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ON THE EARTHQUAKE GENERATED RESPONSE OF
TORSIONALLY UNBALANCED BUILDINGS

by

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A thesis submitted in partial fulfilment of the
requirements for the degree of
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ABSTRACT

An investigation is made into the coupled lateral-torsional response of torsionally unbalanced "shear" buildings to horizontally directed earthquake excitation. Attention is confined to analytical models that have linearly elastic, viscously damped responses.

The investigation involves three separate analyses:

Firstly, in a preliminary study, the earthquake response of an asymmetric single storey building model is analyzed and general expressions are derived for the location of the centre of stiffness and the orientation of the principal axes.

Secondly, an analysis is made of the coupled lateral-torsional response of a partially symmetric single storey building model to a single component of earthquake excitation. A modal solution of the two equations of motion is developed and a general criterion is derived for the existence of full modal coupling.

By employing the design spectrum concept, together with conservative rules for the combination of modal maxima, analytical results in dimensionless form are evaluated for an equivalent static shear and an equivalent static torque. The combination expressions are then modified to include an allowance for modal coupling before the final results are computed and tabulated. The results substantiate previous findings which have pointed to a possible link between strong modal coupling and severely coupled lateral and torsional responses. In particular, they indicate that those nominally symmetric buildings which exhibit strong modal coupling are liable to respond more strongly in torsion than has hitherto been recognised by most building codes. This effect has not in the past been quantified in analytical terms.

Although the results have practical applications in design, the analysis concerns itself primarily with the determination of realistic estimates for the dimensionless response quantities and no attempt is made to derive design rules.

Finally, the partially symmetric single storey model is extended to a special class of partially symmetric multistorey "shear" buildings. The importance of this final analysis derives from the similarity between the results for the single storey model and those for the continuous multistorey model.
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INTRODUCTION

The interaction in buildings between lateral and torsional components of earthquake response is a phenomenon that has been researched for decades. The present investigation is concerned primarily with the determination of equivalent static actions, the intended role of which is to account for the "worst" dynamic consequences of torsional unbalance.

A multistorey building that is torsionally balanced has, at each floor level, coincident centres of mass and stiffness which lie on a common vertical axis. However, the conditions necessary for torsional balance are so restrictive as to never actually occur (Ayre(3)). Consequently, all buildings are torsionally unbalanced to some extent and respond in torsion as well as translation to any form of horizontal ground shaking (torsional unbalance is also manifested in the natural modes of vibration where the resulting interaction between lateral and torsional motions is referred to as modal coupling).

It is customary to design buildings for the effects of torsion using equivalent static analyses which take account of the eccentricities between the centres of mass and centres of stiffness. However, investigations by Housner and Outinen,(2) Bustamante and Rosenblueth,(3) Rosenblueth and Elorduy,(4) and Hoerner,(5) have shown that current static code requirements are inadequate for some buildings. For example, Hoerner noted that those nominally symmetric buildings which exhibit strong modal coupling are likely to respond more severely to earthquake excitation than is envisaged by most static design approaches. Traditional methods of static analysis do not take modal coupling into account.

Code provisions for torsion are, to a large extent, based on results obtained by analysing partially symmetric single storey building models (see Newmark and Rosenblueth,(6) and Elms(7)). These models have the advantage of being analytically tractable while still displaying many of the major dynamic characteristics of torsionally unbalanced response. Of the three building models studied in this work, two are partially symmetric. One of these is a discrete single storey model while the other is a continuous multistorey model. Both are concerned with the evaluation of realistic estimates for equivalent static actions. The study shows the extent to which results, that hold strictly only for single storey buildings, may be extrapolated to multistorey buildings.

A brief outline of each of the remaining chapters follows:

In Chapter 2 the earthquake response of a completely asymmetric single storey building model is analysed. The equations of motion are formulated using Lagrange's equations and general expressions are derived which locate the centre of stiffness and determine the orientation of the principal axes. The stiffness matrix is also generally defined and has applications in the lateral load analysis of single storey buildings. The coupling between the equations of motion is seen to be dependent on the generalized co-ordinates that are used to describe the response (the same is later found to be true of modal coupling).
Chapters 3 and 4 are devoted to an analysis of the earthquake response of a partially symmetric single storey building model. Kan and Chopra\(^{(6)}\) have recently studied a completely asymmetric model along similar, but less analytical, lines. The equations of motion follow directly from the work in Chapter 2 and furnish expressions for the translational and torsional components of displacement at the centre of mass. The modal coupling aspect is examined in some detail and general conditions are derived for the existence of full modal coupling. The results on modal coupling confirm observations that have been made during forced vibration tests of actual buildings.\(^{(9)}\) The design spectrum concept and several modal maxima combinations are then employed in the formulation of two equivalent static actions - a horizontal shear that is applied through the centre of stiffness, and a horizontal torque. Results for these are obtained as functions of two independent parameters.

In Chapters 5 and 6 the earthquake response of a partially symmetric continuous shear beam model is analysed. The equations of motion follow from Hamilton's principle and are solved using the modal analysis - separation of variables technique. Similar multistorey building models have been studied by Hoerner,\(^{(5)}\) Gibson, Moody, and Ayre,\(^{(10,11)}\) and Kan and Chopra.\(^{(12,13)}\) Although the present model is less general than the models studied in their works it is easier to handle analytically. The model is representative of a special class of "shear" buildings which have uniform distributions of mass with height and centres of stiffness which lie on one vertical axis. The distribution of elastic shearing resistance with height is arbitrary. The principal objective in focusing attention on this model is to explore the correspondence between single storey buildings and multistorey buildings. Results are evaluated for three non-dimensional equivalent static actions - a shear, a torque, and an overturning moment.

Finally, it is noted that a simple procedure evolves for the evaluation of realistic estimates for non-dimensional earthquake response quantities. This procedure, which employs idealized acceleration spectra, is used in Chapters 4 and 6.
II

ON THE EQUATIONS OF MOTION GOVERNING THE SEISMIC RESPONSE OF TORSIONALLY UNBALANCED SINGLE STOREY BUILDINGS

2.1 SYNOPSIS

In this chapter the equations of motion governing the undamped response of a torsionally unbalanced single storey building model to two horizontal, perpendicular, translatory components of ground motion are derived using Lagrange's equations. In addition, expressions are derived which define the location of the centre of stiffness and the orientation of the principal axes.

2.2 INTRODUCTION

The model studied consists simply of a rigid diaphragm that is confined to motion in a horizontal plane. Lateral resistance to movement is provided by free-standing, vertical resisting elements that have straight-line relationships of load to deflection. The spatial arrangement of these elements beneath the diaphragm is arbitrary, as is the orientation of the principal axes of stiffness of each element. The ground displacement time history is assumed to be the same for every element. Because only translatory ground motions are considered, any torsional response is attributable solely to the torsional unbalance inherent in the model.

A definition plan view of the model is shown in Figure 1. In this figure the small circle (o) and star (+) denote the centre of mass and centre of stiffness, respectively; the Roman numerals I and II denote the principal axes of stiffness. Principal axis I is taken to be the major stiffness direction. The Greek letter \( \varepsilon \) denotes the eccentricity - the distance separating the centres of mass and stiffness. By definition, the model is torsionally unbalanced when \( \varepsilon > 0 \). If the line joining the centre of mass and centre of stiffness is collinear with one of the principal axes, the model is said to be partially symmetric.\(^1\)

The centre of stiffness is the point in the plane of the diaphragm which remains stationary when the diaphragm is subjected to a statically applied horizontal torque. Similarly, a static\(^+\) horizontal force applied through the centre of stiffness causes no twist. If the static force is directed along one of the principal axes, the diaphragm translates only in the direction of the force.

Consider the response of the model to earthquake excitation which begins at time \( t = 0 \). The position of the diaphragm at any subsequent time \( t \) is delineated by the solid outline in Figure 2. In this figure:

- the dashed outline defines the equilibrium position of the diaphragm relative to the reference axes,
- \( \tau \) denotes the arbitrary reference point,
- \( \bar{X} \) and \( \bar{Y} \) are the co-ordinates of \( \tau \) when the model is at rest,

\(^+\) The same is not true for dynamic actions.
θ(t) is the angle of rotation of the diaphragm, measured positive in a counterclockwise direction, 

\( x_g(t) \) and \( y_g(t) \) are the components of ground displacement in the reference axes directions, 

\( x_t(t) \) and \( y_t(t) \) are the relative displacements of \( \tau \) - also in the reference axes directions, 

\( \Omega A \) is an orientation reference axis, and 

\( \theta \) is the counterclockwise angle between the orientation axis and the \( X \)-axis.

In the work that follows,

- \( P(X,Y) \) denotes any point on the diaphragm,
- \( x(t) \) and \( y(t) \) are the relative displacements of \( P \) in the reference axes directions,
- \( M \) is the mass of the diaphragm,
- \( \bar{x}_0 \) and \( \bar{y}_0 \) are the co-ordinates of the centre of mass,
- \( \bar{x}_e \) and \( \bar{y}_e \) are the co-ordinates of the centre of stiffness,
- \( J \) is the polar moment of inertia of the diaphragm about the centre of mass,
- the subscripts \( o \) and \( * \) refer to the centre of mass and centre of stiffness, respectively,
- the subscript \( i \) refers specifically to the \( i \)th resisting element,
- the Greek letter \( \Sigma \) denotes the summation over all \( i \), and 
- \( \theta_e \) is the counterclockwise angle between the orientation axis and principal axis \( I \).

Because of the assumed in-plane rigidity of the diaphragm, the relative displacements \( x(t) \) and \( y(t) \) at \( P \) may be related to those at \( \tau \), \( x_t(t) \) and \( y_t(t) \), by the approximate expressions\(^\dagger\)

\[
\begin{align*}
x &= x_t - (Y - \bar{y})\theta , \\
y &= y_t + (X - \bar{x})\theta .
\end{align*}
\]

\(^\dagger\) The angle \( \theta \) is treated as a small quantity.
In matrix notation the expressions for $x$ and $y$ are

$$
\begin{bmatrix}
    x \\
    y \\
\end{bmatrix} =
\begin{bmatrix}
    1 & 0 & -\left(\hat{y} - \tilde{Y}\right) \\
    0 & 1 & \left(\hat{x} - \tilde{X}\right) \\
\end{bmatrix}
\begin{bmatrix}
    x_t \\
    y_t \\
    0 \\
\end{bmatrix}.
$$

(2-1)

The relative displacements $x_t$, $y_t$, and $\theta$ are the generalized co-ordinates for this problem. For a problem which has $p$ generalized co-ordinates, $q_1$, $q_2$, ..., $q_p$, kinetic energy $T$, and potential energy $V$, Lagrange's equations (in the absence of non-conservative forces such as damping) are

$$
\frac{d}{dt} \left( \frac{\partial T}{\partial q_k} \right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} = 0 \quad (k = 1, 2, ..., p).
$$

(2-2)

Expressions for $T$ and $V$ for the system under consideration are derived in Sections 2.3 and 2.4, respectively.

In Sections 2.5 and 2.6 expressions are obtained for the co-ordinates of the centre of stiffness and the orientation of the principal axes. In Section 2.7 the stiffness matrix is redefined; the equations
of motion follow in Section 2.8 and an alternative derivation of the equations of motion in Section 2.9. 
In Section 2.10 an approximate analysis is made of a typical single storey shear-wall structure. The 
chapter concludes with a brief look at the case when the model has one axis of symmetry.

2.3 KINETIC ENERGY DERIVATION

If \( \rho(X,Y) \) is the density of the diaphragm at the point \( P(X,Y) \), and if \( A \) is its plan area, the total 
kinetic energy of the diaphragm, \( T \), is given by

\[
T = \int_A \frac{1}{2} \rho(X,Y) \left\{ (\dot{x} + \dot{x}_g)^2 + (\dot{y} + \dot{y}_g)^2 \right\} \, dA ,
\]

or, if matrix notation is used, by

\[
T = \int_A \frac{1}{2} \rho(X,Y) \begin{bmatrix} \dot{x} + \dot{x}_g \\ \dot{y} + \dot{y}_g \end{bmatrix}^{T} A \begin{bmatrix} \dot{x} + \dot{x}_g \\ \dot{y} + \dot{y}_g \end{bmatrix} \, dA . \tag{2-3}
\]

No account is taken of the kinetic energy associated with the resisting elements.

It follows from equation (2-1) that the absolute velocities of the point \( P \) are related to those 
of the point \( \tau \) by

\[
\begin{bmatrix} \dot{x} + \dot{x}_g \\ \dot{y} + \dot{y}_g \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{X}_\tau - \ddot{X}_\tau \rho \end{bmatrix} = \begin{bmatrix} \ddot{X}_\tau \rho \\ \ddot{Y}_\tau \rho \end{bmatrix} . 
\tag{2-4}
\]

Incorporating equation (2-4) in equation (2-3) and observing that

\[
M = \int_A \rho(X,Y) \, dA ,
\]

\[
\ddot{x}_o = \frac{1}{M} \int_A X \rho(X,Y) \, dA , \quad \ddot{y}_o = \frac{1}{M} \int_A Y \rho(X,Y) \, dA ,
\]

and

\[
J = \int_A \left\{ (X - \ddot{x}_o)^2 + (Y - \ddot{y}_o)^2 \right\} \rho(X,Y) \, dA ,
\]

yields

\[
T = \frac{1}{2} \{ \dot{\mathbf{w}} + \dot{\mathbf{w}}_g \}^T \mathbf{M}_\tau \{ \dot{\mathbf{w}} + \dot{\mathbf{w}}_g \} , \tag{2-5}
\]

where

\[
\mathbf{w}_\tau = \begin{bmatrix} x_\tau \\ y_\tau \end{bmatrix} , \quad \mathbf{M}_\tau = \begin{bmatrix} M & 0 & M^{X_\tau} \\ 0 & M & M^{Y_\tau} \end{bmatrix} , \quad \text{and} \quad \mathbf{w}_g = \begin{bmatrix} x_g \\ y_g \end{bmatrix} .
\]
In the mass matrix,

\[ M^X_i = M^Y_i = M (\ddot{y} - \ddot{y}_0), \]

\[ M^X_i = M (\ddot{x} - \ddot{x}_0), \quad \text{and} \]

\[ J = J + M \left\{ (\ddot{x} - \ddot{x}_0)^2 + (\ddot{y} - \ddot{y}_0)^2 \right\}. \]

The last expression defines the polar moment of inertia of the diaphragm about the point \( i \).

### 2.4 Potential Energy Derivation

Each resisting element has two principal axes of stiffness - these are orthogonal and lie in two vertical planes which intersect at the shear-centre axis.

The principal axes of stiffness of the \( i \)th resisting element are denoted by 1 and 2 - the point at which the \( i \)th shear-centre axis merges with the floor is denoted by \( P_i \). If the element has principal stiffnesses \( k_1^1, k_2^2, \) and \( k_i^0 \), the potential energy \( V_i \) associated with it is given by

\[ V_i = \frac{1}{2} \hat{w}_i^T k_i \hat{w}_i, \quad (2-6) \]

where

\[ \hat{w}_i = \begin{pmatrix} \hat{x}_i \\ \hat{y}_i \\ \theta \end{pmatrix}, \quad \text{and} \quad k_i = \begin{bmatrix} k_1^1 & 0 & 0 \\ 0 & k_2^2 & 0 \\ 0 & 0 & k_i^0 \end{bmatrix}. \]

The variables \( \hat{x}_i(t) \) and \( \hat{y}_i(t) \) denote the relative displacements of the point \( P_i \) in the orthogonal directions defined by axes 1 and 2 when the model is at rest. The relative displacements of \( P_i \) in the reference axes directions are denoted by \( x_i(y) \) and \( y_i(t) \). The two sets of displacements are shown in Figure 3. The angle \( \psi_i \) shown in this figure is equal to \( \alpha_i - \beta \) where \( \alpha_i \) measures the counterclockwise angle between the orientation axis OA and axis 1.

The relationship between the two sets of displacements shown in Figure 3 is defined by the equation

\[ \hat{w}_i = R_i w_i, \quad (2-7) \]

where

\[ R_i = \begin{bmatrix} \cos \psi_i & \sin \psi_i & 0 \\ -\sin \psi_i & \cos \psi_i & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad w_i = \begin{pmatrix} x_i \\ y_i \\ \theta \end{pmatrix}.\]
It can be shown using equation (2-1) that

\[ w_i = T_i w_\tau, \quad (2-8) \]

where

\[ T_i = \begin{bmatrix} 1 & 0 & -(Y_i - \bar{Y}) \\ 0 & 1 & (X_i - \bar{X}) \\ 0 & 0 & 1 \end{bmatrix}. \]

The co-ordinates \( X_i \) and \( Y_i \) define the position of the point \( P_i \) when the model is at rest. The vector \( w_\tau \) has already been defined (see equation (2-5)). Equation (2-7) together with equation (2-8) yields

\[ \hat{w}_i = R_i T_i w_\tau, \quad (2-9) \]

When equation (2-9) is substituted into equation (2-6) the result obtained is

\[ V_i = \frac{1}{2} \ w_i^T K_i w_\tau, \quad (2-10) \]

where

\[ K_i = T_i^T R_i^T k_i R_i T_i. \quad (2-11) \]

Expanding the right-hand side of equation (2-11) reveals that

\[ K_i = \begin{bmatrix} k_{ix}^x & k_{ix}^{xy} & k_{ix}^{x\beta} \\ k_{iy}^{yx} & k_{iy}^{yx} & k_{iy}^{y\beta} \\ k_{ix}^{6x} & k_{ix}^{6y} & k_{ix}^{6\beta} \end{bmatrix}. \quad (2-12) \]
where

\[
\begin{align*}
    k_x^i &= k_x^i \cos^2 \psi_i + k_y^i \sin^2 \psi_i, \\
    k_y^i &= k_x^i \sin^2 \psi_i + k_y^i \cos^2 \psi_i, \\
    k_{xy}^i &= k_{xy}^i = (k_x^i - k_y^i) \sin \psi_i \cos \psi_i, \\
    k_{xy}^i &= k_{xy}^i = -k_x^i (y_i - \bar{y}) + k_{xy}^i (x_i - \bar{x}), \\
    k_{y\theta}^i &= k_{y\theta}^i = -k_{x\theta}^i (y_i - \bar{y}) + k_{y\theta}^i (x_i - \bar{x}), \text{ and} \\
    k_{x\theta}^i &= -k_{x\theta}^i (y_i - \bar{y}) + k_{x\theta}^i (x_i - \bar{x}) + k_{i}^\theta.
\end{align*}
\]

Further expansion reveals that

\[
k_{i}^\theta = k_x^i (y_i - \bar{y})^2 + k_y^i (x_i - \bar{x})^2 - 2 k_{xy}^i (x_i - \bar{x}) (y_i - \bar{y}) + k_{i}^\theta. \tag{2-13}
\]

The total potential energy \( V \) is obtained by summing equation (2-10) over all the elements. The potential energy expression so obtained is

\[
V = \frac{1}{2} \mathbf{k}^T \mathbf{K} \mathbf{k}, \tag{2-14}
\]

where

\[
\mathbf{k} = \mathbf{K} + \mathbf{K}^\theta,
\]

in which

\[
\begin{align*}
    k_x^i &= \Sigma k_x^i, \\
    k_y^i &= \Sigma k_y^i, \\
    k_{xy}^i &= \Sigma k_{xy}^i, \\
    k_{x\theta}^i &= \Sigma k_{x\theta}^i, \\
    k_{y\theta}^i &= \Sigma k_{y\theta}^i, \text{ and} \\
    k_{i}^\theta &= \Sigma k_{i}^\theta.
\end{align*}
\]

2.5 LOCATION OF THE CENTRE OF STIFFNESS

Since the centre of stiffness is the point through which a static lateral force may act without causing torsion, the stiffness terms \( k_{x\theta}^i \) and \( k_{y\theta}^i \) must vanish.†† That is

† The subscript \( i \) signifies a dependence on the location of the point \( i \).

†† Recall that when the point \( i \) is at the centre of stiffness, \( \bar{x} = \bar{x}_i \) and \( \bar{y} = \bar{y}_i \).
\[- \sum k_i^x (y_i - \bar{y}_a) + \sum k_i^{xy} (x_i - \bar{x}_a) = 0, \]

and

\[- \sum k_i^{yx} (y_i - \bar{y}_a) + \sum k_i^y (x_i - \bar{x}_a) = 0. \]

It follows from equation (2-15) that

\[K_\ast = \begin{bmatrix}
K^x & k^{xy} & 0 \\
k^{yx} & K^y & 0 \\
0 & 0 & K^0
\end{bmatrix} ,\]  

(2-17)

where

\[k^0 = \sum k_i^x (y_i - \bar{y}_a)^2 + \sum k_i^y (x_i - \bar{x}_a)^2 - 2 \sum k_i^{xy} (x_i - \bar{x}_a) (y_i - \bar{y}_a) + \sum k_i^0. \]  

(2-18)

Equation (2-18) defines the principal torsional stiffness of the model (note that \(k^0 = k_{xx}^0\)).

Solving equations (2-16) for the co-ordinates of the centre of stiffness yields

\[\bar{x}_\ast = \frac{S_x k^x + S_y k^{xy}}{k^x k^y - (k^{xy})^2} ,\]  

and

\[\bar{y}_\ast = \frac{S_x k^{xy} + S_y k^y}{k^x k^y - (k^{xy})^2} .\]  

(2-19)

where

\[S_x = \sum (x_i k_i^x - y_i k_i^{xy}) ,\]

and

\[S_y = \sum (y_i k_i^x - x_i k_i^{xy}) .\]

When \(k^{xy} = 0\), the expressions in equations (2-19) reduce to

\[\bar{x}_\ast = \frac{S_x (x_i k_i^x - y_i k_i^{xy})}{k^y} ,\]  

and

\[\bar{y}_\ast = \frac{S_y (y_i k_i^x - x_i k_i^{xy})}{k^x} .\]  

(2-20)

If the principal axes of each resisting element are aligned parallel to the X,Y reference axes so that \(k_i^{xy} = 0\) for all \(i\), the co-ordinates of the centre of stiffness are given by
\[
\bar{\chi}_x = \sum \frac{X_i k_y^i}{k_y}, \\
\quad \text{and} \quad \\
\bar{\chi}_y = \sum \frac{Y_i k_x^i}{k_x},
\]
(2-21)

these being the simplest forms possible for \( \bar{\chi}_x \) and \( \bar{\chi}_y \).

