic rules given to them and asked to apply them. Likewise, he argued that it would have been more meaningful if:

*... we were to solve problems that made us think about different ways of proving, and then we could build up the methods for ourselves.* (Able)

In an examination of how tasks can be designed to promote conjecturing and proving, Lin et al. (2012) echo the point of Able's comment. They suggest that through carefully crafted tasks students can be led to, "internalise diverse proof schemes through developing them and becoming aware of their plausibility and feasibility" (p. 308). The proof schemes Lin et al. allude to are those described by Harel and Sowder (2007) and which they refer to as *external conviction*, *empirical*, and *deductive*. By internalising these schemes students can recognise the conditions under which a certain proof scheme is plausible and feasible, as well as distinguish between the requirements of conjecturing and proof. They are also more likely to personalise proof as a dynamic activity of knowledge construction.

Having a working knowledge of definitions was a recurring theme in the students' comments. All of the students' expressed the importance of definitions to proving. As noted by Harel (2008) definitions should be *operable* for an individual. That is, students should be able to focus on the properties of a definition that make it possible to make deductions from them (see also Bills & Tall, 1998). Selden and Selden (1995) note that it is a precondition for successful proving and reading of proof that students should be able to unpack definitions and their logical structure. Similarly a characteristic of being able to reason in Tall's formal world of the TWM is that students should understand definitions and axioms, and be able to use them appropriately in their proofs. We will now consider further the student view of definitions beyond their first year.

## 8.4 The Importance of Definitions

We have already seen above how these students' views of definitions were shaped by their first year courses. However, as they progressed through to their third year in mathematics, their view that definitions were an important dimension of proving became even more grounded.

*When you play with the definition you start identifying the direction you should go because the definition gives you that direction.* (Able)

*I really make sure of my definitions before doing a proof. In my first year I used to take definitions at face value and just use them. But my third year courses have forced me to make sure I understand them. I developed my understanding of the abstract part of them,*

*like the logic. I can now fully understanding everything about definitions presented to me given enough time. (Aaron)*

*Knowledge of definitions were important for me in my first year, because that's the way that I would attack proofs. They still are. But now, I don't always directly state them in proofs. I assume that if something I know to be true is true, then it is true. I have seen some people explicitly state definitions in their work, and at some level I think to myself that person's smart, because they are going to get marks for that. (Cam)*

Tas identified Maths 332 (a third year analysis course) as the first course where, in his view, he really started to use definitions to prove theorems. A significant factor for him was that definitions were used as the root for constructing proofs. It also appeared that Tas equated the understanding of definitions to the understanding of mathematics.

*In [Math]332 we always started to prove from definitions. Everything you want to prove must be based on the definition because that is the root of everything else. So you have got to understand the definitions – what it says and how it works. The crucial thing in mathematics is – you've got to understand the definitions otherwise you pretty much don't understand how mathematics works. (Tas)*

*The way I think about definitions now, is that they are still very important, but I only state them if I really need to justify a step. Sometimes you can solve a problem and some definitions don't need to be referred to because they are sort of self evident – in what you say. But for assignments I made sure to state them. (Tas)*

Cam and Tas were the only ones to make reference to determining whether or not they would explicitly state a definition in a proof. Cam stated that he made a mental note of a definition that he would use, but did not necessarily write it down if he knew it to be true. In his own proving, Tas' decision appeared to be based on whether or not he perceived a definition to be self evident in a proof. However, for these students, it appears that the real determinant of whether a definition should be specified in a proof or not is whether it is to be marked as part of course work or not. There was little evidence from the interviews that in their mathematics course lectures or tutorials the students were encouraged to play with definitions and either extract information from them, give meaning to them, or unpack the logic in them. Aaron was the only one who was given an opportunity to explore theorems – what they did and were unable to do, as well as apply them. He referred to Maths 326 (a combinatorial course), where there was a lot of interaction regarding understanding definitions, theorems and proving between the lecturer and within student groups.

[Maths]326 really strengthened my ability and confidence to prove. It really helped. I enjoyed that part of the course. (Aaron)

However, it should be noted that at the time Maths 326 was taught using Team Based Learning (TBL) strategies (see Sneddon, 2010). During this course the 19 students worked in small teams and co-constructed meanings of definitions and theorems, as well as proofs. They also received some lectures related to course content.

The analysis of the interviews presented in this section indicates that these students did consider definitions important, and this became more pronounced in their second and third year courses. Each of them indicated that they actively took steps to make sense of them. Furthermore, the use of definitions was associated with formal mathematics and abstract thinking, which for all of our students epitomised their third year courses. In only one instance were definitions and theorems discussed in class (Aaron), but this appeared to be an unusual occurrence. Otherwise, all sense making relating to definitions was done individually during personal study.

Over a three year period of studying mathematics, the students perspectives on proof developed considerably. In the next section we will discuss some of the significant changes in their conceptions that occurred.

## 8.5 Changing Conceptions of Proof

### Justifying

*The first purpose of proof for me was that it was just a way of finding an answer to a problem. Proof served as an experience of how to learn to prove. It didn't really serve to convince me whether something was true or not, because you sort of knew the statement was right because that's the way it was in lectures. We never, at least from what I can recall, showed that a statement was incorrect.* (Aaron)

We can make three observations regarding Aaron's comment. The first is that he saw proof as an aspect of problem solving, and so he linked proof to the function of justifying. Secondly, Aaron thought of proving as an intellectual challenge that introduced him to logic, strategies for reasoning in a systematic way, and notation. This was highlighted by Aaron's comment that proof, 'served as an experience of how to learn to prove'. An aspect of learning proof that occurred for most students in Maths 150, was a shift from inductive ways of reasoning. This occurred abruptly as proofs presented in lectures, or in

textbooks relied more on the use of algebra, as well as definitions and theorems from linear algebra and calculus – although there were some aspects of visual reasoning (e.g., some of the students’ recalled diagrams that helped them understand the Mean Value Theorem and Fermat’s Theorem for differentiable functions). Alongside Aaron, other students also commented on the justification role of proof, and its contribution to establishing certain modes of thinking:

*in my first year, proof was about using reasoning in your own way – but using mathematics, to justify something. I remember this from Maths 102. In Maths 150, it [proof] was more about following certain rules of thinking, but it was still used to justify the way we did things (Able)*

*Proof is really about the definition and the structure of a theorem ... Proof is based on definition really, because once you understand the definition, you can prove. (Tas)*

From the students’ comments, we suggest that the aim of enculturation may be the reason for the early establishment of our students’ attitude to definitions as discussed in Section 8.4. During their second year courses such as Maths 255, 250, and 255, the students did not make reference to proof as a form of verification. However, understanding of mathematics was a function of proof that the students highlighted.

### **Understanding**

In their second year courses, and certainly by their third year, the students saw proof as a way of learning, and understanding mathematics, and all came to appreciate the intellectual challenge it provided.

*In my first year, I really thought of proof as a study tool ... At stage one and two, proof was just a way to familiarise myself with whatever they were teaching. Learning mathematics was basically learning proofs. (Cam)*

*The purpose of proof was to give you more understanding so that you see the bigger picture ... When you understand something you start exploring it more and more. (Able)*

As noted in his previous comments, Tas considered proof as a way of understanding theorems and this conception was particularly reinforced in his third year analysis course. Cam also added an interesting view of proof with regard to learning. To him it was a measuring stick that he used to determine whether or not he understood what he was learning, or whether he had enough knowledge to successfully engage with proof (either reading proofs, or proving).

*Usually, it [proof] was just a way of testing whether or not I understood something, or like whether or not I knew enough math to prove whatever. (Cam)*

Only two students referred to conviction as an aspect of their early conception of proof. For example during Maths 102 Cam would use proof to convince himself whether a statement was true or not, and gave three examples of ideas he proved, the quadratic formula, the logarithm rules, and the derivative of the natural logarithm function.

*In Maths 102 I would just go off and prove things in the course just to challenge myself. I also used proof to convince myself that something was true. Like I remember in year 12 seeing the quadratic formula and thinking what is this? Who came up with this? I kinda didn't believe it, and that's why I found it hard to apply in class I think. But when I first proved it I was actually convinced for the first time that it was true. (Cam)*

However, by his second year Cam began to view proof as serving two purposes. Primarily it was a way of equipping him with certain tools with which he could do mathematics, and learn concepts and various methods and techniques. Its secondary purpose was to convince. Conviction was also part of Able's conception of the purpose of proof, as seen in the following statements, where Able speaks about the importance to him of determining whether or not a statement is true.

*... If I am given some result without proof, I ask 'Is this true?', or 'Why is it true?'*  
(Able)

(Interviewer) *Which question would you usually ask first?*

*'Why is it true?' If it's not true, you will find out. It's better to ask why it's true, because you can try and find the reasons – it might be true in certain cases or maybe not ... Can I find something that is not true? But at university you don't have time to do these things. You have to do them yourself. (Able)*

It was not unusual for the students to be given a statement, as seen earlier in Aaron's statement at the beginning of this chapter. However, Able's comment highlights how knowing that a statement is true shifts the attention to determining why it's true.

### **Looking for a Proof**

All of the students said that once they understood the problem, they began a proof by understanding the primary objective, highlighting important information that might be useful to achieving it, and then searching for relevant definitions, or proving methods that would get them there. Finding the right hypothesis from a set of possible explanations

appears to be closely related to an abductive style of reasoning, as noted by Pedemonte and Reid (2011). The approach to proving used by these students appeared to give rise to what Pedemonte and Reid refer to as *undercoded abduction*. This describes a situation where the prover identifies several plausible explanations, each of which can be backed by theory if need be. All the prover has to do is choose one that will solve the problem. Tas mentioned this when he talked about restructuring his mathematical ideas and the difficulty involved:

*It's hard to restructure your mathematical thinking in your proof ... You can have lots and lots of ideas for your proof, but you only have one pathway to your proof. You might try every single method and technique, but you have to choose which one gives you the most logical and convincing proof.* (Tas)

We restate two comments by Aaron and Cam used above because they also indicate an abductive style of reasoning toward a proof. Additionally they also indicate effort required to choose the right elements that will ensure a coherent and correct proof.

