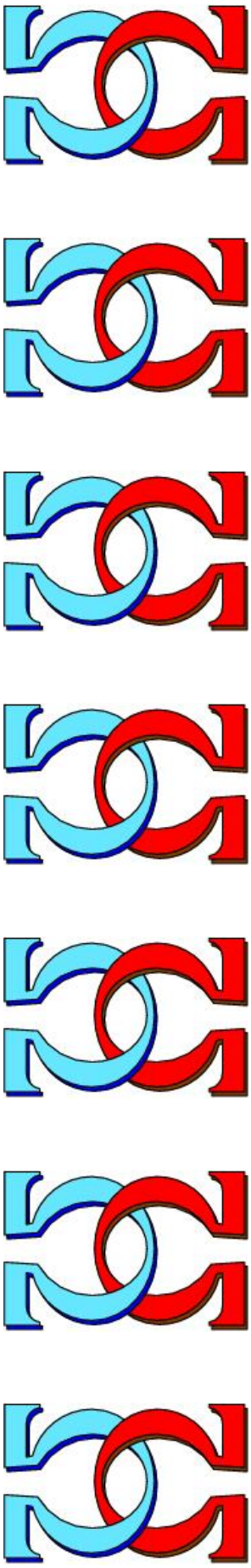

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Possibilistic Functional Dependencies and Their Relationship to Possibility Theory

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Abstract

This short paper introduces possibilistic functional dependencies. These dependencies are associated with a particular possibility distribution over possible worlds of classical database. The possibility distribution reflects a layered view of the database. The highest layer of the (classical) database consists of those tuples which certainly belong to it, while the other layers add tuples that only possibly belong to the database, with different levels of possibility. The relation between the confidence levels associated with the tuples and the possibility distribution over possible database worlds is discussed in detail in the setting of possibility theory. A possibilistic functional dependency is a classical functional dependency associated with a certainty level that reflects the highest confidence level where the functional dependency no longer holds in the layered database. Moreover, the relationship between possibilistic functional dependencies and possibilistic logic formulas is established. Related work is reviewed, and the intended use of possibilistic functional dependencies is discussed in the conclusion.

Keywords: Database decomposition, Functional dependency, Possibilistic logic, Possibility theory, Uncertain data

1 Introduction

Functional dependencies (FDs for short) constitute a core notion in database theory [4], for database decomposition, safe updating, redundancy elimination, and query optimization. This fact has led to a great number of works on *fuzzy* functional dependencies in the fuzzy set literature, especially in the nineties and in the first half of the next decade (see [10, 51] and the related work section of this paper for some overview discussions). This is due to the existence of different views of fuzzy databases, as well as different proposals for fuzzy functional dependencies (FFDs for short). FFDs may be stronger or weaker than classical FDs. They may extend classical FDs to fuzzy databases, or may already differ from classical FDs on classical databases.

The view we investigate in this short note remains close to the one of a classical database where classical FDs hold. We only depart from it by admitting that some tuples may be uncertain, in the sense that we are not sure if some tuple, as it is, belongs or not to the database. This uncertainty may be due to several reasons, for example when the database gathers tuples from different sources with different confidence levels. The uncertainty of some of the tuples will result in levels of certainty associated with classical FDs. The proposal presented here has some similarity in its basic features with an old one, published more than 20 years ago by Kiss [29], and which has had a limited impact in the literature until now. However Kiss' proposal was cast in the setting of multiple-valued logic, while the approach in this short note relies on a possibility theory [24, 52] view. Moreover, the possibilistic view makes more precise the meaning of the weights associated with the tuples and the FDs, respectively. It provides a richer semantic characterization of the weighted FDs. We would like to stress that the simple model we propose may be useful for managing databases with uncertain tuples.

The short note is structured as follows. We start with a motivating example in Section 2. We then discuss the relation between a possibility distribution over possible database worlds and the confidence in the tuples of a database. We make clear that these confidence levels are degrees of possibility. However, the highest one is also associated with a full certainty degree. The uncertain database is then viewed as a layered database. Section 3 introduces possibilistic FDs in this setting and establishes properties for them. It is shown that we can reason with the weighted FDs that hold in an uncertain database using possibilistic logic. We establish soundness and completeness theorems for inference from the weighted FDs with respect to the FDs that hold in the level cuts of the uncertain database, or, in other words, with respect to the possibility distribution over possible database worlds and the

FDs that hold in each world. Section 4 reviews related work that deals either with fuzzy FDs, or with classical FDs in fuzzy tuple databases; it also discusses classical FDs in possibilistic uncertain databases. Section 5 concludes by outlining some potential uses and future developments.