Equations (2-16) were also derived by Benjamin(15) who left them in that form without solving for \( \bar{\chi}_x \) and \( \bar{\chi}_y \).

2.6 ORIENTATION OF THE PRINCIPAL AXES

A static lateral force applied through the centre of stiffness and along one of the principal axes causes translation in the direction of the force without torsion. This observation leads directly to the conclusion that the cross-stiffness term \( k^{xy} \) is zero when \( \beta = \beta_\perp \). That is

\[
\sum \frac{k_1^i - k_2^i}{2} \sin (2\alpha_i - 2\beta_\perp) = 0.
\]
(2-22)

Hence the orientation of the principal axes can be determined from the equation

\[
\tan 2\beta_\perp = \frac{\sum (k_1^i - k_2^i) \sin 2\alpha_i}{\sum (k_1^i - k_2^i) \cos 2\alpha_i}.
\]
(2-23)

This result was also obtained by Ayre\(^1\) and Lin\(^16\). Observe that when \( \beta = \beta_\perp \)

\[
K_\perp = \begin{bmatrix}
K^I & 0 & 0 \\
0 & K^{II} & 0 \\
0 & 0 & K^0
\end{bmatrix},
\]
(2-24)

where

\[
K^I = \sum \left\{ k_1^i \cos^2(\alpha_i - \beta_\perp) + k_2^i \sin^2(\alpha_i - \beta_\perp) \right\}
\]

and

\[
K^I + K^{II} = \sum (k_1^i + k_2^i).
\]

When the reference axes are aligned parallel to the principal axes \( k^x = k^I \) and \( k^y = k^{II} \) (compare equations (2-17) and (2-24)); \( k^I \) and \( k^{II} \) are the principal translational stiffnesses of the model. If \( k^I = k^{II} \), any axis passing through the centre of stiffness is a principal axis.
2.7 REDEFINITION OF $K_T$

The total potential energy $V$ (see equation (2-14)) may be expressed in terms of the principal stiffnesses as

$$V = \frac{1}{2} \begin{bmatrix} x_* & 0 & 0 \\ 0 & K^{II} & 0 \\ 0 & 0 & K^0 \end{bmatrix} \begin{bmatrix} x_* \\ y_* \\ \theta \end{bmatrix}.$$  \hfill (2-25)

The relative displacements in equation (2-25) are related to the generalized co-ordinates $x_*, y_*$, and $\theta$ by

$$\begin{bmatrix} x_* \\ y_* \\ \theta \end{bmatrix} = \begin{bmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -(\bar{y}_* - \bar{y}) \\ 0 & 1 & (\bar{x}_* - \bar{x}) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_* \\ y_* \\ \theta \end{bmatrix},$$  \hfill (2-26)

where $\gamma = \beta_* - \beta$. Substituting equation (2-26) into equation (2-25) and comparing the resulting equation with equation (2-15) reveals that

$$\begin{align*}
K^X &= K^I \cos^2 \gamma + K^{II} \sin^2 \gamma, \\
K^Y &= K^I \sin^2 \gamma + K^{II} \cos^2 \gamma, \\
K^{XY} &= (K^I - K^{II}) \sin \gamma \cos \gamma, \\
K^{X\theta} &= -K^X (\bar{y}_* - \bar{y}) + K^{XY} (\bar{x}_* - \bar{x}), \\
K^{Y\theta} &= -K^{XY} (\bar{x}_* - \bar{x}) + K^Y (\bar{y}_* - \bar{y}), \text{ and} \\
K^\theta &= K^\theta - K^{X\theta} (\bar{y}_* - \bar{y}) + K^{Y\theta} (\bar{x}_* - \bar{x}).
\end{align*}$$  \hfill (2-27)

The last expression in equation (2-27) defines the magnitude of the horizontal torque needed to cause a unit rotation of the diaphragm when the point $\tau$ is held stationary.

2.8 EQUATIONS OF MOTION

On substituting equations (2-5) and (2-14) into equation (2-2) the equations of motion are found to be

$$M_T (\ddot{x}_T + \ddot{g}_T) + K_T \dot{x}_T = 0,$$  \hfill (2-28)

or

$$\begin{bmatrix} M & 0 & M^{X\theta} \\ 0 & M & M^{Y\theta} \\ M^{X\theta} & M^{Y\theta} & J \end{bmatrix} \begin{bmatrix} \ddot{x}_T + \ddot{g}_T \\ \ddot{y}_T + \ddot{g}_T \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} K^X & K^{XY} & K^{X\theta} \\ K^{XY} & K^Y & K^{Y\theta} \\ K^{X\theta} & K^{Y\theta} & K^\theta \end{bmatrix} \begin{bmatrix} x_* \\ y_* \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$  

Both $M_T$ and $K_T$ are symmetric. The equations of motion are always dynamically uncoupled at the centre of mass but are statically uncoupled at the centre of stiffness only when $\beta = \beta_*$. The static response of the model is always torsionally uncoupled at the centre of stiffness. The equations of static equilibrium are, of course, given by...
\[ F_\tau = K_\tau \omega \tau, \]  
\( (2-29) \)

where

\[ F_\tau = \begin{pmatrix} F_\tau^X \\ F_\tau^Y \\ F_\tau^\theta \end{pmatrix}. \]

The actions denoted by \( F_\tau^X \) and \( F_\tau^Y \) are two lateral forces that act through \( \tau \) in the X and Y directions, respectively, while \( F_\tau^\theta \) is a horizontal counterclockwise torque.

2.9 AN ALTERNATIVE DERIVATION OF THE EQUATIONS OF MOTION

The combined effect of all the resisting elements may be simulated by one resisting element at the centre of stiffness if it has principal stiffnesses \( k^I \), \( k^{II} \), and \( k^0 \), and principal axes I and II at the same orientation \( \beta \) as the model. At any instant the reversed inertia forces at the centre of mass and the forces resisting motion at the centre of stiffness are in dynamic equilibrium. If the two sets of forces are resolved into forces at the point \( \tau(\tilde{x}, \tilde{y}) \), the application of D'Alembert's principle results in the equations

\[ M(\ddot{x}_0 + \ddot{x}_g) + K^X x_* + K^{XY} y_* = 0, \]
\[ M(\ddot{y}_0 + \ddot{y}_g) + K^{YX} x_* + K^Y y_* = 0, \]

and

\[ K^{6X}(\ddot{x}_0 + \ddot{x}_g) + K^{6Y}(\ddot{y}_0 + \ddot{y}_g) + J \ddot{\theta} + K^6_x x_* + K^6_y y_* + K^6_\theta = 0. \]

On substituting for \( \ddot{x}_0 + \ddot{x}_g, \ddot{y}_0 + \ddot{y}_g, x_* \), and \( y_* \), using equation (2-1) it is found that these equations are identical to those in equation (2-28).

It is possible, therefore, to construct a lumped-mass representation of the discrete model being analysed. Such a representation appears in the work by Skinner, Skilton, and Laws for the special case when \( \tilde{y}_0 = \tilde{y}_* \) and \( \beta = \beta_* \). This case is discussed at the end of this chapter in anticipation of the modal analysis that follows in Chapter 3.

2.10 APPROXIMATE ANALYSIS OF A SINGLE STOREY SHEAR-WALL STRUCTURE

In order to illustrate the application of the preceding theory, the structure shown in plan view in Figure 4 is analysed \((B = 6m, \ \lambda = 3m)\). The three shear walls are assumed to have full base fixity and are assumed to act as vertical cantilever beams. The diaphragm is assumed to be perfectly flexible normal to its plane. The walls are 0.2m thick and stand 3.5m high. Poisson's ratio for reinforced concrete is taken to be 0.15. In the following analysis the position of the centre of stiffness and the orientation of the principal axes are determined.

The uncracked wall stiffnesses are shown in Table 1. \( E \) denotes Young's modulus and has the units kN/m². The expressions used to calculate these stiffnesses are listed in Appendix A.
TABLE 1

WALL STIFFNESSES

<table>
<thead>
<tr>
<th>Wall</th>
<th>X</th>
<th>Y</th>
<th>k_x/E</th>
<th>k_y/E</th>
<th>k_x/y/E</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0.5B</td>
<td>0</td>
<td>0.0209</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0.5B</td>
<td>B</td>
<td>0.0209</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>B</td>
<td>0</td>
<td>0.0332</td>
<td>0.0332</td>
<td>0.0242</td>
</tr>
</tbody>
</table>

The stiffnesses $k_x$, $k_y$, $k_\theta$, $k_\phi$, and $k_\theta$, are assumed to be negligible.

On adding the stiffnesses in Table 1 it is found that $k^x = k^y = 0.0541 E \text{ kN/m}$ while $k^{xy} = 0.0242 E \text{ kN/m}$.

The co-ordinates of the centre of stiffness are (from equations (2-19)) $\bar{x} = 0.733B \approx 4.4 \text{ m}$ and $\bar{y} = 0.267B = 1.6 \text{ m}$. The principal torsional stiffness is (from equation (2-18)) $k_\theta = 0.0307 B^2 E = 1.105 E \text{ kN/m}$.

A Mohr's circle representation of the first three of equations (2-27) reveals that $\beta = \beta + 45^\circ$, $k_1 = k + k^{xy} = 0.0783 E \text{ kN/m}$, and $k_{11} = k - k^{xy} = 0.0299 E \text{ kN/m}$, where $K = k^x = k^y$. If the centre of mass coincides with the geometric centre of the floor diaphragm, the eccentricity is $e = 0.233B \sqrt{\varepsilon} = 1.9 \text{ m}$. The position of the centre of stiffness and the principal axes are shown in Figure 5.
The above analysis can be compared with similar analyses made by Benjamin,\textsuperscript{(15)} Lin,\textsuperscript{(16)} and Seto.\textsuperscript{(18)}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{Plan view showing the location of the centre of stiffness and the orientation of the principal axes.}
\end{figure}

2.11 Equations of motion for a model that has one axis of symmetry

If the reference axes are aligned parallel to the principal axes, \( \gamma = 0 \) and \( k^{xy} = 0 \). If, in addition, \( \ddot{y}_o = \dot{v}_o \), the model is partially symmetric and the equations of motion at any point \( \tau \) on the axis of symmetry are

\[
\begin{bmatrix}
\dddot{x}_T + \dddot{x}_G \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 \\
0 & K^I & K^{xy} \\
0 & K^{xy} & K^{\theta}
\end{bmatrix} \begin{bmatrix}
x_T \\
y_T \\
\theta_T
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix},
\]

(2.30)

where \( M^{xy} = -M(\ddot{x} - \ddot{x}_o) \), \( K^{xy} = K^{II}(\ddot{x}_e - \ddot{x}) \), \( J_T = J + M(\dddot{x} - \dddot{x}_o)^2 \), and \( K^\theta = K^\theta + K^{II}(\dddot{x}_e - \dddot{x})^2 \).

It is seen that only two of the equations are coupled.

A partially symmetric single storey building model is studied in some detail in the following two chapters.

\( \dagger \) That is, eccentricity exists only in the X-direction.
ANALYSIS OF A PARTIALLY SYMMETRIC SINGLE STOREY BUILDING MODEL

3.1 SYNOPSIS

An analysis is made of the coupled lateral-torsional response of a partially symmetric single storey building model to a single component of earthquake excitation. The model is assumed to have a linearly elastic, viscously damped response. A modal solution of the two coupled equations of motion is outlined before attention is directed toward the question of modal coupling.

3.2 INTRODUCTION

A plan view of the model is shown schematically in Figure 6(a). As in the previous chapter, the small circle and the star denote the centre of mass and centre of stiffness, respectively; $e$ is the calculated eccentricity. A distinguishing and simplifying feature of the model is its partial or one-fold symmetry; the line joining the centre of mass and centre of stiffness is collinear with one of the principal axes. The direction of earthquake attack (hereafter referred to as the $v$-direction) is taken as being parallel to the other principal axis. The model has only two degrees of freedom and therefore provides the simplest means of evaluating the dynamic effects of torsional unbalance on the seismic response of buildings.

If the mass of the resisting elements is neglected, the undamped equations of motion at the centre of mass are (see equation (2-30))

\[
\begin{bmatrix}
M & 0 \\
0 & J
\end{bmatrix}
\begin{bmatrix}
\ddot{v}_0 \\
\ddot{\theta}
\end{bmatrix}
+ \begin{bmatrix}
K^v & K^\theta \\
K^\theta & K^\theta
\end{bmatrix}
\begin{bmatrix}
v_0 \\
\theta
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix},
\]

(3-1)

where $M$ is the mass of the floor diaphragm, $J$ is the polar moment of inertia of the diaphragm about the centre of mass, $v_0(t)$ is the displacement of the centre of mass relative to the ground, $v_g(t)$ is the ground displacement, $\theta(t)$ is the counterclockwise angle of rotation of the diaphragm, $K^v$ is the principal stiffness of the model to translation in the $v$-direction, and $K^\theta$ is the torsional stiffness of the model when the centre of mass is held stationary. It follows from the theory of the preceding chapter that

\[
K^\theta = K^\theta + K^\theta e^2,
\]

(3-2)

where $K^\theta$ is the torsional stiffness when the model is everywhere free to move. A lumped-mass representation of the model is shown in Figure 6(b). Here $r = \sqrt{(J/M)}$ and is the radius of gyration of the diaphragm about a vertical axis through the centre of mass.
The corresponding torsionally balanced model (this model has coincident centres of mass and stiffness but is in every other respect identical to the torsionally unbalanced model) responds to ground shaking without twisting. The governing equation of motion is

$$\ddot{v} + p_v \dot{v} = -\ddot{v}_g,$$

(3-3)

where $p_v = \sqrt{K/M}$ and $v(t)$ denotes the translatory (balanced) response of the floor diaphragm. Although torsional balance is rarely, if ever, encountered in practice the concept provides a useful basis for comparison.

Equation (3-1) can be rearranged and written in matrix notation as

$$\ddot{w}_o + p_v \dot{w}_o \times w_o = -\ddot{z}_g,$$

(3-4)

where

$$w_o = \begin{bmatrix} v_o \\ r \theta \end{bmatrix}, \quad x = \begin{bmatrix} 1 & \delta \\ \delta & \nu^2 \end{bmatrix}, \quad \text{and} \quad z_g = \begin{bmatrix} v_g \\ 0 \end{bmatrix}.$$
in which \( \delta = \omega/r \) and \( \nu_0^2 = K_0^2/(Kr^2) \). The parameter \( \delta \) is a dimensionless eccentricity while \( \nu_0 \) is the ratio of a frequency in which twist is constrained to occur about the centre of mass to that in which only translation is allowed. In general, neither is a natural frequency of the system. Substituting for \( \nu_0^2 \) using equation (3-2) provides the identify

\[
\nu_0^2 = \mu^2 + \delta^2 ,
\]

(3-5)

where \( \mu^2 = K_0^2/(Kr^2) \). Because \( \mu \) is always greater than zero, \( \nu_0 \) is always greater than \( \delta \). It becomes clear as the analysis proceeds that \( \mu \) and \( \delta \) are the fundamental parameters of the system.

3.3 **EIGENVALUES AND EIGENVECTORS OF X**

The eigenvalues of \( X \) (see equation (3-4)) aid in defining the natural frequencies of the system while the eigenvectors of \( X \) define the natural mode shapes. They are defined here as a preliminary to the modal solution of the equations of motion that follows in Section 3.4.

The eigenvalues of \( X \) are denoted by \( \lambda_V^2 \) and \( \lambda_\theta^2 \) and are the roots of the characteristic equation

\[
\lambda^4 - (1 + \nu_0^2) \lambda^2 + \mu^2 = 0 .
\]

(3-6)

Since the system is positive definite, the eigenvalues are always positive and real. It follows from equation (3-6) that

\[
\begin{align*}
\lambda_V^2 + \lambda_\theta^2 &= 1 + \nu_0^2 , \\
\lambda_V \lambda_\theta &= \mu .
\end{align*}
\]

(3-7)

Solving equation (3-6) for \( \lambda_V^2 \) and \( \lambda_\theta^2 \) yields

\[
\begin{align*}
\lambda_V^2 &= \frac{1 + \nu_0^2}{2} + \text{sgn}(1 - \nu_0) \, R , \\
\lambda_\theta^2 &= \frac{1 + \nu_0^2}{2} - \text{sgn}(1 - \nu_0) \, R ,
\end{align*}
\]

(3-8)

where

\[
R^2 = \left\{ \text{sgn}(1 - \nu_0) \left( \frac{1 - \nu_0^2}{2} \right) \right\}^2 + \delta^2 .
\]

The sign function used here may be defined generally as

\[
\text{sgn} \, x = \begin{cases} 
1, & x \geq 0 \\
-1, & x < 0
\end{cases}
\]

(3-9)

Equations (3-8) are consistent in the sense that as \( \delta \to 0 \), \( \lambda_V \to 1 \) and \( \lambda_\theta \to \mu \).
The eigenvectors $\phi_v$ and $\phi_\theta$ associated with the eigenvalues $\lambda_v^2$ and $\lambda_\theta^2$ are the columns of the eigenvector matrix $\phi$ which may be defined as

$$\phi = <\phi_v \phi_\theta> = \begin{bmatrix}
cos \psi & \text{sgn}(1-\mu_0) \sin \psi \\
\text{sgn}(1-\mu_0) \sin \psi & \cos \psi
\end{bmatrix}$$

(3-10)

where

$$\sin^2 \psi = \frac{1}{2} - \text{sgn}(1-\mu_0) \left(\frac{1 - \mu_0^2}{4R}\right)$$
$$\cos^2 \psi = \frac{1}{2} + \text{sgn}(1-\mu_0) \left(\frac{1 - \mu_0^2}{4R}\right).$$

Alternative definitions of $\sin^2 \psi$ and $\cos^2 \psi$ are

$$\sin^2 \psi = \frac{\lambda_v^2 - 1}{\lambda_v^2 - \lambda_\theta^2} \quad \text{and} \quad \cos^2 \psi = \frac{\lambda_\theta^2 - \mu_0^2}{\lambda_v^2 - \lambda_\theta^2}.$$  

(3-11)

The eigenvector matrix satisfies the orthogonality conditions

$$\phi^T \phi = I \quad \text{and} \quad \phi^T \chi \phi = \Lambda,$$

(3-12)

where $I$ is a $2 \times 2$ identity matrix, and

$$\Lambda = \begin{bmatrix}
\lambda_v^2 & 0 \\
0 & \lambda_\theta^2
\end{bmatrix}.$$

Substituting equation (3-10) into the second of equations (3-12) reveals that

$$\lambda_v^2 = \phi_v^T \chi \phi_v = \cos^2 \psi + \mu_0^2 \sin^2 \psi + \text{sgn}(1-\mu_0) 2\delta \sin \psi \cos \psi,$$
$$\lambda_\theta^2 = \phi_\theta^T \chi \phi_\theta = \sin^2 \psi + \mu_0^2 \cos^2 \psi - \text{sgn}(1-\mu_0) 2\delta \sin \psi \cos \psi,$$

and

$$\tan 2\psi = \frac{\text{sgn}(1-\mu_0)}{1 - \mu_0^2} \frac{2\delta}{1}.$$  

(3-14)

Mohr's circle representations of $\lambda_v^2$, $\lambda_\theta^2$, and the angle $\psi$ are shown in Figure 7. Observe that both circles are of radius $R$ (see equations (3-8)). Also, it is apparent that the angle $\psi$ is bounded by $0 \leq \psi \leq \pi/4$. Hence, the diagonal elements of $\phi$ (see equation (3-10)) vary between 1 and $1/\sqrt{\lambda}$.

The off-diagonal elements, on the other hand, vary in magnitude between 0 and $1/\sqrt{\lambda}$. 

3.4 MODAL SOLUTION OF THE EQUATIONS OF MOTION

The equations of motion are uncoupled using the modal transformation

\[ w_0 = \phi \xi, \]  

(3-15)

where \( \phi \) is the eigenvector matrix defined in Section 3.3, and \( \xi \) is a 2 x 1 vector containing the normal co-ordinates \( \xi_v(t) \) and \( \xi_\theta(t) \).

If account is taken of the orthogonality conditions in equation (3-12), the substitution of equation (3-15) into equation (3-4) and subsequent premultiplication by \( \phi^T \) furnishes the uncoupled equations

\[ \ddot{\xi} + \Omega \dot{\xi} = -\phi^T \ddot{z}_g, \]  

(3-16)

where

\[ \Omega = \psi_v^2 \Lambda = \begin{bmatrix} \omega_v^2 & 0 \\ 0 & \omega_\theta^2 \end{bmatrix}. \]

The parameters \( \omega_v = \psi_v \lambda_v \) and \( \omega_\theta = \psi_v \lambda_\theta \) are the natural frequencies of the system. In expanded form, the normal co-ordinate equations of motion - with viscous damping included - are

\[ \begin{align*} 
\ddot{\xi}_v + 2\zeta_v \omega_v \dot{\xi}_v + \omega_v^2 \xi_v &= -\alpha_v \ddot{z}_g, \\
\ddot{\xi}_\theta + 2\zeta_\theta \omega_\theta \dot{\xi}_\theta + \omega_\theta^2 \xi_\theta &= -\alpha_\theta \ddot{z}_g,
\end{align*} \]

(3-17)

where \( \zeta_v \) and \( \zeta_\theta \) are fractions of critical damping. The introduction of damping in this manner tacitly assumes that the damped system possesses classical normal modes.\(^{(19,20)}\) The parameters \( \alpha_v \) and \( \alpha_\theta \) are modal participation factors; \( \alpha_v = \cos \psi \) and \( \alpha_\theta = -\text{sgn}(1-u_0) \sin \psi. \)

With the usual initial conditions, \( \xi(0) = \dot{\xi}(0) = 0, \) and the normal co-ordinates are given by
\[ \xi_v = \alpha_v D_v, \]
\[ \xi_0 = \alpha_0 D_0, \]
\[ (3-18) \]

where
\[ D_v(t) = -D(\xi_v, \omega_v, t, \ddot{\gamma}_g) \quad \text{and} \quad D_0(t) = -D(\xi_0, \omega_0, t, \ddot{\gamma}_g). \]

The notation \( D(\xi, \omega, t, \ddot{\gamma}_g) \) is used to denote Duhamel's integral which is defined as
\[ D(\xi, \omega, t, \ddot{\gamma}_g) = \frac{1}{\omega'} \int_0^t e^{-\omega(t-t')} \sin \omega'(t-t') \ddot{\gamma}_g(t') \, dt', \]
\[ (3-19) \]

where \( \omega' = \omega \sqrt{1-t'^2}. \) In a time-history analysis aimed at investigating the response of the model to a specific earthquake this integral would be evaluated numerically (a new exact routine for the numerical evaluation of Duhamel's integral is presented in Appendix B).

