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The FOCUS Framework

Snapshots of Mathematics Teacher Noticing

Ban Heng Choy

Abstract

This thesis investigates what, and how teachers notice as they plan, teach, and review lessons that support the development of students’ mathematical thinking. Teacher noticing refers to what teachers see and how they interpret these observations to decide their responses to classroom situations. More specifically, this study examines what makes noticing productive with regard to enhancing student reasoning.

The conceptual framework for this study established the centrality of mathematical noticing for teachers to engage in productive practices that enhance student reasoning. An important premise for this research is the assumption that productive noticing is manifested when teachers respond to student thinking through these practices: designing a task that reveals student reasoning; listening and building on student reasoning to orchestrate a mathematically fruitful discussion in class; and analysing student reasoning in order to learn from their practice during post-lesson discussions.

To address the twin challenges of developing theory and documenting productive noticing, a design research paradigm was adopted for this study. A modified Lesson Study protocol, designed to create an environment that would promote more educationally productive noticing, was the main methodology adopted for this research. Data generated from voice and video recordings were analysed using thematic analysis through a commognitive lens to uncover the characteristics of productive noticing.

The FOCUS framework highlights the two key dimensions to promote productive noticing: the need for an explicit focus and the central role of reasoning. Following a photographic metaphor, the framework is used to develop models, which describe and analyse teachers’ noticing from two perspectives: A wide-angle view and a close-up view. The wide-angle view provides a portrayal of teachers’ noticing through the lesson cycle in relation to the productive practices for mathematical reasoning, whereas a close-up view can provide a lens to examine teacher noticing at each stage of the lesson cycle. The FOCUS framework was then applied to develop two portraits of teacher noticing, which demonstrate how a teacher’s noticing can be characterised. The findings suggest that the FOCUS framework is useful for teachers to improve their noticing expertise, and for researchers to analyse teacher noticing.
Acknowledgements

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Chapter 1. Introduction

A central feature of many current mathematics curricula is the emphasis on developing mathematical thinking or reasoning (Ministry of Education-New Zealand, 2007; Ministry of Education-Singapore, 2013; National Council of Teachers of Mathematics, 2000). In Singapore, one of the key aims that frames mathematics education at all levels, from primary to pre-university, is to “develop thinking, reasoning, communication, application and metacognitive skills through a mathematical approach to problem-solving” (Ministry of Education-Singapore, 2013, p. 9). The call for teachers to emphasise engaging *learning experiences* in Mathematics classrooms is not through a change in content coverage; but instead, it focuses on the way Mathematics is taught and learnt (Ministry of Education-Singapore, 2013). Similar to other reform recommendations (Kilpatrick, Swafford, & Findell, 2001; National Council of Teachers of Mathematics, 2009), there is a shift towards a more interactive teaching style that is aimed at developing mathematical competencies by building upon students’ thinking. However, this puts enormous demands on mathematics teachers to strengthen their skills in designing mathematical tasks (Mason & Johnston-Wilder, 2006; Smith & Stein, 2011); orchestrating productive mathematical discussions; assessing students’ mathematical reasoning (Schifter, 2001); and reflecting about their teaching in order to advance students’ learning (Schoenfeld & Kilpatrick, 2008; Stigler & Hiebert, 1999).

Doing this work of *ambitious teaching* (Lampert & Graziani, 2009) is challenging and requires “an unnatural orientation” and “a simultaneous, unusual attention” to both the learners and the content to be taught (Ball & Forzani, 2009, p. 499). It involves a shift of *attention* (Mason, 2010), on the part of the teacher, to listen to and examine students’ thinking carefully in order to identify the key mathematical ideas and address corresponding students’ errors (Ball & Forzani, 2009; Schifter, 2001). Several mathematics educators have advocated for teachers to have opportunities to learn from their teaching to meet these challenges (Hiebert, Morris, Berk, & Jansen, 2007; Lampert, 2009; Mason, 2010; Murata, 2010; Sowder, 2007), but learning from practice can be difficult (Bryk, 2009).
In Singapore, the use of inquiry-based teacher development activities, such as Lesson Study, to support teachers in their attempts at ambitious teaching is prevalent in many mathematics classrooms. And yet, what intrigues me when I worked with mathematics teachers during these professional development activities is that some teachers made orchestrating classroom interactions with students seem so effortless; whereas others did not seem to have the “eyes to see”, nor the “ears to listen” to instances of student reasoning. To illustrate the need for this unnatural orientation to student reasoning, the following vignette, which motivated my interest in this research, is presented in the next section.

1.1. Building on student thinking: A challenging task for teachers

1.1.1. A personal snapshot

A few years ago, I was invited to observe a lesson on Exploring Linear Graphs, and to do a short commentary during the post-lesson discussion. The detailed lesson plan was emailed to me before the lesson, and I saw a noteworthy mathematical point documented in the plan. The lesson objective was to investigate the effect of changing the value of $c$ for the straight-line equation of the form $y = mx + c$ for a Secondary Two class (age 14 years). This was the students’ first encounter with equations of straight lines, and the main teaching idea was to get students to explore the effect of changing $c$ using a graphing calculator. The intent of the lesson was to emphasise the use of multiple representations (Brenner et al., 1997) by focusing on algebraic-graphical-tabular translations. When anticipating students’ possible misconceptions, the teachers highlighted a possible misconception in their lesson plan as shown in Figure 1.1.

![Figure 1.1. A possible misconception about the effect of changing c](image-url)
As seen in Figure 1.1, they expected that students might perceive that the line moved diagonally instead of a vertical translation when the values of $c$ were changed. Despite this being a well-documented error (Chiu, Kessel, Moschkovich, & Muñoz-Nuñez, 2001; Moschkovich, 1999), the teachers’ plan was to explain that the translation is vertical instead of a diagonal one without making any reference to the multiple representations. During the lesson, students used the graphing calculators, as planned, to investigate the effect of changing the value of $c$. However, several students articulated that the line appeared to move diagonally, and some said it was a horizontal translation when they changed the values on the calculators during the whole-class discussion. In order to get a sense of the prevalence of this “error”, the teacher conducted a poll and realised that almost the whole class thought that the movement was not vertical. The teacher then quickly corrected their mistake by telling them that it was supposed to be a vertical translation, and explained that as $c$ varies for a fixed $m$, the lines are parallel and move vertically upward. During the post-lesson discussion, the teachers highlighted that the students were very engaged in the task, and did not discuss the students’ confusion around the vertical translation of the graphs.

1.1.2. The “expert blind spot”

The above vignette raised several questions about what, and how, teachers attend to and make sense of student thinking. Contrary to what the teachers had thought in the vignette, Schoenfeld, Smith, and Arcavi (1993) highlight that the vertical translation of the straight line may not be obvious to students, and may be perceived as a diagonal or horizontal movement instead. While it is commendable that the teachers identified a possible misconception, which was demonstrated by the students during the actual lesson, it appears that they might have overlooked the difficulty posed by the dynamic visual movement of the lines on the calculator. This perceived diagonal or horizontal movement is made even more pronounced through the use of the graphing calculator, in this case, because of the lack of grid lines on the screen (Chiu et al., 2001); the lack of vertical traces (Schoenfeld et al., 1993); and the missed opportunity to switch to other representations, e.g., numerical values of $x$ and $y$ in the table of values, to make the point (Brenner et al., 1997). As pointed out by Chiu et al. (2001), the “presence or absence of a grid may make little difference to those already familiar with linear equations and their graphs”, but it is of “great importance” to a novice learning the concept for the first time.
(p. 247). This overlooking of potential students’ errors as a result of a teacher’s expertise in the subject is sometimes referred to as an “expert blind spot” (Nathan & Petrosino, 2003).

The teachers, in the vignette, also missed the possible perception of a horizontal movement by students during the planning of the lesson. This alternative perception was raised by a number of students during the whole-class discussion. Again, this “unexpected” response has been documented, and this notion may be reasonable in the minds of the students (Chiu et al., 2001; Moschkovich, 1999). Furthermore, it is not entirely wrong because the graph of \( y = mx + c \) can be considered as a translation of the graph of \( y = mx \) in any direction, which can be given as a vector drawn from a point on \( y = mx \) to a point on \( y = mx + c \) (Chiu et al., 2001). A rich discussion would have occurred if the teachers had allowed the students to articulate their reasoning instead (Chiu et al., 2001).

1.1.3. “Seeing the invisible”

The second challenge for the teachers in the vignette was for them to “see the invisible”—what students were thinking in terms of the movement of the straight lines (Schifter, 2001). Mason and Johnston-Wilder (2006) propose that a teacher would have to ask students questions, instead of telling them, if they want to reveal what the students were thinking. However, asking questions is just the first step. In order to make sense of student reasoning, a teacher has to try to attend to the mathematics in what they are saying and doing (Schifter, 2001), and not just what they say and do. Moreover, a teacher needs to evaluate the students’ ideas in terms of their mathematical validity (Schifter, 2001), and to see if there is any “sense” in these ideas (Chiu et al., 2001; Moschkovich, 1999; Schifter, 2001). For example, Chiu et al. (2001) highlight how students’ initial inadequate conceptions can be refined if a teacher can respect their ideas, and support the students to evaluate their own ideas.

In the vignette, the teacher did not press the students for their reasoning (Brodie, 2010f), and had chosen to inform students of the intended answer, albeit with explanations. This indicates that the teacher’s focus was on his own thinking (Mason & Johnston-Wilder, 2006), and that he was listening for what he perceived to be the correct answers (B. Davis, 1997). And when the students gave responses that were not
intended, the teacher did not take up students’ ideas to orchestrate a discussion (Smith & Stein, 2011), but instead conducted a poll. Mathematical ideas are not endorsed through a poll, but they are more readily accepted through a process of negotiation using proofs, explanations or explorations (Nunokawa, 2010; Schoenfeld, 1992; Yackel & Hanna, 2003). Thus, the teacher missed an opportunity to establish the sociomathematical norms for the class (Yackel & Cobb, 1996).

The brief discussion above highlights some of the challenges involved in teaching for mathematical reasoning, and suggests that orchestrating a fruitful lesson is highly attention-dependent (Ainley & Luntley, 2007; Mason & Davis, 2013). Experienced teachers, as Ainley and Luntley (2007) argue, “draw on a repertoire of attentional skills for attending to cognitive and affective aspects of pupil activity”, which enable them to see and read classroom situations better than novice teachers (p. 4). This attention to mathematically relevant details may play a critical role in overcoming one’s expert blind spot to see and understand how students reason mathematically. Therefore, examining how teachers can gain this insight when learning from their practice will be highly consequential for improving teaching.

1.1.4. Gathering teachers together does not lead to teacher learning

Reflecting on teaching and learning has been suggested by many researchers to bring about improvements in teaching (Cochran-Smith & Lytle, 1999; Franke, Carpenter, Levi, & Fennema, 2001; Franke & Kazemi, 2001; Lampert, 2009; Loughran, 2002; Stigler & Hiebert, 1999). This has brought about an emphasis on teacher inquiry-based approaches to improve teaching, and has led to the adoption of collaborative professional activities such as video clubs (Star & Strickland, 2008; van Es & Sherin, 2002) or Lesson Studies (Yoshida, 2005). However, it would be “wishful thinking” to expect that “something good will happen” just because one gathers “teachers together to talk about practice” (Bryk, 2009, p. 599). Instead, it is more crucial for teachers to focus on student reasoning when reflecting on their teaching in order to learn from their practice (Borko, 2004; Breyfogle & Herbal-Eisenmann, 2004; Burns, 2005; Chamberlin, 2005; Clea Fernandez, Cannon, & Chokshi, 2003; Jacobs, Franke, Carpenter, Linda, & Battey, 2007).
However, focusing on student thinking can be challenging for teachers (Clea Fernandez et al., 2003). The teachers, in the vignette, focused their discussion on aspects such as student engagement and use of technology, but missed out on thinking about student reasoning around the anticipated “misconception”. This episode highlights the challenge of adopting a researcher’s lens (Clea Fernandez et al., 2003), and suggests that it is how, and not just what the teachers do together that matters in professional development (Chamberlin, 2005; Darling-Hammond & Richardson, 2009; Lampert, 2009).

1.1.5. The key: Seeing, thinking about, and doing “mathematically worthwhile things”

This brief analysis of the vignette highlights that teachers may find it challenging to focus and maintain their attention on mathematically relevant instructional details when learning from their practice. Although the teachers in the vignette could identify clearly a possible student’s confusion about the targeted concept, they did not plan for a response during lesson planning. Given that these teachers were experienced in teaching, the lack of knowledge may not fully explain their failure to see and interpret student thinking. What did the teachers attend to when they were planning or teaching the lesson? How did they analyse the lesson during post-lesson discussions? What could have hindered them from building on student thinking? What could the teachers do differently in order to respond better to student reasoning? These are some of the questions that prompted me to investigate what teachers attend to when they plan, teach and review lessons. Therefore, in the light of the lessons obtained from the vignette, I gathered that the ability to see and interpret mathematical details in an instructional setting is critical for teachers to learn and improve their practice with regard to teaching for mathematical reasoning. This careful attention and interpretation that results in an instructional response is referred to as teacher noticing.