2 Motivating example

There has been an increase in recognition over recent years that a database may contain uncertain pieces of information, although it has been a concern for a long time [49, 36, 1, 45]. This uncertainty may take different forms. Attribute values may be imprecise or pervaded with uncertainty, or one may just be uncertain about the fact that a tuple, as it is, should be considered or not as belonging to a database. In the following, we take the latter view. The tuples are standard tuples (without null values), but we do not have full confidence in some of them.

To illustrate the idea, let us consider the database in Table I. It consists of a unique relation r with attributes C (*Course*), T (*Time*), L (*Lecturer*), and R (*Room*). As can be seen, each tuple is associated with a weight α_i. These weights α_i belong to a linearly ordered scale $S = \{\alpha_1, \cdots, \alpha_n, \alpha_{n+1}\}$ with $\alpha_1 > \cdots > \alpha_n > \alpha_{n+1}$. They may be encoded numerically, e.g., $\alpha_1 = 1$, $\alpha_2 = 0.8$, $\cdots$, $\alpha_n = 0.2$, $\alpha_{n+1} = 0$, but this is not compulsory. Indeed, a numerical encoding will have no particular meaning beyond the ordering of the numbers. These levels may also receive a linguistic reading. We shall come back to that in the next section.

Clearly, this encoding suggests a layer-based view of the relation r: we have first the tuples with the highest confidence level α_1, followed by those with a smaller confidence level (in the example α_3), and so on (in the example we have a third layer with level α_4). It also implicitly suggests a possibility distribution over possible database worlds. How this distribution can be related to the weights α_i is discussed in the following.

3 Relating possible database worlds and confidence in tuples

The problem we are facing is how to relate a possibility distribution over a *set of* tuples to a distribution over tuples. Although this kind of problem has not been considered very often, it already received an answer many years ago in [23]. We first recall these results using the motivating example used at that time [40], namely the representation of an imprecise and uncertain

Course	*Time*	*Lecturer*	*Room*	*Poss*(t)
DB	Mon, 9am	Ann	Aqua	α_1
IS	Mon, 1pm	Ann	Aqua	α_1
CS	Mon, 1pm	Pete	Buff	α_1
CS	Tue, 2pm	Pete	Buff	α_1
AI	Tue, 4pm	Gill	Buff	α_1
AI	Wed, 3pm	Gill	Cyan	α_1
Math	Thu, 4pm	Mary	Lava	α_3
Logic	Thu, 4pm	Mary	Pink	α_3
HCI	Fri, 9am	Bob	Tan	α_4
OR	Fri, 9am	Bob	Tan	α_4
OR	Fri, 9am	Jack	Tan	α_4

Table 1: Example of an uncertain database

information about a multiple-valued attribute, here, the set of languages spoken by a person.

3.1 Possibility distribution on a power set and its upper and lower approximations

For instance, we have the partial information that "John speaks either English and French, or English and German, and no other languages". In that case, it can be described by a two-valued possibility distribution π defined over the power set $2^{\mathcal{L}}$ of the set of languages $\mathcal{L}$, namely let $A_1 = \{\mathit{English}, \mathit{French}\}$, and let $A_2 = \{\mathit{English}, \mathit{German}\}$, then we have $\pi(A_1) = \pi(A_2) = 1$ and $\pi(A_k) = 0$ for any $k \neq 1, 2$. Clearly, this information has an upper approximation by the set of languages *possibly* spoken by John, here $A^+ = \{\mathit{English}, \mathit{French}, \mathit{German}\}$, and the set of languages *certainly* spoken by John, here $A^- = \{\mathit{English}\}$ is a lower approximation. Note that this is only an approximation of the information conveyed by the original distribution π over $2^{\mathcal{L}}$, since we have lost the information that John speaks (only) two languages. However, the two approximations are now distributions over $\mathcal{L}$. This is simpler, namely, $\mu_{A^+}(l) = 1$ if $l \in \{\mathit{English}, \mathit{French}, \mathit{German}\}$ and $\mu_{A^+}(l) = 0$ otherwise, while $\mu_{A^-}(l) = 1$ if $l = \mathit{English}$ and $\mu_{A^-}(l) = 0$ otherwise.