*I would clarify the problem, identify or highlight key information including relevant definitions, axioms, and theorems, and then determine which method of proof to use – like contradiction, counterexample, direct proof, or induction. If my initial method didn't work, I would either recheck the problem and the information, or change my method.* (Aaron)

*When looking for a way in to proving a statement, I would look through my axioms definitions and theorems and sort out which ones applied to the situation, and then by process of elimination determine which ones were useful, and then which ones were absolutely essential [to making a proof of a statement].* (Cam)

Pedemonte and Reid (2011) suggest that undercoded abductive reasoning is likely to hinder the production of a deductive proof because of the need for the students to coordinate their potential explanations with an appropriate theoretical backing and, at the same time, align each of them with the data in the problem. This appeared to be the situation that Tas, Cam and Aaron confronted, and as noted by Tas was a significant feature in producing a valid proof. It appears from other aspects of the interviews that these students were able to do this, with one of the possible reasons being that they appeared to be well acquainted with the necessary knowledge concepts as well as appropriate definitions and theorems. Given that these students appeared to demonstrate sufficient problem solving skills and knowledge, this observation lead to the following question regarding conjecturing. What opportunities did the students get to develop and prove conjectures of their own?



### Conjecturing and Proof

Conjecturing and proving is seen as an important to initiating mathematical thinking, discovering mathematical knowledge and enhancing mathematical understanding (Lin et al., 2012). By mathematical thinking, Lin et al. (2012) refer to Pedemonte (2007b), who places conjecturing at core of proof (which she situates in problem solving). Even with this brief reference to its significance, we would expect to see elements of conjecturing coming through our students' comments.

*I do not remember being put in a situation where we had to write our own conjectures. All conjectures were given to us in assignments or on the board in lectures. (Aaron)*

*We didn't get any opportunities to [develop and prove] conjecture at stage two or three – no. But I remember stage 2, Maths 250 tutorials. The professor would give us some problems which were like conjectures, which he said would not be examined. He said he wanted us to think. So we would discuss and try and solve the problems. Sometimes we didn't understand some of the ideas. But we were thinking in our own way, and we starting to explore things rather than doing something in a particular way. (Able)*

*Not often. We did a little in [Maths] 328, but the conjecture was given. All you had to do was pick a side whether it was true or not and go in and prove it. Usually I assumed it was true. (Cam)*

These students' comments indicated, that in their view, conjecturing did not feature as an aspect of their mathematics education. However, it does appear that they were given conjectures of others to either explore (as in Able's case), or take a personal viewpoint of and then try to prove them (as in Cam's case).

We now consider how the students' conceptions of proof changed during their third year of studying mathematics.

## 8.6 Summary

There were certain significant features that contributed to the conceptions of proof that the students had built up over three years. Proof appeared to play two markedly different roles. One of these was, contributing to the understanding of mathematical ideas and techniques. An important aspect of this role was learning how to prove, and identifying when to use certain techniques. It was evident that by the second year some of the students came to view mathematics through the lense of proof. This is to be expected since the

second year linear algebra course taken by all of the students included a substantial number of proofs.

Research suggests that conjecturing is an important basis for proof production (Lin, et al. 2012). However, we detected from our conversations with the students that they did not get enough opportunities that afforded this activity. So, how did the students third year course influence their conception of proof?

As they progressed into their third year, there was an increased emphasis on knowing and understanding definitions, and theorems, as well as strategies that could be used to construct a proof. The role of proof was one of explanation and systematisation. The students also commented on the intellectual challenge of proving; being able to identify information, proof strategies, and reasoning and organise them in to a form that will/might lead to a desired result. There also appeared to be some other minor changes in the students' general conception of proof. For example, Aaron initially thought of proof as a learning experience aimed at finding the solution to a problem requiring some aspect of verification and justification. He also noted that proof served as an experience to learn to prove, as well as a means to understand mathematics. The only addition he made to this was proof that contributed to ... *being able to think abstractly*, and ... *not to take stuff at its face value*. Aaron's comments, and those of the other students seem to confirm de Villier's (1990, 1999) claim that various functions of proof are interlaced, and manifest themselves at different times and in problem contexts. At this point it may be worth reminding the reader of Aaron's involvement in the Maths 326 Team Based Learning Research project, which had as one of its focal points constructing statements and justifying them.

It was common to hear students restate that proof was important for learning mathematics. In fact Able prioritised this aspect of proof over conviction. He likened proof to a mind map that has definitions at its centre.

# Chapter 9

## Discussion

### 9.1 Introduction

This research has been concerned with analysing students' written argumentations as they worked on tasks that were set in a problem solving context. The aim of this chapter is to present some of the key findings that emerged from the analysis of the students' argumentation, answer the research questions, and suggest how this research might contribute to the corpus of knowledge that exists in mathematics education. The findings that will be discussed will be based around:

- the nature of the reasoning used by students when writing arguments of conviction;
- the mathematical resources used by students in their argumentations;
- the diachronic development of students' argumentation over three years;
- students' perceptions of proof; and
- the Tertiary Mathematics Argumentation Framework

The research question (RQ) that this study attempted to answer was:

[RQ] How do tertiary students formalise their argumentations of conviction?

To help answer this question, sub-questions RQ 1 and RQ 2 were posed:

[RQ 1] *What changes are evidenced in the structure and rationality of first and second year university mathematics students' argumentations when they change from giving argumentations that secure a personal conviction, to ones that convince someone else?*

[RQ 2] *How do students' argumentations of conviction change as they progress through first, second, and third year courses in university mathematics?*

The tasks that were given to all students who participated in this research allowed them to use various strategies of reasoning. A survey of their argumentations reveals that students certainly did use a variety of reasoning strategies, including inductive, visual, and deductive reasoning. However, there were two main features that emerged regarding students' reasoning in their argumentations. The first was that, in most cases, reasoning used by students in arguments to convince themselves of a result, differed from that used in arguments they wrote to convince an other. Secondly, most students preferred inductive reasoning based on numerical examples in their self-convictions. The following section will consider the nature of the changes in reasoning used in the argumentations of Maths 102 students as they switched from convincing themselves to convincing an other, who in the tasks given to the students was a lecturer. A comparison of Maths 102 argumentations with those of Maths 255 who were given a similar task will also be made.

## 9.2 Reasoning to Establish a Conviction

One of the goals in this research was to answer the following question:

[RQ 1:] *What changes are evidenced in the structure and rationality of first and second year mathematics students' argumentations when they change from giving argumentations that secure a personal conviction, to ones that convince someone else?*

To answer this question, we will consider three sub-questions that will provide a guide to answering RQ 1:

[RQ 1a] *Do students recognise that argumentation aimed at establishing a personal conviction has a different epistemological basis to that directed at convincing an other?*

[RQ 1b] *What changes are evidenced in the shift from students' argumentation for self-*

*conviction to one aimed at convincing an other?*

[RQ 1c] *What obstacles prevent students' writing a mathematically convincing argument?*

Answers to these questions will come from a discussion of findings, expressed as conclusions. They will be denoted by (C1-1a) for example where C1 refers to the conclusion number, and 1a refers to the particular question it relates to (e.g., RQ 1a). This part of the discussion will address the first sub-question, RQ 1a.

All students enrolled in Maths 102 were given two research tasks as part of their coursework. The first task given to the students was taken from elementary number theory. A feature of this task was that it could easily be interpreted algebraically, and so it was expected that many of the students would use algebra to convince themselves of the claim. The second task came from trigonometry and involved mathematics that was already familiar to them. Maths 255 students were given a task that was also drawn from elementary number theory. From an analysis of data we draw our first conclusions (C1-1a):

- First year mathematics students who participated in this research appeared to show a preference for empirical arguments when writing a statement of self-conviction, but a preference for an algebraic argument when writing to convince someone else;
- Most first year participants recognised that they needed to change from an empirical-based argumentation strategy when moving from self-conviction to convincing an other; and
- All second year participants preferred logical reasoning with or without algebra when writing a statement of self-conviction, but switched to using algebra when writing to convince someone else.

Tasks 1 and 2, as well as the task given to Maths 255 participants, are re-presented here as a reminder as they will be the basis of some of the discussion in this section:

### **Task 1**

The set of integers is the set  $I$  where  $I = \{\dots - 3, -2, -1, 0, 1, 2, 3, \dots\}$ . Now, consider the following statement; "The sum of any 3 consecutive integers is always a multiple of 3".

- Investigate this statement, and convince yourself whether it is always true, sometimes true, or never true.
- Write an argument that you would use to convince another person such as a mathematics lecturer of your standpoint.

**Task 2**

A Maths 102 student suggests that  $\cos^2 \theta + \sin^2 \theta = 1$  for all angles  $\theta$  from  $0^\circ$  to  $360^\circ$ .

- What do you think? Write an argument to convince yourself whether the student's suggestion is always true, sometimes true, or never true.
- Write an argument that you would use to convince a your lecturer of your conclusion.

The following task was given to Maths 255 students:

**Task Maths 255**

A student makes a claim that in any set of three consecutive integers, either one of the integers is a multiple of 2 and another a multiple of 3, or one integer is a multiple of 2 and 3.

- What do you think? Write an argument to convince yourself whether this claim is always true, sometimes true, or never true.
- Write an argument that you would use to convince a lecturer of your conclusion.

Evidence in Chapter 6 showed that most first year students reasoned inductively from numerical examples to convince themselves that a claim was true. This was particularly true of the first year students who participated in this research. For example, approximately 66% of the 83 participants in Task 1, and 57% of the 61 participants in Task 2 followed an inductive approach. On the other hand, the evidence presented in Chapter 6 painted a different picture when these students were asked to provide an argument that would convince an other of their conclusion. In Task 1 approximately 76% of the 83 participants used an algebraic argument. Similarly, a high percentage of students (approximately 92%) used algebra in Task 2, but only 31% understood what a generalisation required. Most first year mathematics students who participated in this research recognised that they needed to change their argumentation strategy when moving from self-conviction to convincing a more authoritative other. However, as indicated by the second task, most students did not correctly explain how a result generalised. A probable reason for this was the use of a right angled triangle in conjunction with the Pythagorean theorem (see Section 9.2.2).