Mathematics teacher noticing, which is the process of attending to students’ mathematical ideas, and making sense of the information to make decisions in an instructional context (Jacobs, Lamb, & Philipp, 2010; Mason, 2002; van Es & Sherin, 2008), is central to all mathematical teaching practices (Mason, 2002; Schifter, 2001) and is essential for improving teaching (Schoenfeld, 2011b). Noticing has the potential to bring about significant changes in teaching, and hence, it can be argued that mathematical noticing is an important component of teaching expertise. Despite the
apparent simplicity of the construct of teacher noticing, the ability to “notice productively” during mathematics teaching is both difficult to master, and complex to study (Jacobs, Philipp, & Sherin, 2011, p. xxvii). Mason (2011) also argues that it is difficult to track what teachers attend to, and that one can attend to different things at various levels of details. However, as Erickson (2011, p. 33) contends, mathematics education researchers should “learn more about the what, how and why of teacher noticing” in order to bring about a pedagogy that focuses on developing student reasoning. Moreover, a more complete model to see how teachers notice in-the-moment during these complex classroom interactions can help to further the study of teacher noticing (B. Sherin & Star, 2011). Therefore, what, and how, teachers see, think about, and respond to mathematically worthwhile things, was the focus for this study.

1.2. Research aim

The objective of this study was to investigate what, and how teachers notice as they plan, teach, and review lessons that support the development of students’ mathematical thinking. Firstly, this research set out to examine the characteristics of teachers’ mathematical noticing that result in thinking about, or making decisions that are defined as pedagogically useful. For the purpose of this study, pedagogically useful decisions are those that work towards providing students with opportunities to engage in mathematical reasoning—decisions that encourage students to “think, question, solve problems, and discuss their ideas, strategies, and solutions” (National Council of Teachers of Mathematics, 2000, p. 18). By focusing on what teachers notice when they make these decisions, the researcher aimed to detail the characteristics of productive mathematical noticing.

Secondly, this study sought to explore how teachers notice when they make instructional decisions that are pedagogically productive. The aim of the study was to make visible, understand, and gain insight into teachers’ thinking as they engage in teaching for mathematical reasoning. One possible way to do this is to build a model for teacher noticing to highlight how teachers see, think, and respond to instructional details during the planning, teaching, and reflection of lessons.

This chapter has highlighted the main aim of this study against the backdrop of recent research focus on responsive teaching, learning from teaching, and decomposing
practice. It situates the study in the context of professional development of teachers both in Singapore, and the wider international mathematics education community. This sets the stage to consider the significance of this study, and present the research questions that helped to frame the study.

1.3. Background of the study

The vision for the mathematics curriculum in a changing world challenges teachers to go beyond teaching to the tests, and instead, think about designing learning experiences for students to participate as mathematicians (National Council of Teachers of Mathematics, 2000). To do this effectively, teachers need to develop meaningful or “worthwhile” mathematical tasks (Mason & Johnston-Wilder, 2006; National Council of Teachers of Mathematics, 2000, p. 17; Smith, Bill, & Hughes, 2008), engage students in “doing mathematics” (Smith & Stein, 1998, p. 348), and orchestrate whole class discussions using students’ work (Cengiz, Kline, & Grant, 2011; Hunter, 2010; Smith & Stein, 2011). In recent years, there has also been a focus on assessing students for learning, or formative assessment. In the case of mathematics, there is a push for moment-by-moment assessment of students’ learning to make instructional decisions (Ministry of Education-Singapore, 2013, p. 27; National Council of Teachers of Mathematics, 2000, p. 23). Underpinning these teaching practices is the teachers’ ability to notice mathematical ideas, students’ learning difficulties, students’ thinking and reasoning during instruction. The challenging aspect for teachers is to make sense of students’ thinking even when their explanations do not seem to make sense. This problem is exacerbated by the “blooming, buzzing confusion of sensory data” that teachers face during teaching (M. G. Sherin, Jacobs, & Philipp, 2011a, p. 5). Hence, it is critical for teachers to be able to sieve the “wheat” from the “chaff” as they work towards a more adaptive teaching style. An important factor that can potentially enable teachers to notice better is their knowledge for teaching.

1.3.1. Knowledge for-, in-, and of- practice: The pre-requisite for teacher noticing

Whether teachers can discern the relevant information from the plethora of data depends on their knowledge of mathematics, students’ learning and pedagogy. Ever since Shulman (1986) introduced the notion of pedagogical content knowledge (PCK), there has been an increasing amount of literature on the components of teachers’
knowledge (Ball, Thames, & Phelps, 2008). These developments have tended to centre on what Cochran-Smith and Lytle (1999) call ‘Knowledge-for-Practice’: the “formal knowledge and theory (including codifications of the so-called wisdom of practice) for teachers to use in order to improve practice” (p. 250). Underlying this emphasis on knowledge is the assumption that knowing more leads to better practice (Cochran-Smith & Lytle, 1999). While knowing more is important, what a teacher knows may not be translated to practice in the classroom. Ball and Cohen (1999) argue that teachers need more than content knowledge to teach well. They also need to “learn in and from practice” in order to improve instruction (Ball & Cohen, 1999, p. 10). Cochran-Smith and Lytle (1999) also highlight teacher inquiry as a key to bridge the theory-practice gap through the notions of “knowledge-in-practice” and “knowledge-of-practice”.

"Knowledge-in-practice" is the “practical” knowledge of teaching that is “embedded” in the practice of expert teachers (Cochran-Smith & Lytle, 1999, p. 250). This knowledge is assumed to be developed by teachers when they reflect on how competent teachers react to classroom situations, and capture new insights they gain through examining those incidents (Cochran-Smith & Lytle, 1999). Through inquiry, teachers can deepen their own understanding of teaching to become better decision-makers in curriculum matters (Cochran-Smith & Lytle, 1999). Knowledge-of-practice, on the other hand, is generated by teachers themselves when they use their own classrooms as sites for inquiry to test teaching theories within their own social, cultural and political contexts (Cochran-Smith & Lytle, 1999). According to Cochran-Smith and Lytle (1999), this notion of knowledge-of-practice can help to bridge relationships between theory and practice. Therefore, it is also important to look at what teachers do with their knowledge in actual practice and how they can learn to connect their mathematical knowledge for teaching to their own lessons.

1.3.2. Inquiry-based teacher development: The training ground for teacher noticing

Recently, in the last decade, there has been a shift towards professional development activities that involve some form of job-embedded collaborative teacher inquiry (Darling-Hammond & Richardson, 2009; Timperley, Wilson, Barrar, & Fung, 2007), and teachers learning from their own teaching (Hiebert et al., 2007; Lampert, 2009; Mason, 2010; Smith, 2001; White, Jaworski, Agudelo-Valderrama, & Gooya, 2012). These
professional development programmes include: video clubs where teachers examine, and reflect on practices using video recordings of lessons (Star, Lynch, & Perova, 2011; van Es & Sherin, 2008); study groups where teachers examine classroom artefacts (Goldsmith & Seago, 2011) or analyse lesson plans (Santagata & Angelici, 2010); and lesson study (Lewis, Perry, & Hurd, 2009; Murata, 2010; Stigler & Hiebert, 1999). However, participating in these activities alone does not necessarily help teachers to change or improve their teaching. The contention is that what matters is, not the kind of professional development activities, but what teachers focus on and how they engage with the activities (Bryk, 2009; Lampert, 2009).

In Singapore, the Ministry of Education has emphasised on-going professional development of all teachers as a means to raise the professionalism of teachers. Teachers are entitled to at least 100 hours of professional development annually, and funding is provided by the ministry to attend courses. For mathematics teachers, a two-pronged approach has been adopted to create an environment that encourages quality teaching and learning. In-service mathematics teachers can attend mathematics content and pedagogy courses offered by the National Institute of Education—Singapore’s only teacher training institution—to hone their teaching craft (Lim-Teo, 2009). At the same time, the ministry has supported structures, such as learning communities, for teachers to work collaboratively and learn from their practice (Chua, 2009). Despite the support given to teachers, there are concerns that teachers prefer “generic or pedagogical courses that interest them rather than content courses which address their understanding in areas of weakness” (Lim-Teo, 2009, p. 71). It is also unclear what teachers learn from their participation in learning communities, and how this learning actually helped to improve the quality of mathematics instruction (Chua, 2009). Lastly, there is limited research on mathematics teachers and their teaching in Singapore, even though there have been a lot more studies on students’ learning in mathematics (Lim-Teo, 2009). Therefore, this study makes a contribution towards studying mathematics teaching in Singapore by examining what teachers notice about mathematics, student learning and their teaching strategies.

Besides looking at learning from teaching, there has been a growing interest in decomposing and analysing teaching in order to make teaching practice more learnable by prospective teachers (Ball, Sleep, Boerst, & Bass, 2009; Grossman & McDonald,
The idea is to move away from the notion that teaching is entirely improvisational to a more balanced view, where certain core skills or routines can be mastered by novice teachers (Ball & Forzani, 2009; Lampert & Graziani, 2009). Grossman, Hammerness, and McDonald (2009) characterise these core skills as practices that occur frequently; can be enacted across different contexts; enable teachers to learn about student learning, while honoring the complexity of teaching. Noticing is hypothesised as one of these core skills because it can provide a common language for teachers and researchers to describe, analyse and understand the complexity of teaching, in order to develop teaching expertise (M. G. Sherin, Jacobs, et al., 2011a).

1.3.3. Analysing teaching practice: The aim of teacher noticing

Another related thread of research on mathematics teacher education looks at analysing teaching practice in order to understand its complexity. Lampert (2001), for example, analysed mathematics teaching in terms of the relationships among the teacher, students and mathematics within the context of other factors such as cultural, social and political influences. This complex terrain of teaching highlights the need for teachers to attend and make sense of content, students and teaching simultaneously across different situations at various levels of detail (Ball & Forzani, 2009; Lampert, 2001). Schoenfeld (2011a), on the other hand, views teaching as a goal-oriented decision making process, and analyses teaching as a function of one’s resources, orientations and goals. His framework has been used to examine teachers’ in-the-moment decision making during instruction (Schoenfeld, 2008), teachers’ explanation of mathematical concepts (Schoenfeld, 2010), and other forms of teaching such as lecturing (Hannah, Stewart, & Thomas, 2011). Schoenfeld (2011b) suggests noticing as an important component of this decision making process, and argues that teachers’ ability is influenced by their resources, orientations, and goals.

1.3.4. Teacher noticing during the planning, teaching, and reflecting of a lesson

Noticing thus plays a central role in promoting effective mathematics teaching (Schoenfeld, 2011b; M. G. Sherin, Russ, & Colestock, 2011), and in decomposing teaching practices so that both researchers and practitioners can discuss the various components of teaching expertise (Ball et al., 2009; Grossman & McDonald, 2008). Given the current trends in learning from practice (Chua, 2009; Hiebert et al., 2007), it seems important to
consider how noticing may influence the way teachers study and learn from their practice. Despite the recent advances in the study of noticing, there is still a need to develop a more complete model of how teachers make sense of what they see in the mathematics classrooms (B. Sherin & Star, 2011). Moreover, most research has examined noticing during, or after teaching had taken place. Mason (2002), however, highlights the need for planning to notice so that teachers have a higher likelihood to respond differently to classroom situations. Therefore, it is opportune to investigate what exactly makes teacher noticing productive with regard to planning and designing a task, maintaining a purposeful mathematical discourse with students and learning from their practice.

Considering the central role of noticing in enhancing student reasoning, this study was framed by the following research question:

What makes teachers’ mathematical noticing productive with the goal of enhancing students' mathematical reasoning?

In particular, I aim to answer the following questions in this study:

1. What do mathematics teachers attend to when they notice productively about students’ reasoning?
2. How do teachers interpret and make sense of instructional details to make decisions that are productive with a view to enhance students’ mathematical reasoning?
3. What are the changes in teachers’ resources (mainly knowledge), orientations, and goals with respect to teaching for mathematical thinking when they begin to notice productively?

1.4. Significance of research

This research was designed to contribute to the field of teacher mathematical noticing in two ways. Theoretically, it proposes to clarify what makes teachers noticing productive, and aims to provide a more complete model of what mathematics teachers attend to, and how they interpret teaching events or instructional details, in order to teach for mathematical thinking. This research also seeks to develop a robust framework to analyse teachers’ noticing in order to make teachers’ thinking more
visible, not only during the reflection phase of a lesson, but also during the planning and implementation of the lesson. Such a framework can help to break down the complex processes of teaching to make it more learnable, and at the same time, afford researchers greater insight into the intricacies of practice.

On a more practical note, this study provides a means to incorporate the practices of noticing (Mason, 2002) into the processes of teacher learning to support teacher noticing. It is anticipated that the model developed in this study will be useful for teachers as a self-reflection tool as they examine their own noticing processes.

1.5. Overview of thesis

The thesis is divided into seven chapters. This chapter has set the context, provided the motivation for the study of productive noticing, and highlighted the key questions that guided this study. Chapter Two develops the theoretical framework for this study by reviewing literature on mathematical reasoning and the theoretical perspectives that shape the teaching for reasoning. It also establishes the kind of productive practices that promotes mathematical thinking, and presents some of the key challenges to enact these practices in terms of a teacher’s resources, orientations and goals (Schoenfeld, 2011a). Then, it argues for the centrality of noticing in this study, and refines the questions that framed this research. This is followed by a description of the methodological considerations taken during the study in Chapter Three, before the framework for productive noticing is presented in the following chapter. The framework is then applied to analyse the noticing of two contrasting teachers through a series of snapshots of their noticing in Chapter Five. A discussion chapter then answers the research questions posed in this study, before Chapter Seven completes this thesis with a discussion of the implications, limitations of this research, and possible directions for future research.
Chapter 2. Review of Literature

This chapter develops a conceptual framework for this research by positioning the study of productive noticing as central to teaching for mathematical reasoning. It begins by elaborating on the notion of teaching for mathematical reasoning, and proposes the centrality of reasoning in teaching mathematics. Informed by the four theoretical perspectives of teaching, as presented in Section 2.1.5, this chapter highlights the types of productive practices that can enhance student reasoning in the classroom. The challenges involved in teaching mathematical reasoning are then examined with regard to a teacher’s resources, orientations, and goals for teaching mathematics (Schoenfeld, 2011a), before the notion of teacher noticing is introduced as the key construct, which plays a critical role in making instructional decisions productive for enhancing student reasoning.

