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On Vector Equilibria, Vector Optimisation and Vector Variational Inequalities

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Abstract

It is well-known that, under certain conditions, network equilibrium, optimisation and variational inequality problems are equivalent. Hence, solution algorithms to solve any of the three problems can be used to solve the other problems. Vector network equilibrium (VEQ) problems lead to analogous definitions of vector optimisation (VOP) and vector variational inequality (VVI) problems. Investigating whether a similar equivalence exists in the vector valued case suggests itself, in particular to derive solution algorithms for VEQ. Unfortunately, the three problems are no longer equivalent in the vector valued case. We show under which assumptions a solution of VOP solves VEQ. Even though a solution of VVI is a solution of VEQ, the converse is not true. We demonstrate structural properties of solutions of VEQ that prevent them from being solutions of VVI and show under which assumptions VVI and VEQ are equivalent. We also report on numerous erroneous or inconsistent results within the related literature.

Key words: Vector Equilibrium; Vector Variational Inequality; Vector Optimisation; Traffic Assignment.

1 Introduction

The traffic assignment (TA) problem (e.g. Ortúzar and Willumsen 2002), which models the route choice of network users in a road network, is a prominent equilibrium problem (EQ). This network equilibrium problem has been formulated as optimisation (OP) (e.g. Patriksson 1994) and variational inequality problem (VI) (e.g. Nagurney 1993). Yang and Goh (1997) suggest that a vector equilibrium problem (VEQ) can be used to model the TA problem where network users may have multiple objectives such as finding a minimal travel time path while also paying minimal tolls and / or choosing the safest route (see also Wang...
et al. 2008). Here is where our interest in solving the vector equilibrium problem originates.

In the single-objective case there exists a strong link between the three problems EQ, OP, and VI (and also other problems that are omitted here) as discussed in Section 2. Attempts were made in the literature to establish similar links between VEQ, VOP, and VVI (the vector valued extensions of OP and VI). We introduce the three different problems in Section 3 and report on findings in the literature on this topic in Section 4. We then indicate some erroneous published results in Section 5. In Section 6 we briefly report on the relationship between VEQ and VOP. Then, we confirm that there are significant differences between solutions of a VEQ problem and those of the corresponding VVI problem in Section 7, which indicates that solving VEQ by solving VVI is not possible.

2 The Single-objective Case – Network Equilibrium, Optimisation, and Variational Inequality Problems

In the following, it is assumed that $\mathcal{K}$ is a closed and convex subset of $\mathbb{R}^l$ and there exists a (cost) function $c: \mathcal{K} \mapsto \mathbb{R}^l$.

The variational inequality problem is the problem of finding $f \in \mathcal{K}$ that satisfies

$$c(f)^\top (h - f) \geq 0 \quad \text{for all } h \in \mathcal{K}. \quad (1)$$

This VI has an equivalent optimisation problem (Nagurney 1993, Theorem 1.1). It is assumed that $c$ is continuously differentiable on $\mathcal{K}$ and that its Jacobian matrix $\nabla c$ is symmetric and positive semi-definite. Then there exists a real-valued convex function $B: \mathcal{K} \mapsto \mathbb{R}$ with $\nabla B(f) = c(f)$ such that a solution $f$ of VI (1) is also a solution of the optimisation problem

$$\min \quad B(f)$$
$$\text{s.t.} \quad f \in \mathcal{K}. \quad (2)$$

To define the network equilibrium problem, we need to be more specific. Let $G = (\mathcal{V}, \mathcal{A})$ be a network with a set of $n$ nodes $\mathcal{V}$ and $m$ arcs $\mathcal{A}$, and let $r$ be a path in $G$. Function $c$ assigns to every path $r$ a cost $c_r$. Let $\mathcal{W} \subseteq \mathcal{V} \times \mathcal{V}$ be a set of origin-destination (OD) pairs and $d_w \geq 0$ for $w = (n_1, n_2) \in \mathcal{W}$ be the demand of flow from $n_1$ to $n_2$. For every OD pair $w$ there is a set of paths $\mathcal{R}_w$ connecting the two nodes and all paths are collected in the set $\mathcal{R} = \bigcup_{w \in \mathcal{W}} \mathcal{R}_w$. The flow on each path $r \in \mathcal{R}$ is denoted by $f_r$ and the vector of path flow is $f \in \mathbb{R}^l, l = |\mathcal{R}|$. We denote by $\Phi \in \mathbb{R}^{m \times l}$ the arc-path incidence matrix with elements $\phi_{ar}$. The value of $\phi_{ar}$ is one if arc $a$ is part of path $r$, and zero otherwise. Arc flow $\bar{f}$ is derived from path flow by $\bar{f} = \Phi f \in \mathbb{R}^m$. Flow on arc $a \in \mathcal{A}$ is denoted $f_a$. 2
Demand is satisfied if \( d_w = \sum_{r \in R_w} f_r \) for all \( w \in W \). \( K \) is the set of all demand feasible, non-negative path flow vectors and \( K_A \) denotes the corresponding set of arc flows. A flow vector is said to be in equilibrium if it satisfies the following: The feasible vector \( f^* \in K \) is called equilibrium flow if and only if

\[
\text{(EQ)} \quad \text{for all } w \in W \text{ and for all } r, s \in R_w \quad c_r(f^*) < c_s(f^*) \Rightarrow f^*_s = 0.
\]

The interpretation of EQ is that flow is only sent along paths with minimal cost.

In the TA problem, where \( G \) is a road network and \( c \) is a travel time function, every user wants to choose their minimal travel time path. But as more travellers use a road it becomes congested and its travel time increases and so does the time of every route using this road. Now every network user aims to minimise their own travel time according Wardrop’s first principle (Wardrop 1952).

VI (1) is an equivalent reformulation of EQ (Nagurney 1993). A VI that represents EQ can be formulated in terms of path flow such as (1) but also in terms of arc flow:

\[
\bar{c}(\bar{f})^T (\bar{h} - \bar{f}) \geq 0 \quad \text{for all } \bar{h} \in K_A,
\]

where arc cost function \( \bar{c}_a \) assigns a cost to every arc \( a \). Smith (1979) shows that the path based and the arc based VI (1) and (4) are equivalent if the path costs are additive, i.e. if \( c(f) = \Phi^T \bar{c}(\Phi f) \).

The optimisation problem corresponding to VI (1) is given in (2). This is a path-based formulation, which is not favourable for solving EQ due to the fact that the assumptions on the Jacobian \( \nabla c \) are too strong to be satisfied by a typical EQ problem such as TA. In EQ, as the flow on a path \( r \) changes, this will typically affect the costs of other paths that share arcs with \( r \) and can violate the assumption that \( \nabla c \) is symmetric whenever \( \frac{\partial c(f_r)}{\partial f_s} \neq \frac{\partial c(f_s)}{\partial f_r} \) for some paths \( r \) and \( s \).

Based on the common assumption that arc cost functions \( \bar{c}_a \) are additive and depend only on the flow on arc \( a \), the equilibrium problem EQ (3) can be equivalently solved via an optimisation problem (Beckmann et al. 1956):

\[
\text{(OP)} \quad \min \sum_{a \in A} \int_{0}^{f_a} \bar{c}_a(v) dv \\
\text{s.t. } f \in K_A.
\]

This equivalence as well as existence of a solution of (5) is guaranteed by making the assumptions that for all OD pairs \( w \in W \) it holds that \( |R_w| \geq 1, \ d_w \geq 0 \), and that \( \bar{c}_a \) is positive, continuous, and separable (i.e. arc cost depends on the flow on the arc only). Uniqueness of the (arc flow) solution follows from the additional assumptions that for all \( w \in W \) \( d_w > 0 \) and that \( \bar{c}_a(f_a) \) are strictly increasing (Patriksson 1994), which is a common assumption in TA as arc cost (or time) increases with increasing flow on an arc, as more travellers mean more congestion which leads to higher cost (or time).
Both VI formulations are more general than the optimisation formulation (5) as there are fewer underlying assumptions on arc cost functions: for VI the functions \( \bar{c}(f), c(f) \) may depend on the flow in the entire network. As such a VI formulation is very general (and equivalent to EQ), it is often given as definition of the traffic assignment equilibrium problem (e.g. Oettli 2001) instead of the definition of EQ (3). Due to the equivalence of EQ, OP, and VI, algorithms to solve EQ can be based on either formulation, and several such algorithms have been developed (summaries can be found in Nagurney 1993; Patriksson 1994).

3 Vector Equilibria, Vector Optimisation and Vector Variational Inequalities

We define some order relations on \( \mathbb{R}^p \). Let \( y^1, y^2 \in \mathbb{R}^p \). We write

\[
\begin{align*}
y^1 \geq y^2 & \iff y^1_k \geq y^2_k \text{ for } k = 1, 2, \ldots, p, \\
y^1 \geq y^2 & \iff y^1_k \geq y^2_k \text{ for } k = 1, 2, \ldots, p; \text{ and } y^1 \neq y^2, \\
y^1 > y^2 & \iff y^1_k > y^2_k \text{ for } k = 1, 2, \ldots, p.
\end{align*}
\]

The symbols \( \leq, \leq, < \) are defined analogously. Furthermore, for any of the symbols \( \succ \in \{\geq, \geq, >\} \), we define \( \mathbb{R}^p_\succ \) as \( \mathbb{R}^p_\succ := \{x \in \mathbb{R}^p : x \succ 0\} \). Then, \( y^1 \succ y^2 \iff y^1 - y^2 \in \mathbb{R}^p_\succ \) and \( y^1 \prec y^2 \iff y^1 - y^2 \notin \mathbb{R}^p_\succ \).