From this data, we conclude that Maths 102 participants in this research showed a preference for empirical arguments when writing a statement of self-conviction, but a preference for an algebraic argument when asked to write an argument that would convince someone else. A study by Healy and Hoyles (2000) found that secondary school students (14–15

year olds) preferred to use numerical examples in their arguments of self-conviction, but changed to algebraic ones when asked to convince their teacher. Our research revealed that this behaviour continues into the first year of tertiary mathematics, but appears to change in the second year. Second year students secured a self-conviction by reasoning logically, without numerical examples, with or without the use of algebra.

In a task given to Maths 255 students, that was similar to Task 1, only 2 of the 12 participants gave argumentations for self-conviction that used numerical examples to support inductive reasoning, whereas 3 students reasoned logically in a narrative style and 7 gave an algebraic proof. A possible reason for this can be explained using the phenomenon of *maturing scholarship* (discussed in Chapter 4). These students had already experienced at least 3 mathematics courses and had developed their factual, procedural, conceptual, linguistic and symbolic knowledge of mathematics. An effect of maturing scholarship is that they were also being socialised into ‘traditions’ and ‘membership’ of the culture of university mathematics through his/her engagement in mathematics courses (Habermas, 1998). Thus Maths 255 students had certain expectations about the mathematics and reasoning that arguments should possess, regardless of whether they served the purpose of self-conviction or convincing an other (for example, deductive reasoning is a primary means of communicating a mathematical argument).

A point on which this research differed from that of Healy and Hoyles (2000), was the motivation behind students’ preference to use algebra to convince their teachers. According to Healy and Hoyles, the students’ used algebra because it would give them the best marks from their teacher. However, in this research, every Maths 102 student who wrote something down in both tasks received 5 marks for each (the total mark allocated to each task in their assignments), regardless of whether they had completed them or not. When writing to convince an other, Maths 102 students’ preference to use algebra was based on demonstrating their ability to provide what they perceived to be a logical mathematical argument – they did not think an empirical approach would be theoretical enough to convince a lecturer.

Most participants in this research understood that there was a difference in the epistemological requirements for developing an argument for each type of conviction and, therefore that different cognitive and mathematical resources would be required, where cognitive resources refer to those that involve thinking and reasoning, whilst mathematical resources refer to knowledge, skills, and logic. The use of the TA Framework has so far proven successful in a consideration of students’ argumentations in terms of their cognitive

and mathematical resources. Thus its use will be integral in addressing the second sub-question, RQ 1b which will discuss the changes students made when writing to convince a lecturer (an other).

### 9.2.1 Changes in Resources

An analysis of data related to the research question RQ 1b suggests the following conclusions (C2-1b):

- The changes that students make when they advance from writing an argumentation of self-conviction to one directed at convincing an other, come from a selection of new cognitive and mathematical resources that they have at their disposal at the time;
- A significant change is the selection of algebraic and deductive reasoning, which is sometimes supported by visual/visuospatial reasoning; and
- Changes that are made are evidenced in the structure (data, warrant, backing, modal qualifier) and rationality of students' argumentation.

This section will highlight the changes that occurred through a consideration of a typical student's argumentations mapped to the TA Framework in Table 9.1.

When a student writes an argument to convince a lecturer s/he knows that they are writing to someone who has a measure of mathematical authority. In such a case, experience from their life-world associated with being a student in Maths 102 at university, informs them which elements their argumentation is expected to have (e.g., reasoning with algebra and/or diagrams to make generalisations, and communicating logically and systematically). This research suggests that this causes them to consider other cognitive and mathematical resources they have at their disposal.

Table 9.1 gives an actual example of an argumentation from a Maths 102 student, which is typical of students who went from the use of numerical examples to the use of algebra. When convincing an other, the student in this example chooses to structure his thinking and reasoning around the use of a variable to represent a general integer, rules of algebra, and a general rule about multiples of integers (which is used to deduce the conclusion from an algebraic expression). Hence, new resources from the Symbolic World, and one from the Formal World, are utilised to support his reasoning. Changes that first year students



Table 9.1: The Changes in Argumentation From Convincing Self to Convincing an Other, Exemplified by the TA Framework

Personal Conviction	Convincing an Other
$1 + 2 + 3 = 6, 6/3 = 2$ $4 + 5 + 6 = 15, 15/3 = 5$ $37 + 38 + 39 = 114, 114/3 = 38$ $97 + 98 + 99 = 295, 294/3 = 98$ $21 + 22 + 23 = 66, 66/3 = 22$	<p>The statement is always true.</p> <p>Let <math>a</math> be an integer.</p> <p><math>a, a + 1, a + 2</math> are 3 consecutive integers</p> <p><math>a + a + 1 + a + 2 = 3a + 3</math> or <math>3(a + 1)</math>.</p> <p>If we choose any <math>a</math>, the sum of will be a multiple of 3.</p> <p>This algebraic illustration proves that any 3 consecutive integers is always a multiple of 3.</p>
<p><b>The Framework:</b> Procedural – Numeric: A set of numerical calculations are used to secure a conviction.</p> <p><b>The Three Worlds of Mathematics:</b> Symbolic World of the TWM.</p> <p><b>Toulmin Layout:</b></p> <p><i>Data:</i> 5 sets of 3 consecutive integers</p> <p><i>Warrant:</i> A test on the sum to see whether it is divisible by 3</p> <p><i>Backing:</i> A rule from arithmetic: If a given integer is divisible by 3 without remainder, then it is a multiple of 3</p> <p><i>Modal Qualifier:</i> Absolute (<i>is always</i> right)</p> <p><i>Conclusion</i> The statement is always right</p> <p><b>Rationality:</b></p> <p><i>Epistemological Rationality:</i> Justification is based on sets of numerical examples, which along with an arithmetic rule for determining if an integer is a multiple of 3, are used to support inductive reasoning</p> <p><i>Communicative Rationality:</i> Communication is through the use of arithmetic symbols, and understanding of the claim is demonstrated using an arithmetic procedure</p> <p><i>Teleological Rationality:</i> Goal is to show that sum of each set of integers is divisible by 3</p>	<p><b>The Framework:</b> Generalising – Algebraic: Algebraic reasoning using symbols is used to correctly deduce the claim</p> <p><b>The Three Worlds of Mathematics:</b> Symbolic and Formal Worlds</p> <p><b>Toulmin Layout:</b></p> <p><i>Data:</i> Algebraic: An integer variable <math>a</math> is given, and 3 consecutive integers represented by <math>a, a + 1</math>, and <math>a + 2</math>.</p> <p><i>Warrant:</i> <math>a + a + 1 + a + 2 = 3(a + 1)</math></p> <p><i>Backing:</i> A generalised rule from multiplication: Integers of the form <math>3x</math>, are multiples of 3. Rules of (deductive) logic</p> <p><i>Modal Qualifier:</i> Absolute (<i>is always</i> a multiple of 3)</p> <p><i>Conclusion:</i> Any 3 consecutive integers is always a multiple of 3</p> <p><b>Rationality:</b></p> <p><i>Epistemological Rationality:</i> A variable <math>a</math> represents and integer. Algebraic reasoning is used to get from <math>a + a + 1 + a + 2</math> to <math>3(a + 1)</math> from which, using a generalised rule of multiplication, the conclusion is deduced</p> <p><i>Communicative Rationality:</i> A generalised symbol <math>a</math>, along with algebraic and deductive reasoning are used to communicate an understanding of the claim</p> <p><i>Teleological Rationality:</i> Two subgoals are set up. The first is to algebraically construct 3 consecutive integers. The second is to use algebra to show that the sum is a multiple of 3. Both goals are achieved</p>

made in the move from convincing oneself to convincing an other are summarised from the TA framework as follows:

- (The TWM): Students changed *from* using arithmetic resources including (arithmetic and inductive reasoning) in the Symbolic World *to* algebraic resources (including algebraic and deductive reasoning) in the intersection of the Symbolic and Formal Worlds;
- (Toulmin): Students changed *from* using inductive reasoning based on numerical examples backed by rules of arithmetic, *to* using algebra and deductive reasoning

backed by rules of algebra and logic. Consequently, modal qualifiers changed *from* confidence in numerical data and inductive reasoning *to* confidence in algebraic data and deductive reasoning;

- (Rationality): Students changed *from* grounding the truth of their goals and conclusions in arithmetic and inductive reasoning *to* grounding them in algebraic and deductive reasoning. Consequently, students changed *from* communicating their results (goals and subgoals) and understanding using lines of numerical calculations *to* using lines of algebra.

From these new resources the student forms an argumentation to convince a lecturer, beginning with algebraic data. This is then connected to a conclusion by an algebraic warrant that uses algebraic and deductive reasoning to link the data to the conclusion. The warrant is backed by a rule from mathematics, and a rule of logic (deduction). Thus the backing has a ‘theoretical status’.

The changes in the TWM and the structural content of the student’s argumentations agrees with the changes in aspects of rationality. Consider the first aspect of epistemological rationality. The basis for the validity of the argumentation – the truth of the conclusion, changes from an inductive arithmetic backing using numerical examples, to a deductive algebraic one associated with mathematical theory. In the first argument, the step to the conclusion is essentially an intuitive one, whereas in the second argument, the conclusion is a necessary outcome of deductive reasoning. The second aspect, communicative rationality, is based on reasoning with numerical symbols and arithmetic procedures in the first argumentation. However, in the second one, it is replaced with algebraic and deductive reasoning using a variable. In the third aspect, the change in teleological rationality is characterised by a shift from arithmetic goals to algebraic ones.

The analysis of Maths 102 students’ argumentations in the trigonometry task (Task 2) also showed that they generally preferred to use algebra when convincing an other of a conclusion. Nearly all these students used a right angled triangle (as an embodiment of sine, cosine and the Theorem of Pythagoras) to support their reasoning. In a similar way to Task 1, algebraic data gained from a diagram grounded their argumentations. Further, algebraic and trigonometric warrants were used which were supported by a ‘theoretical’ backing seated in trigonometry.