2.1. Teaching for mathematical reasoning

Mathematical reasoning, conceived as a way of thinking, is the process of developing conclusions, arguments, explanations, judgements and inferences, from mathematical objects such as ideas, concepts, and representations (Ball & Bass, 2003; Brodie, 2010g; National Council of Teachers of Mathematics, 2009). Central to mathematical reasoning is the notion of mathematical thinking as a set of actions, ways of working, or habits of mind of a mathematician (Bass, 2005; Breen & O'Shea, 2010; Henningsen & Stein, 1997; Jeffcoat et al., 2004; Mason, Burton, & Stacey, 1982; Stacey, 2007; Watson & Mason, 1998). This concept of reasoning, therefore, goes beyond the formulation of a formal proof to emphasise a broader conception of mathematical arguments as a means of explanation and justification. Henningsen and Stein (1997), for example, include the ability to make sense of, think about, and reason about mathematical ideas in “flexible ways” as part of what they termed “mathematical dispositions” (p. 525). They describe mathematical sense-making in terms of processes such as “conjecturing, generalizing, justifying, and communicating mathematical ideas”. Similarly, Watson and Mason (1998, p. 7) use words such as “exemplifying, specialising, generalising, explaining”, amongst others, to delineate how mathematicians ask and solve problems. Contrasting mathematical thinking and knowledge, Burton (1984) describes mathematical thinking
in terms of its "operations, processes, and dynamics" (p. 36) instead of "thinking about mathematics". On the other hand, Isoda and Katagiri (2012) have a more encompassing idea of reasoning that describes how mathematical thinking is motivated by "mindsets", and is related to "methods" and "ideas" (p. 49). Despite the different terminology used in the literature (Bergqvist & Lithner, 2012), four fundamental processes—specialising, generalising, conjecturing and convincing—are of particular interest for developing reasoning through school mathematics (Goos, 2014; Jeffcoat et al., 2004; Mason et al., 1982; Stacey, 2007; Watson & Mason, 1998). Hence, this study focuses on these four component skills of mathematical reasoning.

2.1.1. Specialising and Generalising

Specialising refers to examining particular cases and is the "key to an inductive approach to learning" (Burton, 1984, p. 38), whereas generalising looks at finding patterns and establishing relationships (Stacey, 2007). Specialising involves exploring different facets of a mathematical problem to gain some new information, which may lead to a solution (de Villiers, 2010; Nunokawa, 2010). It requires one to search for special cases, experiment with different variables and look for extreme values in order to test a conjecture, or to refute one by producing a counter-example (de Villiers, 2010; Isoda & Katagiri, 2012). The use of specific examples can also provide a way to approach a problem, and serve as a basis for generalisation (Lane, 2011).

Building on specialising, generalising involves identifying patterns (Stylianides, 2009), and seeing the regularity of mathematical information from different cases in order to generate new meaning (Bergqvist & Lithner, 2012; Russell, 1999; Tall, 1991). In the case of advanced mathematics, Tall (1991) views the process of generalising as the ability to see and apply concepts in a wider context. Following this notion of generalisation, Harel and Tall (1991) differentiate three different types of generalisation—expansive, reconstructive, and disjunctive—according to the learner’s schema. In expansive generalisation, the learner expands the range of application of a schema without rebuilding it; while the learner rebuilds an existing schema to broaden the range of application in the case of reconstructive generalisation. On the other hand, disjunctive generalisation occurs when the learner constructs an additional, and separate schema to deal with the new problem. In spite of the context of advanced
mathematics, Harel and Tall’s (1991) notion of generalisation can also be applicable to the elementary grades. For instance, similar conceptions of generalisation, such as “seeing mathematics as a web of interrelated ideas” (Russell, 1999, p. 5), have been used in school mathematics. Russell (1999) posits that generalising from known mathematical results forms the foundation of mathematical reasoning at the early grades. Mason, Graham, and Johnston-Wilder (2005, p. ix) went even further by asserting that “a lesson without learners having the opportunity to express a generality is not a mathematics lesson”. The emphasis on generalising reflects the growing focus on looking at generalising as a core mathematical activity (Ellis, 2007; National Council of Teachers of Mathematics, 2009; Reid, 2002; Stylianides, 2008).

Although Isoda and Katagiri (2012) see specialising as the inverse process of generalising, they acknowledge that specialising is often used in conjunction with generalising. This view is also echoed by de Villiers (2010), who proposes that generalisations are made on the basis of evidence from experimentation. Similarly, Dreyfus and Eisenberg (1996) view generalising and simplifying—a special case of specialising—as two aspects of “reasoning by analogy” (p. 262). They refer to Polya’s demonstration of solving a geometric problem—into how many regions, at most, can five planes split the ordinary three-dimensional Euclidean space—to show how mathematical thinking often involves alternating between specialising and generalising. Hence, specialising and generalising, can be viewed as a pair of inter-related processes that build on one another to advance mathematical reasoning (Stacey, 2007).

2.1.2. Conjecturing and Convincing

Building on the processeses of specialising and generalising, to conjecture means to hypothesise about the possible mathematical relationships and results arising from the examples examined (de Villiers, 2010; Stacey, 2007). A conjecture, therefore, may be true or false, and should be subjected to further testing (Reid, 2002; Stylianides, 2009). What distinguishes a conjecture from an arbitrary guess is the “reasoned” nature of a conjecture (Stylianides, 2009, p. 264). In other words, it is based on some evidence, possibly from specialising, although the evidence base may be incomplete. For example, de Villiers (2010) suggests how students may extend their exploration of Viviani’s Theorem, which states that states that the sum of the distances from any interior point
to the sides of an equilateral triangle equals the length of the triangle’s altitude, to investigate whether this constancy exists in other regular polygons. Here, conjecturing is seen as the intermediate step between specialising and generalising in both inductive and deductive reasoning (Burton, 1984), and can possibly lead to further specialising and generalising.

However, a distinguishing feature of mathematics is to go beyond conjecturing to convince oneself and others why a conjecture is true (Burton, 1984). It involves “finding and communicating the reasons why something is true” (Stacey, 2007, p. 41), which can take the form of a mathematical proof. Convincing or justifying is the mechanism by which a generalisation is made public (Ball & Bass, 2003; Burton, 1984) to convince others of its veracity, and to explain why it is true (Ball & Bass, 2003; Brodie, 2010g; Burton, 1984; Harel & Sowder, 2005; Staples, Bartlo, & Thanheiser, 2012). Given the importance of the explanatory function of justification in school mathematics, this study had adopted a broader conception of argumentation (Ball & Bass, 2003; Brodie, 2010g; Kilpatrick et al., 2001) to include formal proofs, as well as other forms of arguments such as rationales, generic examples, and demonstrations (Dreyfus, Nardi, & Leikin, 2012).

2.1.3. Focusing on generalising and justifying

The four key processes—specialising, generalising, conjecturing, and convincing—do not follow a linear sequence. Instead, Burton (1984) suggests a more cyclical process of mathematical reasoning, consisting of an unspecified series of “helical loops” of actions consisting of “manipulating; getting a sense of pattern; and articulating that pattern symbolically” (p. 39). According to this model of mathematical thinking, the reasoning process starts with an initial exploration (specialising) to get a sense of the pattern (generalising) before articulating the pattern (conjecturing), and testing the pattern (convincing) by looking at particular examples (specialising) or constructing an argument (Burton, 1984; Mason et al., 1982). Although specialising is a possible entry point to mathematical thinking, Lane (2011) argues that reasoning may not always begin with exploration. For instance, students may not begin with looking at specific examples when they make conjectures (Lane & Harkness, 2012). This suggests that mathematical reasoning is a much more dynamic activity than previously thought.
Despite the integrated nature of mathematical reasoning, generalising and justifying seems to stand out as critical processes among the four. As argued by Mason et al. (1982), the purpose of specialising is to generalise. In that respect, they delineate three types of specialisation which leads to generalisation: “random, systematic, and artful” (p. 15). The idea of random specialising is to get an idea of the problem, before systematically specialising to set the stage for generalising. Furthermore, an artful specialising can produce a refutation of the conjecture, and motivate the seeking of another generalisation. Even though researchers, such as Lane and Harkness (2012), may counter Burton’s (1984) assumption that specialising comes before generalising, they acknowledge that generalising without specialising may run the risk of making a false generalisation. On the other hand, if students only generalise based on making sense of empirical cases, then they are also more likely to justify using specific examples (Ellis, 2007). Hence, generalising, which builds on specialising and conjecturing, can be seen as one of the primary processes in reasoning.

Convincing self and others about the truth of a conjecture is another key process in mathematical reasoning that serves to connect different concepts to provide the evidence for the conjectures or generalisations (Brodie, 2010g). Convincing is inextricably linked to generalising (Brodie, 2010g; Ellis, 2007; Lannin, 2005), and as Ellis (2007, p. 196) highlights:

> ... the connection between generalization and justification is bidirectional—engaging in acts of justification may be as likely to influence students’ abilities to generalize as the other way round. Learning mathematics in an environment in which providing justifications for one’s generalizations is regularly expected can promote the careful development of generalizations that make sense and can therefore be explained.

Therefore, given that justifying supports generalising, and vice versa, it is important to focus on developing these two critical processes of mathematical reasoning so that specialising and conjecturing are also encouraged in the classrooms. In light of preceding discussion, and following Blanton and Kaput (2005), this present study thus views mathematical reasoning as a process in which learners conjecture ideas (conjecturing) from specific cases (specialising), and establish these generalisations using explanations and other forms of justifications. The next section will highlight why
mathematical reasoning is a worthwhile goal to pursue, before a description of the theoretical perspectives, which shaped the kind of productive classroom practices that promote students’ reasoning in the context of this study.

2.1.4. Why teach for mathematical reasoning?

Mathematical reasoning, as a way of thinking, is central to doing mathematics. As Thurston (1994, p. 171) puts it:

There is a real joy in doing mathematics, in learning ways of thinking that explain and organize and simplify. One can feel this joy discovering new mathematics, rediscovering old mathematics, learning a way of thinking from a person or text, or finding a new way to explain or to view an old mathematical structure.

It can be argued that mathematical reasoning is the means by which mathematicians make sense of, and create, new mathematical knowledge (Kilpatrick et al., 2001; Pape, Bell, & Yetkin, 2003; Weiss & Moore-Russo, 2012; T. Wood, Williams, & McNeal, 2006). Some researchers view mathematical reasoning and thinking together (Stein, Grover, & Henningsen, 1996); while others see reasoning as part of mathematical thinking (T. Wood et al., 2006). Regardless of their stance, most mathematics educators would agree that developing students’ mathematical reasoning is both an important goal in teaching mathematics, and a means to develop mathematical proficiency (Ball & Bass, 2003; Brodie, 2010g; Kilpatrick et al., 2001; National Council of Teachers of Mathematics, 2009; Stacey, 2007; T. Wood et al., 2006).

Kilpatrick et al. (2001) describe reasoning as the “glue” that binds different mathematical ideas, skills, and concepts together as a coherent whole (p. 129). It allows for conjectures to be proven or countered, and is a way to get a sense of the mathematical meaning behind the procedures. As Ball and Bass (2003) argue, mathematical reasoning is the basis for relational understanding—knowing the what and the why (Skemp, 1978)—and is necessary for recalling, and reproducing forgotten mathematical procedures that were once understood by students. It is also through reasoning about prior mathematical concepts that students are able to learn new ideas (Sfard, 2003) and apply these ideas flexibly in different contexts (Ball & Bass, 2003; Kilpatrick et al., 2001).
Focusing on mathematical reasoning, therefore, has the potential to equip students with the proficiencies to learn mathematics by themselves, and for themselves (Isoda & Katagiri, 2012). Students can learn—through carefully designed activities—the way to approach problems, think about possibilities, generate explanations and discover mathematical ideas (Isoda & Katagiri, 2012; Weiss & Moore-Russo, 2012). For example, Weiss and Moore-Russo (2012) reported how students were engaged in generating questions, investigating their own questions, formulating conjectures, presenting evidence and proving their conjectures, through a research project on the mathematical properties of duals—figures formed by joining the mid-points of adjacent sides of a polygon. This example, and others (Fraivillig, Murphy, & Fuson, 1999; Rigelman, 2007; Romberg, 1994; Stein et al., 1996), demonstrate how teachers can use mathematical reasoning to help students see how different ideas are connected to solve problems in mathematics.

Using instructional strategies that promote mathematical reasoning can also empower learners to “think and make decisions” (Isoda & Katagiri, 2012, p. 32), and engage them to generate new questions in mathematics (Weiss & Moore-Russo, 2012). As Romberg and Kaput (1999) propose, “School mathematics should be viewed as a human activity that reflects the work of mathematicians—finding out why given techniques work, inventing new techniques, justifying assertions, and so forth” (p. 5). Such a view of learning mathematics thus places mathematical reasoning at the centre of teaching. Hence, reasoning acts as a mediating agent between the students and mathematics, as well as between students, as they work together to negotiate new understanding of the subject.

