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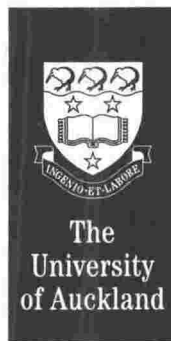
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Solving Variable Coefficient Partial Differential Equations Using The Boundary Element Method

**A thesis submitted in partial fulfilment of the requirements for the
degree of Doctor of Philosophy at the University of Auckland**

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Abstract

The coefficients in a mathematical model of a physical problem typically correspond to the material parameters of the problem. In heterogeneous media the material parameters may vary with position, dependent variable value and/or time. The governing equation of a physical problem in heterogeneous media is therefore likely to involve variable coefficients. For this reason the solution of variable coefficient partial differential equations (PDEs) is an important engineering problem.

In this thesis ways of solving linear variable coefficient PDEs using the boundary element method have been investigated. The application of the boundary element method to these equations is hampered by the difficulty of finding a fundamental solution. In the literature several methods have been proposed to overcome this problem. A survey of these methods has been undertaken in this study from which it is concluded that the most promising approach is the dual reciprocity boundary element method (DR-BEM).

The DR-BEM is tested in this thesis for a range of elliptic variable coefficient PDEs. The results of these test problems indicate that the DR-BEM is a promising method for solving elliptic variable coefficient PDEs. However, in some cases, such as problems in highly heterogeneous media, it is found that a large number of internal solution nodes are necessary to ensure accurate results. This can make the DR-BEM computationally expensive. Some new approaches for improving the efficiency of the DR-BEM are proposed. For problems in highly heterogeneous media a subregion approach is recommended.

The use of the DR-BEM for linear parabolic variable coefficient PDEs is also investigated. It is found that by combining the DR-BEM with the coupled finite difference - boundary element method a wide range of parabolic problems can be solved without requiring domain integration. This time-stepping approach can become expensive for variable coefficient PDEs (particularly for large-time solutions) as it requires the solution of a large

number of associated elliptic problems with large numbers of internal nodes. Also, it is found that for some problems in highly heterogeneous media the error at each time-step can accumulate leading to poor large-time solutions.

To avoid these limitations semi-analytic approaches for solving parabolic equations are investigated. A new semi-analytic method - the separation of variables dual reciprocity method (SOV-DRM) - is proposed which constructs the solution as an eigenfunction expansion. The eigenvalues and eigenvectors are determined using the DR-BEM. This method allows parabolic problems to be solved without requiring time-stepping or domain integration. This method is found to produce accurate results for a range of problems including some problems involving heterogeneous media.

Two other semi-analytic methods are also investigated. These methods are implemented and compared with the SOV-DRM. It is concluded that each method has specific strengths and weaknesses and that the choice of method is largely problem dependent.

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Contents

Abstract	ii
Acknowledgements	iv
Contents	v
List of Symbols	x
List of Abbreviations	xii
1 Introduction	1
2 The Boundary Element Method	5
2.1 Boundary Integral Equation Formulation	5
2.2 Numerical Implementation of the BEM	9
3 Solving Variable Coefficient PDEs Using the BEM	13
3.1 Analytic Methods for Finding a Fundamental Solution	14
3.2 Variable Transformation Methods	17

3.3	Numerical Evaluation of a Fundamental Solution	21
3.4	Piecewise Approximation of a Variable Coefficient PDE	21
3.5	Applying the BEM with Domain Integration	24
3.6	Converting Domain Integrals to Boundary Integrals	28
3.6.1	Removing Domain Integrals due to Inhomogeneous Terms	28
3.6.2	The Dual Reciprocity Boundary Element Method	30
3.6.3	The Particular Integral Method	32
3.6.4	The Secondary Reduction Method	34
3.6.5	The Perturbation Boundary Element Method	35
3.6.6	The Multiple Reciprocity Method	37
3.7	Summary	40
4	The Dual Reciprocity Boundary Element Method	43
4.1	Equation Derivation	43
4.2	The Approximating Function f	46
4.2.1	Radial Basis Functions	46
4.2.2	The Fourier Expansion Method	49
4.3	Inhomogeneous Equations	50
4.4	Elliptic Problems	51
4.5	Transient Problems	55
4.6	Exterior Problems	56

4.7	Nonlinear Problems	57
4.8	The DR-BEM Using Other Operators	58
5	The DR-BEM for Elliptic PDEs	60
5.1	Applying the DR-BEM for Variable Coefficient PDEs	60
5.2	Test Problems	62
5.3	Avoiding Matrix Inversion with the DR-BEM	66
5.4	Constructing an Internal Solution	70
5.5	Using Subregions with the DR-BEM	72
5.6	Summary	77
6	The DR-BEM for Parabolic PDES	78
6.1	Boundary Element Methods for Parabolic Equations	78
6.1.1	Coupled Finite Difference - Boundary Element Method	79
6.1.2	Direct Time-Integration Method	81
6.1.3	Coupling these Methods with the DR-BEM	82
6.2	A Coupled Direct Time-Integration DR-BE Method	83
6.3	A Coupled Finite Difference DR-BE Method	84
6.3.1	The FD-DRM Using the Laplace Operator	85
6.3.2	The FD-DRM Using the Modified Helmholtz Operator	89
6.4	The DR-BEM for Parabolic Equations	92