Thus we can conclude that the first year undergraduate students:

- had the ability to choose mathematical resources that would give a ‘theoretical’ warrant and backing, and thus provide an absolute modal qualifier;
- could reason algebraically and deductively when they perceived it to be required; and
- have a readiness (see Corbishley & Truxaw, 2010, pp. 57–112) for a proving course (this will be discussed further in the Summary).

However, there were some students who continued to give numerical examples in the argumentation to convince a lecturer. In Task 1, there were 11 students who gave argumentations like this. In their life-world of mathematics, writing to convince a lecture appeared to consist of providing a written summary of what was found using their numerical examples, giving more numerical examples to verify their result, or the placement of numerical calculations in to a tabular form or layout that made the result clearer. Six of these 11 students gave a rebuttal to the claim, stating that the claim was true, except when the three consecutive numbers were  $-1, 0, 1$ , since when these are added the answer is 0, which is not divisible by 3 (or that 0 is not a multiple of 3). Seven students who used an algebraic argument to convince a lecturer also stated the same rebuttal.

Some had difficulty trying to communicate their thoughts clearly and concisely. Others maintained an arithmetic view of validity even though they used algebra. The next section will discuss these in further detail and address the question RQ 1c:

### 9.2.2 Obstacles to the Construction of Argumentation

This section will consider a further conclusion (C3-1c) to emerge from an analysis of the data, which we now state.

- Some first year students are prevented from communicating a concise and valid mathematical argument because of obstacles that include: the over-generalisation (of symbols and conclusion); the inclusion of unnecessary information; the ineffective use of mathematics to communicate ideas; and the verification of algebra using numerical examples.

A discussion of this conclusion will centre on real examples of first year students’ argumentations in Table 9.2, with one drawn from the trigonometric argument given in Figure 9.1. These examples represent four obstacles highlighted in C3-1c. Figure 9.1 illustrates

Table 9.2: Examples of weakly framed argumentations

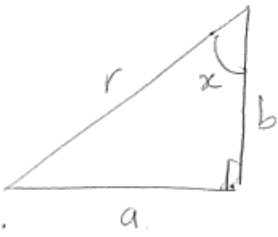
Personal Conviction	Convincing an Other
<p><b>Example 1</b></p> <p>Let <math>n</math> = any natural number (1, 2, 3, ...).  <math>(n + 7) + (n + 8) + (n + 9) = 3(n + 8)</math>  <math>n = 2, 3(2 + 8) = 30</math>  30 is a multiple of 3</p>	<p><math>n</math> = any natural number  <math>a</math> = consecutive add-on, (i.e. next number before or after <math>n</math>)  <math>(n + a) + n + (n + a) = 3n</math>  <math>n = 1, 3(1) = 3</math>  <math>\Rightarrow</math> is a multiple of 3</p>
<p><b>Example 2</b></p> <p><math>n</math> = integer.  <math>(n + 1) + (n + 2) + (n + 3) = 3(n + 1)</math>.  <math>= 3k</math>  <math>3k = 3(n + 1) \Rightarrow k = n + 1</math>  <math>\Rightarrow</math> since <math>n</math> is an integer, <math>k</math> is an integer  <math>\Rightarrow 3(n + 1)</math> is always a multiple of 3</p>	<p>Assume that <math>n</math> is an integer.  Consider the following:  <math>(n + 1) + (n + 2) + (n + 3) = 3n + 3 = 3(n + 1) = 3k</math>.  For <math>3(n + 1)</math> to be a multiple of 3, <math>n + 1</math> or <math>3k</math> must also be an integer  If: <math>3k = 3(n + 1) \Rightarrow k = n + 1</math>  <math>\Rightarrow</math> since <math>n</math> is an integer, <math>k</math> must also be an integer, because the sum of 2 integers can only produce an integer  <math>\Rightarrow 3(n + 1)</math> where <math>n</math> is an integer must always be a multiple of 3</p>
<p><b>Example 3</b></p> <p>1 2 3 = 6  2 3 4 = 9  3 4 5 = 12  10 11 12 = 33  23 24 25 = 72  I can see that no matter which combination I use it is always a multiple of 3.</p>	<p>Because larger numbers do not matter, let any integer be <math>x</math>. To make it into a consecutive, I need to add +0, +1, +2 to the <math>x</math>.  So, <math>x + 0, x + 1, x + 2</math>  Because only the last digit of a figure determines whether the number is divisible by 3, as you can see <math>2 + 1 = 3</math>, therefore 3 consecutive integers will always be divisible by 3.</p>

the first obstacle of over generalising which is linked to communicative rationality. In this example a student argues that  $\sin^2 \theta + \cos^2 \theta = 1$ , is true for all values of  $\theta$  from  $0^\circ$  to  $360^\circ$ . However, the use of a right angled triangle as the primary embodiment of sine, cosine and the Theorem of Pythagoras provides a physical constraint on the values  $\theta$  can take. Nearly all students who used (or focussed on) this representation over-generalised  $\theta$ .

A second obstacle that arose was symbol sense. In Example 1, Convincing an Other (Table 9.2), the student gives the letter  $a$  a dual meaning. It is the number that is added to  $n$  to give the number before it and the number after it. This is used instead for  $n - 1$ , and  $n + 1$ . This is quite likely related to the use of ordinary descriptive language to describe the student's algebraic reasoning.

A third obstacle was the presentation of unnecessary information is illustrated in Example 2 (Table 9.2). The Personal Conviction column shows that this student can reason algebraically and deductively. The problem arises when  $3(n + 1)$  is redefined as  $3k$ , since this student now perceives the need to show that  $k$  is an integer. This leads to an unnecessary explanation in the Convincing an Other column. Another example (not given in

$\cos^2 \theta + \sin^2 \theta = 1$



theorem Pythagoras:  
 $r^2 = a^2 + b^2$

$\sin x = \frac{a}{r} \rightarrow \sin^2 x = \left(\frac{a}{r}\right)^2$

$\cos x = \frac{b}{r} \rightarrow \cos^2 x = \left(\frac{b}{r}\right)^2$

to prove add

$$\sin^2 x + \cos^2 x = \left(\frac{a}{r}\right)^2 + \left(\frac{b}{r}\right)^2$$

$$\sin^2 x + \cos^2 x = \frac{a^2 + b^2}{r^2} \quad \leftarrow \text{(using theorem)}$$

$$\sin^2 x + \cos^2 x = \frac{r^2}{r^2}$$

$$\therefore \sin^2 x + \cos^2 x = 1$$

Figure 9.1: Example 4: Over-generalising

Table 9.2) but explained in the analysis was the inclusion of further algebraic examples. For example, Kerry (Table 6.9) correctly deduced the conclusion about the sum of three consecutive numbers from  $x + 1 + x + 2 + x + 3 = 3(x + 2)$ , but then provided another algebraic example such as  $x + 4 + x + 5 + x + 6 = 3(x + 5)$ .

A fourth obstacle was the inability of some students to communicate their mathematical ideas clearly. Example 3 in Table 9.2 the argumentation to convince an other demonstrates this. The statement, 'only the last digit of a figure determines whether the number is divisible by 3...', does not immediately suggest that he is referring to the last digit in the expression  $3x + 3$ . Rather the following vague statement, '...as you can see  $2+1=3$ , ...' is used instead.

A fifth obstacle was epistemic in origin. This was that for some students, conviction validity was linked to numerical examples regardless of whether an algebraic justification has been given or not.

To restate the conclusion given at the beginning of this section, certain obstacles prevented first year students from providing a concise and valid mathematical argument. This research identified these as: the over-generalisation of symbols and conclusion; the inclusion of unnecessary information; the ineffective use of mathematics to communicate ideas; and the verification of algebra using numerical examples.

According to Pedemonte (2007a) the cause of these obstacles is cognitive, and lies in a student's control structure that enables them to choose which of the mental actions they perform in solving a problem, which are relevant in their representational system and which are efficient in deciding whether a problem is solved or not. Lolli (Arzarello, 2007) places the cause in the formal inscription of text according to shared rules of communication. Both causes suggest that with guidance and knowledge these students could overcome the obstacles that prevent them from writing valid mathematical arguments.

### 9.2.3 Differences in Argumentations Between First and Second Years

The argumentations that Maths 255 students used to convince themselves were either given as a logical narrative, or given algebraically. However the mathematical convictions (those written to convince an other) of Maths 102 students did not differ greatly from those of Maths 255, although there were some areas in which they did. One of these was the provision of mathematical detail. For example, when using a variable Maths 255 students specified it as belonging to a certain number set, using formal notation. They also placed alongside a statement any constraint that was applicable (such as an inequality). Another general area of difference was that argumentations of Maths 255 students were usually more concise than those of Maths 102 students. (e.g., 'Let  $n$  be an integer' is replaced with let  $x \in \mathbb{Z}$ ). Our conjecture is that this is a result of an increased sense of symbol knowledge. Many Maths 102 students explained their algebra, or gave further examples (algebraic or numerical) to validate their algebra. Figure 9.3 compares one of these types of argumentations with a second year student's.

This table (Figure 9.3) demonstrates the conciseness of a second years student's argumentation resulting from a more formal approach. Argumentation from other second year students used in this research also showed that in different contexts in mathematics (differential calculus, functions, and linear algebra), students used a more formal, and wider range of symbolic language, and definitions and theorems were an integral part of reasoning.

Table 9.3: A Comparison of Two Arguments

A First Year Student	Second Year Student
Let $n$ be an integer.	Let $x \in \mathbb{Z}$
Therefore 3 consecutive numbers could be $n$ , $n + 1$ , $n + 2$	$x - 1$ , $x$ , $x + 1$ are 3 consecutive numbers.
Adding these would give the sum $3n + 3$	$x - 1 + x + x + 1 = 3x$
By factorising, we get $3(n + 1)$	The sum of 3 consecutive integers is a multiple of 3
Therefore there are two factors: 3 and $(n + 1)$	
Therefore the sum of any 3 consecutive num- bers is always a multiple of 3	
$x + (x + 1) + (x + 2)$	
$= 3x + 3$	
$= 3(x + 1)$	
Another example:	
$n - 4$ , $n - 3$ , $n - 2$	
$\Rightarrow (3n - 9) = 3(n - 3)$	
We get the same conclusion.	