Back substituting via equations (3-15) and (3-10) yields the displacement response expressions
\[ v_o = D_v \cos^2 \psi + D_0 \sin^2 \psi, \]
\[ r\theta = \text{sgn}(1-\nu_0) (D_v - D_0) \sin \psi \cos \psi. \]
\[ (3-20) \]

Knowing these displacements it is possible to compute the translational response of any point on the diaphragm. For example, the translational response at the centre of stiffness is given by
\[ v_* = v_o + \epsilon \theta. \]
\[ (3-21) \]

### 3.5 MODAL COUPLING

The natural mode shapes are shown in Figures 8 and 9 (the solid line represents the axis of symmetry). In these figures, the nodes \( N_v \) and \( N_\theta \) are natural mode centres of rotation; \( r \cot \psi \) and \( r \tan \psi \) are nodal distances. A static horizontal force applied through \( N_v \) in the \( v \)-direction causes rotation about \( N_\theta \). The same force applied through \( N_\theta \) causes rotation about \( N_v \). During earthquake excitation, modes \( N_v \) and \( N_\theta \) have displacements \( D_v(t) \) and \( D_\theta(t) \), respectively (see equations (3-20) and Figure 10).

The amount of coupling between the \( v_o \) and \( r\theta \) co-ordinates in each of the natural modes of vibration is indicated by the relative magnitudes of \( \cos \psi \) and \( \sin \psi \) in the modal eigenvector matrix. Since \( 0 \leq \psi \leq \pi/4 \), \( \cos \psi \geq \sin \psi \), and the natural mode associated with the eigenvector \( \phi_v \) may be described as being predominantly translational at the centre of mass. Similarly, the natural mode associated with the eigenvector \( \phi_\theta \) may be described as being predominantly torsional at the centre of mass.

The interaction between lateral and torsional motions in the natural modes of vibration is referred
FIGURE 8  MODE SHAPES WHEN $\mu_0 \leq 1$, (a) MODE 1 ($\omega = \omega_0$), (b) MODE 2 ($\omega = \omega_0$)

FIGURE 9  MODE SHAPES WHEN $\mu_0 > 1$, (a) MODE 1 ($\omega = \omega_0$), (b) MODE 2 ($\omega = \omega_0$)

FIGURE 10  SUPERPOSED DAMPED MODAL RESPONSES
to as modal coupling. Figure 11 shows the variation of the modal coupling angle \( \psi \) with \( \mu \) for five values of \( \delta \). These curves were produced using equation (3-14) together with equation (3-5). It is seen that strong modal coupling is encountered at the centre of mass whenever \( \mu_0 \) is close to unity (that is, whenever \( \sqrt{\mu^2 + \delta^2} = 1 \)). There is full modal coupling at the centre of mass, with values of \( \delta < 1 \), when \( \mu_0 = 1 \). Then

\[
\mu = \sqrt{(1 - \delta^2)},
\]

and

\[
\phi = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.
\]

The corresponding eigenvalues are

\[
\lambda^2_y = 1 + \delta,
\]

and

\[
\lambda^2_0 = 1 - \delta.
\]

Note that there can be full modal coupling at the centre of mass even when the eccentricity of the model is small (for example, see the \( \delta = 0.05 \) curve in Figure 11). Intuition would predict that

\[
\phi \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ as } \delta \to 0,
\]

but it is obvious that this is not always the case. When \( \delta \) is small and \( \mu \) is close to unity pronounced modal coupling occurs as a result of the close spacing between the natural frequencies.

In order to obtain results that are more general, consider the response of a point \( r \) on the axis of symmetry and a distance \( h \) to the right of the centre of mass (when \( r \) is to the left of the centre of mass, \( h \) is negative). The equations of motion at the point \( r \) are

\[
\begin{bmatrix} M & -Mh \\ -Mh & J_1 \end{bmatrix} \begin{bmatrix} \ddot{v}_r + \ddot{v}_g \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k^Y & -k^Y(h-c) \\ -k^Y(h-c) & k^\theta \end{bmatrix} \begin{bmatrix} v_r \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]

where \( J_1 = J + Mh^2 \) and \( k^\theta = k^Y + k^Y(h-c)^2 \); \( v_r(t) \) is the relative displacement of the point \( r \) in the \( v \)-direction. These equations can be rewritten in the form

\[
\begin{bmatrix} 1 & -h/r_\tau \\ -h/r_\tau & 1 \end{bmatrix} \begin{bmatrix} \ddot{v}_r + \ddot{v}_g \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 1 & -(h-c)/r_\tau \\ -(h-c)/r_\tau & \mu_\tau^2 \end{bmatrix} \begin{bmatrix} v_r \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]

where \( r_\tau^2 = r^2 + h^2 \) and \( \omega_\tau^2 = k^\theta/(k^Y r_\tau^2) \). When \( \mu_\tau = 1 \), the undamped normal co-ordinate equations of motion at the point \( r \) are
\[ \ddot{v}_v + p_v^2 \frac{1 - \text{sgn}(1-\mu_0) (h-c)/r_\tau}{1 - \text{sgn}(1-\mu_0) h/r_\tau} \dot{v}_v = \frac{1}{\sqrt{2}} \ddot{v}_g, \]

and

\[ \ddot{v}_\theta + p_v^2 \frac{1 + \text{sgn}(1-\mu_0) (h-c)/r_\tau}{1 + \text{sgn}(1-\mu_0) h/r_\tau} \dot{v}_\theta = \text{sgn}(1-\mu_0) \frac{1}{\sqrt{2}} \ddot{v}_g. \]

where

\[
\begin{bmatrix}
    v_x \\
    r_{\tau \theta}
\end{bmatrix}
= \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & -\text{sgn}(1-\mu_0) \\
\text{sgn}(1-\mu_0) & 1
\end{bmatrix}
\begin{bmatrix}
    \ddot{v}_v \\
    \ddot{v}_\theta
\end{bmatrix}.
\]

(3-29)

\[ \psi \]

\( \theta = 1.00 \)
\( \theta = 1.50 \)
\( \theta = 0.50 \)
\( \theta = 0.25 \)
\( \theta = 0.05 \)

\( \theta \)

\( \mu \)

FIGURE 11 VARIATION OF \( \psi \) WITH \( \mu \)

Hence, full modal coupling occurs with the generalized co-ordinates \( v_x \) and \( r_{\tau \theta} \) when \( \mu_x = 1 \).

That is, when

\[
\frac{K_0^x}{\lambda_x^g} = \frac{K_v^v}{M}.
\]

(3-30)

When equation (3-28) is satisfied

\[
\mu^2 = 1 - \delta^2 + 2\delta \Delta,
\]

(3-31)
where $\Delta = h/r$. It follows from equations (3-28) that the corresponding eigenvalues are

$$
\lambda_\psi^2 = 1 + \frac{\text{sgn}(1 - \nu_0) \epsilon / r_\tau}{1 - \text{sgn}(1 - \nu_0) h / r_\tau} = 1 + \delta \Delta + \text{sgn}(1 - \nu_0) \delta \sqrt{1 + \delta^2},
$$

and

$$
\lambda_\theta^2 = 1 - \frac{\text{sgn}(1 - \nu_0) \epsilon / r_\tau}{1 + \text{sgn}(1 - \nu_0) h / r_\tau} = 1 + \delta \Delta - \text{sgn}(1 - \nu_0) \delta \sqrt{1 + \delta^2}.
$$

(3-32)

Since $\nu_0^2 = 1 + 2\delta \Delta$ (when there is full modal coupling), substituting equations (3-32) into equations (3-11) yields

$$
\sin^2 \psi = \frac{1}{2} + \text{sgn}(1 - \nu_0) \frac{\Delta}{2\sqrt{1 + \delta^2}},
$$

and

$$
\cos^2 \psi = \frac{1}{2} - \text{sgn}(1 - \nu_0) \frac{\Delta}{2\sqrt{1 + \delta^2}}.
$$

(3-33)

Putting $h = 0$ for the case when the point $\tau$ is coincident with the centre of mass furnishes the results obtained at the beginning of this section. When the point $\tau$ is coincident with the centre of stiffness, $h = \epsilon$ and

$$
\mu^2 = 1 + \delta^2,
$$

$$
\nu_0^2 = 1 + 2\delta^2,
$$

$$
\lambda_\psi^2 = 1 + \delta^2 - \delta \sqrt{1 + \delta^2},
$$

$$
\lambda_\theta^2 = 1 + \delta^2 + \delta \sqrt{1 + \delta^2},
$$

(3-34)

and

$$
\sin^2 \psi = \frac{1}{2} - \frac{\delta}{2\sqrt{1 + \delta^2}},
$$

$$
\cos^2 \psi = \frac{1}{2} + \frac{\delta}{2\sqrt{1 + \delta^2}}.
$$

when there is full modal coupling. Note that in this case there is no restriction on the magnitude of $\delta$ whereas at the centre of mass there was the requirement that $\delta$ be less than unity. As a last example, consider the case when the point $\tau$ is a distance $\epsilon$ to the other side of the centre of mass, that is, when $h = -\epsilon$. Full modal coupling occurs at this point only for values of $\delta < 1/\sqrt{3}$.

Then

$$
\mu^2 = 1 - 3\delta^2,
$$

$$
\nu_0^2 = 1 - 2\delta^2,
$$

$$
\lambda_\psi^2 = 1 - \delta^2 + \delta \sqrt{1 + \delta^2},
$$

$$
\lambda_\theta^2 = 1 - \delta^2 - \delta \sqrt{1 + \delta^2},
$$

(3-35)

and

$$
\sin^2 \psi = \frac{1}{2} + \frac{6}{2\sqrt{1 + \delta^2}},
$$

$$
\cos^2 \psi = \frac{1}{2} + \frac{6}{2\sqrt{1 + \delta^2}}.$$
Hence, modal motions that appear strongly coupled in one set of co-ordinates may not be as strongly coupled in another, and vice versa. However, when \( \delta \ll 1 \), strong (or even full) modal coupling in one co-ordinate system may be matched in others.

The general conclusion to be inferred from the preceding theory is that modal coupling is a quantity that is largely dependent on the generalized co-ordinates that are used to describe the response. The natural frequencies and mode shapes, on the other hand, are invariant. For the same values of \( \mu \) and \( \delta \) these do not change. This is also true of the eigenvalues \( \lambda_v^2 \) and \( \lambda_\theta^2 \).

Finally, as a further amplification of the results presented in this section, it may be noted that when full modal coupling occurs

\[
\begin{align*}
\nu_t &= (D_v + D_\theta)/2, \\
\mathrm{and} \\
\nu_\theta &= \frac{\text{sgn}(1-\nu_\theta)}{2}(D_v - D_\theta)/2. 
\end{align*}
\]  

If \( D_v(t) = D_\theta(t) = D \), \( \nu_t = D \) and \( \nu_\theta = 0 \). On the other hand, if \( D_v(t) = -D_\theta(t) = D \), \( \nu_t = 0 \) and \( \nu_\theta = \text{sgn}(1-\nu_\theta)D \). The distinction between additive and subtractive combinations of the modal components of response is of importance in the following chapter.
NORMALIZED EQUIVALENT STATIC DESIGN ACTIONS

4.1 SYNOPSIS

By employing the response spectrum concept, together with conservative rules for the combination of the modal maxima, simple analytical expressions are derived for two suitably normalized equivalent static design actions $\tilde{S}_*\, (a\, normalized\, shear)$ and $\tilde{T}_*\, (a\, normalized\, torque)$. Some modifications are then made to the combination expressions (in keeping with the uncertainty of the simultaneous occurrence of peak modal responses) before the final graphical and tabular results for $\tilde{S}_*$ and $\tilde{T}_*$ are presented. Although the analysis is arranged so that the results can be adapted for use in building codes, no undue compromise is made to ensure that simple design formulae are obtained. Examples at the end of the chapter illustrate the practical application of the results.

4.2 FORMULATION OF THE EQUIVALENT STATIC ACTIONS AND THEIR NORMALIZATION

The two maximum responses at the centre of stiffness of the partially symmetric single storey model analysed in the preceding chapter facilitate the determination of two equivalent static design actions. The intended role of these actions - a horizontal shear $S_*$ that acts through the centre of stiffness, and a horizontal torque $T_*$ - is to simulate the "worst" dynamic effects of horizontal torsion during ground shaking. The actions are defined as

\[
\begin{align*}
S_* &= \kappa^v v_*\, (\max), \\
T_* &= \kappa^\theta \theta\, (\max),
\end{align*}
\]

where $v_*\, (\max)$ is the maximum displacement at the centre of stiffness and $\theta\, (\max)$ is the maximum angle of rotation of the diaphragm. The positive directions of $S_*$ and $T_*$ are shown in Figure 12 - positive torques are anticlockwise.

Five independent parameters affect the response of the model to any one earthquake. These are $u$, $\delta$, $p_v$, $\zeta_v$, and $\zeta_\theta$. This number is reduced to four by assuming that the two damping constants are equal (from this point onwards, $\zeta_v = \zeta_\theta = \zeta$). This assumption is in agreement with the findings of experimental research programs that have been carried out in recent years.\[^{(9,21)}\]

Realistic estimates of the displacement maxima in equations (4-1) are obtained using the concept of a design spectrum.\[^{(22,23)}\] With this approach, the ground motion is characterised by the idealized average pseudo-acceleration spectrum shown in Figure 13\(^{\dagger}\), and the maximum value of Duhamel's

\[^{\dagger}\] The implication here is that Figure 13 has been obtained by computing and then normalizing, averaging, and smoothing the pseudo-acceleration response spectra for an ensemble of different earthquake records.
Integral (see equation (3-19)) is defined as

\[ D_{\text{max}} = \frac{S_a(\zeta, \omega)}{\omega^2} , \]  

(4-2)

where, according to Figure 13, the spectral acceleration \( S_a(\zeta, \omega) \) is given by

\[ S_a(\zeta, \omega) = \begin{cases} 
S_a(\zeta, \omega_1), & T \leq T_1 \\
S_a(\zeta, \omega_1) \frac{T}{T_1}, & T \geq T_1
\end{cases} , \]  

(4-3)

in which \( T = 2\pi/\omega \) and \( T_1 = 2\pi/\omega_1 \) is the period at the transition between the flat and hyperbolic portions of the spectrum. Therefore, with reference to equation (3-18)

\[ D_{\nu}(\text{max}) = \frac{S_a(\zeta, \nu \omega)}{\nu^2} , \]  

and

\[ D_\theta(\text{max}) = \frac{S_a(\zeta, \omega_\theta)}{\omega_\theta^2} , \]  

(4-4)

where the spectral accelerations are determined using equation (4-3).

As a preliminary to normalizing the equivalent static actions, consider the effect of restraining the centre of stiffness from translation. The governing undamped equation of motion is then

\[ \text{r} \ddot{\phi} + p_0^2 \text{r} \dot{\phi} = \frac{5}{1+\epsilon^2} \gamma g , \]  

(4-5)

where \( \gamma(t) \) is the counterclockwise angle of rotation of the diaphragm and \( p_0 = \sqrt{\kappa^2/J_s} \); the parameter \( J_s(= J + M_\gamma^2) \) is the polar moment of inertia of the diaphragm about the centre of stiffness. Substitution reveals that \( p_0 = \mu \nu / (1+\epsilon^2) \).
Returning now to the problem at hand, it proves convenient to express the two equivalent static actions in normalized form as

\[
\tilde{S}_a = \frac{S_a}{S_v} \quad \text{and} \quad \tilde{T}_a = \frac{T_a}{S_v} \frac{S_v}{S_0}, \quad (4-6)
\]

where \( S_v = M S_a(\zeta, p_v) \) and \( S_0 = M S_a(\zeta, p_0) \). The importance of normalizing the torque \( T_a \) with respect to a shear that is determined using the frequency \( p_0 \) rather than the frequency \( p_v \) is illustrated clearly by the third example in Section 4.8. An alternative normalization of the torque is

\[
\tilde{T}_a = \frac{T_a}{S_0} = \frac{\tilde{T}_a}{\delta}. \quad (4-7)
\]

However, it will be shown later that \( \tilde{T}_a \) does not always remain finite and for this reason is not a particularly suitable response quantity to work with. Substituting equations (4-1) into equation (4-6) reveals that

\[
\tilde{S}_a = p_v^2 \frac{v_{(\text{max})}}{S_a(\zeta, p_v)} \quad \text{and} \quad \tilde{T}_a = p_0^2 (1+\delta^2) \frac{r_0(\text{max})}{S_a(\zeta, p_0)}. \quad (4-8)
\]

Alternative definitions of \( \tilde{S}_a \) and \( \tilde{T}_a \) are

\[
\tilde{S}_a = \frac{v_{(\text{max})}}{v(\text{max})} \quad \text{and} \quad \tilde{T}_a = \mu^2 \frac{S_a(\zeta, p_v)}{S_a(\zeta, p_0)} \frac{r_0(\text{max})}{v(\text{max})}. \quad (4-9)
\]

where \( v(\text{max}) \) is the maximum displacement in the corresponding torsionally balanced model.
The normalized equivalent static actions $\tilde{S}_a$ and $\tilde{T}_a$ are best evaluated as functions of their individual modal maxima, with the latter being estimated using the design spectrum. It may be shown using equations (3-20), (3-21), and (4-8) that the maximum values of $\tilde{S}_a$ and $\tilde{T}_a$ in the predominantly translational mode of vibration are given by

$$\tilde{S}_{aV} = \frac{1}{\lambda^2} \frac{S_a(\varepsilon, \omega)}{S_a(\varepsilon, p_V)} \left[ \cos^2 \psi + \text{sgn}(1-\nu_0) \delta \sin \psi \cos \psi \right],$$

and

$$\tilde{T}_{aV} = \text{sgn}(1-\nu_0) \frac{\mu^2}{\lambda^2} \frac{S_a(\varepsilon, \omega)}{S_a(\varepsilon, p_V)} \sin \psi \cos \psi,$$

while the corresponding maxima in the predominantly torsional mode of vibration are

$$\tilde{S}_{a\theta} = \frac{1}{\lambda^2} \frac{S_a(\varepsilon, \omega_0)}{S_a(\varepsilon, p_V)} \left[ \sin^2 \psi - \text{sgn}(1-\nu_0) \delta \sin \psi \cos \psi \right],$$

and

$$\tilde{T}_{a\theta} = -\text{sgn}(1-\nu_0) \frac{\mu^2}{\lambda^2} \frac{S_a(\varepsilon, \omega_0)}{S_a(\varepsilon, p_V)} \sin \psi \cos \psi.$$

The values of $\tilde{S}_a$ and $\tilde{T}_a$ evaluated using equations (4-10) and (4-11) together with equation (4-3) are dependent on four independent parameters. These are $\mu, \delta, p_V$, and $T_1$. Neither the level of damping nor the level of earthquake excitation (that is, the spectral acceleration $S_a(\varepsilon, \omega_0)$) need be specified. Both are implicit in the normalizing shears $S_V$ and $S_\theta$.

To begin with, the modal maxima are combined by algebraic addition and algebraic subtraction. In the additive case

$$\tilde{S}_a = \tilde{S}_{aV} + \tilde{S}_{a\theta},$$

and

$$\tilde{T}_a = \tilde{T}_{aV} + \tilde{T}_{a\theta},$$

while, in the subtractive case

$$\tilde{S}_a = |\tilde{S}_{aV} - \tilde{S}_{a\theta}|,$$

and

$$\tilde{T}_a = \begin{cases} 
\text{sgn}(\tilde{S}_{aV} - \tilde{S}_{a\theta}) (\tilde{T}_{aV} - \tilde{T}_{a\theta}), & \tilde{S}_a > 0 \\
|\tilde{T}_{aV} - \tilde{T}_{a\theta}|, & \tilde{S}_a = 0 
\end{cases},$$

where the modulus signs in equation (4-13a) and the sign function in equation (4-13b) are used to ensure that potentially negative shears are made positive (for convenience) and that the correct sense attaches to each of the associated torques. The modulus signs in equation (4-13b) ensure that positive torques accompany zero shears. The results obtained using equations (4-12) and (4-13) represent two extremes between which the actual peak response must lie.
4.3 \textbf{MAXIMUM VALUES OF } \tilde{S}_a \textbf{ AND } \tilde{T}_a \textbf{ COMPUTED NUMERICALLY USING THE DESIGN SPECTRUM}

It has already been pointed out that \( \tilde{S}_a \) and \( \tilde{T}_a \) are functions of four independent parameters (\( \mu, \delta, p_v \) and \( T_1 \)) when the modal maxima are estimated using the design spectrum. In an effort to reduce this number to two and thereby facilitate the graphical presentation of results it was decided to compute maximum values of \( \tilde{S}_a \) and \( \tilde{T}_a \) as functions of \( \mu \) for five values of \( \delta \) (0.05, 0.25, 0.5, 1.0, and 1.5). In so doing the dependence of the results on the parameters \( p_v \) and \( T_1 \) was obscured. The amount of numerical computation involved was large and necessitated the use of a digital computer. For any pair of values of \( \mu \) and \( \delta \), the values of \( \tilde{S}_a \) and \( \tilde{T}_a \) computed represent maxima obtained by sweeping over three period ranges, namely

\[
0.05 \text{ sec} \leq \frac{2\pi}{P_v} \leq 4.0 \text{ sec}, \quad 0.05 \text{ sec} \leq \frac{2\pi}{P_0} \leq 4.0 \text{ sec},
\]

and

\[
0.05 \text{ sec} \leq T_1 \leq 2.0 \text{ sec},
\]

with care being taken to maintain the correct relationship between \( p_v \) and \( p_0 \) (\( p_0/p_v = \mu/(1+\delta^2) \)).

