CDMTCS Research Report Series

A Quest For Algorithmically Random Infinite Structures, II

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CDMTCS-470
October 2014

Centre for Discrete Mathematics and
Theoretical Computer Science

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October 6, 2014

1 Motivation

The history of algorithmic randomness on strings goes back to the work of Kolmogorov [7], Martin-Löf [12], Chaitin[3], Schnorr[16] [17] and Levin[18]. In the last two decades the area has attracted a good amount of attention among experts in computability, algorithmic randomness, complexity, and reverse mathematics. Many notions of algorithmic randomness for strings have been introduced and studied. These include Martin Löf tests, Schnorr tests, 1-generic sets, prefix free complexity, K-triviality, martingales, connections to differentiability, Solovey and related reducibilities. The monographs by Downey and Hirschfeldt [4] and by Nies [14] expose recent advances in the area. Standard textbooks in the area are Calude [2], Li and Vitanyi [10].

The concept of Martin-Löf tests, which we write ML-tests for short, is central for defining algorithmic randomness in the setting of infinite strings. The Martin-Löf tests are defined in the Cantor space $\{0,1\}^\omega$. The natural measure in the Cantor space plays a central role in the definition. In spite of much work, research on randomness of infinite strings excluded the investigation of algorithmic randomness for infinite structures such as graphs, trees, universal algebras, monoids, etc. The main obstacle in introducing algorithmic randomness for infinite structures, like the ones we mentioned, was that these classes of structures lack measure. It was unclear how one would define a meaningful measure through which it would be possible to introduce algorithmic randomness for infinite structures.

A possible way to introduce algorithmic randomness for infinite structures is to identify structures with infinite binary strings that code the atomic diagrams. Under such identification, one calls a structure *string-random* if the string that codes it is ML-random. The author in [8] shows that string-random structures are all isomorphic to Fraïsse limit of the class of finite structures. Hence, each string-random structure has the following properties [6]: (1) The theory of the structure is $\aleph_0$-categorical and is decidable; (2) All string-random structures are isomorphic; Moreover, they are isomorphic to a computable structure; (3) The isomorphism type of the structure is described by extension axioms. All these properties defy the intuitive notion of algorithmic randomness for infinite structures, and suggest that an alternative approach should be taken to define algorithmic randomness for infinite structures.

The author in [8] introduces a natural measure in such classes as finitely generated universal algebras, finitely generated monoids, connected graphs and trees of bounded degree. Consequently one can define algorithmic randomness, such as ML-randomness, in these classes of structures. For this paper, we need to mention several results from [8]. First, in each of the mentioned classes there are continium ML-random structures. This represents randomness as a property of collective, the idea that goes back to von Mises [13]. Moreover, for these classes of structures ML-randomness is the isomorphism invariant property. Second, for the classes of finitely generated universal algebras ML-randomness does not depend on the generators; this shows robustness of ML-randomness for the class. Third, there exist ML-random universal algebras that are computable in the halting set. Furthermore, ML-random computable universal algebras do not exist. Finally, there are ML-random graphs and trees of bounded degree that are computable in the halting set. In the case of trees one can prove a stronger result; there exists a ML-random tree such that the equality predicate and the edge relation on the tree are both computably enumerable. These results are the starting point for development of the theory of algorithmic randomness for infinite structures. Here we continue our study of ML-random structures and provide new results.

2 Contributions of this paper

1. We provide a computability theoretic and algebraic framework that introduces algorithmic randomness for various classes of infinite algebraic structures. The central concept is the notion of a *branching class*, or *B-class* for short, of finite structures. Given a branching class $\mathcal{K}$ of finite structures, one can construct a computable finitely branching infinite tree $T(\mathcal{K})$ without leaves such that there is a bijective mapping that associates nodes of the tree $T(\mathcal{K})$ with the structures from the class $\mathcal{K}$. Moreover, each such class $\mathcal{K}$ uniquely determines the class $\mathcal{K}_\omega$ of infinite structures. The structures from the class $\mathcal{K}_\omega$ are obtained as direct limits of the structures from $\mathcal{K}$. These infinite structures can be viewed as paths through the tree $T(\mathcal{K})$. It is then observed that the class $\mathcal{K}_\omega$ possess a natural measure, metric and topology. Hence, one can define ML-tests and various other algorithmic tests in the class $\mathcal{K}_\omega$. Consequently, the number of ML-random structures in $\mathcal{K}_\omega$ is the cardinality of the continuum. Moreover, the class contains ML-random structures computable in the halting set. Examples of branching classes include finitely generated universal algebras, various classes of trees and graphs, relational structures of bounded degree, partially ordered sets, and plethora of various types of structures such as ordered trees. See Sections 3, 4, 5 and 6.

2. Given a B-class $\mathcal{K}$, the existence of ML-random structures from $\mathcal{K}_\omega$ that are computable in the halting set is an expected phenomenon. Roughly such structures correspond to the leftmost path of a computable finitely branching trees. The leftmost paths are computed with an oracle for the halting set. In the case of infinite binary strings ML-randomness of a string implies immunity property: ML-random strings contain no infinite computable substrings. It turns out that in the setting of infinite algebraic structures the situation could be drastically different. We present a result that, on the surface, goes against our intuition. We construct a B-class $\mathcal{S}$ such that the class $\mathcal{S}_\omega$ contains a computable yet ML-random algebraic structure $\mathcal{A}$. Intuitively, the reason for a such phenomenon is that any ML-test that contains the computable structure $\mathcal{A}$ needs to use an oracle for the halting set. More formally, to build a Martin-Löf test for the computable structure $\mathcal{A}$ one needs to have access to the existential theory of the structure. In terms of our notation above, while building the structure $\mathcal{A}$ we exploit the fact that constructing the path in the tree $T(\mathcal{S})$ that corresponds to $\mathcal{A}$ requires the jump of the open diagram of the structure. In particular, structures $\mathcal{A}$ whose open diagrams compute the paths in the tree $T(\mathcal{S})$ corresponding to $\mathcal{A}$ should possess stronger computability-theoretic properties. For instance, for graphs and trees these stronger properties imply that one is able to compute the number of adjacent elements for any given vertex of the graph. Sections 6 and 7 provide an example of a B-class $\mathcal{S}$ and a computable yet ML-random structure from the class $\mathcal{S}_\omega$.

3. We consider the class of all finitely generated universal algebras. We prove that the class of all finitely generated universal algebras that belong to a non-trivial variety (that is, the class of algebras closed under sub-algebras, homomorphisms, and ultra products) has effective measure 0. For instance, the class of all two generated groups in the class of finitely generated algebras has effective measure 0. As a consequence, no free finitely presented universal algebra in any finitely axiomatised variety is ML-random. In particular, no finitely presented group or semi-group is ML-random. Intuitively, the result states that no finite set of equations can describe the isomorphism types of ML-random algebras. This confirms the intuition that any effective reasonable attempt to formally describe the isomorphism type of an algorithmically random structure should fail. This, in some ways, represents the idea going back to von Mises that randomness passes selection rules [13]. These are described in the last section, Section 8.