### 9.2.4 Summary

Students who take the Maths 102 course do so for a variety of reasons. All of them have one thing in common, preparation for another first year mathematics course that allows them to continue studying mathematics. Thus as a preparatory course, it is not what would be considered a mainstream mathematics course. However, as has been shown, most of the students who participated in this research knew the difference between what is personally convincing and what is mathematically convincing, and they were able to use algebraic, and deductive reasoning to write what could be referred to as a statement of proof. Although this was not the case with the trigonometry task, evidence showed that the students could still engage with algebraic, trigonometric, and deductive reasoning. In this particular case, the problem did not appear to lie with the students, but with the task. What was needed in nearly all cases was further guidance with respect to generalising the range for  $\theta$  given the domain for the function. As shown in this discussion and in Chapter 6, students could have benefitted from guidance in their selection of mathematical resources and in structuring argumentation.

At the tertiary institution where this research took place, techniques of proof and formal reasoning are introduced in second year mathematics courses. A survey by Thomas et al. (2012) showed that proof is not taught as a first year course in New Zealand tertiary institutions, and neither is it in other parts of the world – although it is done in some countries (e.g., France and Germany). However, the conclusions from this research suggests that first year mathematics students have the readiness (Corbishley & Truxaw, 2010)

to engage with proving activities and proof. These students were aware that in mathematics, something is certain if it is mathematically convincing, that this is different from being personally convinced; they were aware that in a mathematical argument, one statement leads to another; they demonstrated that they can compute algebraically if required, and communicate their understanding of the conclusions in a rational way which may involve re-organising communicative action from a personal viewpoint to a mathematical one (Zaslavsky et al., 2012). However, this research also suggests that because of various obstacles, good professional guidance is required. Evidence from mathematics education suggests various theoretical and practical contributions necessary for establishing proof in the school curriculum (Jones & Herbst, 2012; Hsieh, Horng, & Shy, 2012; Lin et al., 2012). This research recommends that similar considerations be given to the construction of a curriculum including proof and proving for first year university mathematics students.

The next section considers the manner in which diachronic changes in students' argumentations occurred.

### 9.3 Diachronic Changes in Student Argumentations

This section will discuss the key results arising from the analysis of students' mathematical argumentations over a three year period, and how these guided an answer to the second question that helped to shape this research:

[RQ 2] *How do students' argumentations of conviction change as they progress through first, second, and third year courses in university mathematics?*

Five students were tracked over three years and examples taken from four of them that best represented their growth in argumentation from each year. These were analysed using the TA Framework. Four of the students began their study of mathematics at university by taking Maths 102. Of the original five students that were tracked, four agreed to be interviewed to enable the researcher to ascertain the changes they had made as they moved to writing of formal proofs, and what they had come to consider as important features of being able to prove.

#### 9.3.1 Argumentations in the First Year

The argumentations selected from their first year came from Maths 102, as these represented some of the first experiences with argumentation in mathematics at university.



These students' argumentations were based on Problems 1 and 2, as follows.

Problem 1:

If a square number  $n^2$  is a multiple of 3 then  $n$  must be multiple of 3.

- Investigate this problem, and convince yourself whether it is always true, sometimes true, or never true.
- Write an argument that you would use to convince another person such as a mathematics lecturer of your conclusion.

Problem 2:

Show that all integers that are multiples of 3 and divisible by 4, are also multiples of 12.

- Investigate this statement, and convince yourself whether it is always true, sometimes true, or never true.
- Write an argument that you would use to convince another person such as a mathematics lecturer of your conclusion.

Conclusions (C4-2) that emerged from an examination of these students' first year argumentations were that these students:

- could reason deductively and algebraically, and construct argumentations that were logically coherent;
- make more advanced use of symbols;
- were aware of the mathematical ideas that are embedded in the mathematics of the problem they were given; and
- would attempt to provide a mathematical backing to the argumentations.

### Reasoning and Logical Coherence

In their first year, the cognitive and mathematical resources these students drew on came from the Symbolic World of the TWM, or an intersection of the Symbolic and Formal Worlds. They were all able to reason deductively and focus on constructed arguments that were logically coherent, which was particularly noticeable in the students' explicit (and implicit) use of If-then statements. For example:

If  $3x$  is also divisible by 4, then  $x$  must be divisible by 4 (Lear)

If  $a \in \mathbb{Z}$  then  $3|a$ , or  $3 \nmid a$ . If  $3 \nmid a$ , then ... (Tas)

If  $n^2 = 3k$  then  $n \notin \mathbb{Z}$  (an implied example) (Cam)

The use of more formal symbols and algebraic reasoning featured in all of the students' argumentations. The main symbol used was the variable representation, others included logical, relational and membership symbols such as,  $\rightarrow$ ,  $\nmid$ , and  $\in$ . Algebra was used to construct expressions in terms of a variable that could be used to reason with.

Logical coherence was evidenced by the strategies used by students to compose their argumentations. For example most students used a direct proof approach, while some used proof by contradiction or by contraposition (the students that used these last two approaches had a knowledge of them and their use). All students were generally aware of whether or not a proof strategy was effective in attaining a desired goal. For example, in one argumentation a student recognised that the initial approach he used did not satisfy the conditions of the problem, and changed to develop a more logical approach (given in narrative form rather than using symbols and algebra). However, there was an instance when this did not occur, and another student presented a circular argument as a proof. However, during his Maths 150 course – a course that gives an introduction to the use of careful mathematical language and reasoning in the context of calculus and linear algebra, this same student gave a formal proof that indicated an advanced use of symbols and algebraic and deductive reasoning. One student (Tas) demonstrated these attributes in Maths 102, and showed knowledge of mathematics and mathematical techniques that was beyond the scope of this course.

### **Mathematical Ideas and Backing an Argumentation**

The students were aware (or became aware) of the mathematical properties or algebraic structures that were embedded in the mathematics of the problems. In some cases an understanding of the problem was clarified or confirmed through inductive reasoning using arithmetic examples. However, these students quickly went on to reason algebraically. Knowledge of mathematical properties and algebraic structure informed the students' backings and hence warrant. The backings used by the students included one or more of the following: a general rule comprising a combination arithmetic and algebra; known rules of algebra; a method certain of proof; or a known rule from mathematics. Evidence shows that all students attempted to demonstrate the use of a mathematical backing, and that, along with the warrant they used, played a central role in giving epistemological status to their argumentations.

General features that were evident in these students first year argumentations are mapped to the TA Framework in Table 9.4

Table 9.4: General Features of First Year Argumentations

Three Worlds of Mathematics	Toulmin Model	Rationality
<p><b>Symbolic World, or the intersection of Symbolic and Formal Worlds:</b>            Numeric and algebraic symbols are used            Language used is logically coherent            A known theorem or result in mathematics is used            The conclusion is reasoned deductively</p>	<p><b>Data:</b> Numeric – self-conviction/understanding the problem, otherwise algebraic data  <b>Warrant:</b> Algebraic reasoning, or reasoning with a known result in mathematics.  <b>Backing:</b> A known result in mathematics, rules of algebra, or a mathematical theorem  <b>Qualifier</b> Absolute qualifier (e.g. ‘must’, ‘is’)  <b>Rebuttal</b> (None given)  <b>Conclusion:</b> Is provided</p>	<p><b>Epistemic:</b> A mathematical warrant is used to secure a statement of truth. The warrant is paired with an absolute qualifier            Use of If-then statements indicate logical construction of statements            Deductive reasoning is used to construct conclusion  <b>Communicative:</b> Text uses symbolic language, as well as logical language. A sequence of deductive steps are shown  <b>Teleological:</b> Goals and subgoals a set, and achieved            Text is illustrative of deductive reasoning</p>

Table 9.4 details how the conclusion (C4-2) was arrived at using the TA Framework. It provides a baseline of cognitive and mathematical resources used by the first year participants; how they structured their argumentations (based on elements of the Toulmin Model); and what general features of their argumentations gave them a rational grounding (Habermas’ Rationality). This baseline will be used to compare the students’ second and third year argumentations. The next section will consider related to these students’ second year argumentations.

### 9.3.2 The Changing Nature of Argumentation

In this section general features that were different from previous ones and were common to students’ second argumentations will be highlighted. The argumentations presented were marked by the use of known facts, linguistic and symbolic formality, and deductive reasoning. Mathematical knowledge was an important factor in the construction of argumentations. The students in this part of the research also demonstrated familiarity with relevant content, including definitions and theorems. From an analysis of the students’ second year argumentations, the following conclusions were made:

- Definitions appeared as important aspects of argumentation;
- Students’ cognitive and mathematical resources were drawn from an intersection of the Symbolic and Formal Worlds of the TWM; and

- An increased emphasis was placed on symbolic knowledge.

### The Use of Definitions

A significant change from the first year argumentations was the use of mathematical definitions. From the interviews conducted with these students, it appears that Maths 150 was the first course where they encountered formal definitions. It also appears that they got to perceive their importance to proof. All students took this course in the second semester of their first year, which might explain why the use of a formal definition only appeared in the Maths 150 example given by Cam.

Formal definitions were usually implied in the structure of students' argumentations and seldom stated explicitly. For example, in Tas' argumentation which showed that if  $f(x) = x+1$ , and  $g(x) = (x-1)^2$ , then  $f \geq g$  whenever  $x$  is taken from the set  $S = \{x \in \mathbb{R} \text{ with } 0 \leq x \leq 3\}$ , he appeared to rely on an implicit definition that for given functions (of a single variable)  $f$  and  $g$ ,  $f \geq g$  on a set  $S$  if and only if for all  $x_0 \in S$ ,  $f(x_0) \geq g(x_0)$ . In another example, Cam gave a condition necessary for a bifurcation to occur at a certain point. On the other hand, definitions were stated more explicitly in other argumentations. For example, in showing a result involving a matrix and its eigenvalue, Lear began by giving a (definitional) statement that, ' $\lambda$  is an eigenvalue of matrix  $A$  if and only if it is a solution to  $|A - \lambda I| = 0$ '. In another example, Able referred to the definitions of an even and odd function in one of his argumentation.

A key point that we wish to make here is that students appeared to be aware of the importance of definitions in providing a theoretical backing to a claim, and this guided their choice of warrants.