Values of \( \tilde{S}_a \) and \( \tilde{T}_a \) were computed in this way using both additive and subtractive combinations of the modal maxima. The latter are defined by equations (4-13), (4-10), and (4-11). Because the transition period \( T_1 \) was varied over such a wide range, the values of \( \tilde{S}_a \) and \( \tilde{T}_a \) are also maxima for almost all soil conditions (see, for example, Seed, Ugas, and Lysmer\textsuperscript{24}).

(i) \textbf{Algebraic addition of the modal maxima}

When the modal maxima were added algebraically in accordance with equations (4-12), the results of the computer study were simply

\[
\begin{cases}
\tilde{S}_a &= 1, \\
\tilde{T}_a &= -\delta
\end{cases}
\]

for every value of \( \mu \) considered. The corresponding equivalent static actions are \( S_a = S_v \) and \( T_a = -S_0 \varepsilon \).

(ii) \textbf{Algebraic subtraction of the modal maxima}

When the modal maxima were subtracted algebraically in accordance with equations (4-13), the results of the computer study were as shown in Figures 14 and 15. In the previous case every value of \( \tilde{T}_a \) computed was negative. In this case both positive and negative values of \( \tilde{T}_a \) occurred. The maximum values of \( \tilde{S}_a \) and \( \tilde{T}_a \) shown in Figure 14 are the results obtained by only considering negative values of \( \tilde{T}_a \).

Similarly, the maximum values of \( \tilde{S}_a \) and \( \tilde{T}_a \) shown in Figure 15 are the results obtained by only considering positive values of \( \tilde{T}_a \).

Apart from noting that positive torques (corresponding to positive values of \( \tilde{T}_a \)) have potentially important ramifications for resisting elements to the right of the centre of stiffness, discussion of these results will be deferred until the analytical results have been presented.
Figure 14 MAXIMUM NORMALIZED SHEARS AND NEGATIVE TORQUES OBTAINED NUMERICALLY USING THE COMPOSITE DESIGN SPECTRUM TOGETHER WITH THE ALGEBRAIC SUBTRACTION OF NODAL MAXIMA
FIGURE 15  MAXIMUM NORMALIZED SHEARS AND POSITIVE TORQUES OBTAINED NUMERICALLY USING THE COMPOSITE DESIGN SPECTRUM TOGETHER WITH THE ALGEBRAIC SUBTRACTION OF MODAL MAXIMA

FIGURE 16  VALUES OF $\bar{T}_n$ EVALUATED USING THE SECOND OF EQUATIONS (4-17)
4.4 **ANALYTICAL RESULTS FOR \( \mathbf{S}_a \) AND \( \mathbf{T}_a \) OBTAINED USING FLAT AND HYPERBOLIC ACCELERATION SPECTRA**

For reasons that will become clear as the analysis proceeds the flat and hyperbolic portions of the design spectrum (see Figure 13) are now extrapolated and analysed separately. The results presented below are derived in Appendix C and are functions of only two independent parameters, namely \( \mu \) and \( \delta \).

(i) **Additive modal maxima combinations**

The algebraic addition of the modal maxima for the flat spectrum yields

\[
\begin{align*}
\tilde{S}_a &= 1, \\
\tilde{T}_a &= -\delta .
\end{align*}
\]

Hence, the computer results in equation (4-14) correspond to the four spectral accelerations \( S_\alpha(\zeta, P_v) \), \( S_\alpha(\zeta, P_0) \), \( S_\alpha(\zeta, w_v) \), and \( S_\alpha(\zeta, w_0) \) all lying on the flat portion of the design spectrum.

The algebraic addition of the modal maxima for the hyperbolic spectrum yields

\[
\begin{align*}
\tilde{S}_a &= \frac{1+\mu}{[(1+\mu)^2 + \delta^2]^{1/2}}, \\
\tilde{T}_a &= -\frac{\delta\sqrt{(1+\delta^2)}}{[(1+\mu)^2 + \delta^2]^{1/2}}.
\end{align*}
\]

It can be seen by inspection that the expressions in equation (4-16) yield smaller shears and torques than are given by equations (4-15).

In both of the above cases, negative torques (corresponding to negative values of \( \tilde{T}_a \)) occur for all values of \( \mu \) and \( \delta \).

(ii) **Subtractive modal maxima combinations**

The algebraic subtraction of the modal maxima for the flat spectrum yields

\[
\begin{align*}
\tilde{S}_a &= \cos 2\psi , \\
\tilde{T}_a &= \text{sgn}(1-\mu_0)(1+\mu_0^2) \sin \psi \cos \psi .
\end{align*}
\]

In this case, both positive and negative torques occur. Positive torques occur when \( \mu_0 \leq 1 \) and negative torques occur when \( \mu_0 > 1 \).

The algebraic subtraction of the modal maxima for the hyperbolic spectrum yields

\[
\begin{align*}
\tilde{S}_a &= \frac{|1-\mu|}{[(1-\mu)^2 + \delta^2]^{1/2}}, \\
\tilde{T}_a &= \text{sgn}(1-\mu) \frac{\delta\sqrt{(1+\delta^2)}}{[(1-\mu)^2 + \delta^2]^{1/2}}.
\end{align*}
\]
Positive torques occur when $\mu \leq 1$ and negative torques occur when $\mu > 1$. Hence, unlike the previous case, positive torques occur for values of $\mu_0$ which are sometimes greater than unity (for example, when $\delta = 1$, $\mu_0 = \sqrt{1+\mu^2}$). A further point to note is that whereas positive torques occur in the present case for all values of $\delta$ when $\mu \leq 1$, they occur in the previous case only for values of $\delta < 1$ and then only when $0 < \mu \leq \sqrt{1-\delta^2}$.

On plotting the expressions in equation (4-17) as functions of $\mu$ for values of $\mu_0 > 1$ and values of $\delta$ equal to 0.05, 0.25, 0.50, 1.00, and 1.50, it is found that almost all the curves obtained agree exactly with the corresponding curves in Figure 14. The only curves that differ are those for $\tilde{\lambda}_e$ and even then only slightly. (The curves obtained for $\tilde{\lambda}_e$ are shown in Figure 16.) On plotting the expressions in equation (4-18) for values of $\mu \leq 1$ and the same five values of $\delta$, it is found that all the curves obtained agree exactly with the corresponding curves in Figure 15. Hence, nearly all of the computer results in Figures 14 and 15 are furnished analytically by equations (4-17) and (4-18).

The correspondence between the analytical results and the computer results is taken as being sufficient justification for not utilizing the design spectrum any further. Instead, all subsequent values of $\tilde{S}_e$ and $\tilde{T}_e$ are obtained by analysing the flat and hyperbolic spectra separately. The decision to dispense with the design spectrum is important because it means that the final results for $\tilde{S}_e$ and $\tilde{T}_e$ are functions of only two independent parameters ($\mu$ and $\delta$). While this is a most useful simplification (from four independent parameters to two), it should not be overlooked that the model has but two degrees of freedom. An increase in the number of degrees of freedom, consistent with a more realistic building model, may lead to situations where two independent parameters will not suffice to describe the maximum response. This point is explored in Chapters 5 and 6.

### 4.5 MODIFIED RESULTS FOR THE SUBTRACTIVE COMBINATION OF MODAL MAXIMA

So far in the analysis it has been assumed that the modal maxima may be combined either by algebraic addition or by algebraic subtraction. It is well recognised, however, that results obtained in this way are likely to be overly conservative when the natural frequencies are well separated. In the work that follows the results obtained by algebraic subtraction are improved by recombining the modal maxima in accordance with modified versions of the square-root-of-the-sum-of-the-squares (SRSS) combination function.

(1) **SRSS combination**

When the natural frequencies are well separated the most probable (though not necessarily a conservative) estimate of any earthquake response quantity is given by the square-root-of-the-sum-of-the-squares of the corresponding modal maxima.\(^{25,26}\) In the present case the SRSS combination functions are

\[
\begin{align*}
S_e &= \left(\tilde{S}_e^2 + \tilde{S}_e^2\right)^{1/2}, \\
\tilde{T}_e &= \left(\tilde{T}_e^2 + \tilde{T}_e^2\right)^{1/2}
\end{align*}
\] (4-19)
The results computed using these functions are shown in Figure 17. They are the maxima of two sets of analytical results, one for the flat spectrum and the other for the hyperbolic spectrum. It is obvious that the results have more in common with the algebraic subtraction of the modal maxima than with their algebraic addition. The reason for this may be traced back to the equations defining the modal maxima.

A major disadvantage of the SRSS approach is that on its application all knowledge of the sign of the response quantity concerned is lost. For a given direction of \( \hat{S}_e \), the distinction between negative and positive values of \( \hat{T}_e \) is clearly of importance. This problem has been overcome in this case by inferring what the sign should be from the results of the previous analysis. Hence, in Figure 17, \( \hat{T}_e \) is negative for \( v_0 > 1 \) and positive for \( v < 1 \). Negative values of \( \hat{T}_e \) are indicated by full lines, positive values by dashed lines and dots indicate values of \( \hat{T}_e \) which are taken as being both positive and negative. The same procedure is adopted in the modified SRSS combination used below.

A further disadvantage of the SRSS approach is that its application can result in considerable error when the natural frequencies are close. When \( v \) is close to unity and \( \delta \) is small (conditions under which the natural frequencies are close) it is felt that the results in Figures 14 and 15 are more accurate than the corresponding results in Figure 17. What is needed is an SRSS approach which takes account of the spacing between the natural frequencies.

(ii) Modified SRSS combination

According to the modified SRSS approach derived by Rosenblueth and Elorduy\(^{4,27}\), an estimate of the maximum value of any earthquake response quantity \( q(t) \), when there are \( N \) modes, is given by \( Q \), where

\[
Q^2 = \sum_{i=1}^{N} \frac{Q_i^2}{1 + \varepsilon_{ij}^2} + \sum_{j=1}^{N} \frac{2Q_i Q_j}{1 + \varepsilon_{ij}^2} \tag{4-20}
\]

in which \( Q_i \) and \( Q_j \) are the \( i \)th and \( j \)th modal maxima, and

\[
\varepsilon_{ij} = \left| \frac{\omega_i - \omega_j}{\omega_i^2 + \omega_j^2} \right| \tag{4-21}
\]

The parameters \( \omega_i \) and \( \omega_j \) are the \( i \)th and \( j \)th "equivalent" percentages of damping, while \( \omega_i' = \omega_i \sqrt{1-\varepsilon_i^2} \) and \( \omega_j' = \omega_j \sqrt{1-\varepsilon_j^2} \) are the \( i \)th and \( j \)th "damped" natural frequencies. Each modal maximum is taken with the sign that its unit impulse response function has when it attains its maximum numerical value. In using this approach, Kan and Chopra\(^{6,12,13}\) define \( \varepsilon_{ij} \) as

\[
\varepsilon_{ij} = \frac{\sqrt{(1-\varepsilon_j^2)}}{\varepsilon} \frac{\omega_i - \omega_j}{\omega_i + \omega_j} \tag{4-22}
\]

and it is apparent that their results are damping dependent entirely as a result of the method used to combine the modal maxima.

So that damping is not reintroduced into the present problem, improved estimates of \( \hat{S}_e \) and \( \hat{T}_e \) are

\^ Having reduced the problem to two independent variables, there is no point in introducing a third at this juncture.
FIGURE 17  MAXIMUM NORMALIZED SHEARS AND TORQUES EVALUATED USING THE SRSS COMBINATION RULE.
INFERRED TORQUE SIGNS: --- POSITIVE (THAT IS, ANTICLOCKWISE), —— NEGATIVE, ... BOTH
POSITIVE AND NEGATIVE
obtained using the combination formulae

\[
\tilde{S}_* = \left( \tilde{S}_{sv}^2 + \tilde{S}_{s0}^2 - 2e^{-4\alpha^2} |\tilde{S}_{sv}| |\tilde{S}_{s0}| \right)^{1/2}
\]

and

\[
\tilde{T}_* = \left( \tilde{T}_{sv}^2 + \tilde{T}_{s0}^2 + 2e^{-4\alpha^2} |\tilde{T}_{sv}| |\tilde{T}_{s0}| \right)^{1/2},
\]

where

\[
\alpha = \frac{\omega_v - \omega_0}{\omega_v + \omega_0} = \text{sgn}(1-\nu) \left( 1 - \frac{4\nu}{(1+\nu)^2 + \delta^2} \right)^{1/2}.
\]

The expressions in equation (4-23) were derived by trial and error; they yield accurate estimates of realistic maxima for the normalized design actions for all frequency spacings.

4.6 FINAL RESULTS FOR \( \tilde{S}_* \) AND \( \tilde{T}_* \)

Because of their simplicity, the results obtained by the algebraic addition of modal maxima, and presented in equations (4-14) and (4-15), (namely, \( \tilde{S}_* = 1 \) and \( \tilde{T}_* = -\delta \)) are left as they are. The other set of final results, corresponding to the algebraic subtraction of modal maxima and computed using equations (4-23), are shown in Figure 18. A further set of results, also computed using equations (4-23), are presented in tabular form (see Tables 2, 3, and 4).

The corresponding values of \( \tilde{T}_* = \tilde{T}_*/\delta \) (see equation (4-7) are shown plotted in Figure 19. It is clear that when \( \nu = 1 \), \( \tilde{T}_* \to \infty \) as \( \delta \to 0 \). This observation is bound up with the striking result that, in the limit, as \( \delta \to 0 \)

\[
\tilde{S}_* = 0 \quad \text{and} \quad \tilde{T}_* = 1, \quad \text{when} \ \nu = 1,
\]

while, for all other values of \( \nu \)

\[
\tilde{S}_* = 1 \quad \text{and} \quad \tilde{T}_* = 0.
\]

Hence, the singular nature of \( \tilde{T}_* \) when \( \nu = 1 \) and \( \delta \to 0 \) is not surprising. Equations (4-24) and (4-25) may be viewed in the context of the comments made on full modal coupling after equations (3-36). In addition, they provide limiting values for the response as \( \delta \) becomes vanishingly small and while equation (4-25) is intuitively obvious, equation (4-24) is not. Perhaps the other feature most readily of note is that when \( \nu \) is not near unity, the values of \( \tilde{S}_* \) are to a large degree independent of \( \nu \) and \( \delta \) while \( \tilde{T}_* \), though largely independent of \( \nu \), tends to be linearly dependent on \( \delta \).

4.7 EQUIVALENT STATIC ANALYSIS PROCEDURE

The vertical and horizontal scales of the design spectrum (see Figure 13) are decided on in practice by taking account of the underlying soil conditions and the level of earthquake excitation considered appropriate; once these factors have been taken into account it is possible to specify the transition period \( T_1 \) and the spectral acceleration at the transition \( S_{a}(\xi, \nu_1) \). It is then possible to determine the spectral accelerations \( S_{a}(\xi, \nu_1) \), \( S_{a}(\xi, \nu_0) \), and the maximum displacement \( v(\text{max}) = S_a(\xi, \nu_1) / \nu_1 \) using equation (4-3). Remember that the frequency \( \nu_0 \) is given by \( \nu_0 = \nu_0 \sqrt{(1+\xi^2)} \).
**Figure 18** Final results for the maximum normalized shears and torques evaluated using modified forms of the SRSS combination rule: --- Positive, —— Negative, --- both positive and negative.
FIGURE 19  PLOT OF AN ALTERNATIVE TORQUE NORMALIZATION: ----- POSITIVE, —— NEGATIVE, ... BOTH POSITIVE AND NEGATIVE

FIGURE 20  EXAMPLE DIAGRAMS
In an equivalent static analysis of a partially symmetric building, aimed at determining its worst response to a single component of earthquake excitation, the two equivalent static actions \( S_a \) and \( T_a \) associated with each set of final results for \( S_a \) and \( T_a \) are applied simultaneously at the centre of stiffness. The resulting displacements at the centre of stiffness are given by

\[
\begin{align*}
\nu_a(\text{max}) &= \tilde{S}_a \nu(\text{max}), \\
r\theta(\text{max}) &= \frac{\tilde{T}_a S_s(\zeta_p \rho)}{\mu^2 S_s(\zeta_p \rho)} \nu(\text{max}).
\end{align*}
\]

(4-26)

The examples in the following section illustrate some of the points that arise.

### 4.8 ILLUSTRATIVE EXAMPLES

It is apparent from Figure 18 that both positive and negative torques may have to be considered when \( \mu \leq 1 \). In these cases, two values of \( \tilde{S}_a \) and three values of \( \tilde{T}_a \) (two negative and one positive) are needed in order to be able to determine the worst response. When \( \mu > 1 \), positive torques do not occur and two sets of values for \( \tilde{S}_a \) and \( \tilde{T}_a \) are sufficient.

**Example 1** \( (\mu = 1, \delta = 0) \)

Buildings with an "even" distribution of vertical resisting elements fall into this category. From equations (4-14) and (4-24) it is apparent that the two cases to consider are when the base shear \( S_v \) is applied without any accompanying torque and when a torque equal to \( S_v \) is applied without any accompanying base shear (note that \( S_\theta = S_v \) since \( \rho_\theta = \rho_v \)). It may readily be shown that if \( \nu(\text{max}) \) is the displacement resulting from the application of \( S_v \), then the torsional displacement \( \Delta_x \) of a point a distance \( x \) from the (coincident) centre of mass and stiffness is given by

\[
\Delta_x = \frac{x}{r} \nu(\text{max}),
\]

where \( r \) is the radius of gyration about the centre of mass and the direction of \( \Delta_x \) is perpendicular to the line of length \( x \). Thus, if the building is uniform and rectangular in plan the torsional displacements at the corners are equal to \( \sqrt{3} \nu(\text{max}) \).

Torsional displacements of a similar magnitude occur in the uniform L-shaped building shown in plan in Figure 20(a). Here the centres of mass and stiffness coincide at \( 0 \) and it is assumed that the vertical stiffness elements are arranged so that \( \mu = 1 \). The radius of gyration is \( r = \frac{5b}{2\sqrt{3}} \), so that the distances \( OA, OB, \) and \( OC \) are \( \sqrt{3} r \), \( \sqrt{3} \frac{r}{5} \) and \( \sqrt{87} \frac{r}{5} \), respectively. As a result, \( \Delta_A = 1.73 \nu(\text{max}), \Delta_B = 1.25 \nu(\text{max}), \) and \( \Delta_C = 1.87 \nu(\text{max}) \).

**Example 2** \( (\mu = \sqrt{2}, \delta = 1) \)

The building shown in Figure 20(b) is torsionally stiff due to the placing of the two shear walls. The only other resisting element is a shear core around the lift shaft. The columns are assumed to
behave as props accepting gravity load only. The calculated values of \( \mu \) and \( \delta \) are \( \sqrt{2} \) and 1.0 respectively. Consequently, only negative (that is, clockwise) torques have to be considered.

From Table 2, \( \hat{S}_a = 0.87 \) and \( \hat{T}_a = -1.22 \). On applying the corresponding shear and torque the in-plane displacement of wall A and the total displacement of wall B are, using equations (4-26) and noting that \( \rho_B = \rho_v \)

\[
\Delta_A = \left(\sqrt{6} \frac{T_a}{2 \mu_v^2}\right) v(\text{max}) = 0.75 v(\text{max}) ,
\]
and

\[
\Delta_B = \left\{\left(\frac{S_a - (\sqrt{6} + 26) T_a/2 \mu_v^2}{2 \mu_v^2}\right)^2 + \left[\sqrt{6} \frac{T_a}{2 \mu_v^2}\right]^2\right\}^{1/2} v(\text{max}) = 2.35 v(\text{max}) .
\]

The other set of normalized design actions \( \hat{S}_a = 1 \) and \( \hat{T}_a = -\delta \) yield values of

\[
\Delta_A = 0.61 v(\text{max}) \quad \text{and} \quad \Delta_B = 2.20 v(\text{max}),
\]

which are appreciably less than the previous results. In general, the compensation owing to a reduction in the base shear is not sufficient to offset the gain due to the increase in the torque. Note, however, that when \( \mu \gg 1 \), \( \hat{S}_a + 1 \) and \( \hat{T}_a + -\delta \) (see Figure 18).

Example 3 \((\mu = 0, \sqrt{6}/2 \leq \delta < \sqrt{3})\)

The third example is simply a rectangular building that has a shear wall at one end as the primary resisting element. The floor diaphragm is uniform and the centre of mass and centre of stiffness are separated by almost half the length of the building. As with the other examples it is assumed that the direction of earthquake attack is perpendicular to the axis of symmetry. Because the translational stiffness of the end wall is large, \( \mu = 0 \) and \( \hat{T}_a = \pm \delta \) (see Table 3B). Since \( v(\text{max}) \) is small, little interest attaches to \( \hat{S}_a \). Hence, only one equivalent static action is required and that is the torque \( T_a = \pm M S_a(c, p_B) c \) (in this case the frequency \( p_B \) is best thought of as being equal to \( \sqrt{(K^0/J_a)} \)).

The problem is essentially of one degree of freedom involving rotation about the centre of stiffness. Consequently, the same result is furnished by equation (4-5). The maximum value of the rotation co-ordinate \( \theta(t) \) in that equation is given by

\[
\theta(\text{max}) = \pm \frac{1}{r} \frac{\delta}{1 + \delta^2} \frac{S_a(c, p_0)}{p_B^2} ,
\]
from which it follows that

\[
T_a = K^0 \theta(\text{max}) = \pm M S_a(c, p_B) c ,
\]

which agrees with the result obtained above.

As a final exercise, consider the effect of normalizing the torque \( T_a \) with respect to the shear \( S_v, T_a/(S_v r) \) say. For most buildings this would be a practical normalization. However, for the extreme case just considered one ends up with a value of \( T_a \) equal to \( \pm M S_a(c, p_v) c \) and because the frequency \( p_v \) is large, the spectral acceleration \( S_a(c, p_v) \) is likely to always lie on the flat portion of the design spectrum. In other words, no account would be taken of the torsional flexibility of the building and an unnecessarily conservative result would be obtained.
### TABLE 2A

**Final results for $\ddot{r}_e$**

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<th>$\delta$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
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<td>0.892</td>
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### TABLE 3A

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TABLE 4A

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TABLE 4B

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5.1 SYNOPSIS

The coupled lateral-torsional response of multistorey "shear" buildings to earthquake excitation is investigated using a continuous shear beam model. The undamped equations of motion are formulated using Hamilton's principle and solved using the standard modal analysis technique for continuous systems. Allowance is made for viscous damping and the response is linearly elastic. Because of the similarity of notation the continuous model lends itself to a comparison with the discrete single storey model analysed in Chapter 3.