Therefore, developing students’ ability to think mathematically can be considered as one of the most important goals of teaching mathematics. The National Council of Teachers of Mathematics (NCTM) advocates that instructional programmes pre-kindergarten through grade 12 should provide opportunities for all students to see reasoning as “fundamental aspects of mathematics” (National Council of Teachers of Mathematics, 2000, p. 56); The New Zealand mathematics curriculum views mathematics and statistics as “different ways of thinking” that equip students with “effective means for investigating, interpreting, explaining, and making sense of the world in which they live” (Ministry of Education-New Zealand, 2007, p. 26); And in
Chapter 2 – Review of Literature

Singapore, one of the key aims of teaching mathematics is to “develop thinking, reasoning, communication, application and metacognitive skills through a mathematical approach to problem-solving” (Ministry of Education-Singapore, 2013, p. 10). Underlying these goals is the belief that reasoning is vital to learning and using mathematics.

While the vision for teaching mathematical reasoning is compelling, how teachers can enact this in the classrooms is less clear, and may be challenging (Ball & Bass, 2003; Brodie, 2010g; Chamberlin, 2005; Schifter, 2001). Emphasising reasoning in the teaching of mathematics, like developing other mathematical competencies, is critical and is not a good-to-have (Ball & Bass, 2003; Brodie, 2010g; National Council of Teachers of Mathematics, 2009). However, doing so places enormous intellectual demands on teachers. As Ball and Bass (2003, p. 42) argue:

Simply posing open-ended mathematical problems that require mathematical reasoning is not sufficient to help students learn to reason mathematically. Neither is merely asking students to explain their thinking.

Hence, the key to teaching for mathematical reasoning lies in how teachers make sense of students’ thinking. It is, therefore, crucial for this study to gain a better understanding of how teachers can design tasks and engage with students in productive discussions that enhance their understanding of mathematics.

2.1.5. Theoretical perspectives on mathematical reasoning

The design of this study was informed by three main theoretical perspectives of learning and teaching of mathematics: constructivist, socio-cultural and situated theories. In addition, a commognitive view of mathematical reasoning (Sfard, 2008), which sees mathematical thinking as communication, was adopted to frame this present study. These theories helped shaped the kinds of productive classroom practices that enhance students' reasoning, the methods used to study and analyse thinking, as well as the interactions between teachers, students, and mathematics, during this research. Even though purists may argue that these various perspectives are incompatible based on their underlying assumptions, other researchers view them as being complementary (Brodie, 2010g; Cobb & Bowers, 1999; Sfard, 2003) or “incommensurable” (Sfard, 2003, p. 355). Incommensurable perspectives cannot be reconciled using any agreed set of
criteria, but they are not necessarily mutually exclusive (Sfard, 2003), while complementary perspectives provide different views of the same thing (Cobb & Bowers, 1999; Sfard, 2003). This study, took a more pragmatic position, and followed similar arguments by Brodie (2010g), and Lester (2010) that various theories can be adapted to account for teaching and learning at different levels in different combinations, depending on the issues at hand. Hence, the purpose of these different perspectives here is to provide a richer view of what constitutes mathematical reasoning, and how teachers can enact practices that enhance student thinking in the classrooms.

2.1.5.1. Constructivist perspectives

A key tenet of constructivism is that all knowledge is constructed (Brodie, 2010g; Noddings, 1990). It assumes the existence of cognitive structures in the mind of the learner, which are actively constructed, organised, and restructured, through *reflective abstraction* (Piaget, 1978). As explained by Noddings (1990), reflective abstraction differs from abstracting from observations of objects in that it occurs through an interiorisation of operations on objects, which is to derive properties of objects through ways of working with them, instead of the objects themselves. Dubinsky extended this notion, and developed the APOS (Actions-Process-Object-Schema) Theory to explain the learning of concepts in advanced mathematics (Arnon et al., 2014). Despite the different varieties of constructivism, the emphasis on the construction mechanisms of mental schema highlights two key principles to consider for the teaching and learning of mathematics:

1. What learners learn is highly dependent on their prior knowledge and experiences (Ausubel, 1963, 1968; Brodie, 2010g; Cobb, 1994; Confrey, 1990; R. B. Davis, Maher, & Noddings, 1990); and

2. Teachers need to think about how learners understand the concepts because what “learners think, say, and do make sense to them in relation to what they know” (Ausubel, 1963; Brodie, 2010g, p. 12; Skemp, 1971).

Therefore, a key area of research in constructivism is the study of misconceptions, which provides the lenses to examine students’ mental construction of mathematical concepts (Brodie, 2010g; Confrey, 1990). Misconceptions are persistent, systematic, and “rational errors” (Ben-Zeev, 1998, p. 366), which stem from students’ attempts at
making sense of the concepts. According to Julie and Julian (2007), these errors may occur when learners have a wrong or incomplete representation of the concepts, when they overgeneralise the rules and procedures, when they attend to irrelevant properties as a result of their prior or informal encounters with the concepts or when they are unable to link flexibly the process-object perspectives (Gray & Tall, 1994) of the concepts. The formation of misconceptions, which is part and parcel of the learning process (Brodie, 2010g), may be explained by a conflict between a learner’s existing cognitive structures, such as concept images (Tall & Vinner, 1981), with the concept definitions. More recently, McGowen and Tall (2010) propose the notion of a met-before as a “mental structure” (p. 171), which a learner possesses as a consequence of prior experiences with the concept. However, unlike a misconception, a met-before can support or impede learning of a concept.

According to constructivists, the notion that misconceptions or met-befores can arise from mental schemata of the learners as they attempt to construct their understanding is an important one (Brodie, 2010g). This idea is critical in orientating teachers’ teaching towards student reasoning, in order to understand what the learners are thinking about the concepts (Ball & Cohen, 1999; Brodie, 2010g; Nesher, 1987). In other words, teachers who focus their attention on students’ thinking would aim to understand the reasons for the learners’ responses, including their mistakes in the classroom, and see these mistakes as possible transitions to forming correct conceptions of mathematical ideas (Brodie, 2010g). Given that misconceptions can also give rise to correct responses, it is crucial that teachers ask learners to explain their reasoning when they give correct, as well as wrong contributions during lessons (Nesher, 1987). This would enable teachers to gain a better understanding of student reasoning about the concept of interest. In order to respond appropriately to students’ contributions, it is critical, therefore, that teachers listen carefully to how students think, in order to make sense of students’ schema of the concept. How teachers can do this in classrooms is less clear. Therefore, investigating what and how teachers listen and respond will be an important area of investigation in this study.
2.1.5.2. Socio-cultural perspectives

While a constructivist view of learning focuses on cognitive processes on mental objects in the mind of an individual student, socio-cultural theorists see social interaction in a culturally-organised context as the key mechanism in learning (Brodie, 2010g; Cobb, 1994; Lerman, 2013). Forman (2003) highlights three important aspects underpinning a socio-cultural perspective of teaching and learning:

1. The focus is on the process of change, instead of the outcome;
2. The centrality of social processes in learning; and
3. Thinking is mediated by signs and symbols.

These ideas are encapsulated in the notion of Zone of Proximal Development (ZPD) (Chaiklin, 2003; Del Río & Álvarez, 2007; Vygotsky, 1978), which is defined as the “distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers” (Vygotsky, 1978, p. 86). Even though Chaiklin (2003) critiques the common view about ZPD in terms of the importance of support from those who are more capable, the notion of ZPD suggests a central role for teachers to coordinate teacher-student and student-student interactions in the context of a learning community (Brodie, 2010g; Forman, 2003).

Of particular interest in this study is how a teacher may structure the mathematical activity in the classroom in order to facilitate these productive interactions. This may involve establishing the sociomathematical norms, which are regulative aspects of a discussion specific to mathematics and mathematical activities, in the classrooms (Yackel & Cobb, 1996). These norms are crucial for modelling the mathematical discourse that allows students and teachers to negotiate meanings collectively during mathematics lessons, and open up the way for a more participatory notion of reasoning (Forman, 2003; Smith & Stein, 2011; Yackel & Cobb, 1996).

2.1.5.3. Situative perspectives

As presented in the preceding paragraphs, both constructivist and sociocultural perspectives posit that learning takes place through participation. However, a situative perspective views learning as participation (Brodie, 2010g; Cobb & Bowers, 1999;
Greeno, 2003; Lave, 1993, 1996). That is, “to learn is to participate better” (Brodie, 2010g, p. 16). In other words, becoming a member of the classroom mathematical community, and participating in mathematical classroom practices, is fundamental to students’ learning of the subject (Brodie, 2010g; Greeno, 2003). The emphasis in the classroom, thus, shifts from knowledge, tools and other resources for learning, to ways of being (Lave, 1996). As Lave (1996) argues, the emphasis on “learning mechanisms” in a more individualistic cognitive paradigm may diminish, and give way to “practices” under a situated learning perspective (p. 157):

Rather than particular tools and techniques for learning as such, there are ways of becoming a participant, ways of participating, and ways in which participants and practices change. In any event, the learning of specific ways of participating differs in particular situated practices. The term "learning mechanism" diminishes in importance; in fact it may fall out altogether, as "mechanisms" disappear into practice. Mainly, people are becoming kinds of persons.

This notion of ways of being gives rise to a learner's identity as a mathematician to various degrees (Greeno, 2003; Lave, 1996). Therefore, as (Greeno, 2003, p. 317) describes, “learning mathematics is like learning one's way around in an environment”, where one learns to look for and make use of resources to reach the intended destination. A situated perspective of learning will necessarily involve students in formulating and solving problems, proposing, evaluating and explaining solutions. Hence, an important area to consider, for this study, will be the role of mathematical discourses in shaping student participation in a community of mathematics learners (Brodie, 2010g; Forman, 2003; Greeno, 2003).

Another important consideration is that there are two communities of practice involved in this study: students and teachers. The teachers, in this study, were part of a localised professional learning community in their respective schools. Therefore, their participation as teachers, taking on the inquiry stance (Cochran-Smith & Lytle, 1999; Hiebert, Morris, & Glass, 2003), can also be viewed through a situative perspective of learning (Borko, 2004). Similar to learners of mathematics, teachers who participate in professional learning communities can be understood as learners trying to participate in the practice of teaching through the process of being a member of this community, in order to know more in and about teaching. Just as in the case of students, where
evidence of learning can be found in the conversations (Brodie, 2010g), the case for teacher learning can also be located in the professional mathematical discourse undertaken during their practice. This notion of teacher participation in a learning community puts forth the analysis of discourses at the centre of this study.

2.1.5.4. A Commognitive perspective

The various perspectives of teaching and learning mathematics, presented thus far, can be broadly positioned along a spectrum, with “learning-as-acquisition” on one end, and “learning-as-participation” on the other (Sfard, 2003, p. 355). The former describes learning in terms of acquiring mental structures such as schemata, models, concept images, or misconceptions; while the latter views learning as reorganising activities that follow from one being part of a community of practice (Sfard, 2003, 2008). These two metaphors for learning have something to offer in the discussion of mathematical reasoning (Sfard, 2003) even though such a dichotomy may pose challenges in examining student thinking (Sfard, 2008). In an attempt to dismantle the perceived dichotomies in the various perceptions of teaching and learning, Sfard (2008) developed the commognition framework, which focuses on discourse as practiced by a community as the unit of analysis in research on mathematical discourse (Forman, 2012; Güçler, 2013; Sfard, 2008, 2012; Sfard & Kieran, 2001).

The premise of the commognition framework is that thinking is an “individualised version of interpersonal communication” (Sfard, 2008, p. 81). Building on a learning-as-participation metaphor, Sfard proposed that mathematical thinking can be seen as a form of discourse (Sfard, 2008, 2012), and hence, learning is conceptualised as changes in one’s discourse through participating in a community (Güçler, 2013). According to this theory, mathematical discourse can be characterised by four elements: word use, visual mediators, narratives, and routines (Sfard, 2008, 2012).

The first distinctive characteristic of mathematical discourses is the keywords used in the communication of mathematical ideas. These keywords have special meanings, which are used in mathematically unique ways (Sfard, 2012). The meanings of these words are negotiated by communities of interest, such as schools and academia, and are necessary for communication. Next, visual mediators refer to the “visible objects that are operated upon as a part of the process of communication” (Sfard, 2008, p. 133). These
can include different mathematical representations such as symbols, graphs, and diagrams (Sfard, 2012). Third, a narrative is any set of utterances, which describes mathematical objects and their relationships, and are subject to endorsement or rejection (Güçler, 2013; Sfard, 2012). Mathematical theorems, definitions, algorithms, and proofs are examples of endorsed narratives, which are accepted as true by the mathematical community.

Last but not least, the notion of a routine refers to “patterned ways in which mathematical tasks are being performed” (Sfard, 2012, p. 2). This refers to typical practices used in specific ways by the community, and includes mathematical processes such as specialising, conjecturing, generalising, and convincing (Nardi, Ryve, Stadler, & Viirman, 2014), which can lead to an endorsed narrative, or mathematical theory (Sfard, 2008). Such routines are termed “explorations” by Sfard (2008, p. 224). However, not all routines ends up in an endorsed narrative. Some routines are characterised by actions that result in a physical change in objects; while others may be geared towards aligning oneself with others to gain some form of social acceptance (Sfard, 2008). These routines are called deeds and rituals respectively.

