Robust $H_{\infty}$ static output feedback control of fuzzy systems: an ILMI approach

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Robust \( \mathcal{H}_\infty \) Static Output Feedback Control of Fuzzy Systems: An ILMI Approach

Dan Huang and Sing Kiong Nguang

Abstract—This paper examines the problem of robust \( \mathcal{H}_\infty \) static output feedback control of a Takagi-Sugeno fuzzy system. The proposed robust \( \mathcal{H}_\infty \) static output feedback controller guarantees the \( L_2 \) gain of the mapping from the exogenous disturbances to the regulated output to be less than or equal to a prescribed level. The existence of a robust \( \mathcal{H}_\infty \) static output feedback control is given in terms of the solvability of bilinear matrix inequalities. An iterative algorithm based on the linear matrix inequality is developed to compute robust \( \mathcal{H}_\infty \) static output feedback gains. To reduce the conservatism of the design, the structural information of membership function characteristics is incorporated. A numerical example is used to illustrate the validity of the design methodologies.

Index Terms—BMIs, ILMI, Robust \( \mathcal{H}_\infty \) control, T-S fuzzy model.

I. INTRODUCTION

In the past three decades, considerable attention has been devoted to the problem of nonlinear \( \mathcal{H}_\infty \) control; see, for instance, [1]–[5]. This problem can be stated as follows. Given a dynamic system with the exogenous input and measured output, design a control law such that the \( L_2 \) gain of the mapping from the exogenous input to the regulated output is minimized or no larger than some prescribed level. In general, there are two common methods to solve the nonlinear \( \mathcal{H}_\infty \) control problems. One is based on the dissipativity theory and theory of differential games (see [6] and [1]), and the other is based on the nonlinearity version of the classical bounded real lemma as developed by Willems [9] and Hill and Moylan [7]; see, e.g., [5], [8], and [4]. Both of these approaches convert the problem of nonlinear \( \mathcal{H}_\infty \) control to the solvability of the so-called Hamilton-Jacobi equation (HJE). However, until now, it is still very difficult to find a global solution to the HJE.

The static output feedback problem has also attracted attentions of many researchers over the past two decades, [10]–[16]. The problem can be stated as follows: given a system, find a static output feedback so that the closed-loop system is stable. Normally, the existence of a full order output feedback control law is given in terms of to the solvability of two convex problems. However, the synthesis of a static output feedback gain or a fixed order controller is much more difficult. The main rationale is that the separation principle does not hold in such cases. A comprehensive survey on static output feedback can be found in [16]. The static output feedback problem is important in its own right, because static controllers are less expensive to be implemented and more reliable in practice. In [15] for a linear system, the authors show that any dynamic output-feedback problem can be transformed into a static output-feedback problem. Hence, for a linear system the static output feedback formulation is more general than the full order dynamic output feedback formulation, that is, static output formulation can be utilized to design a full order dynamic controller, but the converse is not true.

A great amount of researches have been focused on describing a nonlinear system using a Takagi-Sugeno (T-S) fuzzy model in recent years; see [17]–[32]. In this fuzzy model, local dynamics in different state space regions are represented by local linear systems. The overall model of the nonlinear system is obtained by “blending” of these linear models through nonlinear fuzzy membership functions. Unlike conventional modeling techniques which use a single model to describe the global behavior of a nonlinear system, fuzzy modeling is essentially a multi-model approach in which simple submodels (typically linear models) are fuzzily combined to describe the global behavior of a nonlinear system. The T-S fuzzy model has been proved to be a very good representation for a certain class of nonlinear dynamic systems. Motivated by the fact that any smooth nonlinear dynamic system can be approximated by a T-S fuzzy model with linear models as fuzzy rule consequences [17], recently, in [23], the problem of \( \mathcal{H}_\infty \) static output feedback control of T-S fuzzy systems has been investigated.

The major drawback of the above mentioned [17]–[32] papers is that their design methodologies do not incorporate membership function characteristics, which may lead to conservative design methodologies. Motivated by this drawback and the simplicity of static output feedback controller, in this paper, input membership function characteristics are incorporated into our static output feedback control design. We show that the existence of a robust \( \mathcal{H}_\infty \) static output feedback control can be expressed in terms of the solvability of bilinear matrix inequalities. To compute a solution to these BMIs, an iterative algorithm [10] based on the linear matrix inequality has been developed. We show that the proposed static output feedback approach can also be used to design a fuzzy dynamic output feedback controller.

The rest of this paper is organized as follows. In Section II, system description and problem formulation are given. Main results are presented in Section III. The validity of our approach is demonstrated by an example from literatures in Section IV. Finally, conclusions are given in Section V.

II. SYSTEM DESCRIPTION AND PROBLEM FORMULATION

Consider a nonlinear dynamic plant whose operation space can be partitioned into several regimes according to premise variables. The \( i \)th plant local linear model in the T-S fuzzy model is as follows.

Plant Rule \( \varphi_1 \):

\[
\text{if } \psi_1(t) \geq M_{11} \text{ and } \ldots \text{ and } \psi_r(t) \geq M_{1r}, \text{ then }
\]

\[
\begin{align*}
\dot{x}(t) & = [A_1 + \Delta A_1]x(t) + B_1w(t) + [B_{11} + \Delta B_{11}]u(t) \\
x(0) & = 0 \\
z(t) & = [C_{11} + \Delta C_{11}]x(t) + [D_{111} + \Delta D_{111}]u(t) \\
y(t) & = C_{21}x(t)
\end{align*}
\]  

(2.1)

where \( i = 1, \ldots, r \) is the number of fuzzy rules; \( \psi_1(t) \) are premise variables, \( M_{ab} \) is fuzzy sets, \( k = 1, \ldots, p, p \) is the number of premise variables; \( x(t) \in \mathbb{R}^n \) is the state vectors, \( u(t) \in \mathbb{R}^m \) is the input, \( z(t) \in \mathbb{R}^l \) and \( y(t) \in \mathbb{R}^p \) are controlled and measured output, respectively. \( w(t) \in \mathbb{R}^p \) is the disturbance which belongs to \( L_2(0, \infty) \), \( A_i, B_{11}, B_{12}, C_{11}, D_{111}, C_{21} \) are of appropriate dimensions. \( \Delta A_i, \Delta B_{11}, \Delta C_{11}, \Delta D_{111} \) are uncertainties in the system and satisfy the following assumption.

Assumption 2.1: The parameter uncertainties considered here are norm-bounded, in the form

\[
\begin{bmatrix}
\Delta A_i \\
\Delta B_{11} \\
\Delta C_{11} \\
\Delta D_{111}
\end{bmatrix} =
\begin{bmatrix}
H_{11} \\
H_{12} \\
H_{21} \\
H_{22}
\end{bmatrix} F_i(t) 
\begin{bmatrix}
E_{11} \\
E_{12} \\
E_{21} \\
E_{22}
\end{bmatrix}
\]  

(2.2)

where \( H_{11}, H_{12}, E_{11}, \) and \( E_{22} \) are known real constant matrices of appropriate dimensions, and \( F_i(t) \) is an unknown matrix function with Lebesgue-measurable elements and satisfies \( \| F_i(t) \| \leq 1 \).
By a center-average defuzzifier, product inference and singleton fuzzifier, the local models can be integrated into a global nonlinear model

\[ \dot{x}(t) = \sum_{i=1}^{r} \mu_i(x(t))[A_i + \Delta A_i]x(t) + B_{1i}u(t) + (B_{2i} + \Delta B_{2i})w(t) \]

\[ z(t) = \sum_{i=1}^{r} \mu_i(x(t))[C_{1i} + \Delta C_{1i}]x(t) + (D_{12i} + \Delta D_{12i})u(t) \]

\[ y(t) = C_{2i}x(t) \] (2.3)

where \( \mu_i(x(t)) = [\mu_1(x(t)), \mu_2(x(t)), \ldots, \mu_r(x(t))]^T \), \( \omega_i(x(t)) = \prod_{k=1}^{m} M_{ik}(\nu_k(t)) \), \( \mu(r)(x(t)) = \omega_i(x(t))/\sum_{k=1}^{m} \omega_i(x(t)) \), \( \mu_i(x(t)) \geq 0 \), and \( \sum_{i=1}^{r} \mu_i(x(t)) = 1 \). Here, \( M_{ik}(\nu_k(t)) \) denote the grade of membership of \( \nu_k(t) \) in \( M_{ik} \).

For the nonlinear plant represented by (2.3), the fuzzy static output feedback controller is inferred as follows:

\[ u(t) = \sum_{i=1}^{r} \mu_i(x(t))K_{iy}(t) \] (2.4)

where \( K_i \) is the local controller gain for each plant rule.

For the convenience of notations \( \mu_i(x(t)) \) is denoted as \( \mu_i \), and \( \ast \) as the region for terms that are induced by symmetry.

**Problem Formulation:** Given a prescribed \( \mathcal{H}_\infty \) performance \( \gamma > 0 \), design a fuzzy controller (2.4) such that

\[ \int_0^\infty z^T(z)dt \leq \gamma^2 \int_0^\infty w^T(w)dt \] (2.5)

and the closed system (2.3) with (2.4) is asymptotically stable.

**Remark 2.1:** Non-zero initial condition \( x(0) \neq 0 \) can easily be dealt with by using the technique described in [35].

### III. MAIN RESULTS

In this section, we shall present our procedure for designing a robust \( \mathcal{H}_\infty \) static output feedback control gain for the system (2.3). In particular, we are interested in finding a controller of the form (2.4) that ensures (2.5). The closed-loop system of (2.3) with (2.4) can be written as follows:

\[ \dot{x}(t) = \sum_{i=1}^{r} \sum_{j=1}^{m} \mu_{ij} \left\{ (A_{ij} + \Delta A_{ij})x(t) + B_{1j}u(t) + B_{2j}w(t) \right\} + E_{2j}K_{ij}C_2x(t) \]

\[ z(t) = \sum_{i=1}^{r} \sum_{j=1}^{m} \mu_{ij} \left\{ C_{1i} + \Delta C_{1i} \right\} x(t) + (D_{12i} + \Delta D_{12i})u(t) + H_{2i}F_i(t)(E_{1i} + E_{2i}K_{ij}C_2) \] (3.2)

Input membership functions’ characteristics are crucial in many cases and may render less conservative results. Based on the so-called (outer) ellipsoid approximation algorithm, a new design is proposed which incorporates the membership function characteristics. Before we proceed with the development, the following definition is needed.

**Definition 3.1:** Define \( \mathcal{R}_{ij} \) as the region where the fuzzy rule \( i \) and fuzzy rule \( j \) are activated.

\[ \mathcal{R}_{ij} = \{ x | \mu_i(x) > 0 \} \]

We assume that each region \( \mathcal{R}_{ij} \) can be outer approximated by a union of ellipsoids [34] \( \mathcal{E}_{ijk} \) for \( k = 1, \ldots, m \), where \( m \) is the number of ellipsoids. That is, matrices \( T_{ijk} \) and \( f_{ijk} \) exist such that

\[ \mathcal{R}_{ij} \subseteq \bigcup_{k=1}^{m} \mathcal{E}_{ijk} \quad \text{where} \quad \mathcal{E}_{ijk} = \{ x | ||T_{ijk}x + f_{ijk}|| \leq 1 \}. \] (3.3)

Note that \( ||T_{ijk}x + f_{ijk}|| \leq 1 \) can also be represented as the following LMI form:

\[ \begin{bmatrix} x(t) \end{bmatrix}^T \begin{bmatrix} T_{ijk} & f_{ijk} \end{bmatrix} \begin{bmatrix} x(t) \end{bmatrix} \leq 1. \]

**Theorem 3.1:** Given a prescribed \( \mathcal{H}_\infty \) performance \( \gamma > 0 \), for \( i = 1, \ldots, r \) and \( j = 1, \ldots, r \), if there exist symmetric matrices \( P \) and \( Y_{ij} \), matrices \( K_i \) and \( X \), scalars \( \lambda_{ij} \geq 0 \) and \( \varepsilon > 0 \) satisfying the following conditions:

\[ \Phi_{ii} < 0 \quad \text{when} \quad 0 \notin \mathcal{E}_{ii} \]

\[ \Phi_{ij} + \Phi_{ji} < 0 \quad \text{when} \quad 0 \notin \mathcal{E}_{ij} \]

\[ \Psi_{ij} < 0 \quad \text{when} \quad 0 \notin \mathcal{E}_{ij} \]

\[ \Psi_{ij} + \Psi_{ji} < 0 \quad \text{when} \quad 0 \notin \mathcal{E}_{ij} \]

\[ \begin{bmatrix} Y_{11} & \ldots & Y_{1r} \\ \vdots & \ddots & \vdots \\ Y_{r1} & \ldots & Y_{rr} \end{bmatrix} > 0 \quad \text{and} \quad P > 0 \]

where we also have the equation at the bottom of the page and \( U = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \), then inequality (2.5) holds and the closed system is asymptotically stable.

**Remark 3.1:** When \( 0 \in \mathcal{E}_{ij} \), the term \( 1 - f_{ijk}^Tf_{ijk} \) becomes positive which implies that for the LMI to hold, \( \lambda_{ij} \) must be zero.
However, the LMIs are no longer strictly feasible. Hence, to avoid non-
strictly feasible, when \( 0 \in \mathcal{E}_{i,k} \), the membership function characteris-
tics are not incorporated. Notice that conditions (3.5) and (3.6) imply that
the origin is locally stable.

**Proof of Theorem 3.1:** Let us choose our Lyapunov function can-
didate as \( V(x(t)) = x^T(t)P(x(t)) \). The time derivative of \( V(x(t)) \)
along the closed-loop system (3.1) is as shown in (3.10), at the bottom
of the page. Adding and subtracting \( -z^T(t)z(t) + \gamma^2 w^T(t)w(t) \) to
and from (3.10), we have (3.11), shown at the bottom of the next page.
Using the fact \( (X - P)^T B_2 B_3^T (X - P) \geq 0 \) or \(-X^T B_2 B_3^T X \geq -P B_2 B_3^T P \) for any \( X \) and \( P \) of the
same dimension, we obtain

\[
\dot{V}(x(t)) \leq \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_{i,j} \left[ x(t) \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \right]^T \Omega_{i,j} \left[ x(t) \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \right] - z^T(t)z(t) + \gamma^2 w^T(t)w(t) \tag{3.12}
\]

where we have (3.13), as shown at the bottom of the next page. Now,
we need to show that

\[
\left[ \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \right]^T \Omega_{i,j} \left[ \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \right] < 0 \quad \text{for } x(t) \in \mathcal{R}_{i,j}.
\]

Applying S-Procedure, Schur complements, (3.4), and (3.5)-(3.8), we
learn that

\[
\left[ \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \right]^T \Omega_{i,j} \left[ \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \right] < -x^T(t)Y_{i,j}x(t), \quad \forall x(t) \in \mathcal{R}_{i,j}. \tag{3.14}
\]

Using (3.14), we obtain

\[
\dot{V}(x(t)) \leq -\sum_{i=1}^{r} \sum_{j=1}^{r} \mu_{i,j} x^T(t)Y_{i,j}x(t) - 2 \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_{i,j} x^T(t)Y_{i,j}x(t) - z^T(t)z(t) + \gamma^2 w^T(t)w(t)
\]

Thus, if (3.9) holds, we have

\[
\dot{V}(x(t)) \leq -z^T(t)z(t) + \gamma^2 w^T(t)w(t).
\]

Integrating both sides of this inequality yields

\[
\int_0^\infty \dot{V}(x(t))dt \leq \int_0^\infty [-z^T(t)z(t) + \gamma^2 w^T(t)w(t)]dt
\]

Thus, if \( x(0) = 0 \) and \( V(x(\infty)) \geq 0 \), we obtain

\[
\int_0^\infty z^T(t)z(t)dt \leq \gamma^2 \int_0^\infty w^T(t)w(t)dt.
\]

Hence, (2.5) holds and the \( \mathcal{H}_\infty \) performance is fulfilled. Now we prove
the closed system is asymptotically stable. To prove the stability, we set
the disturbance \( w(t) = 0 \), that is

\[
\dot{x}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_{i,j} \left[ A_i + B_2 K_j C_2 \right] x(t) + H_1(t) \left[ E_1 + E_2 K_j C_2 \right] x(t).
\]
The time derivative of Lyapunov function candidate $V(x(t)) = x^T(t)Px(t)$ along (3.15) is as shown in (3.16), at the bottom of the next page. Using the Schur complement and (3.5)–(3.8), it can show that $V(x(t))$ is less than zero for all $x(t) \neq 0$. Hence, the closed-loop system (3.1) is asymptotically stable.

**Iterative linear matrix inequality (ILMI) algorithm [10]**

**Step 1.** For $i = 1, \ldots, r$, solve the following LMIs that is jointly convex in $(Q,F)$

$$
Q = Q^T > 0 \quad (3.17)
$$

$$
\begin{bmatrix}
QA^T_i + A_iQ - F_i^T B_{21} B_{21}^T + B_{21} P - I
\end{bmatrix} < 0. \quad (3.18)
$$

Set $t = 1$ and $X_t = FQ^{-1}$ and choose a larger $\alpha_0$.

Minimize $\alpha_t$

subject to $P > 0$, $\lambda_{ij} \geq 0$, $\varepsilon > 0$, (3.9) > 0

$$
\Phi_{ii} - \alpha_t U^T P U < 0 \quad \text{when } 0 \notin \mathcal{E}_{ii}
$$

$$
\Phi_{ii} + \Phi_{ji} - 2\alpha_t U^T P U < 0 \quad \text{when } 0 \notin \mathcal{E}_{ij}
$$

$$
\Psi_{ijk} - \alpha_t \begin{bmatrix}
U^T P U \\
\varepsilon T
\end{bmatrix} < 0 \quad \text{when } 0 \notin \mathcal{E}_{ijk}
$$

$$
\Psi_{ijk} + \Psi_{jik} - 2\alpha_t \begin{bmatrix}
U^T P U \\
\varepsilon T
\end{bmatrix} < 0
$$

when $0 \notin \mathcal{E}_{ijk}$.

(3.19)

**Step 2.** For $i = 1, \ldots, r$, $i < j \leq r$, and $k = 1, \ldots, m$, solve the following optimization problem in $P$, $K_i$, $Y_{ij}$, $\varepsilon$ and $\lambda_{ijk}$ using the auxiliary variable $X_t$ determined in the previous step to obtain $\alpha_t$:

$$
V(x(t)) \leq \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{\mu_{ij}} \left[ x^T(t) \begin{bmatrix}
A_i^T P + PA_i - PB_{21} B_{21}^T P + \left( B_{21}^T P + K_i C_i \right)^T \left( B_{21}^T P + K_i C_i \right) \\
+ PH_i H_i^T P \quad + \left( E_{ii} + E_{ii} K_i C_i \right)^T \left( E_{ii} + E_{ii} K_i C_i \right) \\
\left( C_{ii} + D_{12} K_i C_i \right)^T \left( I - \varepsilon H_2 H_2^T \right)^{-1} \left( C_{ii} + D_{12} K_i C_i \right) \\
\left( E_{ii} + E_{ii} K_i C_i \right)^T \left( E_{ii} + E_{ii} K_i C_i \right)
\end{bmatrix} \right] X(t) + w^T(t) P B_1 w(t) - \gamma^2 w^T(t) w(t)
$$

$$
- z^T(t) z(t) + \gamma^2 w^T(t) w(t)
$$

$$
\leq \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{\mu_{ij}} \left[ x^T(t) \begin{bmatrix}
A_i^T P + PA_i - PB_{21} B_{21}^T P + \left( B_{21}^T P + K_i C_i \right)^T \left( B_{21}^T P + K_i C_i \right) \\
+ PH_i H_i^T P + \left( E_{ii} + E_{ii} K_i C_i \right)^T \left( E_{ii} + E_{ii} K_i C_i \right) \\
\left( C_{ii} + D_{12} K_i C_i \right)^T \left( I - \varepsilon H_2 H_2^T \right)^{-1} \left( C_{ii} + D_{12} K_i C_i \right) \\
\left( E_{ii} + E_{ii} K_i C_i \right)^T \left( E_{ii} + E_{ii} K_i C_i \right)
\end{bmatrix} \right] X(t) + w^T(t) P B_1 w(t) - \gamma^2 w^T(t) w(t)
$$

$$
- z^T(t) z(t) + \gamma^2 w^T(t) w(t)
$$

$$
= \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{\mu_{ij}} \left[ x^T(t) \begin{bmatrix}
A_i^T P + PA_i - PB_{21} B_{21}^T P \\
+ \left( B_{21}^T P + K_i C_i \right)^T \left( B_{21}^T P + K_i C_i \right) \\
+ PH_i H_i^T P + \left( E_{ii} + E_{ii} K_i C_i \right)^T \left( E_{ii} + E_{ii} K_i C_i \right) \\
\left( C_{ii} + D_{12} K_i C_i \right)^T \left( I - \varepsilon H_2 H_2^T \right)^{-1} \left( C_{ii} + D_{12} K_i C_i \right) \\
\left( E_{ii} + E_{ii} K_i C_i \right)^T \left( E_{ii} + E_{ii} K_i C_i \right)
\end{bmatrix} \right] X(t) + w^T(t) P B_1 w(t) - \gamma^2 w^T(t) w(t)
$$

$$
= \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{\mu_{ij}} \left[ x^T(t) \begin{bmatrix}
A_i^T P + PA_i - PB_{21} B_{21}^T P \\
+ \left( B_{21}^T P + K_i C_i \right)^T \left( B_{21}^T P + K_i C_i \right) \\
+ PH_i H_i^T P + \left( E_{ii} + E_{ii} K_i C_i \right)^T \left( E_{ii} + E_{ii} K_i C_i \right) \\
\left( C_{ii} + D_{12} K_i C_i \right)^T \left( I - \varepsilon H_2 H_2^T \right)^{-1} \left( C_{ii} + D_{12} K_i C_i \right) \\
\left( E_{ii} + E_{ii} K_i C_i \right)^T \left( E_{ii} + E_{ii} K_i C_i \right)
\end{bmatrix} \right] X(t) + w^T(t) P B_1 w(t) - \gamma^2 w^T(t) w(t)
$$

$$
\Omega_{ij} = \begin{bmatrix}
A_i^T P + PA_i - PB_{21} B_{21}^T P \\
+ \left( B_{21}^T P + K_i C_i \right)^T \left( B_{21}^T P + K_i C_i \right) \\
+ PH_i H_i^T P + \left( E_{ii} + E_{ii} K_i C_i \right)^T \left( E_{ii} + E_{ii} K_i C_i \right) \\
\left( C_{ii} + D_{12} K_i C_i \right)^T \left( I - \varepsilon H_2 H_2^T \right)^{-1} \left( C_{ii} + D_{12} K_i C_i \right) \\
\left( E_{ii} + E_{ii} K_i C_i \right)^T \left( E_{ii} + E_{ii} K_i C_i \right)
\end{bmatrix}
$$

(3.11)

(3.13)
Step 3. If $\alpha_t < 0$, $P$, $K_i$, $Y_{ij}$ and $\lambda_{ijk}$ obtained in Step 2 are a feasible solution to the BMIIs and stop.

Step 4. For $i = 1, \ldots, r$, $i < j \leq r$, and $k = 1, \ldots, m$, solve the following optimization problem in $P$, $K_i$, $Y_{ij}$, $\varepsilon$, and $\lambda_{ijk}$ using $\alpha_t$ and $X_t$ determined in the previous steps.

\[
\begin{align*}
\text{Minimize} & \quad \text{trace}(P) \\
\text{subject to} & \quad P > 0, \quad \lambda_{ijk} \geq 0, \quad \varepsilon > 0 \quad (3.9), \quad \text{and} \quad (3.19)
\end{align*}
\]

Step 5. If $\|X_t - P\| < \delta$, go to Step 6, where $\delta$ is a predetermined small value. Else set $t = t + 1$ and $X_t = P$, and go to Step 2.

Step 6. The system may not have a feasible solution and stop.

**Remark 3.2:**

1) Note that (3.5) implies

\[ G_{ii} = A_i^T P + PA_i - X^T B_{21}B_{21}^T P - PB_{21}B_{21}^T X + X^T B_{22}B_{22}^T X < 0. \quad (3.20) \]

which provides an initial guess for $X$. Premultiplying and post-multiplying (3.20) with $Q = P^{-1}$ and defining $F = XQ$, one can see that $G_{ii} < 0$ is equivalent to LMI (3.18) by the Schur complement. What Step 1 does is to seek an $X$ corresponding to the necessary condition of (3.5) and use it as an initial guess for the iterative algorithm.

2) $\alpha P$ is introduced in (3.5)-(3.8) to relax the LMIs, and it corresponds to the following Lyapunov inequality:

\[ V(x(t)) = x^T(t)Px(t), \quad V(x(t)) \leq \alpha V(x(t)) - z^T(t)z(t) + \gamma^2 w^T(t)w(t). \]

Thus, if $\alpha$ is negative, the BMIs (3.5)-(3.8) are feasible and the design performance of fuzzy system (3.1) is realized.

3) The optimization problem in Step 2 is a generalized eigenvalue minimization problem. This step guarantees the progressive reduction of $\alpha_t$. Step 4 is to guarantee the convergence of the algorithm.

4) Sometimes, $\alpha_{t+1}$ can be greater than $\alpha_t$ in Step 2 which may due to the implementation problem of the LMI program. In this situation, we must set $\alpha_{t+1} = \alpha_t$. Owing to the effect of some numerical errors in Step 4, the optimization problem in Step 4 may be infeasible. In such case, let $\alpha_t = \alpha_t + \Delta \alpha_t$, where $\Delta \alpha_t$ is a small positive real number, and go to Step 4 to solve the optimization problem again.

The following remark shows that a dynamic output-feedback problem for the system (2.3) can be transformed into a static output-feedback problem.

**Remark 3.3:** For the system (2.3), a fuzzy dynamic output feedback controller is inferred as follows:

\[
\begin{align*}
\dot{x}(t) & = \sum_{i=1}^{r} \mu_i(t) \left[ A_i \dot{x}(t) + B_i y(t) \right] \\
\dot{u}(t) & = \sum_{i=1}^{r} \mu_i(t) \left[ C_i \dot{x}(t) + D_i y(t) \right]
\end{align*}
\]

where $A_i, B_i, C_i$, and $D_i$ are the controller’s matrices.

Substituting (3.21) into (2.3), we have the following closed-loop system:

\[
\dot{x}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j \left[ A_i + B_2 K_i C_j + \bar{B}_{ij} F_i(t) \right] x(t) \times \left( E_{ii} + E_{ij} K_i C_j \right) x(t) + B_{ij} w(t)
\]

\[
\dot{z}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j \left[ \bar{C}_{ij} + D_{ij} K_i C_j + H_{2i} F_i(t) \right] x(t) \times \left( \bar{E}_{ii} + \bar{E}_{ij} K_i C_j \right) x(t)
\]

where $\dot{x}(t) = x(t)^T \dot{x}(t), \dot{z}(t) = x(t)^T \dot{z}(t), A_i = \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}, B_{ij} = \begin{bmatrix} B_{ij} \\ 0 \\ 0 \end{bmatrix}, C_i = \begin{bmatrix} C_i \\ 0 \\ I \end{bmatrix}, D_{ij} = \begin{bmatrix} D_{ij} \\ 0 \\ 0 \end{bmatrix}, \bar{C}_i = \begin{bmatrix} C_2 \\ 0 \\ I \end{bmatrix}, \bar{K}_i = \begin{bmatrix} D_{ij} \\ \bar{B}_{ij} \\ \bar{A}_j \end{bmatrix}, \bar{H}_i = \begin{bmatrix} H_i \\ 0 \\ 0 \end{bmatrix}, \bar{E}_{ii} = \begin{bmatrix} E_{ii} \\ 0 \\ 0 \end{bmatrix},$ and $\bar{E}_{ij} = \begin{bmatrix} E_{ij} \\ 0 \\ 0 \end{bmatrix}.$

From (3.22), design gain matrices $\bar{K}_i$ such that (2.5) holds and the closed-loop system is asymptotically stable is a static output feedback problem. Hence, we have shown that a dynamic output-feedback problem for the system (2.3) can be transformed into a static output-feedback problem. Therefore, Theorem 3.1 can be utilized to design a dynamic output feedback controller.
IV. NUMERICAL EXAMPLE

To illustrate the validation of the results obtained previously, we consider the following problem of balancing an inverted pendulum on a cart. The equations of motion of the pendulum [18] are

\[
\begin{align*}
    x_1' &= x_2 \\
    x_2' &= \frac{g \sin(x_1) - a \sin(2x_1)}{x_1 \cos(x_1)} + w
\end{align*}
\]  

(4.1)

where \(x_1\) denotes the angle of the pendulum from the vertical position, and \(x_2\) is the angular velocity. The initial state \(x_1 = 0, x_2 = 0, g = 9.8 \text{ m/s}^2\) is the gravity constant, \(m\) is the mass of the pendulum, \(a = 1/(m+M)\), \(M\) is the mass of the cart, \(2l\) is the length of the pendulum, and \(w\) is the force applied to the cart. In the simulation, the pendulum parameters are chosen as \(m = 2 \text{ kg}, M = 8 \text{ kg}, \text{ and } 2l = 1.0 \text{ m}\).

We approximate the system (4.1) by the following T-S fuzzy model [23]:

**Rule 1**: IF \(x_1(t)\) is \(M_1\), THEN

\[
\begin{align*}
    \dot{x}(t) &= (A_1 + \Delta A_1)x(t) + B_1w(t) \\
    &+ (B_{21} + \Delta B_{21})u(t) \\
    z(t) &= C_1x(t) + D_{12}u(t) \\
    y(t) &= C_2x(t)
\end{align*}
\]

**Rule 2**: IF \(x_1(t)\) is \(M_2\), THEN

\[
\begin{align*}
    \dot{x}(t) &= (A_2 + \Delta A_2)x(t) + B_1w(t) \\
    &+ (B_{22} + \Delta B_{22})u(t) \\
    z(t) &= C_1x(t) + D_{12}u(t) \\
    y(t) &= C_2x(t)
\end{align*}
\]

where

\[
\begin{align*}
    A_1 &= \begin{bmatrix} 0 & 1 \\ \frac{g}{x_1 \cos(x_1)} & 0 \end{bmatrix}, & B_{21} &= \begin{bmatrix} 0 \\ -\frac{g}{x_1 \cos(x_1)} \end{bmatrix} \\
    A_2 &= \begin{bmatrix} 0 & 1 \\ \frac{g}{x_1 \cos(x_1)} & 0 \end{bmatrix}, & B_{22} &= \begin{bmatrix} 0 \\ -\frac{g}{x_1 \cos(x_1)} \end{bmatrix} \\
    B_1 &= \begin{bmatrix} 1 & 0 \end{bmatrix}, & C_1 &= \begin{bmatrix} 1 & 0.3 \end{bmatrix}, & D_{12} &= 0.01 \\
    C_2 &= \begin{bmatrix} 0.1 \end{bmatrix}
\end{align*}
\]

and \(\beta = \cos(88^\circ)\). The disturbance attenuation level \(\gamma\) is set to be equal to 1 in this example. The membership functions for Rule 1 and Rule 2 are shown in Fig. 1.

Let the uncertain terms be given as

\[
\begin{align*}
    H_{11} &= H_{12} = \begin{bmatrix} 0 & 0 \\ 0.15 & 0 \end{bmatrix}, & E_{11} &= E_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
    E_{21} &= E_{22} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & H_{21} &= H_{22} = 0, & \|F(t)\| \leq 1.
\end{align*}
\]

From the membership functions given in Fig. 1, we have \(\mathcal{R}_{12} = \bigcup_{i=1}^{2} \bar{\mathcal{E}}_{12i} \bar{\mathcal{E}}_{22i} \) and \(\mathcal{R}_{22} = \bigcup_{i=1}^{2} \bar{\mathcal{E}}_{12i} \bar{\mathcal{E}}_{22i} \) where \(\bar{\mathcal{E}}_{121} = \bar{\mathcal{E}}_{221} = (x_1(t) - (\pi/4))(x_1(t) - (\pi/2)) \leq 0\) and \(\bar{\mathcal{E}}_{122} = \bar{\mathcal{E}}_{222} = (x_1(t) + (\pi/4))(x_1(t) + (\pi/2)) \leq 0\). In the form given in (3.4), we obtain

\[
\begin{align*}
    T_{121} &= T_{221} = \frac{8}{\pi} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & f_{121} &= f_{221} = 3 \\
    T_{122} &= T_{222} = \frac{8}{\pi} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & f_{122} &= f_{222} = -3.
\end{align*}
\]

Fig. 1. Membership functions.

Fig. 2. Ratio of the regulated output energy to the disturbance input noise energy.

Note that \(R_{11}\) is not computed because it contains the origin. Applying the proposed ILMI method, a solution to the above BMIs is found after five iterations with \(\alpha = -0.3211, \varepsilon = 1\)

\[
\begin{bmatrix} 36.838 & -6.7879 \\ -6.7879 & 2.2662 \end{bmatrix}
\]

\[
\begin{bmatrix} 201.71 & -49.604 \\ -49.604 & 12.204 \end{bmatrix}
\]

\[
\begin{bmatrix} -22.348 & 5.4985 \\ 5.4985 & -1.3544 \end{bmatrix}
\]

\[
\begin{bmatrix} 96.283 & -31.205 \\ -31.205 & 12.1 \end{bmatrix}
\]

and the static output feedback gains

\[
\begin{bmatrix} K_1 = 400.49, & K_2 = 47.466. \end{bmatrix}
\]

Remark 4.1: Without incorporating the membership functions’ characteristics implies that the conditions given in Theorem 3.1 must hold for \(x(t) \in \mathbb{R}^n\). Clearly, it will lead to a very conservative result. In this example, we found that no feasible solution can be found when the membership functions’ characteristics are neglected. A square wave with frequency of 1 Hz and amplitude of 0.3 is used to simulate the disturbance input noise \(w(t)\). The ratio of the regulated output energy to the disturbance input noise energy \(\int_0^T z_v^2(t) z_w(t) dt / \int_0^T w^2(t) w(t) dt \) obtained using the fuzzy controller gain (4.2) is given in Fig. 2. We can see that after 5 s, the ratio of the regulated output energy to the disturbance input noise energy given in this example tends to be a constant value which is about 0.1.
Hence, the disturbance attenuation level $\gamma$ is about 0.3, which is less than the prescribed level $\gamma_1$.

V. CONCLUSION

Design of a robust $\mathcal{H}_\infty$ static output feedback controller for a T-S fuzzy system has been provided in this paper. The existence of a static output feedback control law has been expressed in terms of the solvability of bilinear matrix inequalities. To compute a solution to the BMIs, an iterative algorithm based on the linear matrix inequality has been proposed. The conservatism of the design has been reduced by incorporating the input membership structural information. A numerical example has been given to illustrate the validity of our design.

REFERENCES