It is now assumed that there is more than one objective function that determines path choice within the network given by a matrix-valued function \( C : \mathbb{R}^l \mapsto \mathbb{R}^{p \times l} \). The function \( C_r : \mathbb{R} \mapsto \mathbb{R}^p \) is the \( p \)-dimensional path cost function for path \( r \). Efficient paths are all those paths \( r \) such that there does not exist any other path with path cost vector that dominates that of path \( r \), i.e. there does not exists \( s \) with \( C_s(f^*) \leq C_r(f^*) \).

A solution vector \( h \in \mathcal{K} \) is in vector equilibrium if all dominated paths have zero flow and only efficient paths may have positive flow. A flow vector \( f^* \in \mathcal{K} \) is said to be in vector equilibrium (VEQ) if and only if

\[(\text{VEQ}) \quad \text{for all } w \in \mathcal{W} \text{ and for all } r, s \in \mathcal{R}_w \quad C_s(f^*) \geq C_r(f^*) \Rightarrow f^*_s = 0.\]

This vector equilibrium problem VEQ appeared in (Chen and Yen 1993; Yang and Goh 1997; Goh and Yang 1999; Chen et al. 1999; Yang and Goh 2000; Khan and Raciti 2005; Yang and Yu 2005; Li et al. 2007, 2008). Given the equivalence of EQ, optimisation problem (5), and VI in the single-objective case we are interested in similar relationships between VEQ and related vector valued problems.

The vector valued extension of a VI is known as vector variational inequality (VVI) problem, first introduced in Gianessi (1980). The VVI problem is the problem of finding \( f \in \mathcal{K} \) that satisfies

\[(\text{VVI}) \quad C(f)(h - f) \not\leq 0, \text{ for all } h \in \mathcal{K}.\]

Similar to the arc flow based VI problem (4), we can also define a VVI based on arc flow. The path flow and the arc flow based VVI are equivalent problems.
provided that the path cost function is additive, which can be shown similar to Smith (1979).

For the optimisation problem the assumptions that path cost functions $\tilde{C}_a$ are additive and depend only on flow on arc $a$ are made. Then, the vector optimisation problem (VOP) related to VEQ is defined analogously to OP (5):

$$(VOP) \quad \min \quad z(\tilde{f}) = \begin{cases} z_1(\tilde{f}) = \sum_{a \in A} \tilde{f}_a \int_0^{\tilde{f}_a} \tilde{c}_a^1(v) dv \\ \vdots \\ z_p(\tilde{f}) = \sum_{a \in A} \tilde{f}_a \int_0^{\tilde{f}_a} \tilde{c}_a^p(v) dv \end{cases}$$

s.t. $\tilde{f} \in K_A$.

By solutions of VOP we mean the set of efficient solutions of this vector valued optimisation problem with objective vector $z$.

There also exist weak versions of VEQ, VOP, and VVI that are obtained by replacing $\leq, \geq$ by $<, >$. The corresponding weak problems are denoted WVEQ, WVOP, and WVVI. For instance, dominance in VEQ and VOP becomes weak dominance.

4 Literature

The literature on VEQ and related problems is of mainly theoretical nature. Many articles discuss existence of solutions and attempt to relate (W)VEQ to other problems such as (W)VVI and (W)VOP.

Before we discuss the literature on VEQ, VVI, and VOP, we need to introduce more notation. Given a feasible flow solution $f \in K$, the following denote the set of (non-dominated) path cost vectors and the set of efficient paths.

**Definition 4.1** For a feasible $f \in K$, the set of all path cost vectors for OD pair $w$ is given by

$$Z_w(f) = \{ z \in \mathbb{R}^p : \text{there exists } r \in R_w \text{ with } z = C_r(f) \}.$$  

We define the set of non-dominated points for an OD pair $w \in W$ by

$$Z^*_w(f) = \{ z \in Z^w(f) : z \text{ is non-dominated in } Z^w(f) \},$$

while the corresponding set of efficient paths is

$$X^*_w(f) = \{ r \in R_w : \text{there exists } z \in Z^*_w(f) \text{ with } z = C_r(f) \}.$$  

The concept of VEQ was first introduced in a report by Chen and Yen (1993). A special VVI $(C(f)(h - f) \not\geq 0$, for all $h \in K)$ and VEQ are shown to be equivalent problems given that there is only a single efficient solution for each OD pair.

5
Yang and Goh (1997) apply the VEQ principle to the TA problem with multiple objectives. It is shown that a solution of VVI is a solution of VEQ, and also that a solution of WVVI is a solution of WVEQ. In the remainder of their paper, VVI, WVVI, VOP, and WVOP are related to each other but none of the different formulations can be shown to be equivalent.

Goh and Yang (1999) aim at establishing a link between VEQ and VOP. Given a scalarised VI$_{\xi}$ where $f \in \mathcal{K}$ is a solution of VI$_{\xi}$ if $f$ satisfies

$$(VI_{\xi}) \quad \left(\sum_{i=1}^{p} \xi_i C_i(f)\right) (h - f) \geq 0, \quad \text{for all } h \in \mathcal{K},$$

with $\xi \in \mathbb{R}_+^p$. They show that the obtained solution of VI$_{\xi}$ with $\xi \in \mathbb{R}_+^p$ satisfies VEQ. They also propose the re-formulation of VI$_{\xi}$ as a parametric complementarity problem. They show under which conditions a properly efficient solution of VOP solves VEQ, see also Remark 6.1. They study the special case of separable, affine, and monotone arc cost functions. As shown by Li et al. (2006), only parts of the results in Goh and Yang (1999) are correct, in particular VEQ and VVI are not equivalent problems. It should be noted that incorrectness of Theorem 2.1.(i) in Goh and Yang (1999) entails incorrectness of Theorems 3.2.(i) and 3.3. The corresponding Theorems in Yang and Goh (2000, Theorems 4(i),7,9), Chen et al. (2005, Chapter 6), and Yang and Yu (2005, Proposition 5.2) are also incorrect.

Chen et al. (1999) apply a non-linear scalarisation function $\xi_{ea}$ to a VVI and to the VEQ principle:

$$\xi_{ea} = \min \{t \in \mathbb{R} : y \in a + te - \mathcal{K}\},$$

where $K$ is a closed, convex, and pointed cone. They relate the scalarised VEQ to a scalarised VVI. They also show that solutions satisfying this (non-linear) scalarised VEQ principle also satisfy VEQ, but unfortunately the converse claim is not true as shown in Li et al. (2006).

Yang and Goh (2000) develop the relationship between VVI, VOP, VEQ, their weak versions, and the scalarised versions based on orderings defined by a closed convex cone (rather than simply $\mathbb{R}_+^p$ as in previous papers).

Oettli (2001) studies TA in a general setting that allows for interpretation as capacitated equilibrium with user classes. User classes represent groups of network users that may each have different cost functions. The author takes a different approach to establishing an equivalence relation between vector variational inequalities and vector equilibria by simply modifying the equilibrium conditions to yield the desired equivalence. We comment on inconsistencies in the paper in Section 5.1.

Yang and Yu (2005) show existence of a solution of VVI, relate the problem to VOP, and present some so-called gap functions. A dynamic traffic equilibrium is introduced where flow and demand are time dependent. This is extended to a dynamic vector equilibrium for which sufficient conditions are given.
Konnov (2005) studies VVI with set-valued functions $C$. There is a section on the application of vector equilibrium principles and traffic assignment, where it is assumed that $C(f)$ always takes a single value. It is shown that a solution of WVVI is always a solution of WVEQ. A scalar reformulation of WVVI is given.

Khan and Raciti (2005) study a time-dependent vector equilibrium problem, where flow, capacity constraints, and demand may depend on time. The concept of (weak) vector equilibrium is formulated. Corresponding VVIs are also given, solutions of which are shown to solve the related (W)VEQ.

Chapter 6 of Chen et al. (2005) is on vector network equilibrium problems. Findings from previous papers are collected in a section on WVEQ, VEQ, and dynamic VEQ.

Li et al. (2006) show incorrectness of a weighted sum scalarisation based on VI$_{\xi}$ by Goh and Yang (1999) and a scalarisation based on the function $\xi_{ea}$ by Chen et al. (1999). Li et al. (2006) give correct re-formulations of those scalarisations, which give rise to the concept of weakened parametric (weighted sum) equilibrium and weak $\xi_{ea}$-equilibrium. They show that a solution of VEQ is always a solution according to the weakened parametric equilibrium concept. They omit an essential assumption for this, which we comment on in Section 5.2. They also show that a solution of the WVEQ is in weak $\xi_{ea}$-equilibrium. Unfortunately, the two corrected relationships do not seem to give rise to a solution algorithm, whereas the incorrect original versions of the theorems in Goh and Yang (1999) and Chen et al. (1999) reduced the problem to scalar equilibrium problems for which solution algorithms are well-known.

Cheng and Wu (2006) study an equilibrium problem where multiple products are transported through a network and each product (which can also be interpreted as user class) incurs different flow-dependent multi-objective costs. They initially attempt to formulate a single-objective equilibrium problem and relate it to a VI problem and then extend this approach to the multi-objective case. Both are erroneous, see Section 5.3.

Li et al. (2007) extend the concept of $\xi_{ea}$-equilibrium to vector equilibrium problems with path capacity constraints, elastic demand, and different user classes. First, a system of VVIs is introduced, a solution of which always satisfies (W)VEQ. It is shown that a feasible solution is in weak $\xi_{ea}$-equilibrium (see also Li et al. 2006) if and only if it satisfies WVEQ. They also show that a solution satisfying the original concept of (non-weak) $\xi_{ea}$-equilibrium (from Chen et al. 1999) does also satisfy VEQ, but not vice versa. It is furthermore shown that from a weak $\xi_{ea}$-equilibrium principle a VVI formulation can be derived, which is equivalent to WVEQ. For the (non-weak) $\xi_{ea}$-equilibrium a VVI is derived that implies WVEQ, but not vice versa.