This can be generalized to multiple-valued possibility distributions [40]. Let π be a mapping from a power set $2^{\mathcal{L}}$ (we keep the same notation, but $\mathcal{L}$ now denotes any set) to a linearly ordered scale S where 1 and 0 continue to denote the top and the bottom element, respectively. We assume that π is

normalized, i.e., $sup_{i\in I}\pi(A_i) = 1$ (where I is an index set for the subsets in $2^{\mathcal{L}}$). The upper and lower approximations are defined respectively by

$$\mu_{A^+}(l) = \sup_{i:l\in A_i} \pi(A_i) \tag{1}$$

$$\mu_{A^-}(l) = 1 - \sup_{i:l\notin A_i} \pi(A_i) = \inf_{i:l\notin A_i} (1 - \pi(A_i)) \tag{2}$$

where $1-(\cdot)$ denotes a mapping from $S = \{\alpha_1, \cdots, \alpha_n, \alpha_{n+1}\}$ with $\alpha_1 = 1 > \cdots > \alpha_n > \alpha_{n+1} = 0$ into scale $S' = \{\beta_1, \cdots, \beta_n, \beta_{n+1}\}$ with $\beta_1 = 1 > \cdots > \beta_n > \beta_{n+1} = 0$, such that $\beta_1 = 1-(\alpha_{n+1})$, $\cdots$ $\beta_i = 1-(\alpha_{n+2-i})$, $\cdots$, $\beta_{n+1} = 1-(\alpha_1)$. When S is a subset of $[0,1]$, $1-(\cdot)$ is just the complementation to 1, otherwise it is the order-reversing map of the scale S (for S finite). Since S is a possibility scale, S' is a certainty scale.

Equation 2 means that we are all the more certain that $l \in \mathcal{L}$ belongs to the ill-known set A described by π, i.e. $\mu_{A^-}(l)$ is all the higher, as it is impossible to find an A_i such that $l \notin A_i$. Similarly, it is all the more possible that $l \in \mathcal{L}$ belongs to the ill-known set A, i.e. $\mu_{A^+}(l)$ is all the higher, as there exists an A_i such that $l \in A_i$ having a high possibility level. The quantity $1-\mu_{A^+}(l)$ is called by Yager [50] "rebuff measure", since it expresses to what extent l is impossible to be an element of A.

3.2 Some linkage with evidence theory

The construction made here is reminiscent of Shafer's [43] setting for his evidence theory, where he starts with a mass function m, called "basic probability assignment" defined over the subsets A_i of some referential, say $\mathcal{L}$, which is such that $\sum_i m(A_i) = 1$. Then m is nothing but the representation of a random subset A of $\mathcal{L}$. Then, a so-called *contour function* can be defined as $c(l) = \sum_{i:l\in A_i} m(A_i)$, which represents the plausibility that l belongs to A. Due to the probabilistic normalization of m, note that we also have $c(l) = 1 - \sum_{i:l\notin A_i} m(A_i)$. Here, the construct is similar, except that m is replaced by a *possibilistic mass function* π, and $\sum$ is replaced by sup to agree with the idea of possibility. Such a qualitative counterpart of Shafer evidence theory was first suggested in [22] (see [39] for recent developments). Then the contour function splits into upper and lower approximation functions, i.e., μ_{A^+} and μ_{A^-}, respectively, which no longer coincide. Still, the following strong inclusion of the fuzzy set A^- in A^+ can be checked:

$$\forall l \in \mathcal{L}, \mu_{A^-}(l) > 0 \Rightarrow \mu_{A^+}(l) = 1.$$

It can also be observed that if $\mu_{A^-}(l)$ is interpreted as the certainty that l belongs to A (the ill-known set represented by π), namely $\mu_{A^-}(l) = cert(l \in$

A), the expected duality between possibility and certainty holds, namely, $\mu_{A^+}(l) = 1 - cert(l \in \overline{A})$, since if the ill-known set A is represented by $\{(A_i, \pi(A_i))|i \in I\}$ (where I is an index set), then its complement $\overline{A}$ should be represented by $\{(\overline{A_i}, \overline{\pi}(\overline{A_i}))|i \in I\}$ with $\forall i \in I, \overline{\pi}(\overline{A_i}) = \pi(A_i)$. Indeed then $1-cert(l \in \overline{A}) = 1-\mu_{\overline{A}^-}(l) = 1-(1-\sup_{i:l\notin\overline{A_i}} \overline{\pi}(\overline{A_i}) = \sup_{i:l\in A_i} \pi(A_i) = \mu_{A^+}(l)$.