Mathematical learning, according to the commognition framework, can thus be object-level or meta-level. According to Sfard (2008), object-level learning results in the “growth in the number and complexity of endorsed narratives and routines” (p. 300); whereas meta-level learning occurs when there is a change in the metarules of the discourse. A metarule is a “rule that defines patterns in the activity of the discursants” (Sfard, 2008, p. 299), which may include rules that regulate participation in the classroom (such as social norms); those that portray participants’ intentions (engaging in mathematical activities to learn versus engaging to avoid trouble with teachers); as well as those that regulate object-level rules of mathematics, such as fraction as equipartitioning of a whole (Güçler, 2013; Sfard & Kieran, 2001). On the other hand, when words, mediators, or routines, are used differently in a discourse, they may result in an incommensurable discourse, which leads to a commognitive conflict that hinders communication, and in turn, learning (Sfard, 2008). A simplified model to represent mathematical learning from a commognitive perspective is summarised and shown in Figure 2.1.
In this study, teachers’ mathematical discourses with students, which are emphasised in both sociocultural and situative perspectives of learning (Forman, 2003; Greeno, 2003), as well as the mathematical and pedagogical discourses among teachers, will be critical for examining how teachers think about student reasoning. Moreover, as highlighted by Forman (2012), the commognition framework is useful for orchestrating mathematical discussions in the classrooms, and this is critical for examining student reasoning. The commognition perspective of learning and teaching can therefore provide a means to do a fine-grained analysis of the interactions that occur in this study. However, while the commognition framework has been used to study discourses between teachers and students (Güçler, 2013; Nachlieli & Tabach, 2012; Sfard, 2008), student discourses (M. B. Wood & Kalinec, 2012) and curricula (Newton, 2012), it has not been used to study the professional discussions of teachers, which plays a critical role in this study. How the commognition framework can be extended to analyse discourse between teachers is discussed in Section 3.5.5.

![Diagram of the commognitive perspective]

Figure 2.1. Summary of the commognitive perspective.

2.2. Productive practices for teaching mathematical reasoning

The four perspectives—constructivist, socio-cultural, situative and commognitive—discussed in the previous section are important for shaping the kind of practices that are seen to be productive for enhancing students’ mathematical reasoning. Drawing on
the various perspectives, this section reviews some of these productive practices in three aspects: task design, classroom interaction, and reflecting about student thinking.

2.2.1. Designing tasks for mathematical reasoning

Mathematical tasks are important for developing mathematical competency. Engaging students with appropriate tasks is thus critical for developing students’ mathematical reasoning (Anthony & Walshaw, 2009; Brodie, 2010e; Sullivan, Clarke, & Clarke, 2013; Yankelewitz, Mueller, & Maher, 2010), and hence, the design of mathematics tasks plays a key role in facilitating and encouraging student thinking (Ball & Bass, 2003; Breen & O'Shea, 2010; Doerr, 2006; Mason & Johnston-Wilder, 2006; Smith & Stein, 1998, 2011). Tasks and activities are sometimes taken to mean the same thing, but Mason and Johnston-Wilder (2006) distinguish them by highlighting that “the purpose of a task is to initiate activity by learners” (p. 5), and hence a task is a “prompt” or set of instructions for students’ work (Sullivan et al., 2013, p. 13). According to them, learners engage in mathematical processes such as mathematical reasoning, which act upon physical, symbolic, or mental objects, in order to become sensitised to important features in the concept (Mason & Johnston-Wilder, 2006). Therefore, teachers need to design, select, and adapt tasks thoughtfully so that they can provide ample opportunities for students to generalise, explain, and justify their mathematical ideas (Ball & Bass, 2003; Breen & O'Shea, 2010; Smith & Stein, 2011; Sullivan et al., 2013). Different researchers have proposed various principles for designing a mathematics task (Ball & Bass, 2003; Fahlgren & Brunström, 2014; Mason & Johnston-Wilder, 2006; Smith & Stein, 1998; Sullivan et al., 2013), and they can be broadly classified into the following three key considerations.

2.2.1.1. The Mathematical Focus

The first goal of a mathematical task is to engage students in developing mathematical concepts and processes. To do so requires teachers to maintain a clear focus on the mathematics involved (Anthony & Walshaw, 2009; Sullivan et al., 2013). Having a mathematical focus for a task is not simply asking students to work on a series of computations, but rather, as Anthony and Walshaw (2009) highlight, it is to engage learners in thinking with and about mathematical ideas. Teachers, therefore, need to identify the key ideas behind the mathematical concepts, and weave these ideas into the
task (Sullivan et al., 2013). A good starting point is for teachers to understand the common misconceptions, and think about how best to connect learners’ current conceptions to the task, so that they can provide opportunities for students to learn from their mistakes (Anthony & Walshaw, 2009).

To this end, Yang and Ricks (2013) detail how Chinese teachers think about the design of a task in a lesson using the “three points”: the “Key Point”, “Difficult Point”, and the “Critical Point” (p. 54). The Key Point refers to the mathematical concept targeted in the lesson, which is sometimes known as the “Big Idea” (Askew, 2013, p. 6). The Difficult Point is the cognitive obstacle or stumbling block that students face when learning the Key Point. This can refer to persistent errors or common misconceptions that are associated with the concepts to be taught. By anticipating the students’ Difficult Point, teachers can design lessons that are targeted at the challenging aspects of learning the concept. The Critical Point is then the “heart of the lesson”, which highlights the approach that teachers can use to support students in their efforts to overcome the Difficult Point, in order to learn the Key Point (Yang & Ricks, 2012, p. 43). Together, the Three Points can provide a useful way, similar to the notion of milieu of mathematics-student-teacher (Brousseau, 1997; Mason & Johnston-Wilder, 2006) or the idea of the instructional triangle (Cohen, Raudenbush, & Ball, 2003), to think about mathematics teaching, and more specifically, the design of a task before its implementation.

As an example, to teach fraction-decimal conversion at Primary 4 (age 10), a teacher may identify the key concept as the fact that common fractions and decimal fractions are different representations of the same number (Key Point); highlight students’ confusion in terms of their inability to relate fractions with denominators other than 10 to decimals (Difficult Point), that is, they may put 1/5 as 0.15 because the digits “1” and “5” appeared in 1/5; and the proposed course of action (Critical Point) is to create tasks where students can relate fractions such as 1/5 to fractions with denominators 10, 100 or 1000. This example illustrates how the Three Points can be used to direct teachers’ attention to the relationship between specific aspects of the concept (Key Point and Difficult Point) to the design of the task (Critical Point). However, the ability to describe the details of the Three Points is dependent on a good understanding of mathematics as well as the experience in teaching the subject. Hence, this ability has been used as a distinguishing mark between highly and less proficient teachers in China (Yang & Ricks,
2013). In this study, the Three Point Framework was considered a good starting point for teachers in Singapore to think about the mathematics focus when designing a task.

2.2.1.2. The Mathematical Activity

What matters most in a task is the mathematical way of thinking that is embedded in the instructions (Mason & Johnston-Wilder, 2006). The intended mathematical activity of a task, which consists of the actions and thinking, may not correspond to what students do. For example, primary school students may be engaged in drawing different triangles (action), measuring the interior angles in a triangle (action), and calculating the total interior angle sum (action), without realising that their teacher intended them to conjecture about the constant sum of the three angles (action and thinking). As highlighted by Mason and Johnston-Wilder (2006, p. 70), a mathematical activity may not occur even if the tasks are completed correctly:

> It [mathematical activity] must be more than learners busily getting on with something or producing pages of written work. Although learners may be happily engaged in social interaction through discussion, or fully occupied in using scissors or drawing up tables, they may still not be undertaking mathematical activity. On the other hand, they may be sitting quietly, apparently staring out of the window, and yet be thinking deeply: this could be mathematical activity. For learners’ activity to be mathematical, there have to be elements of mathematical thinking...

These elements of mathematical thinking can be seen in tasks, in which learners use their thinking or “natural powers” (Mason & Johnston-Wilder, 2006, p. 74), such as generalising and convincing, to engage with mathematical concepts and relationships (Anthony & Walshaw, 2009; Stein et al., 1996). Mathematics tasks should be designed to encourage students to consider the mathematical ideas and structures as they act on the tasks; decide on the problem solving approaches to take; and reason about the quality of their own responses (Stein et al., 1996), without always relying on the teacher to provide the directions (Anthony & Walshaw, 2009). Moreover, these types of tasks are not only characterised by their “higher-level cognitive demands” (Smith & Stein, 1998, p. 348), but also take into consideration in their designs, the learners’ current mathematical background and given resources, in order for them to realise the possibilities for learning (Brodie, 2010e; Mason & Johnston-Wilder, 2006).
Table 2.1: Characteristics of tasks (Smith & Stein, 1998, p. 348)

<table>
<thead>
<tr>
<th>Lower-level Cognitive Demands</th>
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<tbody>
<tr>
<td><strong>Memorisation</strong></td>
<td>Involves either reproducing previously learned facts, rules etc. or committing them to memory.</td>
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<tr>
<td></td>
<td>Cannot be solved by using procedures, because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure.</td>
</tr>
<tr>
<td></td>
<td>Are not ambiguous. Such tasks involve exact reproduction of previously seen material and what is to be reproduced is clearly and directly stated.</td>
</tr>
<tr>
<td></td>
<td>Have no connection to the concepts or meaning that underlies the facts, rules etc.</td>
</tr>
<tr>
<td><strong>Procedures without Connection</strong></td>
<td>Are algorithmic. Use of the procedure is either specifically called for or is evident from prior instruction, experience or placement of task.</td>
</tr>
<tr>
<td></td>
<td>Require limited cognitive demand for successful completion. Little ambiguity exists about what needs to be done or how to do it.</td>
</tr>
<tr>
<td></td>
<td>Have no overt connection to the concepts or meaning that underlies the procedure being used.</td>
</tr>
<tr>
<td></td>
<td>Are focused on producing correct answers instead of developing mathematical thinking.</td>
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<td></td>
<td>Require no explanations or explanations that focus solely on describing the procedure that was used.</td>
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<tr>
<th>Higher-level Cognitive Demands</th>
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<tbody>
<tr>
<td><strong>Procedures with Connection</strong></td>
<td>Focus students’ attention on the use of procedures for the purpose of developing deeper levels of understanding of math concepts and ideas.</td>
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<tr>
<td></td>
<td>Suggest, explicitly or implicitly, pathways to follow that are broad general procedures that have close connections to underlying concepts as opposed to narrow algorithms that are opaque with respect to underlying concepts.</td>
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<tr>
<td></td>
<td>Usually represented in multiple ways, such as visual diagrams, manipulative, symbols, and problem situations. Making connections among multiple representations helps develop meaning.</td>
</tr>
<tr>
<td></td>
<td>Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with conceptual ideas that underlie the procedures to complete the task successfully.</td>
</tr>
<tr>
<td><strong>Doing Mathematics</strong></td>
<td>Require complex and non-algorithmic thinking—a predictable, well-rehearsed approach or pathway is not explicitly suggested by the task, task instructions, or a worked-out example.</td>
</tr>
<tr>
<td></td>
<td>Require students to explore and understand the nature of math concepts, processes or relationships.</td>
</tr>
<tr>
<td></td>
<td>Demand self-monitoring or self-regulation of one’s cognitive processes.</td>
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<tr>
<td></td>
<td>Require students to access relevant knowledge and experiences and make appropriate use of them in working through the task.</td>
</tr>
<tr>
<td></td>
<td>Require students to analyse the task and actively examine task constraints that may limit possible solution strategies and solutions.</td>
</tr>
<tr>
<td></td>
<td>Require considerable cognitive effort and may involve some level of anxiety for the student because of the unpredictable nature of the solution processes required.</td>
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</table>
For this study, a useful way for teachers to think about these elements of mathematical thinking is to recognise the characteristics of tasks corresponding to the different cognitive demands. Using the categorisation by Stein et al. (1996), tasks that focus on memorisation, and procedures without connections to the underlying concepts, as lower-level cognitive demands; and those that require students to work on procedures and think about the underlying mathematics, as well as those that encourage students to participate as a mathematician, to involve higher-level cognitive demands. The detailed characteristics of tasks with different cognitive demands are shown in Table 2.1 (Smith & Stein, 1998, p. 348).

2.2.1.3. The Mathematical Documentation

In order to see whether learners have successfully engaged with learning the mathematical ideas through the mathematical activity initiated by the task, it is essential for learners to make visible their mathematical thinking as much as possible (Lesh, Hoover, Hole, Kelly, & Post, 2000). Therefore, they should have opportunities to express their thinking explicitly (Doerr, 2006) using multiple representations (Anthony & Walshaw, 2009; Sullivan et al., 2013). As Anthony and Walshaw (2009) argue, giving students opportunities to work with different representations can help them develop their mathematical proficiencies. This emphasis on the role of representations in developing mathematical reasoning is also supported by the notion of “representational versatility—the ability to work seamlessly within and between representations, and to engage in procedural and conceptual interactions with representations” (Thomas, 2006, p. 233; 2008)—as an indicator of student understanding. Hence, accomplished teachers not only provide their students opportunities to reason using a wide range of representations, but also emphasise the development of fluency in using and translating between representations (Anthony & Walshaw, 2009; Sullivan et al., 2013).

Moreover, enhancing student reasoning is not just about incorporating representations within the task design, but more importantly, the choice of representations or models should relate directly to the mathematical concept, which forms the focus of the task (Sullivan et al., 2013). However, creating or adapting these high-quality tasks is not easy, and given the new range of technologies available for teachers to use, it may be helpful for teachers to work collaboratively using students’ misconceptions as a starting
point (Sullivan et al., 2013). By doing so, teachers can engage students to use these models and representations to articulate what they are thinking explicitly, so that teachers can ‘see’ what and how the students are thinking about the mathematical ideas (Doerr, 2006). The use of representations to express thinking can then provide evidence of student reasoning, which the teacher can choose to take up during the implementation of the task.