3 Structures with height function

Let $\sigma = (R_1^{n_0}, \ldots, R_m^{n_m}, c_1, \ldots, c_k)$ be a relational signature where each $R_i^{n_i}$ is a relational symbol of arity n_i and each c_j is a constant symbol. We postulate that the signature contains at least one constant symbol. Algebraic structures that contain functional operations can be turned into relational structures by replacing operations with their graphs. When we consider such structures, we identify them with their relational counterparts that we have just descried. We identify structures up to isomorphisms.

Definition 3.1. An *embedded system* of structures is a sequence $\{(\mathcal{A}_i, f_i)\}_{i\in\omega}$ such that each $\mathcal{A}_i$ is a finite structure of the signature and each f_i is a proper into embedding from $\mathcal{A}_i$ into $\mathcal{A}_{i+1}$. We call the sequence $\mathcal{A}_0, \mathcal{A}_1, \ldots$ *the base* of the embedded system.

Each embedded system $\{(\mathcal{A}_i, f_i)\}_{i\in\omega}$ determines the following infinite structure. On the set $\cup_{i\in\omega} A_i$ consider the equivalence relation $\sim$: $a \sim b$ iff either $a = b$ or there are i and j such that $a \in A_i, b \in A_j$ and $f_{j-1}(f_{j-2}(\ldots(f_i(a))\ldots)) = b$. The atomic relations in the sequence naturally induce relations on $\cup_{i\in\omega} A_i$ and they are well behaved with respect to the equivalence relation $\sim$. Hence we have the structure $\mathcal{A}$ with the domain $A = \cup_{i\in\omega} A_i/\sim$ called the *direct limit* of the sequence, and we denote it by $\lim_i(\mathcal{A}_i, f_i)$.

Definition 3.2. An embedded system $\{(\mathcal{A}_i, f_i)\}_{i\in\omega}$ is *strict* if its direct limit is isomorphic to the direct limit of any embedded system with the same base.

It is not too hard to give examples of non-strict embedded systems. Below we define classes of finite structures called *branching classes*, or B-classes for short. All embedded systems in such classes will necessarily be strict.

Let $\mathcal{K}$ be a decidable class of finite structures, that is, there is an algorithm that given a finite structure $\mathcal{A}$ decides if the structure belongs to the class $\mathcal{K}$. Assume that we have a computable function $h : \mathcal{K} \to \omega$ that we call the *height function for* $\mathcal{K}$. We say that structure $\mathcal{A} \in \mathcal{K}$ *has height* i if $h(\mathcal{A}) = i$. Note that since we identify structures up to isomorphism, the function h is an isomorphism invariant. Assume that the height function $h : \mathcal{K} \to \omega$ satisfies the following properties:

1. The set $h^{-1}(i)$ is finite for all $i \in \omega$, and we can compute the cardinality of $h^{-1}(i)$ of every i.

2. For every $\mathcal{A} \in \mathcal{K}$ of height i there is a substructure $\mathcal{A}[i-1]$ of height $i-1$ such that all substructures of $\mathcal{A}$ of height $\leqslant i-1$ are contained in $\mathcal{A}[i-1]$. Hence, the substructure $\mathcal{A}[i-1]$ is the largest substructure of $\mathcal{A}$ of height $i-1$.

3. For every $\mathcal{A} \in \mathcal{K}$ of height i and every subset C of the set $A \setminus A[i-1]$ the height of the substructure with domain $A[i-1] \cup C$ is i.

The above implies that for all $\mathcal{A} \in \mathcal{K}$ of height i and $j \leqslant i$ there exists a substructure $\mathcal{A}[j]$ of height j such that all substructures of $\mathcal{A}$ of height $\leqslant j$ are contained in $\mathcal{A}[j]$. Also, $\mathcal{A}[0] \subset \mathcal{A}[1] \subset \ldots \subset \mathcal{A}[i]$, where $\mathcal{A}[i] = \mathcal{A}$. The next lemma follows easily:

Lemma 3.3. *Let $\mathcal{K}$ and h be as above. Then for all $\mathcal{A}, \mathcal{B} \in \mathcal{K}$, the structures $\mathcal{A}$ and $\mathcal{B}$ are isomorphic iff $h(\mathcal{A}) = h(\mathcal{B})$ and $\mathcal{A}[j] = \mathcal{B}[j]$ for all $j \leqslant h(\mathcal{A})$.* □

More interesting lemma is the following:

Lemma 3.4. *Let $\mathcal{K}$ and h be as above. Every embedded system of structures from the class $\mathcal{K}$ is strict.*

Proof. Let $\{(\mathcal{A}_i, f_i)\}_{i\in\omega}$ and $\{(\mathcal{B}_i, g_i)\}_{i\in\omega}$ be two embedded systems with the same base, that is $\mathcal{A}_i \cong \mathcal{B}_i$ for all i. Let $\mathcal{A}$ and $\mathcal{B}$ be the direct limits of these two systems, respectively. We want to prove that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic. If need be, by selecting subsequences, we can assume that $\mathcal{A}_j[i] = \mathcal{A}_i[i]$ and $\mathcal{B}_j[i] = \mathcal{B}_i[i]$ for all $j \geqslant i$. This can be done since the number of structures of height i is finite. For all j consider all isomorphisms $\alpha : \mathcal{A}_j \to \mathcal{B}_j$. Note that $\alpha(\mathcal{A}_j[i]) = \mathcal{B}_j[i]$ for all $i \leqslant j$. The set of all such isomorphisms α forms a tree T under extension of mappings. Namely, nodes of this tree at level j are isomorphism from $\mathcal{A}_j$ onto $\mathcal{B}_j$. Also, note that any such isomorphism at level j of the tree T extends an isomorphism from $\mathcal{A}_{j-1}$ into $\mathcal{B}_{j-1}$ at level $j-1$. This tree T is finitely branching. Hence, by König's lemma, there exists an infinite path on the tree. That path induces an isomorphism from $\mathcal{A}$ to $\mathcal{B}$. □

As a consequence, every direct limit of embedded systems of structures from the class $\mathcal{K}$ is the direct limit of a "canonical" embedded system as explained:

Corollary 3.5. *Let $\mathcal{K}$ and h be as above. For every embedded system $\{\mathcal{A}_i, f_i\}_{\imath\in\omega}$ of structures from the class $\mathcal{K}$ there exists an embedded system $\{\mathcal{B}_i, g_i\}_{\imath\in\omega}$ of structures from the same class such that:*

1. *The direct limits $\lim_i(\mathcal{A}_i, f_i)$ and $\lim_i(\mathcal{B}_i, g_i)$ are isomorphic.*
2. *The height of each $\mathcal{B}_i$ is i.*
3. *The embeddings g_i are identity embeddings.*
4. *For all $i \leqslant j$ we have $\mathcal{B}_j[i] = \mathcal{B}_i[i] = \mathcal{B}_i$.* □

Now we introduce the concept of branching classes of structures (or B-classes for short):

Definition 3.6. The class $\mathcal{K}$ together with the height function h is called a *B-class* if for all $\mathcal{A} \in \mathcal{K}$ of height i there exist distinct structures $\mathcal{B}, \mathcal{C} \in \mathcal{K}$ such that $h(\mathcal{B}) = h(\mathcal{C}) > h(\mathcal{A})$ and $\mathcal{B}[i] = \mathcal{C}[i] = \mathcal{A}$.

Now we provide many examples of B-classes.

4 Examples of branching classes

We present several examples here for a better exposition and understanding B-classes. Some of the examples here are taken from [8].