3.3 Recovering the possibility distribution on the power set

We have shown how a normalized possibility distribution π over $2^{\mathcal{L}}$ induces upper and lower approximation functions over $\mathcal{L}$. Conversely, since (A^-, A^+) is only an approximation of the information contained in $\{(A_i, \pi(A_i))|i \in I\}$, there are several possibility distributions over $2^{\mathcal{L}}$ in general that agree with (A^-, A^+) in the sense of Equations 1 and 2. However, it can be shown that there exists a unique possibility distribution which is the largest one in the sense of the fuzzy set inclusion defined on $2^{\mathcal{L}}$ ($\pi \subseteq \pi' \iff \forall i \in I, \pi(A_i) \leq \pi'(A_i)$). This is the least committed one (since it does not arbitrarily weaken the possibility level of any subset). This possibility distribution is defined by

$$\pi^*(B) = \min(\inf_{l\in B} \mu_{A^+}(l), \inf_{l\notin B}(1 - \mu_{A^-}(l))) \tag{3}$$

This equation is easy to understand, a subset B is all the more possible, as both all its elements are possible, and no elements outside B are certain. Entering π^* in Equations 1 and 2, we recover μ_{A^+} and μ_{A^-}.

3.4 Application to layered databases

We can now apply these results to our original problem. Here, we consider subsets of tuples $t \in \mathcal{T}$, so these subsets are in $2^{\mathcal{T}}$, which plays the role of $2^{\mathcal{L}}$ in the previous subsections. The possibility distribution π_r associated with the relation r is now defined as (denoting B a subset of tuples)

$$\pi_r(B) = \alpha_i \text{ if } \exists i, B = r_{\alpha_i}; \pi_r(B) = 0 \text{ otherwise, with } r_{\alpha_i} = \{t \in r|c(t) \geq \alpha_i\} \tag{4}$$

where r_{α_i} is the cut of level α_i of the relation r and $c(t)$ is the confidence level associated with tuple t. Thus, the different possible database worlds are precisely the level cuts of the fuzzy relation induced by the confidence weights. Any other possible database world that would not coincide with such level cuts has a possibility level equal to $\alpha_{n+1} = 0$. Note that the level cuts are nested, i.e. $r_{\alpha_i} \subseteq r_{\alpha_{i+1}}$, and thus r_{α_1} is included in any possible database world that has a non zero possibility level.

Applying Equations 1 and 2 to the distribution defined by Equation 4, we get

$$c^+(t) = \sup_{B:t\in B} \pi_r(B) = \alpha_i \text{ if } t \in r_{\alpha_i} \text{ but } t \notin r_{\alpha_{i-1}} \tag{5}$$

$$c^-(t) = \inf_{B:t\notin B}(1-\pi_r(B)) = \alpha_1 = 1 \text{ if } t \in r_{\alpha_1} \text{ and } c^-(t) = \alpha_{n+1} = 0 \text{ otherwise.} \tag{6}$$

This means that with the exception of the tuples that are in r_{α_1}, which are certainly in the database, the other tuples are only *possibly* in the database r, the possibility levels $c^+(t)$ then corresponding exactly to the confidence levels, i.e. $c^+(t) = c(t)$.

Now applying Equation 3, we get

$$\pi^*(B) = \min(\inf_{t\in B} c^+(t), \inf_{t\notin B}(1 - c^-(t))) \tag{7}$$

and recover the original distribution π_r, i.e., $\forall B, \pi^*(B) = \pi_r(B)$. Indeed $\inf_{t\notin B}(1 - c^-(t)) = 0$ for any B that fails to include some t in r_{α_1}, otherwise, $\inf_{t\notin B}(1 - c^-(t)) = 1$. Thus, in our case, the information conveyed by the distribution π_r over $2^{\mathcal{T}}$ is *equivalent* to the pair (c^+, c^-) of upper and lower contour functions defined on $\mathcal{T}$.

The α_i's may now receive a proper linguistic counterpart. Since they are possibility levels, one may interpret them on a linguistic scale such that (taking, e.g., $n = 4$) $\alpha_1 =$ '*fully possible*', $\alpha_2 =$ '*quite possible*', $\alpha_3 =$ '*medium possible*', $\alpha_4 =$ '*somewhat possible*', $\alpha_5 =$ '*not at all possible*'.

Since a database whose tuples are associated with confidence levels has now received a clear interpretation in the setting of possibility theory, we are in a position to study what the concept of a functional dependency means in this setting. This approach promotes the idea to keep confidence levels fully qualitative in practice.

