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# CHEBYSHEV CONSTANTS, LINEAR ALGEBRA, AND COMPUTATION ON ALGEBRAIC CURVES

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ABSTRACT. In [9], [1], directional Chebyshev constants were studied on a complex algebraic curve. Using linear algebra, we study further properties of these constants. We show that Chebyshev constants in different directions are proportional if their corresponding polynomial classes satisfy certain algebraic relations. If  $V$  is a quadratic curve, this condition is equivalent to  $V$  being irreducible. We conjecture that it is equivalent to irreducibility in general.

## 1. INTRODUCTION

An important quantity associated to a compact set  $K \subset \mathbb{C}$  is its Chebyshev constant  $\tau(K)$ , which is defined as follows. Let

$$\mathcal{M} = \{p \in \mathbb{C}[z] : p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0, n = 1, 2, \dots\}$$

be the family of monic polynomials. Then put

$$(1.1) \quad \tau_n(K) := \inf\{\|p\|_K : p \in \mathcal{M}, \deg(p) \leq n\}^{1/n}, \quad \text{and} \quad \tau(K) := \lim_{n \rightarrow \infty} \tau_n(K).$$

In the above equation  $\|p\|_K = \max\{|p(z)| : z \in K\}$  denotes the sup (or uniform) norm. The fact that the above limit exists is due to algebraic properties of the family  $\mathcal{M}$ .

In several variables, since there is more than one monomial for each degree there is no single analog of a monic polynomial. One can, however, construct families of polynomials whose leading terms have a specified form. For such a family  $\mathcal{U}$  of polynomials, one can try to obtain Chebyshev-type constants by inserting  $\mathcal{U}$  in place of  $\mathcal{M}$  in (1.1). In his study of the Fekete-Leja transfinite diameter in  $\mathbb{C}^N$  ( $N > 1$ ), Zaharjuta [14] studied Chebyshev-type constants on compact sets that are associated to certain specific families of polynomials in  $\mathbb{C}^N$ ,  $N > 1$ ; his constants came to be known as *directional Chebyshev constants*. Zaharjuta's results were later generalized to homogeneous polynomials [7] and weighted polynomials [2],[3].

Later, in [9],[1], Chebyshev-type constants were studied on complex algebraic curves. Starting with an algebraic curve  $V$ , computational properties of polynomials on  $V$  (i.e., as elements of  $\mathbb{C}[V]$ , cf. Section 2) were used to construct natural families of polynomials associated to  $V$ . These families of polynomials have specific leading terms which may be computed explicitly, and are closely connected to the asymptotic directions of the curve (see Section 2.4). The constants obtained using these families were called *directional Chebyshev constants on  $V$* . As with Zaharjuta,

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such constants were used to study a notion of transfinite diameter on  $V$ . We remark that this transfinite diameter on  $V$  is likely to be closely related to the notion of *sectional capacity* on  $V$ , which is used in algebraic number theory by Rumely and others (cf. [11], [8]).\*

The explicit computation of polynomials in  $\mathbb{C}[V]$  is facilitated by linear algebra, where polynomials are represented with respect to some basis on  $\mathbb{C}[V]$ ,<sup>†</sup> and polynomial multiplication is considered as a linear map on  $\mathbb{C}[V]$ . While  $\mathbb{C}[V]$  is infinite-dimensional, the computations in this paper require only standard (i.e. finite-dimensional) linear algebra, since only certain specific coefficients are of interest.

The contents of the paper are as follows. In Section 2 we review computation on an algebraic curve  $V$ , especially the use of linear algebra to compute the leading terms of a product. The eigenvalues and eigenvectors of certain multiplication matrices are closely related to the asymptotic directions of  $V$ . This is illustrated in some specific examples on curves in  $\mathbb{C}^2$ .

In Section 3, we recall the Chebyshev constant  $\tau(K, Q)$  associated to a compact set  $K$  and homogeneous polynomial  $Q$ . The case of interest is when  $K$  is a compact subset of the curve  $V$  and  $Q$  is a homogeneous polynomial that is related (in a dual sense) to a given asymptotic direction  $\lambda$  of  $V$ . (Precisely, take the polynomial  $Q = \mathbf{v}_\lambda$  of Proposition 2.10.) This gives the *directional Chebyshev constant*  $\tau_V(K, \lambda)$ . For completeness, we also review Zaharjuta's directional Chebyshev constants and their relation to the Fekete-Leja transfinite diameter in  $\mathbb{C}^N$ .

In Sections 2 and 3 we omit most of the proofs, referring the reader to [1], [9].

In Section 4, we study Chebyshev-type constants constructed using arbitrary families of polynomials. If two families of polynomials are related to each other via multiplication on the curve  $V$ , then their respective constants are also related (Proposition 4.2). We use this relation to give a condition under which directional Chebyshev constants  $\tau_V(K, \lambda)$  and  $\tau_V(K, \mu)$  are proportional when  $\lambda \neq \mu$  (Corollary 4.7). If  $V$  is a quadratic curve, we show by explicit calculation that this condition is equivalent to  $V$  being irreducible (Theorem 4.4). The calculation is carried out using linear algebra in the spirit of Section 2.

In Section 5, we demonstrate the discontinuous behaviour of directional Chebyshev constants. We close the paper by conjecturing that the result on quadratic curves holds in general: directional Chebyshev constants on an irreducible curve are proportional.

## 2. COMPUTATION AND LINEAR ALGEBRA

Write  $z = (z_1, \dots, z_N)$  for coordinates on  $\mathbb{C}^N$ , and write  $\mathbb{C}[z] = \mathbb{C}[z_1, \dots, z_N]$  for the ring of polynomials over  $\mathbb{C}$  in these variables. We use standard multi-index notation: if  $\alpha = (\alpha_1, \dots, \alpha_N)$  is a multi-index then

$$z^\alpha = z_1^{\alpha_1} \cdots z_N^{\alpha_N} \quad \text{and} \quad |\alpha| = \alpha_1 + \cdots + \alpha_N.$$

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\*This has not yet been studied on curves; however, the close relationship between the Fekete-Leja transfinite diameter in  $\mathbb{C}^N$  and sectional capacity has been studied in some detail (see [12], [6]).

<sup>†</sup>i.e. represented as a *normal form*, cf. Section 2.1.

Given a (nonempty) algebraic variety  $V = \{z \in \mathbb{C}^N : P_1(z) = \cdots = P_k(z) = 0\}$ , where  $P_1, \dots, P_k$  are polynomials in  $\mathbb{C}[z]$ , let

$$\mathbf{I}(V) := \{p \in \mathbb{C}[z] : p(z) = 0 \text{ for all } z \in V\}$$

be the ideal of  $V$ . The polynomials restricted to  $V$  can be identified with elements of the factor ring  $\mathbb{C}[z]/\mathbf{I}(V) =: \mathbb{C}[V]$ , called the *coordinate ring of  $V$* .

Define the degree on  $V$  of a polynomial  $p$  by

$$(2.1) \quad \deg_V(p) = \min\{\deg(q) : q(z) = p(z) \text{ for all } z \in V\},$$

where  $\deg$  denotes the usual degree in  $\mathbb{C}[z]$ , i.e.,  $\deg(c_\alpha z^\alpha) := |\alpha|$  ( $c_\alpha \in \mathbb{C} \setminus \{0\}$ ), and for any polynomials  $p_1, p_2$ ,  $\deg(p_1 + p_2) := \max\{\deg p_1, \deg p_2\}$ .

Next, for a non-negative integer  $s$  write

$$\mathbb{C}[z]_{\leq s} := \{p \in \mathbb{C}[z] : \deg(p) \leq s\} \text{ and } \mathbb{C}[V]_{\leq s} := \{p \in \mathbb{C}[V] : \deg_V(p) \leq s\}$$

for the polynomials of degree at most  $s$ . As a vector space over  $\mathbb{C}$  we have  $\dim(\mathbb{C}[z]_{\leq s}) = \binom{N+s}{s}$  as can be seen by counting the monomials of degree  $\leq s$  in  $z$ , and  $\dim(\mathbb{C}[V]_{\leq s}) \leq \dim(\mathbb{C}[z]_{\leq s})$ .