Li et al. (2008) aim at relating VEQ with additional arc capacities to a multi-objective minimum cost flow problem. They aim to demonstrate that an optimal solution of the flow problem is always an optimal solution of their VEQ problem, which is incorrect as shown in Section 5.4. A generalisation of the VEQ concept to ensure equivalence of the two problems does also not yield the reported result.
5 Incorrect and Inconsistent Results

As mentioned above, some incorrect results can be found in the literature. Some of them have been reported before, whereas we provide counter examples to previously unreported false claims.

5.1 Inconsistencies in Oettli (2001)

Oettli (2001) studies EQ in a general setting that allows for interpretation as capacitated equilibrium with user classes. He aims to derive a vector equilibrium principle from a vector variational inequality and to establish equivalence. Hence the author takes a different approach to establishing an equivalence relation between vector variational inequalities and vector equilibria by simply modifying the equilibrium conditions to yield the desired equivalence. Oettli (2001, p.224) states

But rather the variational equilibrium, which remains invariant if one passes to the vectorial setting, should be the basic notion, and the definition of a Wardrop equilibrium must be adapted in each case.

Unfortunately, it remains unclear to what extent the variational inequalities (which variational equilibria are defined on) are invariant.

Oettli (2001) introduces an equilibrium problem with user classes and multiple objectives as \( f \in \mathcal{K} \) satisfying

\[
\sum_{r \in \mathcal{R}} C_r(f)(h_r - f_r) \in \mathbb{R}_{\geq} \quad \text{for all } h \in \mathcal{K},
\]  

(10)

where the vectors \( h_r \) and \( f_r \) are column vectors in \( \mathbb{R}^q \) and \( C_r(f) \) is a \( p \times q \) matrix. For \( q = 1 \) this is VI\(_\xi\) (8) with weighting factors \( \xi_i = 1 \). Oettli (2001) attempts to derive equilibrium conditions that are equivalent to (10).

The obtained equilibrium conditions model the problem with capacitated path flow. The aim is to find \( f \in \mathcal{K} \) that, for any fixed cost vector \( \tilde{C} = C(f) \), satisfies

\[
\tilde{C}_r - \tilde{C}_s \in -\mathbb{R}_{\geq} \Rightarrow f_s - l_s \notin \mathbb{R}_{>0} \quad \text{or} \quad f_r - u_r \notin -\mathbb{R}_{>0}.
\]  

(11)

Oettli (2001, Proposition 4.3) derives necessary and sufficient conditions for a solution \( f \in \hat{\mathcal{K}} \) of equilibrium conditions (11) also satisfying (10) with fixed cost vector \( \tilde{C} \). While the necessary condition is straight forward, the sufficient condition has strong assumptions. Unfortunately, they appear to be contradicting when \( p > 1 \). In the setting of the paper \( \mathcal{Y} \) is the space of linear mappings between \( \mathcal{X} \) (space of path flow) and \( \mathcal{Z} \) (objective space). Each cost vector \( C_r(f) \) is such a linear mapping. The convex ordering cone \( P_\mathcal{Y} \) of \( \mathcal{Y} \) is assumed to be pointed so that \( P_\mathcal{Y} \cap (-P_\mathcal{Y}) = \{0\} \). On the other hand, the proposition assumes that \( P_\mathcal{Y} \cup (-P_\mathcal{Y}) = \mathcal{Y} \). The two assumptions are trivially true for the single-objective problem with \( \mathcal{Z} = \mathbb{R} \) which makes the space of linear mappings the set of scalars \( \mathcal{Y} = \mathbb{R} \), and the ordering cone \( P_\mathcal{Y} = \mathbb{R}_{\geq} \).
If \( Z = \mathbb{R}^p, p > 1 \) and \( Y = \mathbb{R}^p \) it is not clear how to find such an ordering cone with \( P_Y \cap (-P_Y) = \{0\} \) and \( P_Y \cup (-P_Y) = Y \). A cone \( P_Y \subset \mathbb{R}^p \) is called acute if and only if its closure is contained in an open half-space \( H \) and the origin, i.e. \( \text{cl}(P_Y) \subset H \cup \{0\} \). The cone \( P_Y \) is acute if and only if \( \text{cl}(P_Y) \) is pointed, see Yu (1985). For an acute cone \( P_Y \) we have \( P_Y \cup (-P_Y) = Y \). If the cone \( P_Y \) is closed and pointed, then \( \text{cl}(P_Y) = P_Y \) and therefore it is also acute. Hence, \( P_Y \cup (-P_Y) = Y \) can not hold. A pointed open cone must be an open half-space together with the origin or an open subset thereof: \( P_Y \subseteq H \cup \{0\} \).

If the cone \( P_Y \) is open, it represents only weak dominance.

5.2 Correction of Theorem 2.4 in Li et al. (2006)

Li et al. (2006) identify errors in previous results and propose corrections. One of the corrections, Theorem 2.4 according to which a solution of VEQ is always a solution of a weakened parametric equilibrium is incomplete. Without the additional assumption that for all OD pairs all non-dominated path cost vectors are supported, i.e. that \( Z^w_N(f) \subset \text{bd} (\text{conv}(Z^w_N(f) + \mathbb{R}_e^p)) \), it is incorrect. The theorem is corrected by including this assumption. Otherwise, the counterexample given by Li et al. (2006) themselves can be used as a counterexample for their Theorem 2.4.

5.3 Incorrectness of Cheng and Wu (2006)

Cheng and Wu (2006) formulate a problem where multiple products traverse a network and each product incurs different multi-objective costs. Initially, the multi-product problem is considered with a single objective, i.e. \( p = 1 \). This can be interpreted as a network in which certain goods are produced by suppliers and need to be shipped to certain destination points. There also are warehouses, i.e. nodes through which products can be shipped, without staying there. There is a certain demand for every product. The cost of transporting different products along an arc may differ. Let \( q \) be the number of different products, then the amount of product \( j = 1, \ldots, q \) to be shipped between OD pair \( w \) is \( d^j_w \) and the flow of product \( j \) on arc \( a \) is \( f^j_a \). Similarly, the cost of product \( j \) on arc \( a \) is \( c^j_a \), and the corresponding cost on path \( r \) is \( c^j_r = \sum_{a \in r} c^j_a \).

Cheng and Wu (2006) define \( f^j_r \), the flow of product \( j \) on path \( r \in \mathcal{R}_w \), by

\[
 f^j_r = \min \{ \bar{f}^j_a : a \in r \}. \tag{12}
\]

Then, they state a flow satisfies the demand if

\[
 \sum_{r \in \mathcal{R}_w} f^j_r = d^j_w \quad \text{for all } w \in \mathcal{W}, j = 1, \ldots, q. \tag{13}
\]

Considering the following example, this definition seems to be counter-intuitive:

**Example 5.1** Consider the following network with the amount of flow of a single product, so \( q = 1 \), indicated next to each arc. The demand for this
product between OD pair $1 = (1, 5)$ is $d_{1}^{1} = 8$, so intuitively satisfied by the arc flow indicated in the network diagram.

For the OD pair $(1, 5)$, there are the following paths: $r_1 = (1, 2), (2, 3), (3, 5)$, $r_2 = (1, 2), (2, 3), (3, 4), (4, 5)$, $r_3 = (1, 3), (3, 5)$, and $r_4 = (1, 3), (3, 4), (4, 5)$. Evaluating (12) for the four paths yields

$$f_{r_1}^{1} = 3, \quad f_{r_2}^{1} = 3, \quad f_{r_3}^{1} = 3, \quad f_{r_4}^{1} = 5,$$

and therefore (13) yields $\sum_{r \in R_{w}} f_{r}^{1} = 14$, which does not satisfy $d_{1}^{1} = 8$. Instead of trying to derive path flow from arc flow as in (12), one should simply introduce a variable for the flow of every product $j$ on every path $r$. It is well known for TA problems that arc flow can be uniquely defined in terms of path flow, but not vice versa.

The set $K$ now denotes all demand feasible solutions. Cheng and Wu (2006) define $m_{w}^{j}(f)$ as minimum path cost of product $j$ between OD pair $w$, i.e. $m_{w}^{j} = \min_{r \in R_{w}} c_{r}^{j}(f)$. Both $c_{r}^{j}(f)$ and $m_{w}^{j}(f)$ are grouped into vectors by $c_{k}(f) = (c_{k}^{1}(f), c_{k}^{2}(f), \ldots, c_{k}^{q}(f))$ and $m_{w}(f) = (m_{w}^{1}(f), m_{w}^{2}(f), \ldots, m_{w}^{q}(f))^{\top}$.

Subsequently, an equilibrium flow pattern is introduced. A vector $v \in D$ is called equilibrium flow pattern if and only if

$$\text{for all } w \in W \text{ and for all } r \in R_{w} \quad c_{r}(f) - m_{w}(f) \begin{cases} = 0 & \text{if } f_{r} \in \mathbb{R}_{+}^{q}, \\ \geq 0 & \text{if } f_{r} = 0. \end{cases} \quad (14)$$

According to Cheng and Wu (2006, Proposition 2.1), (14) is equivalent to

$$c_{r}(f) - c_{s}(f) \in \mathbb{R}_{+}^{q} \Rightarrow f_{r} = 0 \quad \text{for all } w \in W \text{ and for all } r, s \in R_{w}. \quad (15)$$

Note that $f_{r} = (f_{r}^{1}, \ldots, f_{r}^{q})^{\top}$ here is the vector of path flow of products 1, . . . , $q$ on path $r$. We present a simple example to show that that (15) and (14) are not equivalent.