6.5	Test Problems	94
6.6	Comparison Between Time-Stepping Methods	109
7	The Separation of Variables DR-BEM	111
7.1	The SOV-DRM for the Diffusion Equation	112
7.1.1	The Separation of Variables Method	112
7.1.2	Eigenvalue Analysis Using the DR-BEM	116
7.1.3	Evaluation of the Separation Constants	120
7.1.4	Calculation of Flux Values	122
7.1.5	Treatment of Nonhomogeneous Boundary Conditions	151
7.2	The SOV-DRM for Other Parabolic Problems	154
7.2.1	Convection-Diffusion Problems	154
7.2.2	Variable Diffusivity	171
7.2.3	Time-Dependent Boundary Conditions	175
7.3	Summary	176
8	Alternative Semi-Analytic Methods for Parabolic Problems	179
8.1	Analytic Integration Method	179
8.1.1	Equation Development	179
8.1.2	Numerical Application	183
8.1.3	Test Problems	185

8.2	Laplace Transform Method	190
8.2.1	Equation Development	190
8.2.2	Laplace Transform Inversion	192
8.2.3	Test Problems	194
8.3	Comparison of these Semi-Analytic Approaches	198
8.3.1	Diffusion Problems	198
8.3.2	A Constant Coefficient Convection-Diffusion Problem	202
8.3.3	A Variable Coefficient Convection-Diffusion Problem	202
8.3.4	Observations	204
9	Conclusions and Future Work	206
	Bibliography	213

List of Symbols

$\{\alpha\}$	Coefficients in DR-BEM global shape function (defined on Page 44)
Δt	Time step length
δ	Dirac delta function (defined on Page 7)
γ	Inhomogenous function, $Lu = \gamma$
Γ	Boundary of problem domain
λ	Eigenvalue
Ω	Problem domain
ω	Fundamental solution (defined on Page 7)
ϕ	Time-scheme parameter
ξ	Source point of fundamental solution
a	Separation constant
b	General forcing function
$\{b\}$	Vector of nodal values of b
$[C]$	DR-BEM matrix for parabolic equations, $[C] = -[S][K]$
E_a	Absolute error (defined on Page 62)
E_r	Relative error (defined on Page 62)
f	DR-BEM approximating function (discussed in Section 4.2)
$[F]$	Matrix of nodal values of f
$[H], [G]$	BEM influence coefficient matrices (defined on Page 11)
I	Number of internal nodes
$[K]$	DR-BEM matrix (defined on Page 54)
L	Linear operator
N	Number of boundary nodes
n	Unit outward normal
q	Normal derivative of dependent variable ($q = \frac{\partial u}{\partial n}$)
\hat{q}	Normal derivative of DR-BEM particular solution ($\hat{q} = \frac{\partial \hat{u}}{\partial n}$)
$[\hat{Q}]$	Matrix of nodal values of \hat{q}

r	Euclidean distance between two points
$[R]$	DR-BEM matrix (defined in Section 4.4)
S	Eigenfunction
$\{S\}$	Eigenvector
$[S]$	DR-BEM matrix, $[S] = ([H] [\hat{U}] - [G] [\hat{Q}]) [F]^{-1}$
t	Time
u	Dependent variable
\hat{u}	DR-BEM particular solution (discussed in Section 4.2)
$[\hat{U}]$	Matrix of nodal values of \hat{u}
\mathbf{x}	Spatial location, $\mathbf{x} = (x, y, z)$

List of Abbreviations

AI-DRM	Analytic Integration DR-BEM (see Section 8.1)
BEM	Boundary Element Method (see Chapter 2)
DR-BEM	Dual Reciprocity Boundary Element Method (see Chapter 4)
FD-BEM	Finite Difference BEM (see Section 6.1.1)
FD-DRM	Finite Difference DR-BEM (see Section 6.3)
FDR-BEM	Fourier Expansion DR-BEM (see Section 4.2.2)
LFD-DRM	FD-DRM based on Laplace operator (see Section 6.3)
LT-DRM	Laplace Transform DR-BEM (see Section 8.2)
MHFD-DRM	FD-DRM based on Modified Helmholtz operator (see Section 6.3)
MRM	Multiple Reciprocity Method (see Section 3.6.6)
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
SOV-DRM	Separation Of Variables DR-BEM (see Chapter 7)
SRM	Secondary Reduction Method (see Section 3.6.4)