4 Possibilistic functional dependencies

A functional dependency (FD for short) $X \to Y$, where X and Y are sets of attributes, is a constraint of the form $t.X = t'.X \Rightarrow t.Y = t'.Y$. It is obvious that if an FD holds in a database, it also holds in any subpart of the original database. Here our layered set of tuples results in a nested sequence of possible database worlds. So, if an FD holds in $r_{\alpha_{i+1}}$, the FD also holds in r_{α_i}. Conversely, if an FD does not hold in r_{α_i} then the FD does not hold in $r_{\alpha_{i+1}}$.

Thus, if we examine the example of Table 2, we can check, that $CT \to R$ holds everywhere, namely in r_{α_4}, $C \to L$ and $RT \to C$ holds in r_{α_3}, and $LT \to C$ in $r_{\alpha_1} = r_{\alpha_2}$ only. This suggests to attach a certainty level to an FD, such that the FD is all the more certain as it holds in a larger database world provided that it is possible to some extent. This suggests the following definition for the certainty level of an FD.

4.1 Defining possibilistic FDs

$$Cert_r(fd) = 1 - \sup\{\pi(r_{\alpha_i}) | fd \text{ does not hold in } r_{\alpha_i}\} \tag{8}$$

where fd denotes an FD, and we have $\pi(r_{\alpha_i}) = \alpha_i$. Equation 8 is nothing but the *necessity* of the event "fd holds in r" with respect to the possibility distribution π_r, since by definition the necessity $N(p)$ of a statement p is equal to $1 - \Pi(\neg p)$, which corresponds to 1 minus the possibility, i.e., to the impossibility of the opposite event "fd does not hold in r". Thus, if fd fails to hold in $r_{\alpha_{i+1}}$, but holds in r_{α_i}, $Cert_r(fd) = 1 - \pi(r_{\alpha_{i+1}}) = 1 - \alpha_{i+1} = \beta_{n+1-i}$ (since the possibility that fd fails is the greatest possibility to be in a database world where fd fails). Thus, $Cert_r(fd) = \beta_1 = 1 - \alpha_{n+1} = 1$ if fd holds for any level cut of r. Note also that in particular, we get $Cert_r(fd) = 0$ if fd fails to hold in r_{α_1}; in fact since the tuples in r_{α_1} are not only fully possible, but also fully certain, there is no possibility at all that fd holds in a database world having a non-zero possibility level, and thus it is fully certain that fd fails to hold. Besides, in case the scales S and S' are included in $[0, 1]$, we just have $Cert_r(fd) = 1 - \alpha_{i+1}$ where now $1 - (\cdot)$ is the usual complementation to 1. We call a classical FD associated with a certainty level a *possibilistic functional dependency*.

Coming back to our example, it can be checked that the set Σ made up of the previously mentioned FDs associated with their certainty weights is:

$\Sigma = \{(CT \to R, \beta_1); (C \to L, \beta_2); (RT \to C, \beta_2); (LT \to C, \beta_3)\}$

Then the following proposition can be stated

$$\vDash_r (X \to Y, c) \Leftrightarrow \vDash_{r_{\underline{1-c}}} X \to Y \tag{9}$$

where $\vDash_r (X \to Y, c)$ means that $Cert_r(X \to Y) \geq c$ and where $r_{\underline{\alpha}}$ denotes the *strict* α-level cut of r, namely $r_{\underline{\alpha}} = \{t | c(t) > \alpha\}$. This proposition is easy to prove. First observe that $\vDash_r (X \to Y, c)$ entails $\vDash_r (X \to Y, c')$ as soon as $c \geq c'$. If $Cert_r(X \to Y) \geq c$, it follows from definition 8 that $X \to Y$ may be violated at most in r_{1-c}, but certainly not in $r_{\underline{1-c}}$. Conversely, if $X \to Y$ holds for any level cut of r of level strictly greater than $1 - c$, $Cert_r(X \to Y)$ cannot be less than $1 - (1 - c)) = c$.

The careful definition of the concept of a possibilistic FD in fully justifiable in terms of possibility theory is also of great potential in database practice. In particular, it allows us to take full advantage of previous results on classical dependencies, which we will explore in future work. For example, if a relation satisfies a classical FD, then that relation can be decomposed into two of its projection without loss of information [47, 2]. More generally, if a possibilistic relation satisfies a possibilistic FD, then the level cuts of the possibilistic relation can be decomposed into two of their projections without loss of information.