Example 1 (Pointed graphs). A *pointed graph of degree d*, where $d > 2$, is a connected finite graph G with a fixed vertex c such that the degree of every vertex of G is bounded by d. Here the edge relation is a symmetric relation E with no self-loops. Given a vertex v, one computes the shortest path-distance from c to v. For each pointed graph G, set $h(G)$ be the maximum of all path-distances from c to vertices $v \in G$. Consider the following class:

$$PG(d) = \{G \mid G \text{ is a finite pointed graph with at least one vertex } v \text{ such that the distance from } c \text{ to } v \text{ is } h(G) \text{ and the degree of } v \text{ is less than } d\}$$

Lemma 4.1. *The class $PG(d)$ is a B-class.*

Proof. The properties (1)-(3) for the function h are clearly satisfied. Let $G \in PG(d)$ be a pointed graph of height i. Let v be a vertex of G at distance i from c such that the degree of v is less than d. Using vertex v, we construct two extensions G_1 and G_2 as follows. The extension G_1 is obtained by adding two new vertices x and y and new edges $\{v, x\}$ and $\{x, y\}$ to G. The extension G_2 is obtained by adding three new vertices a, b, and c, and new edges $\{v, a\}$, $\{a, b\}$, and $\{a, c\}$ to the graph G. The extensions G_1 and G_2 have height $i + 2$ and $G \ncong G_2$. □

Example 2 (Rooted trees of bounded degree). Fix an integer $d > 1$. Consider the class $Tree_d$ of all rooted trees T such that every node of T has not more than d immediate successors. The height $h(T)$ of T is the length of the longest path in the tree. Rooted trees of bounded degree are also rooted graphs. So, $Tree_d \subset PG(d)$. The following is now easy:

Lemma 4.2. *The class $Tree_d$ is a B-class.* □

Example 3 (Structures of bounded degree). Let $\mathcal{A}$ be a relational structure with exactly one constant symbol. The Gaifman graph of $\mathcal{A}$ is the graph $G(\mathcal{A})$ with vertex set A and an edge between a and b, where $a \neq b$, if there is an atomic relation R and a tuple $\bar{x} \in R$ that contains a and b. Say that $\mathcal{A}$ *is of bounded degree d* if $G(\mathcal{A})$ is of bounded degree d. Set $h(\mathcal{A})$ be the maximum of all path-distances from c to vertices $v \in G(\mathcal{A})$. Consider the class

$$Str(d) = \{\mathcal{A} \mid \text{ There is a vertex } v \text{ such that the distance from } c \text{ to } v \text{ is } h(G(\mathcal{A})) \text{ and the degree of } v \text{ is less than } d\}$$

Lemma 4.3. *The class $Str(d)$ is a B-class.* □

Example 4 (Algebras). A *universal algebra* (or *algebra*), is a tuple $(A; f_1, \ldots, f_n, c_1, \ldots, c_m)$, where $A \neq \emptyset$ is the *domain* of $\mathcal{A}$, each $f_i : A^{k_i} \to A$ is a *total* function called a *atomic operation* of arity k_i, and each c_j is a *distinguished element* (or a *constant*) of $\mathcal{A}$. The *signature* of $\mathcal{A}$ is $f_1, \ldots, f_n, c_1, \ldots, c_m$ of symbols representing the operations and constants. The algebra $\mathcal{A}$ is *c-generated* if every element a of

$\mathcal{A}$ is the interpretation of some ground term t; in other words, if every element of $\mathcal{A}$ is obtained from constants by a chain of atomic operations of $\mathcal{A}$. Call the term t a *representation* of a in $\mathcal{A}$. The element a might have more than one representations.

The *height* $h(t)$ of a ground term t is defined by induction. If $t = c_i$ then $h(t) = 0$. If t is of the form $f_i(t_1, \ldots, t_{k_i})$, then $h(t) = \max\{h(t_1), \ldots, h(t_{k_i})\} + 1$. For a c-generated algebra $\mathcal{A}$ the *height* $h(a)$ of an element $a \in A$ is the minimal height among the heights of all the ground terms representing a.

Definition 4.4. The *height* $h(\mathcal{A})$ of the algebra $\mathcal{A}$ is the supremum of all the heights of its elements.

For any ground term t of height n and $i \leqslant n$ there is an algebra $\mathcal{A}$ such that the height of the element represented by t is i. A c-generated algebra is finite iff there exists an n such that all elements of $\mathcal{A}$ have height at most n. Therefore, infinite c-generated algebras all have height ω.

For a c-generated algebra $\mathcal{A}$ and $n \in \omega$, set:

$$A(n) = \{a \in A \mid h(a) \leqslant n\}.$$

Each k_i-ary atomic operation f_i of $\mathcal{A}$ defines a *partial operation* $f_{i,n}$ on $A(n)$ as follows. For all $a_1, \ldots, a_{k_i} \in A(n)$ the value of $f_{i,n}(a_1, \ldots, a_{k_i})$ is $f_i(a_1, \ldots, a_{k_i})$ if $h(a_i) < n$ for $i = 1, \ldots, k_i$; and $f_{i,n}(a_1, \ldots, a_{k_i})$ is undefined otherwise. Thus, we have the partial algebra $\mathcal{A}(n)$ on the domain $A(n)$. We call this algebra *proper partial algebra*. Clearly, every infinite c-generated algebra $\mathcal{A}$ is the direct limit of the embedded system $\{\mathcal{A}(n)\}_{n\in\omega}$. Two c-generated algebras $\mathcal{A}$ and $\mathcal{B}$ are isomorphic if and only if $\mathcal{A}(n)$ is isomorphic to $\mathcal{B}(n)$ for all $n \in \omega$. Define the following class:

$$PAlg = \{\mathcal{B} \mid \text{there exists an infinite } c\text{-generated algebra and } n \in \omega \text{ such that } \mathcal{B} = \mathcal{A}(n)\}.$$

The height function h defined on algebras induces a computable mapping from $PAlg$ into ω. The following is now easy to check:

Lemma 4.5. *The class $PAlg$ is a B-class.* □

Example 5 (Orders with the least element). We fix $d > 1$. Let $(P; \leqslant)$ be a partially ordered set with the least element. For $p, q \in P$, say that q *covers* p if $p \leqslant q$, $p \neq q$ and no element x exists strictly between p and q in the partial order. Call a partially ordered set $\mathcal{P} = (P; \leqslant, C)$, which is expanded with the cover relation C, of bounded degree d if every element in it has at most d covers. The height $h(\mathcal{P})$ of the partial order $\mathcal{P}$ is the length of the longest chain in it. Clearly, the chain contains the least element. Consider the following class:

$$PO(d) = \{\mathcal{P} \mid \mathcal{P} \text{ has the least element, has height } h \text{ and is of bounded degree } d\}$$

Lemma 4.6. *The class $PO(d)$ is a B-class.* □

Example 6 (Binary rooted ordered trees). Here our class $OT(2)$ consists of binary rooted trees where the children of every node are ordered. Since the trees are binary, every node has either left-child or right-child. We view these trees in the signature $\sigma = (L, R, c)$, where c represents the root, $L(x, y)$ represents the fact that y is the left child of x, $R(u, v)$ means that v is the right child of u. The hight function is defined in a natural way. As examples above, we clearly have the following lemma:

Lemma 4.7. *The class $OT(2)$ is a B-class.* □

Example 7 (Monoids). A structure $\mathcal{M} = (M; \circ, e)$ is *monoid* if $\circ$ is an associative binary operation on M and e is the identity element. We consider two generated monoids with generators x and y. Given an element $m \in M$, the length of the shortest string from $\{x, y\}^\star$ representing m is the *height of* m. A *proper partial monoid* of height n is a partial monoid such that (1) For every element $m \in M$ there is a word $\{x, y\}^\star$ representing m (2) For every $m \in M$ of height strictly less than n, we have $m \circ x$, $m \circ y$, $y \circ m$, and $x \circ m$ are all defined, (3) For every $m \in M$ of height n, we have none of $m \circ x$, $m \circ y$, $y \circ m$, and $x \circ m$ is defined. Consider the following class:

$$Mon = \{\mathcal{M} \mid \text{there exists an } n \geqslant 1 \text{ such that } \mathcal{M} \text{ is a proper partial monoid of height } n\}$$

Lemma 4.8. *The class Mon is a B-class.*

Proof. We prove that for every proper partial monoid $\mathcal{M}$ of height n there are non-isomorphic infinite 2-generated monoids that extend $\mathcal{M}$. In particular, $\mathcal{M}$

has two non-isomorphic proper partial monoid extensions of the same height.

Building $\mathcal{M}_1$: Set $M_1 = M \cup \{u\}^\star$, where $u \notin M$. For all $x, y \in M_1$, if $x \circ y$ is defined in $\mathcal{M}$, then preserve the value; otherwise, set $x \circ y = u$. If $x \in \{u\}^\star$ and $y \in M$ then $x \circ y = y \circ x = x$. If both $x, y \in \{u\}^\star$ then $x \circ y = xy$.

Building $\mathcal{M}_2$: Set $M_2 = M \cup \{u_1, u_2\} \cup \{a_1, a_2\}^\star$, where $u_1, u_2, a_1, a_2 \notin M$. On M_2, we define $\circ$ as follows. For all $x, y \in M$ we preserve $x \circ y$ if this was already defined. Otherwise we set $x \circ y = u_1$. Set $u_1 \circ u_1 = u_2$, $u_1 \circ u_2 = a_1$, and $u_2 \circ u_1 = a_2$. The rest is defined similar as above. For instance, If $x \in \{a_1, a_2\}^\star$ and $y \in M$ then $x \circ y = y \circ x = x$. If both $x, y \in \{a_1, a_2\}^\star$ then $x \circ y = xy$. □

5 Computable Tree Lemma

Let $\mathcal{K}$ be a B-class of algebras of the signature σ with height function h. Let $r_{\mathcal{K}}(n)$ be the number of all structures from $\mathcal{K}$ of height n. It is clear that $r_{\mathcal{K}}(n) < r_{\mathcal{K}}(n+1)$ for all $n \in \omega$. Also note that the function $n \to r_{\mathcal{K}}(n)$ is computable.

Let $\mathcal{K}_\omega$ be the class of all structures obtained as the direct limits of structures from the class $\mathcal{K}$. Every structure from $\mathcal{K}_\omega$ is an infinite structure. We show that the class $\mathcal{K}_\omega$ can be viewed as paths through a finitely branching tree $\mathcal{T}(\mathcal{K})$. This allows us to introduce measure and metric into the set $\mathcal{K}_\omega$.

We formally define the tree $\mathcal{T}(\mathcal{K})$ as follows. The root of the tree $\mathcal{T}(\mathcal{K})$ is the empty set. This is level -1 of $\mathcal{T}(\mathcal{K})$. The nodes of the tree $\mathcal{T}(\mathcal{K})$ at level $n \geqslant 0$ are all structures from $\mathcal{T}(\mathcal{K})$ of height n. There are exactly $r_{\mathcal{K}}(n)$ of them. Let $\mathcal{B}$ be a structure from $\mathcal{K}$ of height n. Its successor on the tree $\mathcal{T}(\mathcal{K})$ is any relational structure $\mathcal{C}$ of height $n+1$ such that $\mathcal{B}$ and $\mathcal{C}[n]$ coincide at level n. The last part of the lemma below follows from Corollary 3.5.

Lemma 5.1 (**Computable Tree Lemma**). *The tree $\mathcal{T}(\mathcal{K})$ satisfies the following properties:*

1. *Given any node x of $\mathcal{T}(\mathcal{K})$, we can effectively compute the structure $\mathcal{B}_x$ associated with the node x. We identify the nodes x of $\mathcal{T}(\mathcal{K})$ and the structures $\mathcal{B}_x$.*
2. *For each node x in $\mathcal{T}(\mathcal{K})$, the structure $\mathcal{B}_x$ has an immediate successor. Moreover, we can compute the number of immediate successors of x.*
3. *For each path $\eta = \mathcal{B}_0, \mathcal{B}_1, \ldots$ in $\mathcal{T}(\mathcal{K})$ we have: $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \ldots$. Thus, the union of this chain determines the structure $\mathcal{B}_\eta = \cup_i \mathcal{B}_i \in \mathcal{K}_\omega$.*
4. *The mapping $\eta \to \mathcal{B}_\eta$ is a bijection between all the infinite paths of $\mathcal{T}(\mathcal{K})$ and the class $\mathcal{K}_\omega$.* □

Definition 5.2. For every structure $\mathcal{A} \in \mathcal{K}_\omega$, where $\mathcal{K}$ is a B-class, we set $\mathcal{A}[i]$ to be the largest substructure of $\mathcal{A}$ of height i.

Note that the substructure $\mathcal{A}[i]$ is correctly defined and is unique. The substructure $\mathcal{A}[i]$ contains all substructures of $\mathcal{A}$ of height i.

6 Martin-Löf randomness in $\mathcal{K}_\omega$

We consider the class of structures $\mathcal{K}_\omega$, where $\mathcal{K}$ is a B-class. Due to the previous section, we view the structures from $\mathcal{K}_\omega$ as paths through the tree $\mathcal{T}(\mathcal{K})$.

6.1 Topology, measure and metric

Using the tree $\mathcal{T}(\mathcal{K})$ we can introduce the topology and measure into the class $\mathcal{K}_\omega$

Definition 6.1 (**Topology).** Let $\mathcal{B}$ be a relational structure of height n. The **cone** of $\mathcal{B}$ is:

$$Cone(\mathcal{B}) = \{\mathcal{A} \mid \mathcal{A} \in K_\omega, \text{ and } \mathcal{A}[n] = \mathcal{B} \}.$$

Declare the cones $Cone(\mathcal{B})$ to be the *base open sets* of the topology on $\mathcal{K}_\omega$. We refer to the structure $\mathcal{B}$ as the *base of the cone.*

The measure μ of the cone $Cone(\mathcal{B})$ is defined by induction as follows.

Definition 6.2 (**Measure).** The measure of the cone based at the root is 1. Let $\mathcal{B}_x$ be a structure of height n. Assume that the measure $\mu(Cone(\mathcal{B}_x))$ has been defined. Let e_x be the number of structures of height $n+1$ that are immediate successors of $\mathcal{B}_x$ in the tree. Then for any immediate successor y of x we set $\mu(Cone(\mathcal{B}_y)) = \mu(Cone(\mathcal{B}_x))/e_x$.