It is well-known in algebraic geometry that when  $V$  is an algebraic curve, then  $\dim(\mathbb{C}[V]_{\leq s})$  is linear in  $s$  for  $s$  large:

$$(2.2) \quad \dim(\mathbb{C}[V]_{\leq s}) = c + sd, \quad c, d \in \mathbb{Z}, d > 0.$$

We will work exclusively with algebraic curves in this paper.

**2.1. Computation in  $\mathbb{C}[V]$ .** In one variable, monomials are naturally ordered by degree:  $1, z, z^2, z^3, \dots$ , while in higher dimensions, there are many different orderings on monomials. In this paper we will use the *grevlex ordering*, defined as follows.

Let  $\alpha, \beta$  be multi-indices in  $N$  variables. Denoting grevlex by  $\prec$ , it is defined by setting  $z^\alpha \prec z^\beta$  whenever:

- $|\alpha| < |\beta|$ ; or
- $|\alpha| = |\beta|$  and there exists  $i \in \{1, \dots, n\}$  such that  $\alpha_i > \beta_i$  and  $\alpha_j = \beta_j$  for any positive integer  $j < i$ .

Grevlex is an example of a *graded ordering*, which is an ordering that satisfies the first condition,  $|\alpha| < |\beta| \Rightarrow z^\alpha \prec z^\beta$ .

For example, the first few monomials in  $\mathbb{C}[z_1, z_2]$  listed according to  $\prec$  are

$$1, z_1, z_2, z_1^2, z_1 z_2, z_2^2, z_1^3, z_1^2 z_2, z_1 z_2^2, z_2^3, \dots$$

For a polynomial  $p(z) = \sum_\alpha a_\alpha z^\alpha$ , define the *leading term*  $\text{LT}(p) = a_\beta z^\beta$  to be the term for which  $a_\beta \neq 0$  and for all  $\alpha$  such that  $a_\alpha \neq 0$ ,  $z^\alpha \prec z^\beta$ .

To obtain a basis of monomials for a curve  $\mathbb{C}[V]$ , we go through the monomials of  $\mathbb{C}[z]$  in increasing order (according to grevlex), throwing out linearly dependent monomials as they arise. Let  $\mathcal{B} = \mathcal{B}(V) := \{z^{\alpha_j}\}_{j=1}^\infty$  denote this reduced collection of monomials, which forms a basis of  $\mathbb{C}[V]$  by definition.

Let  $I = \mathbf{I}(V)$ ,  $\text{LT}(I) = \{\text{LT}(p) : p \in I\}$ , and  $\langle \text{LT}(I) \rangle$  the ideal of  $\mathbb{C}[z]$  generated by  $\text{LT}(I)$ . In general, angle brackets  $\langle \rangle$  will denote the polynomial ideal generated by the (set of) polynomials written inside them. A standard result in computational algebraic geometry is the following.

**Proposition 2.1** (Algebraic computation in  $\mathbb{C}[V]$ ). *Let  $I = \mathbf{I}(V)$ . Then*

- (1)  $\mathcal{B} = \{z^\alpha : z^\alpha \notin \langle \text{LT}(I) \rangle\}$ .

(2) There is a finite collection of polynomials  $G = \{g_1, \dots, g_k\}$  with

$$(2.3) \quad \langle G \rangle = I \quad \text{and} \quad \langle \text{LT}(G) \rangle = \langle \text{LT}(I) \rangle.$$

(3) For any polynomial  $p$  there is a unique polynomial  $r$  such that

$$(2.4) \quad p(z) = \sum_{j=1}^k q_j(z)g_j(z) + r(z)$$

where  $q_1, \dots, q_k \in \mathbb{C}[z]$  and all terms of  $r$  are in  $\mathcal{B}$ .  $\square$

Since  $\mathcal{B}$  is a basis for  $\mathbb{C}[V]$ , every polynomial has a unique representative written in terms of  $\mathcal{B}$ ; it must therefore coincide with  $r(z)$  in (2.4). The representation (2.4) is computed using a generalized division algorithm in which  $\{g_1, \dots, g_k\}$  are the divisors,  $\{q_1, \dots, q_k\}$  the quotients, and  $r$  is the remainder. A collection of polynomials satisfying (2.3) is called a Groebner basis. Groebner bases are in general, not unique, but they always give the remainder  $r$  under division. (See [4] for an introduction to this theory.)

We call  $r$  the *normal form* of  $p$ . It is the unique representative of the class of polynomials equal to  $p$  on  $V$  (i.e., element of  $\mathbb{C}[V]$ ) that can be expressed as a linear combination of elements of  $\mathcal{B}$ .

**Example 2.2.** For the most simple example, consider an affine set in  $\mathbb{C}^N$  given by the equations  $z_j = c_j$  for  $j = k+1, \dots, N$  ( $k < N$ ) where the  $c_j$ 's are constants. Then  $z_j \in \langle \text{LT}(I) \rangle$ , so no monomials containing  $z_{k+1}, \dots, z_N$  are in  $\mathcal{B}$ . Rearranging terms, it is easy to see that any polynomial may be written as

$$p(z) = z_{k+1}q_{k+1}(z) + \dots + z_Nq_N(z) + r(z_1, \dots, z_k).$$

**2.2. Multiplication of polynomials.** Rather than considering  $\mathbb{C}[V]$  as a factor ring, by Proposition 2.1 we will consider  $\mathbb{C}[V]$  as the collection of normal forms.<sup>‡</sup> A calculation also shows that for a normal form  $r$ ,  $\deg(r) = \deg_V(r)$  where the latter is defined as in (2.1). This is due to the fact that grevlex is a graded ordering.

Multiplication of polynomials in  $\mathbb{C}[V]$  is a bilinear map  $*$ :  $\mathbb{C}[V] \times \mathbb{C}[V] \rightarrow \mathbb{C}[V]$  given by the composition

$$(p, q) \mapsto pq \mapsto \mathcal{B}(pq) =: p * q,$$

where for convenience  $\mathcal{B}(pq)$  denotes the normal form of the product  $pq$ .

We want to model this with linear algebra. In this section we are interested in the leading homogeneous part of the product. For  $p(z) = \sum_{|\alpha| \leq d} a_\alpha z^\alpha$ , write  $\widehat{p}(z) = \sum_{|\alpha|=d} a_\alpha z^\alpha$  for the leading homogeneous part.

Next, for a nonnegative integer  $d$ , write  $\mathbb{C}[V]_{=d}$  for the collection of homogeneous polynomials (in normal form) of degree  $d$ . We expect  $q \mapsto \widehat{p * q}$  to be a linear map  $\mathbb{C}[V]_{=d} \rightarrow \mathbb{C}[V]_{=d+\deg p}$ . This is not always the case as cancellations may occur. For example:

**Example 2.3.** Let  $V = \{z_2^2 - z_1^2 - z_1 - 1 = 0\}$ , and take  $p = z_1 + z_2$  and  $q = z_1 - z_2$ . Then  $\widehat{p * q} = z_1$  which is not of degree 2.

<sup>‡</sup>This is analogous to considering  $\mathbb{Z}_p$  as the set  $\{0, \dots, p-1\}$  rather than as the collection of equivalence classes mod  $p$ .

To account for cancellation, define

$$p \widehat{*} q = \begin{cases} \widehat{p * q} & \text{if } \deg p + \deg q = \deg_V(pq) \\ 0 & \text{otherwise} \end{cases}.$$

Note that we adjoin the zero element to  $\mathbb{C}[V]_{=n}$ , which turns it into a vector space over  $\mathbb{C}$ . It is easy to see the following.

**Lemma 2.4.** *Let  $p$  be a fixed homogeneous polynomial. Then  $q \mapsto p \widehat{*} q$  is a linear map from  $\mathbb{C}[V]_{=n}$  to  $\mathbb{C}[V]_{=n+\deg p}$ .  $\square$*

When  $V$  is a curve, (2.2) says that there is a fixed positive integer  $d$  such that for sufficiently large degrees  $n$ ,  $\mathbb{C}[V]_{=n}$  is of dimension  $d$ . In what follows, we will take  $n$  sufficiently large enough to ensure this. Proposition 2.1(1) says that  $\mathbb{C}[V]_{=n}$  has basis  $\{z^\alpha \notin \langle \text{LT}(I) \rangle : |\alpha| = n\}$ .

Define a representation of  $\mathbb{C}[V]_{=n}$  as follows: if the basis of  $\mathbb{C}[V]_{=n}$  listed in (increasing) grevlex order is  $\{z^{\alpha_1}, \dots, z^{\alpha_d}\}$ , then match  $z^{\alpha_j}$  with the standard  $j$ -th coordinate in  $\mathbb{C}^d$ . Using this, we form vector and matrix representations of homogeneous polynomials. For  $p \in \mathbb{C}[V]_{=n}$  given by  $p(z) = a_{\alpha_1} z^{\alpha_1} + \dots + a_{\alpha_d} z^{\alpha_d}$ , set

$$[p] := (a_{\alpha_1}, \dots, a_{\alpha_d}) \in \mathbb{C}^d.$$

Similarly, let  $[[p]]$  denote the representation of  $p$  as a  $d \times d$  matrix, i.e., representing the linear map  $q \mapsto p \widehat{*} q$ , so that

$$[[p]][q] = [p \widehat{*} q].$$

(Here we view  $[q] \in \mathbb{C}^d$  as a column matrix.) Note that if  $V \subset \mathbb{C}^N$  with coordinates  $(z_1, \dots, z_N)$ , it is sufficient to compute the representation of each coordinate  $[[z_j]]$ , since a calculation shows that for any polynomial  $p(z) = p(z_1, \dots, z_N)$ ,

$$[[p]] = \widehat{p} \left( [[z_1]], \dots, [[z_N]] \right).$$

**2.3. Illustrations in  $\mathbb{C}^2$ .** Let  $(z_1, z_2)$  denote coordinates in  $\mathbb{C}^2$ .

**Example 2.5.** Let  $V$  be the (complexified) circle given by  $z_1^2 + z_2^2 = 1$ . Then

$$\mathcal{B} = \{1, z_1, z_2, z_1^2, z_1 z_2, z_2^3, \dots, z_1^n, z_1^{n-1}, \dots\}.$$

An element of  $\mathbb{C}[V]_{=n}$  is a polynomial of the form  $p(z) = az_1^n + bz_1^{n-1}z_2$ , represented by the vector  $(a, b)$  (or by  $\begin{bmatrix} a \\ b \end{bmatrix}$  as a column matrix). A calculation shows that

$$[[z_1]] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } [[z_2]] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \text{ Hence } [[p]] = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

**Example 2.6.** Let  $V$  be the hyperbola given by  $z_1 z_2 = 1$ . Then

$$\mathcal{B} = \{1, z_1, z_2, z_1^2, z_2^2, \dots, z_1^n, z_2^n, \dots\},$$

so in this case  $(a, b)$  represents  $az_1^n + bz_2^n \in \mathbb{C}[V]_{=n}$ . We then have

$$[[z_1]] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad [[z_2]] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

**Example 2.7.** Let  $V = \{z = (z_1, z_2) \in \mathbb{C}^2 : P(z) = 0\}$  for some polynomial  $P(z)$  of degree  $d$ , whose leading homogeneous part  $\widehat{P}(z)$  has the form

$$\widehat{P}(z) = \widehat{P}(z_1, z_2) = c_0 z_1^d + c_1 z_1^{d-1} z_2 + \cdots + c_{d-1} z_1 z_2^{d-1} + z_2^d.$$

Then  $\mathcal{B} = \{z^\alpha : z^\alpha \notin \langle z_2^d \rangle\}$ ,  $[[z_1]] = I$  and

$$[[z_2]] = \begin{bmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{bmatrix}.$$

(This is the well-known *companion matrix* of  $p(t) := \widehat{P}(1, t)$ .) One can also factor the leading homogeneous part:

$$(2.5) \quad \widehat{P}(z) = \prod_{j=1}^d (z_2 - \lambda_j z_1)$$

for some numbers  $\lambda_1, \dots, \lambda_d$ . When all the  $\lambda_j$  are distinct, the curve  $V$  has  $d$  linear asymptotes of the form  $z_2 = \lambda_j z_1 + a_j$  ( $a_j \in \mathbb{C}$ ) for each  $j$ . This situation was studied in [9].

**2.4. Geometry and algebraic computation.** Let coordinates in  $\mathbb{C}^N$  be given by  $z = (z_1, \dots, z_N)$ , and let  $V \subset \mathbb{C}^N$  be an algebraic curve of degree  $d$ . In [1], curves with  $d$  distinct linear asymptotes were studied, and the following was shown.

**Theorem 2.8.** *Let  $V \subset \mathbb{C}^N$  be an affine algebraic curve of degree  $d$  with the property that all its linear asymptotes have a parametrization of the form*

$$t \mapsto \lambda_k t + c_k, \quad k = 1, \dots, d$$

where  $c_k = (c_{k1}, \dots, c_{kN}) \in \mathbb{C}^N$  and  $\lambda_k = (1, \lambda_{k2}, \dots, \lambda_{kN})$ .

- (1) *There are  $d - 1$  multi-indices  $\alpha^{(2)}, \dots, \alpha^{(d)}$  such that for sufficiently large  $n$ , the monomials in the basis  $\mathcal{B}$  of degree  $n$  are given by*

$$z_1^n, z_1^{n_2} z^{\alpha^{(2)}}, \dots, z_1^{n_d} z^{\alpha^{(d)}} \quad (n_k = n - |\alpha^{(k)}| \quad \forall k).$$

- (2)  *$[[z_1]] = I$ , the  $d \times d$  identity matrix.*

- (3) *If for  $j \in \{2, \dots, N\}$  we have  $\lambda_{kj} \neq \lambda_{k'j}$  whenever  $k, k' \in \{1, \dots, d\}$  and  $k \neq k'$ , then  $[[z_j]]$  is a  $d \times d$  matrix with eigenvalues  $\lambda_{kj}$ ,  $k = 1, \dots, d$ .  $\square$*

**Example 2.9.** The complexified circle  $z_1^2 + z_2^2 = 1$  in Example 2.5 gives a simple illustration of the above theorem. The linear asymptotes are given by  $z_2 = \pm i z_1$ , or  $t \mapsto t(1, \pm i)$  in parametric form. Observe that the eigenvalues of  $[[z_2]] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  are  $\pm i$ .

The following result was proved in [1].

**Theorem 2.10.** *Suppose the algebraic curve  $V \subset \mathbb{C}^N$  is as in Theorem 2.8, with the additional property that if  $\tilde{\mu} = (1, \tilde{\mu}_2, \dots, \tilde{\mu}_N)$  and  $\mu = (1, \mu_2, \dots, \mu_N)$  are distinct asymptotic directions of  $V$  (i.e.,  $V$  has linear asymptotes given by  $t \mapsto \tilde{\mu}t + \tilde{c}$ ,  $t \mapsto \mu t + c$  for  $c, \tilde{c} \in \mathbb{C}^N$ ), then  $\tilde{\mu}_k \neq \mu_k$  for all  $k = 2, \dots, N$ .*

*Then for any asymptotic direction  $\lambda = (1, \lambda_2, \dots, \lambda_N)$  of  $V$  there is a homogeneous polynomial  $\mathbf{v}_\lambda(z)$  with the following properties:*

- (1)  $\mathbf{v}_\lambda(\lambda) = 1$  and  $\mathbf{v}_\lambda(\mu) = 0$  for any other asymptotic direction  $\mu \neq \lambda$ .
- (2)  $\mathbf{v}_\lambda(z) \widehat{*} \mathbf{v}_\lambda(z) = z_1^{\deg \mathbf{v}_\lambda} \mathbf{v}_\lambda(z)$ .
- (3)  $\mathbf{v}_\lambda(z)$  is unique up to a multiple of  $z_1$ . □

In the rest of the paper, we will take  $\mathbf{v}_\lambda(z)$  to be the polynomial in the above theorem of lowest degree, which is unique. It can be obtained as follows. Write  $\lambda = (1, \lambda_2, \dots, \lambda_N)$ ; then for any  $j \in \{2, \dots, N\}$ ,  $\lambda_j$  is an eigenvalue of  $[[z_j]]$ . Find an eigenvector for  $\lambda_j$  and let  $v_j(z)$  be the corresponding homogeneous polynomial represented by this eigenvector. Do this for each  $j$ , and put  $\mathbf{w}(z) := \prod_{j=2}^N v_j(z)$ . Then  $\mathbf{w}(z) = cz_1^q \mathbf{v}_\lambda(z)$ , where we factor out as many powers of  $z_1$  as possible, and a constant  $c$  so that the normalization  $\mathbf{v}_\lambda(\lambda) = 1$  holds.

**Example 2.11.** As an illustration, consider Example 2.9 with  $\lambda = (1, \lambda_2) = (1, i)$ . An eigenvector of  $[[z_2]]$  for the eigenvalue  $\lambda_2 = i$  is given by  $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ , which corresponds to the homogeneous polynomial  $z_1 - iz_2$ . Set  $\mathbf{w}(z) := z_1 - iz_2$ ; in this case, the product consists of one term. Normalizing, we have  $2 = \mathbf{w}(1, i) = c\mathbf{v}_{(1,i)}(1, i) = c$ , so that  $\mathbf{v}_{(1,i)}(z) = \frac{1}{2}(z_1 - iz_2)$ .

**Remark 2.12.** By doing a projective change of coordinates, the results of this section can be rewritten in terms of multiplication matrices for zero dimensional varieties, which have been studied by several authors (see e.g. chapter 2§4 of the book [5] and the references therein).

### 3. CHEBYSHEV CONSTANTS

**3.1. Classical Chebyshev constant.** Let  $K \subset \mathbb{C}$  be compact, and let  $\mathcal{M}$  denote the set of monic polynomials as defined in the introduction. A *Chebyshev polynomial for  $K$  of degree  $d$*  is a monic polynomial  $t_d(z)$  for which

$$(3.1) \quad \|t_d\|_K = \inf\{\|p\|_K : p \in \mathcal{M}\}$$

and  $\tau_d(K) := (\inf\{\|p\|_K : p \in \mathcal{M}\})^{\frac{1}{d}}$  is called the *Chebyshev constant for  $K$  of order  $d$* .

The existence and uniqueness of Chebyshev polynomials of order  $d$  is well-known (see e.g. chapter 5 of [10]).

**Proposition 3.1.** *The limit  $\tau(K) := \lim_{d \rightarrow \infty} \tau_d(K)$  exists.*

*Proof.* It suffices to show that  $\limsup_{d \rightarrow \infty} \tau_d(K) \leq \liminf_{d \rightarrow \infty} \tau_d(K)$ . Let  $\epsilon > 0$  and choose  $d_0 \in \mathbb{N}$  such that  $\tau_{d_0}(K) \leq \liminf_{d \rightarrow \infty} \tau_d(K) + \epsilon$ . Given  $m > d_0$ , use the division algorithm to find  $q, r$  with  $m = d_0q + r$  with  $0 \leq r < d_0$ . Take a polynomial  $p_0 \in \mathcal{M}$  with  $\deg(p_0) = d_0$  such that  $\|p_0\|_K \leq \tau_{d_0}(K) + \epsilon$ .

We have  $p := z^r p_0^q \in \mathcal{M}$  with  $\deg(p) = m$ , and so

$$\tau_m(K) \leq \|z^r p_0^q\|_K^{1/m} \leq \|z\|_K^{r/m} \|p_0\|_K^{d_0q/m} \leq \|z\|_K^{r/m} (\tau_{d_0}(K) + \epsilon)^{d_0q/m},$$

and hence  $\tau_m(K) \leq \|z\|_K^{r/m} (\liminf_{d \rightarrow \infty} \tau_d(K) + 2\epsilon)^{d_0q/m}$ .

We take the limsup as  $m \rightarrow \infty$  of the inequality. Since  $\frac{r}{m} \in [0, \frac{d_0}{m})$  and  $\frac{d_0q}{m} \in (\frac{m-d_0}{m}, 1]$ , we have  $\frac{r}{m} \rightarrow 0$  and  $\frac{d_0q}{m} \rightarrow 1$ . So  $\limsup_{m \rightarrow \infty} \tau_m(K) \leq \liminf_{d \rightarrow \infty} \tau_d(K) + 2\epsilon$ .

Letting  $\epsilon \rightarrow 0$  yields the result. □



The limiting constant  $\tau(K)$  is called the *Chebyshev constant of  $K$* . The classical example is when  $K$  is the interval  $[-1, 1] \subset \mathbb{R} \subset \mathbb{C}$ . The Chebyshev polynomials in this case are given by  $t_d(z) = \cos(d \cos^{-1}(z))$ , and the Chebyshev constant is  $\tau([-1, 1]) = \frac{1}{2}$ . The proof of the above result is classical, but shows the main idea in the more general constructions that follow.

**3.2. Chebyshev constants in higher dimensions.** We return to  $\mathbb{C}^N$ ,  $N > 1$ . Let  $Q(z) = \sum_{|\alpha|=d_0} a_\alpha z^\alpha$  be a homogeneous polynomial of degree  $d_0$ . Define the polynomial class  $\mathcal{M}(Q)$  to be the collection of all polynomials  $p(z)$  with the property that  $\hat{p}(z) = Q(z)^k$  for some positive integer  $k$  (i.e.,  $p(z) = Q(z)^k + q(z)$ ,  $\deg(q) < d_0 k$ ).

Now fix a compact set  $K \subset \mathbb{C}^N$ . Define

$$\tau_d(K, Q) = (\inf\{\|p\|_K : p \in \mathcal{M}(Q), \deg p \leq d\})^{\frac{1}{d}}.$$

**Proposition 3.2.** *The limit  $\tau(K, Q) := \lim_{k \rightarrow \infty} \tau_{d_0 k}(K, Q)$  exists.*  $\square$

The proof of this proposition is exactly the same as for monic polynomials in Proposition 3.1: simply replace powers of  $z$  there by powers of  $Q$  here.

**Remark 3.3.** A homogeneous polynomial in one variable is of the form  $Q(z) = a_n z^n$ , and  $\tau(K, Q) = |a_n|^{1/n} \tau(K)$  for any compact set  $K$ , i.e., we get a scaled version of the classical Chebyshev constant. In higher dimensions, Chebyshev constants associated to linearly independent homogeneous polynomials are not comparable, e.g. in  $\mathbb{C}^2$  with coordinates  $(z_1, z_2)$ , consider  $\tau(K, z_1)$  and  $\tau(K, z_2)$ . These are instances of Zaharjuta's directional Chebyshev constants.

Zaharjuta's directional Chebyshev constants are described as follows. Let  $S$  be the simplex in  $\mathbb{R}^N$  given by

$$(3.2) \quad S = \left\{ \theta = (\theta_1, \dots, \theta_N) \in \mathbb{R}^N : \sum_{j=1}^N \theta_j = 1, \theta_j \geq 0 \forall j \right\}.$$

Next, given  $\theta \in S \cap \mathbb{Q}^N$ , write  $\theta = \frac{\alpha}{|\alpha|}$  where  $\alpha = (\alpha_1, \dots, \alpha_N)$  is a multi-index. We then set  $\tau(K, \theta) := \tau(K, z^\alpha)$ . This is well-defined as it is easy to see that  $\tau(K, z^\alpha) = \tau(K, z^\beta)$  whenever  $\frac{\alpha}{|\alpha|} = \frac{\beta}{|\beta|}$ .

More generally, for  $\theta \in S$ , Zaharjuta showed in [14] that the following definition of  $\tau(K, \theta)$  is well-defined: take a sequence of multi-indices  $\alpha_1, \alpha_2, \dots$  with  $\lim_{j \rightarrow \infty} \frac{\alpha_j}{|\alpha_j|} = \theta$  and  $\lim_{j \rightarrow \infty} |\alpha_j| = +\infty$ ; then set  $\tau(K, \theta) := \lim_{j \rightarrow \infty} \tau_{|\alpha_j|}(K, z^{\alpha_j})$ .

We call the parameter  $\theta$  a *direction* and  $\tau(K, \theta)$  the *directional Chebyshev constant for  $K$  in the direction  $\theta$* .

**3.3. Chebyshev constants on curves.** On a compact set  $K \subset V \subset \mathbb{C}^N$ , where  $V = \{P_1(z) = \dots = P_k(z)\}$  is an algebraic curve, one can compute Chebyshev constants  $\tau(K, Q)$  as above, by considering  $K$  as a subset of  $\mathbb{C}^N$ . Given a normal form  $Q \in \mathbb{C}[V]$ , the normal form of  $Q^k$  (i.e.  $Q * \dots * Q$  ( $k$  times)) will usually have a complicated pattern of coefficients as  $k \rightarrow \infty$ .

Using properties of computation on  $\mathbb{C}[V]$ , we seek polynomials that behave nicely under multiplication. When  $V \subset \mathbb{C}^N$  satisfies the conditions of Theorem 2.8, one can find a homogeneous polynomial  $Q$  that satisfies

$$(3.3) \quad Q \hat{*} Q = \widehat{Q^2} = z_1^{\deg Q} Q.$$

The simplest example is of course when  $Q = z_1$ . The above equation says that the matrix  $[[Q]]$  satisfies  $[[Q]]^2 = [[Q]]$ .

Important examples of polynomials satisfying (3.3) are given by the polynomials  $\mathbf{v}_\lambda$  of Theorem 2.10 (see part (2) of the theorem).

**Definition 3.4.** Let

$$\mathcal{M}(\lambda) = \{p(z) \in \mathbb{C}[V] : \widehat{p}(z) = z_1^k \mathbf{v}_\lambda(z) \text{ for some integer } k \geq 0\},$$

and for  $d \geq \deg(\mathbf{v}_\lambda)$ , define the *directional Chebyshev constant of order  $d$  (for the direction  $\lambda$ )* by

$$\tau_d(K, \lambda) := \inf\{\|p\|_K : p \in \mathcal{M}(\lambda), \deg(p) \leq d\}^{\frac{1}{d}}.$$

The following is shown in [1].

**Proposition 3.5.** *The limit  $\lim_{d \rightarrow \infty} \tau_d(K, \lambda) =: \tau(K, \lambda)$  exists and, in the notation of Proposition 3.2,  $\tau(K, \lambda) = \tau(K, \mathbf{v}_\lambda)$ .*  $\square$

We call  $\tau(K, \lambda)$  the *directional Chebyshev constant of  $K$  for the direction  $\lambda$* .

**Remark 3.6.** When  $V$  is a line (i.e., irreducible of degree 1), it may be parametrized by  $t \mapsto t\lambda + a$ . Suppose  $\lambda = (1, \lambda_2, \dots, \lambda_N)$ . Then there is only one directional Chebyshev constant  $\tau(K, \lambda)$  which coincides with the classical one given in Section 3.1. Precisely, we have  $\tau(K, \lambda) = \tau(K_1)$  where  $K_1 = \{z_1 \in \mathbb{C} : (z_1, z_2) \in K\}$  is the projection of  $K$  to the  $z_1$ -axis. Thus quadratic curves are the first non-trivial ones to study for this notion.

**3.4. Transfinite diameter formulas.** An important application of Chebyshev constants is to the study of transfinite diameter. We recall the definition of the Fekete-Leja transfinite diameter in  $\mathbb{C}^N$ . Let  $\{z^{\alpha_j}\}_{j=1}^\infty$  be an enumeration of the monomials according to a graded ordering (e.g. grevlex). Given an  $m$ -tuple of points  $(\zeta_1, \zeta_2, \dots, \zeta_m)$ ,  $\zeta_j \in \mathbb{C}^N$ , define

$$\text{Van}(\zeta_1, \dots, \zeta_m) = \det[z^{\alpha_i}(\zeta_j)]_{i,j=1,\dots,m} = \det \begin{bmatrix} z^{\alpha_1}(\zeta_1) & z^{\alpha_1}(\zeta_2) & \cdots & z^{\alpha_1}(\zeta_m) \\ z^{\alpha_2}(\zeta_1) & z^{\alpha_2}(\zeta_2) & \cdots & z^{\alpha_2}(\zeta_m) \\ \vdots & \vdots & \ddots & \vdots \\ z^{\alpha_m}(\zeta_1) & z^{\alpha_m}(\zeta_2) & \cdots & z^{\alpha_m}(\zeta_m) \end{bmatrix}.$$

For a positive integer  $n$ , let  $m_n$  be the number of monomials of degree  $\leq n$ , and let  $l_n = \sum_{\nu=1}^n \nu(m_\nu - m_{\nu-1})$  be the sum of the degrees. We then define the  $n$ -th order diameter of a compact set  $K \subset \mathbb{C}^N$  by

$$(3.4) \quad d_n(K) := \sup \left\{ |\text{Van}(\zeta_1, \dots, \zeta_{m_n})| : \zeta_j \in K, \forall j = 1, 2, \dots, m_n \right\}^{\frac{1}{l_n}}.$$

Zaharjuta's main theorem in [14] is the following.

**Theorem 3.7.** *The limit  $d(K) = \lim_{n \rightarrow \infty} d_n(K)$  exists and*

$$d(K) = \exp \left( \frac{1}{v(S)} \int_S \log(\tau(K, \theta)) dv(\theta) \right). \quad \square$$

Here ‘ $v(S)$ ’ denotes the usual (Lebesgue) volume of  $S$  ( $S$  is defined in (3.2)), and  $dv(\theta)$  is integration with respect to volume in the variable  $\theta$ .

In an analogous way, one can define transfinite diameter on a complex algebraic curve  $V$  that satisfies the assumptions of Theorem 2.8. Let  $\lambda_1, \dots, \lambda_d$  be the asymptotic directions of  $V$  and  $\tau(K, \lambda_j)$  the corresponding directional Chebyshev constants. Let now  $\{z^{\alpha_j}\}$  be an enumeration according to a graded ordering of the monomial basis  $\mathcal{B} = \mathcal{B}(V)$  as defined in Section 2.1. Define  $\text{Van}(\zeta_1, \dots, \zeta_m) = \det[z^{\alpha_i}(\zeta_j)]_{i,j=1,\dots,m}$  as above. Also, for a positive integer  $n$ , let  $m_n$  be the number of monomials in  $\mathcal{B}$  of degree  $\leq n$  and  $l_n = \sum_{\nu=1}^n \nu(m_\nu - m_{\nu-1})$ . Define the  $n$ -th order diameter  $d_n(K)$  of a compact set  $K \subset V$  as in (3.4). Then the following theorem holds ([9],[1]).

**Theorem 3.8.** *The limit  $d(K) = \lim_{n \rightarrow \infty} d_n(K)$  exists and*

$$d(K) = \left( \prod_{j=1}^d \tau(K, \lambda_j) \right)^{\frac{1}{d}},$$

where  $\tau(K, \lambda_1), \dots, \tau(K, \lambda_d)$  denote the directional Chebyshev constants for the asymptotic directions  $\lambda_1, \dots, \lambda_d$  of  $V$ .  $\square$

#### 4. IRREDUCIBILITY AND QUADRATIC CURVES

It will be convenient to distinguish Chebyshev constants on different algebraic curves, so we will use subscripts to do this:  $\tau_V(\cdot)$  will denote a Chebyshev constant defined on the curve  $V$ .

We need the fact that directional Chebyshev constants behave nicely under affine changes of coordinates. The proof of this result is in [1].

**Theorem 4.1.** *Suppose  $W = a + T(V) := \{a + T(z) : z \in V\}$  where  $a \in \mathbb{C}^N$  and  $T = (T_1, \dots, T_N) : \mathbb{C}^N \rightarrow \mathbb{C}^N$  is a non-singular complex linear map. Suppose  $T_1(\lambda) \neq 0$  for any asymptotic direction  $\lambda$  of  $V$ .*

*Given a compact  $K \subset V$ , and  $L := \{a + T(z) : z \in K\} \subset W$ , we have*

$$\tau_V(K, \lambda) T_1(\lambda) = \tau_W(L, \eta),$$

where  $\eta = \frac{T(\lambda)}{T_1(\lambda)} = \left(1, \frac{T_2(\lambda)}{T_1(\lambda)}, \dots, \frac{T_N(\lambda)}{T_1(\lambda)}\right)$  is the corresponding asymptotic direction for  $W$ .  $\square$

We will now study, rather generally, Chebyshev-type constants defined by families of polynomials. Start with two arbitrary collections of polynomials  $\mathcal{U}, \mathcal{V}$ . For convenience, assume that each collection contains the zero polynomial, hence is nonempty. For a compact set  $K$  and a positive integer  $d$ , define

$$\tau_d(K, \mathcal{U}) = \inf\{\|p\|_K : p \in \mathcal{U}, \deg(p) \leq d\}^{1/d},$$

and define  $\tau_d(K, \mathcal{V})$  similarly.

Let us also define

$$\bar{\tau}(K, \mathcal{U}) = \limsup_{d \rightarrow \infty} \tau_d(K, \mathcal{U}), \quad \text{and} \quad \underline{\tau}(K, \mathcal{U}) = \liminf_{d \rightarrow \infty} \tau_d(K, \mathcal{U}).$$

So  $\lim_{d \rightarrow \infty} \tau_d(K, \mathcal{U}) =: \tau(K, \mathcal{U})$  exists if and only if  $\bar{\tau}(K, \mathcal{U}) = \underline{\tau}(K, \mathcal{U})$  (and equals  $\tau(K, \mathcal{U})$ ). Let us call it the *Chebyshev constant of  $K$  defined by  $\mathcal{U}$* .

**Proposition 4.2.** *Suppose there exists a polynomial  $P$  such that  $\|P\|_K > 0$  and for any  $q \in \mathcal{U}$ ,  $Pq \in \mathcal{V}$ . Then*

$$(4.1) \quad \bar{\tau}(K, \mathcal{V}) \leq \bar{\tau}(K, \mathcal{U}), \quad \underline{\tau}(K, \mathcal{V}) \leq \underline{\tau}(K, \mathcal{U}).$$

Hence  $\tau(K, \mathcal{V}) \leq \tau(K, \mathcal{U})$  if both quantities exist.

*Proof.* Given an integer  $d$ , take  $q_d$  to be polynomial of degree  $d$  for which  $\|q_d\|_K = \tau_d(K, \mathcal{U})^d$ . Then  $Pq_d \in \mathcal{V}$  with  $\deg(Pq_d) = d + \deg(P) =: d'$  and so

$$\tau_{d'}(K, \mathcal{V})^{d'} \leq \|Pq_d\|_K \leq \|P\|_K \|q_d\|_K = \|P\|_K \tau_d(K, \mathcal{U})^d.$$

Hence  $\tau_{d'}(K, \mathcal{V}) \leq \|P\|_K^{1/d'} \tau_d(K, \mathcal{U})^{d/d'}$ . Equation (4.1) now follows upon taking first the lim sup, then lim inf, of this inequality as  $d \rightarrow \infty$ , since  $1/d' \rightarrow 0$  and  $d/d' \rightarrow 1$ .  $\square$

**Remark 4.3.** The proof of Proposition 4.2 uses the same basic idea as the classical proof given for Proposition 3.1: modify polynomials in one class to get polynomials in another class. It was a key idea in Zaharjuta's study of directional Chebyshev constants in [14].

Let us now introduce some convenient notation.

- Notation.** (1) When  $q \in \mathcal{U} \Rightarrow Pq \in \mathcal{V}$  for some polynomial  $P$ , we write this as  $P\mathcal{U} \subset \mathcal{V}$ .  
 (2) If there exists a polynomial  $P$  with  $\|P\|_K > 0$  and  $P\mathcal{U} \subset \mathcal{V}$ , we write  $\mathcal{U} \rightarrow \mathcal{V}$ . Note that by Proposition 4.2,  $\tau(K, \mathcal{U}) \geq \tau(K, \mathcal{V})$  if both quantities are defined.

If  $\tau(K, \mathcal{U}_0)$  exists for some family  $\mathcal{U}_0$  of polynomials, and (with the above notation) there is a chain of finite length

$$\mathcal{U}_0 \rightarrow \mathcal{U}_1 \rightarrow \mathcal{U}_2 \rightarrow \cdots \rightarrow \mathcal{U}_n$$

where  $\mathcal{U}_j$  ( $j = 1, \dots, n$ ) is a family of polynomials for each  $j$ , then  $\tau(K, \mathcal{U}_j) \geq \tau(K, \mathcal{U}_k)$  whenever these quantities exist and  $k > j$ .

We specialize now to quadratic curves. The main result of this section is the following.

**Theorem 4.4.** *Suppose  $V \subset \mathbb{C}^2$  is a quadratic curve with asymptotic directions  $\lambda_1$  and  $\lambda_2$  that satisfy the hypotheses of Theorem 2.8. Then  $V$  is irreducible if and only if the directional Chebyshev constants  $\tau(K, \lambda_1)$  and  $\tau(K, \lambda_2)$  are proportional: there is a fixed constant  $C_V > 0$  such that for any compact set  $K \subset V$ ,*

$$\tau(K, \lambda_1) = C_V \cdot \tau(K, \lambda_2).$$

We first make a complex linear change of coordinates  $T : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  so that in these new coordinates,  $V$  is transformed to the curve  $W$  given by the equation  $z_2^2 - z_1^2 = 4\epsilon$ . The asymptotic directions of  $W$  are given by the vectors  $(1, \pm 1)$ ; we will simply label them as  $\pm 1$ . Using the notation of the previous section, we write

$$\mathbf{v}_{-1}(z) = -\frac{1}{2}(z_2 - z_1), \quad \mathbf{v}_1(z) = \frac{1}{2}(z_2 + z_1).$$

We have the following explicit calculations in  $\mathbb{C}[W]$ :

$$(4.2) \quad \mathbf{v}_1 \mathbf{v}_{-1} = -\epsilon, \quad (\mathbf{v}_{-1})^2 = -z_1 \mathbf{v}_{-1} + \epsilon, \quad \mathbf{v}_1^2 = z_1 \mathbf{v}_1 + \epsilon, \quad z_1 = \mathbf{v}_1 + \mathbf{v}_{-1}.$$

Consider the basis of  $\mathbb{C}[W]$  given by

$$1, \mathbf{v}_{-1}, \mathbf{v}_1, z_1 \mathbf{v}_{-1}, z_1 \mathbf{v}_1, \dots, z_1^k \mathbf{v}_{-1}, z_1^k \mathbf{v}_1, \dots$$

We will use linear algebra to facilitate calculations in  $\mathbb{C}[W]$  with respect to this basis, in a similar way as done in Section 2. The main difference here is that coefficients of lower degree terms will be incorporated into the linear algebra, i.e., our multiplication matrices will be formed with respect to  $\mathbb{C}[W]_{\leq n}$ , not just  $\mathbb{C}[W]_{=n}$ . Given a positive integer  $n$ , on the finite dimensional subspace  $\mathbb{C}[W]_{\leq n}$  we represent the polynomial

$$p(z) = c + \sum_{k=0}^{n-1} (a_k z_1^k \mathbf{v}_{-1} + b_k z_1^k \mathbf{v}_1) \quad \text{by} \quad [p] = (c, a_0, b_0, \dots, a_{n-1}, b_{n-1}).$$

Multiplication by  $\mathbf{v}_1$  and  $\mathbf{v}_{-1}$  is then given by linear maps  $\mathbb{C}[W]_{\leq n} \rightarrow \mathbb{C}[W]_{\leq n+1}$  which we represent as  $(2n+1) \times (2n+3)$  matrices  $[[\mathbf{v}_1]]$  and  $[[\mathbf{v}_{-1}]]$ . Precisely, using the identities (4.2), the bottom right corner of these matrices is given by

$$[[\mathbf{v}_1]] = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & -\epsilon & \epsilon \\ \cdots & 0 & 0 & -\epsilon & \epsilon \\ \cdots & 0 & 0 & 0 & 0 \\ \cdots & 0 & 1 & 0 & 0 \\ \cdots & 0 & 0 & 0 & 0 \\ \cdots & 0 & 0 & 0 & 1 \end{bmatrix}, \quad [[\mathbf{v}_{-1}]] = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & \epsilon & -\epsilon \\ \cdots & 0 & 0 & \epsilon & -\epsilon \\ \cdots & -1 & 0 & 0 & 0 \\ \cdots & 0 & 0 & 0 & 0 \\ \cdots & 0 & 0 & -1 & 0 \\ \cdots & 0 & 0 & 0 & 0 \end{bmatrix}.$$

For both matrices, all entries above the  $6 \times 2$  block in the bottom right corner are zero. They have a diagonal pattern whose entries are given by pasting a copy of this block two steps up and two steps to the left, then filling in above and below with zeros.<sup>§</sup>

We will now construct families of polynomials that are related via polynomial multiplication as in Proposition 4.2. Let us use the notation  $D(n)$  to denote an arbitrary polynomial of degree  $\leq d$ . Recalling Definition 3.4, we have

$$\mathcal{M}(\pm 1) = \{p \in \mathbb{C}[V] : p(z) = z_1^n \mathbf{v}_{\pm 1}(z) + D(n), n = 0, 1, 2, \dots\}.$$

Let  $\mathcal{U} \subset \mathbb{C}[V]$  be the collection of polynomials

$$\mathcal{U} := \{p \in \mathbb{C}[V] : p(z) = \mathbf{v}_{-1}(z_1^n + a z_1^{n-1}) + D(n-1), n \in \mathbb{N}, a \in \mathbb{C}\}.$$

A polynomial in  $\mathcal{U}$  is therefore represented by a vector of the form  $[p] = (\dots, 0, 1, 0)$ . (Here all entries inside the dots are arbitrary.)

Next, define the class  $\mathcal{U}_0$  by

$$\mathcal{U}_0 := \{p \in \mathcal{U} : p(z) = \mathbf{v}_{-1}(z_1^n + a_1 z_1^{n-1} + a_2 z_1^{n-2}) - \epsilon \mathbf{v}_1 z_1^{n-2} + D(n-2), a_1, a_2 \in \mathbb{C}\}.$$

So  $p \in \mathcal{U}_0$  means  $[p] = (\dots, -\epsilon, *, 0, 1, 0)$ . (Here '\*' also denotes an arbitrary entry.)

Notice that  $\mathcal{U}_0 \subset \mathcal{U} \subset \mathcal{M}(-1)$ , so that  $\mathcal{U}_0 \rightarrow \mathcal{U} \rightarrow \mathcal{M}(-1)$ .

**Proposition 4.5.** *We also have  $\mathcal{M}(-1) \rightarrow \mathcal{U} \rightarrow \mathcal{U}_0 \rightarrow \mathcal{M}(1)$ .*

<sup>§</sup>We will ignore what happens when we reach the top left corner, as we are only interested in terms of sufficiently high degree. Multiplication matrices of a similar form are studied in Chapter 11 of [13].

*Proof.* For  $\mathcal{M}(-1) \rightarrow \mathcal{U}$ , we show that  $-\mathbf{v}_{-1}\mathcal{M}(-1) \subset \mathcal{U}$ . Note that a polynomial  $p$  in  $\mathcal{M}(-1) \cap \mathbb{C}[W]_{\leq n}$  is given by a vector  $[p]$  of length  $2n+1$  of the form  $(\dots, 1, 0)$ , and a polynomial in  $\mathcal{U}$  is of the form  $(\dots, 0, 1, 0)$ . Using our matrix representation to compute the leading entries, we have for  $p \in \mathcal{M}(-1)$ ,

$$[[-\mathbf{v}_{-1}]] [p] = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & 0 \\ \cdots & 0 & 1 & 0 \\ \cdots & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \vdots \\ * \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

and hence  $-\mathbf{v}_{-1}p \in \mathcal{U}$ . So  $\mathcal{M}(-1) \rightarrow \mathcal{U}$ .

Similarly, we show that  $\mathcal{U} \rightarrow \mathcal{U}_0$  by showing that  $-\mathbf{v}_{-1}\mathcal{U} \subset \mathcal{U}_0$ :

$$\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & 0 & -\epsilon & \epsilon \\ \cdots & 0 & 1 & 0 & 0 & 0 \\ \cdots & 0 & 0 & 0 & 0 & 0 \\ \cdots & 0 & 0 & 0 & 1 & 0 \\ \cdots & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \vdots \\ * \\ * \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \vdots \\ -\epsilon \\ * \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Hence  $\mathcal{U} \rightarrow \mathcal{U}_0$ .

Finally, we show that  $\mathcal{U}_0 \rightarrow \mathcal{M}(1)$  by showing that  $-\frac{1}{2\epsilon}\mathbf{v}_1^2\mathcal{U}_0 \subset \mathcal{M}(1)$ : multiplying first by  $\mathbf{v}_1$ , then by  $-\frac{1}{2\epsilon}\mathbf{v}_1$ , yields

$$\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 1 & 0 & 0 & -\epsilon & \epsilon \\ \cdots & 0 & 0 & 0 & 0 & 0 \\ \cdots & 0 & 0 & 1 & 0 & 0 \\ \cdots & 0 & 0 & 0 & 0 & 0 \\ \cdots & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vdots \\ -\epsilon \\ * \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \vdots \\ -2\epsilon \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \ddots & \vdots & \vdots \\ \cdots & 0 & 0 \\ \cdots & 0 & -\frac{1}{2\epsilon} \end{bmatrix} \begin{bmatrix} \vdots \\ * \\ -2\epsilon \end{bmatrix} = \begin{bmatrix} \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Hence  $\mathcal{U}_0 \rightarrow \mathcal{M}(1)$ .

Altogether, this shows that  $\mathcal{M}(-1) \rightarrow \mathcal{U} \rightarrow \mathcal{U}_0 \rightarrow \mathcal{M}(1)$ .  $\square$

Applying Proposition 4.2 immediately gives  $\tau(K, -1) \geq \tau(K, 1)$ . By a similar process as above, we can find polynomial families  $\mathcal{V}$  and  $\mathcal{V}_0$  with the property that  $\mathcal{M}(1) \rightarrow \mathcal{V} \rightarrow \mathcal{V}_0 \rightarrow \mathcal{M}(-1)$ , and get the reverse inequality. This gives the following.

**Corollary 4.6.** *We have  $\tau(K, -1) = \tau(K, 1)$ .*

We can now complete the proof of Theorem 4.4.

*Proof of Theorem 4.4.* Let  $V \subset \mathbb{C}^2$  be a general irreducible quadratic curve as in Theorem 2.8 with asymptotic directions  $(1, \lambda_1)$  and  $(1, \lambda_2)$ . There is a map  $z \mapsto T(z) + b$  sending  $V$  to the curve  $W = \{z_2^2 - z_1^2 = \epsilon\}$  (with  $\epsilon \neq 0$ ), where  $T = (T_1, T_2) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  is linear with  $T_1(1, \lambda_1), T_1(1, \lambda_2) \neq 0$ , and  $b \in \mathbb{C}^2$ . Suppose  $(1, \lambda_1)$  is the direction of the linear asymptote of  $V$  that maps to the line  $z_2 = z_1$  while  $(1, \lambda_2)$  is the direction of the linear asymptote that maps to  $z_2 = -z_1$ . Given a compact set  $K \subset V$ , Theorem 4.1 gives

$$\tau_V(K, (1, \lambda_2)) = \tau_W(T(K), -1)T_1(1, \lambda_2), \quad \tau_V(K, (1, \lambda_1)) = \tau_W(T(K), 1)T_1(1, \lambda_1);$$

and since  $\tau_W(T(K), -1) = \tau_W(T(K), 1)$ , Corollary 4.6 gives

$$\frac{\tau_V(K, (1, \lambda_2))}{T_1((1, \lambda_2))} = \frac{\tau_V(K, (1, \lambda_1))}{T_1(1, \lambda_1)}.$$

So the quantities are proportional with  $C_V = \frac{T_1(1, \lambda_1)}{T_1(1, \lambda_2)}$ .

Conversely, suppose  $V$  is reducible. Then it is a union of two lines  $V = L_1 \cup L_2$ , which we assume are given by equations  $z_2 - \lambda_1 z_1 + c_1 = 0$  and  $z_2 - \lambda_2 z_1 + c_2 = 0$  respectively. In this case, one can check by direct calculation that the asymptotic directions are given by  $(1, \lambda_1)$  and  $(1, \lambda_2)$  and that

$$\mathbf{v}_{(1, \lambda_1)}(z) = \frac{z_2 - \lambda_2 z_1}{\lambda_1 - \lambda_2} \quad \text{and} \quad \mathbf{v}_{(1, \lambda_2)}(z) = \frac{z_2 - \lambda_1 z_1}{\lambda_2 - \lambda_1}.$$

It follows that the polynomials  $p_d(z) = z_1^{d-1} \mathbf{v}_{(1, \lambda_2)}(z) + z_1^{d-1} \frac{c_1}{\lambda_2 - \lambda_1} \in \mathcal{M}((1, \lambda_2))$  are identically zero on  $L_1$ . Hence for any compact set  $K \subset L_1$ , we have  $\tau_d(K, \lambda_2)^d \leq \|p_d\|_K = 0$ ; so that  $\tau_V(K, \lambda_2) = 0$ .

On the other hand, given  $p \in \mathcal{M}(1, \lambda_1)$ , then for any  $z = (z_1, z_2) \in L_1$  a calculation (i.e. eliminate  $z_2$ ) gives  $p(z) = t(z_1)$ , where  $t$  is a monic polynomial. By classical potential theory (see e.g. [10], Chapter 5), the Chebyshev constant of the unit disk  $\{z_1 \in \mathbb{C} : |z_1| \leq 1\}$  is 1. It follows easily that  $\tau_V(K, (1, \lambda_1)) = 1$  where  $K = \{(z_1, z_2) \in L_1 : |z_1| \leq 1\}$ . Since  $\tau(K, (1, \lambda_2)) = 0$  by the previous paragraph, the quantities  $\tau(K, \lambda_1)$  and  $\tau(K, \lambda_2)$  are not proportional.  $\square$

More generally, the proof of Proposition 4.5 gives the following.

**Corollary 4.7.** *Let  $\lambda, \mu$  be asymptotic directions for an algebraic curve  $V \subset \mathbb{C}^n$ . If there are collections of polynomials  $\mathcal{U}_0, \dots, \mathcal{U}_k$  and  $\mathcal{V}_0, \dots, \mathcal{V}_m$  such that*

$$(4.3) \quad \mathcal{M}(\lambda) \rightarrow \mathcal{U}_0 \rightarrow \dots \rightarrow \mathcal{U}_k \rightarrow \mathcal{M}(\mu) \quad \text{and} \quad \mathcal{M}(\mu) \rightarrow \mathcal{V}_0 \rightarrow \dots \rightarrow \mathcal{V}_m \rightarrow \mathcal{M}(\lambda),$$

then  $\tau(K, \lambda) = \tau(K, \mu)$ .

*If  $V$  is an algebraic curve that can be transformed by an affine change of coordinates to a curve with the above property, then the Chebyshev constants  $\tau(\cdot, \lambda)$  and  $\tau(\cdot, \mu)$  are proportional.*

## 5. CONCLUDING REMARKS

**Reducibility and discontinuity.** Theorem 4.4 applies only to an irreducible quadratic curve. It cannot hold for reducible curves because of the following result in [1].

**Theorem 5.1.** *Suppose  $V = V_1 \cup V_2$  is a union of algebraic curves and  $K \subset V$ . Put  $K_1 = K \cap V_1$ . Suppose  $\lambda$  is an asymptotic direction of  $V_1$ . Then*

$$\tau_V(K, \lambda) = \tau_V(K_1, \lambda) = \tau_{V_1}(K_1, \lambda). \quad \square$$

A reducible quadratic curve  $V$  is the union of two complex lines  $V_1$  and  $V_2$ , say with directions  $\lambda_1$  and  $\lambda_2$ . (So  $V_j$  may be parametrized by  $t \mapsto t(1, \lambda_j)$ .)

Let  $K$  be a set in  $V_1$  for which  $\tau_{V_1}(K, \lambda_1) > 0$ . By the above theorem,  $\tau_V(K, \lambda_1) = \tau_{V_1}(K, \lambda_1) > 0$ . On the other hand,  $K \cap V_2$  is either empty or contains at most  $V_1 \cap V_2$  which is a single point  $\{a\}$ ; then the above theorem shows that  $\tau_V(K, \lambda_2) = \tau_{V_2}(\{a\}, \lambda_2) = 0$ , i.e.,  $\tau_V(K, \lambda_1) \neq \tau_V(K, \lambda_2)$ . This agrees with the calculations in the proof of Theorem 4.4.

As a consequence, directional Chebyshev constants can be “discontinuous under perturbation” because a reducible quadratic curve may be approximated by irreducible curves.

**Example 5.2.** Let

$$D = \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_2 - z_1| \leq 1 \text{ and } |z_2 + z_1| \leq \delta\}.$$

For each  $\epsilon \geq 0$  we put  $V_\epsilon = \{z : z_2^2 - z_1^2 = 4\epsilon\}$ . Let  $K_\epsilon = D \cap V_\epsilon$ . Then in the Hausdorff metric on compact subsets of  $\mathbb{C}^2$  it is easy to see that  $V_\epsilon \rightarrow V_0$  locally uniformly and  $K_\epsilon \rightarrow K_0$  uniformly as  $\epsilon \rightarrow 0$ .

We have  $V_0 = L_1 \cup L_{-1}$  where  $L_1$  is the line  $z_2 = z_1$  and  $L_{-1}$  is the line  $z_2 = -z_1$ ,

$$K_0 \cap L_1 = \{z \in L_1 : |z_1| \leq \frac{1}{2}\} =: K_{0,1},$$

and

$$K_0 \cap L_{-1} = \{z \in L_{-1} : |z_1| \leq \frac{\delta}{2}\} =: K_{0,-1}.$$

When  $\delta < 1$ , Theorem 5.1 says that  $\tau_{V_0}(K_0, 1) \neq \tau_{V_0}(K_0, -1)$ ; precisely,

$$\tau_{V_0}(K_0, 1) = \tau_{L_1}(K_{0,1}, 1) = \frac{1}{2}, \quad \tau_{V_0}(K_0, -1) = \tau_{L_{-1}}(K_{0,-1}, -1) = \frac{\delta}{2}. \quad \spadesuit$$

On the other hand, by the proof of Theorem 4.4, we have

$$\tau_{V_\epsilon}(K_\epsilon, 1) = \tau_{V_\epsilon}(K_\epsilon, -1) \quad \text{for all } \epsilon > 0.$$

Therefore, either  $\lim_{\epsilon \rightarrow 0} \tau_{V_\epsilon}(K_\epsilon, 1) \neq \tau_{V_0}(K_0, 1)$  or  $\lim_{\epsilon \rightarrow 0} \tau_{V_\epsilon}(K_\epsilon, -1) \neq \tau_{V_0}(K_0, -1)$ .

**Reducibility and Corollary 4.7.** The key property for directional Chebyshev constants to be proportional is given by Corollary 4.7: after a linear change of coordinates, the corresponding polynomial families for these directions are related as in (4.3). For the quadratic curve  $z_2^2 - z_1^2 = \epsilon$ , it is easy to see that this fails when  $\epsilon = 0$ , which is precisely when the curve is reducible.

What about an algebraic curve  $V \subset \mathbb{C}^n$  of higher degree? By Theorem 5.1, irreducibility is a necessary condition for its directional Chebyshev constants to be proportional. A plausible conjecture is that irreducibility is also sufficient.

**Conjecture.** *The following two conditions are equivalent.*

- (1)  $V \subset \mathbb{C}^n$  is an irreducible algebraic curve of degree  $d \geq 1$  with  $d$  linear asymptotes in distinct directions.
- (2) For any directions  $\lambda$  and  $\mu$  of  $V$ , one can find an affine change of coordinates such that the polynomial classes  $\mathcal{M}(\lambda)$  and  $\mathcal{M}(\mu)$  satisfy the hypotheses of Corollary 4.7.

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<sup>¶</sup>Here we use the fact that the classical Chebyshev constant of a disk in the complex plane is its radius (see e.g. [10], Chapter 5).



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