The model analysed is a simplified version of the continuous model that was analysed by Hoerner. However, its simplicity allows a more encompassing study to be made of the torsional interaction phenomenon and facilitates the evaluation of equivalent static actions - the latter are formulated in Chapter 6.

5.2 INTRODUCTION

The model studied is a continuous vertical cantilevered shear beam. The beam is prismatic and its mass is uniformly distributed along its length. The centres of mass lie on a single vertical axis as do the centres of stiffness. A distance \( \varepsilon \) separates the two axes. The ground motion is horizontal and purely translatory - torsional ground motion is not considered.

The model is depicted schematically in Figure 21. In this figure, the \( x \) and \( y \) reference axes are aligned parallel to the principal directions of stiffness, the \( z \) axis is coincident with the centroidal axis, and the small circle and star denote the centre of mass and centre of stiffness, respectively, at height \( z \). The moduli of rigidity in the \( x-y \) planes and the moduli of elasticity in the \( x \), \( y \), and \( z \) directions are assumed to be infinite. Consequently, each cross-section is restricted to horizontal plane motion without any in-plane distortion. Since the line joining the centre of mass and centre of stiffness of each cross-section is collinear with one of the principal axes the model is partially symmetric. The direction of earthquake attack is taken as being parallel to the other principal axis. The foregoing assumptions ensure that the model is analytically tractable but limit its response to horizontal shear in the lateral and torsional directions.

The coupled lateral-torsional response of the model is described by the relative displacements \( v_0(z,t) \) and \( \theta(z,t) \) of the centroidal axis. The directions of these displacements are shown in Figure 21(a). The relative displacements \( u \) and \( v \) of any point \( P(x,y) \) of the cross-section at height \( z \) are

\[
\begin{align*}
  u &= -y\theta , \\
  v &= v_0 + x\theta ,
\end{align*}
\]
FIGURE 21  DEFINITION DIAGRAM OF CONTINUOUS MODEL, (a) PLAN VIEW, AND (b) ELEVATION
in the x and y directions, respectively. The relative translational displacement $v_*(z, t)$ at the centre of stiffness of the cross-section is given by

$$v_* = v_0 + e\theta .$$

(5-2)

5.3 DERIVATION OF THE EQUATIONS OF MOTION

As a preliminary to formulating the undamped equations of motion, the total kinetic energy $T$ and the total strain energy $V$ are determined. The former quantity is given by

$$T = \int_0^L \int_A \frac{1}{2} \rho(x,y) \left\{ u^2 + (v + \dot{v}_g)^2 \right\} \, dA \, dz ,$$

(5-3)

where $L$ is the length of the model, $A$ is its cross-sectional area, $\rho(x,y)$ is the mass density at the point $P(x,y)$, and $v_g(t)$ is the ground displacement at time $t$. Substituting equations (5-1) into equation (5-3) leads to

$$T = \int_0^L \left\{ \frac{1}{2} m \left( \dot{v}_o + \dot{v}_g \right)^2 + \frac{1}{2} m \, r^2 \, \dot{\theta}^2 \right\} \, dz ,$$

(5-4)

where

$$m = \int_A \rho(x,y) \, dA$$

and

$$r^2 = \frac{1}{m} \int_A \left( x^2 + y^2 \right) \rho(x,y) \, dA .$$

The quantity $m$ is the mass per unit length and $r$ is the radius of gyration of a unit length of the beam about the centroidal axis.

Since the only contribution to strain energy is from shearing deformation the total strain energy $V$ is given by (5)

$$V = \int_0^L \left\{ \frac{1}{2} k^{yV} G_{xz}(x,y,z) \left[ \frac{\partial u}{\partial z} \right]^2 + \frac{1}{2} k^{yV} G_{yz}(x,y,z) \left[ \frac{\partial v}{\partial z} \right]^2 \right\} \, dA \, dz ,$$

(5-5)

where $G_{xz}(x,y,z)$ and $G_{yz}(x,y,z)$ are the moduli of rigidity in the $x$-z and $y$-z planes, respectively. Substituting equations (5-1) into equation (5-5) yields

$$V = \int_0^L \left\{ \frac{1}{2} k^{yV} G_{xz}(x,y,z) \left[ \frac{\partial u}{\partial z} \right]^2 + k^{yV}(z) e \, v_o^2 + \frac{1}{2} k^{yV}_0(z) \, \dot{\theta}^2 \right\} \, dz ,$$

(5-6)

where

$$k^{yV}(z) = \int_A G_{yz}(x,y,z) \, dA ,$$

and

$$e = \frac{1}{k^{yV}(z)} \int_A x \, G_{yz}(x,y,z) \, dA ,$$

(5-7)

and

$$k^{yV}_0(z) = \int_A \left\{ x^2 \, G_{yz}(x,y,z) + y^2 \, G_{xz}(x,y,z) \right\} \, dA .$$

The distribution of elastic shearing resistance is assumed to be such that the eccentricity $e$ is constant. Since the $y$ co-ordinate of the centre of stiffness is zero
\[
\int_A y \, G_{xz}(x, y, z) \, dA = 0 .
\]  
(5-8)

It may also be shown that
\[
V = \int_0^L \left\{ \frac{1}{2} k'(z) \, \nu^2 + \frac{1}{2} k'(z) \, \theta^2 \right\} \, dz ,
\]  
(5-9)
where
\[
k'(z) = \int_A \left\{ (x-z)^2 \, G_{yz}(x, y, z) + y^2 \, G_{xz}(x, y, z) \right\} \, dA .
\]  
(5-10)

It is easily verified that
\[
k'(z) = k'(z) + k'(z) \, \nu^2 .
\]  
(5-11)

The functions \( k'(z) \) and \( k'(z) \) define the stiffness of the model to the unit shearing displacements \( \nu = 1 \) and \( \theta = 1 \), respectively. The function \( k'(z) \) defines the stiffness of the model to the unit shearing displacement \( \theta = 1 \) when the centroidal axis is constrained to remain in its undeformed position.

It follows from equations (5-4) and (5-6) that the Lagrangian \( L = T - V \) is given by
\[
L = \int_0^L F(z, t, \dot{v}, \dot{v}, \theta) \, dz ,
\]  
(5-12)
where
\[
F(z, t, \dot{v}, \dot{v}, \theta) = \frac{1}{2} \, (\dot{v} + \dot{v})^2 + \frac{1}{2} \, m \, \nu^2 - \frac{1}{2} \, k'(z) \, \nu^2 - k'(z) \, \theta^2 .
\]  
(5-13)

It is useful at this stage to note that if the Lagrangian \( L \) is of the general form
\[
L = \int_{x_1}^{x_2} G(x, t, \dot{q}_1, ..., \dot{q}_n ; q_1, ..., q_n) \, dx ,
\]  
(5-14)
the application of Hamilton's principle furnishes the dual requirement that
\[
\frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{q}_k} \right) + \frac{\partial}{\partial x} \left( \frac{\partial L}{\partial \dot{q}_k} \right) = 0 \quad (k = 1, 2, ..., n) ,
\]  
(5-15)
and
\[
\sum_{k=1}^{n} \frac{\partial L}{\partial q_k} \left. \frac{\partial L}{\partial \dot{q}_k} \right|_{x_1}^{x_2} = 0 .
\]  
(5-16)

In this case, \( n = 2 \) and the application of equations (5-15) and (5-16) yields
\[
\frac{\partial}{\partial t} \left( \frac{\partial F}{\partial \dot{v}_0} \right) + \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial \dot{v}_0} \right) = 0 ,
\]  
(5-17)
and
\[
\frac{\partial}{\partial t} \left( \frac{\partial F}{\partial \dot{v}_0} \right) + \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial \dot{v}_0} \right) = 0 ,
\]  
(5-18)
and
\[ \left[ \frac{\partial F}{\partial v_o} \delta v_o + \frac{\partial F}{\partial \theta} \right]_0^2 = 0. \] (5-18)

Substituting for F in equations (5-17) provides the two coupled equations of motion. These may be written using matrix algebra as
\[
\begin{bmatrix}
 m & 0 \\
 0 & m \alpha^2
\end{bmatrix}
\begin{bmatrix}
 \ddot{v}_o + \ddot{v}_g \\
 \ddot{\theta}
\end{bmatrix}
- \begin{bmatrix}
 k^V(z) & k^V(z) e \\
 k^V(z) e & k^V(z)
\end{bmatrix}
\begin{bmatrix}
 v_o' \\
 \theta'
\end{bmatrix}
= \begin{bmatrix}
 0 \\
 0
\end{bmatrix}
\] (5-19)

The corresponding equations of motion in terms of \( v_0(z,t) \) and \( \theta(z,t) \) are
\[
\begin{bmatrix}
 m - mc \\
 -mc & m(r^2 + c^2)
\end{bmatrix}
\begin{bmatrix}
 \ddot{v}_o + \ddot{v}_g \\
 \ddot{\theta}
\end{bmatrix}
- \begin{bmatrix}
 k^V(z) & 0 \\
 0 & k^\theta(z)
\end{bmatrix}
\begin{bmatrix}
 v_o' \\
 \theta'
\end{bmatrix}
= \begin{bmatrix}
 0 \\
 0
\end{bmatrix}
\] (5-20)

Substituting for F in equation (5-18) provides the boundary conditions term
\[
\left[ \{ k^V(z) \ (v_o' + \epsilon \theta') \} \ \delta v_o + \{ k^V(z) \ \epsilon \ v_o' + k^\theta(z) \ \epsilon \theta \} \ \delta \theta \right]_0^Z = 0. \] (5-21)

Since \( \delta v_o = \delta v_0 - \epsilon \ \delta \theta \) equation (5-21) can be rewritten as
\[
\left[ \{ k^V(z) \ v_o' \} \ \delta v_o + \{ k^V(z) \ \epsilon \} \ \delta \theta \right]_0^Z = 0. \] (5-22)

Besides being zero at the base, the variations \( \delta v_o \), \( \delta v_0 \), and \( \delta \theta \) are arbitrary. The boundary conditions for this problem are therefore
\[
\begin{align*}
v_o(0,t) &= 0(0,t) = 0, \\
k^V(z) \left\{ v_o'(z,t) + \epsilon \theta'(z,t) \right\} &= k^\theta(z) \theta'(z,t) = 0.
\end{align*}
\] (5-23)

The first of equations (5-23) defines the geometric or imposed boundary conditions, while the second defines the natural or additional boundary conditions. Physically speaking, the natural boundary conditions correspond to there being zero shear and torque at the free end of the beam.

In order to ensure that \( \epsilon \) is constant as assumed, the shearing rigidities \( G_{xz}(x,y,z) \) and \( G_{yz}(x,y,z) \) are defined as
\[
\begin{align*}
G_{xz}(x,y,z) &= G_{xz}(x,y,0) \eta(z), \\
G_{yz}(x,y,z) &= G_{yz}(x,y,0) \eta(z).
\end{align*}
\] (5-24)

where \( \eta(z) \geq 0 \), \( z > 0 \), and \( \eta(0) = 1 \). It follows from the preceding theory that
The equations of motion linking the generalized co-ordinates \( v_0(z,t) \) and \( \theta(z,t) \) (see equation (5-19)) can be restated now in terms of the dimensionless length \( z_\ast = z/z_0 \) as

\[
\ddot{w}_0(z_\ast,t) - c_0^2 \chi \left\{ \eta(z_\ast) w'_0(z_\ast,t) \right\} = - \ddot{z}_g(t),
\]

where

\[
w_0(z_\ast,t) = \left\{ v_0(z_\ast,t) \right\}, \quad \chi = \left[ \begin{array}{cc} 1 & \delta \\ \delta & \delta^2 \end{array} \right], \quad z_0(t) = \left\{ v_0(t) \right\},
\]

\[
c_0^2 = \frac{k_V}{m z_0^2}, \quad \delta = \frac{E}{r}, \quad \text{and} \quad \nu_0^2 = \frac{k_0^0}{k_V r_0^2}.
\]

The boundary conditions associated with equation (5-27) are (from equation (5-23))

\[
z_0(0,t) = \eta(1) z_0'(1,t) = 0.
\]

Since \( k_0^0 = k_0^0 + k_V e^2 \) (see equation (5-11)) the parameter \( \nu_0^2 \) defined above can be redefined as

\[
\nu_0^2 = \mu^2 + \delta^2,
\]

where \( \mu^2 = k^0/(k_V r_0^2) \). The parameter \( \delta \) is a dimensionless eccentricity while \( \mu \) is a dimensionless stiffness ratio. Because \( \mu \) is always greater than zero, \( \nu_0 \) is always greater than \( \delta \). The identity in equation (5-29) was also derived in Chapter 3 while studying the single storey model (see equation (3-5)).

The undamped equation of motion governing the response of the corresponding torsionally balanced model (this model responds to earthquake excitation without twisting) is

\[
\ddot{v}(z_\ast,t) - c_0^2 \left\{ \eta(z_\ast) v'(z_\ast,t) \right\} = - \ddot{v}_b(t),
\]

where \( v(z_\ast,t) \) denotes the balanced translational response, and

\[
v(0,t) = \eta(1) v'(1,t) = 0.
\]
5.4 MODAL SOLUTION OF THE EQUATIONS OF MOTION

The modal solution of equation (5.27) is given by

\[ w_0(z_*, t) = \sum_{i=1}^{\infty} \rho_i(z_*) \phi_i(t), \tag{5-32} \]

where

\[ \rho(0) = \eta(1) \rho'(1) = 0. \tag{5-33} \]

The eigenfunction \( \rho_i(z_*) \) defines the \( i \)th natural mode shape of the corresponding torsionally balanced model, and \( \phi_i(t) \) is a 2x1 vector containing the normal co-ordinates \( \xi_{v1}(t) \) and \( \xi_{b1}(t) \). The matrix \( \phi \) is a 2x2 orthonormal matrix of modal components which satisfies the orthogonality conditions

\[ \phi^T \phi = I \quad \text{and} \quad \phi^T X \phi = \Lambda, \tag{5-34} \]

where \( I \) is a 2x2 identity matrix, and

\[ \Lambda = \begin{bmatrix} \lambda^2_v & 0 \\ 0 & \lambda^2_0 \end{bmatrix}. \]

The eigenvalues \( \lambda^2_v \) and \( \lambda^2_0 \) and the modal matrix \( \phi \) have already been defined (see Section 3.3).

When the corresponding torsionally balanced model is vibrating in its \( i \)th natural mode of vibration \( v(z_*, t) = \rho_i(z_*) f_i(t) \), where \( f_i(t) \) is the \( i \)th time function of free-vibration, and

\[ \ddot{v}(z_*, t) - c_v^2 \left( \eta(z_*) \nu'(z_*, t) \right)' = 0. \tag{5-35} \]

The application of the separation of variables technique yields the equations

\[ \left\{ \begin{array}{l} \left( \eta(z_*) \rho'_i(z_*) \right)' + \beta^2_i \rho_i(z_*) = 0, \\ \ddot{f}_i(t) + p_{v1}^2 f_i(t) = 0, \end{array} \right. \tag{5-36} \]

where \( p_{v1} = \beta_i c_v \) is the \( i \)th natural frequency. Since the system is positive definite the natural frequencies are real and positive, and the time functions of free vibration are harmonic. It is easily shown that the eigenfunctions are orthogonal to one another, that is

\[ \int_0^1 \rho_i(z_*) \rho_j(z_*) \, dz_* = 0 \quad (i \neq j), \tag{5-37} \]

and that

\[ \beta^2_i = \frac{\int_0^1 \eta(z_*) [\rho'_i(z_*)]^2 \, dz_*}{\int_0^1 \rho^2_i(z_*) \, dz_*}. \tag{5-38} \]

Substituting equation (5-32) into equation (5.27) with the subscript \( j \) instead of \( i \) yields

\[ \sum_{j=1}^{\infty} \left\{ \rho_j(z_*) \phi_j(t) - c_v^2 \left( \eta(z_*) \rho'_j(z_*) \right)^2 \phi_j(t) + \chi \phi_j(t) \right\} = - \ddot{z}_g(t). \tag{5-39} \]
Premultiplying equation (5-39) by $\phi^T$, substituting for $(n(z_a) p_1(z_a))^T$ from the first of equations (5-36), and making use of the orthogonality conditions in equation (5-34) yields

$$
\sum_{j=1}^{\infty} \left\{ \ddot{e}_j(t) + p_{Vj}^2 \Lambda \dot{e}_j(t) \right\} \rho_j(z_a) = - \phi^T \ddot{z}_g(t). \tag{5-40}
$$

Multiplying equation (5-40) by $\rho_i(z_a)$, integrating from 0 to 1 with respect to $z_a$, and using the orthogonality condition in equation (5-37) reveals that

$$
\ddot{e}_1(t) + \Omega_1 e_1(t) = - \alpha_1 \phi^T \ddot{z}_g(t), \tag{5-41}
$$

where

$$
\Omega_1 = p_{V1}^2 \Lambda = \begin{bmatrix} \omega_{V1}^2 & 0 \\ 0 & \omega_{01}^2 \end{bmatrix} \quad \text{and} \quad \alpha_1 = \frac{1}{\int_0^1 \rho_1(z_a) \, dz_a}, \frac{1}{\int_0^1 \rho_1^2(z_a) \, dz_a}.
$$

The parameters $\omega_{V1}$ ($= p_{V1} \lambda_1$) and $\omega_{01}$ ($= p_{V1} \lambda_0$) are the $i$th natural frequencies; there are two natural modes of vibration associated with every value of $i$. The two normal co-ordinate equations of motion obtained by expanding equation (5-41) and introducing classical modal damping at the same time are

$$
\begin{align*}
\ddot{e}_{V1}(t) + 2\zeta_{V1} \omega_{V1} \dot{e}_{V1}(t) + \omega_{V1}^2 e_{V1}(t) = - \alpha_{V1} \ddot{v}_g(t), \\
\ddot{e}_{01}(t) + 2\zeta_{01} \omega_{01} \dot{e}_{01}(t) + \omega_{01}^2 e_{01}(t) = - \alpha_{01} \ddot{v}_g(t),
\end{align*} \tag{5-42}
$$

where $\alpha_{V1}$ and $\alpha_{01}$ are the $i$th modal participation factors, and $\zeta_{V1}$ and $\zeta_{01}$ are the $i$th modal damping constants; $\alpha_{V1} = \alpha_i \cos \psi$ and $\alpha_{01} = - \text{sgn}(1-\nu_0) \alpha_i \sin \psi$. The solution of these equations for zero initial conditions is

$$
\begin{align*}
e_{V1}(t) &= \alpha_{V1} D_{V1}(t), \\
e_{01}(t) &= \alpha_{01} D_{01}(t),
\end{align*} \tag{5-43}
$$

where

$$
D_{V1}(t) = - D(\zeta_{V1}, \omega_{V1}, t, \ddot{v}_g) \quad \text{and} \quad D_{01}(t) = - D(\zeta_{01}, \omega_{01}, t, \ddot{v}_g).
$$

Substituting back into equation (5-32) yields the two response expressions

$$
V_o(z_a, t) = \sum_{i=1}^{\infty} \alpha_i \rho_i(z_a) \left\{ D_{V1}(t) \cos^2 \psi + D_{01}(t) \sin^2 \psi \right\}, \tag{5-44}
$$

and

$$
r_o(z_a, t) = \sum_{i=1}^{\infty} \alpha_i \rho_i(z_a) \text{sgn}(1-\nu_o) \left\{ D_{V1}(t) - D_{01}(t) \right\} \sin \psi \cos \psi.
$$

$D(\zeta, \omega, t, \ddot{v}_g)$ is defined in equation (3-19).
Equations (5-44) evaluate the dynamic response of the continuous model at any height $z = z_a$ and any time $t$. They are the solution of two coupled equations of motion which were uncoupled using the eigenfunctions $\rho_i(z_a) (i = 1, 2, \ldots)$ of the corresponding torsionally balanced model.

The modal solution of the equation of motion for the corresponding torsionally balanced model (see equation (5-30)) is given by

$$v(z_a,t) = \sum_{i=1}^{\infty} \alpha_i \rho_i(z_a) \, d_{\psi_i}(t) \, , \tag{5-45}$$

where $d_{\psi_i}(t) = -D(z_{v_i}, \rho_{v_i}, t, \psi_0) \, . \,$

An analogous model to the corresponding torsionally balanced model is the pure torsion model. In this model the centres of stiffness are restrained from translating and only twisting deformation occurs. The equation of motion governing the twist $\tilde{\theta}(z_a,t)$ is

$$r \, \tilde{\theta}(z_a,t) - c_0^2 \left\{ n(z_a) \, r \, \tilde{\theta}'(z_a,t) \right\}' = \frac{g}{1 + \delta^2} \, \tilde{v}_g(t) \, , \tag{5-46}$$

where

$$c_0^2 = \frac{k \delta}{m \, k^2 r^2 (1 + \delta^2)} \, . \,$$

The natural frequencies of vibration of the pure torsion model are given by $\omega_{01} = \beta_i c_0$ where $\beta_i$ is determined from the first of equations (5-36). The modal solution of equation (5-46) is

$$r \, \tilde{\theta}(z_a,t) = \frac{g}{1 + \delta^2} \sum_{i=1}^{m} \alpha_i \rho_i(z_a) \, d_{\theta_i}(t) \, , \tag{5-47}$$

where $d_{\theta_i}(t) = D(c_{\theta_i}, \rho_{\theta_i}, t, \psi_0) \, . \,$

The two associated models, the torsionally balanced model and the pure torsion model, are employed in the formulation of dimensionless equivalent static actions in the next chapter.
ON THE EVALUATION OF EQUIVALENT STATIC ACTIONS FOR THE MULTISTOREY MODEL

6.1 SYNOPSIS

In Chapter 5 expressions were derived for the earthquake response of a partially symmetric, multistorey "shear" building modelled as a continuous beam. In this chapter, the same building model is analysed and expressions are derived for the equivalent static shear, torque, and overturning moment at any level. These actions are then non-dimensionalized with respect to equivalent static actions which pertain to the two idealized models, namely, the torsionally balanced model and the pure torsion model. The translational and torsional components of displacement at the centre of stiffness axis are also suitably non-dimensionalized. The resulting dimensionless earthquake response quantities are evaluated using the response spectrum concept with account being taken of the coupling between the translational and torsional modes of vibration. The results obtained show an exact correspondence with those in Chapter 4 thereby completing the analogy between the discrete single storey model and the continuous multistorey model.