As expected, we can define a metric in the class $\mathcal{K}_\omega$.

Definition 6.3 (Metric). Let $\mathcal{A}$ and $\mathcal{C}$ be structures from $\mathcal{K}_\omega$. Let n be the maximal level at which $\mathcal{A}[n]$ and $\mathcal{C}[n]$ coincide. Let $\mathcal{B}$ be the node of the tree such that $\mathcal{A}[n] = \mathcal{B}$. The distance $d(\mathcal{A}, \mathcal{B})$ between $\mathcal{A}$ and $\mathcal{C}$ is then: $d(\mathcal{A}, \mathcal{B}) = \mu(Cone(\mathcal{C}))$.

The next lemma shows that the distance d determines a metric in $\mathcal{K}_\omega$. Note, we identify structures up to isomorphism. So, an isomorphism maps values of constants c in one structure to the values of the same constants c in the other structure.

Lemma 6.4. *The function d is a metric on $\mathcal{K}_\omega$.*

Proof. Lemma 3.4 implies that a structure $\mathcal{A}$ is isomorphic to structure $\mathcal{B}$ if and only if $d(\mathcal{A}, \mathcal{B}) = 0$. It is obvious that $d(\mathcal{A}, \mathcal{B}) = d(\mathcal{B}, \mathcal{A})$. We need to show that the triangle inequality

$$d(\mathcal{A}, \mathcal{B}) \leqslant d(\mathcal{A}, \mathcal{C}) + d(\mathcal{C}, \mathcal{B})$$

holds for all structures $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ all in $\mathcal{K}_\omega$. If two of these three structures are isomorphic then the triangle inequality obviously holds. So, assume that the structures $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are pairwise not isomorphic. Let $x = n(\mathcal{A}, \mathcal{B})$ be the maximal level such that $\mathcal{A}[x] = \mathcal{B}[x]$. Similarly, consider $n(\mathcal{A}, \mathcal{C})$ and $n(\mathcal{C}, \mathcal{B})$.

If $n(\mathcal{A}, \mathcal{B}) > n(\mathcal{A}, \mathcal{C})$ then clearly the triangle inequality holds. If $n(\mathcal{A}, \mathcal{B}) \leqslant n(\mathcal{A}, \mathcal{C})$ then $\mathcal{A}[x]$ equals $\mathcal{C}[x]$. This implies that $n(\mathcal{C}, \mathcal{B}) = n(\mathcal{A}, \mathcal{B})$. Therefore, $d(\mathcal{A}, \mathcal{B}) \leqslant d(\mathcal{A}, \mathcal{C}) + d(\mathcal{C}, \mathcal{B})$. □

The following proposition follows from the lemmas and definitions above.

Proposition 6.5. *The space $\mathcal{K}_\omega$ has the following properties: (1) $\mathcal{K}_\omega$ is compact; (2) Finite unions of cones form clo-open sets in the topology; (3) The set of all μ-measurable sets is a σ-algebra.* □

6.2 ML-randomness

The set-up above allows us to formally define ML-random structures in the class $\mathcal{K}_\omega$. We provide several standard definitions from algorithmic randomness suited for our setting.

A class $C \subseteq \mathcal{K}_\omega$ is a Σ^0_1-*class* if there is computably enumerable (c.e.) sequence $\mathcal{B}_0, \mathcal{B}_1, \ldots$ of structures from $\mathcal{K}$ such that $C = \cup_{i \geqslant 1} Cone(\mathcal{B}_i)$. Computable enumerability of $\mathcal{B}_0, \mathcal{B}_1, \ldots$ implies that given i we can compute the the open diagram of $\mathcal{B}_i$; in particular, we know the cardinality of $\mathcal{B}_i$.

Recall that $r_{\mathcal{K}}(n)$ is the number of structures of height n in the the class $\mathcal{K}$. We use the measure μ given in Definition 6.2.

Definition 6.6. Let $\mathcal{K}$ be a B-class. Consider the class $\mathcal{K}_\omega$ of infinite structures.

1. A *Martin-Löf test* is a uniformly c.e. sequence $\{G_n\}_{n \geqslant 1}$ of Σ^0_1-classes such that $G_{n+1} \subset G_n$ and $\mu_m(G_n) < 1/r_{\mathcal{K}}(n)$ for all $n \geqslant 1$.

2. A structure $\mathcal{A}$ from $\mathcal{K}_\omega$ *fails* a *Martin-Löf test* $\{G_n\}_{n \geqslant 1}$ if $\mathcal{A}$ belongs to $\cap_n G_n$. Otherwise, we say that the structure $\mathcal{A}$ *passes* the test.

3. A structure $\mathcal{A}$ from $\mathcal{K}$ is *ML-random* if if it passes every Martin-Löf test.

If a class $C \subset \mathcal{K}_\omega$ is contained in a ML-test, then we also say that C has *effective measure* 0.

We refer to Martin-Löf tests as ML-tests. It turns out that there exists a *universal* ML-test in the sense that passing that test is equivalent to passing all ML-tests. Formally, an ML-test $\{U_n\}_{n \geqslant 1}$ is *universal* if for any ML-test $\{G_m\}_{m \geqslant 1}$ it is the case that $\cap_m G_m \subseteq \cap_n U_n$. A construction of a universal ML-test is easy. Indeed, enumerate all ML-tests $\{G^e_k\}_{k \geqslant 1}$, where $e \geqslant 1$, uniformly on e and k, and set $U_n = \cup_e G^e_{n+e+1}$. It is not hard to see that $\{U_n\}_{n \geqslant 1}$ is an ML-test and for any ML-test $\{G_m\}_{m \geqslant 1}$ we have $\cap_m G_m \subseteq \cap_n U_n$. Therefore, to prove that a structure $\mathcal{A} \in \mathcal{K}_\omega$ is ML-random it suffices to show that $\mathcal{A}$ passes the universal ML-test $\{U_n\}_{n \geqslant 1}$. Note the class of not ML-random structures has μ-measure 0. Thus, we have the following simple corollary:

Corollary 6.7. *Let $\mathcal{K}$ be a B-class. Then the number of ML-random structures in the class $\mathcal{K}_\omega$ is continuum. In particular, each of the following classes of infinite structures contains continuum ML-random structures: (a) connected graphs of bounded degree,* *(b) c-generated algebras, (c) rooted trees of bounded degree, (d) structures of bounded degree, (e) partially ordered sets of bounded degree and with the least element, (f) binary ordered trees, and (e) two generated monoids.* □

6.3 Randomness in the Halting set

We study computable aspects of ML-random structures from the classes $\mathcal{K}_\omega$, where $\mathcal{K}$ is a B-class. For this we start with the following definition that originally goes to Malcev [11] and Rabin [15].

Definition 6.8. An infinite structure $\mathcal{A}$ is *computable* if it is isomorphic to a structure with domain ω such that all atomic operations and relations of the structure are computable.

Thus, computability is an isomorphism property of structures. Below is a bit stronger definition that involves the height function of the class $\mathcal{K}$.

Definition 6.9. Let $\mathcal{K}$ be a B-class and $\mathcal{A}$ be a computable structure from $\mathcal{K}_\omega$. We call $\mathcal{A}$ *strictly computable* if the size of the substructure $\mathcal{A}[i]$ can be effectively computed from any $i \in \omega$.

The following proposition gives examples of strictly computable structures:

Proposition 6.10. *The following are true:*

1. *Every computable c-generated algebra is strictly computable.*
2. *A connected computable pointed graph of bounded degree d is strictly computable if and only if there exists an algorithm that for every v vertex of the graph computes the degree of v.*
3. *A computable rooted tree of bounded degree d is strictly computable if and only if there exists an algorithm that for every v node of the tree computes the number of immediate successors of v.*
4. *A computable d-bounded partial order with the least element is strictly computable if and only if there exists an algorithm that for every v element of the partial order computes all covers of v.*
5. *Every computable two generated monoid is strictly computable.* □

Strict computability implies the following fact:

Proposition 6.11. *If $\mathcal{A}$ is strictly computable then $\mathcal{A}$ is not ML-random.*

Proof. There exists an effective procedure that given n computes the structure $\mathcal{A}[n]$. Hence, the sequence of cones $Cone(\mathcal{A}[0])$, $Cone(\mathcal{A}[1])$, $Cone(\mathcal{A}[2])$, ... forms a Martin-Löf test that the structure $\mathcal{A}$ fails. □

Corollary 6.12. *Let $\mathcal{A}$ be either an infinite pointed graph or tree or partial order of bounded degree. If $\mathcal{A}$ is computable and its $\exists$-diagram, that is the set*

$$\{\phi(\bar{a}) \mid \bar{a} \in A \text{ and } \mathcal{A} \models \phi(\bar{a}) \text{ and } \phi(\bar{x}) \text{ is an existential first-order formula}\},$$

is decidable then $\mathcal{A}$ is not ML-random. □

A natural class that contains the class of all computable structures is the class of structures computable in the Halting set. For instance, finitely presented groups are computable in the halting set. We denote the halting set by $\mathcal{H}$. Here is a definition.

Definition 6.13. A structure $\mathcal{A}$ is *$\mathcal{H}$-computable* if the domain of $\mathcal{A}$ is ω and all atomic relations and operations of $\mathcal{A}$ are computable in $\mathcal{H}$.

The next theorem shows that ML-random $\mathcal{H}$-computable structures exist. Thus, the proposition above can't be extended to $\mathcal{H}$-computable structures.

Theorem 6.14. *For every B-class $\mathcal{K}$ there exist an $\mathcal{H}$-computable ML-random structure in class $\mathcal{K}_\omega$.*

Proof. Consider the tree $T(\mathcal{K})$. Let $\{U_n\}_{n\geqslant 1}$ be the universal ML–test for $\mathcal{K}_\omega$. So, $\mu(U_n) < 1/r_{\mathcal{K}}(n)$ for all $n \geqslant 1$. We construct a path η on this tree such that $\mathcal{A} \notin \cap_n U_n$ by building η such that $\eta \notin U_1$.

Suppose that we have constructed a finite path $\eta_n = x_0, \ldots, x_n$. If $Cone(B_{x_n}) \subseteq U_n$, then we find the largest prefix $x_0, \ldots, x_m$ of η_n and its extension $\eta_{n+1} = x_0, \ldots, x_m, x'_{m+1}, \ldots, x'_{n+1}$ such that $\eta_{n+1} \notin U_1 \cup \ldots \cup U_n$. Such extensions exist.

Let η be the limit of the sequence $\eta_1, \eta_2 \ldots$. It is standard to show that η is computable in the halting set. The path η is ML-random by construction. The structure $\mathcal{B}_\eta$, as in part (4) of Lemma 5.1, is thus ML-random and $\mathcal{H}$-computable. □

Corollary 6.15. *The following classes of infinite structures contains $\mathcal{H}$-computable ML-random structures: (a) connected graphs of bounded degree, (b) c-generated algebras, (c) rooted trees of bounded degree, (d) structures of bounded degree, (e) partially ordered sets of bounded degree and with the least element, (f) binary ordered trees, and (e) two generated monoids.* □

In [8] the author constructs a computably enumerable ML-random tree in the class of trees of bounded degree. This is a stronger result for the class of trees.

7 Computable ML-random structures

Let $\mathcal{K}$ be a B-class and $\mathcal{A} \in \mathcal{K}_\omega$. Clearly, there exists a (unique) path $\eta \in T(\mathcal{K})$ such that the structure $\mathcal{B}_\eta$ is isomorphic $\mathcal{A}$. The path η can be constructed in the jump of the open diagram of $\mathcal{A}$. In particular, if $\mathcal{A}$ is a computable structure then the path η is computable in the halting set $\mathcal{H}$. This observation suggests that for some B-classes $\mathcal{K}$ there might exist ML-random yet computable structures in $\mathcal{K}_\omega$. This intuition is confirmed in the following theorem:

Theorem 7.1. *There exists a B-class $\mathcal{S}$ such that the class $\mathcal{S}_\omega$ has a computable ML-random structure.*

Proof. Consider the B-class $OT(2)$ of all finite ordered trees as in **Example 6**. Given a tree $\mathcal{B}$ in the class $OT(2)$ we can order all the nodes of the tree on the same level (from left to right) in a natural way. We refer to this order as the level order of the nodes. We define the following subclass $\mathcal{S}$ of $OT(2)$. A binary tree $\mathcal{B} \in OT(2)$ belongs to $\mathcal{S}$ if it has the following properties:

1. All leaves of $\mathcal{B}$ are of the same height,
2. If v in $\mathcal{B}$ has the right child then all nodes left of v on the v's level-order including v have the left and the right children, and
3. At every level i of $\mathcal{B}$ there exists at most one node such that it is the left child of its parent, and the parent does not have a right child in $\mathcal{B}$.

We provide several properties of the class $\mathcal{S}$.

Lemma 7.2. *If $\mathcal{B}$ belongs to $\mathcal{S}$ and has height n then there are exactly two non-isomorphic extensions of $\mathcal{B}$ of height $n+1$ both in $\mathcal{S}$.*

To prove the lemma, let $a_1, \ldots, a_k$ be all the leaves of $\mathcal{B}$ from the left to the right order. One desired extension of $\mathcal{B}$ is obtained by adding the left and the right children to the nodes $a_1, \ldots, a_{k-1}$ and by adding the left child of a_k and omitting the right child of a_k. The second desired extension is obtained by adding to all the leaves $a_1, \ldots, a_k$ their left and right children. These two extensions are obviously in $\mathcal{S}$.

Let $\mathcal{A}$ be an extension of $\mathcal{B}$ such that $\mathcal{A} \in \mathcal{S}$. Part (1) of the definition of $\mathcal{S}$ implies that for each leave a_i of $\mathcal{B}$ either the left child of a_i or the right child of a_i must belong to $\mathcal{A}$. Assume that $i < k$ and the node a_i has exactly one child, say v, in $\mathcal{A}$. Then v must be the left child. Otherwise, by Part (2), both children of a_i belong to $\mathcal{A}$. Consider a_{i+1}. The node must have a child in $\mathcal{A}$. If the child is the right child then a_i must have two children in $\mathcal{A}$, again by Part (2). Hence, in $\mathcal{A}$ the node a_{i+1} has the left child only. But this contradicts Part (3) of the definition of $\mathcal{S}$. The lemma is proved.

Corollary 7.3. *The class $\mathcal{S}$ is a B-class. Moreover, the tree $T(\mathcal{S})$ is isomorphic to the infinite binary tree.*

The next lemma provides a nice algebraic property of trees in $\mathcal{S}$ that have the same height. The property is expressed in terms of embeddings.

Lemma 7.4. *For every $n \geqslant 0$, the set of all trees in $\mathcal{S}$ of height n form a chain of embedded structures.*

The lemma is proved by induction. For $n = 1$, the lemma is obviously true. By Corollary 7.3, there are exactly 2^n trees $\mathcal{A}_1, \ldots, \mathcal{A}_{2^n}$ from $\mathcal{S}$, all of height n. So, we can assume that $\mathcal{A}_i$ is embedded into $\mathcal{A}_{i+1}$ for all i with $1 \leqslant i \leqslant 2^n - 1$. Also, by induction we can assume that each $\mathcal{A}_k$ has exactly k leaves. Let $\mathcal{A}'_k$ and $\mathcal{A}''_k$ be two the left and the right immediate successors of $\mathcal{A}_k$. Clearly, from the proof of Lemma 7.2, the tree $\mathcal{A}'_k$ is embedded into $\mathcal{A}''_k$. Consider $\mathcal{A}'_{k+1}$, the left successor of $\mathcal{A}_{k+1}$. Let $a_1, \ldots, a_k$ be the leaves of $\mathcal{A}_k$ and $b_1, \ldots, b_{k+1}$ be the leaves of $\mathcal{A}_{k+1}$. Clearly, the embedding of $\mathcal{A}_k$ into $\mathcal{A}_{k+1}$ maps each a_i into b_i, $i = 1, \ldots, k$. In $\mathcal{A}'_{k+1}$ each of the elements b_i, with $1 \leqslant i \leqslant k$ has exactly two children and b_{k+1} has exactly one child, the left child. This shows that $\mathcal{A}''_k$ is embedded into $\mathcal{A}'_{k+1}$. We have proved the lemma.

The last lemma and Corollary 7.3 imply that there is a natural bijection $\alpha \to \mathcal{A}_\alpha$ from the set of all binary strings to the class $\mathcal{S}$. Moreover, we have:

Corollary 7.5. *Let $\alpha \preceq \beta$, where $\preceq$ is the lexicographical order on binary strings. Then:*

1. *If $|\alpha| \leqslant |\beta|$ then $\mathcal{A}_\alpha$ is embedded into $\mathcal{A}_\beta$.*
2. *If $|\alpha| > |\beta|$ then $\mathcal{A}_\alpha$ is embedded into $\mathcal{A}_{\beta\gamma}$ for all γ such that $|\alpha| \leqslant |\beta\gamma|$*

Thus, we identify the tree $T(\mathcal{S})$ with the infinite binary tree and use the mapping $\alpha \to \mathcal{A}_\alpha$ in our construction of ML-random computable tree in $\mathcal{S}_\omega$.

Let $\{U_n\}_{n\in\omega}$ be the universal ML-test. We want to construct a computable binary ordered tree $\mathcal{C}$ in the class $\mathcal{S}_\omega$ such that $\mathcal{C}$ passes the ML-test $\{U_n\}_{n\in\omega}$. It suffices to construct $\mathcal{C}$ so that $\mathcal{C} \notin U_1$. Informally, this follows a standard construction of a ML-random string. Since $\mu(U_1) < 1/2$, the complement $S_\omega \setminus U_1$ can be thought of as the set of all infinite paths through a computable tree. The leftmost infinite path η of this tree determines the structure $\mathcal{C}$. One needs to note then the structure $\mathcal{C}$ that corresponds to η is a computable structure even if η is not computable (in fact, η is a ML-random binary string). The machinery above developed, through the lemmas, guarantee that $\mathcal{C}$ is computable. We present our argument more formally now.

Construction. Start building our structure as the direct limit of $\mathcal{A}_0 \subset \mathcal{A}_{00} \subset \mathcal{A}_{000} \subset \ldots$. At stage n of our construction suppose that we have built the structure $\mathcal{A}_n$ of the form $\mathcal{A}_{\alpha_n}$. Assume that we have $\sigma_1, \ldots, \sigma_n$ all appeared in U_1 such that $\mathcal{A}_{\alpha_n} \notin \mathcal{U}_{1,n}$, where $\mathcal{U}_{1,n} = Cone(\mathcal{A}_{\sigma_1}) \cup \ldots \cup Cone(\mathcal{A}_{\sigma_n})$. By induction, $\mu(S_\omega \setminus \mathcal{U}_{1,n}) = 1 - \mu(\mathcal{U}_{1,n}) > 1/2$. Consider σ_{n+1}. If $\mathcal{A}_{\alpha_n} \notin Cone(\mathcal{A}_{\sigma_{n+1}})$ then set $\mathcal{A}_{n+1} = \mathcal{A}_{\alpha_n 0}$. Otherwise, find the largest prefix α of α_n and a string β such that $\alpha_n \preceq \alpha\beta$ and $|\alpha_n| + 1 = |\alpha\beta|$ and $\mathcal{A}_{\alpha\beta} \notin \mathcal{U}_{1,n} \cup Cone(\mathcal{A}_{\sigma_{n+1}})$. Such string α and β exist. Declare $\alpha_{n+1} = \alpha\beta$. The inductive assumptions are preserved. By Corollary 7.5, $\mathcal{A}_{\alpha_n}$ is embedded into $\mathcal{A}_{\alpha_{n+1}}$. The structure $\mathcal{C} = \lim_n \mathcal{A}_{\alpha_n}$ is thus a desired computable ML-random structure. □

Note that the class $\mathcal{S}_\omega$ is a subclass of $OT(2)_\omega$. Moreover, the class $\mathcal{S}_\omega$ has effective measure 0 in the class $OT(2)_\omega$. So, ML-randomness in a class does not imply randomness in larger classes. So, ML-randomness for infinite structures is context dependent.

8 Measures of varieties

In this section our interest is in the class of finitely generated algebras $PAlg_\omega$; see Example 4 of Section 4. Thus, our signature σ is functional (yet we view structures from $PAlg$ as relational). A class of algebras of signature σ is called a *variety* if its closed under sub-algebras, homomorphisms, and ultra-products. It is a classical result of model theory that a class of algebras is variety if and only if is axiomatised by a set E of universally quantified equitations [5]. Recall that an equation is of the form $p(\bar{x}) = q(\bar{x})$ where p and q are terms whose variables are among the variables $\bar{x}$. Call an equation $p(\bar{x}) = q(\bar{x})$ *non-trivial* if at least one of the terms contains a variable and $p \neq q$ syntactically. If E contains at least one non-trivial equation then we call the variety of algebras satisfying E a *non-trivial variety.*

Let V be a variety defined by a set of equation E and let R be a set of defining relations on the generators $\bar{c}$. Defining relations are of the form $t_1(\bar{c}) = t_2(\bar{c})$, where t_1 and t_2 are ground terms. The set of all c-generated algebras that satisfy E and R contains the free c-algebra $\mathcal{A}(E, R)$. This algebra is unique and any c-generated algebra from V that satisfies R is a homomorphic image of $\mathcal{A}(E, R)$. In this sense, one can view the pair E and R as a description of the free algebra $\mathcal{A}(E, R)$. Immediate examples of such algebras include finitely presented groups, semigroups, rings, lattices, etc. A natural question is if finitely presented algebras are ML-random. We present our answer to this question in a strongest form:

Theorem 8.1. *The class of all infinite c-algebras that belong to a non-trivial variety has an effective measure zero. In particular, no finitely presented algebra of a non-trivial variety is ML-random.*

Proof. For the proof we use notations and definitions, such as proper partial algebra, from Example 4 of Section 4. We also denote the tree $T(PAlg)$ by T.

Let V be a nontrivial variety defined by set E. Let $p(\bar{x}) = q(\bar{x})$ be a nontrivial equation in E. Say that $p = f(t_1(\bar{x}), \ldots, t_k(\bar{x}))$ and $q = g(r_1(\bar{x}), \ldots, r_l(\bar{x}))$. It suffices to show that the variety V' defined by only one equation $p(\bar{x}) = q(\bar{x})$ has effective measure zero.

For each node x at level n of the tree T consider the structure $\mathcal{A}_x$ that corresponds to node x. Define:

$$U_x = \{\bar{b} \mid \text{ either } p(\bar{b}) \text{ or } q(\bar{b}) \text{ is not defined}\}.$$

Call a node x *potential* if $p(\bar{b}) = q(\bar{b})$ in $\mathcal{A}_x$ for all $\bar{b} \notin U_x$. Potentiality indicates that $Cone(\mathcal{A}_x)$ might contain an algebra from the variety V. If x is not potential then the cone $Cone(\mathcal{A}_x)$ contains no algebra from V. Note that if a tuple $\bar{b}$ has an element of the same height as the level of x in T then $\bar{b} \in U_x$.

Roughly, probability that a proper partial algebra extends $\mathcal{A}_x$, where x is potential, and satisfies the equation $p(\bar{b}) = q(\bar{b})$ for some $\bar{b} \in U_x$ is small. Intuitively, this is the underlying reason as to why V has effective measure 0. We formalise this reasonning.

Consider the term $t(z_1, \ldots, z_s)$ such that we can write p as $t(p_1(\bar{x}), \ldots, p_s(\bar{x}))$ and we can write q as $q = t(q_1(\bar{x}), \ldots, q_s(\bar{x}))$, where for each i either p_i and q_i are both equal variables or the first symbols of p_i and q_i are unequal syntactically. Informally, in the tree representations of p and q, the term t is the largest common tree (starting from the root) that the trees p and q share. For instance, for the terms $p = f(f(f(x,y),y), f(z,y))$ and $q = f(f(y,x), f(f(z,y), f(x,y)))$ it is easy to see that $t = f(f(z_1, z_2), f(z_3, z_4))$.

To simplify our exposition, suppose that p starts with f and q starts with g, two different functional symbols. In this case the term t described above is just a variable term z. The case when t is not a variable term is treated in a similar manner but with a bit more care.

Let x be a potential node. Call a node $y \in T$ a *border* node (with respect to x) if (1) y extends x in T, (2) y is a potential node, (3) there exists a $\bar{b} \in U_x$ such that either $p(\bar{b})$ or $q(\bar{b})$ is undefined in $\mathcal{A}_y$, and (4) for all $\bar{b} \in U_x$ and all immediate successor nodes z of y both $p(\bar{b})$ and $q(\bar{b})$ are defined in $\mathcal{A}_z$.

Let y be a border node. Consider a tuple $\bar{b}$ such that either $p(\bar{b})$ or $q(\bar{b})$ is undefined in $\mathcal{A}_y$. To simplify our writing assume that both f and g are binary operations. Write $d_1 = p_1(\bar{b})$, $d_2 = p_2(\bar{b})$, $e_1 = q_1(\bar{b})$, $e_2 = q_2(\bar{b})$. One of the elements among d_1, d_2, e_1, e_2 has height that is equal to the level of the node y (by the definition of algebras $\mathcal{A}_y$). For simplicity assume that d_1 and e_1 are the elements of $\mathcal{A}_y$ that have height equal to the level of y.

Consider $\mathcal{A}_z$, where z is a successor of y such that z is potential. Assume that $f(d_1, d_2) = g(e_1, e_2)$ in $\mathcal{A}_z$. Construct $\mathcal{A}_{z'}$ such that $\mathcal{A}_{z'}$ is the same as $\mathcal{A}_z$ with exactly one difference: in $\mathcal{A}_{z'}$ we have $f(d_1, d_2) \neq g(e_1, e_2)$. For instance, if $g(e_1, e_2)$ has height smaller then the height of e_1, we set $f(d_1, d_2)$ to be of height the height of d_1 plus 1. If both $f(d_1, d_2)$ and $g(e_1, e_2)$ have height equal to the level of z, then we set $f(d_1, d_2)$ not to be equal to $g(e_1, e_2)$. The following lemma is now easy:

Lemma 8.2. *The node z' is an immediate successor of y which is not potential. Furthermore, if z_1 and z_2 are immediate successor of y and are potential nodes then $\mathcal{A}_{z_1'}$ and $\mathcal{A}_{z_2'}$ are not isomorphic.*

This implies that among all immediate successors of y the number of potential nodes is at most as the number of non-potential nodes. Hence, we have the following:

$$\mu(Cone(\mathcal{A}_x)) \geqslant 2 \sum_{y} \mu(Cone(\mathcal{A}_y)),$$

where the sum is taken over all nodes y in T that are border nodes.

We construct effective measure 0 set that contains V. Set $X_1 = \{x \mid x \text{ is a potential node whose level in the tree } T \text{ equals the maximum of the nights of terms } p \text{ and } q\}$. Let U_1 be the union of all $Cone(\mathcal{A}_x)$ where $x \in X_1$. Set X_{i+1} to be the set of all y such that y is a border node for some $x \in X_i$. Let U_{i+1} be the union of all $Cone(\mathcal{A}_y)$ where $y \in X_{i+1}$. By construction we have $V \subset U_i$ for all i. Moreover, from the inequality above we have $\mu(U_{i+1}) \leqslant 2 \cdot \mu(U_i)$.

In the case when the term t has height $s \geqslant 1$, the sequence $U_1 \supset U_2 \supset \ldots$ can be constructed such that $\mu(U_{i+1}) \leqslant (2^{s-1}/2^s) \cdot \mu(U_i)$ for all $i \geqslant 1$. □

Corollary 8.3. *No finitely generated ML-random algebra exists that satisfies a nontrivial set of equations. Hence, no ML-random group, monoid, or lattice exist in the class of all c-generated algebras.* □

Another corollary describes finitely axiomatized varieties that have non-zero measure:

Corollary 8.4. *A finitely axiomatised variety V has either measure 0 or measure that is equal to a rational number > 0. The latter case occurs if and only if the variety is axiomatised by a trivial set of equations.*

Proof. Let E be a finite set of equations for the variety V. If E is not trivial then the theorem above implies the corollary. Otherwise, E is a trivial set of equations. By Kozen [9] the free c-generated algebra in V is a free extension of some proper partial algebra $\mathcal{A}_x$. This implies that V is a finite union of cones in the space of all infinite c-generated algebras. □

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