However the design of the task, as Henningsen and Stein (1997) point out, is necessary but not sufficient for enhancing student reasoning. Based on their analysis of 58 tasks (out of 144) that may afford ‘doing mathematics’ (Smith & Stein, 1998), they found that it is critical for teachers to support student reasoning by “pressing” them to “provide meaningful explanations or make meaningful connections”, without “reducing the complexity and cognitive demands of the tasks” (p. 546). This is what Mason and Johnston-Wilder (2006) termed as “scaffolding and fading” (p. 83). Therefore, how a teacher engages students with the task during task implementation is of utmost importance in supporting their mathematical reasoning.

### 2.2.2. Listening and responding to student reasoning

Henningsen and Stein (1997) point out that it is possible for a high-level cognitive demand task to be implemented as a task with lower cognitive demands. This possibility highlights the importance of the participatory nature of mathematics over the cognitive perspective. One significant aspect of this situative perspective of mathematics is the type of discourses that take place during the implementation of the task (Franke et al., 2009; Greeno, 2003; Henningsen & Stein, 1997; Mason, 2000; O'Connor & Michaels, 1993; Yackel & Cobb, 1996). For the task to be implemented in a way that realises its design potential, it is useful to direct teachers’ attention to how students (and teacher) discuss mathematics (Yackel & Cobb, 1996), the patterns of conversations that may hinder or promote student participation (Greeno, 2003; Mehan, 1979; O'Connor & Michaels, 1993) and the type of questions and prompts used in these discourses (Herbal-Eisenmann & Breyfogle, 2005; Mason, 2000; Mason & Johnston-Wilder, 2006). The following three sections review some of the literature that informed the nature of the classroom discourses envisioned in this study.
2.2.2.1. Orchestrating whole-class discussions

An important consideration for orchestrating whole-class discussions is the distinction between two patterns of discourse, which occur frequently in the classrooms (Forman, 2003; Franke, Kazemi, & Battey, 2007; Greeno, 2003). The first, found by Mehan (1979), is called IRE or IRF as characterised by the three key components of this pattern: *Initiation* (I), which usually comes in the form of a question by the teacher; *Response* (R), usually by a student responding to the question posed; and *Evaluation* (E) or *Feedback* (F), in which the teacher evaluates the student’s response. The second pattern, revoicing, is initiated by the student with a statement or question that is followed by the teacher's rephrasing of what was understood by the statement, before inviting the student initiator, or others, to explain or justify the initiating statement. This provides an opening for students to respond to the statement. The teacher will then coordinate the negotiation between students in order to arrive at an accepted answer or explanation (O'Connor & Michaels, 1993).

Adapting the notion of conversation schema (Clark & Schaefer, 1989), Greeno (2003) highlights the differences between these two patterns, as shown in Figure 2.2 and Figure 2.3.

![Figure 2.2. An IRE discourse schema (Greeno, 2003, p. 322).](image)

As represented in Figure 2.2 and Figure 2.3, the key difference between the two patterns lies in the role of student and teacher. In the schema, to contribute means to participate in the discourse, which usually refers to a collective action by participants on some piece of information (Clark & Schaefer, 1989). In the simple case, a conversation presents some idea and is accepted by the participants, and if the idea is
not accepted, then it starts another contribution via a new presentation (Clark & Schaefer, 1989). Greeno (2003) extends the schema to include this negotiation phase when an idea is not accepted. As seen in Figure 2.2, the teacher is the initiator and the evaluator of the contribution, whereas students not only initiate the conversation, but more importantly, they take an active part in negotiating and accepting the responses (See Figure 2.3). Hence, in a revoicing pattern, the teacher acts as the mediator to direct the conversation using students’ presentation as a focus.

\[
\text{Figure 2.3. A revoicing discourse schema (Greeno, 2003, p. 324).}
\]

This present study posits that the revoicing pattern is more in line with the notion of productive mathematical discussions (Smith & Stein, 2011), in which the students participate actively in the reasoning processes as supported by both the sociocultural and situative perspectives (Brodie, 2010f; Forman, 2003; Greeno, 2003). Therefore, the role of the teacher during whole-class discussions is to support students’ learning of the mathematical concepts (Henningsen & Stein, 1997), by inducting them into the mathematics discourse practices (Yackel & Cobb, 1996). As argued by Walshaw and Anthony (2008), students learn how to reason mathematically through the patterns of discourse and interaction created by teachers in the classrooms. However, orchestrating such discussions can be very challenging for teachers (Brodie, 2010f, 2010g; Lampert, 2001; Stein, Engle, Smith, & Hughes, 2008). For example, Brodie (2010f) suggests that teachers often find it difficult to work with discussions where students give partially correct ideas, and it is not trivial for teachers to deal with interactions involving incorrect answers. To support teachers in orchestrating productive discussions, Franke
et al. (2007) propose that certain “routines of practice” (p. 249), such as the five practices developed by Stein et al. (2008), may be helpful.

### 2.2.2.2. Five practices for orchestrating productive mathematical discussions

According to Stein et al. (2008), using students’ correct, partially correct, and incorrect responses to the tasks as the initiator of classroom discourses, and teachers facilitating such interactions in order to shape students' mathematical reasoning is the mark of a well-orchestrated discussion. This study presupposes that orchestrating such discussions is “deliberate work” (Franke et al., 2007, p. 228), and certain aspects of this teaching expertise can be planned for in advance (Stein et al., 2008). In order to provide a frame for teachers to think about classroom discussions, Stein et al. (2008, p. 321) introduce the following five practices:

1. “Anticipating” possible student responses to a task;
2. “Monitoring” their responses when students work with the task;
3. “Selecting” students purposefully to present their work;
4. “Sequencing” their presentations carefully to build up mathematical ideas; and
5. “Connecting” different students’ responses to one another, as well as, to relate these responses to the mathematical concepts underlying them.

As Stein et al. (2008) highlight, each of these practices draws from and depends on the practices before it (See Figure 2.4). Smith and Stein (2011) emphasise that these five practices highlight the importance of planning for moderating improvisation during in-the-moment decision making in the classrooms.

![Diagram of five practices](image)

**Figure 2.4. The five practices (Stein et al., 2008, p. 322).**
The first practice is to anticipate how students may think about the task mathematically (Lampert, 2001; Smith & Stein, 2011; Stein et al., 2008), and to interpret the task by taking on a student’s perspective (Clea Fernandez et al., 2003). It involves going beyond analysing the level of difficulty to interpreting how students may understand the task, how they approach the task, the kind of strategies they use, the possible cognitive obstacles they may face and how these issues relate to the mathematical concepts underlying the task (Smith & Stein, 2011; Stein et al., 2008). This is similar to how the Chinese teachers think about the lesson in terms of the Three Points (Yang & Ricks, 2013). Anticipating requires teachers to draw on their mathematical knowledge for teaching (Ball et al., 2008), knowledge of what their students know about the topic (Stein et al., 2008), and possibly their students’ common misconceptions (Difficult Point) or met-befores (McGowen & Tall, 2010). Another useful strategy for anticipating student responses, as well as teachers’ possible responses to students, is the use of a lesson play (Zazkis, Liljedahl, & Sinclair, 2009). This involves teachers writing an imaginary script about their interactions with students, which serves as a thinking tool for teachers to reason about student thinking. It focuses on the problematic aspects of the concepts that students are likely to encounter, how these may arise during the task, and possible ways to resolve them. This helps to prepare teachers to monitor their students’ reasoning during the lesson.

Monitoring student reasoning entails teachers’ careful attention to the mathematical aspects of how students work on the task (Schifter, 2001; Stein et al., 2008) when teachers circulate around the classroom (Baxter & Williams, 2009). More specifically, it involves teachers focusing on the underlying mathematical thinking of the students from what they do and say (Grossman et al., 2009; Schifter, 2001), beyond the surface features of student participation (Stein et al., 2008). For example, instead of noting whether students are actually working on the task, teachers should also listen to students’ discussions, in order to assess the mathematical ideas that these students have, and attempt to make sense of students’ thinking even when their reasoning is unclear (Schifter, 2001). This practice is further strengthened if students are familiar with the practices (Lampert, 2001), or the sociomathematical norms (Yackel & Cobb, 1996) associated with a community of mathematics learners. By engaging in these
practices or norms, students’ thinking is made public and visible (Ball & Bass, 2003) for teachers to access in order to coordinate a whole-class discussion of the students’ work.

Orchestrating a whole-class discussion of the task hinges on a purposeful selection of students’ work so that the teacher can get “a particular piece of mathematics on the table” (Lampert, 2001, p. 146). This allows the teacher to create windows of opportunity for students to engage with the concepts through participation in communicative practices of mathematics (Lampert, 2001; Smith & Stein, 2011). This practice of selecting students’ responses privileges the active role of a teacher in whole-class discussions, in contrast to some ideas amongst teachers that they should minimise control in order for a discussion to be focused on student thinking (Lobato, Clarke, & Ellis, 2005). Instead of a laissez-faire approach in selecting students for presentation, a teacher can invite students to volunteer, but thoughtfully select the students in order to make a mathematical point (Lampert, 2001; Stein et al., 2008). The careful selection ensures that both important mathematical ideas as well as common misconceptions are addressed during the discussion (Stein et al., 2008), and provide opportunities for the teacher to share useful alternative strategies that were not presented by the students (Baxter & Williams, 2009).

Beside selecting students to present their ideas, it is also equally important for teachers to sequence the order of presentations so that they can increase the possibility of attaining the lesson objectives (Smith & Stein, 2011; Stein et al., 2008). Sequencing can be done in various ways depending on the objective of the discussion. For instance, the teacher can start with the most common strategy; or they can begin with a more concrete representation before moving on to a more abstract presentation; or even address a common misconception or difficulty that most students face. They can also consider comparing two contrasting strategies to illustrate the different underlying mathematical perspectives afforded by the task (Smith & Stein, 2011; Stein et al., 2008).

The motivation for carefully selecting and sequencing responses is to lay the groundwork for the teacher to connect these different responses to important mathematical ideas. By directing students’ attention to the connections between different strategies, and by shifting their focus from solutions to mathematical ideas,
teachers can begin to support students’ efforts in understanding the concepts targeted in the lesson (Smith & Stein, 2011; Stein et al., 2008).

As seen from the preceding discussion, these five practices build on one another, and are critical in supporting a teacher’s preparation to better respond to student thinking. This advance preparation helps to reduce the cognitive load of a teacher during the actual lesson, so that he or she can begin to listen attentively to student reasoning, in order to control the direction of the whole-class discussions.

2.2.2.3. Listening to student reasoning

To enact teaching practices that promote student reasoning, it is important for the teacher to listen to what their students say during the classroom interactions (Carpenter, Fennema, & Franke, 1996; B. Davis, 1997; Mason & Johnston-Wilder, 2006). For example, Mason and Johnston-Wilder (2006) argue that a listening teacher can help students to connect seemingly different mathematical ideas (one of the five practices), in order to draw learners’ attention to important features of the concepts. However, listening to students’ responses is not sufficient for promoting student reasoning in classrooms. Instead, it is also critical for teachers to adopt an appropriate listening stance during the classroom interactions. B. Davis (1997) distinguishes three types of listening stance amongst teachers: evaluative listening; interpretive listening; and hermeneutic listening (B. Davis, 1997; B. Davis & Renert, 2014), to interpret practice and teaching actions.

Evaluative listening is “listening for something” (B. Davis, 1997, p. 359) in order to evaluate the contribution in terms of an established knowledge. It often involves listening for a particular “correct answer in mind” (p. 360), and may manifest itself through the teacher filling in “the blanks” when the “correct” answer is not given by students. There is little wait time prior to teacher’s response, and the teacher seems to follow some pre-determined trajectory without pausing to think about student reasoning. In some ways, this mode of listening is demonstrated during a IRE-type of classroom interaction (See Section 2.2.2.1).

On the other hand, interpretive listening means listening to in order to seek interpretations, and make sense of student understanding (B. Davis & Renert, 2014). This form of listening is anchored in the belief that students’ utterances are meaningful.
As described by B. Davis and Renert (2014), interpretive listening express itself through a teacher pausing to ask questions; and giving opportunities for students to explain their reasoning, in order to see where students stand in terms of their current understanding. It is a deliberate act of attention on the part of the teacher (B. Davis, 1997). Interpretive listening is similar to the skills highlighted by Schifter (2001) as essential for a more discursive approach to teaching. She highlights the importance of seeing mathematical concepts in what students say and do, evaluating the mathematical correctness of student thinking and “listening for sense in students’ mathematical ideas even when something is amiss” (p. 126). While the interaction patterns for an interpretive listening mode are more interactive than that of an evaluative mode, both modes position the teacher as the sole authority on mathematics in the classroom (B. Davis & Renert, 2014).

Last but not least, is the notion of hermeneutic listening, which involves “listening in order to” (Mason & Johnston-Wilder, 2006, p. 119) participate in mathematical sense-making (B. Davis & Renert, 2014). What separates hermeneutic listening from the other two is the “negotiated and participatory nature” of the teacher-student interactions (B. Davis, 1997), which seeks to find variations, rather than one specific optimal answer. It is guided by the premise that there are different and new ways of talking about mathematics (B. Davis & Renert, 2014). Such interaction patterns shift the centralised authority of the teacher to a shared participation model depicting the teacher and students as a community to negotiate meanings. As B. Davis and Renert (2014, p. 87) put it:

... the lesson trajectory of the evaluative listener-teacher is largely unaffected by student articulations, the lesson trajectory of the interpretative listener-teacher is modified by them, and the lesson trajectory of the hermeneutic listener-teacher is defined by them.