6.2 FORMULATION OF THE EQUIVALENT STATIC ACTIONS AND THEIR NON-DIMENSIONALIZATION

The present formulation of equivalent static actions begins by defining the distributed shear \( f^V(z,t) \) and distributed torque \( f^\theta(z,t) \) which, when applied statically at the centre of stiffness axis, cause the model to deform into its displaced configuration at time \( t \). The definitions follow from equation (5-20) and are

\[
\begin{align*}
  f^V(z,t) &= -m(\ddot{\nu}_o + \ddot{v}_g) = -\{k^V(z) \nu'_o(z,t)\}, \\
  f^\theta(z,t) &= -\{m^2 \ddot{v}_o - m(\ddot{\nu}_o + \ddot{v}_g)e\} = -\{k^\theta(z) \theta'(z,t)\}.
\end{align*}
\]

The shears, torques and overturning moments that are caused by the application of the distributed shear and torque are given by

\[
\begin{align*}
  S(z,t) &= \int_z^t f^V(z,t) \, dz = k^V(z) \nu'_o(z,t), \\
  T(z,t) &= \int_z^t f^\theta(z,t) \, dz = k^\theta(z) \theta'(z,t), \\
  M(z,t) &= \int_z^t \dot{T}(\tau,t) \, (\tau-z) \, d\tau = \int_z^t S(z,t) \, dz.
\end{align*}
\]

Differentiating the last of equations (6-2) and then integrating by parts provides the relation

\[
\frac{\partial M}{\partial z}(z,t) = -S(z,t).
\]
Equation (6-3) has an analogous counterpart in the bending theory of simple beams.

Equations (6-2) can be rewritten in terms of the dimensionless length \( z_\ast = z/\ell \) as

\[
\begin{align*}
S(z_\ast, t) & = \frac{k^V}{\ell} \eta(z_\ast) \nu'(z_\ast, t), \\
T(z_\ast, t) & = \frac{k^\theta}{\ell} \eta(z_\ast) \theta'(z_\ast, t), \\
M(z_\ast, t) & = k^V \int_{0}^{1} \eta(z_\ast) \nu'(z_\ast, t) \, dz_\ast,
\end{align*}
\]

(6-4)

and

where \( \eta(z_\ast) \geq 0 \) for \( z_\ast > 0 \) and \( \eta(0) = 1 \) (review equations (5-24) to (5-27)). Thus, the equivalent static actions at height \( z_\ast \) are

\[
\begin{align*}
S(z_\ast)_{\text{max}} & = \frac{k^V}{\ell} \eta(z_\ast) \left\{ \nu'(z_\ast, t) \right\}_{\text{max}} , \\
T(z_\ast)_{\text{max}} & = \frac{k^\theta}{\ell} \eta(z_\ast) \left\{ \theta'(z_\ast, t) \right\}_{\text{max}}, \\
M(z_\ast)_{\text{max}} & = k^V \left\{ \int_{0}^{1} \eta(z_\ast) \nu'(z_\ast, t) \, dz_\ast \right\}_{\text{max}}.
\end{align*}
\]

(6-5)

The application of these actions at the centre of stiffness axis causes the level under consideration to undergo its "worst" dynamic response.

In order to facilitate a comparison with the single storey model analyzed in Chapters 3 and 4, the equivalent static actions are non-dimensionalized in accordance with the equations

\[
\tilde{S}(z_\ast) = \frac{S(z_\ast)_{\text{max}}}{S_b(z_\ast)_{\text{max}}}, \quad \tilde{T}(z_\ast) = \frac{T(z_\ast)_{\text{max}}}{T_b(z_\ast)_{\text{max}}}, \quad \text{and} \quad \tilde{M}(z_\ast) = \frac{M(z_\ast)_{\text{max}}}{M_b(z_\ast)_{\text{max}}},
\]

(6-6)

where \( S_b(z_\ast)_{\text{max}} \) and \( M_b(z_\ast)_{\text{max}} \) are the corresponding equivalent static shear and overturning moment in the torsionally balanced model, and \( T_b(z_\ast)_{\text{max}} \) is equal to the corresponding equivalent static torque in the pure torsion model divided by \( \delta \). In addition, the two components of displacement at the centre of stiffness axis are non-dimensionalized according to

\[
\tilde{v}(z_\ast) = \frac{v(z_\ast)_{\text{max}}}{v(z_\ast)_{\text{max}}} \quad \text{and} \quad \tilde{\theta}(z_\ast) = \frac{\theta(z_\ast)_{\text{max}}}{\theta(z_\ast)_{\text{max}}},
\]

(6-7)

where \( v(z_\ast)_{\text{max}} \) is the maximum displacement at height \( z_\ast \) in the torsionally balanced model.

The maxima in equations (6-6) and (6-7) are evaluated as functions of their individual modal maxima. The latter are estimated using idealized pseudo-acceleration response spectrum curves. Consequently, the maximum values of the four time functions involved (see equations (5-44), (5-45), and (5-47) are defined in terms of pseudo-spectral accelerations as (it is assumed that \( \zeta_{vi} = \zeta_{\theta i} = \xi \))

\[
d_{vi}(\text{max}) = \frac{S_A(z, p_{vi})}{p_{vi}^2} \quad \text{and} \quad d_{\theta i}(\text{max}) = \frac{S_A(z, p_{\theta i})}{p_{\theta i}^2},
\]

(6-8a)
and
\[ D_{\nu}^{\text{(max)}} = \frac{S_a(c,\nu_{\nu})}{\omega_{\nu}^2}, \quad D_{\beta}^{\text{(max)}} = \frac{S_a(c,\nu_{\beta})}{\omega_{\beta}^2}. \] (6-8b)

Recall that the frequencies \( \nu_{\nu} \), \( \nu_{\beta} \), \( \omega_{\nu} \), and \( \omega_{\beta} \) are defined as 
\[ \nu_{\nu} = \beta_1 \mathcal{C}_v, \quad \nu_{\beta} = \beta_1 \mathcal{C}_\beta, \quad \omega_{\nu} = \nu_{\nu} \lambda_v, \] 
and \( \omega_{\beta} = \nu_{\beta} \lambda_\beta \). The frequencies \( \nu_{\beta} \) and \( \nu_{\nu} \) are related by
\[ \nu_{\beta} = \mu \nu_{\nu} / \sqrt{(1 + \epsilon_2)}. \] (6-9)

The modal maxima are now defined as a preliminary to evaluating results.

The modal maxima associated with the denominators of equations (6-6) and (6-7) are from equations (5-45) and (6-5)
\[
\begin{align*}
\nu_1(z_*)^{\text{(max)}} &= \alpha_1 \rho_1(z_*) \frac{S_a(c,\nu_{\nu})}{p_{\nu}^2}, \\
S_{\beta_1}(z_*)^{\text{(max)}} &= \frac{k^2}{k^2} \eta(z_*) \left\{ \nu_1(z_*,t) \right\}^{\text{max}}, \\
T_{\nu_1}(z_*)^{\text{(max)}} &= \frac{1}{\delta} \frac{k^2}{k^2} \eta(z_*) \left\{ \Theta_1(z_*,t) \right\}^{\text{max}}, \\
M_{\beta_1}(z_*)^{\text{(max)}} &= k \left\{ \int_{z_*}^{1} \eta(z_*) \nu_1(z_*,t) \, dz_* \right\}^{\text{max}}.
\end{align*}
\] (6-10)

Substituting for \( \nu_1(z_*,t) \) and \( \Theta_1(z_*,t) \) from equations (5-45) and (5-47) and using equation (6-8) yields
\[
\begin{align*}
\nu_1(z_*)^{\text{(max)}} &= \frac{\alpha_1 \rho_1(z_*)}{p_{\nu}^2} S_a(c,\nu_{\nu}), \\
S_{\beta_1}(z_*)^{\text{(max)}} &= \frac{\alpha_1 \rho_1(z_*)}{\beta_1^2} \eta(z_*) \frac{m\epsilon}{S_a(c,\nu_{\nu})}, \\
T_{\nu_1}(z_*)^{\text{(max)}} &= \frac{\alpha_1 \rho_1(z_*)}{\beta_1^2} \eta(z_*) \frac{m\epsilon}{S_a(c,\nu_{\nu})}, \\
M_{\beta_1}(z_*)^{\text{(max)}} &= \frac{\alpha_1 \Gamma_1(z_*)}{\beta_1^2} \frac{m\epsilon^2}{S_a(c,\nu_{\nu})},
\end{align*}
\] (6-11)

and
\[
\Gamma_1(z_*)^{\text{(max)}} = \int_{z_*}^{1} \eta(z_*) \rho_1(z_*) \, dz_*.
\] (6-12)

Each of the numerators in equations (6-6) and (6-7) has two modal maxima associated with it for every value of \( i \). One of the maxima corresponds to the \( i \)th predominantly translational mode of vibration while the other corresponds to the \( i \)th predominantly torsional mode of vibration. The two maxima are distinguished by the subscripts \( \nu \) and \( \beta \). The expressions defining the modal maxima are furnished by equations (5-44), (6-5), and (6-8) and are
The evaluation of results in this section parallels the computation of final results in Chapter 4.

Two sets of final results for $S(z_*)_{\text{max}}$, $T(z_*)_{\text{max}}$, $M(z_*)_{\text{max}}$, $v_*(z_*)_{\text{max}}$, and $r_0(z_*)_{\text{max}}$ are evaluated by first determining an intermediate set of maxima, $S_1(z_*)_{\text{max}}$, $T_1(z_*)_{\text{max}}$, $M_1(z_*)_{\text{max}}$, $v_1(z_*)_{\text{max}}$, and $r_0(z_*)_{\text{max}}$, from the corresponding modal components in equation (6-13). These are then combined using the square-root-of-the-sum-of-the-squares (SRSS) combination function. With this function

\[
S(z_*)_{\text{max}} = \left\{ \sum_{i=1}^{n} s_i^2(z_*)_{\text{max}} \right\}^{1/2}, \quad T(z_*)_{\text{max}} = \left\{ \sum_{i=1}^{n} t_i^2(z_*)_{\text{max}} \right\}^{1/2},
\]

\[
M(z_*)_{\text{max}} = \left\{ \sum_{i=1}^{n} m_i^2(z_*)_{\text{max}} \right\}^{1/2},
\]

\[
v_*(z_*)_{\text{max}} = \left\{ \sum_{i=1}^{n} v_i^2(z_*)_{\text{max}} \right\}^{1/2} \quad \text{and} \quad r_0(z_*)_{\text{max}} = \left\{ \sum_{i=1}^{n} r_i^2(z_*)_{\text{max}} \right\}^{1/2},
\]

where $n$ is the number of mode pairs taken into account.
In the first case, the modal maxima are determined using a flat acceleration spectrum and then added algebraically. It follows from equations (6-13) and Appendix C that the intermediate maxima are given by

\[
\begin{align*}
S_i(z_*)_{\text{max}} &= S_{vi}(z_*)_{\text{max}} + S_{\theta i}(z_*)_{\text{max}} = S_{bi}(z_*)_{\text{max}}, \\
T_i(z_*)_{\text{max}} &= T_{vi}(z_*)_{\text{max}} + T_{\theta i}(z_*)_{\text{max}} = -\delta T_i(z_*)_{\text{max}}, \\
M_i(z_*)_{\text{max}} &= M_{vi}(z_*)_{\text{max}} + M_{\theta i}(z_*)_{\text{max}} = M_{bi}(z_*)_{\text{max}}, \\
v_{si}(z_*)_{\text{max}} &= v_{vi}(z_*)_{\text{max}} + v_{\theta i}(z_*)_{\text{max}} = v_i(z_*)_{\text{max}}, \\
\text{and} \quad r_{\theta i}(z_*)_{\text{max}} &= r_{vi}(z_*)_{\text{max}} + r_{\theta i}(z_*)_{\text{max}} = -\frac{\delta}{\mu^2} v_i(z_*)_{\text{max}}.
\end{align*}
\] (6-15)

Substituting equations (6-15) into equations (6-14) and putting the results in dimensionless form yields

\[
\bar{s}(z_*) = 1, \quad \bar{T}(z_*) = -\delta, \quad \bar{R}(z_*) = 1, \quad \bar{V}(z_*) = 1, \quad \text{and} \quad \bar{r}(z_*) = -\frac{\delta}{\mu^2}, \quad (6-16)
\]

where it is understood that the non-dimensionalizing quantities \(S_{bi}(z_*)_{\text{max}}\), \(T_i(z_*)_{\text{max}}\), \(M(z_*)_{\text{max}}\), and \(v(z_*)_{\text{max}}\) have been evaluated using the SRSS combination function (see equation (6-14)).

In the second case, the translational-torsional interaction between each \(i\)th pair of natural modes is taken into account. The intermediate maxima are determined using the modified SRSS combination functions derived in Chapter 4 (see equations (4-23)). That is

\[
\begin{align*}
S_i(z_*)_{\text{max}} &= Q(S_{vi}(z_*)_{\text{max}}, S_{\theta i}(z_*)_{\text{max}}), \\
T_i(z_*)_{\text{max}} &= Q(T_{vi}(z_*)_{\text{max}}, T_{\theta i}(z_*)_{\text{max}}), \\
M_i(z_*)_{\text{max}} &= Q(M_{vi}(z_*)_{\text{max}}, M_{\theta i}(z_*)_{\text{max}}), \\
v_{si}(z_*)_{\text{max}} &= Q(v_{vi}(z_*)_{\text{max}}, v_{\theta i}(z_*)_{\text{max}}), \\
\text{and} \quad r_{\theta i}(z_*)_{\text{max}} &= Q(r_{vi}(z_*)_{\text{max}}, r_{\theta i}(z_*)_{\text{max}}), \quad \quad \quad \quad (6-17)
\end{align*}
\]

where

\[
Q(Q_{vi}, Q_{\theta i}) = Q_{vi}^2 + Q_{\theta i}^2 + 2 e^{-4Dx_2} |Q_{vi} Q_{\theta i}|, \quad (6-18)
\]

in which

\[
\alpha = \frac{\omega_{vi} - \omega_{\theta i}}{\omega_{vi} + \omega_{\theta i}} = \frac{\lambda_v - \lambda_0}{\lambda_v + \lambda_0} = \text{sgn}(1-\mu) \left\{ 1 - \frac{4\mu}{(1+\mu)^2 + \delta^2} \right\}^{1/2}. \quad (6-19)
\]
It is interesting to note that the parameter $\alpha$ retains the definition that it had in Chapter 4. The plus sign in equation (6-18) is used only in the evaluation of $T_{i}(z_{a})_{\max}$. It needs to be emphasized here that the modified SRSS combination function merely accentuates the results yielded by the SRSS combination function in the region of strong modal coupling (where the latter function is generally held to be inaccurate). Returning now to the computation of results, in this case two sets of results were evaluated. One set was evaluated using a flat acceleration spectrum while the other was evaluated using the hyperbolic spectrum. The final results were obtained by taking the maxima from the two sets and when put in dimensionless form were found to be given by

$$
\tilde{S}(z_{a}) = \tilde{S}_{a}, \quad \tilde{T}(z_{a}) = \tilde{T}_{a}, \quad \tilde{M}(z_{a}) = \tilde{S}_{a}, \quad \tilde{v}(z_{a}) = \tilde{S}_{a}, \quad \text{and} \quad \tilde{r}(z_{a}) = \frac{\tilde{T}_{a}}{\mu^{2}}, \quad (6-20)
$$

where $\tilde{S}_{a}$ and $\tilde{T}_{a}$ denote the final results computed in Chapter 4 and presented graphically in Figure 18.

Note that these non-dimensional quantities do not depend on the height $z_{a}$ or the variation of stiffness $n(z_{a})$. Equations (6-20) carry implications that would probably have been missed had a more general multistorey model been analyzed; they clearly illustrate the advantage of concentrating on this particular model. An immediate inference is that the results for the dimensionless shear and torque obtained for the single storey model carry over essentially without modification to the multistorey model. The analogy between the two models is complete.

6.4 ON THE EVALUATION OF $S_{b}(z_{a})_{\max}$, $T_{a}(z_{a})_{\max}$, $M_{b}(z_{a})_{\max}$, AND $v(z_{a})_{\max}$

In order to arrive at results for $S(z_{a})_{\max}$, $T(z_{a})_{\max}$, $M(z_{a})_{\max}$, $v(z_{a})_{\max}$, and $r(z_{a})_{\max}$ once the dimensionless quantities $\tilde{S}(z_{a})$, $\tilde{T}(z_{a})$, $\tilde{M}(z_{a})$, $\tilde{v}(z_{a})$, and $\tilde{r}(z_{a})$ are known, it is necessary to be able to evaluate $S_{b}(z_{a})_{\max}$, $T_{a}(z_{a})_{\max}$, $M_{b}(z_{a})_{\max}$, and $v(z_{a})_{\max}$. This requires an assumption to be made regarding the distribution of elastic shearing resistance, or, in other words, a definition of the function $n(z_{a})$. This section specifically considers the case where $n(z_{a})$ is a linearly decreasing function, that is, when

$$
n(z_{a}) = 1 - \kappa z_{a}, \quad (6-21)
$$

where the parameter $\kappa$ controls the fractional reduction in stiffness. With this assumption the eigenproblem equation in equation (5-36) becomes

$$
\left\{ \left(1-\kappa z_{a}\right)c_{1}^{i}(z_{a}) \right\}^{\prime} + \beta_{1}^{2} \phi_{1}(z_{a}) = 0. \quad (6-22)
$$

The quantities that have to be determined, once $\phi_{1}(z_{a})$ and $\beta_{1}^{2}$ are known, are, from equations (6-11), $\phi_{1}^{i}(z_{a})$, $\alpha_{1}$, and $T_{1}(z_{a})$.

A perturbation solution of equation (6-22) is outlined in Appendix D - although this solution method treats $\kappa$ as a small quantity, expressions are obtained for $\phi_{1}(z_{a})$ and $\beta_{1}^{2}$ which are accurate for values of $\kappa$ up to $\frac{1}{4}$ or more. The accuracy of the perturbation solution is ascertained in Appendix E where an exact solution for $\phi_{1}(z_{a})$ is derived and exact results for $\beta_{1}^{2}$ are presented. The perturbation expressions for $\phi_{1}(z_{a})$ and $\beta_{1}^{2}$ are, from equations (D-19) and (D-20)
\[ \rho_1(z_*) = \sqrt{2} \sin \lambda_0 z_* + \kappa \sqrt{2} \left\{ 2 z_* \sin \lambda_0 z_* - \left\{ 1 + \frac{1}{2} \right\} \sin \lambda_0 z_* \\
+ 2 \lambda_0 z_* \cos \lambda_0 z_* - 2 \lambda_0 \left\{ 1 - \frac{1}{2} \right\} z_* \cos \lambda_0 z_* \right\} + O(\kappa^2), \]  
(6-23)

and

\[ \rho_1^2 = \lambda_0^2 \left\{ 1 - \frac{\kappa}{2} \left\{ 1 - \frac{1}{\lambda_0^2} \right\} \right\} + O(\kappa^2), \]  
(6-24)

where \( \lambda_0 = (21-1)\pi/2 \). The parameter \( \lambda_0 \) corresponds to the unperturbed solution, that is, to \( \kappa \) being equal to zero. The first derivative of \( \rho_1(z_*) \) is given by

\[ \rho_1'(z_*) = \sqrt{2} \lambda_0 \cos \lambda_0 z_* + \kappa \sqrt{2} \left\{ 2 \sin \lambda_0 z_* - \lambda_0 \left\{ 3 - \frac{3}{2} \right\} \cos \lambda_0 z_* \\
+ 2 \lambda_0 \left\{ 1 - \frac{1}{2} \right\} z_* \sin \lambda_0 z_* + 6 \lambda_0 z_* \cos \lambda_0 z_* \\
- 2 \lambda_0^2 z_*^2 \sin \lambda_0 z_* \right\} + O(\kappa^2). \]  
(6-25)

Because the eigenfunctions are orthogonal (see equation (6-4)), the definition for \( a_i \) that follows equation (5-41) reduces to

\[ a_i = \int_0^1 \rho_i(z_*) \, dz_*. \]  
(6-26)

Substituting equation (6-23) into equation (6-26) yields

\[ a_i = \frac{\sqrt{2}}{\lambda_0} + \kappa \frac{\sqrt{2}}{8} \left\{ 1 - \frac{3}{\lambda_0^2} \right\} + O(\kappa^2). \]  
(6-27)

Substituting equation (6-21) into equation (6-12) and integrating by parts reveals that

\[ \Gamma_i(z_*) = (1-\kappa) \rho_i(1) - (1-\kappa z_*) \rho_i'(z_*) + \kappa \int_{z_*}^1 \rho_i(z_*) \, dz_* = \sqrt{2} \sin \lambda_0 \\
+ \kappa \frac{\sqrt{2}}{8} \left\{ \frac{8}{\lambda_0^2} \cos \lambda_0 z_* - \left\{ 7 + \frac{1}{\lambda_0^2} \right\} \sin \lambda_0 \right\} - (1-\kappa z_*) \rho_i'(z_*) + O(\kappa^2). \]  
(6-28)

Because the perturbation expressions are accurate only to first order, terms of \( O(\kappa^2) \) are neglected. Observe that \( \sin \lambda_0 = (-1)^{i+1} \). Table 5 shows values of the coefficients necessary for the evaluation of the base actions \( S_b(0)_{\text{max}} \), \( T_s(0)_{\text{max}} \), and \( H_b(0)_{\text{max}} \) for three different values of \( \kappa \) and for the first five modes. It is clear that the rather dramatic reduction in stiffness between the top of the model and the base that is implied by a value of \( \kappa = 1/2 \) has very little effect on the magnitude of the base quantities.