4.2 Relation with possibilistic propositional logic

It is well known [25, 27, 28] that FDs in classical databases have a simple propositional logic counterpart in terms of Horn clauses. In fact, the following holds

$$\vDash_r \{A_1, \cdots, A_k\} \rightarrow B \Leftrightarrow \ \forall t, t' \in r, \vDash_{\omega_{\{t,t'\}}} \neg A'_1 \vee \cdots \vee \neg A'_k \vee B' \qquad (10)$$

where $A'_1, \cdots, A'_k, B'$ are propositional variables associated with attributes $A_1, \cdots, A_k, B$, respectively, and $\omega_{\{t,t'\}}(A') = \texttt{True}$ if $t.A = t'.A$ and $\omega_{\{t,t'\}}(A') = \texttt{False}$ otherwise.

This result extends to our setting, just as propositional logic extends to possibilistic logic [21]. Let us first have a brief refresher on possibilistic logic. A (standard) propositional possibilistic logic formula is a pair (p, β) where p is proposition and β is a certainty level. At the semantic level it corresponds to the semantic constraint $N(p) \geq \beta$, where N is a necessity measure, associated with a possibility distribution π in the following way $N(p) = \inf_{\omega \not\vDash p} 1 - \pi(\omega)$. The lower the possibility of an interpretation that makes p `False`, the higher the necessity degree of p. So, given a formula (p, β), an interpretation ω that makes p `True` is possible at the maximal level in the scale S, say 1, while an interpretation ω that makes p `False` is at most possible at level $1 - \beta$. A possibilistic logic knowledge base K is a collection of possibilistic logic formulas, namely $K = \{(p_i, \beta_i) | i = 1, \cdots, n\}$, whose semantic counterpart is $\pi_K(\omega) = \min_{i=1,\cdots,n} \max(1 - \beta_i, [p_i](\omega))$ where $[p](\omega) = 1$ if $\omega \vDash p$ and where $[p](\omega) = 0$ otherwise. Then in possibilistic logic, the following soundness and completeness theorem holds

$$\vDash_K (p, \beta) \ \Leftrightarrow \vdash_K (p, \beta) \Leftrightarrow \vdash_{K_\beta} p \Leftrightarrow \vDash_{K_\beta} p$$

where $\vDash_K (p, \beta)$ means $\forall \omega, \pi_K(\omega) \leq \pi_{\{(p,\beta)\}}(\omega)$, and $K_\beta = \{p_i | (p_i, \beta_i) \in K \text{ and } \beta_i \geq \beta\}$. So the last half of the above expression reduces to the soundness and completeness theorem of propositional logic, applied to each level

cut of K, which is an ordinary propositional logic knowledge base. Lastly, $\vdash_K (p, \beta)$ refers to the syntactic part of possibilistic logic, which relies on the repeated use of the resolution rule $(\neg p \vee q, \beta), (p \vee r, \gamma) \vdash (q \vee r, \min(\beta, \gamma))$. It is also interesting to notice that, due to the characteristic property of necessity measures, i.e., $N(p \wedge q) = \min(N(p), N(q))$, a possibilistic logic base can be easily put in clausal form.

Thus, we have seen that the semantics for the possibilistic logic formula (p, β) amounts to rank-order interpretations according to the possibility distribution $\pi_{\{(p,\beta)\}}$, where $\pi_{\{(p,\beta)\}}(\omega) = 1$ if $\omega \vDash p$ (i.e., ω makes p `True`) and $\pi_{\{(p,\beta)\}}(\omega) = 1 - \beta$ if ω is an interpretation that makes p `False` (i.e., $\omega \nvDash p$). Going back to possibilistic FDs, interpretations now refer to pairs of tuples, but one may have a similar construct. The counterpart of Equivalence 10 can be stated in the following way:

$$\vDash_r (\{A_1, \cdots, A_k\} \rightarrow B, \beta) \Leftrightarrow \ \forall t, t' \in r^*, \vDash_{\pi_{\{t,t'\}}} (\neg A'_1 \vee \cdots \vee \neg A'_k \vee B', \beta) \tag{11}$$

where r denotes a possibilistic database (in the sense of this paper), r^* is the same database without the levels, and $\pi_{\{t,t'\}}$ is the possibility distribution over logical interpretations that accounts for the possible failure of the FD in the possibilistic database. Indeed, the distribution $\pi_{\{t,t'\}}$ is defined in the following way

- $\pi_{\{t,t'\}}(\omega_{\{t,t'\}}) = \min(\alpha, \alpha')$, with $c(t) = \alpha$, $c(t') = \alpha'$, if (t, t') violates $\{A_1, \cdots, A_k\} \rightarrow B$ in $r_{\min(\alpha,\alpha')}$
- $\pi_{\{t,t'\}}(\omega_{\{t,t'\}}) = 0$, if (t, t') satisfies $\{A_1, \cdots, A_k\} \rightarrow B$ in r^*
- $\pi_{\{t,t'\}}(\omega) = 1$ for all $\omega \neq \omega_{\{t,t'\}}$.