**Example 5.2** In this example we assume $q = 2$, $W = \{1 = (1, 2)\}$ and that there are exactly two paths connecting the OD pair $w = 1$, namely $r_1 = a_1$ and $r_2 = a_2$. Furthermore we assume $d_{1}^{1} = 1$ and $d_{1}^{2} = 1$. 

![Diagram](attachment:image.png)

1

$a_1$

2

$a_2$
For the two paths \( r_1 = a_1, r_2 = a_2 \) and the two products \( j = 1, 2 \) we assume the following costs:
\[
\begin{align*}
    c_{r_1}^1(f) &= 10(f_{a_1}^1 + f_{a_2}^1) \\
    c_{r_1}^2(f) &= 7(f_{a_2}^1 + f_{a_2}^2) \\
    c_{r_2}^1(f) &= 3(f_{a_1}^1 + f_{a_2}^2) \\
    c_{r_2}^2(f) &= 7(f_{a_1}^1 + f_{a_2}^2)
\end{align*}
\]
Choosing \( h_{r_1}^1 = 0, h_{r_1}^2 = 1, h_{r_2}^1 = 1, h_{r_2}^2 = 0 \), which is a demand feasible solution, we obtain
\[
c_{r_1}(h) = (10, 3)^\top, \quad c_{r_2}(h) = (7, 7)^\top, \quad m_1(h) = (7, 3)^\top.
\]
Clearly, the above solution \( h \) is not an equilibrium flow pattern according to (14) as
\[
c_{r_1}(h) - m_1(h) = (3, 0)^\top \geq 0 \quad \text{and} \quad c_{r_2}(h) - m_1(h) = (0, 4)^\top \geq 0,
\]
but both \( h_{r_2} \in \mathbb{R}_\geq^q \) and \( h_{r_2} \in \mathbb{R}_\geq^q \). The vector \( h \) does, however, satisfy (15) as both \( c_{r_1} - c_{r_2} \notin \mathbb{R}_\geq^q \) and \( c_{r_2} - c_{r_1} \notin \mathbb{R}_\geq^q \). In fact, an equilibrium flow pattern satisfying (14) can only exist if the cost for each product is minimal along the same path, so if there exists a unique optimal path. In Example 5.2 there exists no feasible solution satisfying the equilibrium flow pattern (14).

Cheng and Wu (2006, Theorem 2.1.) state the following: A vector flow \( f^* \in \mathcal{K} \) is an equilibrium flow pattern as in (14) if and only if \( f^* \) is a solution to the following vector variational inequality:
\[
\begin{align*}
\text{find } f^* \in \mathcal{K}, \ \text{s.t. } \ c(f^*)(h - f^*)^\top &\in \mathbb{R}^{q \times q}_\geq \quad \text{for all } h \in \mathcal{K}. \quad (16)
\end{align*}
\]
For \( l = |\mathcal{R}| \), the cost matrix is defined as \( c(f) := (c_1(f), \ldots, c_l(f)) \in \mathbb{R}^{q \times l} \), and the flow matrix by \( f := (f_1, \ldots, f_l) \in \mathbb{R}^{q \times l} \), where every \( c_r(f) \) and \( f_r \) is a column vector of class cost and class flow, respectively. Throughout the proof the term equilibrium flow pattern as defined in (15) is used (which is supposedly equivalent to the original definition (14)), so we demonstrate their claim is incorrect with this definition. Again, we use the feasible solution from Example 5.2, which satisfies (15), and show that the variational inequality (16) is not satisfied:

\[
\begin{align*}
    c(h)(u - h)^\top &= \begin{pmatrix} 10 & 7 \\ 3 & 7 \end{pmatrix} \begin{pmatrix} u_{r_1}^1 - 1 \\ u_{r_2}^2 - 1 \end{pmatrix} \\
    &= \begin{pmatrix} 10u_{r_1}^1 + 7u_{r_2}^1 - 7 \\ 3u_{r_1}^1 + 7u_{r_2}^2 - 7 \end{pmatrix} \begin{pmatrix} 10u_{r_2}^2 - 10 + 7u_{r_2}^2 \\ 3u_{r_1}^1 - 3 + 7u_{r_2}^2 \end{pmatrix}
\end{align*}
\]
Choosing the feasible solution \( \hat{u}_{r_1} = 1, \hat{u}_{r_2}^1 = 0, \hat{u}_{r_2}^2 = 0, \hat{u}_{r_2}^2 = 1 \), we obtain:
\[
    c(h)(\hat{u} - h)^\top = \begin{pmatrix} 3 & -3 \\ -4 & 4 \end{pmatrix} \notin \mathbb{R}^{q \times q}_\geq.
\]
More importantly in the context of this paper, Cheng and Wu (2006) consider the multi-product, multi-objective equilibrium problem defined similar to (15)
by extending scalar-valued cost functions to \( p \)-dimensional cost vectors for every product \( j \) (Cheng and Wu 2006, Definition 3.1): A vector \( f^* \in K \) is said to be an equilibrium pattern flow in the generalised context of the multi-product supply-demand network equilibrium problem with vector-valued cost function if and only if

\[
C_r(f^*) - C_s(f^*) \in \mathbb{R}^{q \times p} \geq 0 \quad \text{for all } w \in W \text{ and for all } r, s \in R_w. \quad (17)
\]

They claim that an equilibrium according to this is equivalent to a \( \xi_e \)-equilibrium pattern. The function \( \xi_e \) is just \( \xi_{ea} \) (9) without the \( a \) component: Given a fixed \( e \in \mathbb{R}^p \), \( \xi_e : \mathbb{R}^p \mapsto \mathbb{R} \) is (9) with \( a = 0 \) and \( K = \mathbb{R}^p \). The usage of \( \xi_e \) gives rise to the definition of \( \xi_e \)-equilibrium. This was attempted similarly by Chen et al. (1999), and shown to be incorrect by Li et al. (2006). A vector \( f^* \in K \) is said to be a \( \xi_e \)-equilibrium pattern flow in the vector valued network equilibrium problem for multiple products if there exists an \( e \in \text{int}(\mathbb{R}^p) \) such that (Cheng and Wu 2006, Definition 3.3):

\[
\xi_e \circ C_r(f^*) - \xi_e \circ C_s(f^*) \in \mathbb{R}^{q \times p} \geq 0 \quad \text{for all } w \in W \text{ and for all } r, s \in R_w. \quad (18)
\]

Despite their attempts to prove the contrary (Cheng and Wu 2006, Theorem 3.1), the two concepts (17) and \( \xi_e \)-equilibrium (18) are not equivalent. This can be shown by the same counterexample used by Li et al. (2006).

### 5.4 Incorrectness of Li et al. (2008)

Li et al. (2008) aim at relating a capacitated version of VEQ to a special multi-objective minimum cost flow problem defined in the following. Li et al. assume that there are lower and upper bounds \( \bar{l}, \bar{u} \) on arc flow, and limits for path flow on path \( r \) are derived as

\[
l_r = \max \{ \bar{l}_a \phi_{ar} : a \in \mathcal{A} \} \quad \text{and} \quad u_r = \max \{ \bar{u}_a \phi_{ar} : a \in \mathcal{A} \}. \quad (19)
\]

However, this does not guarantee that the resulting arc flow is within the bounds \( \bar{l}, \bar{u} \) as demonstrated in the following example.

**Example 5.3** Let the demand for the single OD pair \( (1,3) \) be 2 and the upper bound for flow on each arc \( 1 \), i.e. \( \bar{u} = (1, 1, 1, 1) \). There are three paths, \( r_1 = a_1, a_4 \), \( r_2 = a_2, a_4 \), and \( r_3 = a_3 \). According to (19), the upper bound for flow on each path is also 1, i.e. \( u = (1, 1, 1) \).
Clearly, the path flow solution \( f_{r_1} = 1, f_{r_2} = 1, f_{r_3} = 0 \) satisfies \( f \leq u \), but the resulting arc flow has \( \bar{f}_{a_4} = 2 \), which violates the upper bound on arc flow.

Li et al. (2008) formulate the following optimisation problem

\[
\begin{align*}
& \min \sum_{r \in \mathcal{R}} C_r(f_r) \\
& \text{s.t.} \quad l \leq f \leq u \quad \forall r \in \mathcal{R}, \quad \sum_{r \in \mathcal{R}_w} f_r = d_w \quad \forall w \in \mathcal{W},
\end{align*}
\]

(20)

where path cost is assumed to be additive, i.e. \( C_r(f_r) = \sum_{a \in \mathcal{A}} \overline{C}_a(f_a) \). Also, arc costs are assumed to be given by linear functions depending on arc flow. A feasible \( f^* \) is in capacitated vector equilibrium if and only if:

\[
\text{for all } r, s \in \mathcal{R}_w \quad C_{s-r}(f^*) \geq C_{t-r}(f^*) \Rightarrow f^*_s = l_s \text{ or } f^*_t = u_t.
\]

(21)

Li et al. (2008, Theorem 2.1) claim that an efficient solution of (20) always satisfies (21). Within the proof, the assumption that path cost functions are linear is made (without explicitly stating it), which generally does not follow from the fact that arc cost functions are linear. In Example 5.4 linear arc costs \( \overline{C}_a(f) \) do not lead to linear path costs \( C_r(f_r) \). The next example presents a network with cost functions where a solution of (20) does not satisfy (21).