### Cognitive and Mathematical Resources, and Symbol Knowledge

The mathematical behaviour evidenced in students' argumentations, indicated that they were using mental and mathematical resources drawn from the intersection of the Symbolic and Formal Worlds. Deductive reasoning was a key feature of students' argumentation.

The nature of symbols used differed from their first year, where the variable was the main symbol used to reason with. New symbols were included that were constitutive of certain concepts in mathematics. For example, Tas used  $x_0$  to signify the idea of taking an arbitrary  $x$  from a particular set; Cam used  $\frac{\partial f_a}{\partial y} = 0$  to show that he was interested in a parameter  $a$ , where the graph of  $f_a(y)$  is tangential to the  $y$  axis; and Lear referred to  $\lambda$  as an eigenvalue of a matrix that must possess a certain property. These symbols

which appeared as procepts (since they indicated an associated process) placed a higher demand on the mathematical knowledge of the reader in order to understand the students' argumentation. In addition the repertoire of logic symbols was added to, with the inclusion of the 'if and only if' symbol,  $\Leftrightarrow$ .

Hence based on the TA Framework, the primary changes in student argumentations from their first year can be summarised as follows:

- (The TWM): Cognitive and mathematical resources now come primarily from the intersection of the Symbolic and Formal Worlds. The variable is no longer the only symbol on which to reason, other more formal, or advanced symbols are introduced that are proceptual in nature. A known definition (or theorem) is used (usually implicitly);
- (Toulmin): Algebraic and deductive reasoning is now not only backed by rules of algebra and logic, but also by a definition (or theorem) from mathematics. The modal qualifier is paired with a theoretical warrant; and
- (Rationality): Truth of a conclusion is now grounded by algebraic and deductive reasoning, including mathematical theory (definitions). Consequently, results (goals and subgoals) and understanding are communicated symbolically through the use of algebra.

An important feature not mentioned explicitly in this summary, is the growth in mathematical knowledge. In the students' argumentations, this was evidenced in the areas of differential calculus and linear algebra where knowledge included definitions, and theorems. [Note: some second year mathematics course also include vector spaces, and subspaces and consider their axioms.]

### 9.3.3 The Nature of Advanced Argumentation

In their third year, mathematics students have an option of taking either applied, or pure mathematics courses. Two of the four students in this part of the research studied pure mathematics, while the other two pursued studies in applied mathematics. Another third year student who agreed to be interviewed, but who did not submit examples of argumentation in each of his three years, also majored in pure mathematics. A survey of the third year argumentations of these students showed the following differences from their second year:

- The students appeared to possess an in-depth, connected knowledge of mathematics content and representations underlying given problems:
- Argumentations made extensive use of more advanced symbolic notation to support deductive reasoning;
- The use of definitions (or theorems) were more explicit in their statements; and
- Subgoals were clearly indicated.

### Mathematical Knowledge and Understanding

An important factor in these students' argumentations was their mathematical knowledge and understanding. Unlike their second year, they presented their knowledge as a connected complex of mathematical ideas. For example, the proof of  $\pi(e-1) < \int_0^\pi e^{|\cos 4x|} dx < 2\left(e^{\frac{\pi}{2}} - 1\right)$ , involved having a knowledge of a trigonometric function, absolute value, and the definite Riemann integral and its area representation. In another example, knowledge of induction was connected to properties of an odd function and a definition. The complexity of knowledge was also expressed in another way. Some statements such as, 'Q is dense in R', or 'x is a singular point' were unexplained. However, the way they were implemented suggested the students understood how they connected to aspects of their argumentation, and the impact they had on achieving a desired goal. These students were able to synthesise their knowledge of various aspects of mathematics in order to provide an understanding of how they were able to reach their goals, and on what bases their conclusion was justified.

### Extensive Use of Symbolic Notation

Compared with their second year argumentations, all students made extensive use of symbolic notation to support their reasoning. There appeared to be an increase in complexity associated with the understanding, and use of, symbolic notation such as the linear operators  $L(U)$  and  $\nabla^2 u$ , the contour integral  $\oint_C$ , and the statement  $t \in \overline{\mathbb{Q}}$  where each has a set of concepts embedded in it. As an example of embedded concepts consider  $\oint_C \frac{f(z)}{z-a} dz$ . Among other things, this assumes that  $f(z)$  is analytic in a simply connected domain, and that a piecewise, smooth, simple closed curve  $C$  is oriented anticlockwise in the simply connected domain. In each case, the symbols were linked to a definition that the student used to make a deductive inference.

### Use of Definitions

Students' connected knowledge included definitions and theorems that related to the problem they were engaged with. By their third year, the students appeared to be more explicit in their use of definitions and theorems which were often directly stated, or could be clearly identified in their argumentations. For example, one student stated the Cauchy Integral Formula (Theorem) and checked that each of its conditions were satisfied in order to use it. In another a theorem of linearity of a partial differential operator was stated in a conclusion.

If definitions or theorems students were using were not stated, they were clearly identifiable in their argumentation. For example, in one student's argumentation he stated, 'Since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ ,  $t \in \overline{\mathbb{Q}}$  ...', which is a theorem from analysis that was crucial to deriving his conclusion. In another example, a student wrote two statements, ' $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy} = 0$ ' followed by, 'Also, functions are  $C^2$ '. These two statements combine to give a theorem for harmonic functions. Definitions and theorems were clearly instrumental in achieving the goals of either a proof, or a solution to a problem.

The argumentation of one student (Tas) did not exhibit explicit or clearly discernible definitions or theorems, rather, they were deeply embedded in his reasoning. An analysis of his argumentation showed that definitions and theorems he used related to the definite integral, absolute value, and linear and trigonometric functions which are certainly known to most Maths 150 students. However, showing  $\pi(e - 1) < \int_0^\pi e^{|\cos 4x|} dx < 2(e^{\frac{\pi}{2}} - 1)$  was a very non-routine problem, and Tas' focus was on developing a problem solving strategy and then marshalling together the mathematics required to solve it. As evidenced in his argumentation, the strategy he used (encapsulated in two diagrams) clearly involved insight.

The students' responses in their interviews revealed that, as they progressed through to third year courses in mathematics, definitions became a significant factor in proving. In their view definitions:

- provided direction through deductions that can be made from them, or the theorems they lead to;
- validated a proof and one's understanding of mathematics; and
- were not always necessary to record in a proof, although steps in a proof should suggest their use.

As noted in their interviews, students used definitions explicitly (as a backing) to convince themselves or others that a particular step was justified, or implied their use by integrating them as a natural part of text so as to preserve a logical flow.

### Clear Subgoals

Another feature of third year argumentations was that they exhibited clear subgoals. Some stated their subgoals explicitly, for example Cam, whose proof relied on showing certain conditions held in each of the number systems, while Able's subgoal was to check that he could use a certain theorem to solve a problem. In other argumentations, subgoals were not explicitly given, but were clearly observable in the layout of argumentation. For example, Tas' subgoals were to show  $\pi(e - 1) < \int_0^\pi e^{|\cos 4x|} dx$ , and  $\int_0^\pi e^{|\cos 4x|} dx < 2 \left( e^{\frac{\pi}{2}} - 1 \right)$ . These were clearly implied by the way he separated his step by step argumentation into two parts. In another example, Lear's subgoals were to construct  $L(U)$ , and  $L(V)$ ,  $L(\alpha U + \beta V)$ , and  $\alpha L(U) + \beta L(V)$ , and her penultimate goal was to show an equality. In all cases, subgoals involved goals of their own.

In the light of the above we can summarise the changes from the students' second year argumentations using the TA Framework as follows.

- (The TWM): Although cognitive and mathematical resources come primarily from the intersection of the Symbolic and Formal Worlds, the students are able to work in the Formal World when necessary. Axioms, definitions and theorems are more visible, or used explicitly;
- (Toulmin): Data, warrants, and backings are mathematically based giving the argumentation a theoretical status; and
- (Rationality): The truth of a conclusion is grounded in mathematical theory. Argumentations are complex, involving multiple steps, and are therefore sorted into goals and subgoals. These are communicated in a way that demonstrate the student's theoretical knowledge of the problem and its solution. The primary mode of reasoning is deductive, although other modes may also be present.

### 9.3.4 Summary

The purpose of tracking students' argumentation over three years was to establish a portrait of growth in argumentation from their first year. Using the TA Framework enabled



the researcher to consider this growth in terms of cognitive and mathematical resources (TWM), and how these were manifest in the structural components of their argumentation that consisted of data, warrant, backing, a modal qualifier, and rebuttal (if used). The third dimension that was used to form a portrait was Habermas' three types of rationality. As the students progressed through the three years of mathematics, their argumentations became more formal. This was influenced by development of students' knowledge of mathematics, which lead to the introduction of definitions in their second year argumentations. These were usually implicit in their argumentations, but became more visible in their third year. By the third year, the primary backing for warrants were definitions or mathematical theory that could be connected to a definition or theory. Thus the warrant was given a theoretical status, served as a warrant for truth and paired with an absolute qualifier (Rodd, 2000). Assisting in the development of formality were the growing complexity of mathematical objects (e.g., a bifurcation), symbols (e.g.,  $\left(\oint_C \frac{f(z)}{z-a} dz\right)$ ), and 'definition like' statements. The strand that held the structural components of argumentation together was reasoning. Although algebraic and deductive reasoning were used in first year argumentations, the second and third year saw these forms of reasoning being applied to increasingly complex problems. Students also used other forms of reasoning such as visual reasoning, but this was used to explain or verify a particular thinking strategy. Solving these complex problems often involved multiple steps, which became more clearly differentiated as subgoals by the third year.

The most noticeable change in rationality was the use of mathematical theory to give an epistemological status to an argumentation. This became more pronounced in their third year argumentations through the use of definitions or a mathematical theory to back warrants. In their third year, students often used mathematical knowledge that connected different domains of mathematics, which was not evident in their first and second year argumentations. Changes in communicative rationality were made to include new symbols (mathematical, and logical), and signal that subgoals were being addressed. For example, an argumentation would be separated into sections, each one dealing with a particular subgoal, however this change was dependent on the complexity of the problem the student were solving. The students continued to demonstrate their understanding of the mathematical ideas in their argumentation. Changes in teleological rationality were manifest in the increased logical structuring of the students' arguments.