Each of three modes of listening has a role to play in teaching, but a classroom practice that promotes student reasoning is likely to involve more interpretive and hermeneutic listening than evaluative listening alone. Moreover, listening and questioning go hand-in-hand (Brodie, 2010a). In order to find out what students are thinking, a teacher needs to ask questions and listen to students’ answers; and from listening to how
students answer the questions, the teacher can decide on the next course of action to support student thinking.

2.2.2.4. Responding to student reasoning

Knowing exactly how to respond to student reasoning in order to support their thinking is an important, but challenging task for teachers (Brodie, 2010f; Franke et al., 2009; Herbal-Eisenmann & Breyfogle, 2005; Mason & Johnston-Wilder, 2006). For example, Franke et al. (2009), in their analysis of 66 classroom interaction segments taken from part of a larger-scale study, found that asking follow-up questions when students gave wrong responses did not always support them in giving better responses. However, the researchers highlighted that teachers who asked a series of focused questions (Herbal-Eisenmann & Breyfogle, 2005), which were crafted to guide students based on what they were thinking, instead of teachers’ intended solutions, had a higher likelihood of success in helping students correct their initial wrong explanations. Their research, as well as others’ (Kazemi & Stipek, 2001; Sfard & Kieran, 2001), highlight the key role of questioning in supporting student reasoning.

Questioning is distinguished from telling in terms of what teachers are attending to (Mason & Johnston-Wilder, 2006). When teachers attend to their own thinking, they are more likely to tell students without listening to their reasoning. On the other hand, when teachers try to uncover what learners are thinking, they tend to ask questions after listening to what learners have to say. More importantly, it is the kind of questions that determines whether teachers’ responses to student reasoning are productive for supporting students to learn from their engagement with tasks.

To raise teachers’ awareness of their own questioning patterns, Herbal-Eisenmann and Breyfogle (2005) highlights two patterns of questioning: funnelling and focusing (T. Wood, 1998). According to T. Wood (1998), funnelling happens when a teacher attempts to channel the students, through a series of guided questions, to know a procedure or other desired ends. This type of questioning engages a teacher in thinking rather than the students who are just responding to the questions without seeing the relationship between the series of questions (Herbal-Eisenmann & Breyfogle, 2005). As Mason and Johnston-Wilder (2006) argue, funnelling questions are neutral, and the
appropriateness of these questions is dependent on the context, and the objectives that the teacher wishes to achieve.

Focusing questions, in contrast, requires a teacher to listen to student responses, and direct them using students’ thinking, rather than the teacher’s own thinking (Herbal-Eisenmann & Breyfogle, 2005; T. Wood, 1998), with the aim of focusing students’ attention on specific aspects of the concept (Mason & Johnston-Wilder, 2006). This pattern of questioning also values student reasoning, and provides a means to make student thinking more visible, giving them opportunities to articulate what they are thinking (Herbal-Eisenmann & Breyfogle, 2005). It presses students to justify and explain their reasoning so that others in the classroom can access their thinking. This kind of questioning is also similar to the press move described by Brodie (2010f). In order to orchestrate a productive mathematical discussion, teachers will then need to sensitise themselves to be more aware of student reasoning (Mason & Johnston-Wilder, 2006). To support the teachers in using questioning to enhance student reasoning, Mason and Johnston-Wilder (2006) suggest a number of questions and prompts, which are similar to the questions used by mathematicians. These questions are shown in Table 2.2, and were a useful resource for the teachers in this study (See Section 3.4.3).

2.2.3. Reflecting about student reasoning

Teaching for mathematical reasoning does not end in the implementation of a well-designed high-level cognitive demand task. Instead, evidence from research suggests a central role for teacher reflection in shaping both knowledge (Ball & Cohen, 1999; Chamberlin, 2005; Cochran-Smith & Lytle, 1999; Franke et al., 2001; Loughran, 2002; M. G. Sherin, 2002) and beliefs (Franke et al., 2001; Goodell, 2006; Jaworski, 2006; T. Wood, Cobb, & Yackel, 1991), with regard to enhancing student reasoning. For example, Ball and Cohen (1999, p. 10) argue for teachers to reflect and learn from their practice, in order to teach mathematics in the way envisioned by the reformers (Kilpatrick et al., 2001; Lampert, 2001; National Council of Teachers of Mathematics, 2000):

Teachers could not do such work unless they knew how to learn in the contexts of their work. That would require the capability to attend to and learn about individual students’ knowledge, ideas, and intentions. It also would require the capability to stand back from and analyze their own teaching, to ask and answer such questions as: What is working? What is not working? For whom
are certain things working or not working? To teach well, given reformers’ ambitions and the situational and uncertain nature of teaching and learning, teachers would need to use what they learn to correct, refine, and improve instruction.

Table 2.2: Useful questions and prompts (Mason & Johnston-Wilder, 2006, p. 110)

<table>
<thead>
<tr>
<th>Mathematical Processes</th>
<th>Questions and Prompts</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exemplifying</td>
<td>Give me one or more examples of...</td>
</tr>
<tr>
<td>Specialising</td>
<td>Describe (show, choose, draw, find,...) an example of...</td>
</tr>
<tr>
<td></td>
<td>Is ... an example of ...? What makes ... an example?</td>
</tr>
<tr>
<td></td>
<td>Find a counter-example of ...</td>
</tr>
<tr>
<td></td>
<td>Are there any special examples of ...?</td>
</tr>
<tr>
<td>Completing</td>
<td>What must added (removed, altered) in order to allow (ensure, contradict) ...?</td>
</tr>
<tr>
<td>Deleting</td>
<td>What can be added (removed, altered) without affecting ...?</td>
</tr>
<tr>
<td>Correcting</td>
<td>What needs to be changed so that ...?</td>
</tr>
<tr>
<td></td>
<td>Tell me what is wrong with ...</td>
</tr>
<tr>
<td>Comparing</td>
<td>What is the same and what is different about ...?</td>
</tr>
<tr>
<td>Sorting</td>
<td>Is it or is it not ...?</td>
</tr>
<tr>
<td>Organising</td>
<td>Sort or organise or group the following according to ...</td>
</tr>
<tr>
<td>Changing</td>
<td>Change ... in response to ...</td>
</tr>
<tr>
<td>Varying</td>
<td>What if ...?</td>
</tr>
<tr>
<td>Reversing</td>
<td>Do ... in two (or more ways. Which is quickest, easiest, ...?)</td>
</tr>
<tr>
<td>Altering</td>
<td>If this is the answer to a similar question, what was the question?</td>
</tr>
<tr>
<td></td>
<td>Alter an aspect of something to see the required effect.</td>
</tr>
<tr>
<td>Generalising</td>
<td>What happens in general?</td>
</tr>
<tr>
<td>Conjecturing</td>
<td>Of what is this a special case?</td>
</tr>
<tr>
<td></td>
<td>Is it always, sometimes, never ...?</td>
</tr>
<tr>
<td></td>
<td>Describe all possible ... as concisely as you can.</td>
</tr>
<tr>
<td></td>
<td>What can change and what has to stay the same so that ... is still true?</td>
</tr>
<tr>
<td>Explaining</td>
<td>Explain why ...</td>
</tr>
<tr>
<td>Justifying</td>
<td>How is ... used in ...? Explain the role or use of ...</td>
</tr>
<tr>
<td>Verifying</td>
<td>Give a reason ... (using or not using ...)</td>
</tr>
<tr>
<td>Convincing</td>
<td>How can you be sure that ...?</td>
</tr>
<tr>
<td>Refuting</td>
<td>Convince me that ...</td>
</tr>
<tr>
<td></td>
<td>Tell me what is wrong with ...</td>
</tr>
<tr>
<td></td>
<td>Is it ever false that ... (always true that ...)</td>
</tr>
</tbody>
</table>
Findings from research on Cognitively Guided Instruction (Carpenter et al., 1996; Carpenter, Fennema, Franke, Levi, & Empson, 1999; Franke et al., 2001; Franke et al., 2007; T. Wood et al., 1991) also support the need for teachers to engage and think about their practice by focusing on their students’ thinking. More recently, Kullberg, Runesson, and Martensson (2013) have highlighted how teachers’ participation in structured reflection practices, such as Lesson Study or Learning Study, can provide time for them to reflect about the mathematics and tasks used. Similarly, Yang and Ricks (2012) allude to the need for teachers to see beyond the implementation of a single lesson to derive more general pedagogical principles that can be applied in future lessons. This kind of reflection can help to raise teachers’ awareness about how they can implement tasks more effectively in the classrooms (Clea Fernandez et al., 2003).

An impediment to learning from practice is the perception that teaching is solely about improvisation (Ball & Forzani, 2009). The five practices (See Section 2.2.2.2) and the emphasis on learning from practice shift the centre of teaching expertise from in-the-moment performance in the classroom to preparation and reflection (Hiebert et al., 2007; Smith & Stein, 2011; Sowder, 2007). In order to “respond rather than to react” during lessons as events unfold in the classrooms, Mason (2002) emphasises the role of “systematic practices” in “preparing, supporting and enhancing teachers’ sensitivities” during lesson preparation (p. 87). In addition, Hiebert et al. (2007) propose that the key construct for learning from practice is to analyse teaching from the perspective of student learning. They suggest four essential skills for teachers to develop:

1. Specifying the learning goal(s) for instruction;
2. Observing instances of student learning during teaching;
3. Interpreting student learning with respect to the teaching strategies; and
4. Analysing student learning to propose improvements.

These skills, part of an inquiry stance (Cochran-Smith & Lytle, 1999; Hiebert et al., 2007), place an emphasis on collecting data on students’ mathematical thinking (Hiebert et al., 2007; Hiebert, Morris, et al., 2003). In an earlier work, Hiebert and his colleagues highlighted the usefulness of focusing on student thinking, instead of narrowly defined measures such as behavioural objectives (Hiebert, Morris, et al., 2003). Rather, they suggested that the teacher should focus on what the students are
thinking about the mathematics before the task, and how their thinking may have changed after the task is implemented. Furthermore, the analysis of student reasoning is contingent upon the development and implementation of a task that reveals student reasoning (Hiebert et al., 2007). This provides a feedback loop into the task design, which can serve as a means to improve the design of the task, so as to enhance student reasoning in the next implementation of the task.

One productive way to analyse student reasoning is to examine classroom artefacts or “physical records that capture aspects of the work that happens during mathematics lessons” (Chamberlin, 2005; Goldsmith & Seago, 2011; 2013, p. xvii; van Es & Sherin, 2008). Some of the artefacts that are suitable for this purpose include samples of students’ work (Ball & Cohen, 1999; Goldsmith & Seago, 2011, 2013; Hiebert, Morris, et al., 2003), video clips of classroom interactions (Miller & Zhou, 2007; M. G. Sherin, 2007; van Es & Sherin, 2008), and transcripts of students’ discussion (B. Davis, 1997). By engaging teachers in studying and discussing these artefacts, they can have opportunities to compare different interpretations and perspectives in the context of a professional learning community, as in the case of this study (Ball & Cohen, 1999). This can help teachers to “validate” what they learn as they compare their own experiences with the “world of observations and theories” as well as the experiences of others (Mason, 2002, p. 94).

However, engaging in reflecting about student thinking using artefacts or other means do not necessarily lead to improvement in practice. While there are frameworks to help teachers analyse student thinking (Goldsmith & Seago, 2011; Hiebert et al., 2007; Santagata, 2011; Yang & Ricks, 2012), it has been unclear how teachers can be supported in making progress towards this endeavour (Chamberlin, 2005; Hiebert et al., 2007). Chamberlin (2005), for example, suggests that while engaging teachers in mini-inquiries has the potential to support them in gaining insights about student thinking; how they can be supported to do this, is still an agenda for future research. This study investigated how teachers can be supported to see, and interpret student thinking from classroom artefacts in order to make instructional decisions that enhance mathematical reasoning.
2.3. Challenges in teaching mathematical reasoning

Teaching for mathematical reasoning is challenging work. Brodie (2010b) uses Lampert’s (1985) notion of a teaching dilemma to highlight some of the challenges a teacher may face when teaching students to reason. To resolve dilemmas such as connecting learners to mathematical ideas; coordinating between an individual learner and a group; to take up or to let go of a mathematical point, requires a lot from a teacher in terms of knowledge and skills (Brodie, 2010b; Lampert, 2001). As Schifter (2001) has demonstrated, there are mathematical skills besides content knowledge that a teacher needs to draw on in order to orchestrate a fruitful mathematical discussion. How a teacher designs and implements a task is not only influenced by one's knowledge, but also one’s beliefs and intentions in teaching mathematics (Sullivan et al., 2013). According to Schoenfeld (2011a), teaching is a goal-oriented decision making process, which is shaped by one's orientations and perceptions of the resources available. Using this idea, he has modelled teaching practices of teachers in terms of the decisions they make in class (Schoenfeld, 2000, 2008), the kind of instructional explanations and how they are given (Schoenfeld, 2010). This section draws on the conceptual framework of seeing teaching as a function of one’s resources, orientations, and goals (Schoenfeld, 2011a), to review some of these challenges in teaching mathematical reasoning.

2.3.1. Resources

A key component of the resource dimension in Schoenfeld’s framework is knowledge. Schoenfeld (2011a) defines knowledge as “the information that he or she has potentially available to bring to bear in order to solve problems, achieve goals or perform other such tasks” (p. 25). In the case of teaching, Shulman (1987) connects knowledge to teaching using the construct of pedagogical content knowledge (PCK), which proposes the existence of a special category of content knowledge that is unique to the work of teaching. This special body of knowledge includes different representations of content knowledge that make it accessible to learners, and an understanding of common conceptions and misconceptions that may enhance or hinder a student's learning (Shulman, 1986).

