Solving Variable Coefficient Partial Differential Equations Using The Boundary Element Method

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Abstract

The coefficients in a mathematical model of a physical problem typically correspond to the material parameters of the problem. In heterogeneous media the material parameters may vary with position, dependent variable value and/or time. The governing equation of a physical problem in heterogeneous media is therefore likely to involve variable coefficients. For this reason the solution of variable coefficient partial differential equations (PDEs) is an important engineering problem.

In this thesis ways of solving linear variable coefficient PDEs using the boundary element method have been investigated. The application of the boundary element method to these equations is hampered by the difficulty of finding a fundamental solution. In the literature several methods have been proposed to overcome this problem. A survey of these methods has been undertaken in this study from which it is concluded that the most promising approach is the dual reciprocity boundary element method (DR-BEM).

The DR-BEM is tested in this thesis for a range of elliptic variable coefficient PDEs. The results of these test problems indicate that the DR-BEM is a promising method for solving elliptic variable coefficient PDEs. However, in some cases, such as problems in highly heterogeneous media, it is found that a large number of internal solution nodes are necessary to ensure accurate results. This can make the DR-BEM computationally expensive. Some new approaches for improving the efficiency of the DR-BEM are proposed. For problems in highly heterogeneous media a subregion approach is recommended.

The use of the DR-BEM for linear parabolic variable coefficient PDEs is also investigated. It is found that by combining the DR-BEM with the coupled finite difference - boundary element method a wide range of parabolic problems can be solved without requiring domain integration. This time-stepping approach can become expensive for variable coefficient PDEs (particularly for large-time solutions) as it requires the solution of a large number of associated elliptic problems with large numbers of internal nodes. Also, it is found that for some problems in highly heterogeneous media the error at each time-step can accumulate leading to poor large-time solutions.

To avoid these limitations semi-analytic approaches for solving parabolic equations are investigated. A new semi-analytic method - the separation of variables dual reciprocity method (SOV-DRM) - is proposed which constructs the solution as an eigenfunction expansion. The eigenvalues and eigenvectors are determined using the DR-BEM. This method allows parabolic problems to be solved without requiring time-stepping or domain integration. This method is found to produce accurate results for a range of problems including some problems involving heterogeneous media.

Two other semi-analytic methods are also investigated. These methods are implemented and compared with the SOV-DRM. It is concluded that each method has specific strengths and weaknesses and that the choice of method is largely problem dependent.

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Contents

List of Symbols

$\{\alpha\}$	Coefficients in DR-BEM global shape function (defined on Page 44)
Δt	Time step length
δ	Dirac delta function (defined on Page 7)
γ	Inhomogenous function, $Lu = \gamma$
Γ	Boundary of problem domain
λ	Eigenvalue
Ω	Problem domain
ω	Fundamental solution (defined on Page 7)
ϕ	Time-scheme parameter
ξ	Source point of fundamental solution
a	Separation constant
b	General forcing function
$\{b\}$	Vector of nodal values of b
$[C]$	DR-BEM matrix for parabolic equations, $[C] = -[S][K]$
E_a	Absolute error (defined on Page 62)
E_r	Relative error (defined on Page 62)
f	DR-BEM approximating function (discussed in Section 4.2)
$[F]$	Matrix of nodal values of f
$[H],[G]$	BEM influence coefficient matrices (defined on Page 11)
I	Number of internal nodes
$[K]$	DR-BEM matrix (defined on Page 54)
L	Linear operator
N	Number of boundary nodes
n	Unit outward normal
q	Normal derivative of dependent variable ($q = \frac{\partial u}{\partial n}$)
$\hat{q}$	Normal derivative of DR-BEM particular solution ($\hat{q} = \frac{\partial \hat{u}}{\partial n}$)
$[\hat{Q}]$	Matrix of nodal values of $\hat{q}$

r	Euclidean distance between two points
$[R]$	DR-BEM matrix (defined in Section 4.4)
S	Eigenfunction
$\{S\}$	Eigenvector
$[S]$	DR-BEM matrix, $[S] = \left([H]\left[\hat{U}\right] - [G]\left[\hat{Q}\right]\right)[F]^{-1}$
t	Time
u	Dependent variable
$\hat{u}$	DR-BEM particular solution (discussed in Section 4.2)
$\left[\hat{U}\right]$	Matrix of nodal values of $\hat{u}$
$\mathbf{x}$	Spatial location, $\mathbf{x} = (x, y, z)$

List of Abbreviations

AI-DRM	Analytic Integration DR-BEM (see Section 8.1)
BEM	Boundary Element Method (see Chapter 2)
DR-BEM	Dual Reciprocity Boundary Element Method (see Chapter 4)
FD-BEM	Finite Difference BEM (see Section 6.1.1)
FD-DRM	Finite Difference DR-BEM (see Section 6.3)
FDR-BEM	Fourier Expansion DR-BEM (see Section 4.2.2)
LFD-DRM	FD-DRM based on Laplace operator (see Section 6.3)
LT-DRM	Laplace Transform DR-BEM (see Section 8.2)
MHFD-DRM	FD-DRM based on Modified Helmholtz operator (see Section 6.3)
MRM	Multiple Reciprocity Method (see Section 3.6.6)
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
SOV-DRM	Separation Of Variables DR-BEM (see Chapter 7)
SRM	Secondary Reduction Method (see Section 3.6.4)