## 9.4 Third Year Students' Conceptions of Proof

Four students who were tracked through three years of their mathematics study were interviewed to find out about their views of proof, and how these changed over time.

In their first year, the students regarding proof as part of a problem solving process that involved finding a solution that required justification. Proof did not have a convincing function for these students, but rather, it was a learning experience that introduced them (in Maths 150) to a logical commentary that included mathematical deduction and induction, and formal vocabulary and justifications. By the second year, proof was seen as a learning tool and viewed as a primary means to understanding mathematics. It was at this stage that they began to appreciate the role of definitions and theorems in proofs. All of the students pointed to the second year as a time when they were formally introduced to methods of proof which proved to be essential tools of mathematics in their third year.

The conception of proof as a means to understanding mathematical concepts continued through to the third year. All students commented on the importance of knowing definitions, which included understanding their logical structure. Thus it was essential for them to know what a definition can and cannot do. For example, one student spent time matching diagrams illustrating the formal definition of the limit with its written form in her attempt to understand proofs involving the limit definition. She also substituted numerical data for epsilon and delta to get an idea of how the definition worked. Thus, although students saw deduction as a primary source of mathematical reasoning in their third year, they were also versatile reasoners integrating other forms of reasoning as well (e.g., Tas' example in Section 7.5.3).

A notable aspect missing from the students' conception of proof was conjecturing. Three of the students said that they were not given an opportunity to develop conjectures and then attempt to write a proof for them. The one student that did, did so as part of a third year research project. In his third year, this student also came to view proof as having a role of personal conviction, a view that did not resonate with the other students.

## 9.5 A Refined Model of Argumentation

An unexpected outcome of this study was a refinement of the original viewpoint of argumentation given in Section 3.4. This occurred as a result of reflecting on the analyses of students' argumentations provided in Chapters 6 to 8, and surrounding literature. With

backing from the works of Balacheff (2010), Brousseau (1997), and Pirie and Kieren (1994), a model of argumentation was developed consisting of three layers: a core layer embedding argumentation in the theories underpinning the TA Framework; another that considers the impact of the teacher on the milieu of teaching/learning in a problem solving environment; and a third layer linking argumentation to thinking, understanding, and communicating. The layers reflect a dynamic dimension to argumentation. Some examples of this are the transition of students through the three worlds of mathematics in the first layer, the reflexive relationships in the second layer that occur between the teacher and the students as argumentation is constructed in the classroom, and the responsiveness of argumentation to different audiences in the third layer.

The first layer shown in Figure 9.2 consists of Tall's three worlds of mathematics, Toulmin's layout of an argument, and an aspect of Habermas' communicative action comprising his theories of rationality and the life-world. Other theories also support this theoretical level, for example Harel and Sowder's theory of proof schemes (Harel & Sowder, 1998) and Pedemonte's characterisation of argumentation (Pedemonte, 2007b), which are both discussed in the review of literature. Although the next layer (Fig. 9.3) was not an explicit

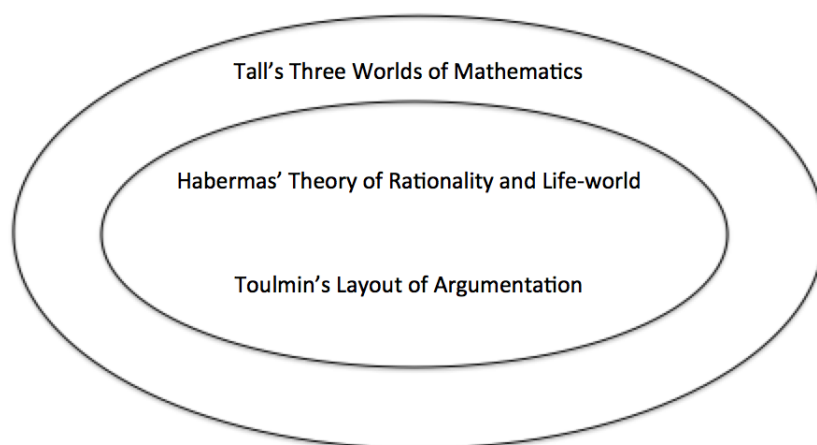


Figure 9.2: A refined model of argumentation: A theoretical basis

part of this study, the role of the lecturer and the nature of the course taken appeared to have an increasing influence on the way in which students crafted their argumentations. Thus the focus of the second layer is on the role of the teacher (denoted by the letter T) in using all teaching resources to orchestrate a learning environment where students can be guided toward developing a mathematical disposition. In this respect, Yackel and Cobb (1996) have discussed the importance of establishing socio-mathematical norms; Brousseau (1997) has argued the pedagogic value of the concepts of milieu and the didactic contract;

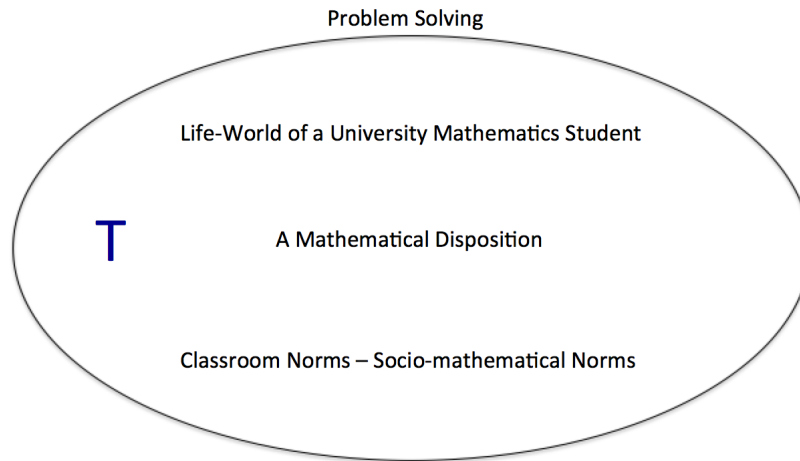


Figure 9.3: A refined model of argumentation: The teacher's role

and Lin et al. (2012) have recommended ways of guiding students to proof writing. This layer is important as it constructs the life-world of a first, second, or third year student.

The third layer in this model (Fig. 9.4) is developed from two claims made by Balacheff (2010). The first of these is that the explanatory power of a mathematical text is directly related to the quality and density of a student's understanding of the mathematics involved. The second is the claim that mathematical understanding is enhanced as a student searches for certainty, understanding, and more sophisticated means of communication. Additionally, Balacheff also notes that the status of an argument rests on the current standards of the mathematics community (which relates to the second layer). The third layer, considers argumentation as actions students take in constructing an acceptable argument. It is divided into three levels. The first of these (the outer level) highlights mathematical knowledge, understanding and thinking whereas the second level focusses on communication to specific types of audiences. The main concern of the third level is specialisation; the specialisation of one's form of communication to coincide with the specific demands of an 'ordinary' undergraduate type of proof. The final level concerns the production of proof as a highly specialised form of logical argumentation. Facility with this level is seldom reached by first, second or third mathematics students.

Evidence from this study of students' argumentations suggests that, when asked to determine whether a mathematical claim is true, they will examine it with the intention of understanding inherent mathematical ideas. They will then look for ways of thinking and reasoning that will either shore up their confidence in the conclusion they have drawn, or prompt them to re-evaluate it. This aspect requires they access certain elements of

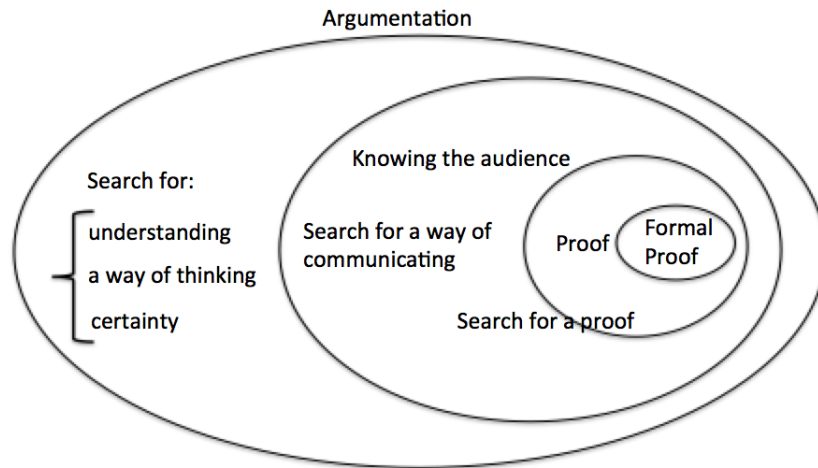


Figure 9.4: A refined model of argumentation: The terrain

their mathematical knowledge as well as problem solving and reasoning strategies. When writing to convince another of their conclusion, the means of communicating they use is shaped by the audience to whom they direct their argumentation. If the audience is seen as a representative of the community of mathematicians, such as their lecturer, then it is likely that the students will search for a means of communicating based on socio-mathematical norms and other expectations gained in the second layer of the model. This was demonstrated in this research, when students produced arguments using algebra and deductive reasoning in a style similar to, or consistent with, what was modelled and encouraged in lectures, assignments, tutorials, and solutions.

From observations of students' argumentations in this research, there appeared many instances where they moved back and forth from one level to another – usually as a means of developing a thickened understanding of their argument (or elements within it). Hence, this model also includes a recursive feature as described by Pirie and Kieren in their recursive theory of mathematical understanding (Pirie & Kieren, 1994). In our model, a backward move enriches and refines the inner (previous) layer. Our observations suggest that when this occurred, it led to a heightened awareness of students' intentions and goals. Also from our observations, a forward move often led to the production of new knowledge, or an elaborated understanding of existing knowledge. Thus in this model, proof can be used as part of one's communicative strategy for an argumentation that itself is not intended to be a proof, or it can be used as a tool for an individual's search for understanding, a way of thinking, and certainty. An effect of being able to move back and forth in a recursive manner is that an argument can be made more formal, robust, or

more accessible to others.

## 9.6 Answering The Research Question

The question that shaped this research was, ‘How do students formalise their argumentations of conviction?’