**Example 5.4** We consider two OD pairs, (1,3) and (1,4) each with demand 1. The paths \( r_1 = a_1, a_3 \) and \( r_2 = a_2, a_3 \) connect the first OD pair and paths \( r_3 = a_1, a_4 \) and \( r_4 = a_2, a_4 \) connect the second one. We assume the lower bounds are zero and upper bounds are \( u_{r_i} > 1, i = 1, 2, 3, 4 \).

\[
\begin{array}{c}
\text{(1)} \\
\text{\quad a_1} \\
\text{\quad a_2} \\
\text{\quad a_3} \\
\text{\quad a_4} \\
\text{(4)}
\end{array}
\]

We consider cost functions \( \overline{C}_{a_i} = (\bar{f}_{a_i}, 1)^\top; i = 1, 3, 4 \) and \( \overline{C}_{a_2} = (2\bar{f}_{a_2}, 1)^\top \). Now the objective in (20) is \( \sum_{r \in \mathcal{R}} C_r(f_r) = (2f_{a_1} + 4f_{a_2} + 2f_{a_3} + 2f_{a_4}, 8)^\top \). Clearly, an efficient solution with minimal first component is \( f_{r_1} = 1, f_{r_2} = 0, f_{r_3} = 1, f_{r_4} = 0 \), as this solution avoids the most expensive arc \( a_2 \). The corresponding path costs are

\[
\begin{align*}
C_{r_1}(f_{r_1}) &= (3, 2)^\top \\
C_{r_2}(f_{r_2}) &= (1, 2)^\top \\
C_{r_3}(f_{r_3}) &= (3, 2)^\top \\
C_{r_4}(f_{r_4}) &= (1, 2)^\top.
\end{align*}
\]

(22)

For both OD pairs, the equilibrium conditions are violated as there is positive flow on non-efficient paths \( r_1, r_3 \). Even in the special case of constant path costs \( C_r(f_r) = C_r \), a solution of the optimisation problem does not have to satisfy...
the equilibrium conditions: constant path costs mean that the objective of (20) is constant and therefore any feasible solution is optimal, whereas a solution that satisfies the equilibrium conditions should only use paths with non-dominated costs.

Li et al. (2008, Theorem 2.1) claim that an efficient solution of (21) satisfies (20) when $|R_w| \leq 2$ holds for all $w \in \mathcal{W}$. The network given in Example 5.4 satisfies those assumptions. The feasible path flow solution $h_{r_1} = \frac{2}{3}, h_{r_2} = \frac{1}{3}, h_{r_3} = \frac{2}{3}, h_{r_4} = \frac{1}{3}$ has path cost $C \cdot r_1(h_{r_1}) = \left(\frac{7}{3}, 2\right)^\top, C \cdot r_2(h_{r_2}) = \left(\frac{7}{3}, 2\right)^\top$ and $C \cdot r_3(h_{r_3}) = \left(\frac{7}{3}, 2\right)^\top, C \cdot r_4(h_{r_4}) = \left(\frac{7}{4}, 2\right)^\top$, (23) and therefore satisfies the equilibrium conditions (21). The objective function value of (20) for this solution is $(\frac{28}{3}, 8)^\top$, whereas the objective function value of feasible solution $f$ in Example 5.4 is $(8, 8)^\top \leq (\frac{28}{3}, 8)^\top$. Therefore, $h$ is not an efficient solution of the optimisation problem. It also follows that Li et al. (2008, Theorem 2.3 and 2.4) are incorrect as they are similar to the above theorems addressing weak efficiency and weak vector equilibrium. Also, Li et al. (2008, Proposition 2.1) state that the scalar version of problems (20) and (21) are equivalent is incorrect, which can be seen by considering Example 5.4 without the second objective.

Li et al. (2008) generalise (21): a feasible $f^*$ is in generalised capacitated vector equilibrium if and only if for all OD pairs $w \in \mathcal{W}$ and for any positive integers $n_1, n_2$ with $n_1 + n_2 \leq |R_w|:

$$
\sum_{i=1}^{n_1} \lambda_i C_{s_i}(f^*_{s_i}) \geq \sum_{j=1}^{n_2} \mu_j C_{t_j}(f^*_{t_j})
\Rightarrow \text{there exists } i_0 \in \{1, \ldots, n_1\}, f^*_{s_{i_0}} = l_{s_{i_0}} \text{ or } j_0 \in \{1, \ldots, n_2\}, f^*_{t_{j_0}} = u_{t_{j_0}},
$$

for all $r_j, s_i \in R_w$ and $\lambda_i, \mu_j > 0, i = 1, \ldots, n_1, j = 1, \ldots, n_2$ and $\sum_{i=1}^{n_1} \lambda_i = 1, \sum_{j=1}^{n_2} \mu_j = 1$. In Li et al. (2008, Theorem 3.1), it is claimed that with this definition the optimisation problem and the generalised vector equilibrium problem are equivalent. However, the counter example presented in Example 5.4 still applies with $n_1 = n_2 = 1$.

5.5 Inconsistencies in Raciti (2008)

Raciti (2008) studies TA with user classes as well as multi-objective cost functions for each of the user classes on each path. The models initially include capacity constraints, which are later omitted. The aim is establishing a vector equilibrium principle that is equivalent to a VVI formulation, similar to the previous work of Oettli (2001). In the following we present some of the mentioned variational equilibrium concepts and attempt an economic interpretation similar to Wardrop’s.

Raciti (2008) defines a vector variational equilibrium according to (10). Note that in his paper, Raciti (2008) actually presents problem (10) in a more general framework assuming the space of flows (here $\mathbb{R}^l$), and the space of costs (here...
\( \mathbb{R}^p \) are topological vector spaces and also that order relations are given by ordering cones. In (10) and everything that follows, we replace those general vector spaces by the set of \( q \) class flows, \( \mathbb{R}^q \), and the cost space by \( \mathbb{R}^p \) so that \( p \) objectives are considered.

Two other equilibrium concepts are also introduced by Raciti (2008), which are basically VEQ and WVEQ with additional user classes and the WVEQ concept is also formulated with capacity constraints on path flow.

Raciti (2008) shows that (10) is equivalent to the following equilibrium conditions:

\[
\text{for all } i \in \{1, \ldots, p\}, \text{ for all } w \in W \text{ and for all } r, s \in R_w \quad C^j_r(f)_{i} > C^j_s(f)_{i} \Rightarrow f^j_r = 0, \text{ for all } j = 1, \ldots, q. \tag{24}
\]

Entry \( i \) of the \( p \)-dimensional path cost vector \( C^j_r(f) \) for user class \( j \) is denoted by \( C^j_r(f)_{i} \). Therefore, the aim of identifying equilibrium conditions that are equivalent to a variational inequality formulation is achieved. However, a closer look at (24) reveals that those equilibrium conditions can only be satisfied if there is exactly one efficient path for every OD pair, i.e. there exists a unique minimal solution. Fixing a product \( j \), (24) is a set of scalar VIs for every component \( i \) of the objective vector. For each \( j \), (24) represents \( p \) scalar VIs, each of which implies that path flow can only be positive if the \( i^{th} \) cost component is minimal on the corresponding path. So a solution exists only if all components attain their minimal cost for the same path(s). In this case, however, it is not necessary to consider a vector equilibrium.

Let us show that (24) does not have a solution if any OD pair \( w \) with positive demand for a user class \( j \), \( d^j_w > 0 \), has more than one non-dominated path cost vector \( C^j_r(f) \) for all \( f \in K \). Now, for user class \( j \) and for every pair of efficient paths \( r, s \) with \( C^j_r(f) \neq C^j_s(f) \), there exist two indices \( i_1, i_2 \in \{1, \ldots, p\} \) such that

\[
C^j_r(f)_{i_1} < C^j_s(f)_{i_1} \text{ and } C^j_r(f)_{i_2} > C^j_s(f)_{i_2}. \tag{25}
\]

From (25) and (24) we can conclude that

\[
C^j_r(f)_{i_1} > C^j_s(f)_{i_1} \Rightarrow h^j_s = 0 \\
C^j_r(f)_{i_2} > C^j_s(f)_{i_2} \Rightarrow h^j_r = 0.
\]

Repeating this process for every pair of efficient paths \( r, s \) with \( C^j_r(f) \neq C^j_s(f) \), the flow on all those paths must become zero to satisfy (24). But then, demand \( d^j_w > 0 \) is not satisfied. Therefore, there exists no feasible solution satisfying (24) whenever there is more than one efficient path per OD pair \( w \) and class \( j \).

### 6 Relationship Between VEQ and VOP

Before the relation of VOP and VEQ is discussed, we need to introduce the term properly efficient solutions for a VOP problem with objective vector \( z(f) = (z_1(f), \ldots, z_p(f)) \).
Definition 6.1 (Geoffrion 1968). A feasible solution $\bar{f}^*$ is called properly efficient, if it is efficient and if there is a real number $M > 0$ such that for all $i$ and $\bar{f} \in K_A$ for which $z_i(\bar{f}) < z_i(\bar{f}^*)$, there exists an index $j$ such that
\[
\frac{z_i(\bar{f}^*) - z_i(\bar{f})}{z_j(\bar{f}) - z_j(\bar{f}^*)} \leq M.
\]
(26)

Properly efficient solutions are efficient solutions with bounded trade-offs between the objectives. Properly efficient solutions can be obtained as optimal solutions of minimisation problems with weighted sum objective, given that all weights are positive (Geoffrion 1968, Theorem 1).

Theorem 6.1 Every properly efficient solution of the multi-objective optimisation problem VOP with convex objectives
\[
z_i = \sum_{a \in A} \int_0^{\bar{f}_a} \bar{c}_a^i(v)dv, i = 1, \ldots, p
\]
based on additive, separable, positive and continuous functions $\bar{c}_a^i$, is a VEQ solution.

Proof For every properly efficient solution $\bar{f}^*$ there exist positive weights $\omega_1, \ldots, \omega_p$ such that $\bar{f}^*$ is optimal for the single-objective optimisation problem
\[
\min \omega_1 \sum_{a \in A} \int_0^{\bar{f}_a} \bar{c}_a^1(v)dv + \ldots + \omega_p \sum_{a \in A} \int_0^{\bar{f}_a} \bar{c}_a^p(v)dv
\leq
\] (27)
s.t. $\bar{f} \in K_A$.

Problem (27) can be re-written as
\[
\min \sum_{a \in A} \int_0^{\bar{f}_a} (\omega_1 \bar{c}_a^1(v) + \ldots + \omega_p \bar{c}_a^p(v)) dv
\leq
\] s.t. $\bar{f} \in K_A$.

This is the equivalent optimisation formulation corresponding to a standard EQ problem with arc cost function $\omega_1 \bar{c}_a^1(\bar{f}) + \ldots + \omega_p \bar{c}_a^p(\bar{f})$. This cost function is positive and continuous as all its components $\bar{c}_a^i$ are positive and continuous, and the weights are positive. At equilibrium all used paths for any OD pair $w$ have the same minimal weighted sum cost value, $\eta_w$:
\[
\omega_1 c_1^i(f^*) + \ldots + \omega_p c_p^i(f^*) = \eta_w \quad \text{if } f_r^* > 0,
\omega_1 c_1^i(f^*) + \ldots + \omega_p c_p^i(f^*) \geq \eta_w \quad \text{if } f_r^* = 0.
\] (28)

If we now assume that there exists a path $s \in R_w$ with positive flow $f_s^* > 0$ that is dominated by another path $t \in R_w$, then
\[
c_1^i(f^*) \leq c_1^i(f^*), \quad c_2^i(f^*) \leq c_2^i(f^*), \ldots, \quad c_p^i(f^*) \leq c_p^i(f^*)
\]
holds with at least one strict inequality. From this and $\omega_i > 0$ it follows that
\[
\omega_1 c_1^i(f^*) + \ldots + \omega_p c_p^i(f^*) < \omega_1 c_1^i(f^*) + \ldots + \omega_p c_p^i(f^*),
\]
which contradicts (28), and therefore $\bar{f}^*$ satisfies VEQ. □
Remark 6.1 Let $\Omega$ be an open subset of $\mathbb{R}^m$ and $h$ a function differentiable on $\Omega$. Furthermore, $K$ is a convex subset of $\Omega$. Then, $h$ is convex on $K$ if and only if its gradient $\nabla h$ is monotone, i.e. satisfies

$$\left(\nabla h(x) - \nabla h(y)\right)^\top (x - y) \geq 0,$$

for all $x, y \in K$ (Hiriart-Urruty and Lemaréchal 2001, Theorem 4.1.4). Therefore, Theorem 6.1 is valid assuming that each cost function $\bar{c}_i$ is positive, continuous, monotone, and separable as this implies that the objectives $z_i$ are convex functions. A similar theorem appears in Goh and Yang (1999).

Remark 6.2 A solution with one or more of the weights equal to zero does not necessarily satisfy VEQ. If, for example, the only two objectives are $\bar{c}_1$ and $\bar{c}_2$, then the solution of EQ with cost function $\omega_1 \bar{c}_1 + 0 \cdot \bar{c}_2$ is not necessarily a solution that satisfies VEQ. At equilibrium, all used paths for an OD pair will have the same $c_1$-value, but if the $c_2$-values for those paths differ, then VEQ is not satisfied. To satisfy VEQ, there can only be flow on the path with lowest $c_2$-value as this path dominates the other paths with same $c_1$-value.

Next, we show that the reverse of Theorem 6.1 is not true, even for convex functions $z_i$. We give an example in which there exists a solution of VEQ (with $p = 2$) that can not be obtained as solutions of a VOP problem even though the objectives of VOP are convex. Consider the following example.

Example 6.1 Three-arc network

A network with two nodes and three arcs is used as shown below:

![Three-arc network diagram]

The total demand from $o$ to $d$ is 1000, so that the set of feasible path flows is given by $K = \left\{ f \in \mathbb{R}^3_+ : f_1 + f_2 + f_3 = 1000 \right\}$. Arc cost functions are:

$$C(f) = \begin{pmatrix} 10 \left[ 1 + 0.15 \left( \frac{f_1}{200} \right)^4 \right] & 20 \left[ 1 + 0.15 \left( \frac{f_2}{300} \right)^4 \right] & 25 \left[ 1 + 0.15 \left( \frac{f_3}{300} \right)^4 \right] \end{pmatrix}.$$  

In TA, the first cost component could represent travel time, whereas the second one could represent a toll. Thus, path 1 is the fastest route with the highest toll while path 3 is toll free and the slowest.

Here solution

$$f^* = (0, 607, 393)^\top$$

with $C(f^*) = \begin{pmatrix} 19 & 35.9 & 36 \\ 20 & 15 & 0 \end{pmatrix}$. 

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satisfies VEQ. But there exists a solution $h$ that dominates solution $f^*$ in VOP:

$$h = (117.5, 450, 432.5)^\top$$ with $$C(h) = \begin{pmatrix} 19.3 & 24.8 & 41.2 \\ 20 & 15 & 0 \end{pmatrix}.$$ 

The VEQ solution $f^*$ is not an efficient solution of VOP (7), as the objective vector $z(f^*)$ of VOP is dominated by $z(h)$:

$$z(h) = \begin{pmatrix} 22825.4 \\ 9100 \end{pmatrix} \leq \begin{pmatrix} 24764.4 \\ 9105 \end{pmatrix} = z(f^*).$$

In the next section, we explore why the problems VVI and VEQ are significantly different by characterising solutions of VVI and showing which solutions of VEQ can not be obtained by a VVI.

7 Relationship Between VEQ and VVI

It is well-known that a solution of VVI also satisfies VEQ, which we repeat here. Equivalence of VVI and VEQ is established under very strong assumptions, namely that for each OD pair all efficient paths have the same path cost vector. We then characterise which properties of VEQ solutions prohibit them from satisfying VVI. This answers the question whether solving VVI can help us understand and solve VEQ. Then, we give assumptions which are weaker than those previously made in the literature, that guarantee that a solution of VEQ also solves VVI.

It is well-known that every solution of VVI is a solution of VEQ, as the following theorem confirms. However, the reverse is not true in general.

**Theorem 7.1** (Yang and Goh 1997) If $f^* \in \mathcal{K}$ is a solution of VVI, then $f^*$ is also a solution of VEQ.

Chen and Yen (1993) show the following equivalence provided that the set $Z^w_N(f)$ is singleton, i.e. $|Z^w_N(f)| = 1$, for all $w \in \mathcal{W}$.

**Theorem 7.2** (attributed to Chen and Yen 1993) Let $Z^w_N(f)$ be singleton for all $w \in \mathcal{W}$. If $f \in \mathcal{K}$ satisfies VEQ, then $f$ satisfies the following modified VVI:

$$C(f)(h - f) \not\geq 0 \quad \text{for all } h \in \mathcal{K}. \quad (30)$$

Therefore, equivalence of the modified VVI in (30) and VEQ is established. Unfortunately, the singleton assumption renders this equivalence worthless. If for each OD pair there is always a single path that is optimal for each of the two or more objectives, we do not have to consider a vector valued problem at all.

Lee et al. (1998) show how the sets of solutions of VVI, WVVI, and VI_ξ are related. They give a proof of the following theorem, where the set of solutions of VVI (WVVI, VI_ξ) is denoted by $\text{sol}(\text{VVI})$ ($\text{sol}(\text{WVVI})$, $\text{sol}(\text{VI}_\xi)$):

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Theorem 7.3 (Lee et al. 1998) The following properties hold:

\[
\bigcup_{\xi \in \mathbb{R}^p_\geq} \text{sol} (V\!I_{\xi}) \subset \text{sol} (VVI) \subset \text{sol} (WVVI) = \bigcup_{\xi \in \mathbb{R}^p_\geq} \text{sol} (V\!I_{\xi}).
\]

Theorem 7.3 shows that all solutions of VVI can be obtained by applying a weighted sum scalarisation of the objectives \(\sum_{i=1}^p \xi_i C_i (f)\). It appears that this gives a first indication that VVI is not suitable to solve VEQ. The theorem indicates that path cost vectors \(C_r\) that are not optimal for a generalised cost function with some weighting factors cannot be included (i.e. have positive \(f_r\)) into a solution of VVI. We confirm this conjecture in the following.

Theorem 7.3 shows that \(\text{sol} (VVI)\) is a subset of the solutions of WVVI, which in turn can be obtained via \(\text{sol} (V\!I_{\xi})\) when \(\xi \in \mathbb{R}^p_\geq\). This indicates that VVI and WVVI do not permit solutions in which for some \(w \in W\) one of the efficient paths is non-supported. We give a proof of this result for \(p \geq 2\).

Theorem 7.4 Assume \(p \geq 2\), \(f\) is a solution of VEQ and there exists \(w \in W\) such that the set of non-dominated points for \(w\), \(Z_N^w (f)\), contains at least one non-supported point \(z^r\) with corresponding path \(r \in R^w\) such that \(z^r = C_r(f)\) and \(f_r > 0\) (\(C_r(f)\) denotes column \(r\) of the path cost matrix). Then \(f\) does not solve VVI.

Proof We know that \(\text{conv} (Z_N^w (f))\) is a polyhedron. All supported solutions lie on the boundary of \(\text{conv} (Z_N^w (f) + \mathbb{R}^p_\geq)\), whereas the non-supported solutions lie in the interior of \(\text{conv} (Z_N^w (f) + \mathbb{R}^p_\geq)\). We need to establish that the non-supported solution \(z^r\) is dominated by (at least) one point on a face of the polyhedron. This point on the face is, however, infeasible for the problem of finding shortest paths with given costs \(C(f)\).

A set \(Z\) is called \(\mathbb{R}^p_\geq\)-compact if for all \(z \in Z\) the section \((z - \mathbb{R}^p_\geq) \cap Z\) is compact (Ehrgott 2005, Def. 2.13). The set \(Z_N\) of all non-dominated points of \(Z\) is called externally stable if for each \(z \in Z \setminus Z_N\) there is \(\hat{z} \in Z_N\) such that \(z \in \hat{z} + \mathbb{R}^p_\geq\), i.e. for every non-dominated point \(z\) there is always a point \(\hat{z}\) that dominates \(z\) (Ehrgott 2005, Def. 2.20).

The set \(Z_N^w (f)\) is discrete and finite, therefore \(\text{conv} (Z_N^w (f))\) is a compact set. Clearly, \(\text{conv} (Z_N^w (f))\) is a \(\mathbb{R}^p_\geq\)-compact set. As \(\text{conv} (Z_N^w (f)) \subset \mathbb{R}^p_\geq\) is nonempty and \(\mathbb{R}^p_\geq\)-compact, Theorem 2.21 in Ehrgott (2005) implies that \(\text{conv} (Z_N^w (f))\) is externally stable.

Therefore, the point \(z^r\) is dominated by some point on one of the faces of \(\text{conv} (Z_N^w (f))\), we call this face the face associated with \(z^r\) and we denote by \(d(z^r)\) the point that dominates \(z^r\). The distance between \(d(z^r)\) and \(z^r\) is \(\|d(z^r) - z^r\| > 0\) as \(z^r\) is non-supported. Every face is defined by a set of supported extreme points \(z^{s_1} = C_{s_1}(f), \ldots, z^{s_q} = C_{s_q}(f) \in Z_N^w (f)\) with \(q \geq 2\). The point \(d(z^r)\) can be obtained as a convex combination of \(z^{s_1}, \ldots, z^{s_q}\). For an illustration in case \(p = 3\), refer to Figure 1.
We construct a feasible solution $h \neq f$ so that the VVI condition is violated. Choose

\[
\begin{align*}
h_u &= f_u \text{ for } u \neq s_1, \ldots, s_q, r \\
h_{s_1} &= f_{s_1} + a_1 \mu \\
& \vdots \\
h_{s_q} &= f_{s_q} + a_q \mu \\
h_r &= f_r - \mu,
\end{align*}
\]

for some $0 \leq a_1, \ldots, a_q$ with $\sum_{i=1}^{q} a_i = 1$ and $0 < \mu \leq f_r$. Solution $h$ is feasible as for OD pair $w$ a flow of $\mu$ is removed from path $r$ and re-distributed to paths $s_1, \ldots, s_q$. As $h_u - f_u = 0$ for $u \neq s_1, \ldots, s_q, r$ the LHS of VVI reduces to

\[
C(f)(h - f) = \mu (a_1 z^{s_1} + \ldots + a_q z^{s_q} - z^r)
\]

As the point $d(z^r)$ can be expressed as a convex combination of $z^{s_1}, \ldots, z^{s_q}$, factors $0 \leq a_1, \ldots, a_q \leq 1$ can be chosen so that $d(z^r) = a_1 z^{s_1} + \ldots + a_q z^{s_q}$ and $\sum_{i=1}^{q} a_i = 1$, which simplifies the previous equation to

\[
C(f)(h - f) = \mu (d(z^r) - z^r).
\]

The vector $d = z^r - d(z^r)$ is in $\mathbb{R}_z^p$ as $z^r$ is dominated by $d(z^r)$. Replacing $z^r$
by \( d(z^r) + d \) in (32) yields

\[
C(f)(h - f) = \mu(d(z^r) - (d(z^r) + d))
\]

\[
= -\mu \frac{d}{\epsilon R^p_{\geq}}
\]

\( \square \)

Even when there are only supported objective vectors in a solution of VEQ, this solution might not be a solution of VVI as we show next for the bi-objective case.

**Theorem 7.5** Assume \( p = 2 \), \( f \) is a solution of VEQ and there exists \( w \in W \) such that the set of non-dominated points for \( w \), \( Z^w_N(f) \), contains at least three supported points \( z^r, z^s, z^t \) that are not optimal for a weighted sum problem with the same weighting factor. Furthermore, we assume for the paths \( r, s, t \) \( \in R^w \) that \( z^u = C_u(f) \) and \( f_u > 0, u = r, s, t \) (\( C_u(f) \) denotes column \( u \) of the path cost matrix). Then \( f \) does not solve VVI.

**Proof** The location of the path cost vectors of the three paths is indicated in Figure 2.

As all path cost vectors are non-dominated, we have

\[
C_{1s}(f) < C_{1r}(f) < C_{1t}(f) \quad \text{and} \quad C_{2s}(f) > C_{2r}(f) > C_{2t}(f)
\]

(33)

For the slopes we have \( m^{st} < m^{rt} \) and therefore

\[
m^{st} = \frac{C_{2t}(f) - C_{2s}(f)}{C_{1t}(f) - C_{1s}(f)} < \frac{C_{2t}(f) - C_{2r}(f)}{C_{1t}(f) - C_{1r}(f)} = m^{rt}.
\]

(34)
We proceed by constructing \( h \in \mathcal{K} \) such that \( C(f)(h - f) \in -\mathbb{R}_+^2 \). Let path-flow vector \( h \) be given as
\[
\begin{align*}
  h_u &= f_u \text{ for } u \neq r, s, t \\
  h_s &= f_s - a\mu \\
  h_r &= f_r + \mu \\
  h_t &= f_t - (1 - a)\mu,
\end{align*}
\]
with appropriate choice of \( 0 < a < 1 \) and \( \mu > 0 \), which we will comment on later. As \( h_u - f_u = 0 \) for \( u \neq r, s, t \) the LHS of VVI reduces to
\[
C(f)(h - f) = \left(\begin{array}{c} -a\mu C_{1s}(f) + \mu C_{1r}(f) - (1 - a)\mu C_{1t}(f) \\ -a\mu C_{2s}(f) + \mu C_{2r}(f) - (1 - a)\mu C_{2t}(f) \end{array}\right).
\]
To show that \( f \) does not satisfy VVI, i.e. \( C(f)(h - f) \in -\mathbb{R}_+^2 \), it suffices to show that \( a \) and \( \mu \) exist so that
\[
-\mu C_{1s}(f) + \mu C_{1r}(f) - (1 - a)\mu C_{1t}(f) = 0 \tag{36}
\]
and
\[-\mu C_{2s}(f) + \mu C_{2r}(f) - (1 - a)\mu C_{2t}(f) < 0. \tag{37}\]
From (36) we conclude \( a = \frac{C_{1r}(f) - C_{1t}(f)}{C_{1r}(f) - C_{1s}(f)} \) and \( 0 < a < 1 \) by (33). It remains to verify that (37) holds for this choice of \( a \):
\[
\begin{align*}
-\mu C_{2s}(f) + \mu C_{2r}(f) - (1 - a)\mu C_{2t}(f) &= a(C_{2t}(f) - C_{2s}(f)) + (C_{2r}(f) - C_{2t}(f)) \\
&= \frac{C_{1r}(f) - C_{1t}(f)}{C_{1r}(f) - C_{1s}(f)}(C_{2t}(f) - C_{2s}(f)) + (C_{2r}(f) - C_{2t}(f)) \\
&= \frac{C_{2r}(f) - C_{2s}(f)}{C_{1r}(f) - C_{1s}(f)}(C_{1r}(f) - C_{1t}(f)) + (C_{2r}(f) - C_{2t}(f)) \\
&< \frac{C_{2r}(f) - C_{2s}(f)}{C_{1r}(f) - C_{1s}(f)}(C_{1r}(f) - C_{1t}(f)) + (C_{2r}(f) - C_{2t}(f)) = 0
\end{align*}
\]
Therefore (37) is true when choosing \( a = \frac{C_{2r}(f) - C_{2s}(f)}{C_{1r}(f) - C_{1s}(f)} \). It remains to choose \( \mu \) so that \( h \) is feasible. As \( f_s, f_t > 0 \), by choosing \( \mu \leq \min \{ \frac{f_s}{a}, \frac{f_t}{1 - a} \} \), we obtain a feasible \( h \in \mathcal{K} \) as defined in (35) such that (36) and (37) hold, therefore \( f \) does not satisfy VVI.

Solutions of VEQ that satisfy the assumptions of Theorems 7.4 and 7.5 do exist. For instance, in Example 6.1 we consider a network in which the first cost function increases with flow. The second cost function, however, remains constant. Here, non-supported solutions may occur as well as more than two supported solutions, which are not optimal for the same weighting factors.

In Example 6.1 the solution \( \hat{f} \) has three supported points, whereas solution \( \hat{h} \) has a non-supported point. Clearly both solutions satisfy VEQ as there is only flow on efficient paths. This is illustrated in Figure 3.

\[
\hat{f} = (300, 300, 400)^	op \text{ with } C(\hat{f}) = \begin{pmatrix} 17.59 & 20.95 & 36.85 \\ 20 & 15 & 0 \end{pmatrix}
\]
\[
\hat{h} = (300, 400, 300)^	op \text{ with } C(\hat{h}) = \begin{pmatrix} 17.59 & 23.00 & 28.75 \\ 20 & 15 & 0 \end{pmatrix}
\]
In the following Theorem 7.5 is extended for \( p > 2 \) but with slightly stronger assumption on the supported points, see Remark 7.1.

**Theorem 7.6** Assume \( p \geq 2 \), \( f \) is a solution of VEQ and there exists \( w \in W \) such that the set of non-dominated points for \( w \), \( Z_{\lambda}^w(f) \), contains at least \( p+1 \) extreme supported points \( z^{s_1}, \ldots, z^{s_p}, z^r \) with corresponding paths \( s_1, \ldots, s_p, r \in R^w \) such that \( z^u = C_u(f) \) and \( f_u > 0, u = s_1, \ldots, s_p, r \) (by \( C_u(f) \) we mean column \( u \) of the path cost matrix). Furthermore, assume the objective vectors \( z^{s_1}, \ldots, z^{s_p} \) lie on the same facet of \( \text{conv}(Z_{\lambda}^w) \) and all points are optimal for exactly one common weighting factor \( \xi \in \Lambda \{ \lambda \in R^p : \lambda_i \geq 0, \sum_{i=1}^{p} \lambda_i = 1 \} \) (there may exist weighting factors for which some of the vectors are optimal, but only a unique common one). If the solution \( z^r \) is not optimal for this weighting factor \( \xi \), then \( f \) does not solve VVI.

**Proof** The objective vectors \( z^{s_1}, \ldots, z^{s_p} \) lie on the same facet \( F \) of \( \text{conv}(Z_{\lambda}^w) \) and all are optimal for exactly one common weighting factor \( \xi \in \Lambda \). Therefore, we have \( \xi^T z^{s_i} = \ldots = \xi^T z^{s_p} \) and \( \xi^T z^r > \xi^T z^{s_i}, i = 1, \ldots, p \).

The points \( z^{s_1}, \ldots, z^{s_p} \) generate a hyperplane \( H \) of dimension \( p - 1 \) in \( R^p \) (as \( \xi \in \Lambda \) with \( \xi^T z^{s_1} = \ldots = \xi^T z^{s_p} \) is unique). The point \( z^r \) is dominated by a point \( d(z^r) \) in \( H \), but not in \( F \) (as \( z^r \) is supported). As the point \( z^r \) does not lie in the hyperplane, it follows that there exists \( d \in R^p \) such that \( z^r = d(z^r) + d \).

A \( p - 1 \) dimensional simplex is obtained by \( \text{conv}\{ z^{s_1}, \ldots, z^{s_p} \} \). The point \( d(z^r) \) does not lie within this simplex (as \( \xi^T z^r > \xi^T z^{s_i}, i = 1, \ldots, p \), i.e. it can not be obtained as a convex combination of \( z^{s_1}, \ldots, z^{s_p} \). We choose one of the vertices \( z^{s_i} \) of the simplex and then obtain a cone \( C_i = \{ z \in R^p : z = z^{s_i} + \sum_{j \neq i} a_j (z^{s_j} - z^{s_i}) \} \) with apex \( z^{s_i} \). The corresponding opposite cone \( -C_i \) is obtained by only allowing coefficients \( a_j \leq 0 \). Now \( d(z^r) \) lies within at least one of the cones \( C_i, -C_i, i = 1, \ldots, p \). Without loss of generality we now assume that \( d(z^r) \in C_1 \).
For an illustration of $C_1$ and $-C_1$ in case $p = 3$, refer to Figure 1. We can write
\[
d(z^r) = z^{s_1} + a_2(z^{s_2} - z^{s_1}) + \ldots + a_p(z^{s_p} - z^{s_1})
\]
\[
= (1 - a_2 - \ldots - a_p)z^{s_1} + a_2z^{s_2} + \ldots + a_p z^{s_p}.
\] (38)

Along the lines of the proof of Theorem 7.5, we construct a feasible solution $h \neq f$ so that the VVI condition is violated. Choose
\[
h_u = f_u \text{ for } u \neq s_1, \ldots, s_p, r
\]
\[
h_r = f_r - \mu,
\]
\[
h_{s_1} = f_{s_1} + (1 - a_2 - \ldots - a_p)\mu
\]
\[
h_{s_2} = f_{s_2} + a_2\mu
\]
\[
\vdots
\]
\[
h_{s_p} = f_{s_p} + a_p\mu
\]
for some $0 \leq a_2, \ldots, a_p$ and $0 < \mu \leq \min \left\{ f_r, \frac{f_{s_1}}{1 - a_2 - \ldots - a_p} \right\}$. Note that $a_2 + \ldots + a_p > 1$ as $d(z^r)$ does not lie within the simplex, so it can not be obtained as a convex combination. Solution $h$ is feasible as for OD pair $w$ a flow of $\mu$ is removed from paths $r$ and $s_1$ and re-distributed to paths $s_2, \ldots, s_p$. As $h_u - f_u = 0$ for $u \neq s_1, \ldots, s_p, r$ the LHS of VVI reduces to
\[
C(f)(h - f) = \mu((1 - a_2 - \ldots - a_p)z^{s_1} + a_2z^{s_2} + \ldots + a_p z^{s_p} - z^r),
\]
where the details of this step are analogous to the corresponding step in the proof of Theorem 7.4. Using (38) and $z^r = d(z^r) + d$, we obtain
\[
C(f)(h - f) = \mu(d(z^r) - z^r)
\]
\[
= - \mu \frac{d}{d_{>0} \in \mathbb{R}_+^p} \in -\mathbb{R}_+^p,
\]
which confirms that $f$ is not a solution of VVI. □

Figure 4: Illustration of the situation described in Theorem 7.6 for $p = 3$. 
Remark 7.1 We conjecture that it is possible to drop the assumption that points $z^1, \ldots, z^p$ must lie on the same facet in Theorem 7.6. We believe that it may be possible to show a version of this theorem based on the assumption that there exist $p+1$ points that are not all optimal for the same weighting factor $\xi \in \Lambda$.

To summarise, we established that VVI does not yield any solutions with positive flow on non-supported efficient paths in Theorem 7.4. Furthermore, solutions with positive flow on at least $p$ efficient paths that are not optimal for the same weighting factor (so they do not lie on the same face in objective space) can not be obtained, see Theorem 7.6. VVI only provides solutions that can be obtained by using a single weighting factor (for every OD pair). We can now state under which conditions we can obtain equivalence of VEQ and VVI.

Theorem 7.7 Assume that $f \in \mathcal{K}$ is a solution of VEQ. Also assume there exists $\xi \in \Lambda \cap \mathbb{R}_{>0}$ so that for each $w \in \mathcal{W}$ among all the efficient paths $r \in \mathcal{X}_{\mathcal{E}}^w(f)$ there is only positive flow on those paths $r$ with weighted cost value $\xi^\top C \cdot r(f)$ equal to $C_{\text{min}} = \min \{\xi^\top C \cdot r(f) : r \in \mathcal{X}_{\mathcal{E}}^w(f)\}$, i.e. those paths with minimal weighted cost. Then $f$ is a solution of VVI.

Proof We have to show that $f$ is a solution of VVI. First observe that the assumptions imply the following

for all $w \in \mathcal{W}$ and for all $r, s \in \mathcal{R}_w$ \[ \xi^\top C \cdot r(f) < \xi^\top C \cdot s(f) \Rightarrow f_s = 0. \]

Therefore, EQ is satisfied for the single-objective cost function $\xi^\top C$. Equivalence of EQ and VI (1) (see Smith 1979) implies that $f$ also satisfies

\[ (\xi^\top C(f)) \cdot (h - f) \geq 0 \quad \text{for all } h \in \mathcal{K}. \]

The latter can be re-written as $\text{VI}_\xi$:

\[ \left( \sum_{i=1}^{p} \xi_i C_i(f) \right) (h - f) \geq 0 \quad \text{for all } h \in \mathcal{K}. \]

As $f$ satisfies $\text{VI}_\xi$ with $\xi \in \mathbb{R}_{>0}$, it follows by Theorem 7.3 that $f$ is a solution of VVI. \qed

Remark 7.2 Note that $\xi$ needs to be identical for all $w \in \mathcal{W}$. Also, the assumptions of Theorem 7.7 do not require that all non-dominated points $\mathcal{Z}_{\mathcal{K}}^w(f)$ lie on the boundary of the convex hull $bd(\text{conv}(\mathcal{Z}_{\mathcal{K}}^w(f) + \mathbb{R}_\mathcal{P}^p))$ – it is sufficient that no efficient path with objective vector lying in the interior of $\text{conv}(\mathcal{Z}_{\mathcal{K}}^w(f) + \mathbb{R}_\mathcal{P}^p)$ has positive flow.

Remark 7.3 Clearly, the assumptions of Theorem 7.7 are weaker than those of Theorem 7.2, and no modification of the VVI is required. With VEQ and VVI defined as in this paper (and throughout the related literature), it appears that Theorem 7.7 is the strongest link between VEQ and VVI that can be established. This is because we know from Theorem 7.3 that all $f$ that satisfy VVI can be obtained by solving $\text{VI}_\xi$ with varying parameter $\xi \in \mathbb{R}_{>0}^p$. In particular, the value of $\xi$ must be equal for all $w$. 

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8 Conclusion

In conclusion, we have seen that, although a VVI solution is always a VEQ solution, solutions of VVI have very limited structural properties. They can not have non-supported efficient paths and also no efficient paths that solve different weighted sum problems. This shows that efforts to relate the two problems VEQ and VVI are not promising. Also, they cannot lead to solution algorithms that solve VEQ by solving VVI.

Instead research efforts should go into the development of (possibly heuristic) solution algorithms that directly solve VEQ.

References


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