There still remains the problem of evaluating the pseudo-spectral accelerations \( S_a(\zeta, p_{\phi 1}) \) and \( S_a(\zeta, p_{\phi 1}) \). As in Chapter 4 (see Section 4.7) there is a slight anomaly between the evaluation of the non-dimensionalizing quantities in the computation of final results and their evaluation in practice. This anomaly arises from the tacit assumption in this chapter, based on a similar assumption in Chapter 4, that final results for the dimensionless response quantities could be obtained by analyzing the flat and hyperbolic spectra separately. The advantage of this assumption is that final results have been
obtained in both chapters which are dependent only on two parameters. However, when it comes to actually evaluating $S_b(z_s)_{\text{max}}$, $T_b(z_s)_{\text{max}}$, $M_b(z_s)_{\text{max}}$, and $v(z_s)_{\text{max}}$, it is felt that a more realistic design spectrum should be used (such as the composite design spectrum introduced in Chapter 4).

**TABLE 5**

Coefficients necessary for the evaluation of $S_b(0)_{\text{max}}$, $T_s(0)_{\text{max}}$, and $M_b(0)_{\text{max}}$

<table>
<thead>
<tr>
<th>$\kappa$ = 0</th>
<th>$i$</th>
<th>$\alpha_i$</th>
<th>$p_i(0)$</th>
<th>$\Gamma_i(0)$</th>
<th>$\alpha_i$ $p_i(0)/\sigma_i^2$</th>
<th>$\alpha_i$ $\Gamma_i(0)/\sigma_i^2$</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>0.900</td>
<td>2.221</td>
<td>1.414</td>
<td>0.811</td>
<td>0.516</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.300</td>
<td>6.664</td>
<td>-1.414</td>
<td>0.090</td>
<td>-0.019</td>
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</tr>
<tr>
<td>3</td>
<td>0.180</td>
<td>11.107</td>
<td>1.414</td>
<td>0.032</td>
<td>0.004</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.129</td>
<td>15.550</td>
<td>-1.414</td>
<td>0.017</td>
<td>-0.002</td>
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</tr>
<tr>
<td>5</td>
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<td>19.993</td>
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<td>0.010</td>
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<th>$\kappa$ = 1/4</th>
<th>$i$</th>
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<th>$p_i(0)$</th>
<th>$\Gamma_i(0)$</th>
<th>$\alpha_i$ $p_i(0)/\sigma_i^2$</th>
<th>$\alpha_i$ $\Gamma_i(0)/\sigma_i^2$</th>
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<td>1.312</td>
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<tr>
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<td>0.035</td>
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</tr>
<tr>
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</tr>
<tr>
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<td>18.122</td>
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<td>-0.001</td>
<td></td>
</tr>
</tbody>
</table>

<table>
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<th>$p_i(0)$</th>
<th>$\Gamma_i(0)$</th>
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<th>$\alpha_i$ $\Gamma_i(0)/\sigma_i^2$</th>
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<td>0.845</td>
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SUMMARY AND CONCLUSIONS

The use of two partially symmetric building models has enabled a detailed examination to be made of the problem of torsional unbalance in buildings subjected to earthquake excitation. The development of modal solutions, coupled with the adoption of the response spectrum concept and certain reasonable rules for the combination of modal maxima, made it possible to obtain realistic estimates for the associated equivalent static actions. These actions - a shear and a torque in one case, and a shear, torque and overturning moment in the other - account for the worst dynamic effects of torsional interaction.

During the development of the modal solution to the equations of motion governing the unbalanced response of the single storey model some time was spent on the question of modal coupling. The important result here has to do with the conditions necessary for full modal coupling, which exists when the components of the modal transformation matrix are all equal in magnitude. Full modal coupling was shown to arise when, for a co-ordinate system taken about any point on the axis of symmetry, the frequency of rotation that is constrained to occur about this point is equal to a frequency in which only translation is allowed. Although, in general, neither is a natural frequency of the system, the result gives a degree of formalism to the well known, but loosely stated, assertion that strong modal coupling occurs when the natural frequencies are close (which in turn generally requires the eccentricity to be small).

The non-dimensional earthquake response quantities for the single storey model were denoted by $\tilde{S}_x$ and $\tilde{T}_x$ (corresponding to the equivalent static shear and torque, respectively). For a torsionally balanced building (in practice buildings are invariably torsionally unbalanced) $\tilde{S}_x = 1$ and $\tilde{T}_x = 0$. The first sets of results for $\tilde{S}_x$ and $\tilde{T}_x$ were computed in a thorough numerical study using a composite design spectrum. A subsequent comparison with the maxima obtained analytically by considering flat and hyperbolic acceleration spectra in turn, showed that the latter were almost always identical to the former and could, therefore, be used in lieu of them. Coupled with this was the important conclusion that the dimensionless shear and torque ($\tilde{S}_x$ and $\tilde{T}_x$) were primarily dependent on two parameters: a "frequency" ratio $\mu$ and a dimensionless eccentricity $\delta$. This greatly facilitated the evaluation of the final set of results (see Figure 18 and Tables 2, 3, and 4). In deriving these results (which should be of considerable use if current torsional provisions in building codes require amendment) the somewhat conservative algebraic subtraction rule for combining the modal maxima was replaced by new modified forms of the SRSS combination rule. The latter approximate the results obtained by algebraic subtraction when the natural frequencies are close (that is, when $\mu = 1$ and $\delta \ll 1$), but tend to yield the same results as the SRSS combination rule when the frequency spacing grows.

The notable trends in the results were that for $\mu = 1$ and $\delta = 0$, either

$$\tilde{S}_x + 1 \quad \text{and} \quad \tilde{T}_x + 0,$$

or

$$\tilde{S}_x + 0 \quad \text{and} \quad \tilde{T}_x + 1,$$

while elsewhere the magnitude of $\tilde{T}_x$ was never appreciably greater than $\delta$, although the effect of this will
be partially offset by $\ddot{S}_s$ never being much less than unity. The idea of applying a torque without an accompanying shear (this is implied by the second set of actions above) appears to be a new one and was explored in the first example. The general results were that $\ddot{S}_s \leq 1$ and $|\ddot{T}_s| \geq \delta$. The abatement of shear and amplification of torque are most prominent for small values of $\delta$ when $\mu$ is close to unity - a region where modal coupling is strong.

At this stage of the analysis the results for $S_s$ and $T_s$ applied specifically only to the very simple, two degree-of-freedom, single storey model. The extent to which the results applied to the equivalent static analysis of multistorey buildings (a practice which is still widespread) was not known. In order to investigate this aspect an analysis was made of the coupled lateral-torsional response of a multistorey partially symmetric building model. Here the non-dimensional response quantities were denoted by $\ddot{S}(z_s)$, $\ddot{T}(z_s)$, $\ddot{M}(z_s)$, $\ddot{v}(z_s)$, and $\ddot{r}(z_s)$. The first three of these are the dimensionless shear, torque, and overturning moment at any level, while the remaining two are dimensionless replacement ratios about the centre of stiffness axis. A surprising outcome of the analysis was that $\ddot{S}(z_s)$, $\ddot{T}(z_s)$, and $\ddot{M}(z_s)$ were found to be given by

$$\ddot{S}(z_s) = \ddot{S}_s, \quad \ddot{T}(z_s) = \ddot{T}_s, \quad \ddot{M}(z_s) = \ddot{M}_s,$$

where $\ddot{S}_s$ and $\ddot{T}_s$ are the quantities that pertain to the single storey model. Hence, it is possible, by using a carefully developed analysis of a single storey building model, to obtain results which apply to certain classes of multistorey buildings.

In applying the modal analysis technique to the continuous model use was made of the same modal transformation matrix that was employed in the previous analysis. Consequently, many of the comments made previously in connection with modal coupling apply as well to the multistorey model. The evaluation of results for the multistorey model made use of knowledge gained in Chapter 4. The modified SRSS combination functions were used to take account of the interconnection between translational and torsional modes of vibration, before the final results were determined using the flat and hyperbolic acceleration spectra. These final results show no dependence on the dimensionless height $z_s$ nor do they show any dependence on the distribution of elastic stiffness throughout the height of the model.

Finally, as a general conclusion, it is noted that the present study supports the view that torsional unbalance gives rise to amplified torques and to reduced shears and overturning moments.
APPENDIX A

ELEMENT STIFFNESS EXPRESSIONS

The following expressions furnish the element stiffnesses tabulated in Section 2.8 (\( v = 0.15, \ t = 0.2 \ \text{m}, \ z = 3.0 \ \text{m}, \ \text{and} \ h = 3.5 \ \text{m} \)).

\[ k_{2,3}^X = \frac{E}{\frac{h^3}{3I_{2,3}} + \frac{2(1 + v)h}{A'}} \]

\[ k_{1,3}^Y = k_{2,3}^X \], and

\[ k_{3}^{XY} = \left[ \frac{I_2^2}{I_3} \right] k_3^X \].

In these expressions,

\[ A' = \frac{5}{6} \ \& \ t, \]

\[ I_2 = \frac{t^3}{12} \], and

\[ I_3 = \frac{1}{3} \left\{ t (t - c)^3 + \frac{t}{2} c^3 - (t - t) (c - t)^3 \right\}, \]

where

\[ c = \frac{t^2 + 2t t - t^2}{2(2t - t)}. \]

The expression for \( k_{3}^{XY} \) was obtained by analysing a unit-displacement equilibrium state in a manner similar to that outlined by Benjamin.\(^{(15)}\)
APPENDIX B

ON THE NUMERICAL EVALUATION OF DUAHEMEL'S INTEGRAL

B.1 SYNOPSIS

General expressions are derived for the numerical evaluation of Duhamel's integral and its derivative. The work comprises an extension (to unequal time steps) and an application (to a piecewise linear forcing function) of the numerical integration approach adopted by Cronin. The application is particularly relevant to the digital computation of response spectra from strong motion earthquake records.

B.2 PROBLEM STATEMENT

If \( \zeta \) and \( \omega \) are constants, the general solution of the equation

\[
\ddot{y} + 2\zeta \omega \dot{y} + \omega^2 y = f(t),
\]

is

\[
y(t) = e^{-\zeta \omega t} \left( y_0 \cos \omega t + \frac{\dot{y}_0 - \zeta \omega y_0}{\omega} \sin \omega t \right) + \int_0^t h(t-\tau)f(\tau) \, d\tau,
\]

where

\[
h(t-\tau) = e^{-\zeta \omega (t-\tau)} \sin \omega (t-\tau),
\]

and where \( y_0 = y(0) \), \( \dot{y}_0 = \dot{y}(0) \), and \( m = \sqrt{1-\zeta^2} \). The integral in equation (B-2) is Duhamel's integral and is denoted henceforth by \( D(t) \). The expression obtained for \( y(t) \) by differentiating equation (B-2) is

\[
\dot{y}(t) = -\zeta \omega \dot{y}(t) + \omega \frac{\dot{y}_0 - \zeta \omega y_0}{m} \sin \omega t + D(t),
\]

where

\[
\dot{D}(t) = -\zeta \omega D(t) + \omega \int_0^t k(t-\tau) f(\tau) \, d\tau,
\]

in which

\[
k(t-\tau) = e^{-\zeta \omega (t-\tau)} \cos \omega (t-\tau).
\]

This appendix is concerned primarily with the problem of obtaining values of \( D(t) \) and \( \dot{D}(t) \) when the convolution of \( h \) and \( f \), \( h(t-\tau)f(\tau) \), is not integrable analytically. The corresponding values of \( y(t) \) and \( \dot{y}(t) \) follow immediately from equations (B-2) and (B-3).

The general expressions derived below for \( D(t) \) and \( \dot{D}(t) \) are subsequently applied to the particular case where \( f(t) \) is a piecewise linear function.
B.3 DERIVATION OF A GENERAL EXPRESSION FOR $D(t)$

Figure (B-1) shows a general function $f(t)$ and three discrete times $t_i$, $t_j$, and $t_k$. The general recursive relationship derived in this section expresses the value of Duhamel's integral at time $t_k$ in terms of its values at times $t_i$ and $t_j$.

The derivation begins by noting that the value of Duhamel's integral at time $t_i$ is

$$D_i = D(t_i) = \int_0^{t_i} h(t_i - \tau) f(\tau) \, d\tau = \int_0^{t_j} h(t_i - \tau) f(\tau) \, d\tau - \int_{t_i}^{t_j} h(t_i - \tau) f(\tau) \, d\tau$$

$$= \int_0^{t_j} h(t_i - \tau) f(\tau) \, d\tau - \int_0^{\Delta t} h(-\tau) f(t_i + \tau) \, d\tau ,$$  \hspace{1cm} (B-5)

while at time $t_k$ it is

$$D_k = D(t_k) = \int_0^{t_k} h(t_k - \tau) f(\tau) \, d\tau = \int_0^{t_j} h(t_k - \tau) f(\tau) \, d\tau + \int_{t_j}^{t_k} h(t_k - \tau) f(\tau) \, d\tau$$

$$= \int_0^{t_j} h(t_k - \tau) f(\tau) \, d\tau + \int_0^{\Delta t'} h(\tau) f(t_k - \tau) \, d\tau .$$  \hspace{1cm} (B-6)

Note that

$$h(-\tau) = \frac{e^{-\sigma t} \sin m\omega t}{m\omega} \quad \text{and} \quad h(\tau) = \frac{e^{-\sigma t} \sin m\omega t}{m\omega} .$$
The functions $h(t_j - \tau)$ and $h(t_k - \tau)$ may be expanded to give

$$h(t_j - \tau) = \left( e^{\omega \Delta t} \cos \omega \Delta t \right) h(t_j - \tau) - \left( e^{\omega \Delta t} \sin \omega \Delta t \right) k(t_j - \tau) , \quad (B-7)$$

and

$$h(t_k - \tau) = \left( e^{-\omega \Delta t' \cos \omega \Delta t'} \right) h(t_j - \tau) + \left( e^{-\omega \Delta t' \sin \omega \Delta t'} \right) k(t_j - \tau) . \quad (B-8)$$

Substituting equations (B-7) and (B-8) into equations (B-5) and (B-6), respectively, provides the results

$$D_j = \left( e^{\omega \Delta t \cos \omega \Delta t} \right) D_j - \left( e^{\omega \Delta t \sin \omega \Delta t} \right) \int_0^{t_j} k(t_j - \tau) f(\tau) d\tau$$

$$- \int_0^{\Delta t} h(\tau) f(t_j + \tau) d\tau , \quad (B-9)$$

and

$$D_k = \left( e^{-\omega \Delta t' \cos \omega \Delta t'} \right) D_j + \left( e^{-\omega \Delta t' \sin \omega \Delta t'} \right) \int_0^{t_j} k(t_j - \tau) f(\tau) d\tau$$

$$+ \int_0^{\Delta t'} h(\tau) f(t_k - \tau) d\tau , \quad (B-10)$$

where $D_j = D(t_j)$. Eliminating the integral common to both equation (B-9) and equation (B-10) yields

$$D_k = \left[ \alpha e^{-\omega \Delta t \cos \omega \Delta t} \right] D_j + \left[ 2\beta e^{-\omega \Delta t' \cos \omega \Delta t'} \right] D_j - \alpha e^{-\omega \Delta t} \int_0^{\Delta t} h(\tau) f(t_j + \tau) d\tau$$

$$+ \int_0^{\Delta t'} h(\tau) f(t_k - \tau) d\tau , \quad (B-11)$$

where

$$\alpha = \frac{e^{-\omega \Delta t' \sin \omega \Delta t'}}{e^{-\omega \Delta t \sin \omega \Delta t}} \quad \text{and} \quad \beta = \frac{1}{2} \left( 1 + \tan \omega \Delta t' \frac{\tan \omega \Delta t}{\tan \omega \Delta t} \right) .$$

When the time steps $\Delta t$ and $\Delta t'$ are equal, $\alpha$ and $\beta$ are both equal to unity. Equation (B-11) then differs only in notation from the result previously obtained by Cronin.\(^{(28)}\)

The required values of $D(t)$ are obtained by the repeated application of equation (B-11). Initially it should be kept in mind that $D(t) = 0$ for $t \leq 0$ and that $f(t) = 0$ for $t < 0$. The expression for the first value of $D(t)$, $D_1 = D(t_1)$, is simply a formal statement of Duhamel's integral:

$$D_1 = \int_0^{t_1} h(\tau) f(t_1 - \tau) d\tau . \quad (B-12)$$

### B.4 DERIVATION OF A GENERAL EXPRESSION FOR $\dot{D}(t)$

It follows from equation (B-4) that

$$\int_0^{t_j} k(t_j - \tau) f(\tau) d\tau = \frac{\dot{D}(t_j) + \omega D(t_j)}{m} .$$

On incorporating this equation in (B-10) it is found that the value of $D(t)$ at time $t_j$, $\dot{D}_j$, is given by

$$\dot{D}_j = m \omega e^{\omega \Delta t'} \cosec \omega \Delta t' \left\{ D_j - e^{-\omega \Delta t'} \left( \cos \omega \Delta t' + \frac{\Delta t'}{m} \sin \omega \Delta t' \right) \right\} - \int_0^{\Delta t'} h(\tau) f(t_k - \tau) d\tau .$$

$$\dot{D}_j = m \omega e^{\omega \Delta t'} \cosec \omega \Delta t' \left\{ D_j - e^{-\omega \Delta t'} \left( \cos \omega \Delta t' + \frac{\Delta t'}{m} \sin \omega \Delta t' \right) \right\} - \int_0^{\Delta t'} h(\tau) f(t_k - \tau) d\tau . \quad (B-13)$$
B.5 APPLICATION TO A PIECEWISE LINEAR FORCING FUNCTION

An inspection of Figure (B-2) shows that when \( f(t) \) is piecewise linear

\[
f(t_i + \tau) = F_i + \frac{\tau}{\Delta t} (F_j - F_i), \quad 0 \leq \tau \leq \Delta t,
\]

and

\[
f(t_k - \tau) = F_k - \frac{\tau}{\Delta t'} (F_j - F_k), \quad 0 \leq \tau \leq \Delta t',
\]

in which the notation \( F_i = f(t_i) \) (\( i = 0, 1, 2, \ldots \)) is employed. Although time \( t_1 \) is not shown, it follows from equation (B-14b) that

\[
f(t_1 - \tau) = F_1 - \frac{t_1}{\Delta t_1} (F_1 - F_0), \quad 0 \leq \tau \leq t_1.
\]

Incorporating equation (B-14c) in equation (B-12), equations (B-14a) and (B-14b) in equation (B-11), and equation (B-14b) in equation (B-13) yields

\[
D_1 = c(\zeta, \omega, t_1) F_0 + d(\zeta, \omega, t_1) F_1,
\]

\[
D_k = - \left\{ \alpha e^{-2\Omega \Delta t} \right\} D_j + \left\{ 2B e^{-\gamma t} \cos \Omega \Delta t \right\} D_j + \alpha a F_i + (c - ab) F_j + d F_k,
\]

and

\[
\dot{D}_j = m \omega e^{\Omega \Delta t} \cos \Omega \Delta t \left\{ D_k - |1 - \omega^2 (c + d)| D_j - c F_j - d F_k \right\},
\]

where

\[
a = a(\zeta, \omega, \Delta t) = e^{-2\Omega \Delta t} \left\{ \frac{1}{\Delta t} \int_0^{\Delta t} \tau h(-\tau) \, d\tau - \int_0^{\Delta t} \frac{h(-\tau)}{\Delta t} \, d\tau \right\},
\]

\[
b = b(\zeta, \omega, \Delta t) = e^{-2\Omega \Delta t} \frac{1}{\Delta t} \int_0^{\Delta t} \tau h(-\tau) \, d\tau.
\]
\[ c = c(t, \omega, \Delta t) = \frac{1}{\Delta t} \int_{0}^{\Delta t} \tau h(\tau) \, d\tau, \]

and

\[ d = d(t, \omega, \Delta t) = \int_{0}^{\Delta t} h(\tau) \, d\tau - \frac{1}{\Delta t} \int_{0}^{\Delta t} \tau h(\tau) \, d\tau. \]

On evaluating the above integrals it is found that

\[ \omega_a^2 = e^{-\xi \omega t} \left\{ \left[ 1 + \frac{2\xi}{\omega} \right] e^{-\xi \omega t} - \left( \frac{2\xi}{\omega} \right) \cos \omega t + \frac{1-2\xi^2}{\omega^2} \sin \omega t \right\}, \]

\[ \omega_b^2 = \omega_c^2 + \left[ \left( \frac{\omega}{\omega_t} \right) \sin \omega t - \left( \frac{\xi}{m} \right) \sin \omega t \right] - e^{-\xi \omega t}, \]

\[ \omega_c^2 = \frac{2\xi}{\omega_t} - e^{-\xi \omega t} \left\{ \left[ 1 + \frac{2\xi}{\omega_t} \right] \cos \omega t' - \left( \frac{1-2\xi^2}{\omega_t^2} - \frac{\xi}{m} \right) \sin \omega t' \right\}, \]

and

\[ \omega_d^2 = \left[ 1 - e^{-\xi \omega t} \left( \cos \omega t' + \frac{\xi}{m} \sin \omega t' \right) \right] - \omega_c^2. \]

Equation (B-15) provides the first value of \( D(t) \), while equation (B-16) provides the subsequent values \( (k = 2, 3, \ldots) \).

### B.6 CONCLUDING REMARKS

Equations (B-15), (B-16), and (B-17) facilitate the numerical evaluation of Duhamel's integral and its derivative for the particular case where \( f(t) \) is a piecewise linear function. These equations together with equations (B-2) and (B-3) constitute an exact solution routine for evaluating \( y(t) \) and \( \dot{y}(t) \) at discrete points in time. As such they complement the routine described by Nigam and Jennings.\(^{(29)}\)

Both routines are exact in the sense that exact answers are obtained if \( f(t) \) is truly piecewise linear. However, in connection with the computation of response spectra from digitised earthquake records, it should be noted that the answers obtained will be affected by a discretisation error. This is kept within acceptable limits by a suitable choice of the interval of integration.\(^{(29)}\)

A practical advantage of the method presented in this appendix is the computational efficiency with which values of \( y(t) \) are evaluated. Often, a displacement response time history is all that is required.
APPENDIX C

ANALYTICAL RESULTS FOR $\bar{S}_a$ AND $\bar{T}_a$

The modal maxima $\bar{S}_{av}$, $\bar{S}_{av}$, $\bar{T}_{av}$, and $\bar{T}_{av}$ are defined in equations (4-10) and (4-11). In deriving the results below use is made of the results presented in Section 3.3.

C.1 ADDITIVE MODAL MAXIMA COMBINATIONS

(i) For the flat acceleration spectrum

$$\bar{S}_{av} + \bar{S}_{av} = \frac{\cos^2 \psi + \sin^2 \psi + \text{sgn}(1-\mu_0) \left( \frac{1}{\lambda_v} - \frac{1}{\lambda_0} \right) \delta \sin \psi \cos \psi}{\lambda_v^2 + \lambda_0^2},$$

and

$$\bar{T}_{av} + \bar{T}_{av} = \text{sgn}(1-\mu_0) \left( \frac{1}{\lambda_v} - \frac{1}{\lambda_0} \right) \sin \psi \cos \psi.$$  

But

$$\lambda_0^2 \cos^2 \psi + \lambda_v^2 \sin^2 \psi = \mu_0^2,$$

and

$$\text{sgn}(1-\mu_0) \left( \lambda_v^2 - \lambda_0^2 \right) \sin \psi \cos \psi = \delta.$$

Hence

$$\bar{S}_{av} + \bar{S}_{av} = 1,$$

and

$$\bar{T}_{av} + \bar{T}_{av} = -\delta.$$  

(ii) For the hyperbolic acceleration spectrum

$$\bar{S}_{av} + \bar{S}_{av} = \frac{\cos^2 \psi + \sin^2 \psi + \text{sgn}(1-\mu_0) \left( \frac{1}{\lambda_v} - \frac{1}{\lambda_0} \right) \delta \sin \psi \cos \psi}{\lambda_v^2 + \lambda_0^2},$$

and

$$\bar{T}_{av} + \bar{T}_{av} = \text{sgn}(1-\mu_0) \left( \lambda_v^2 - \lambda_0^2 \right) \sin \psi \cos \psi,$$

But

$$\lambda_0 \cos^2 \psi + \lambda_v \sin^2 \psi = \mu \left( \lambda_v + \delta_2 \right)^2,$$

and

$$\text{sgn}(1-\mu_0) \left( \lambda_v - \lambda_0 \right) \sin \psi \cos \psi = \frac{\delta}{\lambda_v + \lambda_0},$$

where

$$\lambda_v + \lambda_0 = \left( (1+\mu)^2 + \delta^2 \right)^{\frac{1}{2}}.$$

Hence

$$\bar{S}_{av} + \bar{S}_{av} = \frac{\lambda_v}{\left( (1+\mu)^2 + \delta^2 \right)^{\frac{1}{2}}},$$

and

$$\bar{T}_{av} + \bar{T}_{av} = -\frac{\delta \sqrt{(1+\delta^2)^2}}{\left( (1+\mu)^2 + \delta^2 \right)^{\frac{1}{2}}}.$$
C.2 SUBTRACTIVE MODAL MAXIMA COMBINATIONS

(i) For the flat acceleration spectrum

\[ \tilde{S}_{x_{V}} - \tilde{S}_{x_{0}} = \cos^{2} \psi \frac{\sin^{2} \psi}{\lambda_{V}^{2}} + \frac{\delta}{\lambda_{0}^{2}} \sin \psi \cos \psi, \]

and

\[ \tilde{T}_{x_{V}} - \tilde{T}_{x_{0}} = \text{sgn}(1-\nu_{0}) \mu \left( \frac{1}{\lambda_{V}^{2}} + \frac{1}{\lambda_{0}^{2}} \right) \sin \psi \cos \psi. \]

But

\[ \lambda_{V}^{2} \cos^{2} \psi - \lambda_{0}^{2} \sin^{2} \psi = \text{sgn}(1-\nu_{0}) \left[ \mu^{2} (1-\nu_{0}^{2}) - 2\delta^{2} \right] / 2R, \]

and

\[ (\lambda_{V}^{2} + \lambda_{0}^{2}) \sin \psi \cos \psi = (1+\nu_{0}^{2}) \delta / 2R. \]

Hence

\[ \tilde{S}_{x_{V}} - \tilde{S}_{x_{0}} = \text{sgn}(1-\nu_{0}) (1-\nu_{0}^{2}) / 2R = \cos 2\psi, \]

and

\[ \tilde{T}_{x_{V}} - \tilde{T}_{x_{0}} = \text{sgn}(1-\nu_{0}) (1+\nu_{0}^{2}) \sin \psi \cos \psi. \]

(ii) For the hyperbolic acceleration spectrum

\[ \tilde{S}_{x_{V}} - \tilde{S}_{x_{0}} = \cos^{2} \psi \frac{\sin^{2} \psi}{\lambda_{V}^{2}} + \text{sgn}(1-\nu_{0}) \left( \frac{1}{\lambda_{V}} + \frac{1}{\lambda_{0}} \right) \delta \sin \psi \cos \psi, \]

and

\[ \tilde{T}_{x_{V}} - \tilde{T}_{x_{0}} = \text{sgn}(1-\nu_{0}) \mu \sqrt{(1+\delta^{2}) \left( \frac{1}{\lambda_{V}^{2}} + \frac{1}{\lambda_{0}^{2}} \right)} \sin \psi \cos \psi. \]

But

\[ \lambda_{V}^{2} \cos^{2} \psi - \lambda_{0}^{2} \sin^{2} \psi = \text{sgn}(1-\nu_{0}) \left[ \mu(1-\nu_{0}) - \delta^{2} \right] / \left| \lambda_{V}^{2} - \lambda_{0}^{2} \right|, \]

and

\[ (\lambda_{V} + \lambda_{0}) \sin \psi \cos \psi = \frac{\delta}{|\lambda_{V} - \lambda_{0}|}, \]

where

\[ |\lambda_{V} - \lambda_{0}| = [(1-\nu)^{2} + \delta^{2}]^{1/2}. \]

Hence

\[ \tilde{S}_{x_{V}} - \tilde{S}_{x_{0}} = \text{sgn}(1-\nu_{0}) \frac{1-\nu}{[(1-\nu)^{2} + \delta^{2}]^{1/2}}, \]

and

\[ \tilde{T}_{x_{V}} - \tilde{T}_{x_{0}} = \text{sgn}(1-\nu_{0}) \frac{\delta \sqrt{(1+\delta^{2})}}{[(1-\nu)^{2} + \delta^{2}]^{1/2}}. \]
APPENDIX D

PERTURBATION SOLUTION OF THE EIGENPROBLEM EQUATION \((1-\kappa z_\kappa) \rho_i \frac{d^2 \rho_i}{dz^2} + \beta_i^2 \rho_i = 0\)

D.1 SYNOPSIS

A perturbation procedure is used to solve the eigenproblem equation

\[
\frac{d}{dz^2} \left\{ \left[ 1-\kappa z_\kappa \right] \frac{d \rho_i}{dz} \right\} + \beta_i^2 \rho_i = 0,
\]

(D-1)

for the \(i\)th eigenfunction \(\rho_i\) and the \(i\)th eigenvalue \(\beta_i^2\).

Approximate solutions for \(\rho_i\) and \(\beta_i^2\), for values of \(\kappa < 1\), are obtained in the form of asymptotic expansions by letting

\[
\rho_i = \phi_0 + \kappa \phi_1 + \ldots ,
\]

and

\[
\beta_i^2 = \lambda_0^2 + \kappa \lambda_1^2 + \ldots .
\]

(D-2)

The case that arises when \(\kappa = 1\) is not considered (recall that \(\kappa\) controls the linear reduction of stiffness in the continuous model and is bounded therefore by \(0 \leq \kappa \leq 1\)). The eigenfunctions satisfy the boundary conditions

\[
\rho_i(0) = (1-\kappa) \rho_i(1) = 0,
\]

(D-3)

and are orthogonal, that is

\[
\int_0^1 \rho_i \rho_j dz = \delta_{ij},
\]

(D-4)

where the Kronecker delta function \(\delta_{ij}\) is defined as

\[
\delta_{ij} = \begin{cases} 
1, & i = j \\
0, & i \neq j
\end{cases}
\]

(D-5)

D.2 SOLUTION TO \(O(\kappa)\)

Substituting the perturbation expansions for \(\rho_i\) and \(\beta_i^2\) (see equation (D-2)) in equation (D-1) and collecting coefficients of like powers of \(\kappa\) yields an identity in \(\kappa\). Since each coefficient of \(\kappa\) vanishes independently, the procedure furnishes the equations

\[
L \phi_0 = 0,
\]

(D-6)

and

\[
L \phi_1 = \frac{d}{dz} \left[ \frac{d \phi_0}{dz} \right] - \lambda_1^2 \phi_0,
\]

(D-7)

where \(L\) denotes the linear differential operator

\[
L = \frac{d^2}{dz^2} + \lambda_0^2.
\]
Similarly, substituting the perturbation expansion for $\rho_i$ into equation (D-4) for the case when $i = j$ provides the equations
\[
\int_0^1 \phi_0^2 \, dz_* = 1 , \tag{D-8}
\]
and
\[
\int_0^1 \phi_0 \phi_1 \, dz_* = 0 \tag{D-9}
\]

**Determination of $\lambda^2_1$**

The boundary conditions for $\phi_n$ are
\[
\phi_n(0) = \phi_n'(1) = 0 \quad (n = 0, 1, \ldots) \tag{D-10}
\]

Integrating by parts twice, reveals that
\[
\int_0^1 \phi_m L \phi_n \, dz_* = \int_0^1 \phi_n L \phi_m \, dz_* \tag{D-11}
\]

Equation (D-11) together with equation (D-6) gives
\[
\int_0^1 \phi_0 L \phi_0 \, dz_* = 0 \tag{D-12}
\]

Hence, multiplying equation (D-7) by $\phi_0$ and integrating the resulting equation from 0 to 1 with respect to $z_*$ yields
\[
\lambda^2_1 = - \int_0^1 z_* \left( \frac{d \phi_0}{dz_*} \right)^2 \, dz_* , \tag{D-13}
\]

provided that the function $\phi_0$ satisfies equation (D-8).

**The zeroth-order solution**

The solution of the zeroth-order problem (see equation (D-6)) is given by
\[
\phi_0 = A_0 \sin \lambda_0 z_*, \tag{D-14}
\]

where $\lambda_0 = (2j-1)\pi/2$.

**The first-order solution**

Substituting equation (D-14) into equation (D-7) yields
\[
L \phi_1 = A_0 \lambda_0 \frac{d}{dz_*} \left( z_* \cos \lambda_0 z_* \right) - A_0 \lambda^2_1 \sin \lambda_0 z_* \tag{D-15}
\]

The solution of this equation is given by
\[
\phi_1 = A_1 \sin \lambda_0 z_* + \frac{A_0}{4} z_* \sin \lambda_0 z_* + \frac{A_0}{4} \lambda_0^2 \cos \lambda_0 z_* + \frac{A_0 \lambda^2_1}{2 \lambda_0} z_* \cos \lambda_0 z_* \tag{D-16}
\]
The functions $\phi_0$ and $\phi_1$ satisfy equations (D-8) and (D-9) only if $A_0 = \sqrt{2}$ and

$$A_1 = -\frac{\sqrt{2}}{8} \left(1 + \frac{1}{\lambda_0^2}\right).$$  \hfill (D-17)

Substituting $\phi_0 = \sqrt{2} \sin \lambda_0 z_*$ into equation (D-13) reveals that

$$\lambda_1^2 = -\frac{\lambda_0^2}{2} \left(1 - \frac{1}{\lambda_0^2}\right).$$  \hfill (D-18)

The perturbation solutions for $\rho_1$ and $\rho_1^2$ are therefore

$$\rho_1 = \sqrt{2} \sin \lambda_0 z_* + \kappa \frac{\sqrt{2}}{8} \left\{(2z_0 - 1 - \frac{1}{\lambda_0^2}) \sin \lambda_0 z_*
+ 2\lambda_0 \left(z_* - 1 + \frac{1}{\lambda_0^2}\right) z_0 \cos \lambda_0 z_*\right\} + O(\kappa^2),$$ \hfill (D-19)

and

$$\rho_1^2 = \lambda_0^2 \left\{1 - \frac{\kappa^2}{2} \left(1 - \frac{1}{\lambda_0^2}\right)\right\} + O(\kappa^2),$$ \hfill (D-20)

where $\lambda_0 = (2i-1)\pi/2$. The first five values of $\rho_1$, for values of $\kappa$ equal to 0, 1/4, and 1/2, are listed in Table D-1.

**TABLE D-1**

<table>
<thead>
<tr>
<th>$s$</th>
<th>$\kappa$</th>
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<th>$1/4$</th>
<th>$1/2$</th>
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<td>12.25335</td>
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</tr>
</tbody>
</table>

The problem of obtaining exact solutions for $\rho_1$ and $\rho_1^2$ is considered in Appendix E. A comparison of the approximate values of $\rho_1$ in Table D-1 with the corresponding exact values in Table E-1 reveals that they differ at most by 1.4%.
APPENDIX E

EXACT SOLUTION OF THE EIGENPROBLEM EQUATION \[(1 - \kappa z_e) \rho_i'' + \beta_i^2 \rho_i = 0\]

E.1 SYNOPSIS

An exact treatment is given to the problem considered in Appendix D. Particular attention is paid to the limiting cases \(\kappa \to 0\) and \(\kappa \to 1\).

E.2 SOLUTION OUTLINE

The \(i\)th eigenfunction \(\rho_i\) associated with the eigenproblem equation

\[
\frac{d}{dz_e} \left\{ (1 - \kappa z_e) \frac{d\rho_i}{dz_e} \right\} + \beta_i^2 \rho_i = 0 ,
\]

(E-1)

where \(0 \leq \kappa \leq 1\), is given exactly by

\[
\rho_i = A J_\alpha \left[ \frac{2\beta_i}{\kappa} \sqrt{(1 - \kappa z_e)} \right] + B Y_\alpha \left[ \frac{2\beta_i}{\kappa} \sqrt{(1 - \kappa z_e)} \right] .
\]

(E-2)

The first derivative of \(\rho_i\) is given by

\[
\rho_i' = \frac{\beta_i}{\sqrt{(1 - \kappa z_e)}} \left\{ A J_1 \left[ \frac{2\beta_i}{\kappa} \sqrt{(1 - \kappa z_e)} \right] + B Y_1 \left[ \frac{2\beta_i}{\kappa} \sqrt{(1 - \kappa z_e)} \right] \right\} .
\]

(E-3)

The notations \(J_\nu\) and \(Y_\nu\) denote Bessel functions of order \(\nu\) and of the first and second kinds, respectively.

Since the boundary conditions for \(\rho_i\) are

\[
\rho_i(0) = (1 - \kappa) \rho_i'(1) = 0 ,
\]

(E-4)

it follows from equations (E-2) and (E-3) that

\[
A J_\alpha \left[ \frac{2\beta_i}{\kappa} \sqrt{(1 - \kappa z_e)} \right] + B Y_\alpha \left[ \frac{2\beta_i}{\kappa} \sqrt{(1 - \kappa z_e)} \right] = 0 ,
\]

(E-5)

and

\[
\sqrt{(1 - \kappa)} \left\{ A J_1 \left[ \frac{2\beta_i}{\kappa} \sqrt{(1 - \kappa z_e)} \right] + B Y_1 \left[ \frac{2\beta_i}{\kappa} \sqrt{(1 - \kappa z_e)} \right] \right\} = 0 .
\]

(E-6)

It follows in turn from equations (E-5) and (E-6) that the constants \(A\) and \(B\) are non-trivial only if the parameter \(\beta_i\) satisfies the transcendental equation

\[
\sqrt{(1 - \kappa)} \left\{ J_\alpha \left[ \frac{2\beta_i}{\kappa} \sqrt{(1 - \kappa z_e)} \right] Y_1 \left[ \frac{2\beta_i}{\kappa} \sqrt{(1 - \kappa z_e)} \right] - J_1 \left[ \frac{2\beta_i}{\kappa} \sqrt{(1 - \kappa z_e)} \right] Y_\alpha \left[ \frac{2\beta_i}{\kappa} \sqrt{(1 - \kappa z_e)} \right] \right\} = 0 .
\]

(E-7)

Equations (E-2) and (E-5) together redefine the eigenfunction \(\rho_i\) as

\[
\rho_i = A \left\{ J_\alpha \left[ \frac{2\beta_i}{\kappa} \sqrt{(1 - \kappa z_e)} \right] - \frac{J_\alpha \left(2\beta_i/\kappa\right)}{J_\alpha \left(2\beta_i/\kappa\right)} Y_\alpha \left[ \frac{2\beta_i}{\kappa} \sqrt{(1 - \kappa z_e)} \right] \right\} .
\]

(E-8)

Equation (E-8) is an exact solution for \(\rho_i\) - the exact values of \(\beta_i\) are determined from equation (E-7).
E.3 \textbf{LIMITING SOLUTION AS $\kappa \to 0$}

The principal asymptotic expansions of $J_\nu(z)$ and $Y_\nu(z)$ for large arguments ($z \to \infty$) are given by\(^{30}\)

\[
J_\nu(z) = \left( \frac{2}{\pi z} \right)^{\frac{1}{2}} \cos \left( z - \frac{\nu \pi}{2} - \frac{\pi}{4} \right), \\
Y_\nu(z) = \left( \frac{2}{\pi z} \right)^{\frac{1}{2}} \sin \left( z - \frac{\nu \pi}{2} - \frac{\pi}{4} \right). 
\]

(E-9)

\[
\begin{align*}
J_\nu(z) & = \left( \frac{2}{\pi z} \right)^{\frac{1}{2}} \cos \left( z - \frac{\nu \pi}{2} - \frac{\pi}{4} \right), \\
Y_\nu(z) & = \left( \frac{2}{\pi z} \right)^{\frac{1}{2}} \sin \left( z - \frac{\nu \pi}{2} - \frac{\pi}{4} \right).
\end{align*}
\]

Hence, as $\kappa \to 0$, equations (E-7) and (E-8) reduce to

\[
- \frac{\kappa}{\pi \beta_1} (1-\kappa)^{\frac{1}{2}} \cos \left\{ 2\beta_1 \left[ 1 - \sqrt{1-\kappa} \right] \right\} = 0, \\
\]

(E-10)

and

\[
\rho_1 = A \left( \frac{\kappa}{\pi \beta_1} \right)^{\frac{1}{2}} (1-\kappa)^{-\frac{1}{2}} \csc \left\{ \frac{2\beta_1}{\kappa} - \frac{\pi}{4} \right\} \sin \left\{ 2\beta_1 \left[ 1 - \sqrt{1-\kappa} \right] \right\}. \\
\]

(E-11)

Thus, when $\kappa$ is small

\[
\rho_1 = A \sin \beta_1 z_*,
\]

where

\[
\beta_1 = (2l-1)\frac{\pi}{2} \quad \text{since} \quad \cos \beta_1 = 0.
\]

(E-12)

(E-13)

E.4 \textbf{LIMITING SOLUTION AS $\kappa \to 1$}

The limiting forms of $J_\nu(z)$, $J_1(z)$, $Y_\nu(z)$, and $Y_1(z)$ for small arguments ($z \to 0$) are given by\(^{30}\)

\[
J_0(z) = 1, \quad J_1(z) = z/2, \\
Y_0(z) = \frac{2}{\pi} \ln z, \quad \text{and} \quad Y_1(z) = -\frac{2}{\pi z}.
\]

(E-14)

Hence, as $\kappa$ approaches unity, equation (E-7) reduces to

\[
- \frac{\kappa}{\pi \beta_1} J_0 \frac{2\beta_1}{\kappa} - \beta_1 (1-\kappa) Y_0 \frac{2\beta_1}{\kappa} = 0.
\]

(E-15)

Therefore, in the limit as $\kappa \to 1$

\[
J_0 \frac{2\beta_1}{\kappa} = 0.
\]

(E-16)

An alternative form of equation (E-8) is, from equations (E-2) and (E-6)

\[
\rho_1 = A \left\{ J_0 \left( \frac{2\beta_1}{\kappa} \right) \sqrt{(1-\kappa)z_*} - \frac{J_1 \left( \frac{2\beta_1}{\kappa} \right) \sqrt{(1-\kappa)} Y_0 \left( \frac{2\beta_1}{\kappa} \right)}{Y_1 \left( \frac{2\beta_1}{\kappa} \sqrt{(1-\kappa)z_*} \right)} \right\}. \\
\]

(E-17)

Again, as $\kappa$ approaches unity, equation (E-17) tends to

\[
\rho_1 = A \left\{ J_0 \left( \frac{2\beta_1}{\kappa} \sqrt{(1-\kappa)z_*} \right) + \frac{\pi \beta_1^2}{\kappa^2} (1-\kappa) Y_0 \left( \frac{2\beta_1}{\kappa} \sqrt{(1-\kappa)z_*} \right) \right\}.
\]

(E-18)
Thus, the limiting solution as $\kappa \rightarrow 1$ is given by

$$\rho_1 = A J_0 (2\beta_1 \sqrt{(1-z_\kappa)}), \quad \text{(E-19)}$$

where $\beta_1$ is the $i$th root of the transcendental equation $J_0 (2\beta_1) = 0$.

It is interesting to note that the first derivative of equation (E-19) is

$$\rho_1' = A \frac{\beta_1}{\sqrt{(1-z_\kappa)}} J_1 (2\beta_1 \sqrt{(1-z_\kappa)}). \quad \text{(E-20)}$$

The limiting form for $J_1 (2\beta_1 \sqrt{(1-z_\kappa)})$ for $z_\kappa \rightarrow 1$ is $\beta_1 \sqrt{(1-z_\kappa)}$. Hence, when $\kappa = 1$ (this is the only value of $\kappa$ for which $\rho_1'(1) \neq 0$)

$$\rho_1'(1) = \beta_1^2 A. \quad \text{(E-21)}$$

However, the boundary condition $(1-\kappa)\rho_1'(1) = 0$ is satisfied because $1-\kappa$ is zero.

### E.5 TABULATED VALUES

The first five exact values of $\beta_1$ for values of $\kappa$ equal to 0, $\frac{1}{4}$, and $\frac{1}{2}$, and 1, are presented in Table E-1.

#### TABLE E-1

<table>
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<tr>
<th>$\kappa$</th>
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<th>$\frac{1}{2}$</th>
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REFERENCES