A detailed discussion of the conclusions drawn from this research have been given, therefore this part of the discussion will provide a general summary of the main points. This summary is based on the participants in this research and is not meant to be a generalisation to all students taking mathematics courses at this, or any other, university.

- When formalising their argumentation, students make a conscious choice of what cognitive and mathematical resources to use from those that are available to them at the time. These are deliberately chosen to give, what they perceive to be, an epistemological status to their arguments. Thus, mathematical ideas are selected from their knowledge base that highlight mathematical theory.
- Students choose mathematical symbols and symbolic tools to support algebraic and deductive reasoning. Communicative resources are selected to assist with explanations, conciseness of statements, and logical flow.
- With their selected resources, students’ formalisations involve a teleological element. Their ideas are placed into a logical structure which this research details (for the purpose of analysis) as, data, warrant, backing, modal qualifier, rebuttal, and conclusion. This logical structure reflects sub-goals, and reasoning that is aimed at achieving and communicating understanding.
- When formalising their argumentation students attempt to choose data that consists of formal mathematical objects (such a variable or an algebraic/calculus/analysis expression). An affect of this on their argumentation is that the warrant that they use to link the data to the conclusion becomes backed by mathematical theory.
- The primary form of reasoning students use is deductive, although other forms are also used in supporting aspects of their thinking.

Although this research has managed to draw these conclusion, there are recommendations for further refinement and investigation that can be made. These will be addressed in the following chapter.

## Chapter 10

# Limitations and Recommendations for Further Research

The aim of this research was to see how students formalise their argumentations in mathematics. A theoretical framework was developed, and used to analyse students' written argumentations. Participants were drawn from an introductory mathematics course, as well as a second year course that had mathematical thinking and proof as a major part of its curriculum. Besides these participants, the changing nature of argumentation was analysed, by considering argumentations from four students over a three year period. It is important to note that it was not the intention of this research to produce results that would generalise to all mathematics students from this, or any other, university, but to present the findings and then use them to suggest possible opportunities to investigate further some areas for research into argumentation.

This research primarily focussed on written argumentation. This was chosen for three reasons. The first of these came about through the review of literature where there seemed to be little research on students' written argumentation as a 'historical' artefact. Secondly, the main avenues students use to communicate a mathematical argument to their lecturer and tutor(s), are written assignments, tests and examinations. The third reason was organisational. A preliminary study indicated that focussing on voice recordings as a method of data gathering would be time intensive in analysing a large sample of students. However, this also partly limited the potential scope of this research. Thus if a similar study were to be repeated, it is suggested that a smaller group of students should be used, with greater use of interviews. Analysing students' argumentations of self-conviction required the researcher to look at them through their eyes. This was not a problem with

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convincing an other because, as the reader, it was the researcher they were attempting to convince.

Another limitation in this research was the manner in which the tasks were administered. An essential facet of the students' participation in this research was honesty. Student participation was voluntary, and any work that was done had to be their work without assistance from any other source. This was carefully explained in class, and in their participation form (see Appendix D). It is possible that some students might have used outside assistance. However, an examination and comparison of scripts indicated that this did not appear to be a problem.

This research involved a relatively large number of first year students who were taking a preparatory course in mathematics. It would have been helpful to use other first year mathematics courses, including students from a general mathematics course (e.g., Maths 108), as well as those from an advancing first year mathematics course such as Maths 150. While this research included an analysis of argumentations of a group of second year students who were taking a course in which thinking mathematically and proof was a significant aspect, this was not necessarily helpful in providing a general picture of second year students' argumentation. A suggestion here would be to include students from a wider range of second year mathematics courses such as Maths 208, Maths 250, and Maths 253. This might provide more definition to the terrain that students traverse when developing mathematical processes of logic and reasoning and communicating mathematically in the first two years of studying mathematics at the tertiary level. Some other areas that might be worthy of investigation that arise from, and would extend this research study, include:

- An examination of how the TA Framework could be adapted to analyse collaborative argumentation; and
- How the TA Framework could be used to help inform tertiary pedagogical practice.

Although there were some deficiencies in the design of this research, there were some positive facets to emerge. One of these was the usefulness of the framework in being able to talk about students' argumentations, in terms of what thinking and reasoning they were using; the mathematical knowledge that their argumentations reflected; and what changes they made when engaging with an other. Another was that students who were doing a preparatory course in mathematics seemed to possess the ability to do mathematics that included proving. At the third year level, it was a challenge to connect students'



mathematical knowledge to the TA Framework, although giving them an opportunity to talk through their argumentation gave much greater clarity to the analysis.

A contribution to mathematics education that is also made by this thesis is linking argumentation and proof. This is summarised in the model of argumentation presented in Section 9.5, where proof – including formal logic proofs, are examples of highly specialised argumentations requiring specific linguistic, and reasoning strategies. This notion is supported by Tall et al. (2012), and Mejia-Ramos (2006), who argue that argumentation ability and the nature of childrens'/students' proofs are developmental.

Another contribution to mathematics education is in the area of pedagogy. The second level of the model of argumentation (Section 9.5) considered the role of the teacher in implementing and sustaining a classroom environment that fostered a disposition toward mathematics that includes an inclination and ability to engage with mathematical inquiry, argumentation and proof. In this level these are introduced to a student's life-world. Through a range of experiences, students should come to see the functions of creating an argument; its purposes and its usefulness. Related to this contribution is the communicative component of the TA Framework, namely Habermas' functions of language with respect to rationality. This thesis suggests that a focus on students' life-worlds, and the role of language in communicating rationality can sharpen a pedagogical focus on designing tasks and interactions particularly for minority cultural groups, so that they can build their confidence and ability to engage successfully with argumentation and transition to proving and proof. One practical step is to look and listen for knowledge, reasoning, and communicative strategies a student uses when providing a mathematical argument.

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# Appendices

# Appendix A

## Tasks for Maths 102 Students

### A.1 Task 1: The Sum of Consecutive Numbers

The set of integers is the set  $\mathbb{Z}$  where  $\mathbb{Z} = \{\dots - 3, -2, -1, 0, 1, 2, 3, \dots\}$ .

Now, consider the following statement;

“The sum of any 3 consecutive integers is always a multiple of 3”.

- Investigate this statement, and convince yourself whether it is always true, sometimes true, or never true.
- Write an argument that you would use to convince another person such as a mathematics lecturer of your conclusion.

### A.2 Task 2: A Trigonometrical Identity

A Maths102 student suggests that  $\cos^2 \theta + \sin^2 \theta = 1$  for all angles  $\theta$  from  $0^\circ$  to  $360^\circ$ .

- What do you think? Write an argument to convince yourself whether the student’s suggestion is always true, sometimes true, or never true.
- Write an argument that you would use to convince a mathematics lecturer of your conclusion.

## Appendix B

# Questions From a Preliminary Study

### B.1 Problem 1

If a square number  $n^2$  is a multiple of 3 then  $n$  must be multiple of 3.

- Investigate this problem, and convince yourself whether it is always true, sometimes true, or never true.
- Write an argument that you would use to convince another person such as a mathematics lecturer of your conclusion.

### B.2 Problem 1 (Variation)

A postgraduate mathematics student suggests that there may be a large number that is not a multiple of 3 but whose square is a multiple of 3. Is he correct, or incorrect?

### B.3 Problem 2: The Derivative of an Even Function

The derivative of an even function is an odd function.

- Is this statement always true, sometimes true, never true?
- Write an argument that you would use to convince another person such as a mathematics lecturer of your conclusion.

## Appendix C

### Task for Maths 255 Students

A student makes a claim that in *any* set of three consecutive integers, either one of the integers is a multiple of 2 and another a multiple of 3, or one integer is a multiple of 2 and 3.

- What do you think? Write an argument to convince yourself whether this claim is always true, sometimes true, or never true.
- Write an argument that you would use to convince a lecturer of your conclusion.

# Appendix D

## Interview Questions

- (1) At what stage in your university study of mathematics did you first start proving?  
(Course?)
  - (i) How were you introduced to proving?
  - (ii) How was this way of thinking about justification different or similar to what you had previously experienced with regard to justifying mathematics ideas?
- (2) In your first experiences with proof, what aspects of it did you find easy/hard? As you developed in mathematics, how did these change?
- (3) Thinking about your first experiences with proof at university, what purpose did you think it served?
- (4) As a 3rd year mathematics student, has that purpose changed in any way? If so, how?
- (5) What has contributed to any changes you have made in the way you go about constructing and presenting a proof?
- (6) What role do you think definitions play in proving? How important were they in your first year? Second Year? How are they important (if they are) in your third year?
- (7) What things do you do when you first begin to prove a mathematical statement?
- (8) From your experiences in mathematics what changes would you suggest to the way that we go about teaching proof (reading, constructing, and writing)? (Explain)

- (9) How important is conjecturing to proving? (Give an example if the student doesn't know the term 'conjecturing').
- (i) Is conjecturing part of your lectures?
  - (ii) What about exploring definitions/theorems such as what they can do, can't do, can nearly do if you make an adjustment to it?

# Appendix E

## Participant Consent Form

Understanding how undergraduate students formalise their written argumentation in mathematics.

**Researcher:** Garry Nathan

Please place your initials at the beginning of each bullet point statement to indicate your agreement with it. I understand that:

- if I participate in this research, I will need to commit a maximum of 3.5 hours of my own time to it;
- my name will not be used in any data that will be recorded and in the analyses of data;
- if I participate in this research, I may be audio-taped during follow-up conversations with the researcher if required;
- any notes that I make during each research question will be included in responses given to the researcher;
- my consent form and all audio-tapes (if made), notes, written work, and associated computer files will be stored securely for 6 years and then appropriately destroyed (shredded, and erased);
- I am aware that the researcher may also be my course lecturer or tutor. I am satisfied that participation in this study will not have an adverse affect on my role as a student in this course;



- All written work that I will be required to present for the purposes of this research will be **my own work** without assistance from any electronic media, or other person(s), which includes the internet, computer or cell phone software, other students, tutors, teachers, and lecturers.

I have read through the requirements for consent to participate in this research, and agree to all of them. I therefore give my consent to be a participant in this research.

First Name:

Surname:

Signature:

Date: