Topics in Generalised Inverse Limits

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Abstract

Generalised inverse limits are a new topic of study in the area of continuum theory. Although similarly defined to the more traditional inverse limits of continuous functions on continua, they have a much richer structure. It was realised early on that many of the theorems relating to the inverse limits of continuous functions do not carry over to generalised inverse limits, which use set valued functions. Much of the research in generalised inverse limits to this point has been to attempt to characterise their structure.

This thesis contains three main chapters, each of which is based on a topic in generalised inverse limits research. In the first of these we explore the structure of a particular generalised inverse limit known as $K_{(0,1)}$. This is the inverse limit of a generalised tent map. The inverse limits of tent maps have played an important role in continuum theory in the past, and with the introduction of generalised inverse limits we can include generalised tent maps that are not continuous functions. The only such generalised tent map whose inverse limit does not have a very simple structure is $K_{(0,1)}$. In this chapter we prove a number of topological properties of $K_{(0,1)}$, and give an embedding of $K_{(0,1)}$ into $\mathbb{R}^3$.

For the second topic we characterise a certain kind of disconnection in generalised inverse limits over Hausdorff continua. This generalises a result by Greenwood and Kennedy for generalised inverse limits over intervals. Connectedness is a property that has attracted much interest in generalised inverse limits, as these are not necessarily connected, unlike the inverse limits of continuous functions, which are.

In the final chapter we prove a result relating to path connectedness in generalised inverse limits. Path connectedness in generalised inverse limits is in some ways a very different property to connectedness. For example, a generalised inverse limit may not be path connected even though all its finite approximants are path connected. This cannot happen if we replace the words "path connected" with "connected" in the previous sentence. The result proved in this chapter links the path connectedness of a generalised inverse limit with path connected properties of its finite approximants.

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The main body of work in Chapter 3: Connected Generalised Inverse Limits is based on work published in Topology and its Applications, 160, pp 513 - 523, titled Connected generalised inverse limits over Hausdorff continua, by Sina Greenwood and Michael Lockyer.

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CO-AUTHORS

Name	Nature of Contribution
Sina Greenwood	- Many of the details in the proofs of the main theorem and earlier lemmas, and most of the final draft - Some modifications to the examples

Certification by Co-Authors

The undersigned hereby certify that:

- the above statement correctly reflects the nature and extent of the PhD candidate's contribution to this work, and the nature of the contribution of each of the co-authors; and
- in cases where the PhD candidate was the lead author of the work that the candidate wrote the text.

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Chapter 1

Introduction

This thesis is on the topic of generalised inverse limits, a relatively new area of study in continuum theory. Generalised inverse limits are defined in a very similar manner to the inverse limits of continuous functions on continua that have been central to continuum theory for the last 50 years, but their structure is much richer. Generalised inverse limits use upper semicontinuous set valued functions as bonding functions, rather than continuous single valued functions that have been more commonly used in the past. Many of the standard theorems regarding inverse limits of continuous functions on continua do not carry over to the generalised case, and much of the work that has been done to this point has been to look for theorems for generalised inverse limits analogous to those of the inverse limits of continuous functions.

This thesis contains three fairly self contained main chapters, each exploring a different topic in the field of generalised inverse limits. In Chapter 2, we explore the structure of a particular generalised inverse limit known as $K_{(0,1)}$. This inverse limit is a very complicated continuum that has a single bonding function: a generalised tent map. This is the only inverse limit of a generalised tent map that is difficult to describe, despite many inverse limits of tent maps which are continuous functions having a very complicated structure. In Chapter 2 we prove a number of topological properties of $K_{(0,1)}$, including that it has the fixed point property, and we also give an embedding into $\mathbb{R}^3$, which will give the reader a good intuition into its structure.

In Chapter 3, we give a result characterising a certain type of disconnection in generalised inverse limits on continua, by defining what is called an HC-sequence. This is a generalisation of a result by Greenwood and Kennedy [GK], we extend this result to work for compact Hausdorff factor spaces from the intervals used in the original theorem. Although the extended result looks similar in nature, the proof uses quite different ideas. A corollary of the main result of this section is that if an HC-sequence exists, the generalised inverse limit is disconnected. Connectedness is an area where generalised inverse limits on continua differ from inverse limits of continuous functions on continua, the latter being always connected.

In Chapter 4, we look into path connectedness in generalised inverse limits. This property is very different to the connectedness explored in Chapter 3. It is possible for an inverse limit to be path connected for each finite approximation, and then not path connected in the limit (this is not possible for connectedness). This gives two somewhat distinct problems when trying to characterise path connectedness in generalised inverse limits. Firstly, we need to know whether all the finite approximations are path connected, and secondly, if all the finite approximants are path connected, we have the distinct question of whether the inverse limit itself is path connected. Chapter 4 focuses on the latter question, and gives a result that to some extent characterises path connectedness in the inverse limit, assuming all finite approximants are path connected.

For the remainder of this chapter we will firstly introduce some notation and basic definitions that will be used throughout the thesis. We will then look into the current state of knowledge in generalised inverse limits, giving an overview of results obtained to date.

1.1 Notation and Definitions

The field of generalised inverse limits is a relatively new area of study, and not all notation is standard. In this section we present preliminaries and a few basic results for generalised inverse limits that will be used in following chapters.

The subset notation $\subset$ will allow for equality of sets when used throughout this thesis. For example if A and B are sets, then if we write $A \subset B$ then it is possible that $A = B$. If we want strict inclusion, ie $A \neq B$, then we will write $A \subsetneq B$.

We take $\mathbb{N}$ to be the set of natural numbers $\{0, 1, 2, \dots\}$.

$\mathfrak{c}$ is the cardinality of the continuum.

A point in $\mathbb{R}^2$ is denoted with angle brackets, as $\langle a, b\rangle$.

We include the following basic topological definitions, due to their importance in this thesis.

Definition 1.1.1. If X is a topological space, we say X is *connected* if X does not contain a proper non-empty clopen subset. Otherwise we say X is disconnected. If $x \in X$, the largest connected subspace of X containing x is called the *component of x in X*.

Definition 1.1.2. A *path* in X is a continuous function $\gamma : [0, 1] \to X$. We say X is *path connected* if for all pairs of points x, $y \in X$, there exists path $\gamma : [0, 1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. If $x \in X$, the largest path connected subspace of X containing x is called the *path component of x in X*.

Definition 1.1.3. A *continuum* is a compact, connected, metric space. A *Hausdorff continuum* is a Hausdorff, compact, connected topological space.

Note that if a space X is a continuum, then X is also a Hausdorff continuum.

The main topic of this thesis is inverse limits, which we will begin to define now.

Definition 1.1.4. An *inverse sequence* is a sequence of pairs (X_i, f_{i+1}) for $i \geq 0$, where for each i, X_i is a Hausdorff continuum, and $f_{i+1} : X_{i+1} \to X_i$ is a continuous function.

From this we can define the inverse limit of the inverse sequence.

Definition 1.1.5. If (X_i, f_{i+1}) is an inverse sequence, then the *inverse limit* of the inverse sequence, denoted $\varprojlim(X_i, f_i)$, is defined as:

$$\varprojlim(X_i, f_i) = \{\mathbf{x} = (x_0, x_1, x_2, \dots) \in \Pi_{i\in\mathbb{N}} X_i : \text{ for all } i \in \mathbb{N}, x_i = f_{i+1}(x_{i+1})\}.$$

If only one function, f, is being used for every function, then often we will shorten $\varprojlim(X_i, f_i)$ to $\varprojlim f$.

The spaces X_i are called *factor spaces*, and the functions f_i are called *bonding functions*.

To avoid confusion, often we will refer to the inverse limits of continuous functions defined above as *classical* inverse limits, as opposed to *generalised* inverse limits, the topic of this thesis.

If X is a topological space, we use 2^X to denote the collection of nonempty closed subsets of X.

Definition 1.1.6. If X and Y are continua, a function $f : X \to 2^Y$ is called *upper semicontinuous* at a point $x \in X$ if for each open set V in Y containing $f(x)$, there is an open set U in X containing x such that if y is in U, then $f(y) \subset V$. If f is upper semicontinuous at x for all $x \in X$, then we say f is *upper semicontinuous.*

If $f : X \to 2^Y$ is a function, then the *graph* of f is the subset of $X \times Y$, denoted $G(f)$, defined as

$$G(f) = \{\langle x, y\rangle \in X \times Y : y \in f(x)\}.$$

The following theorem gives an equivalent definition for upper semicontinuity, and this is usually easier to use as a working definition. See [IM, Theorem 2.1] for the proof.

Theorem 1.1.7 (Ingram and Mahavier)**.** *If X and Y are continua, then $f : X \to 2^Y$ is upper semicontinuous if and only if the graph of f is closed in $X \times Y$.*

An upper semicontinuous set valued function $f : X \to 2^Y$ is said to be *surjective* if for every $y \in Y$, there exists $x \in X$ such that $y \in f(x)$.

Now we can define a generalised inverse limit. The definition looks very similar to that of classical inverse limits, but we need to take into account the set valued functions. So for a point $\mathbf{x} = (x_0, x_1, x_2, \dots)$ to be in the generalised inverse limit, we only require that each coordinate x_i is *contained* in the image of the following coordinate, x_{i+1}, by the function f_{i+1}.

Definition 1.1.8. An *generalised inverse sequence* is a sequence of pairs (X_i, f_{i+1}) for $i \geq 0$, where for each i, X_i is a Hausdorff continuum, and $f_{i+1} : X_{i+1} \to 2^{X_i}$ is an upper semicontinuous set valued function.

With a slight abuse of notation, often we will refer to a (generalised) inverse sequence by (X_i, f_i).

If (X_i, f_{i+1}) is a generalised inverse sequence, then the *generalised inverse limit*, denoted $\varprojlim(X_i, f_i)$, is defined by

$$\varprojlim(X_i, f_i) = \{\mathbf{x} = (x_0, x_1, x_2, \dots) \in \Pi_{i\in\mathbb{N}} X_i : \text{ for all } i \in \mathbb{N}, x_i \in f_{i+1}(x_{i+1})\}.$$

Again, the spaces X_i are called *factor spaces*, and the functions f_i are called *bonding functions*, and if only one function, f, is being used for every bonding function, then often we will shorten $\varprojlim(X_i, f_i)$ to $\varprojlim f$. In this thesis, when an inverse sequence (I_i, f_i) is written, it can be assumed that the factor spaces are intervals, that is $I_i = [0, 1]$ for all $i \in \mathbb{N}$.

In the above definitions, the factor spaces are indexed over the natural numbers, $\mathbb{N}$. This is a special case of a more general indexing which is done over a directed set. The details of this will not be included here, and we note that unless it is mentioned otherwise, all inverse limits in this thesis will be indexed over the natural numbers, $\mathbb{N}$.

Often it is useful to look at finite restrictions of the inverse limit. With this in mind, we have the following definition.

Definition 1.1.9. If (X_i, f_i) is a generalised inverse sequence, and $1 \leq m \leq n$, then we define

$$\mathcal{G}(f_m, \ldots, f_n) = \{\langle x_{m-1}, x_m, \ldots, x_n\rangle \in \Pi_{m-1\leq i\leq n} X_i : x_i \in f_{i+1}(x_{i+1}) \text{ for all } m-1 \leq i < n\}.$$

$\mathcal{G}(f_1, \ldots, f_n)$ is known as the *Mahavier product* from f_1 to f_n. If $m = 1$, then often we will abbreviate $\mathcal{G}(f_1, \ldots, f_n)$ as $\mathcal{G}_n$. These are sometimes called *finite approximants* of an inverse limit.

We also denote the graph of a set valued function $f_n : X_n \to 2^{X_{n-1}}$ by G_n, that is, $G_n = \{\langle x_n, x_{n-1}\rangle \in X_n \times X_{n-1} : x_n \in X_n \text{ and } x_{n-1} \in f_n(x_n)\}$. Alternatively, if f is a set valued function, its graph can be denoted $G(f)$.

Note that if $m = n$ then $\mathcal{G}(f_m)$ is the graph of f_m^{-1}, so $\mathcal{G}(f_m)$ is homeomorphic to $G_m = G(f_m)$.

The following notation will be used for projection functions.

Definition 1.1.10. Given a generalised inverse sequence (X_i, f_i) and $j \in \mathbb{N}$, we define $\pi_j : \varprojlim(X_i, f_i) \to G_j$ to be the projection map into the graph G_j, ie $\pi_j((x_0, x_1, x_2, \ldots)) = \langle x_j, x_{j-1}\rangle$. Given a graph G_j, we also define $\rho_{j,j} : G_j \to X_j$ and $\rho_{j,j-1} : G_j \to X_{j-1}$ to be projection maps into factor spaces, ie $\rho_{j,j}(\langle x_j, x_{j-1}\rangle) = x_j$, and $\rho_{j,j-1}(\langle x_j, x_{j-1}\rangle) = x_{j-1}$. If a function $f : X \to 2^Y$ with graph G is not indexed, the notation $\rho_X : G \to X$ and $\rho_Y : G \to Y$ will denote the projection maps into X and Y respectively.

1.2 Background

In this section we present some background on previous work done in the area of generalised inverse limits. We begin with a brief introduction into classical inverse limits.

Inverse limits have been shown to be a very useful tool in continuum theory. One application is that inverse limits give an easy way to construct complicated continua from simple bonding functions. A famous example of this is a tent map as a bonding function gives as an inverse limit the buckethandle continuum, an example of an indecomposable continuum (there will be more on this later in the thesis). Another example by Henderson [H] is a pseudo-arc being the inverse limit of a single bonding function. A pseudo-arc is a one dimensional *arc-like* continuum that is hereditarily indecomposable, that is, all of its subcontinua are indecomposable.

Inverse limits describe a type of 'backward dynamics' - the bonding functions are defined as going backwards in 'time'. Inverse limits of continuous functions have been used to model a variety of different phenomena. Often a phenomenon being modelled by discrete dynamics will be modelled by relations which are not functions, but their inverses are functions. So an inverse limit in some sense represents the forward dynamics of the inverses of the relations. Some examples of applications include the following.

In physics, inverse limits play a central role in the theory of tiling spaces, which are used to model aperiodic crystals, see [S]. In economics, the 'overlapping generations model' and the 'cash in advance model' both use inverse limits to model economic phenomena. See for example [AR] and [K]. The Christiano - Harrison model [CH] is an example of a model in economics utilising relations that are not functions in either the forward or reverse direction, so requires the use of generalised inverse limits. Inverse limits also appear in areas such as game theory and functional analysis, see [P], [TY].

The field of generalised inverse limits is relatively new, but in the last few years there has been considerable research in the area. The first paper published on the subject was by Mahavier in 2004 [M]. This paper introduced many of the basic theorems that are used, and concentrated exclusively on inverse limits of set valued functions over intervals. A little later, Ingram and Mahavier published an article [IM] which generalises many of these results to Hausdorff continua.

The first two results given here from [IM] confirm that the generalised inverse limits of compact Hausdorff spaces are indeed compact (and Hausdorff).

Theorem 1.2.1 (Ingram and Mahavier). *Suppose for each $i \geq 0$, X_i is compact and Hausdorff, and for $i \geq 1$, $f_i : X_i \to 2^{X_{i-1}}$ is upper semicontinuous. Then $\mathcal{G}_n$ is nonempty and compact.*

Theorem 1.2.2 (Ingram and Mahavier). *Suppose for each $i \geq 0$, X_i is compact and Hausdorff, and for $i \geq 1$, $f_i : X_i \to 2^{X_{i-1}}$ is upper semicontinuous. Then $\varprojlim(X_i, f_i)$ is nonempty and compact.*

The following theorems also appear in [IM], and are the first theorems regarding connectedness in generalised inverse limits, a topic that will be explored in more detail in Chapter 3.

Theorem 1.2.3 (Ingram and Mahavier). *Suppose that for each $i \geq 0$, X_i is a Hausdorff continuum, and for each $i \geq 1$, $f_i : X_i \to 2^{X_{i-1}}$ is an upper semicontinuous function, and for each $x \in X_i$, $f_i(x)$ is connected. Then $\varprojlim(X_i, f_i)$ is a Hausdorff continuum.*

Theorem 1.2.4 (Ingram and Mahavier). *Suppose that for each $i \geq 0$, X_i is a Hausdorff continuum, and for each $i \geq 1$, X_i is a Hausdorff continuum, $f_i : X_i \to 2^{X_{i-1}}$ is an upper semicontinuous function, and for each $x \in X_{i-1}$, $\{y \in X_i : x \in f_i(y)\}$ is non-empty and connected. Then $\varprojlim(X_i, f_i)$ is a Hausdorff continuum.*

A very useful theorem regarding connectedness in generalised inverse limits was given by Nall in [Nal2] (it appears as Lemma 3.2).

Theorem 1.2.5 (Nall)**.** *Suppose X is a Hausdorff continuum, and $f : X \to 2^X$ is a surjective upper semicontinuous set valued function. Then $\varprojlim f$ is connected if and only if $\mathcal{G}_n$ is connected for every $n \in \mathbb{N}$.*

The proof given for this theorem allows for the spaces and functions to be different, and it will be generalised as Lemma 3.1.1 in Chapter 3.

Theorem 1.2.5 is very powerful in that it means that if an inverse limit is disconnected, then there will be some finite n such that $\mathcal{G}_n$ is disconnected.

This illustrates a common problem in generalised inverse limits - knowing whether if $\mathcal{G}_n$ has a certain property for all n implies that $\varprojlim(X_i, f_i)$ has the same property. The above result says that it does for the property of connectedness, but as we will see in Chapter 4, Theorem 1.2.5 does not have an analogy for path connectedness. There are upper semicontinuous functions $f : [0,1] \to 2^{[0,1]}$ such that $\mathcal{G}_n$ is path connected for all n, but $\varprojlim f$ is not path connected (the usual surjective tent map has this property).

Much of the early research into generalised inverse limits has been in the property of indecomposability.

Definition 1.2.6. A continuum X is *decomposable* if X contains proper subcontinua A and B such that $X = A \cup B$. Otherwise X is said to be *indecomposable.*

For example, the interval $[0,1]$ is decomposable as $[0,1] = [0, \frac{1}{2}] \cup [\frac{1}{2}, 1]$. It is not clear *a priori* from the definition that an indecomposable continuum even exists. Two examples of indecomposable continua are the buckethandle continuum and the pseudo-arc that were mentioned earlier in this section. These are difficult to define, however the following theorem, appearing in [Nad] as Theorem 2.7, links inverse limits of certain continuous functions with indecomposable continua.

We say that an inverse sequence (X_i, f_i) is an *indecomposable inverse sequence* if for each $i \geq 0$, whenever A_{i+1} and B_{i+1} are subcontinua of X_{i+1} such that $X_{i+1} = A_{i+1} \cup B_{i+1}$, then $f_{i+1}(A_{i+1}) = X_i$ or $f_{i+1}(B_{i+1}) = X_i$.

Theorem 1.2.7. *If (X_i, f_i) is an indecomposable inverse sequence with inverse limit $\varprojlim(X_i, f_i)$, then $\varprojlim(X_i, f_i)$ is an indecomposable continuum.*

This gives an easy way to create indecomposable continua - by defining them as the inverse limit of a particular indecomposable inverse sequence. A simple bonding function can be used to give rise to a complex continuum. Perhaps the most famous example is the buckethandle continuum, which will be described in detail in Chapter 2. The buckethandle continuum is homeomorphic to the continuum defined as the inverse limit of the tent map shown in Figure 1.1 [Nad, 2.9]. The usual description of the buckethandle as given in Chapter 2 requires a rather involved definition and defining the continuum using the inverse limit is a much simpler method.

The function shown in Figure 1.1 is an example of a tent map. The inverse limits of tent maps have been studied extensively, and are the subject of Chapter 2. Tent maps have played a huge role in the study of classical inverse limits and in topological dynamics. See for example [BBS], [BM], [H]. One of the major areas of research in classical inverse limits has been to classify the inverse limits of tent maps. A tent map is defined as follows.

Definition 1.2.8. For $a \in (0,1)$ and $b \in [0,1]$, we define a *tent map* $f_{(a,b)} : [0,1] \to [0,1]$ to be a continuous function with a graph consisting of two lines: one from $\langle 0,0\rangle$ to $\langle a,b\rangle$, and another from $\langle a,b\rangle$ to $\langle 1,0\rangle$.

The classification of the inverse limits of tent maps is still incomplete, but following a large amount of work on the specific question, Barge, Bruin, and Štimac proved the following theorem, which was known as the *Ingram Conjecture.*

Theorem 1.2.9 (Barge, Bruin, Štimac)**.** *Let $\frac{1}{2} \leq a < b \leq 1$, and let $f_{(\frac{1}{2},a)}$ and $f_{(\frac{1}{2},b)}$ be tent maps. Then $\varprojlim f_{(\frac{1}{2},a)}$ and $\varprojlim f_{(\frac{1}{2},b)}$ are not homeomorphic.*

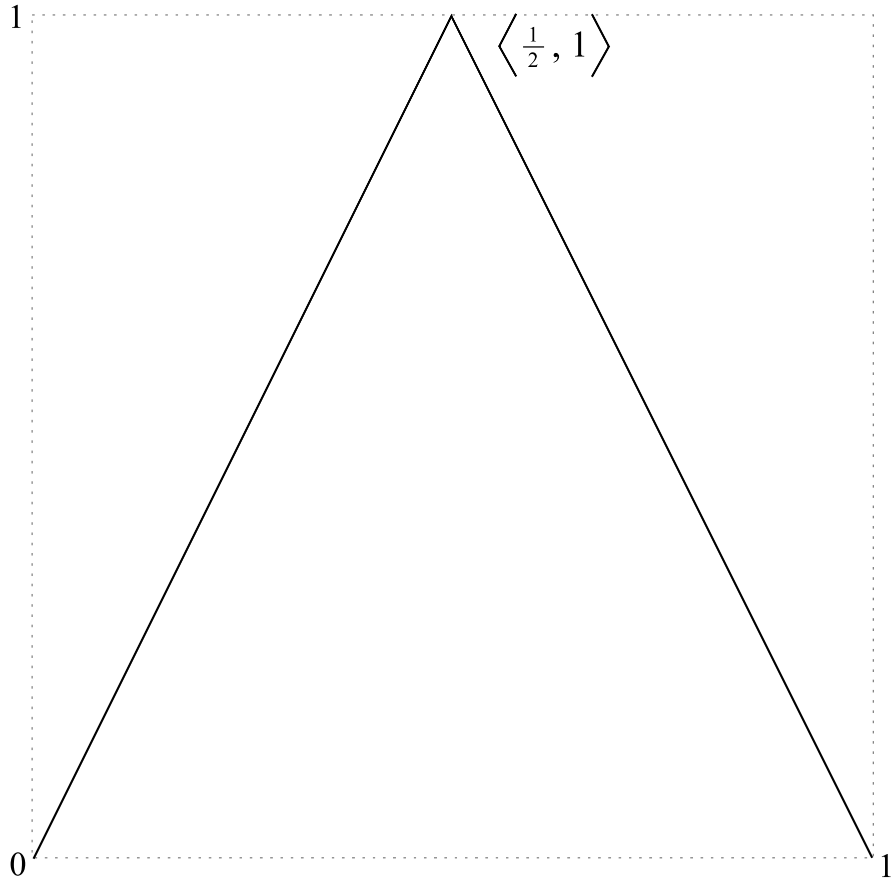

Figure 1.1: The graph of $f_{(\frac{1}{2},1)}$.

There have been numerous results concerning indecomposability for generalised inverse limits. One of the properties of classical inverse limits is that any subcontinuum of an inverse limit whose projection onto a factor space is the whole space for infinitely many factor spaces is the entire inverse limit. This is called the *full projection property*, and in general it does not hold for a generalised inverse limit; this is a topic of research in its own right.

Ingram gives a result regarding indecomposability in [In1], where a sufficient condition called the 'two pass condition' is given. This condition is similar to the condition of an indecomposable inverse sequence defined earlier in the section. The two pass condition, along with the full projection property, guarantees indecomposability of the resultant inverse limit. If $f : [0,1] \to 2^{[0,1]}$ is upper semicontinuous, f is said to satisfy the *two-pass condition* provided there exist two mutually exclusive connected open subsets U and V of $[0,1]$ such that $f|_U$ and $f|_V$ are continuous functions and $\overline{f(U)} = \overline{f(V)} = [0,1]$.

Theorem 1.2.10 (Ingram). *If for each $i \geq 0$ $X_i = [0,1]$, and for each $i \geq 1$, $f_i : [0,1] \to 2^{[0,1]}$ is upper semicontinuous and satisfies the two-pass condition, and $\varprojlim(X_i, f_i)$ has the full projection property, then $\varprojlim(X_i, f_i)$ is indecomposable.*

As can be seen, one premise of the theorem is that $\varprojlim(X_i, f_i)$ has the full projection property. It remains open as to what conditions on the bonding functions will guarantee the full projection property.

Theorem 1.2.10 has been generalised by Varagona [Va], and then further again by Kelly and Meddaugh [KM]. An example of an upper semicontinuous set valued function that gives an indecomposable inverse limit is shown in Figure 1.2. It contains a Cantor set of vertical lines, with endpoints of the Cantor set joined by lines whose inverses are functions. Further details can be found in [KM].

This represents one of the most complicated classes of bonding functions whose inverse limits are guaranteed to be indecomposable. A full characterisation of indecomposability in generalised inverse limits still remains some way away.

Inverse limits (in the classical sense at least) derive from the category theory notion of an inverse limit. For a good introduction to category theory see [M]. Banič and Sovič investigated generalised inverse limits in terms of category theory [BS]. Although Hausdorff continua (as objects) and upper semicontinuous set valued functions (as arrows) form a category, generalised inverse limits as defined in Definition 1.1.8 are not always inverse limits in this category. The authors of [BS] introduced the notion of a *weak inverse*

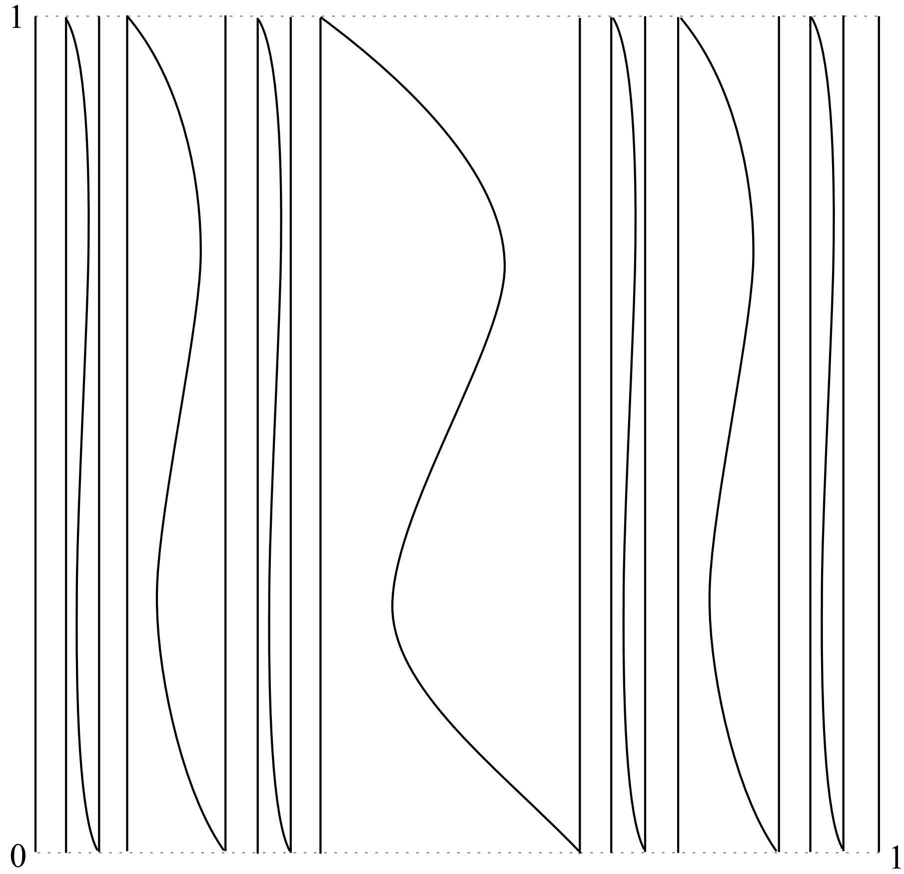

Figure 1.2: A bonding function giving an indecomposable inverse limit.

limit, and showed that generalised inverse limits are weak inverse limits in this category.

One of the reasons for this work in category theory was to study generalised inverse limits that were indexed over arbitrary directed sets. Some of the basic theorems mentioned earlier in this chapter have been proved for directed set indices in [IM2]. To this point in time, the majority of results regarding generalised inverse limits have been done indexing over the positive integers. An exception to this is the work of Vernon in [Ve], where indexing over the integers in their entirety is investigated. One of the advantages of indexing over the integers is that the shift map on the inverse limit space is a homeomorphism.

This fact is exploited by Vernon in his proof that if a single bonding function is used, and a generalised inverse limit is a finite graph, then it contains no loops. This is a generalisation of Theorem 1.2.12 by Illanes below.

Indexing over the integers allows for continua to be constructed that cannot be constructed when indexing over only the positive integers. For example, Vernon shows how to construct a two-cell (a space homeomorphic to a unit disc) using a single bonding function where the factor spaces are intervals. The graph of the bonding function is shown below in Figure 1.3.

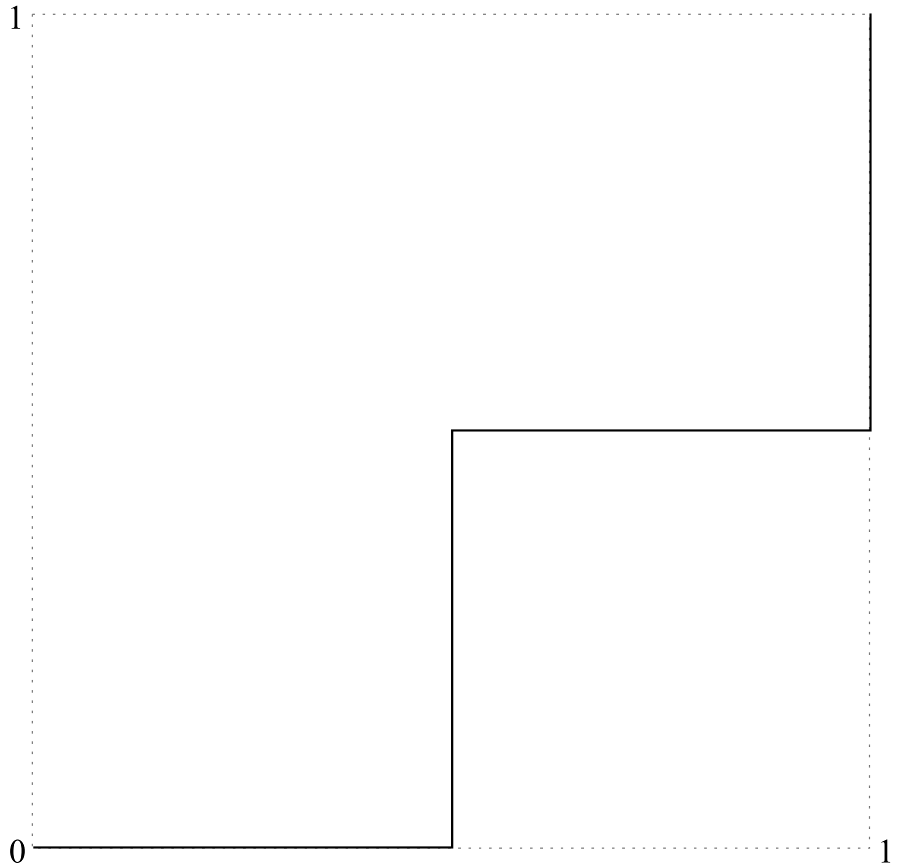

Figure 1.3: A bonding function giving an inverse limit that is a two-cell, when indexed over $\mathbb{Z}$.

Nall has shown in [Nal1] that this is not possible when indexing by $\mathbb{N}$. It is not known whether it is possible when indexing over the integers to construct an n cell for $n > 2$. The method that Vernon uses cannot be generalised to produce such an n cell.

There has been research into what spaces can be obtained as the inverse limit of a single bonding function. To date, this work has focussed on the factor spaces being intervals. Nall showed in [Nal3] that the only finite graph that can be obtained as a generalised inverse limit of a single bonding function over intervals is an arc.

Theorem 1.2.11 (Nall). *If $f : [0,1] \to 2^{[0,1]}$ is a surjective upper semicontinuous function such that $\varprojlim f$ is a finite graph, then $\varprojlim f$ is an arc.*

Illanes showed that a circle cannot be a generalised inverse limit of a single bonding function over intervals in [Il].

Theorem 1.2.12 (Illanes). *There is no upper semicontinuous set valued function $f : [0,1] \to 2^{[0,1]}$ such that $\varprojlim f$ is a simple closed curve.*

These results were both replicated for the case when indexing over the integers is used by Vernon [Ve].

In classical inverse limits, if the factor space is an arc, then every finite approximation $\mathcal{G}(f_1, \ldots, f_n)$ for each $n \geq 1$ will be an arc, so these finite approximations are not very interesting to study. With generalised inverse limits however, a finite approximation can be much more interesting. Even if the factor spaces are arcs, the first bonding function could have a graph that is, for example, a pseudo-arc. If we allow different bonding functions, then any space that can be produced by a finite number of bonding functions, ie any $\mathcal{G}(f_1, \ldots, f_n)$, can be produced as an inverse limit by extending the sequence of functions to include f_m to be the identity function for $m > n$.

Because it seems somewhat unnecessary to have to include these identity functions, we often simply refer to the Mahavier product $\mathcal{G}(f_1, \ldots, f_n)$.

As can be seen from the selection of results given above, this area of study is very new (the first mention of generalised inverse limits was in 2004) To date there has been much research into a number of areas. The results so far indicate that there is a rich structure inherent in generalised inverse limits and there is still a long way to go before the structure is fully explained.

Chapter 2

Generalised Tent Maps: $K_{(0,1)}$

Recall from Definition 1.2.8 that a tent map is a continuous function $f_{(a,b)} : [0,1] \to [0,1]$ with a graph consisting of two lines: one from $\langle 0,0\rangle$ to $\langle a,b\rangle$, and another from $\langle a,b\rangle$ to $\langle 1,0\rangle$. This definition can be extended to allow for set valued functions, which we call *generalised tent maps* (defined in the next section).

This chapter concentrates on a specific generalised inverse limit known as $K_{(0,1)}$, the inverse limit of a generalised tent map. The inverse limits of tent maps have been the subject of much research. Tent maps are very simple functions, yet inverse limits of tent maps can be very complex. In Chapter 1, we introduced the buckethandle continuum, which is homeomorphic to the inverse limit of the tent map $f_{(\frac{1}{2},1)}$, shown in Figure 1.1. We will now give a construction of the buckethandle continuum.

Let $n \geq 1$. If $S \subset \mathbb{R}^n$ is a set of points in $\mathbb{R}^n$, $a \in \mathbb{R}$, and $\mathbf{b} \in \mathbb{R}^n$, then by $a \cdot S + \mathbf{b}$ we mean the set

$$\{\mathbf{x} \in \mathbb{R}^n : \text{ there exists } \mathbf{y} \in S \text{ such that } \mathbf{x} = a \cdot \mathbf{y} + \mathbf{b}\}.$$

Let $C_0 = [0,1]$, and for all $i \geq 1$, let

$$C_i = \frac{1}{3} \cdot C_{n-1} \cup (\frac{1}{3} \cdot C_{n-1} + \frac{2}{3}).$$

Note that $\bigcap_{i \in \mathbb{N}} C_i = C$, where C is the Cantor middle thirds set.

Now, define:

$$D_0 = \left\{ \mathbf{x} = \langle x_1, x_2 \rangle \in \mathbb{R}^2 : \left\| \mathbf{x} - \langle \frac{1}{2}, 0 \rangle \right\| \le \frac{1}{2}, x_2 \ge 0 \right\},$$

and for $n \ge 1$, define

$$D_n = \left\{ \mathbf{x} = \langle x_1, x_2 \rangle \in \mathbb{R}^2 : \left\| \mathbf{x} - \langle \frac{1}{2}, 0 \rangle \right\| \in \frac{1}{3} \cdot C_{n-1}, x_2 \ge 0 \right\},$$

so D_n is a collection of half annuli with thicknesses related to the sets C_{n-1}. For $n \ge 0$, define

$$\begin{aligned} D_n^* = \{ &\mathbf{x} = \langle x_1, x_2 \rangle \in \mathbb{R}^2 : \\ &\text{there exists } \mathbf{y} = \langle y_1, y_2 \rangle \in D_n \text{ such that } y_1 = x_1 \text{ and } x_2 = -y_2 \}, \end{aligned}$$

so D_n^* is the set D_n reflected about the x-axis.

Finally, let $B_0 = D_0$, and for $n \ge 1$, define

$$B_n = D_n \cup \bigcup_{1 \le i \le n} \left(\frac{1}{3^i} \cdot D_{n-i}^* + \left(\frac{5}{2 \cdot 3^i} - \frac{1}{2} \right) \right).$$

We can now define the buckethandle continuum B to be the intersection of all the sets B_n, so

$$B = \bigcap_{n \in \mathbb{N}} B_n.$$

The first few sets B_n are shown in Figure 2.1

The reader can appreciate how the use of an inverse limit with a simple bonding function (in this case $f_{(\frac{1}{2},1)}$) to construct an indecomposable continuum is much simpler than the construction of the buckethandle above.

The problem of the classification of the inverse limits of tent maps is a highly non-trivial, open problem. Much work has been undertaken, and there are many partial results. For example in the article [BCMM], the authors describe curves on $[0,1]^2$ such that if two tent maps have their peaks on the same curve, their respective inverse limits are homeomorphic, and their inverse limits are not homeomorphic if the peaks lie on different curves. Another

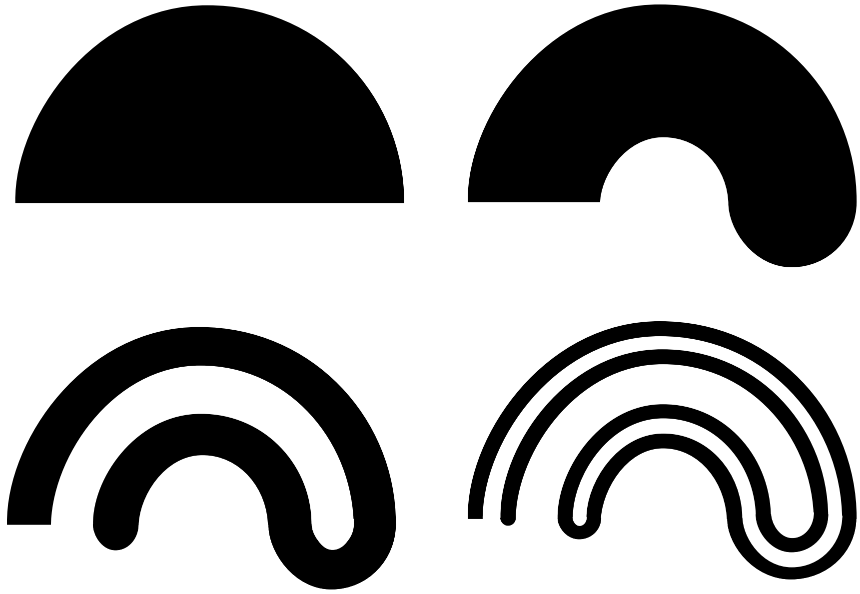

Figure 2.1: The sets B_0, B_1, B_2, and B_3 in the construction of the buckethandle continuum.

major result is found in [BBS], where the Ingram conjecture (Theorem 1.2.9) is proven. The full classification however, is still incomplete.

With the conception of generalised inverse limits, it is natural to extend the problem of the classification of inverse limits of tent maps to the classification of generalised inverse limits of generalised tent maps.

Including these 'extra' generalised inverse limits in this classification makes the problem more difficult, purely due to there being more cases to consider. Fortunately, there are not as many cases as there might appear. All but one of these generalised tent maps that cannot be expressed as continuous functions have inverse limits that are easy to describe. In [BCMM], Banič et al show that $K_{(0,0)}$ and $K_{(1,b)}$ for $b \in [0,1)$ are a single point (the point $(0,0,0,\dots)$), $K_{(0,b)}$ (for $b \in (0,1)$) is homeomorphic to an arc, and $K_{(1,1)}$ is homeomorphic to a harmonic fan (the notation $K_{(a,b)}$ will be defined in Section 2.1).

This leaves only one additional generalised tent map that is not a continuous function, $f_{(0,1)}$. The function $f_{(0,1)}$ that has $K_{(0,1)}$ as an inverse limit is shown below.

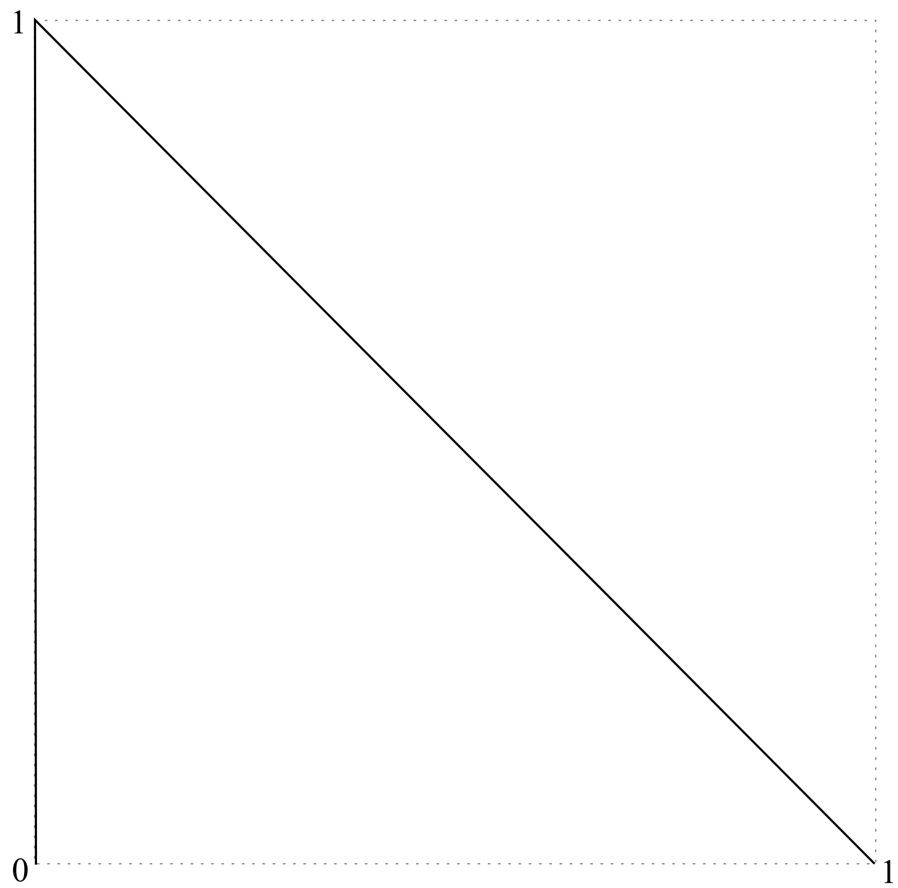

Figure 2.2: The function $f_{(0,1)}$ that has $K_{(0,1)}$ as its inverse limit.

Some work has been done in understanding the structure of $K_{(0,1)}$. In [BCMM] it is shown that $K_{(0,1)}$ contains harmonic fans and $\sin\frac{1}{x}$ continua, and that $K_{(0,1)}$ is one dimensional. In [CR] it is shown that $K_{(0,1)}$ has trivial shape and is thus tree-like. In his book [In2], Ingram gives a basic model for the structure of $K_{(0,1)}$. In this he shows, in addition to what is already mentioned, that $K_{(0,1)}$ is nonplanar, and contains many mutually exclusive n-ods for each positive integer n.

In this chapter we will describe the topological properties of $K_{(0,1)}$ in more detail. In Section 2.1, we will cover preliminary results that will be required for later sections, including the structure of a typical point in $K_{(0,1)}$. In Section 2.2 we prove some of the topological properties of $K_{(0,1)}$. In Section 2.3 we show $K_{(0,1)}$ has the fixed point property. In Section 2.4, we describe an embedding of $K_{(0,1)}$ into $\mathbb{R}^3$.

2.1 The Basic Structure of $K_{(0,1)}$

In this section we develop some notation and explore the basic structure of $K_{(0,1)}$. Tent maps were defined in Chapter 1, Definition 1.2.8. Here we extend this definition to generalised tent maps.

Definition 2.1.1. A *generalised tent map* with a peak at $\langle a, b\rangle$, where $a \in [0,1]$ and $b \in [0,1]$ is a function $f_{(a,b)} : [0,1] \to 2^{[0,1]}$ that has a graph consisting of two straight lines. One from the point $\langle 0,0\rangle$ to the point $\langle a,b\rangle$, and another from the point $\langle a,b\rangle$ to the point $\langle 1,0\rangle$.

If $f_{(a,b)}$ is a generalised tent map, then we denote $\varprojlim f_{(a,b)}$ by $K_{(a,b)}$.

Note that a generalised tent map is an upper semicontinuous set valued function, so $K_{(a,b)}$ is well defined.

The function we are interested to study in this chapter is $f_{(0,1)}$, and more specifically its resulting inverse limit $K_{(0,1)} = \varprojlim f_{(0,1)}$. First we look at what a typical point in $K_{(0,1)}$ looks like.

Suppose that $\mathbf{x} = (x_0, x_1, x_2, \dots) \in K_{(0,1)}$. Then as $K_{(0,1)}$ is surjective, $x_0 \in [0,1]$. Now, to find a suitable x_1, we need to find what values in $[0,1]$ map to x_0. It is clear from the graph that only 0 and $(1 - x_0)$ map to x_0. So there are points in the inverse limit that begin with $(x_0, (1-x_0), \dots)$ and $(x_0, 0, \dots)$. Now consider what x_2 can be. If $x_1 = (1 - x_0)$, then we have the same choice as before, either $x_2 = 1 - (1 - x_0) = x_0$ (in which case we are back where we started), or $x_2 = 0$. If $x_1 = 0$, then we have two choices, either $x_2 = 0$

(and we are back where we started), or $x_2 = 1$. If $x_2 = 1$, then there is only one choice, it must be that $x_3 = 0$. In other words, if $x_i = a$, the choice is always between 0 and $(1 - a)$.

We have now fully described the structure of what the coordinates of a point can look like, so a typical point can be summarised as follows:

- The sequence of coordinates begins with $a \in [0, 1]$, and is followed by a sequence of alternating $1 - a$ and a (note this following sequence may be of length zero).
- If the alternating sequence ends after finitely many terms, say after x_n, then $x_{n+1} = 0$ (unless $n = 0$ and $x_0 = 0$).
- After the first term of 0, each subsequent term is either 0 or 1, but there is also the condition that a 1 is always followed by a 0.

In order to explore the structure of $K_{(0,1)}$ further, we will define a certain class of points that will be particularly useful. The set of *vertices* of $K_{(0,1)}$ (which we denote by V) is the set of all points that have coordinates consisting of only 0s and 1s. This set V is a Cantor set, as mentioned in [BCMM].

The set V is totally disconnected, and from Theorem 1.2.3 we know that $K_{(0,1)}$ is connected [IM, Theorem 4.7], so the other points in $K_{(0,1)}$ must connect the vertices. We saw earlier that a point in $K_{(0,1)}$ consists of two sections. It starts with an alternating sequence of a and $1 - a$ for $a \in [0, 1]$, and then (if at all) there is a 0, followed by a sequence of 1s and 0s (where a 1 is also followed by a 0).

Definition 2.1.2. Two vertices $\mathbf{a}$ and $\mathbf{b} \in K_{(0,1)}$ are *directly connected* if there exists a path in $K_{(0,1)}$ that has $\mathbf{a}$ and $\mathbf{b}$ as end points, and the path does not contain any other vertices.

If either $\mathbf{a}$ or $\mathbf{b}$ (or both) is not a vertex, $\mathbf{a}$ and $\mathbf{b}$ are directly connected if there exists a path in $K_{(0,1)}$ that has $\mathbf{a}$ and $\mathbf{b}$ as end points, and the path does not contain any other vertices. If a vertex is directly connected to only one other vertex, this is an *end vertex*.

Note that any point $\mathbf{a} = (a_0, a_1, a_2, \dots) \in K_{(0,1)}$ that has $a_0 = a_1 = 0$ will be an end vertex.

Above we mentioned that two vertices, $\mathbf{a} = (a_0, a_1, a_2, \dots)$ and $\mathbf{b} = (b_0, b_1, b_2, \dots)$ are connected if there exists $n \in \mathbb{N} \cup \{\infty\}$ such that:

- $a_m \neq b_m$ for $m < n$, and
- $a_k = b_k$ for $k \geq n$.

It is clear that if this property holds the vertices will be directly connected. This statement is also true in the other direction.

Lemma 2.1.3. *Two distinct vertices* $\mathbf{a}$ *and* $\mathbf{b} \in K_{(0,1)}$ *are directly connected if and only if there exists* $n \in \mathbb{N} \cup \{\infty\}$ *such that:*

- $a_m \neq b_m$ *for* $m < n$, *and*
- $a_k = b_k$ *for* $k \geq n$.

Proof. Suppose $\mathbf{a}$ and $\mathbf{b}$ are directly connected, and $\mathbf{a} \neq \mathbf{b}$. If $a_m \neq b_m$ for all $m \in \mathbb{N}$, then the conditions are satisfied ($n = \infty$).

Suppose $a_m = b_m$, m is minimal with respect to this property, and $m \geq 1$. Then it follows that $a_m = b_m = 0$, since if $a_m = b_m = 1$ then either $a_{m-1} = 1$ or $b_{m-1} = 1$, so one sequence will have two consecutive 1s. As m is minimal, this means either $a_m = 1$ or $b_m = 1$. Without loss of generality, suppose $b_m = 1$, so $a_m = 0$. Now suppose $a_n \neq b_n$ for some $n > m$. Then for any path connecting $\mathbf{a}$ and $\mathbf{b}$ will need to contain points $\mathbf{p}^\alpha = (p_0^\alpha, p_1^\alpha, p_2^\alpha \dots) \in K_{(0,1)}$ for every $\alpha \in (0,1)$ such that $p_n^\alpha = \alpha$. Then as a $\mathbf{p}_\alpha \in K_{(0,1)}$, for every such p^α, we have $p_{n-1}^\alpha = 1 - \alpha$, $p_{n-2}^\alpha = \alpha$, etc. Also, if there is a path between $\mathbf{a}$ and $\mathbf{b}$ that does not contain any other vertices, every point in the path must be either $\mathbf{a}$, $\mathbf{b}$, or $\mathbf{p}^\alpha$ for some $\alpha \in (0,1)$.

Then if there is a path in $K_{(0,1)}$ between $\mathbf{a}$ and $\mathbf{b}$, the points $\mathbf{p}^\alpha$ must converge coordinate-wise to $\mathbf{a}$. But as $a_{m-1} = a_m = 0$, if a sequence of points p^α_{m-1} converges to 0, p^α_m converges to 1.

Therefore we conclude such a path cannot exist, so $\mathbf{a}$ and $\mathbf{b}$ are not directly connected.

Finally, suppose $a_0 = b_0$. Then for the same reason above, $a_n = b_n$ for all $n \geq 1$. This means that $\mathbf{a} = \mathbf{b}$, contradicting $\mathbf{a}$ and $\mathbf{b}$ being distinct. This finishes the proof of the forward direction.

For the reverse direction, suppose there exist distinct vertices $\mathbf{a}$ and $\mathbf{b} \in K_{(0,1)}$, and $n \in \mathbb{N} \cup \{\infty\}$ such that $a_m \neq b_m$ for $m < n$ and $a_k = b_k$ for $k \geq n$. Then the path directly connecting $\mathbf{a}$ and $\mathbf{b}$ is given by $\gamma : [0,1] \to K_{(0,1)}$, defined as:

$$\gamma(x) = \begin{cases} (x, 1-x, x, \dots, 0, a_{n+1}, a_{n+2}, \dots) & \text{if } n < \infty \\ (x, 1-x, x, \dots) & \text{if } n = \infty. \end{cases}$$

Then either $\gamma(0) = \mathbf{a}$ and $\gamma(1) = \mathbf{b}$, or $\gamma(0) = \mathbf{b}$ and $\gamma_1 = \mathbf{a}$, meaning $\mathbf{a}$ and $\mathbf{b}$ are endpoints of γ, and furthermore, if $x \neq 0$ and $x \neq 1$, $\gamma(x)$ is not a vertex, so $\mathbf{a}$ and $\mathbf{b}$ are directly connected.

A figure showing how two vertices are directly connected is given in Figure 2.3.

□

Finally, we will introduce some notation that will be helpful in the next section.

Definition 2.1.4. If $\mathbf{a} = (a_0, a_1, a_2, \dots)$ is a vertex, we say $\mathbf{a}$ has a *01-sequence of length* n for some $n \in \mathbb{N}$ if $a_n = a_{n+1} = 0$, and $a_i \neq a_{i-1}$ for all $1 \leq i \leq n$. If $a_i \neq a_{i-1}$ for all $i \geq 1$, then we say $\mathbf{a}$ has a *01-sequence of length* ∞.

This means that if $\mathbf{a}$ has a 01-sequence of length n, then there is an alternating sequence of 0s and 1s at the start of $\mathbf{a}$, up to the nth coordinate.

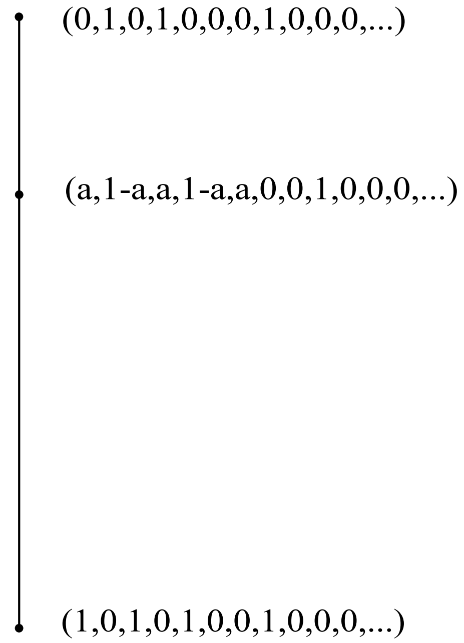

Figure 2.3: Two vertices in $K_{(0,1)}$ connected by an arc.

One important function that is applied to vertices if what is called a *forward move*, defined below. A forward move essentially increases the 01-sequence at the start of a vertex by one.

Definition 2.1.5. Let $\mathbf{a} = (a_0, a_1, a_2, \dots)$ be a vertex with a 01-sequence of length n. Then we define a *forward move* by the function $m : V \to V$, $m(\mathbf{a}) = (b_0, b_1, b_2, \dots)$, where $b_i \neq a_i$ for $i \leq n$, and $b_i = a_i$ for $i > n$.

Note that $m(\mathbf{a})$ is directly connected to $\mathbf{a}$ by Lemma 2.1.3. Also note that m is a well defined function, but it is not injective, so m^{-1} is not well defined. We will however, use the term *backwards move* to denote the reverse of a forwards move (ie given a vertex $\mathbf{a}$, a backwards move on $\mathbf{a}$ gives a vertex $\mathbf{b}$ such that $m(\mathbf{b}) = \mathbf{a}$). It should be remembered that a backwards move is not a well defined function, for example $m((0,1,0,0,0,0,\dots)) = (1,0,1,0,0,0,\dots) =$

$m((0,0,1,0,0,0,\dots))$.

There is another important subset of $K_{(0,1)}$ that is helpful in understanding the structure. That is the arc with endpoints $(0,1,0,\dots)$ and $(1,0,1,\dots)$. We call this the *limit line*, abbreviated *ll*.

Lemma 2.1.6. *Every point in $K_{(0,1)}$ is directly connected to a vertex.*

Proof. Let $\mathbf{a} = (a_0, a_1, a_2, \dots) \in K_{(0,1)}$, and $\mathbf{a}$ is not a vertex. Then the coordinates of $\mathbf{a}$ are of the form $a_0, 1-a_0, a_0$ for length n, where $n \in (\mathbb{N} \setminus \{0\}) \cup \{\infty\}$. Then $\mathbf{a}$ will be directly connected to the vertex $\mathbf{v} = (v_0, v_1, v_2, \dots)$, where

$$\begin{cases} v_0 = 1, & \\ v_i = 1 - v_{i-1} & \text{if } i \leq n, \text{ and} \\ v_i = a_i & \text{if } i > n. \end{cases}$$

This path $\gamma : [0,1] \to K_{(0,1)}$ is defined as $\gamma(x) = (p_0, p_1, p_2)$, where

$$\begin{cases} p_0 = x \cdot (1 - a_0), & \\ p_i = 1 - v_{i-1} & \text{if } i \leq n, \text{ and} \\ p_i = a_i & \text{if } i > n. \end{cases}$$

If $\mathbf{a}$ is a vertex, then $\mathbf{a}$ is directly connected to the vertex obtained under a forward move.

□

We end this section with the model of $K_{(0,1)}$ that was given by Ingram in [In2]. Some notation has been changed slightly to align with the notation in this chapter, but the description is largely taken from the text. Ingram begins by creating subsets of $K_{(0,1)}$ defined as $B_n \coloneqq \{\mathbf{x} \in K_{(0,1)} : x_{n+1} = 0\}$ for each $n \geq 0$. Each of these sets B_n will be the product of a Cantor set with an arc. $\mathcal{B}_n$ denotes the collection of arcs in B_n.

We then partition the vertices V into sets $C_0 \coloneqq \{\mathbf{x} \in V : x_0 = 0\}$ and $C_1 \coloneqq \{\mathbf{x} \in V : x_0 = 1\}$. Then for each $n \geq 1$, let $\mathbf{p_n} \in \{0,1\}^n$ be a string of n

symbols starting with 0 and alternating between 0 and 1, and similarly define $\mathbf{q_n} \in \{0,1\}^n$ to be a string of n symbols starting with 1 and alternating between 0 and 1.

Now define for each $n \geq 0$ the sets $D_n := \{\mathbf{p_n}\} \times C_0$ and $E_n := \{\mathbf{q_n}\} \times C_0$. Also, let $\mathbf{p} = (0,1,0,\dots)$ and $\mathbf{q} = (1,0,1,\dots)$. Then $C_0 = D_1 \cup D_3 \cup D_5 \cup \cdots \cup \{\mathbf{p}\}$, and $C_1 = E_2 \cup E_4 \cup E_6 \cup \cdots \cup \{\mathbf{q}\}$.

Note that for each positive odd integer n, each element of $\mathcal{B}_n$ is an arc directly connecting a point in D_n with a point in $E_n = E_{n+1} \cup E_{n+2}$, and if n is even, each element of $\mathcal{B}_n$ is an arc directly connecting a point in E_n with a point in $D_n = D_{n+1} \cup D_{n+2}$. Also, for $n \in \mathbb{N}$ and $\mathbf{x} \in D_n \cup E_n$, $\mathbf{x}$ is the endpoint for some arc in $\mathcal{B}_n$.

We conclude the construction by picking two skew lines in $\mathbb{R}^3$ and embedding C_0 in one and C_1 in the other. As noted above, $C_0 = D_1 \cup D_3 \cup D_5 \cup \cdots \cup \{\mathbf{p}\}$, and $C_1 = E_2 \cup E_4 \cup E_6 \cup \cdots \cup \mathbf{q}$, so from each point in D_1, directly connect it to an appropriate point in E_1, then for every point in E_2, directly connect it to an appropriate point in D_2, repeating this for all $n \in \mathbb{N}$. Finally, connect $\mathbf{p}$ and $\mathbf{q}$ with the limit line arc.

Figure 2.4 gives a schematic of Ingram's model.

2.2 The Topology of $K_{(0,1)}$

Now that we know the basic structure of $K_{(0,1)}$, we can look at some of its topological properties. We present them here as a series of propositions.

As mentioned earlier, it is easy to see using Theorem 1.2.3 that $K_{(0,1)}$ is a continuum. In Section 2.1, we showed how vertices in $K_{(0,1)}$ are connected by paths: direct connections. By extending this, we will be able to find the path components of $K_{(0,1)}$.

Definition 2.2.1. Let $V_L = \{\mathbf{x} \in V : \text{ there exists } n \in \mathbb{N} \text{ such that } x_{i+1} = 1 - x_i \text{ for all } i \geq n\}$.

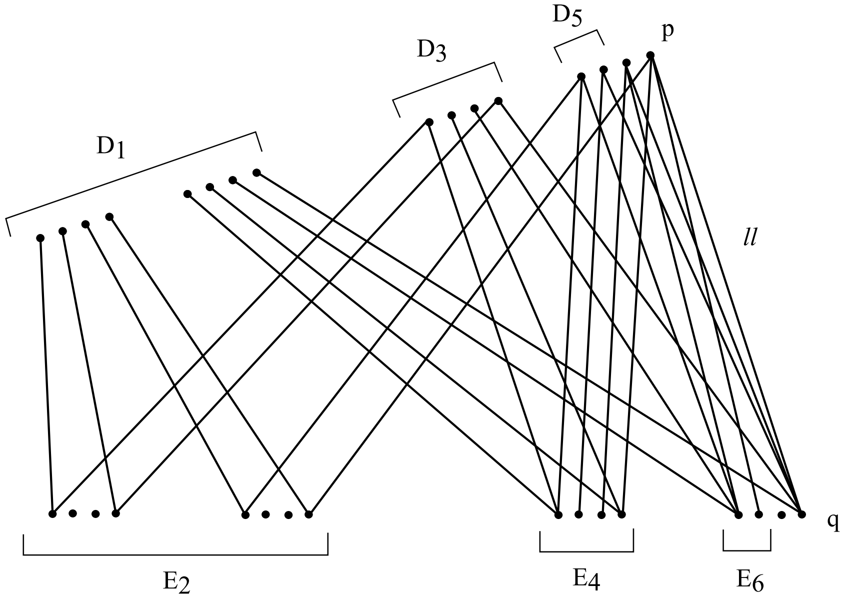

Figure 2.4: A schematic of Ingram's model of $K_{(0,1)}$.

Given a vertex $\mathbf{a} = (a_0, a_1, a_2, \dots)$, if $\mathbf{a} \notin V_L$, we define

$$B_{\mathbf{a}} = \{\mathbf{x} = (x_0, x_1, x_2, \dots) \in K_{(0,1)} : \text{ there exists } \\ n \in \mathbb{N} \text{ such that } x_i = a_i \text{ for all } i \geq n\}.$$

If $\mathbf{a} \in V_L$, we define

$$B_{\mathbf{a}} = \{\mathbf{x} = (x_0, x_1, x_2, \dots) \in K_{(0,1)} : \\ \text{there exists } n \in \mathbb{N} \text{ such that } x_i = 1 - x_{i+1} \text{ for all } i \geq n\}.$$

Then it is easy to check that the relation $\sim$ on $K_{(0,1)}$ defined by $\mathbf{a} \sim \mathbf{b}$ if and only if $B_{\mathbf{a}} = B_{\mathbf{b}}$ is an equivalence relation. We call these equivalence classes the *branches* of $K_{(0,1)}$. There is clearly more than one branch, for example $(0, 0, 0, 0, \dots)$ and $(0, 1, 0, 1, 0, \dots)$ are in different branches. In fact, it is not hard to see there are uncountably many branches.

Lemma 2.2.2. *There are $\mathfrak{c}$ many branches, and each branch contains countably many vertices.*

Proof. First we will show that there are countably many vertices in each branch. Let $\mathbf{c}$ be a vertex, $B_{\mathbf{c}}$ its branch, and let $V_{\mathbf{c}}$ be the set of all vertices in $B_{\mathbf{c}}$. Now we define inductively $A_0 = \{\mathbf{a} = (a_0, a_1, a_2, \dots) \in V_{\mathbf{c}} : a_i = c_i \text{ for all } i > 0\}$, and for all other $n \in \mathbb{N}$, define $A_n = \{\mathbf{a} = (a_0, a_1, a_2, \dots) \in (V_{\mathbf{c}} \setminus \bigcup_{0 \le i < n} A_i) : a_i = c_i \text{ for all } i > n\}$. So we have partitioned $V_{\mathbf{c}}$ into sets defined by the last coordinate at which a vertex differs from $\mathbf{c}$. These sets are clearly disjoint, and $\bigcup_{i \in \mathbb{N}} A_i = V_{\mathbf{c}}$. Also note that each A_i is a finite set, as for each i, there are only finitely many possible combinations of points in the coordinates less than i. Therefore, $V_{\mathbf{c}}$ is a countable union of finite sets, so countable.

As V is a Cantor set, we know there are $\mathfrak{c}$ many vertices, and these are disjointly partitioned into branches, each branch containing only countably many vertices. Therefore, there must be $\mathfrak{c}$ many branches.

□

Lemma 2.2.3. *Two vertices* $\mathbf{a}$ *and* $\mathbf{b}$ *of* $K_{(0,1)}$ *are in* $B_{\mathbf{c}}$ *for some vertex* $\mathbf{c}$ *if and only if there is a finite chain of vertices* $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_q$ *such that* $\mathbf{v}_0 = \mathbf{a}$, $\mathbf{v}_q = \mathbf{b}$, *and for all* $1 \le i \le q$, $\mathbf{v}_i$ *is directly connected to* $\mathbf{v}_{i-1}$.

Proof. Suppose $\mathbf{a}$ and $\mathbf{b}$ are vertices in $B_{\mathbf{c}}$. Then there exists $n_1 \in \mathbb{N}$ such that for $i \ge n_1$ we have $a_i = c_i$, and also there exists $n_2 \in \mathbb{N}$ such that for $i \ge n_2$, $b_i = c_i$. Let $n = \max\{n_1, n_2\}$, then for all $i \ge n$, $a_i = b_i$. Suppose $\mathbf{a}$ has a 01-sequence of length m_1, and $\mathbf{b}$ has a 01-sequence of length m_2 let $k_1 = \min\{n, m_1\}$, and $k_2 = \min\{n, m_2\}$. Now, by applying a total of $n - k_1$ forward moves to $\mathbf{a}$, we have a finite sequence of vertices $\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_l$ where $\mathbf{p}_0 = \mathbf{a}$, $\mathbf{p}_l = \mathbf{d}$ (where $\mathbf{d}$ has a 01-sequence of length n, and $d_i = c_i$ for $i > n$), and $\mathbf{p}_i$ is directly connected to $\mathbf{p}_{i-1}$ for all $1 \le i \le l$. Similarly by applying $n - k_2$ forward moves to $\mathbf{b}$, we have a finite sequence of vertices $\mathbf{p}_l, \mathbf{p}_{l+1}, \dots, \mathbf{p}_j$ (where $j \ge l$), where $\mathbf{p}_l = \mathbf{d}$, $\mathbf{p}_j = \mathbf{b}$, and $\mathbf{p}_i$ is directly connected to $\mathbf{p}_{i-1}$ for all $l + 1 \le i \le j$. Then $\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_j$ is the sequence we require.

For the reverse direction note that by Lemma 2.1.3 if two vertices are directly connected, their coordinate sequences have the same tail. So any finite collection as described by $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_q$ will have the same tail, and hence are in the same branch $B_{\mathbf{c}}$.

□

The next lemma will help us to see what the path components of $K_{(0,1)}$ are.

Lemma 2.2.4. *Suppose* $\mathbf{a}$ *and* $\mathbf{b}$ *are vertices in* $K_{(0,1)}$. *Then there is a path* $\gamma : [0,1] \to K_{(0,1)}$ *between* $\mathbf{a}$ *and* $\mathbf{b}$ *if and only if there is a finite chain of vertices* $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_q$ *such that* $\mathbf{v}_0 = \mathbf{a}$, $\mathbf{v}_q = \mathbf{b}$, *and for all* $1 \le i \le q$, $\mathbf{v}_i$ *is directly connected to* $\mathbf{v}_{i-1}$.

Proof. Let $\gamma : [0,1] \to K_{(0,1)}$ be a path, let $A \subset K_{(0,1)}$ be the image of γ, and suppose A contains κ vertices for some infinite cardinal κ. Let V_0 be the set of all vertices with first coordinate 0 contained in A, and let V_1 be the set of all vertices with first coordinate 1 contained in A. Then either V_0 or V_1 is infinite. Suppose, without loss of generality, that V_0 is infinite. Then as $V_0 \subset K_{(0,1)}$ and $K_{(0,1)}$ is compact, V_0 has an accumulation point, call this $\mathbf{b}$. As A is compact, $\mathbf{b} \in A$, so $\mathbf{b} \in V_0$.

Let $(\mathbf{b}_i : i \in \mathbb{N})$ be a sequence of points in V_0 converging to $\mathbf{b}$. If we take the inverse image of each $\mathbf{b}_i$, and their limit point $\mathbf{b}$, we have a collection of points in $[0,1]$, $B = \{\gamma^{-1}(\mathbf{b}), \gamma^{-1}(\mathbf{b}_0), \gamma^{-1}(\mathbf{b}_1), \gamma^{-1}(\mathbf{b}_2), \dots\}$. Then as we have an infinite collection of points in a compact interval, they will have an accumulation point p. Choose a sequence $(c_i : i \in \mathbb{N})$ in B that converges to p. Then $(c_i : i \in \mathbb{N})$ is convergent, and hence $(\gamma(c_i) : i \in \mathbb{N})$ must be a subsequence of $(\mathbf{b}_i : i \in \mathbb{N})$, hence $\gamma(p) = \mathbf{b}$, so $\gamma(p)$ had first coordinate 0.

By Lemma 2.1.3, for any $i \in \mathbb{N}$, the vertices $\mathbf{b}_i$ and $\mathbf{b}_{i+1}$ cannot be directly connected since they both have first coordinate 0. Therefore, for each $i \in \mathbb{N}$, there exists a point $d_i \in [0,1]$ such that d_i lies between c_i and c_{i+1}, and $\gamma(d_i)$

has first coordinate 1. Then we have a sequence of points $(d_i : i \in \mathbb{N})$ that converges to p, and for each $i \in \mathbb{N}$, $\gamma(d_i)$ has first coordinate 1. Therefore $(\gamma(d_i) : i \in \mathbb{N})$ converges to a point in $K_{(0,1)}$ with first coordinate 1, hence $\gamma(p)$ must have first coordinate 1.

So we have a contradiction, and conclude the image of any path contains only finitely many vertices. That there is a chain with the necessary properties is clear from the existence of a path and the definition of a direct connection.

For the reverse direction, for $i \in \{1, \dots q\}$, let $\gamma_i : [0,1] \to K_{(0,1)}$ be a path between $\mathbf{v}_{i-1}$ and $\mathbf{v}_i$ that exists as they are directly connected. Then let $\gamma_i' : [0, \frac{1}{q}] \to K_{(0,1)}$ be defined as $\gamma_i'(x) = \gamma_i(q \cdot x)$. Then $\gamma : [0,1] \to K_{(0,1)}$ defined as

$$\gamma(x) = \begin{cases} \mathbf{a} & \text{if } x = 0 \\ \gamma_i(x - \frac{i-1}{q}) & \text{if } x \in (\frac{i-1}{q}, \frac{i}{q}] \end{cases}$$

is a path in $K_{(0,1)}$ from $\mathbf{a}$ to $\mathbf{b}$.

□

Corollary 2.2.5. *Two points* $\mathbf{a}$ *and* $\mathbf{b}$ *in* $K_{(0,1)}$ *are connected by a path if and only if they are in the same branch* $B_{\mathbf{c}}$ *for some* $\mathbf{c} \in V$.

Proof. If $\mathbf{a}$ and $\mathbf{b}$ are vertices, this follows from Lemma 2.2.3 and Lemma 2.2.4. If $\mathbf{a}$ or $\mathbf{b}$ is not a vertex, by Lemma (above) it will be directly connected to a vertex in the same branch, so again we can obtain a path.

□

This shows that the path components of $K_{(0,1)}$ are precisely the branches, $\{B_{\mathbf{v}} : \mathbf{v} \in V\}$. As there is more than one branch, we have:

Proposition 2.2.6. $K_{(0,1)}$ *is not path connected.*

In addition to being not path connected, we also have that $K_{(0,1)}$ is not locally connected anywhere.

Proposition 2.2.7. $K_{(0,1)}$ *is not locally connected anywhere.*

Proof. Let $\mathbf{x} = (x_0, x_1, x_2, \dots) \in K_{(0,1)}$. Let

$$U = \left(\left(\left(x_0 - \frac{1}{8}, x_0 + \frac{1}{8}\right) \cap [0,1]\right) \times [0,1]^{\omega}\right) \cap K_{(0,1)}$$

(the actual value $\frac{1}{8}$ used here and later in the proof has no real significance other than it needs to be less than $\frac{1}{4}$). Let $V \subset U$ be open, with $\mathbf{x} \in V$. We will show that V is disconnected. Let V_α be a basic open set in V that contains $\mathbf{x}$. Then as V_α must have as factors $[0,1]$ for infinitely many coordinates, we can conclude that there must exist $\mathbf{y} = (y_0, y_1, y_2, \dots) \in V_\alpha$ such that there is at least one occurrence of consecutive 0s in $\mathbf{y}$, and $\mathbf{y}$ has a different tail to $\mathbf{x}$. Suppose the first occurrence of consecutive 0s begins at coordinate y_n. Let $m > n+2$ be such that $y_m \neq x_m$ (such a point exists as $\mathbf{x}$ and $\mathbf{y}$ have different tails), and let $l^* = |y_m - x_m|$ and $l = \frac{l^*}{2}$. Let $d \in [0,1] \setminus ((x_0 - \frac{1}{8}, x_0 + \frac{1}{8}) \cup (1-(x_0+\frac{1}{8}), 1-(x_0-\frac{1}{8})))$. Now, let

$$\begin{aligned} A_U = ((\Pi_{i<n}((x_i - \tfrac{1}{8}, x_i + \tfrac{1}{8}) \cap [0,1])) \times [0,d) \times [0,d) \times \\ [0,1]^{m-(n+1)} \times ((y_i - l, y_i + l) \cap [0,1]) \times [0,1]^{\omega}) \cap K_{(0,1)}, \end{aligned}$$

and

$$\begin{aligned} A_U^* = ((\Pi_{i<n}([x_i - \tfrac{1}{8}, x_i + \tfrac{1}{8}] \cap [0,1])) \times [0,d] \times [0,d] \times \\ [0,1]^{m-(n+1)} \times ([y_i - l, y_i + l] \cap [0,1]) \times [0,1]^{\omega}) \cap K_{(0,1)}. \end{aligned}$$

Then $A_U \cap U$ is open in U, and $A_U^* \cap U$ is closed in U. $A_U^* \cap U = A_U$, as any point in $K_{(0,1)}$ that has an ith coordinate on the boundary of $(x_i - \frac{1}{8}, x_i + \frac{1}{8}) \cap [0,1]$ for $i < n$, or on the boundary of $[0,d)$ for the the nth and $(n+1)$th coordinate, or on the boundary of $(y_i - l, y_i + l) \cap [0,1]$ for the mth coordinate will not be in U.

Therefore, A_U is clopen , so if we let $B_U = U \setminus A_U$, then A_U is open and nonempty ($\mathbf{y} \in A_U$), B_U is open and nonempty ($\mathbf{x} \in B_U$), and $A_U \cup B_U = U$.

Then if we define $A = A_U \cap V$, $B = B_U \cap V$ then A and B are nonempty, open, and $A \cup B = V$, so V is disconnected.

$\square$

Earlier in this thesis, we encountered the buckethandle continuum, an indecomposable continuum. In addition to the buckethandle continuum being homeomorphic to $\varprojlim f_{(\frac{1}{2},1)}$, it is also true that the buckethandle continuum is homeomorphic to $\varprojlim f_{(a,1)}$ for $a \in (0,1)$. The next proposition shows that although there are tent maps $f_{(a,1)}$ with peaks arbitrarily close to that of $f_{(0,1)}$, $K_{(0,1)}$ is not homeomorphic to the buckethandle continuum, and it is not even indecomposable.

Proposition 2.2.8. *$K_{(0,1)}$ is decomposable.*

Proof. Consider the following decomposition. Let

$$D_1 = [0,1]^\omega \smallsetminus \left(\left[0, \frac{1}{4}\right) \times \left[0, \frac{1}{4}\right) \times \left[0, \frac{1}{4}\right) \times [0,1]^\omega\right).$$

Then D_1 is closed, so $D_1 \cap K_{(0,1)}$ is closed. We can then see that $D_1 \cap K_{(0,1)}$ includes all vertices in $K_{(0,1)}$ except vertices beginning with at least three 0s. Furthermore, we can then see that it will include all points in $K_{(0,1)}$ except those with first three coordinates $a, 0, 0$, where $a \in [0, \frac{1}{4})$. As these excluded points all occur at the ends of the path components, ie we have removed some of the end vertices and a portion of the arc that directly connects them to the next vertex, so what remains is homeomorphic to what we began with. So $D_1 \cap K_{(0,1)}$ is connected, hence it is a continuum, ie a proper subcontinuum of $K_{(0,1)}$.

Now, let

$$D_2 = [0,1]^\omega \smallsetminus \left(\left[0, \frac{1}{4}\right) \times \left[0, \frac{1}{4}\right) \times \left(\frac{3}{4}, 1\right] \times [0,1]^\omega\right),$$

then similarly to above $D_2 \cap K_{(0,1)}$ contains all vertices in $K_{(0,1)}$ except those that start with exactly two 0s, and then it will include all points in $K_{(0,1)}$ except those that start with $(a, 0, 1, \dots)$, where $a \in [0, \frac{1}{4})$. Then similarly to above, $D_2 \cap K_{(0,1)}$ is a proper subcontinuum of $K_{(0,1)}$, and we have $K_{(0,1)} = (D_1 \cap K_{(0,1)}) \cup (D_2 \cap K_{(0,1)})$, so $K_{(0,1)}$ is decomposable.

□

Definition 2.2.9. A *dendrite* is a locally connected continuum that contains no simple closed curves. A *finite dendrite* is a dendrite which can be written as the union of finitely many arcs, any two of which are either disjoint, or intersect at one of their endpoints. A finite dendrite is sometimes called a *tree*.

Definition 2.2.10. If $\mathbf{x} \in K_{(0,1)}$, the *degree* of $\mathbf{x}$ is the maximum number of distinct arcs whose pairwise intersection is $\mathbf{x}$.

As $K_{(0,1)}$ is not locally connected, it is not a dendrite. One could then ask the question "Does $K_{(0,1)}$ *contain* a universal dendrite?". Equivalently "Does $K_{(0,1)}$ contain every dendrite?" - such dendrites exist, for example Wazewski's universal dendrite [Nad, 10.37]. The answer to this question is no. $K_{(0,1)}$ has exactly two points of degree ω, $(0,1,0,1,0,\dots)$ and $(1,0,1,0,1,\dots)$. These are the only points of degree ω, since are the only vertices directly connected to infinitely many other vertices. To see this, suppose a vertex $\mathbf{x} = (x_0, x_1, x_2, \dots)$ is such that $\mathbf{x} \neq (0,1,0,1,\dots)$, and $\mathbf{x} \neq (1,0,1,0,\dots)$. Then $\mathbf{x}$ has a finite 01 sequence, which will be followed by a 00. Say the first 0 in the 00 happens at the nth coordinate. Note that from Lemma 2.1.3, two vertices are directly connected if and only if their initial coordinates are not equal. Then any point directly connected to $\mathbf{x}$ will only be able to differ in every coordinate in the first $n-1$ coordinates (or else there will be a 11), and only finitely many can do this.

If we concentrate instead on finite dendrites, we have the following proposition.

Proposition 2.2.11. $K_{(0,1)}$ *contains every finite dendrite.*

Proof. Let $\mathbf{a} = (0,0,0,0,\dots)$. After applying $4n+1$ forward moves to $\mathbf{a}$, we have $m^{2n+1}(\mathbf{a}) = \mathbf{b}$ is directly connected to at least n vertices. This is because after $4n+1$ forward moves, there will be at least $2n$ 1s in the 01 sequence of $\mathbf{b}$. Then, By Lemma 2.1.3, $\mathbf{b}$ will be be directly connected to at least $2n$ other vertices, each obtained by replacing a 1 in the 01 sequence of $\mathbf{b}$ by a 0, and then changing the coordinates before this to make a 01 sequence. By the same reasoning, the vertices obtained by replacing 1 by 0 for each 1 occurring between the $(n+1)$st occurrence of 1 through to the $2n$th occurrence of 1 will all be directly connected to at least n vertices.

So after $4n+1$ forward moves, we have (at least) n vertices that are directly connected to (at least) n vertices each. Let the point $m^{(4n+1)^n}(\mathbf{a}) = \mathbf{c}$, and let A be the union of the collection of all vertices obtained as backwards moves from $\mathbf{c}$, and the arcs directly connecting these vertices. Then A contains a continuum containing at least n vertices, each of which is directly connected to at least n vertices in A. Moreover A is locally connected (it is a finite portion of one branch). Then given any finite dendrite D with n vertices, we can embed D into A, and hence into $K_{(0,1)}$.

$\square$

Note that the proof of Proposition 2.2.11 shows that each branch actually contains every finite dendrite. This strengthens Ingram's result [In2, Example 2.1.5] of $K_{(0,1)}$ containing many mutually exclusive n-ods for each n (an n-od is is a collection of n arcs, all intersecting at a single point).

2.3 The Fixed Point Property in $K_{(0,1)}$

In this section we show $K_{(0,1)}$ has the fixed point property.

Definition 2.3.1. A continuum A has the *fixed point property* if for every continuous function $f : A \to A$, there is a point $x \in A$ such that $f(x) = x$.

Before proving that $K_{(0,1)}$ has the fixed point property, we first present some background on the fixed point property.

We begin with the definition of $\mathcal{P}$-like, where $\mathcal{P}$ is the collection of spaces that share a particular property, for example the collection of all arcs, or the collection of all trees.

Definition 2.3.2. Let X and Y be compact metric spaces and $f : X \to Y$. Then f is called an *ε-map* provided that f is continuous, and the diameter of $f^{-1}(f(x)) < \varepsilon$ for all $x \in X$.

Let $\mathcal{P}$ be a collection of compact metric spaces. Then X is said to be *$\mathcal{P}$-like* provided that for each $\varepsilon > 0$, there is an ε-map f_ε from X onto some member Y_ε of $\mathcal{P}$.

It is shown in Nadler [Nad, Corollary 12.30], that all arc-like continua have the fixed point property. In [CR, Example 2], it is shown that $K_{(0,1)}$ is tree-like. There are examples of tree-like continua that do not have the fixed point property, for example in [B] and [OR]. It is, however, shown in [Nad, Theorem 10.31] that every dendrite (and hence every tree) has the fixed point property (this result will be used repeatedly in the proof). These results indicate that $K_{(0,1)}$ having the fixed point property is not a trivial result.

We will call the path component that contains the limit line L. That is, L is the collection of all points in $K_{(0,1)}$ that have a $(0,1,0,1,\dots)$ tail (Given two sequences of values in $[0,1]$, $\mathbf{a} = (a_0, a_1, a_2, \dots)$ and $\mathbf{b} = (b_0, b_1, b_2, \dots)$, we say $\mathbf{a}$ has a $\mathbf{b}$-tail if there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_{N+n} = b_n$). Whereas all other path components look essentially the same, L has some different properties.

For any point $\mathbf{x} \in L$, either $\mathbf{x}$ is on the limit line, or after finitely many forward moves, $\mathbf{x}$ will be directly connected to a point in the limit line. We can think of L as being an arc (the limit line) that has (countably) infinitely many trees attached at each end.

For example, from the point $(0,1,0,1,0,1,\dots)$ we can obtain the point $(1,0,1,0,0,1,0,\dots)$ by a backward move. From there we can also (through backwards moves) obtain the vertices $(0,0,1,0,0,1,0,\dots)$, $(0,1,0,0,0,1,0,\dots)$,

$(1, 0, 0, 0, 0, 1, 0, \dots)$, and $(0, 0, 0, 0, 0, 1, 0, \dots)$. These vertices (along with the lines joining them) form one of the trees coming out of $(0, 1, 0, 1, 0, 1, \dots)$. This is illustrated in Figure 2.5.

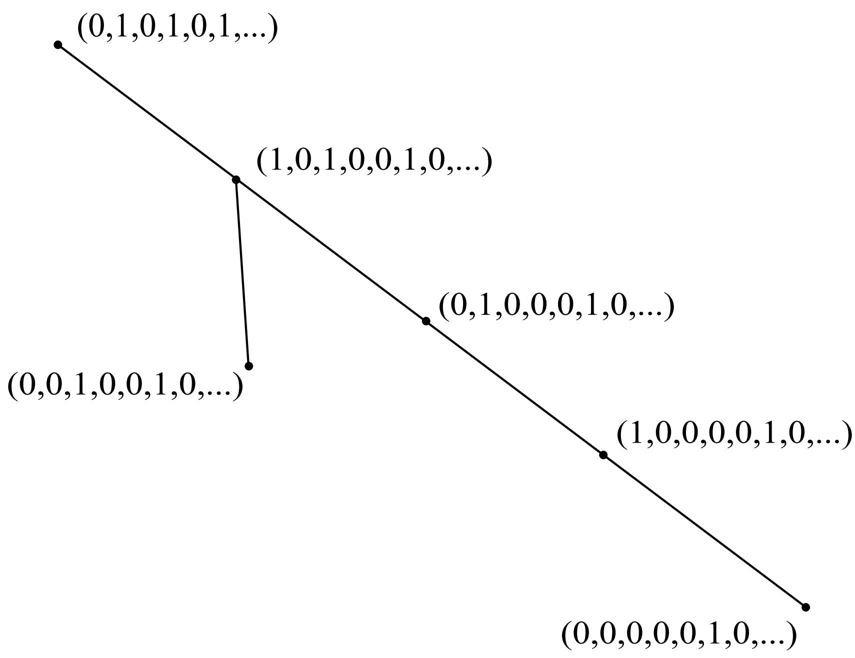

Figure 2.5: A tree in L.

The trees arrange to something similar to a 'harmonic fan' around the limit line (ll). To see this, note that the first line in a tree coming out of one of the endpoints of ll is closer to ll the greater the coordinate that the first backwards move originates from. These lines will then converge to the limit line as this coordinate becomes large.

Note that L is not compact, as we can create a sequence of points $(\mathbf{x}_i)_{i\in\mathbb{N}}$ in L, where $\mathbf{x}_i$ consists of i 0s followed by a 01 sequence. This sequence of points converges to the point $(0, 0, 0, 0, \dots)$ which is not in L.

Figure 2.6 is a schematic of L. The limit line is the centre line, and the trees are emerging from each end of the limit line, converging back to the line itself.

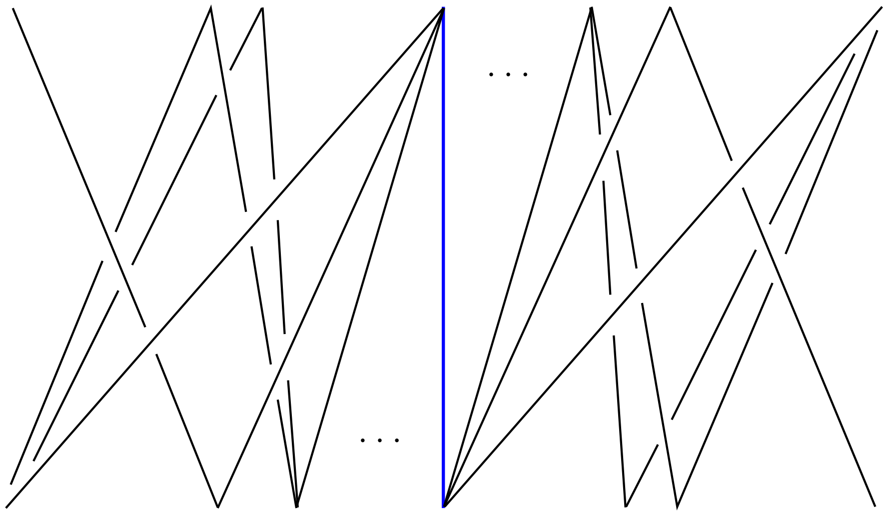

Figure 2.6: A schematic of L. The limit line is shown in blue.

We will proceed first to show that L has the fixed point property, and from there show that $K_{(0,1)}$ has the fixed point property.

The following well known theorem of Brouwer is used in the proof. In particular, we use the case in one dimension.

Theorem 2.3.3 (Brouwer [Br]). *Every compact convex set in $\mathbb{R}^n$ has the fixed point property.*

In the next lemma we use the notion of a ball around a set. In this context, given a set $A \subset K_{(0,1)}$ and $\varepsilon > 0$, we define the closed ball $\overline{B_\varepsilon(A)} = \{\mathbf{x} \in K_{(0,1)} : d(\mathbf{x}, \mathbf{y}) \leq \varepsilon$ for some $\mathbf{y} \in A\}$. Here the distance function on $K_{(0,1)}$ is the metric generated by the norm $||x|| = \sum_{i \geq 0} |\frac{x_i}{i^2}|$, where x_i is the ith coordinate in $\mathbf{x}$. This means that for $\mathbf{x} \in K_{(0,1)}$, $||\mathbf{x}|| < 2$.

The lemma also uses the term *continuous sequence of length m* in reference to a point $\mathbf{x} \in K_{(0,1)}$. This simply refers to an initial subsequence $(x_i)_{0 \leq i \leq m}$ of

$\mathbf{x}$ where for all $1 \le i \le m$ we have $x_i = 1 - x_{i-1}$. This is a generalisation of a 01 sequence.

Lemma 2.3.4. *For every $\varepsilon > 0$, there exists a compact set C in $K_{(0,1)}$ contained in $\overline{B_\varepsilon(ll)}$ that is connected, and a neighbourhood of ll.*

Proof. Let $\varepsilon > 0$. Let

$$C_n = \{\mathbf{x} \in K_{(0,1)} : \mathbf{x} \text{ has a continuous sequence of length at least } n\}.$$

We make two claims about C_n.

1. $C_n \subset \overline{B_\varepsilon(ll)}$ for some suitably large (finite) n, and
2. C_n is homeomorphic to $K_{(0,1)}$.

For the first claim, given $\varepsilon > 0$, note that there exists some $k \in \mathbb{N}$ such that $\sum_{i \ge k} \frac{1}{i^2} < \varepsilon$. Then given $\mathbf{x} \in C_k$, the first k coordinates of x are the same as some $\mathbf{y} \in ll$, so since for all $i > k$ we have $|x_i - y_i| \le 1$, $\|x - y\| \le \sum_{i=k}^{\infty} \frac{1}{i^2} < \varepsilon$. Hence $C_k \subset \overline{B_\varepsilon(ll)}$.

For the second claim, note that C_n restricted to all coordinates after n is homeomorphic to $K_{(0,1)}$, as it contains all possible points in $K_{(0,1)}$. Also, the string of coordinates before the nth coordinate is uniquely determined by the nth coordinate.

C_n is a neighbourhood of ll because it contains the open set

$$\left(\left[0, \tfrac{3}{4}\right) \times \left(\tfrac{1}{4}, 1\right] \times \left[0, \tfrac{3}{4}\right) \times \cdots \times \left[0, \tfrac{3}{4}\right) \times [0,1]^\omega\right)$$
$$\cup \left(\left(\tfrac{1}{4}, 1\right] \times \left[0, \tfrac{3}{4}\right) \times \left(\tfrac{1}{4}, 1\right] \times \cdots \times \left(\tfrac{1}{4}, 1\right] \times [0,1]^\omega\right)$$

(here the products are restricted for the first $n + 1$ coordinates).

So given $\varepsilon > 0$, we can simply define $C = C_n$ for some suitably large n.

□

The following is a slightly modified version of the Brouwer theorem in one dimension.

Lemma 2.3.5. *Suppose $f : K_{(0,1)} \to K_{(0,1)}$ is continuous, and $ll \subset f(ll)$. Then ll has a fixed point.*

Proof. First we make the claim that there exist $\mathbf{x}$ and $\mathbf{y} \in ll$ such that $f[\mathbf{x}, \mathbf{y}] = ll$ (here $[\mathbf{x}, \mathbf{y}]$ denotes the arc in ll between $\mathbf{x}$ and $\mathbf{y}$). To prove this, first note that ll is an arc, and if points $\alpha \in ll$ and $\beta \notin ll$ are path connected, the path connecting them contains one of the endpoints of ll. Call the endpoints of ll $\mathbf{w}$ and $\mathbf{v}$ (so $ll = [\mathbf{w}, \mathbf{v}]$). As $ll \subset f(ll)$, there exists $\mathbf{a}$ and $\mathbf{b} \in ll$ such that $f(\mathbf{a}) = \mathbf{w}$ and $f(\mathbf{b}) = \mathbf{v}$. Suppose that $\mathbf{a} < \mathbf{b}$ (here the ordering comes from the identification of ll with an arc). Then let

$$\mathbf{d} = \inf\{\mathbf{x} \in ll : \mathbf{x} > \mathbf{a} \text{ and } f(\mathbf{x}) = \mathbf{v}\}.$$

As f is continuous $\mathbf{d} > \mathbf{a}$. Now let

$$\mathbf{c} = \sup\{\mathbf{x} \in ll : \mathbf{x} < \mathbf{d} \text{ and } f(\mathbf{x}) = \mathbf{w}\}.$$

As f is continuous $\mathbf{c} < \mathbf{d}$.

Then $f(\mathbf{c}) = \mathbf{w}$, $f(\mathbf{d}) = \mathbf{v}$, so as f is continuous, by the intermediate value theorem, $f([\mathbf{c}, \mathbf{d}]) \supset ll$, but as there is no $\mathbf{z} \in (\mathbf{c}, \mathbf{d})$ such that $f(\mathbf{z}) = \mathbf{w}$ or $\mathbf{v}$, and these are the only points through which the image can 'exit' ll, we conclude $f([\mathbf{c}, \mathbf{d}]) = ll$. The proof works entirely similarly if $\mathbf{b} < \mathbf{a}$.

Now consider the function $f|_{[\mathbf{c},\mathbf{d}]} : [\mathbf{c}, \mathbf{d}] \to ll$. By identifying ll with $[0, 1]$, using the homeomorphism $h : ll \to [0, 1]$ defined by simply taking the first coordinate of a point in ll (it is easy to see this is a homeomorphism), we can define the function $g : [\mathbf{c}, \mathbf{d}] \to [0, 1]$ by $g(\mathbf{x}) = h(f(h^{-1}(\mathbf{x})))$. Then g is continuous, so by the intermediate value theorem the (continuous) function $g(\mathbf{x}) - \mathbf{x}$ has a 0, which corresponds to a fixed point for g, and hence a fixed point for f.

□

Lemma 2.3.6. *Let B be a branch, and suppose $B \neq L$. Then any continuum contained in B is a tree.*

Proof. Let A be a continuum contained in a branch B. We will show that A has only finitely many vertices. Suppose with a view to contradiction A has infinitely many vertices, these being the set $V^* := \{v_i : i \in \mathbb{N}\}$. We claim that as there are infinitely many vertices, there are vertices in V^* with arbitrarily long 01 sequences. Suppose there is a maximum length of a 01 sequence, say n, and let v_m have a 01 sequence of length n. Then v_m cannot be connected to any other vertices with a 01 sequence of length n. For v_m to do so would require at least one forward move and corresponding backward moves (with all vertices associated with these moves included in V^*), and this would mean there is a vertex in A with a 01 sequence of length $n+1$. So the only vertices included in A come from backward moves from v_m, and there are only finitely many of these. So we conclude that if there are infinitely many vertices, there are arbitrarily long 01 sequences in the set of these vertices.

Now consider the open cover of A by the following sets. For $i \geq 0$, let U_i be defined as $\Pi_{j\geq 0} X_j$, where:

$$X_j = \begin{cases} [0, \frac{3}{4}) & \text{if } j = i \text{ or } j = i+1 \text{ or } j = i-k \text{ for } k \text{ even} \\ (\frac{1}{4}, 1] & \text{if } j = i-k \text{ for } k \text{ odd} \\ [0,1] & \text{if } j > i+1 \end{cases}$$

Then a point in $K_{(0,1)} \setminus ll$ will be in some U_i if it has a 01 sequence of length i, so $\{U_i : i \geq 0\}$ covers A, but as there are arbitrarily long 01 sequences of vertices in A, there is no finite subcover of $\{U_i : i \geq 0\}$ for A, so A is not compact.

Therefore, we conclude that there must be only finitely many vertices in A, so by [Nad, Theorem 9.10], we have that A is a tree.

□

Lemma 2.3.7. *If A is a continuum in $K_{(0,1)}$ that does not contain any points in the limit line, then A is contained in a single branch.*

Proof. As A is compact, using a similar argument to Lemma 2.3.6, we have that there exists some $n \in \mathbb{N}$ such that A only contains vertices with 01 sequences of length less than n (if not we can construct the same open cover and show that A is not compact). Therefore, for any point $\mathbf{x} \in A$, the value of x_m for $m > n$ is either 0 or 1.

Suppose that A contains points from two branches, $\mathbf{a} \in B_1$ and $\mathbf{b} \in B_2$. Then as $\mathbf{a}$ and $\mathbf{b}$ have different tails, there exists some $k > n$ such that $a_k = 0$ and $b_k = 1$. Then let

$$U_{\mathbf{a}} = ([0,1]^k \times \left[0, \frac{1}{2}\right) \times [0,1]^{\omega}) \cap A,$$

and

$$U_{\mathbf{b}} = ([0,1]^k \times \left(\frac{1}{2}, 1\right] \times [0,1]^{\omega}) \cap A.$$

Then $U_{\mathbf{a}}$ and $U_{\mathbf{b}}$ are nonempty, open, disjoint, and $A = U_{\mathbf{a}} \cup U_{\mathbf{b}}$, so A is disconnected.

Therefore, we conclude that A is contained in a single branch. □

Now we will show L has the fixed point property.

Lemma 2.3.8. *L has the fixed point property.*

Proof. Given a continuous $f : L \to L$, we will consider what happens to the limit line, ll, under f. There are four cases:

1. $f(ll) \subset ll$. Here we have an arc mapping into itself, so by the Brouwer theorem (Theorem 2.3.3), $f|_{ll}$ has a fixed point, hence so does f.

2. $ll \subset f(ll)$. This time, by Lemma 2.3.5, $f|_{ll}$ has a fixed point, and therefore so does f.

3. One of the end points of the limit line ll (call this point $\mathbf{w}$), has $f(\mathbf{w})$ in a tree that is rooted at $\mathbf{w}$ (one of the trees that arrange as a harmonic fan and are rooted at $\mathbf{w}$, as mentioned earlier in the section). We can assume

$f(\mathbf{w}) \neq \mathbf{w}$, or else $\mathbf{w}$ is a fixed point. Call this tree T, so $f(\mathbf{w}) \in T$. Let $g : T \to T$ be defined as follows:

$$g(\mathbf{x}) = \begin{cases} f(\mathbf{x}) & \text{if } f(\mathbf{x}) \in T \\ \mathbf{w} & \text{if } f(\mathbf{x}) \notin T \end{cases}$$

Then g is continuous on T, since if $\mathbf{x} \in T$, if $g(\mathbf{x})$ is either equal to $f(\mathbf{x})$ (which is continuous), or $\mathbf{w}$, and since the two possible cases agree on the boundary (as f is continuous), g will be continuous.

Then as g is a continuous map from a tree T to itself, and trees have the fixed point property [Nad, Theorem 10.30], g has a fixed point. This fixed point cannot possibly be $\mathbf{w}$, as $f(\mathbf{w}) \neq \mathbf{w}$, so the fixed point must be some $\mathbf{x} \in T \setminus \{\mathbf{w}\}$. Then as $g(\mathbf{x}) = \mathbf{x} \in T$, we have $f(\mathbf{x}) = \mathbf{x} \in T$, so $\mathbf{x}$ is a fixed point for f.

4. In the last case, we have all other possibilities. As none of the other three cases can apply, we conclude that in this case, one of the endpoints of ll (call it $\mathbf{w}$) maps to a tree coming from the other endpoint of ll (call this $\mathbf{v}$), and $\mathbf{v}$ maps inside ll. Then as $\mathbf{v}$ is a disconnection point for L, there exists $\mathbf{x} \in ll$ such that $f(\mathbf{x}) = \mathbf{v}$. Then similarly to the proof in Lemma 2.3.5, we can obtain a maximum value for such a point $\mathbf{x}$, and by the intermediate value theorem $f|_{ll}$ has a fixed point, and hence so does f.

In all cases, f has a fixed point, so we conclude that L has the fixed point property.

□

Now we can prove the main theorem of this section.

Theorem 2.3.9. *$K_{(0,1)}$ has the fixed point property.*

Proof. Let $f : K_{(0,1)} \to K_{(0,1)}$ be continuous. Note that as f is continuous, path components are mapped by f to path components. We have two cases, depending on where f maps the limit line, ll.

1. ll maps into L. In this case L maps to itself, so since L has the fixed point property, $f|_L$ has a fixed point, and hence f has a fixed point.

2. ll maps to a branch B that is not L. In this case, consider $f(ll)$. Here $f(ll)$ is compact and connected, hence by Lemma 2.3.6, $f(ll)$ is a tree. As $f(ll)$ is compact, and maps outside ll, then $d(ll, f(ll)) = a > 0$ (here $d(ll, f(ll)) = \inf_{x \in ll}\{\inf_{y \in f(ll)}\{d(x,y)\}\}$. Then for $\varepsilon = \frac{a}{2}$, as f is (uniformly) continuous, there exists a $\partial > 0$ such that $d(ll, f(\overline{B_\partial(ll)})) > \frac{a}{2} > 0$.

   Therefore $f(\overline{B_\partial(ll)})$ does not contain ll. By Lemma 2.3.4, $\overline{B_\partial(ll)}$ contains a continuum that contains a neighbourhood of ll, call this C. Then $f(C)$ is compact, connected, and does not contain the limit line, so by Lemma 2.3.7, $f(C)$ is contained in a single branch. But as C is a neighbourhood of ll, C meets every path component in $K_{(0,1)}$, ie points from every branch. Then as f is continuous, all these path components must map under f to the same branch (the branch that C maps to), so we conclude the image of $K_{(0,1)}$ under f is contained in a single branch.

   Then as $K_{(0,1)}$ is compact and connected, its image is compact and connected, so by Lemma 2.3.6, $f(K_{(0,1)})$ is a tree, call it T. Then if we restrict f to T, $f|_T : T \to T$ is a continuous function from a tree into itself, and as trees have the fixed point property [Nad, Lemma 10.30], $f|_T$ has a fixed point, and hence so does f.

In either case, f has a fixed point, and hence $K_{(0,1)}$ has the fixed point property.

□

2.4 An Embedding of $K_{(0,1)}$ into $\mathbb{R}^3$

In this section we will describe an embedding of $K_{(0,1)}$ into $\mathbb{R}^3$. This embedding will be particularly useful for visualising the structure of $K_{(0,1)}$. In Section 2.1 we gave a basic model of $K_{(0,1)}$ due to Ingram. In this section we will develop this further into an explicit embedding in $\mathbb{R}^3$.

We know that the set of vertices of $K_{(0,1)}$, which we are calling V, is homeomorphic to the Cantor set, C of all binary sequences. For the purposes of the embedding, it will be helpful to refer to the vertices by their coordinates under a particular homeomorphism that re-labels each vertex as a binary sequence in the Cantor set. This homeomorphism $h : V \to C$ acts on a vertex by (starting at the left of the sequence) removing a 0 that appears directly to the right of each 1, and shifting the remainder of the sequence to the right of the 1 one place to the left. For example: $h((0,1,0,0,1,0,1,0,0,\dots)) = (0,1,0,1,1,0,\dots)$.

We will describe the vertex relations (ie which vertices are joined by direct connections) under the new coordinates. For the remainder of this section, unless otherwise specified, when mentioning the coordinates of a vertex, now we are referring to the coordinates after the homeomorphism is applied, eg $(1,1,1,1,\dots)$ is now a perfectly valid point in $K_{(0,1)}$ (and corresponds to the limiting point whose pre-homeomorphism coordinates are $(1,0,1,0,1,\dots)$). The reason for doing this is that it makes it easier to describe the embedding by describing the position of the point in terms of the usual Cantor middle thirds set.

We now need to redefine some terminology. For any $\mathbf{a} = (a_0, a_1, a_2, \dots) \in C$, we say that $\mathbf{a}$ has a *1 sequence of length n* for $n \in \mathbb{N} \cup \{\infty\}$ if $a_n = 0$, and either:

1. $a_i = 1$ for all $0 \le i < n$, or
2. $a_0 = 0$, and $a_i = 1$ for all $1 \le i < n$.

Note that if $a_0 = a_1 = 0$, then $\mathbf{a}$ has a 1 sequence of length 1. The concept of a 1 sequence is analogous to that of a 01 sequence.

We can describe a forward move by the function $m' : C \to C$ as follows. Suppose an element $\mathbf{a} = (a_0, a_1, a_2, \dots)$ in C has a 1 sequence of length n. We can then define:

$$m'(\mathbf{a}) = \begin{cases} (1, 1, \dots, 1, a_{n+1}, a_{n+2}, \dots) & \text{if } a_0 = 0 \\ (0, 1, 1, \dots, 1, a_{n+1}, a_{n+2}, \dots) & \text{if } a_0 = 1 \end{cases}$$

(in each case there are n 1s before a_{n+1}).

For notational purposes, we will split C into a number (countably infinite) of disjoint subsets. Let $\mathbf{s} = (s_0, 1, \dots, 1, 0)$ be a finite sequence of 0s and 1s of length $n+1$ such that $s_0 = 0$ or 1, $s_n = 0$, and $s_i = 1$ for all $1 \le i < n$. Then let $C_{\mathbf{s}}$ be the set of all binary sequences such that the first $n+1$ coordinates match the coordinates of $\mathbf{s}$. Also, define $C_{(0,1,1,1,\dots)} = \{(0, 1, 1, 1, \dots)\}$, and $C_{(1,1,1,1,\dots)} = \{(1, 1, 1, 1, \dots)\}$. It is easy to see that the sets $C_{\mathbf{s}}$ form a disjoint union to give C - the Cantor set. We call these sets *partition sets of* C. In doing this we have partitioned the points C by the 1 sequence of the coordinates at the start of the point.

The concept of a direct connection is still used in the same way as before. The end vertices (vertices directly connected to only one other vertex) have sequences that begin with at least two 0s, ie the end vertices are $C_{(0,0)}$. These vertices have the natural structure of a Cantor set (we simply ignore the first two 0s, and we can represent them on a compact interval as the points of the usual middle thirds Cantor set).

In representing $C_{(0,0)}$ in this way, we see that it can be divided into two classes - those whose third coordinate is a 1, and those whose third coordinate is a 0. Consider what happens if the next coordinate is a 0. Suppose $\mathbf{a} = (0, 0, 0, a_3, a_4, \dots)$. After applying a forward move to $\mathbf{a}$ we have $m'(\mathbf{a}) = (1, 0, a_3, a_4, \dots)$. This means the 1 sequence of $m'(\mathbf{a})$ also has length 1, and $m'(\mathbf{a})$ has first coordinate 1, and second coordinate 0. So $m'(\mathbf{a}) \in C_{(1,0)}$.

Now suppose that $\mathbf{a} \in C_{(0,0)}$, but $a_2 = 1$. Then $m'(\mathbf{a}) = (1, 1, a_3, a_4, \dots)$. We cannot tell yet what partition set of C $m'(\mathbf{a})$ is in. To do this we need to know what a_3 is. If $a_3 = 0$, then $m'(\mathbf{a}) = (1, 1, 0, a_4, \dots)$, so in this case $m'(\mathbf{a}) \in C_{(1,1,0)}$. If however, $a_3 = 1$, we still don't know what partition set $\mathbf{a}$ is in, so we need to keep repeating this process until $a_n = 0$ for some n, or if $a_i = 1$ for all $i \geq 2$, then $\mathbf{a} \in C_{(1,1,1,\dots)}$.

This means that given an end vertex $\mathbf{a}$ (in $C_{(0,0)}$), we can easily find which partition set $m'(\mathbf{a})$ is in. We simply find where the first 0 is (after the initial two 0s). Suppose it occurs at a_n, then $\mathbf{a}$ will be directly connected to a vertex in $C_{(1,1,\dots,1,0)}$, where the 0 occurs at s_n.

Suppose that we have two points in $C_{(0,0)}$ that are sufficiently close (where they have the same beginning to their sequences for n coordinates for some arbitrarily large n - say at least until the first 0 if it exists). Then the images of these points are similarly close under m'. In this sense, the Cantor set structure is preserved under m'.

We can now consider what partition sets will be 'close' together. Each partition set $C_{\mathbf{s}}$, where $\mathbf{s}$ is a finite sequence, has the natural structure of a Cantor set, so for points within the partition set we know which ones are 'close'. The partition sets themselves also have a natural 'closeness' relation based on the points that are in them. Firstly, suppose $\mathbf{a} = (0, a_1, a_2, \dots)$, and $\mathbf{b} = (1, b_1, b_2, \dots)$. Then $\mathbf{a}$ and $\mathbf{b}$ are not particularly close, as they will always differ at the first coordinate. Therefore $C_{(0,1,1,\dots,1,0)}$ and $C_{(1,1,1,\dots,1,0)}$ can never be close, no matter how many 1s each has (including infinitely many).

Also, suppose $C_{\mathbf{s_1}}$ and $C_{\mathbf{s_2}}$ have the same first coordinate, but different (finite) lengths (suppose $\mathbf{s_1}$ has length m, and $\mathbf{s_2}$ has length n, where $m < n$). Then as two points, one in each partition set, will differ at the mth coordinate, $C_{\mathbf{s_1}}$ and $C_{\mathbf{s_2}}$ will also not be close (they get closer as m gets larger). On the other hand, $C_{(0,1,1,\dots)}$ has sets of the form $C_{(0,1,1,\dots,1,0)}$ arbitrarily close (as the 1 sequence of the second set becomes arbitrarily large). Similarly, $C_{(1,1,1,\dots)}$ has sets of the form $C_{(1,1,1,\dots,1,0)}$ arbitrarily close.

Now that we know how the vertices are directly connected, and how the vertices and partition sets relate to each other in terms of distance, we can begin to look for an embedding of $K_{(0,1)}$ into $\mathbb{R}^3$. We will put the embedding into a unit cube $[0,1] \times [0,1] \times [0,1]$ in $\mathbb{R}^3$.

We start with the end vertices, ie $C_{(0,0)}$. We will embed these points into the line in the unit cube which has coordinates $(0, x, 1)$, where $x \in [0,1]$. We order the points in the way described above (as a middle thirds Cantor set), by putting the point $(0,0,1,1,1,\dots)$ at $(0,0,1)$, and the point $(0,0,0,0,\dots)$ at $(0,1,1)$. This is what we mean by a *canonical embedding* of a Cantor set into $\{0\} \times [0,1] \times \{1\}$, ie the point with all coordinates 0 to the right, and the point with all coordinates 1 to the left. These points all have a first coordinate of 0, and we know that points with first coordinate 0 will converge to the point $(0,1,1,1,\dots)$ as part of a $\sin \frac{1}{x}$ curve. Now we need to consider what happens to these points after a forward move. As mentioned above, if we have a point $\mathbf{a}$, where $\mathbf{a} = (0, 0, a_2, \dots)$, and $a_2 = 0$, then $m'(\mathbf{a}) \in C_{(1,0)}$. Therefore all points with coordinates $(0, x, 1)$, where $x \in [\frac{1}{2}, 1]$, will be directly connected to points in $C_{(1,0)}$.

Now we must consider where to place $C_{(1,0)}$. Similarly to the points beginning with 0, the points beginning with 1 converge to the point $(1,1,1,\dots)$ as part of a $\sin \frac{1}{x}$ curve. So we will embed $C_{(1,0)}$ in a different plane to $C_{(0,0)}$, and indeed all points that begin with a 1 in a different plane to those that begin with a 0. As only the points in $C_{(0,0)}$ that have a third coordinate of 0 will directly connect to points in $C_{(1,0)}$, we only need half of the length of Cantor set as for $C_{(0,0)}$. Therefore, we can embed $C_{(1,0)}$ as a Cantor set in the line in the cube with coordinates $(\frac{1}{2}, x, 0)$, where $x \in [\frac{2}{3}, 1]$. The first coordinate of this embedding being $\frac{1}{2}$ is because, as the length of the 1 sequence of the points is increasing, we are proceeding further along the $\sin \frac{1}{x}$ curves, which will have a limiting line that has a first coordinate of 1 (in the cube).

A diagram showing the placement of these sets is shown in Figure 2.7, along with the intended position of the limit line.

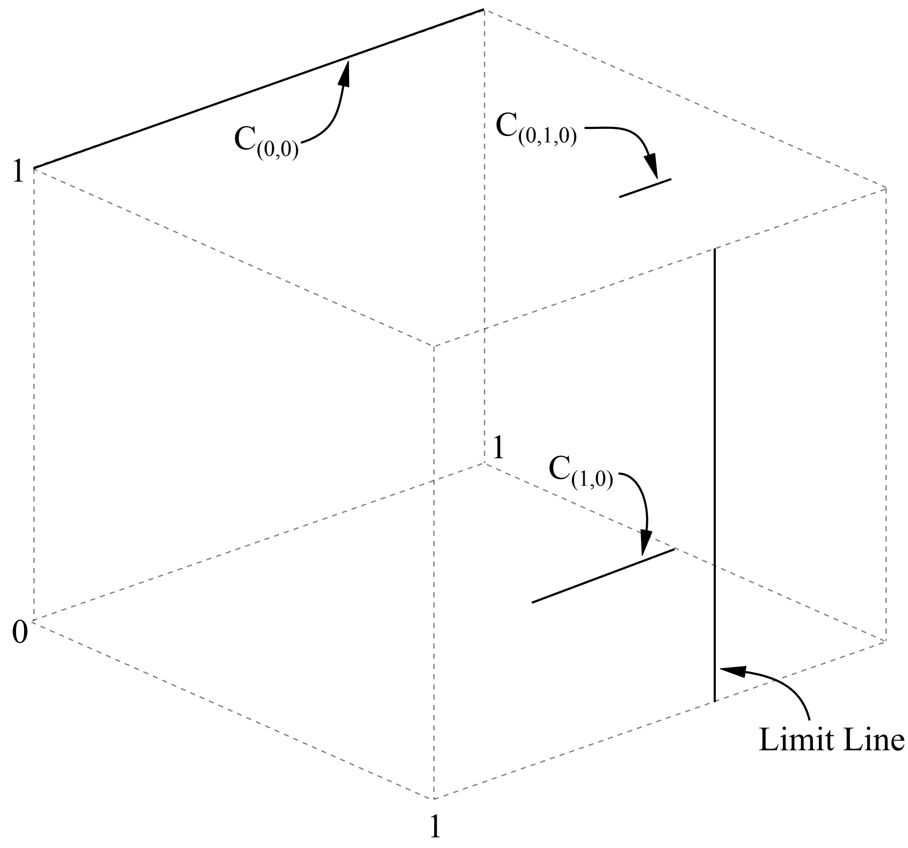

Figure 2.7: The placement of sets $C_{(0,0)}$, $C_{(1,0)}$, and $C_{(0,1,0)}$.

Now we have to consider how to lay out the points of $C_{(0,1)}$ in the interval. Remembering that a point $\mathbf{a} = (0, 0, a_2, a_3, \dots)$ will have a direct connection to a point $\mathbf{b} = (1, 0, b_2, b_3, \dots)$ if and only if $a_i = b_i$ for all $i \geq 2$, there is the obvious way to lay them out. This involves having $(1, 0, 1, 1, 1, \dots)$ at $(\frac{1}{2}, \frac{2}{3}, 0)$, and $(1, 0, 0, 0, \dots)$ at $(\frac{1}{2}, 1, 0)$ so they will be connected with a straight line. This will however cause problems later, so we will lay it out slightly differently. To do this, we will now introduce two different ways to rearrange a Cantor set, called a *type 1 switch*, and a *type 2 switch*. Each of these is a homeomorphism of C. We denote a type 1 switch by the function s_1, and a type 2 switch by the function s_2. They are defined as follows: let $\mathbf{a} = (a_0, a_1, a_2, \dots) \in C$. Then we define:

$$s_1(\mathbf{a}) = (1 - a_0, 1 - a_1, 1 - a_2, \dots)$$

$$s_2(\mathbf{a}) = \begin{cases} (1, a_1, a_2, a_3, \dots) & \text{if } a_0 = 0 \\ (0, 1 - a_1, 1 - a_2, 1 - a_3, \dots) & \text{if } a_0 = 1 \end{cases}$$

As the intent of these functions is to show the way in which we make direct connections, in Figure 2.8 the Cantor set at the top is transformed by s_1 (on the left), and s_2 (on the right), and in each case, for each $\mathbf{a} \in C$, $\mathbf{a}$ is directly connected to $s_i(\mathbf{a})$, for $i \in \{1, 2\}$. This is a representation of how an embedding in $\mathbb{R}^3$ of a particular section would look.

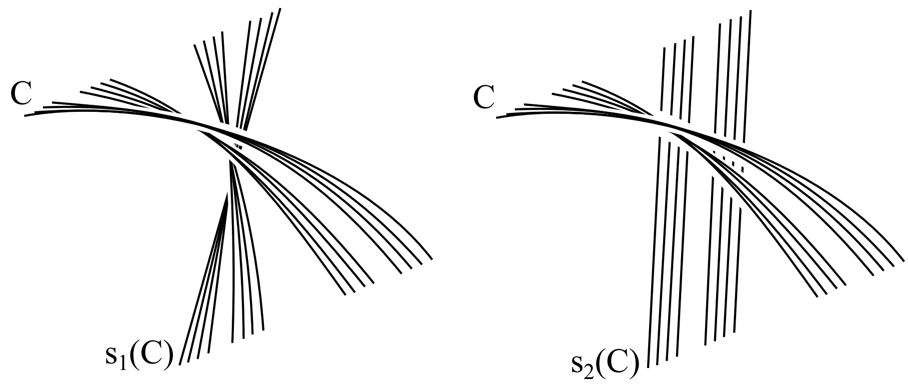

Figure 2.8: The effects of the functions s_1 and s_2 on C.

Now we can see how $C_{(1,0)}$ will be arranged. We apply a type 2 switch (the function s_2) to the canonical arrangement of $C_{(1,0)}$, so we have the point $(1, 0, 0, 1, 1, 1, \dots)$ at $(\frac{1}{2}, \frac{2}{3}, 0)$, and $(1, 0, 1, 1, 1, \dots)$ at $(\frac{1}{2}, 1, 0)$, etc.

Now, similarly to the case for $C_{(0,0)}$, the points in $C_{(1,0)}$ that have a third coordinate of 0 will be directly connected to points in $C_{(0,1,0)}$. We will embed the points of $C_{(0,1,0)}$ in the cube on the line of points $(\frac{3}{4}, x, 1)$, where $x \in [\frac{2}{3}, \frac{7}{9}]$. These are again at the top of the cube (the same plane as $C_{(0,0)}$ - note that in $C_{(0,1,0)}$ we are 'further along' the $\sin \frac{1}{x}$ curve than at $C_{(0,0)}$, but both are converging to the same point in the limit line). $C_{(0,1,0)}$ sits at the left of $C_{(1,0)}$, whereas $C_{(1,0)}$ sat to the right of $C_{(0,0)}$. This time we will embed $C_{(0,1,0)}$ in the canonical way, ie $(0,1,0,1,1,1,\dots)$ is at $(\frac{3}{4},\frac{2}{3},1)$, and $(0,1,0,0,0,\dots)$ is at $(\frac{3}{4},\frac{7}{9},1)$.

We now have embeddings for the first three partition sets. These have created a pattern we will now specify how to follow. Let $\mathbf{s} = (s_0, s_1, \dots, s_{n-1}, 0)$ be a finite sequence of 0s and 1s of length $n+1$ such that $s_0 = 0$ or 1, $s_n = 0$, and $s_i = 1$ for all $1 \le i < n$, as defined above. Now, we can embed $C_{\mathbf{s}}$ into the line in the cube as follows (note that $\mathbf{a} \in C_{\mathbf{s}}$ has a 1 sequence of length n):

- If $n = 1$, then the line has coordinates:

$$\begin{cases} (0, x, 1) & \text{if } s_0 = 0 \\ (\frac{1}{2}, y, 0) & \text{if } s_0 = 1 \end{cases}$$

  where $x \in [0,1]$, and $y \in [\frac{2}{3}, 1]$, or

- If $n > 1$, the line has coordinates:

$$\begin{cases} (\sum_{i=1}^{2n-2} \frac{1}{2^i}, x, 1) & \text{if } s_0 = 0 \\ (\sum_{i=1}^{2n-1} \frac{1}{2^i}, y, 0) & \text{if } s_0 = 1 \end{cases}$$

  where

$$x \in \left[\sum_{i=2}^{n} \frac{2}{3^{2i-3}}, 1 + \sum_{i=2}^{n} \left(-\frac{2}{3^{2i-2}}\right)\right],$$

  and

$$y \in \left[\sum_{i=1}^{n} \frac{2}{3^{2i-1}}, 1 + \sum_{i=2}^{n} \left(-\frac{2}{3^{2i-2}}\right)\right].$$

In terms of the arrangement of the points in each partition set in the line, every partition set whose coordinates begin with a 0 is arranged in the canonical way, and every partition set whose coordinates begin with a 1 is given a type 2 switch (the function s_2).

This embedding of the vertices can be seen in Figure 2.9 (Figure 2.9 also includes the arcs connecting them). The numbers on the left refer to the initial coordinates of the partition set in that row.

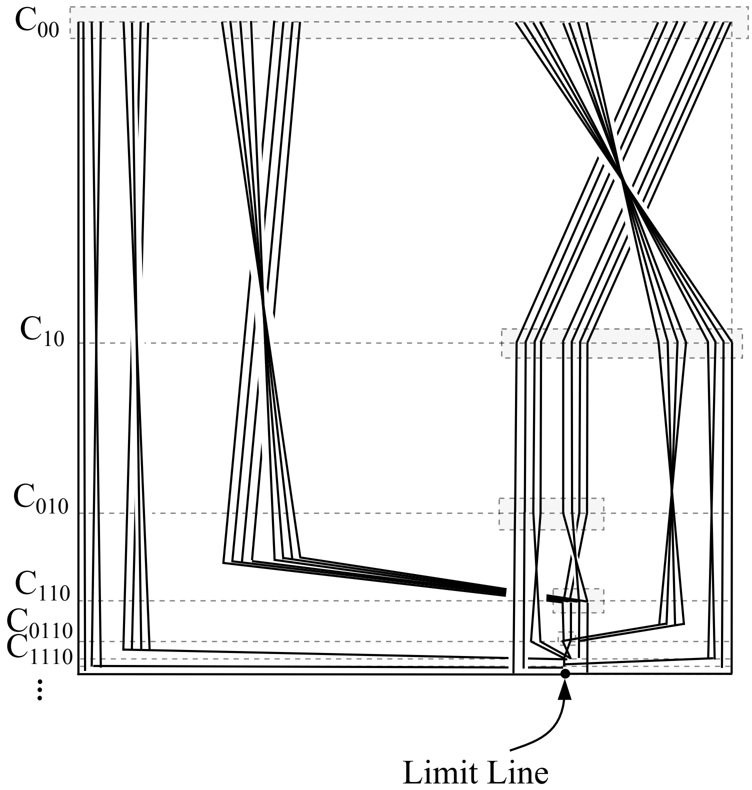

Figure 2.9: A plan view of the embedding in $\mathbb{R}^3$ of $K_{(0,1)}$.

We will now specify how the non-vertex points will be embedded. First we note that any non-vertex point in $K_{(0,1)}$ will be in an arc between two vertices, so we only need to specify how arcs that directly connect vertices are embedded.

Suppose we have two partition sets embedded in the cube, and they are directly connected (or at least parts of them are). How the arcs connecting them will be arranged will depend on the order of the Cantor sets. Suppose that the two Cantor sets are embedded in the line segments with coordinates $\{a\} \times [b,c] \times \{d\}$ and $\{e\} \times [f,g] \times \{h\}$ respectively, where (wlog) $a < e$ and d, $h \in \{0,1\}$. We have the following cases:

- If there has been no change in order (ie no switches relative to each other), the arcs can be embedded in the plane that is bound by the line segments the Cantor sets are embedded in. That is, we have taken $C \times [0,1]$, and resized it appropriately to fit between the two sets of vertices being connected, embedded in a plane.

- If there has been a type 1 switch, then similarly to the first case, we can embed the arcs as $C \times [0,1]$, but the 'strip' that they are embedded into needs to be given a 'twist'. This can be resized to fit into a box $[a,e] \times [\min\{b,f\}, \max\{c,g\}] \times [s,t]$, where $[s,t] \supset [d,h]$. This is essentially the box bounded by the Cantor sets of vertices, the extra height of $[s,t]$ is possibly needed in order to accommodate the 'twist' (but the height of the twist can be resized to be arbitrarily small).

- If there has been a type 2 switch, then this requires a combination of the above two cases. Again, it can be fit into an appropriate sized box as per the previous case.

A picture of what the embedding for type 1 and type 2 switches will look like can be seen in Figure 2.8.

Note that in the embedding of the vertices in each partition set, half the vertices will be directly connected to the partition set one step closer to the limit line, then half the remaining vertices will be directly connected to the partition set a further two steps closer to the limit line, then the next remaining half connect to a partition set a further two steps, and so on. Finally, the last vertex will be directly connected to the limit line.

The way the vertices have been laid out, the limit line is somewhere near the centre of the cube horizontally (as seen in Figure 2.9), and the further toward the edge of the cube the vertex is embedded in its particular partition set, the greater the number of steps the partition set is that it is directly connected to.

This was the reason for introducing the type 1 and type 2 switches, to keep these points on the outside, making the embedding easier.

We will now describe how the various sets of arcs are embedded. Let $\mathbf{x} \in K_{(0,1)}$ be a non-vertex point. Then $\mathbf{x}$ will be in an arc between two vertices that are directly connected. How the arc will be embedded will depend on how many steps apart the partition sets of the vertices are in terms of distance to the limit line. Suppose the vertices are embedded as $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, where $a_1 = 1 - \frac{1}{2^m}$ and $b_1 = 1 - \frac{1}{2^n}$, with $n < m$, and a_3, $b_3 \in \{0, 1\}$ (note that $a_3 \neq b_3$). The following cases apply.

- The distance is one, ie $m = n + 1$. In this case, if $a_3 = 1$ and $b_3 = 0$, then the vertices in the partition set of $\mathbf{a}$ that have next coordinate 0 will be directly connected to the vertices in the partition set of $\mathbf{b}$ with arcs that have been rearranged with a type 2 switch mentioned above. The box that this fits into is spanned by the vertices being connected. If $a_3 = 0$ and $b_3 = 1$, then the vertices in the partition set of $\mathbf{a}$ with next coordinate 0 are directly connected to the vertices in the partition set of $\mathbf{b}$ with arcs that go straight through without a switch, the kind mentioned in the first case above.

- The distance is greater than one, but finite. In this case we proceed in two steps. First suppose $a_3 = 1$ and $b_3 = 0$. Here we embed all the arcs that directly connect vertices in these two partition sets as follows. Firstly, we create a copy of the vertices of the partition set of $\mathbf{a}$ that are to be connected, and embed these in the line $\{1 - \frac{1}{2^n} - \frac{1}{2^{n+m+1}}\} \times [a, b] \times \{1\}$, where $[a, b]$ is the coordinate interval of vertices in the partition set of $\mathbf{a}$ that are being directly connected. These undergo a type 2 switch, and the arcs are embedded as per the description above, in the box $[1 - \frac{1}{2^m}, 1 - \frac{1}{2^n}] \times [a, b] \times [1 - \frac{1}{2^n}, 1]$. These copied vertices are then directly connected to the vertices in the partition set of $\mathbf{b}$, via straight arcs between the sets.

These straight arcs then do not intersect any others in the embedding as they pass underneath any arcs inside them, the arcs inside stay sufficiently close to the top of the cube. The type 2 switch was necessary to cancel the type 2 switch that the vertices in the partition set of **b** had been subject to.

If $a_3 = 0$ and $b_3 = 1$, the process is essentially the same, except the switch occurs at the bottom of the cube, where the vertices of the partition set of **a** are embedded, and are then connected upwards to the vertices in the partition set of **b**. Also in this case a type 1 switch is performed instead of a type 2.

- The distance is greater than one, but infinite. In this case the vertex is directly connected to the limit line, so **b** is a vertex in the limit line. The embedding is done by embedding a straight line from **a** to $\mathbf{c} = (1, a_2, a_3)$, and a straight line from **c** to **b**. This can be viewed as the completion of the previous step.

We have now completely described the embedding of $K_{(0,1)}$ into $[0,1] \times [0,1] \times [0,1]$. A picture of what the three dimensional embedding will look like is shown in Figure 2.10.

A summary of the key structural properties of $K_{(0,1)}$ that can be seen from the embedding are:

- each collection of points that have the same (finite) initial sequence form a Cantor set;
- two vertices that have opposite initial sequences immediately followed by the same sequence are joined by a line; and
- collections of lines having coordinates with the same tail converge to the 'limit line'.

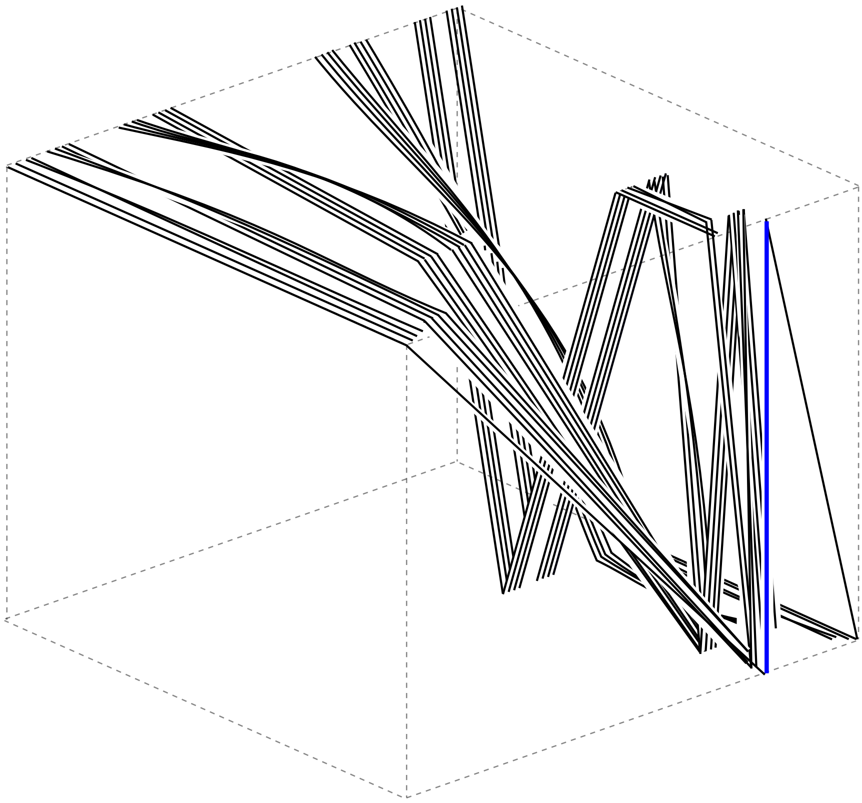

Figure 2.10: A view of the embedding in $\mathbb{R}^3$ of $K_{(0,1)}$. The limit line is shown in blue.

There are similarities of the embedding with Ingram's model in Section 2.1. In particular, we have $D_1 = C_{(0,0)}$, $D_3 = C_{(0,1,0)}$, $E_2 = C_{(1,0)}$, etc. $\mathcal{B}_n$ (for each n) represents all the direct connections between the vertices.

Chapter 3

Connectedness over Hausdorff Continua

As mentioned in Chapter 1, one of the properties of classical inverse limits is that they are always connected [Nad, Theorem 2.4]. It was realised early on that this is not necessarily the case for generalised inverse limits, even with a single bonding map with a connected graph. The standard example of such a generalised inverse limit first appeared in [IM, Example 1]. In this example, each factor space $X_i = [0,1]$, and each bonding function f_i is the (upper semicontinuous) set valued function with the graph shown in Figure 3.1.

To see that this is disconnected, consider the subset of $\varprojlim f$ that is

$$C = \left(\left\{ \frac{1}{4} \right\} \times \left\{ \frac{1}{4} \right\} \times \left\{ \frac{3}{4} \right\} \times \Pi_{i \geq 3}[0,1] \right) \cap \varprojlim f.$$

This subset is closed, nonempty, and a proper subset of $\varprojlim f$. Now consider the open subset

$$U = \left(\left(\frac{1}{8}, \frac{3}{8} \right) \times \left(\frac{1}{8}, \frac{3}{8} \right) \times \left(\frac{5}{8}, \frac{7}{8} \right) \times \Pi_{i \geq 3}[0,1] \right) \cap \varprojlim f.$$

Let $\mathbf{x} = (x_0, x_1, x_2 \dots) \in U$. From the first graph of f, G_1, the first two coordinates of $\mathbf{x}$ must have values such that $\frac{1}{8} < x_0 \leq \frac{1}{4}$ and $\frac{1}{8} < x_1 \leq \frac{1}{4}$. But

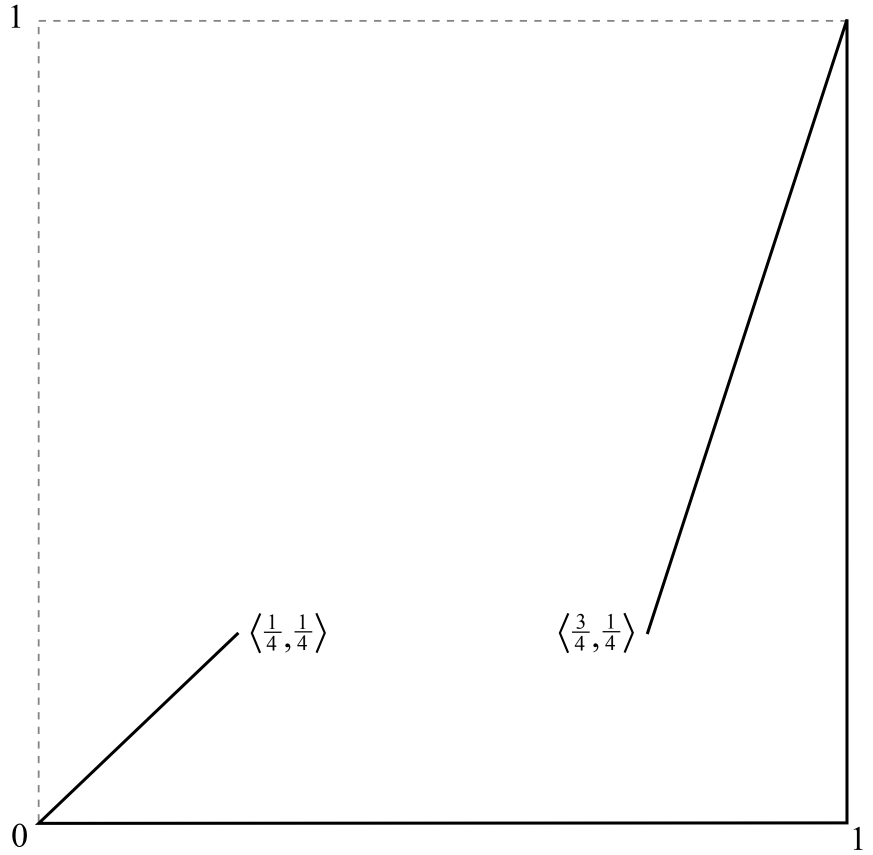

Figure 3.1: A function f whose inverse limit is not connected.

then from the second graph of f, G_2, we have $\frac{1}{4} \leq x_1 < \frac{3}{8}$ and $\frac{3}{4} \leq x_2 < \frac{7}{8}$. Therefore $x_1 = \frac{1}{4}$, so $x_0 = \frac{1}{4}$ and $x_2 = \frac{3}{4}$. Hence $C = U$, so C is a nonempty proper clopen subset of $\varprojlim f$.

This discovery that generalised inverse limits are not necessarily connected led Ingram to pose the following problem in his 2011 paper [In1].

Problem 3.0.1 (Ingram)**.** Suppose for each nonnegative integer i, X_i is a compact Hausdorff (metric) space and $f_{i+1} : X_{i+1} \to 2^{X_i}$ is an upper semicontinuous function. Find necessary and sufficient conditions (preferably on the bonding functions) such that $\varprojlim(X_i, f_i)$ is connected.

In Chapter 1 we gave some partial solutions: Theorem 1.2.3 and Theorem 1.2.4, both appearing in [IM]. Also mentioned in the opening chapter was Theorem 1.2.5 from [Nal2], if a generalised inverse limit $\varprojlim(X_i, f_i)$ is disconnected, then one of its corresponding Mahavier products $\mathcal{G}_n$ will be disconnected for some $n \in \mathbb{N}$. If this is the case we say a *disconnection* happens over the first n functions.

An important result in [Nal2] states that if the graph of an upper semicontinuous set valued function F can be represented as a union of the graphs of upper semicontinuous set valued functions, each of which has the property that at each point in the domain its image is connected, then $\varprojlim F$ is connected.

Theorem 3.0.2 (Nall)**.** *Suppose X is a compact metric space, and $\{F_\alpha\}_{\alpha\in\Lambda}$ is a collection of closed subsets of $X \times X$ such that for each $x \in X$ and each $\alpha \in \Lambda$ the set $\{y \in X : (x, y) \in F_\alpha\}$ is nonempty and connected, and such that $F = \bigcup_{\alpha\in\Lambda} F_\alpha$ is a connected closed subset of $X \times X$ such that for each $y \in X$ the set $\{x \in X : (x, y) \in F\}$ is nonempty. Then $\varprojlim F$ is connected.*

It was mentioned in Chapter 1 that if f is an upper semicontinuous set valued function, then so is f^{-1}. Another result from [Nal2] states that if $\varprojlim f$ is connected, then $\varprojlim f^{-1}$ is connected.

Theorem 3.0.3 (Nall)**.** *Suppose X is a Hausdorff continuum, and $f : X \to 2^X$ is a surjective upper semicontinuous set valued function. Then $\varprojlim f$ is connected if and only if $\varprojlim f^{-1}$ is connected.*

Ingram and Marsh [Imar] give an example of a generalised inverse sequence (X_i, f_i) having a connected inverse limit, but the generalised inverse sequence (X_i, g_i), where for each $i \in \mathbb{N}$, $g_i = f_i^{-1}$, has a disconnected inverse limit. This means that extending Theorem 3.0.3 to allow for different bonding functions is not possible.

Ingram [In3] gives a sequence of upper semicontinuous set valued functions $f_1, f_2, f_3, \ldots$ with the property that the graph of f_n^n is not connected, but the graph of f_n^k is connected for each $1 \leq k < n$. This means that $\varprojlim f_n$ is not connected for each n, but longer sequences of functions need to be examined to see this as n increases.

Throughout this chapter, we assume that for every generalised inverse sequence (X_i, f_i), the graph G_i of each f_i is connected. This is because if there is a graph that is disconnected, then the inverse limit will also be disconnected, as seen in the next lemma.

Lemma 3.0.4. *Let (X_i, f_i) be a generalised inverse sequence, and suppose f_i is surjective for each $i \geq 1$. Then if there exists an $i \in \mathbb{N}$ such that G_i is disconnected, then $\varprojlim(X_i, f_i)$ is disconnected.*

Proof. Suppose G_i is disconnected, then there exist disjoint nonempty open sets A, $B \subset G_i$ such that $G_i = A \cup B$. Then if we let $A^* = (\Pi_{0 \leq j \leq i-2} X_j \times A \times \Pi_{j \geq i+1} X_j) \cap \varprojlim(X_i, f_i)$, and $B^* = (\Pi_{0 \leq j \leq i-2} X_j \times B \times \Pi_{j \geq i+1} X_j) \cap \varprojlim(X_i, f_i)$, we have A^* and B^* are open, disjoint, nonempty (since each f_i is surjective), and $A^* \cup B^* = \varprojlim(X_i, f_i)$, so $\varprojlim(X_i, f_i)$ is disconnected.

□

The main theorem in this chapter, Theorem 3.0.6, is a generalisation of the following theorem from Greenwood and Kennedy [GK]. This theorem essentially selects a sequence of closed sets, subsets of the spaces $I_i \times I_{i-1}$ (each I_i is a closed interval), that are appropriately aligned so that they give rise to what is called a C-sequence. We do not define a C-sequence here but note that it is very similar to an HC-sequence defined in the next section. Theorem 3.0.5 characterises a certain kind of disconnection in the inverse limit, one where there is a nonempty proper 'basic' clopen subset (the term basic is a slight abuse of notation, but its meaning should be clear from the definition).

Theorem 3.0.5 (Greenwood and Kennedy)**.** *Suppose that for each $i \geq 0$, I_i is a closed interval, and for each $i > 0$, $f_i : I_i \to 2^{I_{i-1}}$ is a surjective upper semi-continuous function and the graph, G_i of f_i is connected. There exists*

$m \geq 0$ and $n > m+1$ such that if $m \leq i \leq n$ then there exists an open interval U_i and a closed interval A_i such that $A_i \subset U_i \subset I_i$, $U_i \neq I_i$,

$$\begin{aligned}&\varprojlim (I_i, f_i) \cap (\Pi_{i<m} I_i \times \Pi_{m \leq i \leq n} U_i \times \Pi_{i>n} I_i) \\ &\qquad = \varprojlim (I_i, f_i) \cap (\Pi_{i<m} I_i \times \Pi_{m \leq i \leq n} A_i \times \Pi_{i>n} I_i) \neq \varnothing,\end{aligned}$$

and $\varprojlim (I_i, f_i) \neq \Pi_{i<m} I_i \times \Pi_{m \leq i \leq n} A_i \times \Pi_{i>n} I_i$, if and only if $\{f_i : i > 0\}$ has a C-sequence.

Theorem 3.0.5 deals only with generalised inverse limits over intervals. The aim of this chapter is to generalise this result to generalised inverse limits over Hausdorff continua. The main result of the chapter is formulated as follows:

Theorem 3.0.6. *Let (X_i, f_i) be a generalised inverse sequence, and suppose every function f_i has a connected graph. Then $\{f_i : i > 0\}$ admits an HC-sequence if and only if there exists a connected basic open set*

$$U = \Pi_{0 \leq i < m} X_i \times \Pi_{m \leq i \leq n} U_i \times \Pi_{i>n} X_i \subset \Pi_{i \in \mathbb{N}} X_i$$

containing a closed set

$$A = \Pi_{0 \leq i < m} X_i \times \Pi_{m \leq i \leq n} A_i \times \Pi_{i>n} X_i,$$

such that

$$\varprojlim (X_i, f_i) \cap U = \varprojlim (X_i, f_i) \cap A \neq \varnothing,$$

and $\varprojlim (X_i, f_i) \not\subset U$.

This then gives the following sufficient condition for disconnectedness:

Corollary 3.0.7. *Suppose that for all $i \in \mathbb{N}$, X_i is a Hausdorff continuum, $f_{i+1} : X_{i+1} \to 2^{X_i}$ is a surjective upper semicontinuous function, the graph G_i of f_i is connected. If $\{f_i : i > 0\}$ admits an HC-sequence then $\varprojlim (X_i, f_i)$ is disconnected.*

Subsequently Greenwood and Kennedy [GK3] have given conditions similar to a C-sequence and HC-sequence, known as a CC-sequence, which give a characterisation of disconnectedness of generalised inverse limits over intervals.

Theorem 3.0.8 (Greenwood and Kennedy)**.** *Suppose that for each $i \geq 0$, $I_i = [0,1]$ and f_i is an upper semicontinuous set valued function. Then $\varprojlim(I_i, f_i)$ is disconnected if and only if $\{f_i : i \geq 0\}$ admits a CC-sequence.*

The work presented in this chapter has been published in [GL].

3.1 HC-Sequences

In this section we will give some definitions and basic results specific to this chapter, and in particular, define an HC-sequence.

We begin with Lemma 3.1.1, which generalises Theorem 1.2.5. The proof is the same proof given in [Nal2], which will work in both cases.

Lemma 3.1.1. *Let (X_i, f_i) be a generalised inverse sequence. Then $\varprojlim(X_i, f_i)$ is connected if and only if $\mathcal{G}(f_m, \dots, f_n)$ is connected for every m, n, $0 < m \leq n$.*

Since the inverse of an upper semicontinuous function is also an upper semicontinuous function (the graph is compact) we have:

Lemma 3.1.2. *Let (X_i, f_i) be a generalised inverse sequence. Then for every $n > 0$, $\mathcal{G}(f_1, \dots, f_n)$ connected if and only if $\mathcal{G}(f_n^{-1}, \dots, f_1^{-1})$ is connected.*

Proof. The proof comes from the fact that $(x_0, \dots x_n) \in \mathcal{G}(f_1, \dots, f_n)$ if and only if $(x_n, \dots x_0) \in \mathcal{G}(f_n^{-1}, \dots, f_1^{-1})$, and this gives the obvious homeomorphism.

□

Lemma 3.1.4 generalises [GK, Lemma 2.5]. The aim is to show that without a loss of generality we can assume that the disconnection starts with the first function. In order to state the lemma we require the notation of the following definition.

Definition 3.1.3. Let (X_i, f_i) be a generalised inverse sequence. Let $X_i^m = X_{i+m}$, and $f_i^m = f_{i+m}$ and define $\varprojlim(X_i, f_i)_{i\geq m} = \varprojlim(X_i^m, f_i^m)$.

Lemma 3.1.4. *Let (X_i, f_i) be a generalised inverse sequence, and suppose f_i is a surjective function for all $i \geq 1$. If there exists $m, n \in \mathbb{N}$ such that $m \leq n$ and whenever $m \leq i \leq n$ there exists an open set U_i and a closed set A_i such that $A_i \subset U_i \subset X_i$, then:*

$$\begin{aligned}&\varprojlim(X_i, f_i) \cap (\Pi_{0\leq i<m} X_i \times \Pi_{m\leq i\leq n} U_i \times \Pi_{i>n} X_i)\\ = &\varprojlim(X_i, f_i) \cap (\Pi_{0\leq i<m} X_i \times \Pi_{m\leq i\leq n} A_i \times \Pi_{i>n} X_i)\end{aligned}$$

if and only if

$$\begin{aligned}&\varprojlim(X_i, f_i)_{i\geq m} \cap (\Pi_{m\leq i\leq n} U_i \times \Pi_{i>n} X_i)\\ = &\varprojlim(X_i, f_i)_{i\geq m} \cap (\Pi_{m\leq i\leq n} A_i \times \Pi_{i>n} X_i)\end{aligned}$$

Proof. Let:

$$\begin{aligned}U &= \varprojlim(X_i, f_i) \cap (\Pi_{0\leq i<m} X_i \times \Pi_{m\leq i\leq n} U_i \times \Pi_{i>n} X_i),\\ A &= \varprojlim(X_i, f_i) \cap (\Pi_{0\leq i<m} X_i \times \Pi_{m\leq i\leq n} A_i \times \Pi_{i>n} X_i),\\ V &= \varprojlim(X_i, f_i)_{i\geq m} \cap (\Pi_{m\leq i\leq n} U_i \times \Pi_{i>n} X_i), \text{ and}\\ B &= \varprojlim(X_i, f_i)_{i\geq m} \cap (\Pi_{m\leq i\leq n} A_i \times \Pi_{i>n} X_i).\end{aligned}$$

Suppose $U = A$. Clearly $B \subset V$. Let $(x_i) \in V$. Then as each f_i is surjective, there exists an element $(y_i) \in U$, where for each $i \in \mathbb{N}$, $y_{i+m} = x_i$. But $(y_i) \in A$, and so $(x_i) \in B$, hence $V = B$.

Now suppose $V = B$. Clearly, $A \subset U$. Let $(x_i) \in U$ and for each $i \in \mathbb{N}$ let $y_i = x_{i+m}$. Then $(y_i) \in V$. As this implies $(y_i) \in B$, we conclude that $(x_i) \in A$, so $A = U$.

$\square$

We can now begin to define an HC sequence. An HC sequence is a finite sequence of sets of consecutive factor spaces, which will isolate a proper (nonempty) clopen subset of the inverse limit. The details will be presented over the following three definitions.

Definition 3.1.5. Suppose m, $n \in \mathbb{N}$ with $m+2 \leq n$, for each i, $m \leq i \leq n$, X_i is a Hausdorff continuum, U_i is a connected open subset of X_i, $U_i \neq X_i$, A_i is a closed subset of X_i such that $A_i \subset U_i$, and for $m < j < n$ there exist sets $P_j, S_j \subset U_j \setminus A_j$ such that $P_j \cap S_j = \varnothing$, $\overline{P_j} \cap A_j \neq \varnothing \neq \overline{S_j} \cap A_j$, and $U_j = A_j \cup P_j \cup S_j$. Then let

$$\begin{aligned} L_{S_{m+1}} &= (S_{m+1} \times U_m) \cup (A_{m+1} \times A_m), \\ T_{P_{n-1}} &= (U_n \times P_{n-1}) \cup (A_n \times A_{n-1}), \end{aligned}$$

and for each k, such that $m+1 < k < n$, let

$$TL_{S_k, P_{k-1}} = (U_k \times P_{k-1}) \cup (S_k \times U_{k-1}) \cup (A_k \times A_{k-1})$$

(and if each of the sets S_i and P_i are clear from the context we will simply write L_{m+1}, T_n and TL_k).

The regions can be seen schematically in Figure 3.2

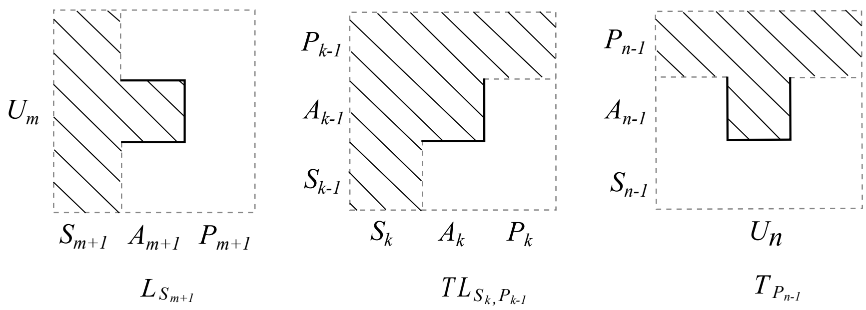

Figure 3.2: A graphical representation of the regions defined in Definition 3.1.5.

Definition 3.1.6. Suppose X and Y are Hausdorff continua, $f : X \to 2^Y$ is an upper semicontinuous function, and $G \subset X \times Y$ is the graph of f. Suppose $U, C, D \subset X \times Y$ and $C \subset D \subset U$. Then $G \Subset_{U,C} D$ if $G \cap U \subset D$ and $G \cap C \neq \varnothing$.

When it is clear from the context what the sets U and C are, we will drop the subscripts and simply write $G \Subset D$.

Definition 3.1.7. Let (X_i, f_i) be a generalised inverse sequence. Then $\{f_i : i > 0\}$ admits a *Hausdorff cropping-sequence*, or *HC-sequence* $\{(U_i, A_i) : m \leq i \leq n\}$, over $[m, n]$, if $m, n \in \mathbb{N}$, $m + 1 < n$, and whenever $m \leq i \leq n$, there exists a connected open set $U_i \subset X_i$, a closed set $A_i \subset U_i \subset X_i$, $U_i \neq X_i$, $\mathcal{G}(f_{m+1}, \ldots, f_n) \cap \Pi_{m \leq i \leq n} A_i \neq \varnothing$, and whenever $m < j < n$, there exist disjoint nonempty sets $P_j, S_j \subset U_j \setminus A_j$ such that $P_j \cap S_j = \varnothing$, $\overline{S_j} \cap A_j \neq \varnothing \neq \overline{P_j} \cap A_j$, and the graphs G_i of f_i have the following properties:

1. $G_{m+1} \Subset_{U_{m+1} \times U_m, A_{m+1} \times A_m} L_{m+1}$;
2. $G_n \Subset_{U_n \times U_{n-1}, A_n \times A_{n-1}} T_n$; and
3. if $m + 1 < i < n$ then $G_i \Subset_{U_i \times U_{i-1}, A_i \times A_{i-1}} TL_i$.

The collection of functions $\{f_i : i > 0\}$ admits an *HC-sequence* if there exist m, $n \in \mathbb{N}$ such that $\{f_i : i > 0\}$ admits an HC-sequence over $[m, n]$.

The definition of an HC-sequence is very similar to that of a C-sequence in [GK]. The main difference is that since we are now working with the more general Hausdorff continua instead of intervals, notions of 'top' (T), 'bottom' (B), 'left' (L), and 'right' (R) no longer make sense, so we only have the more abstract T sets (which cover both T and B in the C-sequence case) and L sets (which cover L and R). This is discussed further in Section 3.3.

The following lemma is a partial extension to Theorem 3.0.3.

Lemma 3.1.8. *Let (X_i, f_i) be a generalised inverse sequence, suppose each f_i is a surjective function, and $\{f_i : i > 0\}$ admits an HC-sequence $\{(U_i, A_i) : m \leq$*

$i \le n\}$ over $[m,n]$. For each i, $m \le i \le n$, let $V_i = U_{n-(i-m)}$ and $B_i = A_{n-(i-m)}$, and or each i, $m < i \le n$, let $g_i = f^{-1}_{n-(i-m)+1}$. Then $\{(V_i, B_i) : m \le i \le n\}$ is an HC-sequence admitted by $\{g_i : m < i \le n\}$.

Proof. This result follows from the symmetry between the sets S_i and P_i in the definition of an HC-sequence and Theorem 3.1.2. □

3.2 Connected Generalised Inverse Limits

We will prove the main result of this chapter, Theorem 3.0.6 by proving each direction of the equivalence separately, as Theorem 3.2.1 and Theorem 3.2.4. The forward direction is relatively simple. Remember that due to Lemma 3.1.4 we can assume that the disconnection starts at the first factor space, X_0.

Theorem 3.2.1. *Let (X_i, f_i) be a generalised inverse sequence, suppose each f_i is a surjective function, the graph G_{i+1} of f_{i+1} is connected, and there exists $n > 1$ such that $\{f_i : i > 0\}$ admits an HC-sequence over $[0,n]$. Then for each $i \le n$, there exists an open set $U_i \subset X_i$, a closed set $A_i \subset U_i \subset X_i$, such that $U_i \ne X_i$, $\mathcal{G}_n \cap \Pi_{i \le n} A_i = \mathcal{G}_n \cap \Pi_{i \le n} U_i \ne \varnothing$ and $\mathcal{G}_n \ne \mathcal{G}_n \cap \Pi_{i \le n} A_i$.*

Proof. Suppose there exists an HC-sequence $\{(U_i, A_i) : i \le n\}$ over $[0,n]$.

Since $G_1 \Subset L_1$, $x_1 \in (S_1 \cup A_1)$, and hence, since $G_2 \Subset TL_2$, $x_2 \in (S_2 \cup A_2)$. Suppose $1 < k < n$. If $x_k \in (S_k \cup A_k)$, then since $G_{k+1} \Subset TL_{k+1}$, $x_{k+1} \in (S_{k+1} \cup A_{k+1})$. Hence by induction $x_i \in S_i \cup A_i$ if $0 < i < n$.

Thus, since $G_n \Subset T_n$, $x_{n-1} \in A_{n-1}$ and $x_n \in A_n$. Suppose $1 < k < n$ and $x_k \in A_k$, then since $x_{k-1} \notin P_{k-1}$ and $G_k \Subset TL_k$, $x_{k-1} \in A_{k-1}$. Thus for each i such that $0 < i \le n$, $x_i \in A_i$, and since $G_1 \Subset L_1$, $x_0 \in A_0$.

Thus $\mathcal{G}_n \cap \Pi_{i \le n} A_i = \mathcal{G}_n \cap \Pi_{i \le n} U_i$. Since we are dealing with an HC-sequence, $\mathcal{G}_n \cap \Pi_{i \le n} A_i \ne \varnothing$, and since each f_i is surjective, there exists $\langle x_0, \dots, x_n \rangle \in \mathcal{G}_n$ such that $x_1 \notin U_1$, and so $\mathcal{G}_n \notin \Pi_{i \le n} U_i$.

□

For the next lemma, we will require a version of the 'Boundary Bumping Theorem' from continuum theory. A proof can be found in [Nad, Theorem 5.6]. Here $Bd(E)$ denotes the boundary of E.

Theorem 3.2.2 (Boundary Bumping Theorem). *Let X be a continuum, and let E be a nonempty proper subset of X. If K is a component of E, then $\overline{K} \cap Bd(E) \neq \emptyset$.*

To prove the backwards direction, first we prove the case where the disconnection happens over the first two functions, so the HC sequence happens over three factor spaces.

Lemma 3.2.3. *Let (X_i, f_i) be a generalised inverse sequence, and suppose each f_i is a surjective function, the graph G_{i+1} of f_{i+1} is connected, and there exists a connected open set $U_i \subset X_i$ and a closed set $A_i \subset U_i \subset X_i$, such that $U_i \neq X_i$, and*

$$\mathcal{G}_2 \cap (A_0 \times A_1 \times A_2) = \mathcal{G}_2 \cap (U_0 \times U_1 \times U_2) \neq \emptyset.$$

Then $\{f_i : i \in \mathbb{N}\}$ admits an HC-sequence.

Proof. Let

$$\begin{aligned}
A'_0 &= A_0 \\
A'_2 &= A_2 \\
B_1 &= f_2(A_2) \cap A_1 \cap f_1^{-1}(A_0) \\
S_1 &= (U_1 \cap f_1^{-1}(U_0)) \setminus B_1 \\
P_1 &= (U_1 \cap f_2(U_2)) \setminus B_1 \\
A'_1 &= U_1 \setminus (S_1 \cup P_1).
\end{aligned}$$

Then we have

$$\begin{aligned}
\mathcal{G}_2 \cap (A_0 \times A_1 \times A_2) &= \mathcal{G}_2 \cap (A_0 \times B_1 \times A_2) \\
&= \mathcal{G}_2 \cap (U_0 \times U_1 \times U_2) \neq \emptyset.
\end{aligned}$$

We claim that $\{(U_i, A'_i) : i \le 2\}$ is an HC-sequence admitted by $\{f_1, f_2\}$.

Observe that $((A'_1 \setminus B_1) \times U_0) \cap G_1 = \varnothing$ and $(U_2 \times (A'_1 \setminus B_1)) \cap G_1 = \varnothing$. Suppose $\langle x, y\rangle \in (G_1 \cap (U_1 \times U_0))$:

- If $x \in B_1$, then there exists $z \in A_2$ such that $x \in f(z)$, hence $\langle y, x, z\rangle \in \mathcal{G}_2$, and so $y \in A_0$.

- If $x \in P_1$, then there exists $z \in U_2$ such that $x \in f(z)$, and hence $\langle y, x, z\rangle \in \mathcal{G}_2 \cap U_0 \times U_1 \times U_2$, but $x \notin B_1$ giving a contradiction, so $x \notin P_1$, and thus $G_1 \Subset L_1$.

We can show similarly, by Lemma 3.1.8, that if $\langle x, y\rangle \in (G_2 \cap (U_2 \times U_1))$, then $y \notin S_1$, and if $y \in B_1$ then $x \in A_2$, and so $G_2 \Subset T_2$.

Suppose that $S_1 = \varnothing$, then $G_1 \cap (U_1 \times U_0) \subseteq A_1 \times A_0$. Since $U_1 \ne X_1$,

$$W := G_1 \cap ((X_1 \times X_0) \setminus (A_1 \times A_0)) \ne \varnothing.$$

Let K be a component of W, then by the Boundary Bumping Theorem (Theorem 3.2.2), there exists a point $p \in \overline{W} \cap \overline{A_1 \times A_0}$, but $A_1 \times A_0$ is a closed subset of the open set $U_1 \times U_0$, so $p \in A_1 \times A_0$ and p does not have a neighbourhood in $U_1 \times U_0$. Thus $S_1 \ne \varnothing$.

Similarly, $P_1 \ne \varnothing$. Also, $P_1 \cap S_1 = \varnothing$, since if there exists $y \in P_1 \cap S_1$, then there exist $x \in U_0$ and $z \in U_2$ such that $\langle y, x\rangle \in G_1$ and $\langle z, y\rangle \in G_2$, and hence $\langle x, y, z\rangle \in \mathcal{G}_2$, but $y \notin B_1$.

Since for every $\langle x, y\rangle \in (G_1 \cap (U_1 \times U_0))$, $x \notin P_1$ and if $x \in B_1$ then $y \in A_0$, and G_1 is connected, $\overline{S_1} \cap B_1 \ne \varnothing$ by the Boundary Bumping Theorem, and hence $\overline{S_1} \cap A'_1 \ne \varnothing$. Similarly, $\overline{P_1} \cap B_1 \ne \varnothing$, and hence $\overline{P_1} \cap A'_1 \ne \varnothing$.

Thus $\{(U_i, A'_i) : i \le 2\}$ is an HC-sequence admitted by $\{f_1, f_2\}$.

□

We can now use Lemma 3.2.3 to complete the proof of the backwards portion of the main theorem.

Theorem 3.2.4. *Let (X_i, f_i) be a generalised inverse sequence, and suppose each f_i is a surjective function with a connected graph G_{i+1}. If for each i, $m \leq i \leq n$, there exists a connected open set $U_i \subset X_i$, and a closed set $A_i \subset U_i \subset X_i$, such that $U_i \neq X_i$, and*

$$\mathcal{G}(f_{m+1}, \dots, f_n) \cap \Pi_{m \leq i \leq n} A_i = \mathcal{G}(f_{m+1}, \dots, f_n) \cap \Pi_{m \leq i \leq n} U_i \neq \varnothing,$$

then $\{f_i : i \in \mathbb{N}\}$ admits an HC-sequence.

Proof. Without loss of generality, assume that $n - m$ is minimal. That is, for any p, q such that $m \leq p < q \leq n$ and either $p \neq m$ or $q \neq n$,

$$\mathcal{G}(f_{p+1}, \dots, f_q) \cap (A_p \times \cdots \times A_q) \neq \mathcal{G}(f_{p+1}, \dots, f_q) \cap (U_p \times \cdots \times U_q).$$

By Lemma 3.1.4 we can also assume that $m = 0$, and by Lemma 3.2.3 assume $n > 2$.

For each j such that $0 < j < n$, let

$$\begin{array}{rcl} B_j &=& \pi_j(\mathcal{G}_n \cap \Pi_{i \leq n} A_i) \\ P_j &=& \pi_j(\mathcal{G}(f_{j+1}, \dots, f_n) \cap \Pi_{j \leq i \leq n} U_i) \smallsetminus B_j \\ S_j &=& \pi_j(\mathcal{G}_j \cap \Pi_{i \leq j} U_i) \smallsetminus B_j \\ Q_j &=& U_j \smallsetminus (B_j \cup P_j \cup S_j) \\ A'_j &=& B_j \cup Q_j. \end{array}$$

Also, let $A'_0 = A_0$, and $A'_n = A_n$.

We claim that $\{(U_i, A'_i) : i \leq n\}$ is an HC-sequence admitted by $\{f_1, \dots, f_n\}$.

Suppose $\langle x, y \rangle \in (G_1 \cap (U_1 \times U_0))$. Then $x \notin B_1$ if and only if $x \in S_1$. Suppose $x \notin B_1$ and $x \in P_1$ then there exists $\langle x_1, \dots, x_n \rangle \in (\mathcal{G}(f_2 \dots, f_n) \cap \Pi_{1 \leq i \leq n} U_i)$ such that $x = x_1$ and hence

$$\langle y, x, x_2, \dots, x_n \rangle \in \mathcal{G}_n \cap \Pi_{i \leq n} U_i),$$

but $x \notin A_1$ so we have a contradiction. Therefore, $x \notin P_1$, and $S_1 \cap P_1 = \varnothing$. If $x \in B_1$, then there exists $\langle x_0, \dots, x_n \rangle \in (\mathcal{G}_n \cap \Pi_{i \leq n} A_i)$ such that $x_1 = x$, hence $\langle y, x, x_2, \ldots, x_n\rangle \in (\mathcal{G}_n \cap \Pi_{i\leq n} U_i$, and so $y \in A'_0$. Observe that for each $\langle x, y\rangle \in (G_1 \cap (U_1 \times U_0))$, $x \notin Q_1$. Thus $G_1 \Subset L_1$.

If $\langle x, y\rangle \in (G_n \cap (U_n \times U_{n-1}))$, then we can show similarly, by Lemma 3.1.8, that $x \notin S_{n-1} \cup Q_{n-1}$, $S_{n-1} \cap P_{n-1} = \varnothing$ and if $y \in B_{n-1}$ then $x \in B_n$, and hence $G_n \Subset T_n$.

Suppose $1 < j < n$ and $\langle x, y\rangle \in G_j \cap (U_j \times U_{j-1})$. Suppose $x \in P_j$. Then there exists

$$\langle x_j, \ldots, x_n\rangle \in (\mathcal{G}(f_{j+1} \ldots, f_n) \cap \Pi_{j\leq i\leq n} U_i)$$

such that $x = x_j$. If $y \in (B_{j-1} \cup S_{j-1})$, then there exists

$$\langle x_0, \ldots, x_{j-1}\rangle \in (\mathcal{G}_{j-1} \cap \Pi_{i<j} U_i)$$

such that $y = x_{j-1}$, but then

$$\langle x_0, \ldots, x_{j-2}, y, x, x_{j+1}, \ldots, x_n\rangle \in (\mathcal{G}_n \cap \Pi_{i\leq n} U_i)$$

giving a contradiction since $x \notin B_j$, so $y \notin (B_{j-1} \cup S_{j-1})$. Furthermore, $y \notin Q_{j-1}$, since $x \in P_j$, and

$$\langle y, x, x_{j+1}, \ldots, x_n\rangle \in (\mathcal{G}(f_j \ldots, f_n) \cap \Pi_{j-1\leq i\leq n} U_i),$$

implies that $y \in P_{j-1}$. Hence

$$G_j \cap (P_j \times (B_{j-1} \cup S_{j-1} \cup Q_{j-1})) = \varnothing.$$

If $x \in B_j$ and $y \in S_{j-1}$, then there exists

$$\langle x_0, \ldots, x_{j-1}\rangle \in (\mathcal{G}_{j-1} \cap \Pi_{i\leq j-1} U_i)$$

such that $y = x_{j-1}$, and there exists

$$\langle y_0, \ldots, y_n\rangle \in (\mathcal{G}_n \cap \Pi_{i\leq n} A_i),$$

such that $y_j = x$. Hence $\langle x_0, \ldots, x_{j-2}, y, x, y_{j+1}, \ldots, y_n\rangle \in (\mathcal{G}_n \cap \Pi_{i\leq n} U_i)$, a contradiction, and so $G_j \cap (B_j \times S_{j-1}) = \varnothing$.

If $x \in Q_j$ and $y \in S_{j-1}$, then again there exists $\langle x_0, \ldots, x_{j-1} \rangle \in \mathcal{G}_{j-1}$ such that $y = x_{j-1}$, but then $\langle x_0, \ldots, x_{j-1}, x \rangle \in \mathcal{G}_j$ and hence $x \in S_j$. Similarly, $\langle x, y \rangle \notin (P_j \times Q_{j-1})$ (in fact, we can show similarly that both $G_j \cap (Q_j \times B_{j-1})$ and $G_j \cap (B_j \times Q_{j-1})$ are empty, but this is not required in order to construct an HC-sequence.)

By the same argument as in the proof of Lemma 3.2.3, $\overline{S_1} \cap B_1 \neq \varnothing$ and $\overline{P}_{n-1} \cap B_{n-1} = \varnothing$.

Suppose $P_j = \varnothing$. Then

$$\pi_j(\mathcal{G}(f_{j+1}, \ldots, f_n) \cap (U_j \times \cdots \times U_n)) \subseteq A_j.$$

Hence

$$(\mathcal{G}(f_{j+1}, \ldots, f_n) \cap (U_j \times \cdots \times U_n)) \subseteq A_j \times \cdots \times A_n,$$

giving a contradiction since

$$\mathcal{G}(f_{j+1}, \ldots, f_n) \cap (A_j \times \cdots \times A_n) \neq \mathcal{G}(f_{j+1}, \ldots, f_n) \cap (U_j \times \cdots \times U_n).$$

Similarly, $S_j \neq \varnothing$, otherwise

$$\pi_j(\mathcal{G}_j \cap (U_1 \times \cdots \times U_j)) \subseteq A_0 \times \cdots \times A_j.$$

Suppose $\overline{P}_j \cap B_j = \varnothing$. Then we can choose an open set $V \subset U_j$ such that $B_j \subset V \subset \overline{V} \subset U_j$ and $V \cap P_j = \varnothing$. But then

$$\mathcal{G}_n \cap \Pi_{i \leq n} A_i = \mathcal{G}_n \cap (U_0 \times \cdots \times U_{j-1} \times V \times U_{j+1} \times \cdots \times U_n) \neq \varnothing,$$

and had we started with V in place of U_j, we would obtain P_j such that $P_j \cap B_j = \varnothing$. Thus, $\overline{P}_j \cap B_j \neq \varnothing$.

Similarly, by Lemma 3.1.8, $\overline{S}_{j+1} \cap B_{j+1} \neq \varnothing$.

Thus it follows that $G_i \Subset TL_{(A'_i \times A'_{i-1}) \times (U_i \times U_{i-1})}$ for each i, and $\{(U_i, A'_i) : i \leq n\}$ is an HC-sequence admitted by $\{f_1, \ldots, f_n\}$. □

We now prove the main theorem of this chapter, Theorem 3.0.6.

Proof. Suppose that for each $i \geq 0$, X_i is a Hausdorff continuum, and $f_{i+1} : X_{i+1} \to 2^{X_i}$ is an upper semicontinuous surjective function with a connected graph G_{i+1}.

If $\{f_i : i > 0\}$ admits an HC-sequence over $[m, n]$, then $\{f_i^m : i > 0\}$ admits an HC-sequence over $[0, n-m]$ and hence by Theorem 3.2.1, for each $i \leq n$, there exists an open set $U_{i+m} \subset X_{i+m}$ and a closed set $A_{i+m} \subset U_{i+m} \subset X_{i+m}$, such that $U_{i+m} \neq X_{i+m}$,

$$\mathcal{G}(f_{m+1}, \dots f_n) \cap \Pi_{m \leq i \leq n} A_i = \mathcal{G}(f_{m+1}, \dots f_n) \cap \Pi_{m \leq i \leq n} U_i \neq \varnothing$$

and

$$\mathcal{G}(f_{m+1}, \dots f_n) \neq (\mathcal{G}(f_{m+1}, \dots f_n) \cap \Pi_{m \leq i \leq n} A_i.$$

Hence

$$U = \Pi_{0 \leq i < m} X_i \times \Pi_{m \leq i \leq n} U_i \times \Pi_{i > n} X_i$$

is a basic open subset of $\Pi_{i \in \mathbb{N}} X_i$ containing the closed set

$$A = \Pi_{0 \leq i < m} X_i \times \Pi_{m \leq i \leq n} A_i \times \Pi_{i > n} X_i,$$

such that $\varprojlim (X_i, f_i) \cap U = \varprojlim (X_i, f_i) \cap A \neq \varnothing$, and $\varprojlim (X_i, f_i) \notin U$.

Suppose there exists a connected basic open set

$$U = \Pi_{0 \leq i < m} X_i \times \Pi_{m \leq i \leq n} U_i \times \Pi_{i > n} X_i$$

in $\Pi_{i \geq 0} X_i$ containing a closed set

$$A = \Pi_{0 \leq i < m} X_i \times \Pi_{m \leq i \leq n} A_i \times \Pi_{i > n} X_i,$$

such that $\varprojlim (X_i, f_i) \cap U = \varprojlim (X_i, f_i) \cap A \neq \varnothing$, and $\varprojlim (X_i, f_i) \notin U$. Then by Lemma 3.1.4,

$$\begin{aligned} &\varprojlim (X_i, f_i)_{i \geq m} \cap (\Pi_{m \leq i \leq n} U_i \times \Pi_{i > n} X_i) \\ &= \varprojlim (X_i, f_i)_{i \geq m} \cap (\Pi_{m \leq i \leq n} A_i \times \Pi_{i > n} X_i) \end{aligned}$$

and hence by Theorem 3.2.4 $\{f_i : i > m\}$ admits an HC-sequence over $[0, n-m]$. Thus $\{f_i : i > 0\}$ admits an HC-sequence over $[m, n]$. $\square$

3.3 Connectedness of Sets U_i, A_i, S_i and P_i

If each set X_i is a closed interval, then a C-sequence as defined in [GK] gives an HC-sequence in the current setting. A C-sequence as defined in [GK] imposes stronger properties on the various sets involved. In particular, each of the sets J_i, K_i (which are the analogue to S_i, P_i) and A_i is connected. The following examples show that in general we can't expect that if there is an HC-sequence then there is an HC-sequence in which any of these sets is connected.

Recall from Chapter 1 that for each $i > n$, f_i is the identity function, then $\varprojlim (X_i, f_i)$ is homeomorphic to $\mathcal{G}_n$. Thus we need only consider finite sequences of functions.

Example 3.3.1. Let T be the triod consisting of three line segments in $\mathbb{R}^2$ that meet at the point $\left\langle \frac{1}{4}, 0\right\rangle$ as shown in Figure 3.3.

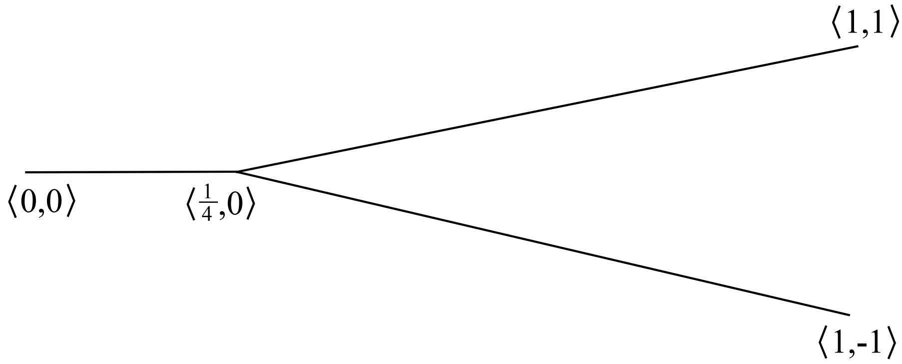

Figure 3.3: The triod space T in Example 3.3.1.

Let $X_0 = X_2 = [0,1]$, and let $X_1 = T$. Define $f_1 : X_1 \to 2^{X_0}$ by

$$f_1(\langle x, y\rangle) = \begin{cases} \{0, x\} & \text{if } x \leq \frac{1}{4} \\ [0,1] & \text{if } x = 1 \\ \{0\} & \text{otherwise,} \end{cases}$$

and $f_2 : X_2 \to 2^{X_1}$ by

$$f_2(x) = \begin{cases} \{\langle 0,0\rangle\} & \text{if } x < \frac{3}{4} \\ \{\langle 0,0\rangle, \langle 3x-2, 4x-3\rangle, \langle 3x-2, 3-4x\rangle\} & \text{if } \frac{3}{4} \le x < 1 \\ T & x = 1. \end{cases}$$

See Figures 3.4 and 3.5 for the graphs G_1 of f_1 and G_2 of f_2.

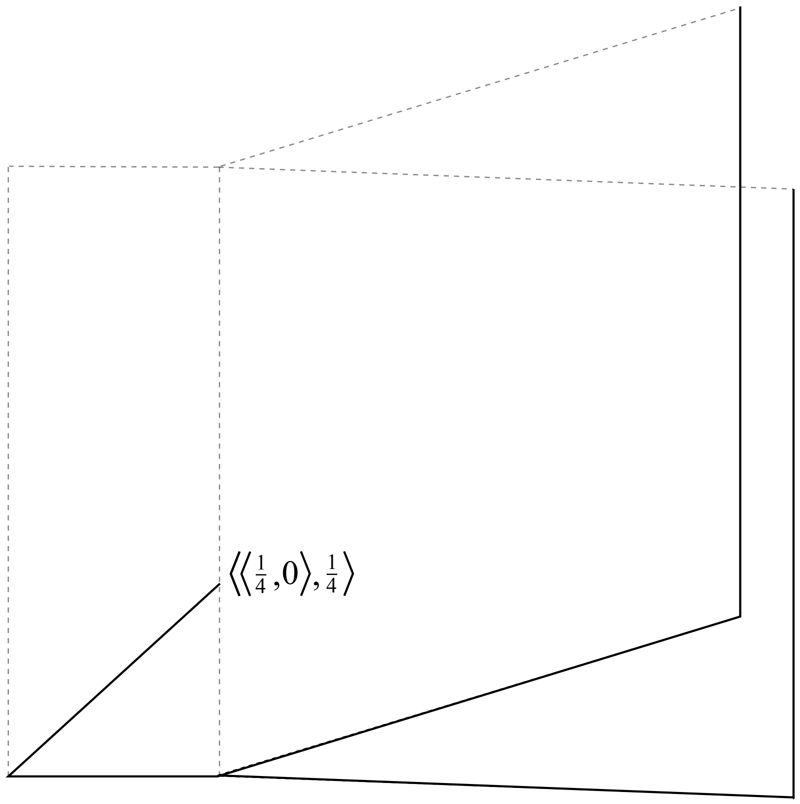

Figure 3.4: The graph G_1 of f_1 in Example 3.3.1.

Let $U_0 = \left(\frac{1}{8}, \frac{3}{8}\right)$, $U_1 = \left(\left(\frac{1}{8}, \frac{3}{8}\right) \times \left(\frac{-1}{8}, \frac{1}{8}\right)\right) \cap T$, $U_2 = \left(\frac{5}{8}, \frac{7}{8}\right)$, $A_0 = \{\frac{1}{4}\}$, $A_1 = \{\langle\frac{1}{4}, 0\rangle\}$, $A_2 = \{\frac{3}{4}\}$, $S_1 = (U_1 \cap f_1^{-1}(U_0)) \setminus A_1 = \{\langle x, 0\rangle : \frac{1}{8} < x < \frac{1}{4}\}$ and

$$\begin{aligned} P_1 &= (U_1 \cap f_2(U_2)) \setminus A_1 \\ &= \left\{\langle 3x-2, 4x-3\rangle : \frac{3}{4} < x < \frac{7}{8}\right\} \cup \left\{\langle 3x-2, 3-4x\rangle : \frac{3}{4} < x < \frac{7}{8}\right\}. \end{aligned}$$

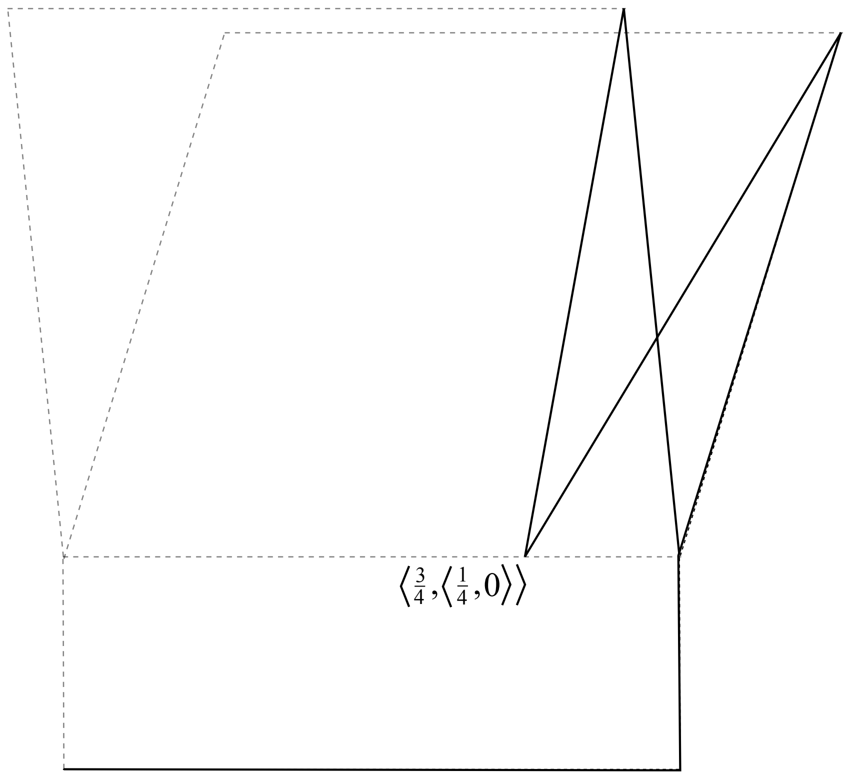

Figure 3.5: The graph G_2 of f_2 in Example 3.3.1.

Then $G_1 \Subset R_1$ and $G_2 \Subset T_2$ with respect to these sets, and hence $\{(U_i, A_i) : i \leq 2\}$ is an HC-sequence where P_1 is the union of the two disjoint sets $\{\langle 3x - 2, 4x - 3\rangle : \frac{3}{4} < x < \frac{7}{8}\}$ and $\{\langle 3x - 2, 3 - 4x\rangle : \frac{3}{4} < x < \frac{7}{8}\}$.

Then if $\{(V_i, B_i)\}$ is any HC-sequence, we have that $B_i \supset A_i$ for each $i \leq 2$, and since V_i is open, and $V_i \supset B_i$, then P_i will contain points from two arcs in the triod, so P_i will be disconnected.

Note that any HC-sequence admitted by $\{f_2^{-1}, f_1^{-1}\}$ will render S_1 disconnected.

Example 3.3.2. Let $C = \{c_\alpha : \alpha \in \mathfrak{c}\} \subset \left\{\frac{1}{4}\right\} \times \left[\frac{1}{8}, \frac{3}{8}\right]$ be a Cantor set embedded in $\mathbb{R}^2$, and let X_0 be the subset of $\mathbb{R}^2$ indicated in Figure 3.6, namely, the union of the the line segments with end points:

- $\langle 0, 0\rangle$ and $\langle 1, 0\rangle$,
- $\langle 1, 0\rangle$ and $\langle 1, 1\rangle$,
- $\langle 1, 1\rangle$ and $\langle \frac{3}{4}, \frac{1}{4}\rangle$, and
- $\langle 0, 0\rangle$ and c_α for each $\alpha \in \mathfrak{c}$.

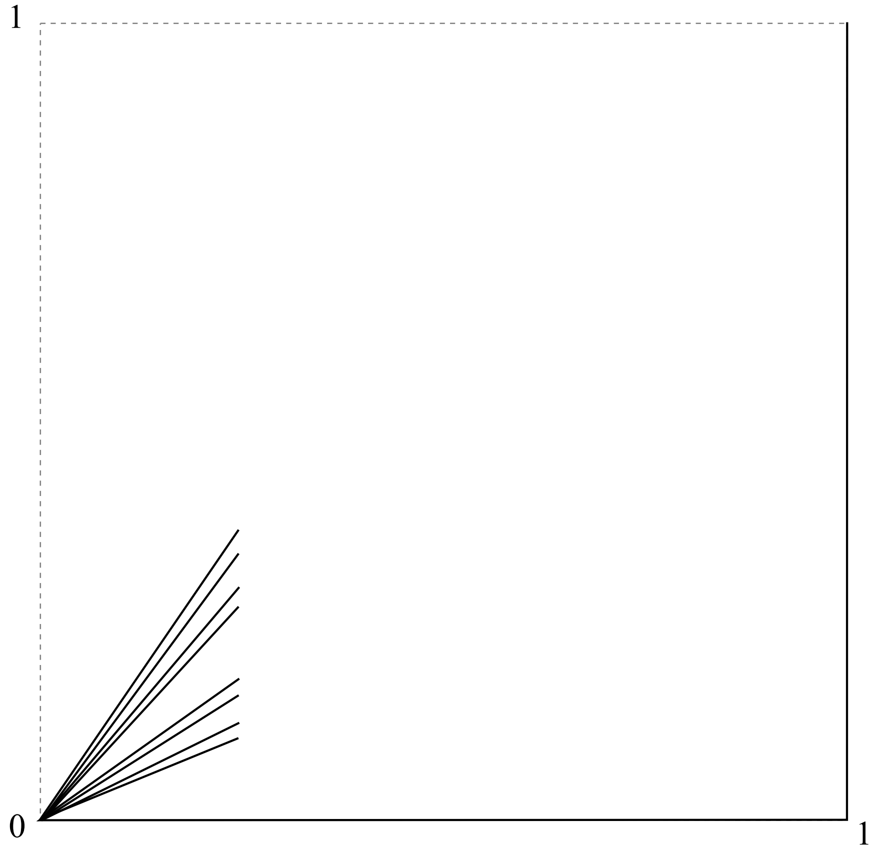

Figure 3.6: The space X_0 in Example 3.3.2.

Let $X_1 = X_2 = [0, 1]$, define $f_1 : X_1 \to X_0$ by $f_1(x) = \{\langle x_0, x_1\rangle \in X_0 : x_0 = x\}$, and let f_2 be the function whose graph is that shown in Figure 3.7.

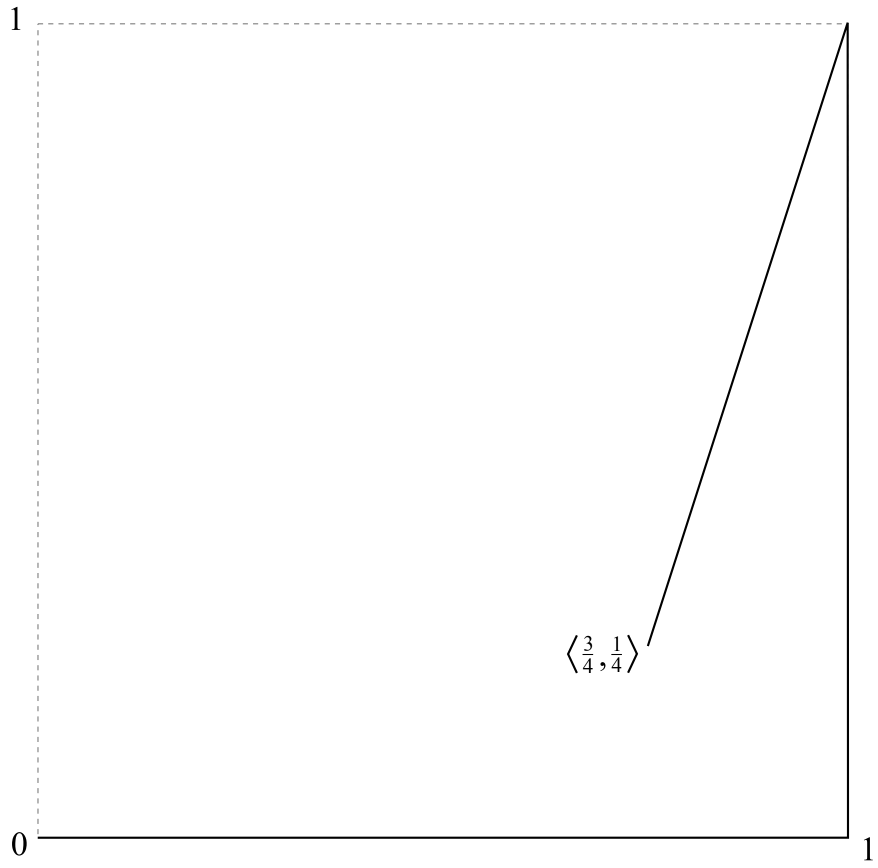

Figure 3.7: The function f_2 in Example 3.3.2.

Let $U_0 = \left(\left(\frac{1}{8}, \frac{3}{8}\right) \times \left(\frac{1}{9}, \frac{4}{9}\right)\right) \cap X_0$, $U_1 = \left(\frac{1}{8}, \frac{3}{8}\right)$, $U_2 = \left(\frac{5}{8}, \frac{7}{8}\right)$, and let $A_0 = \left(\left\{\frac{1}{4}\right\} \times \left[\frac{1}{8}, \frac{3}{8}\right]\right) \cap X_0$, $A_1 = \left\{\frac{1}{4}\right\}$, and $A_2 = \left\{\frac{3}{4}\right\}$. Then clearly $\{(U_i, A_i) : i \leq 2\}$ is an HC-sequence, A_0 is disconnected, and U_0 is disconnected.

Furthermore, for any HC-sequence $\{(V_i, B_i) : i \leq 2\}$ (and hence

$$\{\langle\langle 1/4, c_\alpha\rangle, 1/4, 3/4\rangle : \alpha \in D\} \subseteq (B_0 \times B_1 \times B_2) \cap \mathcal{G}(f_1, f_2)$$

for some $D \subseteq \mathfrak{c}$), it must be the case that B_0 is a neighbourhood of some point $\left\langle \frac{1}{4}, a\right\rangle$ where $a \in \left[\frac{1}{8}, \frac{3}{8}\right]$, and hence both B_0 and V_0 say, are disconnected.

Example 3.3.2 demonstrates that the clopen subset of an inverse limit that is "trapped" by an HC-sequence need not be a component, and can in fact be a Cantor set. The following example shows that this can be the case even if our spaces are intervals.

Example 3.3.3. Let $X_0 = X_1 = X_2 = [0, 1]$, let $f_1 : X_1 \to 2^{X_0}$ be the function whose graph is shown in Figure 3.6 (the graph of f_1 is space X_0 in Example 3.3.2), and let $f_2 : X_2 \to X_1$ be the function whose graph is that shown in Figure 3.7.

The set $W = \left\{\langle x, \frac{1}{4}, \frac{3}{4}\rangle \in \mathcal{G}(f_1, f_2) : x \neq 0\right\}$ is a Cantor set. Any one of the points in W can be captured by an HC sequence $\{(U_i, A_i) : i \leq 2\}$, but the definition of an HC-sequence requires that $A_0 \times A_1 \times A_2$ includes other members of W, and hence for any HC-sequence $\{(V_i, B_i) : i \leq 2\}$, it must be the case that B_0 is a neighbourhood of some point $\langle \frac{1}{4}, a\rangle$ where $a \in [\frac{1}{8}, \frac{3}{8}]$, and hence $(B_0 \times B_1 \times B) \cap \mathcal{G}(f_1, f_2)$ contains a Cantor set.

3.4 Consequences

In this chapter we have generalised the notion of a C-sequence to that of an HC-sequence. This gives a sufficient condition on the graphs that is often easy to recognise for the inverse limit to be disconnected.

Connectedness is a central property in the study of generalised inverse limits. Theorem 1.2.2 states that a generalised inverse limit will always be compact, hence a generalised inverse limit will be a continuum if and only if it is connected. Knowing if a particular generalised inverse limit is connected is therefore very useful to a researcher, even when looking at other properties. For example, one of the main current areas of study is indecomposability, which will require the inverse limit to be connected. Any conditions that guarantee indecomposability will therefore also require the inverse limit to be connected, and so will need to incorporate some result to guarantee connectedness.

Further work is required to extend this result to characterise disconnected generalised inverse limits completely. A starting point is to generalise the main result in [GK3], where the result is given for factor spaces being intervals.

Chapter 4

Path Connectedness

Related to the question of the connectedness of generalised inverse limits is the question of path connectedness. There are important differences between the two properties in this context. As we saw in Chapter 1 from Theorem 1.2.5, if $\varprojlim(X_i, f_i)$ is disconnected, then $\mathcal{G}_n$ must be disconnected for some $n \in \mathbb{N}$. This is not the case for path connectedness, for example the usual tent map $f_{(\frac{1}{2},1)}$, whose graph is shown in Figure 1.1, has $\mathcal{G}_n$ homeomorphic to an arc for all $n \in \mathbb{N}$, but $\varprojlim f_{(\frac{1}{2},1)}$ is homeomorphic to the indecomposable buckethandle continuum, which is not path connected.

Of course, for $\varprojlim(X_i, f_i)$ to be path connected, it is necessary that $\mathcal{G}_n$ is path connected for all $n \in \mathbb{N}$, so this means that in addition to ensuring that $\mathcal{G}_n$ is path connected for all $n \in \mathbb{N}$, there is the separate problem of finding when $\varprojlim(X_i, f_i)$ is path connected. This chapter deals with the latter problem. Throughout this chapter we assume for all n that $\mathcal{G}_n$ is path connected, and look for conditions that will ensure $\varprojlim(X_i, f_i)$ is path connected.

The main theorem that will be proved in this section is the following (some of the terms will be defined in Section 4.1.

Theorem 4.0.1. *For all $i \geq 0$, let $I_i = [0,1]$, and for all $i \geq 1$ let $f_i : I_i \to 2^{I_{i-1}}$ be upper semicontinuous. Let* $\mathbf{a}$ *and* $\mathbf{b} \in \varprojlim(I_i, f_i)$*. Then there exists a path* *in* $\varprojlim(I_i, f_i)$ *between* $\mathbf{a}$ *and* $\mathbf{b}$ *if and only if for every* $n \in \mathbb{N}$*, for every open interval* $U \subset I_n$*, there is an upper bound* $k \in \mathbb{N}$ *such that for all* $l \in \mathbb{N}$*, where* $l \geq n$*, there exists a finite extension sequence* $(\gamma_i^l : i \leq l)$ *for* $(\mathbf{a}, \mathbf{b})$ *such that* γ_n^l *covers* U *fewer than* k *times.*

This theorem links the path connectedness of a generalised inverse limit to a condition on the path connectedness of its finite Mahavier product approximants, $\mathcal{G}_n$ for each $n \geq 1$.

The method for proving Theorem 4.0.1 is as follows. First we show that the question of whether a path exists between two points $\mathbf{a}$ and $\mathbf{b} \in \varprojlim(I_i, f_i)$ can be reduced to whether there exists a sequence of paths in the individual graphs that have the property that the coordinates of these paths will line up in such a way that they will create a path in $\varprojlim(I_i, f_i)$. This sequence is called a limit extension, and is properly defined later. This step is relatively straight forward.

For the second step, we show that there is a limit extension if and only if for each open $U \subset I_n$, there is an upper bound k such that there is an arbitrarily long finite extension sequence that covers U less than k times (many of these terms are more rigorously defined later). The forward direction of this step is trivial. For the reverse direction, we concentrate on individual graphs, and show that if there is an arbitrarily long number of finite extension sequences that cover each open set less than some maximum number of times (for that set), then the paths on each graph can be reparameterised to what are known as well parameterised functions. These well parameterised functions will then converge to a path in each graph, by the Arzelá Ascoli Theorem. These paths have the necessary properties to be a limit extension, and hence correspond to a path in the inverse limit.

In the next section we will introduce some definitions and preliminary lemmas required for proving the main theorem in the following section. In the final section we give some examples of the application of this theorem to deciding whether a generalised inverse limit is path connected.

4.1 Paths and Coverings

In this chapter the norms used are the usual ones, ie if $\mathbf{x} \in \mathbb{R}^2$ then $||\mathbf{x}|| = \sqrt{x_1^2 + x_2^2}$, and if $f : [0,1] \to [0,1] \times [0,1]$ is a continuous function, then $||f|| = \sup\{||f(x)|| : x \in [0,1]\}$.

Throughout this chapter, we will assume all spaces $X_i = I_i = [0,1]$. The concept of a cover of a set is used throughout this chapter, and is defined as follows:

Definition 4.1.1. Let κ be a cardinal, $I_1 = I_2 = [0,1]$, $f : I_2 \to 2^{I_1}$ an upper semicontinuous set valued function with graph G, $U \subset I_1$ (resp. $U \subset I_2$) an open interval, and $\gamma : [0,1] \to G$ a continuous function. Then we say γ *covers* U κ *many times* if there are exactly κ many components $\{C_\alpha : \alpha < \kappa\}$ of $\gamma^{-1}((U \times I_2) \cap G)$ (resp. $\gamma^{-1}((U \times I_1) \cap G)$) such that $\rho_{I_1} \circ \gamma(C_\alpha) = U$ (resp. $\rho_{I_2} \circ \gamma(C_\alpha) = U$). Each of these components C_α is called a *covering of* U.

The following lemmas about coverings will be useful later.

Lemma 4.1.2. *Let $I_1 = I_2 = [0,1]$, $f : I_2 \to 2^{I_1}$ an upper semicontinuous set valued function with graph G, let $\gamma : [0,1] \to G$ be continuous, and let $x, y \in [0,1]$ with $x < y$. Suppose $U = (a,b) \subset (\rho_{I_1} \circ \gamma(x), \rho_{I_1} \circ \gamma(y))$ (or $U \subset (\rho_{I_2} \circ \gamma(x), \rho_{I_2} \circ \gamma(y))$). Then $\gamma|_{[x,y]}$ covers U at least once.*

Proof. Suppose $U \subset I_2$. Define $g : [x,y] \to I_2$ by $g = \rho_{I_2} \circ \gamma|_{[x,y]}$, so g is a continuous function from a compact interval to a compact interval. Suppose $g(x) < g(y)$, then $(a,b) \subset (g(x), g(y))$, and $g(x) < b$. Then by the intermediate value theorem, there exists $s \in (x,y]$ such that $g(s) = b$. Let

$$d = \inf\{s \in (x,y] : g(s) = b\}.$$

Then as g is continuous, $d > x$. Also, we have (by the intermediate value theorem again) there exists $t \in [x, d)$ such that $g(t) = a$. Let

$$c = \sup\{t \in [x, d) : g(t) = a\}.$$

As g is continuous, $g(c) = a$, $g(d) = b$, and $c < d$.

Furthermore, for all $z \in (c, d)$, we have that $g(z) \in (a, b)$. To justify this claim, suppose there exists $z \in (c, d)$ such that $g(z) < a$. Then by the intermediate value theorem, there exists z', where $c < z < z' < d$ such that $g(z') = a$, contradicting $c = \sup\{t \in [x, d) : g(t) = a\}$. Similarly for $g(z) > b$.

Therefore, we have (c, d) is a covering of U.

The proof works entirely the same if $g(y) < g(x)$, or if $U \subset I_1$. □

Lemma 4.1.3. *Let $I_1 = I_2 = [0, 1]$, $f : I_2 \to 2^{I_1}$ an upper semicontinuous set valued function with graph G, let U be an open interval in I_1 or I_2, and let κ be some cardinal. Suppose a continuous function $\gamma : [0, 1] \to G$ covers U κ many times. Then κ is finite.*

Proof. Suppose κ is infinite. Let $U = (a, b)$ for $a, b \in I_2$. Then $\gamma^{-1}(U \times I_1)$ must be open, so it is a union of κ many disjoint intervals. Furthermore, the endpoints of these intervals correspond to (infinite) subsets of $\gamma^{-1}(\{a\} \times I_1)$ and $\gamma^{-1}(\{b\} \times I_1)$, call these A and B respectively. Suppose $(a_0, a_1, \dots)$ is a monotone injective infinite sequence of points in A (one will exist as there are κ many disjoint intervals in $\gamma^{-1}(U)$), then this sequence converges, to a point $c \in [0, 1]$. Then since between any two points a_n and a_{n+1} there is a point in B, call it b_n, we can construct a sequence $(b_0, b_1, \dots)$ that also converge to c. But if γ is continuous, both $(\gamma(a_0), \gamma(a_1), \dots)$ and $(\gamma(b_0), \gamma(b_1), \dots)$ converge to the same point, $\gamma(c)$. But in fact they converge to two distinct points, that are at least a distance of $|a - b|$ apart, so we have a contradiction and γ is not continuous.

The proof works similarly if $U \subset I_1$. □

The next definition introduces the terms 'pairwise joining property' and 'limit joining property'. They are important concepts, and used throughout the chapter.

Definition 4.1.4. Let (I_i, f_i) be a generalised inverse sequence. Suppose $(\gamma_i : [0,1] \to G_i : i \in \mathbb{N})$ is a sequence of continuous functions. We say (γ_i, γ_{i+1}) have the *pairwise joining property* if $\rho_{i,i} \circ \gamma_i(t) = \rho_{i+1,i} \circ \gamma_{i+i}(t)$ for all $t \in [0,1]$. The sequence $(\gamma_i : [0,1] \to G_i : i \in \mathbb{N})$ has the *limit joining property* if (γ_i, γ_{i+1}) have the pairwise joining property for all $i \in \mathbb{N}$.

Lemma 4.1.5. *Let (I_i, f_i) be a generalised inverse sequence. Suppose for all $i \geq 1$, we have $\gamma_i : [0,1] \to G_i$ is a path in G_i where $\gamma_i(0) = \langle a_i, a_{i-1} \rangle$ and $\gamma_i(1) = \langle b_i, b_{i-1} \rangle$, and the sequence (γ_i) has the limit joining property. Then there exists a path $\gamma : [0,1] \to \varprojlim(I_i, f_i)$ where $\gamma(0) = \mathbf{a} = (a_0, a_1, a_2, \dots)$ and $\gamma(1) = \mathbf{b} = (b_0, b_1, b_2, \dots)$.*

Proof. We have that $\gamma : [0,1] \to \varprojlim(I_i, f_i)$ defined by $\gamma(t) = (\gamma_0(t), \dots, \gamma_j(t), \dots)$ is a well defined function, as for all $j \geq 0$, we have $\gamma_j(t) \in f_{j+1}(\gamma_{j+1}(t))$. It remains to show that γ is continuous. Let U be a basic open set in the image of γ. As U is a basic open set, it is restricted at finitely many coordinates, call the projection of U onto the ith coordinate U_i. As each γ_i is a path, the inverse image of each U_i is open in $[0,1]$, and the intersection of the inverse image of the union of the sets U_i for all i is the inverse image of U (for a point $\mathbf{p} \in \varprojlim(I_i, f_i)$ to be in U, each coordinate of $\mathbf{p}$ must be in the corresponding U_i). As each $U_i = I_i$ for all but finitely many spaces, the inverse image of U_i is $[0,1]$ for all but finitely many spaces, and hence the inverse image of U is open, so γ is continuous and hence a path. □

The converse of Lemma 4.1.5 is also true, as if we have a path $\gamma : [0,1] \to \varprojlim(I_i, f_i)$, then the projections $p_i : [0,1] \to G_i$ will be continuous for all $i \geq 0$, and for all $j \geq 1$, $\rho_{j,j} \circ p_j(t) = \rho_{j+1,j} \circ p_{j+1}(t)$. Therefore, we can extend Lemma 4.1.5 to an equivalence:

Lemma 4.1.6. *Let (I_i, f_i) be a generalised inverse sequence. There exists a path $\gamma : [0,1] \to \varprojlim(I_i, f_i)$ where $\gamma(0) = \mathbf{a} = (a_0, a_1, \dots)$ and $\gamma(1) = \mathbf{b} = (b_0, b_1, \dots)$ if and only if for all $i \geq 1$, there exists a path $\gamma_i : [0,1] \to G_i$ in G_i where $\gamma_i(0) = \langle a_i, a_{i-1} \rangle$ and $\gamma_i(1) = \langle b_i, b_{i-1} \rangle$, and the sequence (γ_i) has the limit joining property.*

This lemma then means we only need to look at paths through the graphs of the corresponding bonding functions to establish whether there is a path in the inverse limit. We can now justify our assumption that each graph is path connected, as if it isn't the inverse limit will be not path connected, by Lemma 4.1.6. The following is a corollary of Lemma 4.1.6.

Corollary 4.1.7. *Let (I_i, f_i) be a generalised inverse sequence. Suppose for all $n \in \mathbb{N}$ we have $\mathcal{G}_n$ is both connected and path connected. Then $\varprojlim(I_i, f_i)$ is not path connected if and only if there exists $\mathbf{a} = (a_0, a_1, a_2, \dots)$ and $\mathbf{b} = (b_0, b_1, b_2, \dots)$ such that there is no sequence of paths $(\gamma_i : [0,1] \to G_i)$ with $\gamma_i(0) = \langle a_i, a_{i-1} \rangle$ and $\gamma_i(1) = \langle b_i, b_{i-1} \rangle$ that have the limit joining property.*

Proof. Suppose there is an appropriate sequence of continuous functions $(\gamma_i : [0,1] \to G_i)$ that have the limit joining property. Then by Lemma 4.1.6, $\varprojlim(I_i, f_i)$ is path connected. Conversely (also by Lemma 4.1.6), if $\varprojlim(I_i, f_i)$ is path connected, there exists a sequence of continuous functions $(\gamma_i : [0,1] \to G_i)$ with the limit joining property.

□

4.2 Paths in Generalised Inverse Limits

We begin with some terminology that will be used extensively in this section.

Definition 4.2.1. Suppose (I_i, f_i) is a generalised inverse sequence, and the graph G_i of each function f_i is path connected. Let $\mathbf{a} = (a_0, a_1, a_2, \dots)$, $\mathbf{b} = (b_0, b_1, b_2, \dots) \in \varprojlim(I_i, f_i)$.

Given $n \in \mathbb{N}$, a *finite extension sequence of length n for* $(\mathbf{a}, \mathbf{b})$ is a finite sequence $(\gamma_i^n : [0,1] \to G_i : i \leq n)$ of continuous functions, such that for all $i \leq n$ $\gamma_i^n(0) = \langle a_i, a_{i-1} \rangle$, $\gamma_i^n(1) = \langle b_i, b_{i-1} \rangle$, and γ_i^n and γ_{i+1}^n have the pairwise joining property.

A *limit extension for* $(\mathbf{a}, \mathbf{b})$ is an infinite sequence of paths $(\gamma_i : [0,1] \to G_i : i \in \mathbb{N})$ from $\mathbf{a}$ to $\mathbf{b}$ that have the limit joining property.

Note that a limit extension will not always exist. By Lemma 4.1.6 we have that there is a path between $\mathbf{a}$ and $\mathbf{b}$ if and only if there is a limit extension for $(\mathbf{a}, \mathbf{b})$. But as we are assuming that $\mathcal{G}_n$ is path connected for all $n \in \mathbb{N}$, a finite extension sequence of length n will exist for all $n \in \mathbb{N}$.

The next few lemmas make use of the well known Arzelá Ascoli Theorem. The following (standard) terminology is used.

Given a collection of continuous functions $\mathcal{F} = \{f_\alpha : [0,1] \to [0,1] \times [0,1] : \alpha \in A\}$ for some indexing set A, we say that $\mathcal{F}$ is *equicontinuous* if for every $\varepsilon > 0$, there exists $\partial > 0$ such that for all $x \in [0,1]$, and all $f \in \mathcal{F}$, if $y \in [0,1]$, and $|x - y| < \partial$, then $\|f(x) - f(y)\| < \varepsilon$. Also, a collection $\mathcal{F}$ is *totally bounded* if there exists $M > 0$ such that for all $f \in \mathcal{F}$, and all $x \in [0,1]$, we have $\|f(x)\| < M$.

From the condition on $\mathcal{F}$ given above, that the codomain is $[0,1] \times [0,1]$, it is clear that in this context every collection $\mathcal{F} = \{f_\alpha : [0,1] \to [0,1] \times [0,1] : \alpha \in A\}$ is totally bounded ($M = 2$ will suffice). Now we present the Arzelá Ascoli Theorem (a proof can be found in [Be]). Here $C(I)$ denotes the space of continuous real valued functions defined on a compact interval, however as noted in [Be], the proof is virtually identical if the codomain is changed to $\mathbb{R}^d$, and in particular (for our purposes), $\mathbb{R}^2$.

Theorem 4.2.2 (Arzelá, Ascoli)**.** *If $\mathcal{F}$ is a totally bounded , equicontinuous family of functions in $C(I)$, then every sequence in $\mathcal{F}$ contains a subsequence that converges in norm to an element of $C(I)$.*

To prove the main result of the chapter, we need to introduce the concept of a *well parameterised* function. This will be done over the the next few lemmas and definitions. We begin with the definition of a reparameterisation of a path.

Definition 4.2.3. If X is a topological space and $\gamma : [0,1] \to X$ is a continuous function, we say that $\gamma' : [0,1] \to X$ is a *reparameterisation* of γ if there exists a continuous function $f : [0,1] \to [0,1]$ with fixed points 0 and 1 such that $\gamma' = \gamma \circ f$.

Note that if $\gamma : [0,1] \to I_n \times I_{n-1}$ is a path (where $I_n = I_{n-1} = [0,1]$), and a set $U \subset I_{n-1}$ (or $U \subset I_n$) is covered by a collection of sets $\{C_\alpha : \alpha \in A\}$ for γ, $f : [0,1] \to [0,1]$ is a homeomorphism, and γ' is a reparameterisation of γ where $\gamma' = \gamma \circ f$, then U will be covered by a collection of sets $\{C'_\beta : \beta \in B\}$ (with respect to γ'), and there will be a bijection between $\{C_\alpha : \alpha \in A\}$, and $\{C'_\beta : \beta \in B\}$. This bijection is simply given by the homeomorphism f, ie the bijection $b : \{C'_\beta : \beta \in B\} \to \{C_\alpha : \alpha \in A\}$ is given by $b(C'_\beta) = f(C'_\beta)$. In this case we say that C'_β and $b(C'_\beta)$ are *corresponding covering.*

The next lemmas will help us to reparameterise certain paths into well parameterised paths.

Lemma 4.2.4. *For some $n \in \mathbb{N}$, let S be a collection of n disjoint open subintervals of $[0,1]$, $S = \{U_i = (a_i, b_i) : i \leq n\}$, such that $\overline{\bigcup_{i \leq n} U_i} = [0,1]$. For each $i \leq n$, let $m_i \in \mathbb{R}^+$, and suppose that $\sum_{i \leq n}(b_i - a_i) \cdot \frac{1}{m_i} = 1$. Then there is a piecewise linear homeomorphism $f : [0,1] \to [0,1]$ such that for all $x \in U_i$, where $i \leq n$, the slope of the function f at $f^{-1}(x)$ is m_i.*

Proof. Note that as the indexing of the set S is arbitrary, we can assume (possibly after reindexing) S is indexed such that if $i < j$ then if $x \in U_i$ and $y \in U_j$, then $x < y$, ie they are indexed 'in order'.

Note that as the closure of the union of the sets in S gives the whole line $[0,1]$, $a_1 = 0$, and $b_n = 1$, and the union of the sets in S gives the whole line

$[0,1]$ minus a finite set of points, namely the points $b_1 = a_2, \ldots, b_{n-1} = a_n$.

Let f be the function with the graph $G(f)$, where $G(f)$ is the union of line segments L_i, where $i \leq n$, defined inductively as follows: let L_1 be the line between $\langle 0,0\rangle$ and $\langle (b_1 - a_1)\cdot \frac{1}{m_1}, b_1\rangle$. Then for $k \leq n$, if L_{k-1} is defined, and L_{k-1} is the line segment between $\langle l_{k-1}, a_{k-1}\rangle$ and $\langle t_{k-1}, b_{k-1}\rangle$, let L_k be the line segment between $\langle t_{k-1}, a_k\rangle$ and $\langle t_{k-1} + (b_k - a_k)\cdot \frac{1}{m_k}, b_k\rangle$.

Then $f : [0,1] \to [0,1]$ is well defined (as $\sum_{i\leq n}(b_i - a_i)\cdot \frac{1}{m_i} = 1$), piecewise linear ($n$ pieces), continuous (as for $i < n$ the line segments L_i, L_{i+1} are connected), injective (the slope is always strictly positive), and surjective (as $\sum_{i\leq n}(b_i - a_i)\cdot \frac{1}{m_i} = 1$, $f(1) = 1$ and f is continuous), and the inverse f^{-1} is continuous (as f is piecewise linear and continuous).

Therefore, $f : [0,1] \to [0,1]$ is a homeomorphism with the necessary properties.

□

If $S \subset [0,1]$ is a set of real numbers, we denote by $|S|$ the sum of the lengths of the open intervals in S.

Lemma 4.2.5. *Let $n \in \mathbb{N}$, $I_n = I_{n-1} = [0,1]$, $f_n : I_n \to 2^{I_{n-1}}$ an upper semicontinuous set valued function with graph G_n, and let $\gamma : [0,1] \to G_n$ be a path. Let $\mathcal{U} = \{U_\alpha \subset I_n : \alpha \leq k_1\} \cup \{U_\alpha \subset I_{n-1} : k_1 < \alpha \leq k\}$ be a finite collection of k open intervals of I_{n-1} and I_n. Let $\mathcal{C} = \{C_i : i \leq m\}$ be the collection of all coverings of all sets in $\mathcal{U}$ for γ, with $m \in \mathbb{N}$. To each covering C_i, assign a positive real number l'_i such that $\sum_{i\leq m} l'_i \leq \frac{1}{2}$.*

Then there exists a reparameterisation of γ, call it γ', such that $\gamma' = \gamma \circ f$ where f is a piecewise linear homeomorphism, and for each $i \leq m$, the covering C_i of γ has a corresponding covering C'_i of γ', such that C'_i has length at least l'_i.

Proof. Firstly, note that as there are m many coverings C_i, and the indexing is arbitrary, we can assume that each C_i is indexed by increasing value of $\frac{l_i}{l'_i}$, ie $\frac{l_1}{l'_1} \leq \frac{l_2}{l'_2} \leq \cdots \leq \frac{l_m}{l'_m}$, where $l_i = |C_i|$.

For each $i \leq m$, let $C_i = (a_i, b_i)$. Then let $\{t_j : j \leq p\}$ be the collection of all endpoints of the intervals C_i, such that $t_1 < t_2 < \cdots < t_p$. Now, let $\mathcal{A} := \{(t_j, t_{j+1}) : (t_j, t_{j+1}) \subset C_i \text{ for some } i \leq m\}$. Then $\mathcal{A}$ is a finite disjoint collection of open intervals of $[0,1]$, say $\mathcal{A}$ has cardinality $q \in \mathbb{N}$, and $\mathcal{A} = \{A_i : i \leq q\}$, ordered such that if $x \in A_i$ and $y \in A_{i+1}$, then $x < y$. Then we have $\overline{\bigcup_{\alpha \leq q} A_\alpha} = \overline{\bigcup_{i \leq m} C_i}$, and A_α is the intersection of a number of the coverings C_i, hence a subset of C_i for at least one $i \leq m$. Intuitively, we have 'taken intersections' of the coverings C_i, for $i \leq m$.

Now, for all $\alpha \leq q$, we define $D_\alpha := \{k : A_\alpha \subset C_k\}$, and

$$m'_\alpha := \min_{i \in D_\alpha} \left\{4, \frac{l_i}{l'_i}\right\}$$

Looking at it another way (and in a way that will commonly be used), $\frac{1}{m'_\alpha} = \max_{i \in D_\alpha} \left\{\frac{1}{4}, \frac{l'_i}{l_i}\right\}$.

<u>Claim</u>: $\sum_{\alpha \leq q} \frac{1}{m'_\alpha} \cdot |A_\alpha| < \frac{3}{4}$.

To prove this, first define, for $i \leq m$,

$$B_i := \{\alpha \leq q : A_\alpha \subset C_i, A_\alpha \not\subset C_k \text{ for } k < i\}.$$

Then for all $i, j \leq m$, $B_i \cap B_j = \varnothing$, and $\bigcup_{i \leq m} B_i = \{\alpha : \alpha \leq q\}$.

Now, define $l^*_k = \sum_{\alpha \in B_k} |A_\alpha|$. Then $\sum_{k \leq m} l^*_k \leq 1$, and $l^*_k \leq l_k$. Then we have for $k \leq m$:

$$\sum_{\alpha \in B_k} \frac{1}{m'_\alpha} |A_\alpha| = \max\left\{\frac{1}{4}, \frac{l'_k}{l_k}\right\} \cdot l^*_k = \max\left\{\frac{l^*_k}{4}, \frac{l'_k \cdot l^*_k}{l_k}\right\}.$$

Then as $\sum_{k \leq m} \frac{l'_k \cdot l^*_k}{l_k} \leq \sum_{k \leq m} l'_k < \frac{1}{2}$, and $\sum_{k \leq m} \frac{l^*_k}{4} \leq \frac{1}{4}$, we have:

$$\sum_{\alpha \leq q} \frac{1}{m'_\alpha} \cdot |A_j| = \sum_{k \leq m} \sum_{\alpha \in B_k} \frac{1}{m'_\alpha} \cdot |A_\alpha| = \sum_{k \leq m} \max\left\{\frac{l^*_k}{4}, \frac{l'_k \cdot l^*_k}{l_k}\right\} < \frac{3}{4}.$$

This completes the proof of the claim.

Now, let $L = |\bigcup_{i \leq m} C_i|$. We have two cases:

1. $L < 1$. In this case, we have $U := [0,1] \setminus \bigcup_{j\le q} A_j$ has nonempty interior, so let $\mathcal{V}$ be the collection of components of the interior of U. Then $\mathcal{V} = \{V_i : i \le r\}$ for some $r \in \mathbb{N}$, where each V_i is an open interval. Then $\sum_{i\le r} |V_i| = 1 - L$. Let
$$L^* = \sum_{j\le q} \frac{1}{m'_j} \cdot |A_j| < \frac{3}{4}.$$
Let $p = q + r$, and for $j \le q$, let $D_j = A_j$ and $m_j = m'_j$, and for $q < j \le r$, let $D_j = V_{j-q}$ and $m_j = \frac{1-L}{1-L^*}$. Then
$$\sum_{j\le p} \frac{1}{m_j} \cdot |D_j| = \sum_{j\le q} \frac{1}{m'_j} \cdot |A_j| + \sum_{q<j\le p} \frac{1-L^*}{1-L} \cdot |V_j| = L^* + \frac{1-L^*}{1-L} \cdot (1-L) = 1.$$

2. L=1. In this case, we have $\overline{\bigcup_{j\le m} A_j} = [0,1]$, so for each $j \le q$, let $D_j = A_j$, and let $p = q$. Again, let
$$L^* = \sum_{j\le q} \frac{1}{m'_j} \cdot |A_j| < \frac{3}{4}.$$
Then for all $j < p$, let $m_j = m'_j$, and $m_p = \frac{|A_p|}{\frac{|A_p|}{m'_p} + (1-L^*)}$.

   Then
$$\begin{aligned}\textstyle\sum_{j\le p} \frac{1}{m_j} \cdot |D_j| &= \textstyle\sum_{j<p} \frac{1}{m'_j} \cdot |D_j| + \frac{\frac{|A_p|}{m'_p} + (1-L^*)}{|A_p|} \cdot |A_p| \\ &= \textstyle\sum_{j\le p} \frac{1}{m'_j} |D_j| + (1-L^*) = L^* + (1-L^*) = 1.\end{aligned}$$

In either case, $1 - L^* > \frac{1}{4}$ and $1 - L < 1$, so $\frac{1-L}{1-L^*} < 4$. Also, $\sum_{j\le p} \frac{1}{m_j} \cdot |D_j| = 1$, and $\{D_i : i \le p\}$ is a disjoint collection of open sets such that $\overline{\bigcup_{i\le p} D_i} = [0,1]$. So by Lemma 4.2.4, there is a piecewise linear homeomorphism f, such that f has slope m_j for $x \in A_j$. Also, note that as $m_j \le m'_j$ for all $j \le q$, and $\frac{1-L}{1-L^*} < 4$, the maximum slope of f is 4, and the slope of f is always positive. Then given a covering C_k for γ of length l_k, the slope of f at any point $x \in f^{-1}(C_k)$ is at most $\frac{l_k}{l'_k}$, so $C'_k = f^{-1}(C_k)$ is a corresponding covering for C_k, and (where $A'_i = f^{-1}(A_i)$), we have $|C'_k| = \sum_{\{i:A_i\subset C_k\}} A'_i = \sum_{\{i:A_i\subset C_k\}} \frac{1}{m_i} \cdot A_i \ge \frac{l'_k}{l_k} \sum_{\{i:A_i\subset C_k\}} A_i = \frac{l'_k \cdot l_k}{l_k} = l'_k$, so C'_k has length at least l'_k.

$\square$

If $\gamma : [0,1] \to G_n$ is a path and γ' a reparameterisation of γ such that $\gamma' = \gamma \circ g$ where g is a piecewise linear homeomorphism, then we say that γ' is a *linear reparameterisation* of γ. Note that the reparameterisation constructed in Lemma 4.2.5 is a linear reparameterisation.

We can now define a well parameterised path.

Definition 4.2.6. Given $n \in \mathbb{N}$, suppose $I_n = [0,1]$, then for $i \geq 1$, let S_i^n be the collection of open intervals in I_n of length $\frac{1}{2^i}$ defined by

$$S_i^n = \left\{ (\frac{j}{2^i}, \frac{j+1}{2^i}) \subset I_n : j \in \{0, \dots, 2^i - 1\} \right\}.$$

Definition 4.2.7. Let $n \in \mathbb{N}$, $I_n = I_{n-1} = [0,1]$, $f_n : [0,1] \to 2^{[0,1]}$ an upper semicontinuous set valued function with corresponding graph G_n, a path $\gamma : [0,1] \to G_n$, and an open interval $U \subset I_n$, or $U \subset I_{n-1}$, we define the *covering number* of U for γ, denoted $K_\gamma(U)$ to be the number of times that γ covers U.

Note that from Lemma 4.1.3, the covering number $K_\gamma(U)$ will always be finite.

Definition 4.2.8. Let $n \geq 1$, let $I_n = I_{n-1} = [0,1]$, let $f_n : I_n \to 2^{I_{n-1}}$ be an upper semicontinuous set valued function with graph G_n, and $\gamma : [0,1] \to G_n$ a continuous function. Let $\mathbf{k} = (k_1, k_2, k_3, \dots)$ be a sequence of non-negative integers. Then we say that γ is *well parameterised with respect to* $\mathbf{k}$ if for each $i \in \mathbb{N}$, and for each $S \in S_i^{n-1} \cup S_i^n$, each covering of S for γ has length at least $\frac{1}{2 \cdot 2 \cdot 2^i \cdot 2^n \cdot 2^i \cdot k_i}$. If the sequence $\mathbf{k}$ is obvious, or if we only need to know that a path γ is well parameterised with respect to *some* $\mathbf{k}$, we will often simply say that γ is well parameterised.

The 2^n that appears in the denominator of the fraction appears because eventually we will want to apply the following lemma, which deals with reparameterising a path in one graph, to paths in a sequence of graphs. The sequence $\mathbf{k}$ will be related to the covering numbers of the sets S_i^n.

Lemma 4.2.9. *For some $n \geq 1$, let $I_n = I_{n-1} = [0,1]$, $f_n : I_n \to 2^{I_{n-1}}$ an upper semicontinuous set valued function with graph G_n, and $\gamma : [0,1] \to G_n$ a continuous function. Let $\mathbf{k} = (k_1, k_2, k_3, \dots)$ where for each $i \geq 1$, $k_i \geq \max\{K_\gamma(S) : S \in S_i^{n-1} \cup S_i^n\}$. Then there exists $\gamma' : [0,1] \to G_n$ such that γ' is well parameterised with respect to $\mathbf{k}$, and a reparameterisation of γ.*

Proof. Let $l \in \mathbb{N}$, and let $\mathcal{U}_l := \{S_j^m : 1 \leq j \leq l, m \in \{n, n-1\}\}$. Then $\mathcal{U}_l$ is a collection of $2 \cdot \sum_{1 \leq j \leq l} 2^j$ open sets. Let $\{C_\alpha : 1 \leq \alpha \leq p\}$ be the collection of all coverings of all sets in $\mathcal{U}_l$.

Finally, for each $\alpha \leq p$, let $l'_\alpha = \frac{1}{2 \cdot 2 \cdot 2^j \cdot 2^n \cdot 2^j \cdot k_j}$, where C_α is a covering of γ of a set $S \in S_j^m \in \mathcal{U}_i$. Then we have:

$$\sum_{1 \leq \alpha \leq p} l'_\alpha \leq \sum_{1 \leq j \leq l} 2 \cdot 2^j \cdot k_j \cdot \frac{1}{2 \cdot 2 \cdot 2^j \cdot 2^n \cdot 2^j \cdot k_j} = \frac{1}{2 \cdot 2^n} \cdot \sum_{1 \leq j \leq l} \frac{1}{2^j} \leq \frac{1}{2 \cdot 2^n} < \frac{1}{2}.$$

Here the first inequality comes from possibly overestimating the number of coverings (there are p many), by taking the number of open sets in $\mathcal{U}_l$, and multiplying by the maximum number of coverings a set can have (which is less than or equal to k_j).

So by Lemma 4.2.5, there is a linear reparameterisation $\gamma'_l = \gamma \circ f_l$ of γ, where f_l is a piecewise linear homeomorphism with a maximum slope of 4, and each covering C'_α of γ'_l of a set in S_j^m (where $j \leq l$) has length at least $\frac{1}{2 \cdot 2 \cdot 2^j \cdot 2^n \cdot 2_j \cdot k_j}$.

So if for all $i \in \mathbb{N}$ we construct such a $\gamma'_i = \gamma \circ f_i$, we have a sequence of homeomorphisms (f_i) that are totally bounded and equicontinuous (as the maximum slope is 4), so by the Arzelá Ascoli Theorem (Theorem 4.2.2), there is a subsequence of (f_i), call it (f_{i_k}), such that (f_{i_k}) converges to a continuous function f.

Note that as each f_i is a homeomorphism, it is strictly increasing (ie if $x < y$, $f_i(x) < f_i(y)$), and so f is increasing (ie if $x < y$, $f(x) \leq f(y)$).

Claim: $\gamma' := \gamma \circ f$ is well parameterised with respect to $\mathbf{k}$.

To prove this, for $j \in \mathbb{N}$, let $S \in S_j^m$, and let C_α be a covering of S for γ, and suppose C_α has length l_α. Then C_α has a corresponding covering C'_α for γ'. Suppose (with a view to contradiction) that C'_α has length $l^*_\alpha < l'_\alpha = \frac{1}{2\cdot 2\cdot 2^j\cdot 2^n\cdot 2^j\cdot k_j}$. Let $C_\alpha = (a,b)$, and $f^{-1}((a,b)) = (c,d)$, ie $|(c,d)| = l^*_\alpha < l'_\alpha$.

From the construction of each f_i (from Lemma 4.2.5), for all f_i where $i \geq j$, and for all $x \in (a,b)$, the slope at $f_i^{-1}(x)$ is at most $\frac{l_\alpha}{l'_\alpha}$.

As the sequence (f_i) converges to f (uniformly), given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|f_n(c) - a| < \varepsilon$, and $|f_n(d) - b| < \varepsilon$. Now, let $\varepsilon := \frac{l_\alpha - l^*_\alpha \cdot \frac{l_\alpha}{l'_\alpha}}{2}$ (which is greater than 0 as $l^*_\alpha < l'_\alpha$).

Now, given the ε just defined, we have that there exists $N \in \mathbb{N}$ such that for $n \geq N$, $|f_n(c) - a| + |f_n(d) - b| < 2\varepsilon$, so (by two applications of the triangle inequality) we have:

$$\begin{aligned}|f_n(c) - f_n(d)| \geq |a-b| - |f_n(c) - a| + |f_n(d) - b| > |a-b| - 2\varepsilon \\ = l_\alpha - (l_\alpha - l^*_\alpha \cdot \tfrac{l_\alpha}{l'_\alpha}) = l^*_\alpha \cdot \tfrac{l_\alpha}{l'_\alpha}.\end{aligned}$$

This means that $|f_n(c) - f_n(d)| > |c-d| \cdot \frac{l_\alpha}{l'_\alpha}$, but this is impossible, as the slope of f_n at x for $x \in (c,d)$ is no more than $\frac{l_\alpha}{l'_\alpha}$. Therefore, we have a contradiction, so we conclude C'_α has length at least l'_α, completing the proof of the claim, and the proof of the lemma.

□

The preceding lemma reparameterised a path on one graph to a well parameterised function. The next lemma extends this to reparameterise a finite extension sequence of arbitrary length so each path in the sequence is well parameterised.

Corollary 4.2.10. *Let (I_i, f_i) be a generalised inverse sequence,* $\mathbf{a}$ *and* $\mathbf{b} \in \varprojlim(I_i, f_i)$*, and let $n \in \mathbb{N}$. Suppose $(\gamma_i^n : i \leq n)$ is a finite extension sequence for $(\mathbf{a}, \mathbf{b})$. Let $\mathbf{k} = (k_1, k_2, k_3, \dots)$ where for each $i \geq 1$, $k_i \geq \max_{1\leq j\leq n}\{K_{\gamma_j}(S) : S \in S_i^{j-1} \cup S_i^j\}$. Then there exists a finite extension sequence $(\gamma_i^{n\prime} : i \leq n)$ for*

$(\mathbf{a}, \mathbf{b})$*, such that for each* $i \le n$*, we have* $\gamma_i^{n\prime}$ *is well parameterised with respect to* $\mathbf{k}$*, and a reparameterisation of* γ_i^n*.*

Proof. Let $(\gamma_i^n : i \le n)$ be a finite extension sequence. Then any reparameterisation f given to γ_i^n to obtain a well parameterised $\gamma_i^{n\prime}$ must also be given to γ_j^n for all $j \le n$, in order to maintain the pairwise joining property.

For $1 \le i \le n$ and $l \in \mathbb{N}$, let $\mathcal{U}_l^i := \{S_j^m : 1 \le j \le l, m \in \{i, i-1\}\}$. Then each $\mathcal{U}_l^i$ is a collection of $2 \cdot \sum_{1 \le j \le l} 2^j$ open sets. Let $\{C_\alpha : \alpha \le p\}$ be the collection of all coverings of all sets in $\mathcal{U}_l^i$ for all $i \le n$.

Finally, let $l'_\alpha = \frac{1}{2 \cdot 2 \cdot 2^j \cdot 2^i \cdot 2^j \cdot k_j}$, where C_α is a covering for γ_i^n of a set $S \in S_j^m \in \mathcal{U}_l^i$. Then we have (as in Lemma 4.2.9):

$$\sum_{\alpha \le p} l'_\alpha \le \sum_{1 \le i \le n} \sum_{1 \le j \le i} 2 \cdot 2^j \cdot k_j \cdot \frac{1}{2 \cdot 2 \cdot 2^j \cdot 2^i \cdot 2^j \cdot k_j} \le \sum_{1 \le i \le n} \frac{1}{2 \cdot 2^i} < \frac{1}{2}.$$

Then using the same method as in the proof of Lemma 4.2.9, we can create a sequence of functions (f_l) that converge to f such that $\gamma_i^{n\prime} = \gamma_i^n \circ f$ is well parameterised with respect to $\mathbf{k}$ for all $i \le n$, and as we have used the same function to reparameterise all paths in the finite extension sequence, we have that $(\gamma_i^{n\prime} : i \le n)$ is a well parameterised (with respect to finite extension sequence of length n between $\mathbf{a}$ and $\mathbf{b}$.

□

We now have the necessary tools to prove the following lemma, from which together with Corollary 4.1.7, Theorem 4.0.1 will follow.

Lemma 4.2.11. *Let* (I_i, f_i) *be a generalised inverse sequence, and let* $\mathbf{a} = (a_0, a_1, a_2, \dots)$ *and* $\mathbf{b} = (b_0, b_1, b_2, \dots) \in \varprojlim(I_i, f_i)$*. Then a limit extension for* $\mathbf{a}$ *and* $\mathbf{b}$ *exists if and only if for every* $n \in \mathbb{N}$*, for every open interval* $U \subset I_n$*, there is an upper bound* $k \in \mathbb{N}$ *such that for all* $l \in \mathbb{N}$*, where* $l \ge n$*, there exists a finite extension sequence* $(\gamma_i^l : i \le l)$ *for* $(\mathbf{a}, \mathbf{b})$ *such that* γ_n^l *covers* U *less than* k *times.*

Proof. Suppose $(\gamma_i : i \in \mathbb{N})$ is a limit extension for $(\mathbf{a}, \mathbf{b})$. For every $l \in \mathbb{N}$ there is a finite extension sequence $(\gamma_i^l : i \leq l)$ for $(\mathbf{a}, \mathbf{b})$, which is obtained from the limit extension $(\gamma_i : i \in \mathbb{N})$ by 'forgetting' all coordinates greater that l, so $\gamma_i^l = \gamma_i$ for each $1 \leq i \leq l$. Then as each γ_i^l is a continuous function, by Lemma 4.1.3 for every open interval $U \subset I_i$ there exists a $k \in \mathbb{N}$ such that γ_i covers U less than k times.

For the converse, let $n \in \mathbb{N}$. Let $\mathbf{k} = (k_0, k_1, k_2 \dots)$, where for each $i \in \mathbb{N}$, k_i is the maximum number of times a set in S_i^n or S_i^{n-1} is covered by γ_n^l in a finite extension sequence for $l \geq n$. Then we have that for all $l \in \mathbb{N}$, where $l \geq n$, there exists a finite extension sequence $(\gamma_i^l : i \leq l)$ for $(\mathbf{a}, \mathbf{b})$, and for each $i \in \mathbb{N}$, γ_n^l covers each $S \in S_i^n$ less than k_i times, and also γ_n^l covers $S \in S_i^{n-1}$ less than k_i times. Therefore, by Corollary 4.2.10, we can assume each γ_i^l in the finite extension sequence is well parameterised with respect to $\mathbf{k}$.

Now, for all $l \geq n$ we have a finite extension sequence $(\gamma_i^l : i \leq l)$ that is well parameterised with respect to $\mathbf{k}$. Let $\mathcal{C}_n$ be the sequence of functions defined by $\mathcal{C}_n := (\gamma_n^l : [0,1] \to G_n : l \geq n)$. So $\mathcal{C}_n$ is an infinite sequence of totally bounded functions. We will now show the functions in $\mathcal{C}_n$ are equicontinuous.

Let $\varepsilon > 0$, $f : [0,1] \to G_n \in \mathcal{C}_n$, and $x \in [0,1]$. As $f = \gamma_n^l$ for some $l \geq n$, f is well parameterised with respect to $\mathbf{k}$. Let $i \in \mathbb{N}$ be such that $\frac{1}{2^{i-2}} < \varepsilon$. Let $\partial_\varepsilon = \frac{1}{2 \cdot 2 \cdot 2^i \cdot 2^n \cdot 2^i \cdot k_i}$, and let $y \in [0,1]$ such that $|x - y| < \partial_\varepsilon$.

Suppose $\|f(x) - f(y)\| \geq \varepsilon$, ie $f(y) \notin B_\varepsilon(f(x))$. Then we have that either $|\rho_{n,n} \circ f(y) - \rho_{n,n} \circ f(x)| > \frac{\varepsilon}{2}$, or $|\rho_{n,n-1} \circ f(y) - \rho_{n,n-1} \circ f(x)| > \frac{\varepsilon}{2}$. WLOG, suppose $|\rho_{n,n} \circ f(y) - \rho_{n,n} \circ f(x)| > \frac{\varepsilon}{2}$. Then as the interval $(\rho_{n,n} \circ f(y), \rho_{n,n} \circ f(x)) \subset I_n$ has length greater than $\frac{\varepsilon}{2}$, and $\frac{1}{2^i} < \frac{\varepsilon}{4}$, there exists an interval $S \in S_i^n$ such that $S \subset (\rho_{n,n} \circ f(y), \rho_{n,n} \circ f(x)) \subset I_n$. But then by Lemma 4.1.2, $f|_{[x,y]}$ covers S. But as f is well parameterised, every covering of s must have length at least $\frac{1}{2 \cdot 2 \cdot 2^i \cdot 2^n \cdot 2^i \cdot k_i}$, so $|x - y| \geq \frac{1}{2 \cdot 2 \cdot 2^i \cdot 2^n \cdot 2^i \cdot k_i}$, contradicting our assumption that $|x - y| < \partial_\varepsilon$.

Hence the functions in $\mathcal{C}_n$ are equicontinuous, and so by the Arzelá Ascoli Theorem (Theorem 4.2.2), there is a subsequence of $\mathcal{C}_n$ that converges to a function $\gamma_n : [0,1] \to [0,1] \times [0,1]$. Then we have:

- γ_n is continuous, as the convergence is uniform,
- the image of γ_n is contained in G_n, as this is the case for each function in the sequence, and G_n is closed, and
- $\gamma_n(0) = \langle a_{n+1}, a_n \rangle$, and $\gamma_n(1) = \langle b_{n+1}, b_n \rangle$, as $\gamma_n^l(0) = \langle a_{n+1}, a_n \rangle$, and $\gamma_n^l(1) = \langle b_{n+1}, b_n \rangle$ for all $l \geq n$.

Hence $\gamma_n : [0,1] \to G_n$ is a path from $\langle a_{n+1}, a_n \rangle$ to $\langle b_{n+1}, b_n \rangle$ in G_n. Note that such a path obtained from convergent finite extension sequences is not in general unique, as there may be another subsequence of $\mathcal{C}_n$ that converges to a different path $\gamma' : [0,1] \to G_n$. This means that in general, picking a γ_n for each $n \in \mathbb{N}$ will not give a limit extension $(\gamma_n : n \in \mathbb{N})$. It is, however, possible to select a sequence $(\gamma_n : n \in \mathbb{N})$ that is a limit extension, using the following process.

Firstly, consider γ_1, which is obtained as the limit of a subsequence of $\mathcal{C}_1$. This subsequence is then $(\gamma_1^l : l \in N_1)$, where N_1 is a strictly increasing sequence in $\mathbb{N}$. The sequence $\mathcal{C}_1$ was obtained from a sequence of finite extension sequences of well parameterised functions, and the convergent subsequence used only certain of these finite extension sequences, the finite extension sequences $(\gamma_i^l : l \in N_1)$.

Now, we have a sequence of well parameterised functions in G_2, $(\gamma_2^l : l \in N_1)$. Just like above, these will have a convergent subsequence, $(\gamma_2^l : l \in N_2)$, where $N_2 \subset N_1$. These will converge to a path γ_2 in G_2 between $\langle a_2, a_1 \rangle$ and $\langle b_2, b_1 \rangle$. Furthermore, γ_1 and γ_2 have the pairwise joining property. To see this, note that for each $k \in N_2$, there is a pair of paths γ_1^k and γ_2^k in G_1 and G_2 respectively that have the pairwise joining property (as they are in the same finite extension sequence $(\gamma_i^k : i \leq k)$), so $\rho_{1,1} \circ \gamma_1^l(x) = \rho_{2,1} \circ \gamma_1^l(x)$ for $l \in N_2$, for each $x \in [0,1]$. Therefore in the limit $\rho_{1,1} \circ \gamma_1(x) = \rho_{2,1} \circ \gamma_2(x)$ for all $x \in [0,1]$, so γ_1 and γ_2 have the pairwise joining property.

Repeating this process for all $n \geq 1$, we have a subsequence of finite extension sequences $(\gamma_n^k : k \in N_n)$ that converge to a path $\gamma_n : [0,1] \to G_n$ that will have the pairwise joining property with $\gamma_{n-1} : [0,1] \to G_{n-1}$.

So, by induction over $\mathbb{N}$, we have for all $n \in \mathbb{N}$, each γ_n is a path in G_n from $\langle a_n, a_{n-1}\rangle$ to $\langle b_n, b_{n-1}\rangle$, and γ_n and γ_{n+1} have the pairwise joining property. Therefore, $(\gamma_n : n \in \mathbb{N})$ is a limit extension for $(\mathbf{a}, \mathbf{b})$.

$\square$

The proof of Theorem 4.0.1 then follows from Lemma 4.1.6 and Lemma 4.2.11.

4.3 Examples

In this section we will apply Theorem 4.0.1 to some examples.

Example 4.3.1. The first example is the inverse limit of the function f, where f has as its graph the union of a line between $\langle 0,0\rangle$ and $\langle 1,1\rangle$, and a line between $\langle 0,1\rangle$ and $\langle 1,0\rangle$. The graph of f is shown in Figure 4.1.

Let $\mathbf{a} = (a_0, a_1, a_2, \dots)$ and $\mathbf{b} = (b_0, b_1, b_2, \dots) \in \varprojlim f$. We will define inductively a finite extension sequence of length l for each $l \in \mathbb{N}$. For γ_1^1, define $\gamma_1^1|_{[0,\frac{1}{2}]}$ by the (constant speed) path between $\langle a_1, a_0\rangle$ and $\langle \frac{1}{2}, \frac{1}{2}\rangle$, and $\gamma_1^1|_{[\frac{1}{2},1]}$ by the (constant speed) path between $\langle \frac{1}{2}, \frac{1}{2}\rangle$ and $\langle b_1, b_0\rangle$. Then define $\gamma_1^1 : [0,1] \to G_1$ as the union of $\gamma_1^1|_{[0,\frac{1}{2}]}$ and $\gamma_1^1|_{[\frac{1}{2},1]}$. This is then a finite extension sequence of length 1. Note that each open set in I_0 and I_1 is covered a maximum of two times.

Now suppose we have a finite extension sequence of length n, and there is a path γ_n^n with the property that $\gamma_n^n|_{[0,\frac{1}{2}]}$ is a (constant speed) path between $\langle a_n, a_{n-1}\rangle$ and $\langle \frac{1}{2}, \frac{1}{2}\rangle$, and $\gamma_n^n|_{[\frac{1}{2},1]}$ is a (constant speed) path between $\langle \frac{1}{2}, \frac{1}{2}\rangle$ and $\langle b_n, b_{n-1}\rangle$. Then we define γ_{n+1}^{n+1} as being the union of $\gamma_{n+1}^{n+1}|_{[0,\frac{1}{2}]}$, a (constant speed) path between $\langle a_{n+1}, a_n\rangle$ and $\langle \frac{1}{2}, \frac{1}{2}\rangle$, and $\gamma_{n+1}^{n+1}|_{[\frac{1}{2},1]}$, a (constant speed)

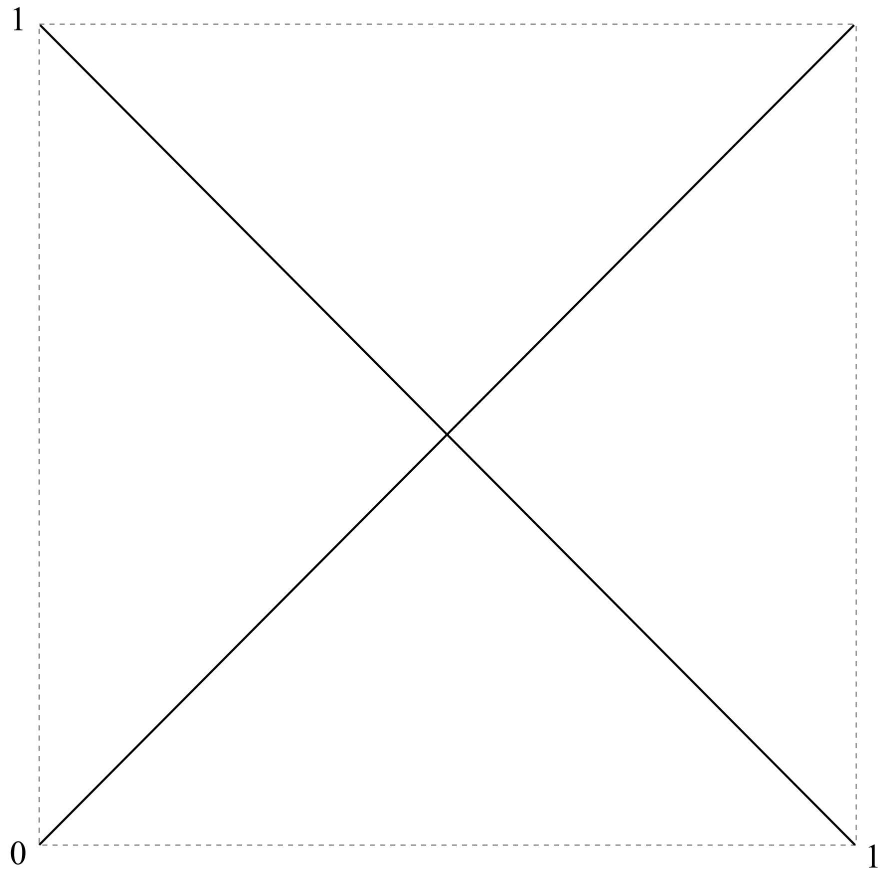

Figure 4.1: The graph of f in Example 4.3.1.

path between $\langle \frac{1}{2}, \frac{1}{2} \rangle$ and $\langle b_{n+1}, b_n \rangle$. Then γ_n^n and γ_{n+1}^{n+1} will have the pairwise joining property (as each section of the path was at constant speed), so we can extend the finite extension sequence $(\gamma_i^n : i \leq n)$ to a finite extension sequence $(\gamma_i^{n+1} : i \leq n+1)$, where for each $i \leq n$ we have $\gamma_i^{n+1} = \gamma_i^n$. Furthermore, each path in the sequence will only cover each open set a maximum of two times.

So for all $l \in \mathbb{N}$ there is a finite extension sequence such that each open set in I_n for $n \leq l$ is covered a maximum of two times. Therefore, by Theorem 4.0.1 we have a path in $\varprojlim f$ between $\mathbf{a}$ and $\mathbf{b}$, and as $\mathbf{a}$ and $\mathbf{b}$ were arbitrary, we can conclude that $\varprojlim f$ is path connected.

That Example 4.3.1 is path connected becomes more obvious when it is known that $\varprojlim f$ is homeomorphic to a 'Cantor star' (ie a cone over a Cantor set) [In2, Example 2.7].

Example 4.3.2. This example is the inverse limit of the function g, where g has as its graph the union of a line between $\langle 0, 0\rangle$ and $\langle 1, \frac{1}{2}\rangle$, and a line between $\langle 0, 1\rangle$ and $\langle 1, \frac{1}{2}\rangle$. The graph of g is shown in Figure 4.2.

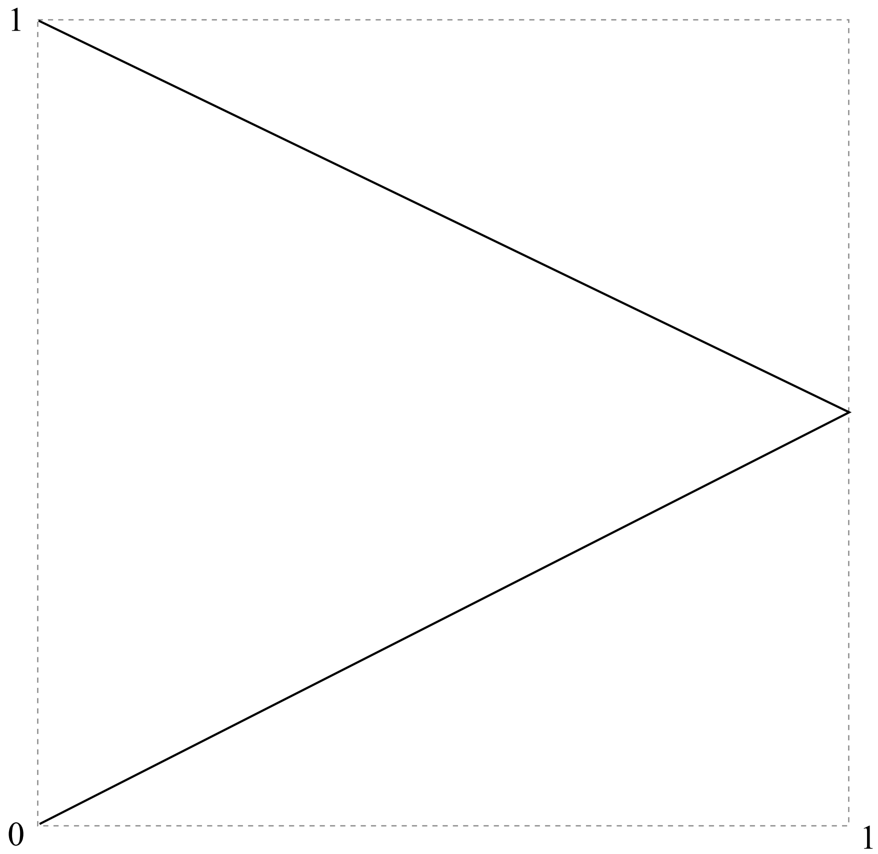

Figure 4.2: The graph of g in Example 4.3.2.

Let $\mathbf{a} = (a_0, a_1, a_2, \dots)$ and $\mathbf{b} = (b_0, b_1, b_2, \dots) \in \varprojlim f$. Again we will inductively define a finite extension sequence of length l for each $l \in \mathbb{N}$. For γ_1^1 we can use any path between $\langle a_1, a_0\rangle$ and $\langle b_1, b_0\rangle$. By Lemma 4.1.3, each open set in I_0 and I_1 is only covered finitely often by γ_1^1.

Suppose we have a finite extension sequence $(\gamma_i^n : i \leq n)$ of length n. As g is injective in the sense that for each $y \in [0,1]$ there is a unique $x \in [0,1]$ such that $y \in f(x)$, we can define a path γ_{n+1}^{n+1} that has the pairwise joining property with γ_n^n, and do this without making any changes to γ_n^n. Therefore, we can extend $(\gamma_i^n : i \leq n)$ to a finite extension sequence of length $n+1$, and keep $\gamma_i^{n+1} = \gamma_i^n$ for all $i \leq n$. As γ_{n+1}^{n+1} is continuous, all open sets in I_{n+1} are covered finitely often.

So although sets in I_n are possibly being covered increasingly often as n increases, for each n there is a limit to how many times any open set $U \subset I_n$ will be covered. Hence, by Theorem 4.0.1, $\varprojlim g$ will be path connected.

The final example is an inverse limit of a single valued bonding function, whose inverse limit is not path connected.

Example 4.3.3. This example is the inverse limit of the function h, where h has as its graph the union of a line between $\langle 0,0\rangle$ and $\langle \frac{1}{4},1\rangle$, a line between $\langle \frac{1}{4},1\rangle$ and $\langle \frac{1}{2},\frac{1}{2}\rangle$, and a line between $\langle \frac{1}{2},\frac{1}{2}\rangle$ and $\langle 1,1\rangle$. The graph of h is shown in Figure 4.3.

Let $\mathbf{a} = (0,0,0,\dots)$ and $\mathbf{b} = \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\dots\right) \in \varprojlim h$, and consider a path between these two points. Consider γ_1 in a limit extension sequence $(\gamma_i : i \geq 1)$ that gives rise to this path, and the interval $U = \left(\frac{1}{2},1\right) \subset I_0$. Then γ_1 clearly must cover U at least twice. Now consider γ_2. This must have the pairwise joining property with γ_1, so for this to occur, we must reparameterise γ_1 to pass through the point $\langle 1,1\rangle$. Then γ_1 must cover U at least four times. Then it is not hard to see that with every additional path we add to increase the length of the finite extension sequence by one, U must be covered at least two more times. So there is no finite k such that U is covered less than k times by an arbitrarily long finite extension sequence. Therefore, by Theorem 4.0.1, $\varprojlim h$ is not path connected.

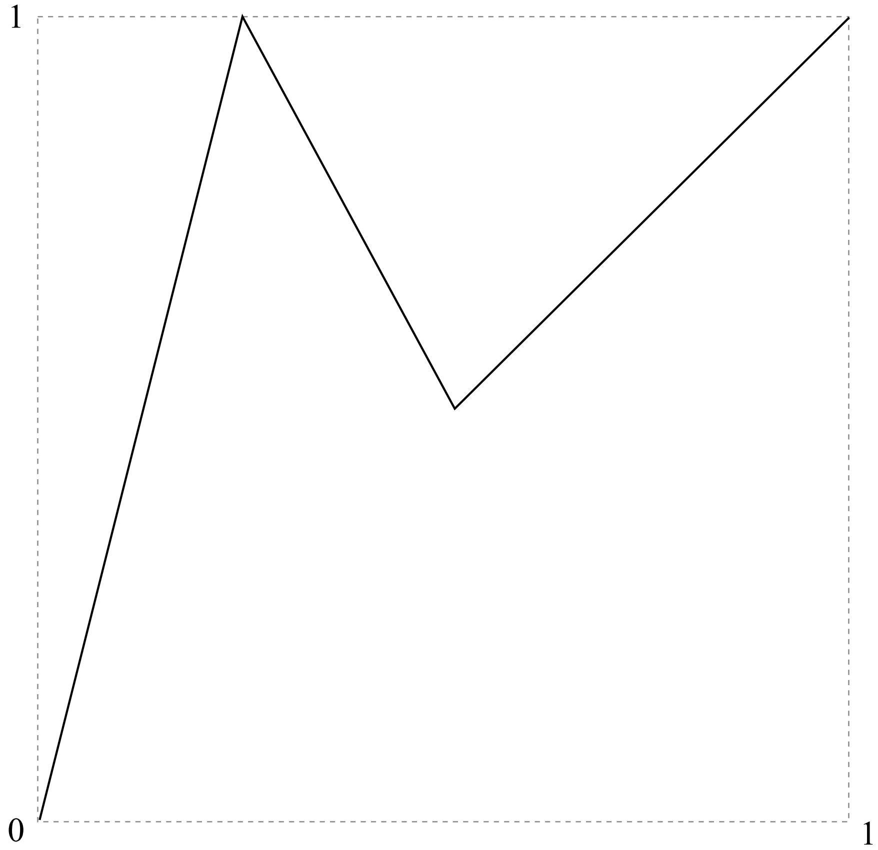

Figure 4.3: The graph of h in Example 4.3.3.

As shown in [IM2, Example 103], $\varprojlim h$ from Example 4.3.3 is homeomorphic to a the closure of a $\sin \frac{1}{x}$ curve.

4.4 Further Work

The main theorem in this chapter gives a condition that reduces the problem of path connectedness in generalised inverse limits to a condition on the finite approximants $\mathcal{G}_n$. It would be of great interest if this could be extended to conditions on the graphs of the functions that will guarantee the conditions on the $\mathcal{G}_n$ approximants.

The other main extension that can be done is to find conditions that guarantee that the finite approximants will be path connected. This is what was assumed throughout this chapter, but it is not a trivial problem. Perhaps this could be incorporated with the proposed conditions mentioned in the previous paragraph.

The factor spaces used in this chapter have all been compact intervals. There does not seem any reason why Theorem 4.0.1 cannot be extended to compact Hausdorff spaces. This may even be possible using a very similar method of proof to the one in this chapter, for example there is an extension of the Arzelá Ascoli Theorem to compact Hausdorff spaces in [DS]. Most other concepts used in the proof can probably be generalised, at least to metric spaces.

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