Here, the interpretations ω are the ones induced by the literals $A'_1, \cdots, A'_k, B'$ (where A'_i is `True` iff $t.A_i = t'.A_i$, and B' is `True` iff $t.B = t'.B$), and $\omega_{\{t,t'\}}$ is the particular interpretation $A'_1 \cdots A'_k \neg B'$ that falsifies $\neg A'_1 \vee \cdots \vee \neg A'_k \vee B'$.

Proof of (11). Let $\varphi = \{A_1, \cdots, A_k\} \rightarrow B$, and $\varphi' = \neg A'_1 \vee \cdots \vee \neg A'_k \vee B'$. When (t, t') violates φ it means that $\min(\alpha, \alpha') \leq 1 - \beta$ assuming $\vDash_r (\varphi, \beta)$. Since $\pi_{\{(\varphi',\beta)\}}(\omega_{\{t,t'\}}) = 1 - \beta$ and $\pi_{\{(\varphi',\beta)\}}(\omega) = 1$ for all $\omega \neq \omega_{\{t,t'\}}$, it is clear that we have $\forall \omega, \pi_{\{t,t'\}}(\omega) \leq \pi_{\{(\varphi',\beta)\}}(\omega)$. Conversely, if this later inequality holds, there cannot exist t, t' such that $\min(\alpha, \alpha') > 1 - \beta$, and thus $Cert_r(\varphi) \geq \beta$, i.e., $\vDash_r (\varphi, \beta)$. Q.E.D.

The above result indicates that possibilistic FDs have counterparts in possibilistic propositional logic just as FDs have counterparts in propositional logic.

5 Related work

The literature on fuzzy FDs is quite abundant. It is not the place here to survey it in detail, and some overview papers exist [9, 10, 51] for the first decade of literature on the topic. We first briefly mention the main existing types of fuzzy FDs, and then compare in detail the proposed approach to a somewhat similar proposal, which originates from a different perspective. In the second part of this section, we discuss FDs in the context of the possible world semantics of another type of possibilistic databases.

5.1 Fuzzy functional dependencies

Fuzzy FDs may refer to a quite large variety of situations. First, we may consider classical databases (where one mines FDs with satisfaction degrees [5], or fuzzy approximate dependencies [7]), or databases with precise attribute values but weighted tuples, or databases with fuzzy attribute values, or still fuzzy similarity-based relational databases (moreover the database may have null values [30]). Then we may either study classical FDs on weighted tuple databases or on fuzzy attribute value databases [12] or even fuzzy values with imprecise membership functions [35], or we may consider fuzzy FDs on classical databases [3] as well as on more general databases allowing for weighted tuples, fuzzy attribute values, or fuzzy values defined by means of fuzzy similarity relations [41, 8, 18, 19, 20, 15, 16, 48, 33, 44, 46, 6]. Fuzzy FDs may be stronger or weaker than classical FDs depending on whether they are adding further constraints to the one conveyed by a classical FD (such as ordered FDs that agree with orderings existing in attribute domains [37], or gradual FDs [42]), or whether they weaken the constraint associated with a classical FD. Clearly, all these different options may serve different goals, which may depart from the role of classical FDs in classical databases (such as data summarization [17], building of linguistic summaries [11], or a Bayesian network [34]).

However, in this short note, we are not dealing with any fuzzy FDs of any kind. The proposal made here is motivated by the idea that FDs may fail to hold in the presence of some tuples in which we have not full confidence. This might be related to the idea of partial FDs [26], where FDs hold up to exceptions whose number may be quantified. However, here, we take advantage of the confidence levels of the tuples for accommodating the exceptions. There has been another proposal made more than two decades ago, by Kiss [29] for dealing with classical FDs in a weighted tuple database, viewed as a fuzzy relation r. The author computes the degree of truth with which an FD $X \to Y$ holds, in the following way (where μ denotes membership functions):

$$\texttt{Truth}(X \to Y) = \min_{\{t,t'\}}(min(\mu_r(t), \mu_r(t'), \mu_=(t.X, t'.X) \Rightarrow_L \mu_=(t.Y, t'.Y)))$$

where $\mu_=$ denotes the exact equality relation, and $\Rightarrow_L$ is Łukasiewicz implication (note however that other implications will give the same result since the righthand part of the implication is equal to 0 or 1). An easy computation leads to

$$\texttt{Truth}(X \to Y) = 1 - \sup_{t,t':t.X=t.X \text{ and } t.Y \neq t'.Y} \min(\mu_r(t), \mu_r(t')).$$

Reorganizing the weighted tuples into layers of decreasing degrees, we see that the above formula coincides with our definition of $Cert_r(X \to Y)$, and indeed $X \to Y$ holds in any level cut r_α of r such that $\alpha > 1 - \texttt{Truth}(X \to Y)$. However, this simple multiple-valued logic view has no clear interpretation from an uncertainty modeling point of view, while a possible database world perspective also enables us to get a possibilistic logic counterpart. Moreover, interestingly enough, the author wrote about his proposal some years after: "The so defined fuzzy relations can be handled mathematically well, but they have less practical importance" [38]. On the point of usefulness, we disagree with this view. Indeed, just as possibilistic logic is a valuable extension of propositional logic, one may expect that certainty-based FDs with a layer-based view of databases can help to control the normalization of the decomposition process of uncertain relations.

5.2 FDs in possibilistic databases. Discussing the meaning of the levels

In this short note, we have emphasized the relationship between the levels attached to the tuples and the associated possibility distribution over possible database worlds. Several authors have pointed out the interest of seeing a possibilistic database as a set of classical databases associated with possibility degrees. When the possibilistic database is a database where attribute values are fuzzy (i.e., for each tuple and each attribute, we have a possibility distribution restricting the possible values), the possibility degrees associated with database worlds can be computed from the possibility degrees attached to the possible attribute values chosen for building each classical database compatible with the possibilistic database. One may then precisely define the possibility degree and the necessity degree with which a particular FD holds in the possibilistic database [13].

As can be seen, we have not used here this view of a possibilistic database. However, consider the particular case where all the attribute values of each tuple t would be precise but uncertain, with the same certainty level β_t, which would correspond to particular possibility distributions equal to 1 for the precise value, and equal to $1 - \beta_t$ everywhere else. Then, the database would contain only certainty-qualified values in the sense studied in [14]. Since here the certainty of all the attribute values is the same for a given tuple, one can associate this certainty level to the whole tuple (without losing any information), in agreement with the min-decomposability of necessity measures. Thus, what is obtained looks a bit like the possibilistic database considered in this note, except that tuples are now associated with certainty levels rather with possibility levels. So, one may wonder, if an approach similar to the one presented here, but with certainty levels would not be interesting as well. The answer is negative. This is because as soon as an FD is violated in $r*$ (the database without the certainty levels here), there would be a fully possible world where the FD is violated, and then the FD would have no certainty, and one cannot reason in a possibilistic logic manner with FDs that are just possible to some extent. Besides, if we only consider relations r where the FDs are not violated in $r*$, we would be in a position to associate a certainty level with the FDs, but it would always be the same, namely the minimal value of all the certainty values attached to tuples in r. This is not very interesting and confirms that our approach with possibility levels is the right one if one does not want to trivialize the approach.

6 Concluding remarks

This short note has introduced the notion of possibilistic functional dependencies based on the idea of a classical database, layered according to possibility levels attached to tuples, and where the first layer is the only certain one. We have shown that in such a case the associated possibility distribution over possible database worlds is uniquely determined by the possibility levels attached to tuples, and vice versa. This has led us to associate certainty levels with FDs in a natural way. Furthermore, this definition allows us to extend the well-known propositional logic counterpart of FDs in the setting of possibilistic logic.

The notion of possibilistic functional dependencies proposed here seems particularly appealing for use in database practice. Indeed, the layered view of the database together with the different levels of certainty of the FDs suggest their use in the control of the decomposition process of relations in Third normal forms, or in Boyce-Codd normal forms, which can then be

layered. The full investigation of these issues, with the study of the weighted counterpart of Armstrong's system of axioms, is the topic of a companion paper [32] and a patent application [31]. Besides, rather than starting with a layered database, and computing the certainty levels of the FDs, one may also think of doing the converse, namely starting with a set of more or less certain FDs that should hold in a classical database, and looking for a stratification of the database which agrees with the certainty levels of the FDs.

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