Building on the notion of PCK (Shulman, 1986), Ball et al. (2008) delineated three subsets of PCK: Knowledge of Content and Student (KCS); Knowledge of Content and
Teaching (KCT); and Knowledge of Content and Curriculum. In addition, Ball and her colleagues expanded the idea of subject-matter knowledge (SMK), by describing a new construct, *specialised content knowledge* (SCK), which consists of mathematical knowledge and skills that are part of the subject-matter knowledge unique to teaching. SCK distinguishes itself from *common content knowledge* (CCK) in that the former is only needed in teaching (Hill et al., 2008), and this may include knowledge of less common algorithms, different representations of concepts, and explanations of rules and procedures (Ball et al., 2008). The third provisional component of SMK is *horizon content knowledge*, which “is an awareness of how mathematical topics are related over the span of mathematics included in the curriculum”, and this may include “vision useful in seeing connections to much later mathematical ideas” (Ball et al., 2008). This characterisation of mathematical knowledge for teaching (MKT) is represented in Figure 2.5.

While MKT presents a possible way to discuss the kind of mathematical knowledge needed by a teacher to design a task, and orchestrate a discussion, there are a couple of issues with regard to this conceptualisation. First, the construct of SCK, the unique specialised knowledge is not clearly distinct from Shulman’s PCK (Petrou & Goulding, 2011). As noted by Petrou and Goulding (2011), the definition of PCK also signals the uniqueness of this knowledge as “a special amalgam of content and pedagogy that is uniquely the province of teachers”, which belongs to teachers’ “special form of professional understanding” (Shulman, 1987, p. 8). In addition, the definition of *horizon content knowledge* is still rather vague (Sullivan et al., 2013) and it is not clear how this relates to SCK.

![Mathematical Knowledge for Teaching (MKT)](image)

**Figure 2.5. Components of MKT.**
Nevertheless, MKT has been useful for examining the knowledge of mathematics needed for the productive practices outlined in Section 2.2. For example, Cengiz et al. (2011), in their study of six experienced elementary school teachers, found that a well-connected MKT is critical in extending student reasoning during whole-class discussions. More specifically, a good understanding of content knowledge (CCK and SCK), together with KCS and KCT, enabled six teachers to make instructional decisions that afforded more opportunities for students to articulate their thinking. Hence, a strong knowledge base has been found to support teachers in their decisions to press students’ thinking or take up the different mathematical points offered by students. On the other hand, their research also highlights that a less developed MKT may hinder a teacher’s ability to orchestrate a whole-group discussion, and hence his or her pursuit to extend student thinking.

Similarly, Sullivan et al. (2013) express their concern that teachers in their study were unable to draw on their SCK to get a sense of the different possible strategies that students might use to approach a fraction task (Which is bigger? 2/3 or 201/301?). Some teachers could not identify the key concept (fraction comparison) underlying the task, and had problems seeing the relationship between this task and the curriculum (Knowledge of Content and Curriculum). Moreover, the grade 5 and 6 teachers, in their study, could not suggest how this task could be set up in a lower secondary classroom. This also signals a possible lack of understanding of the knowledge in the horizon. Therefore, the MKT that a teacher has can clearly influence the design and choice of the task, as well as the way the task is implemented.

2.3.2. Orientations

The second aspect of Schoenfeld’s framework involves orientations, which consist of one’s “dispositions, beliefs, values, tastes and preferences” (Schoenfeld, 2011a, p. 29). Research literature suggest that beliefs, and attitudes influence teaching practice (Barkatsas & Malone, 2005; Philipp, 2007; Schoenfeld, 2011a; Stipek, Givvin, Salmon, & MacGyvers, 2001), but the interactions between beliefs, attitudes, goals and practice are still largely unclear. For example, on one hand, Stipek et al. (2001) findings suggest a positive association between the beliefs that teachers hold and their practice: teachers who hold traditional beliefs about teaching are also more traditional in their teaching;
while on the other hand, other research highlight possible inconsistencies between perceived beliefs and practice (Barkatsas & Malone, 2005; Mansour, 2013; Raymond, 1997). Some of these inconsistencies can be explained by the methods of measuring and attributing beliefs to the participants (Mansour, 2013; Thompson, 1992). To overcome some of these limitations, some researchers attempt to put together a composite picture of the belief system from different data sources (Mansour, 2013); while others assign beliefs that are consistent with the actions (Schoenfeld, 2011a).

Nevertheless, a teacher’s set of orientations plays an important role in shaping one’s teaching practice (Schoenfeld, 2011a). Of particular interest in this study, will be teachers’ beliefs about the nature of mathematics, and ways of teaching and learning the subject. Orientations are also tied to one’s resources, or more specifically knowledge, which one may bring to mind to achieve one’s objective (Schoenfeld, 2011a). For instance, Nathan and Petrosino (2003) found that pre-service teachers with high levels of content expertise have a tendency to underestimate the difficulty of the problems posed to students, who are novices in the subject. They suggest that the notion of “expert blind spot” may help to explain why teachers may sometimes be “blind” to the “learning and developmental profiles” of their students (Nathan & Petrosino, 2003, p. 906). The blind spot occurs in these teachers because they tend to believe that students see the mathematical concepts or tasks in the same way as they do. This notion of expert blind spot may also be applicable for a teacher who is trying to orchestrate a discussion in the classroom because of his or her expectation that students would attend to the same aspects of the discourse.

2.3.3. Goals

Another important factor to consider in the teaching for mathematical reasoning is teachers’ intentions (Sullivan et al., 2013) or goals, which refer to what teachers want to achieve during the lesson (Schoenfeld, 2011a). This follows from Schoenfeld (2011a) theory that models teaching as a goal-oriented activity. In the context of teaching, a teacher’s goals can come in different sizes, and may have different priority levels, depending on his or her orientations (Schoenfeld, 2008, 2011a; Zimmerlin & Nelson, 2000). To add to the complexity, goals can change in the middle of a lesson, and teachers have to grapple with short term and long term goals simultaneously (Lampert,
These changes in decisions made during a lesson can be attributed to conflicts between the orientations and goals held by the teacher (Hannah et al., 2011; Paterson, Thomas, & Taylor, 2011; Thomas & Yoon, 2013). For example, Thomas and Yoon (2013) demonstrates how a teacher, Adam, decided to use a teacher-led discussion over a more student-centred approach as a result of competing priorities (e.g., to prepare students for success in future tasks, engage in student-centric teaching, fulfill requirements of curriculum etc.) and his beliefs (e.g., teacher-led discussions can bring about student understanding in a shorter time).

In the context of this study, it was important to examine the possible goals or intentions of the teacher conducting the lesson because the task implementation may not be as intended. For example, Henningsen and Stein (1997) detail how a high-level cognitive problem solving task declined into a low level task during implementation. The teacher involved, Ms. Capra, over-regulated the problem solving process and there was no discussion on the different methods or strategies. Instead, the teacher was focused on getting the students to arrive at the right answers, rather than to engage in the process. A possible explanation could be the teacher’s focus on maintaining order in her classroom management, and other factors such as time allocation for the task (Henningsen & Stein, 1997), as in the case of Adam described by Thomas and Yoon (2013)

Moreover, the design of the task can also be changed as a result of one’s perceived goal of instruction. For example, Tzur (2008) suggest that a teacher may think that the task is not going to achieve its purpose, and modify the design to match his/her expectation of the students’ learning. It is also possible that a teacher may decide to change the task midway through the lesson because he/she is not getting the desired responses from the students. Whatever the reasons, what teachers want to achieve with the task and discussion will be an critical aspect to examine in this study.

2.3.4. A hypothetical trajectory for teaching reasoning

Following Schoenfeld (2011a), this study, which aims to examine what teachers see, and how they interpret instructional details, considered the teachers’ orientations, goals and resources, in relation to the decisions taken while planning, implementing and reflecting about a task. As shown in Figure 2.6, Schoenfeld (2011a) proposes three
planes of teaching activity: managing classroom dynamics, implementing engaging tasks; engaging in diagnostic teaching. These three planes of activity, in that order, reflect the increasing demands on a teacher's clusters of resources, orientations and goals (ROG clusters). He further hypothesises that a teacher’s professional development is modelled by the changing ROG clusters as they work on improving their practice.

![Diagram](image)

**Figure 2.6. A teacher's hypothetical growth trajectory (Schoenfeld, 2011a).**

Since the productive practices detailed in Section 2.2 are similar to the idea of engaging in diagnostic teaching, Schoenfeld's theory, may provide a possible way to model, or explain in part, a teacher's growth from classroom management to teaching for mathematical reasoning.

### 2.3.5. Noticing: A means to overcome these challenges?

As seen from the preceding discussion, doing the work of diagnostic teaching to enhance student reasoning is demanding, and requires teachers to focus their attention on noteworthy aspects of their teaching practice. They need to attend to aspects of student thinking from classroom artefacts; student explanations; and discourses, and interpret them using a mathematical perspective before, during, and after a lesson (Goldsmith & Seago, 2013; Jacobs et al., 2010; Schifter, 2001; Smith & Stein, 2011). Expert teachers, who are highly proficient in this work, can perceive meaningful patterns from what they see, and connect these observations to what they know, to make productive instructional decisions in the midst of a complex classroom environment (Berliner, 2001). These teachers are more sensitive and attuned to the task demands and the social contexts, and are more able to call upon different but useful
strategies to solve their problems in practice (Berliner, 2001; Mason, 2002). This high level of attention is more active and intentional, rather than passive or spontaneous (Erickson, 2011; Mason, 2011; Miller, 2011; M. G. Sherin, Jacobs, et al., 2011a), and constantly seeks to use experience as evidence to form new ideas that can inform future practice (Schön, 1991). This specialised seeing, sense-making, and decision making is a set of three inter-related skills referred to by researchers as noticing (Mason, 2002; M. G. Sherin, Jacobs, & Philipp, 2011b).

2.4. Mathematics Teacher Noticing

2.4.1. What is mathematics teacher noticing?

People notice all the time. But this notion of noticing is more closely related to the idea of “professional vision, which consists of socially organized ways of seeing and understanding events that are answerable to the distinctive interests of a particular social group” (Goodwin, 1994, p. 606). Mason (2002) sees noticing as a discipline, and highlights mathematical noticing as “the heart of all practice” (p. 1) and offers it as a means by which teachers can “do something about it [teaching] in a practical and disciplined manner” (p. 1). Noticing not only occurs frequently but also across different classroom contexts (Jacobs, Lamb, Philipp, & Schappelle, 2011; Santagata, 2011; van Es, 2011). It enables teachers to know more about learning and teaching (Schoenfeld, 2011b) and yet preserves the complexity of teaching (M. G. Sherin, Jacobs, et al., 2011a). The processes of noticing help teachers break down and analyse their practice in order to learn from their teaching (Mason, 2009, 2011; M. G. Sherin, Jacobs, et al., 2011a). Therefore, noticing can be considered as one of Grossman et al.’s (2009) “high-leverage core practices (Mason, 2002; M. G. Sherin, Jacobs, et al., 2011a; M. G. Sherin & van Es, 2003), and can have a direct positive impact on the quality of teaching (Schoenfeld, 2011b; M. G. Sherin, Jacobs, et al., 2011a).

Noticing can be viewed as a set of “practices” that work together to improve teachers’ sensitivity to act differently during teaching situations (Mason, 2002). He distinguishes disciplined noticing from spontaneous noticing by emphasising its systematic aspect (Mason, 2002, p. 61):
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The idea is simply to work on becoming more sensitive to notice opportunities in the moment; to be methodical without being mechanical. This is the difference between ‘finding opportunities’ and ‘making them’. Instead of being caught up in moment by moment flow of events according to habits and pre-established patterns, the idea is to have the opportunity to respond freshly and creatively yet appropriately, every so often.

There are two main ways, as Mason (2002) puts it, to raise the possibility of noticing in order to respond freshly or have a different act in mind for the future: advance preparation and using past experience. Moreover, noticing does not necessarily occur at an individual level, but rather, the practices of disciplined noticing lie in the “merging” of three worlds of experience—“the world of personal experience, the world of one’s colleagues’ experience and the world of observations, accounts, and theories” (Mason 2002, p. 93). Professional development, according to Mason (2002), takes place in the world of personal experience, and is supported by one’s colleagues while drawing on the world of theories, which informs how noticing can take place. See Figure 2.7 for a representation of how the practices fit into the three worlds.

![Diagram](image)

**Figure 2.7. Noticing in the three worlds of experience (Mason, 2002, p. 94).**

As seen from Figure 2.7, the ability to recognise possibilities to act differently lies at the intersection of the three worlds. By reflecting on personal experiences systematically, one can prepare to notice by developing sensitivity to common threads or themes that might emerged from one’s experiences. These personal accounts of events are often interpreted in light of the theories and observations; and sometimes validated with other people, in order to distinguish and recognise the possibility to act differently. While these actions usually happen retrospectively, the essence of noticing is to bring
these “moments of noticing from the retrospective to the spective” (Mason, 2002, p. 87) through four inter-related actions: systematic reflection; preparing and noticing; recognising choices; and validating (See Table 2.3).

<table>
<thead>
<tr>
<th>Practices</th>
<th>Description</th>
<th>Pictorial representation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Systematic Reflection</strong></td>
<td>Keeping and using accounts, through noticing, marking, and recording brief-but-vivid moments, and considering what might have been done, in retrospect; looking back over a period to find common threads, themes, and issues.</td>
<td><img src="image1" alt="Diagram" /></td>
</tr>
<tr>
<td><strong>Recognising</strong></td>
<td>Being alert to the action of others, as distinct from the effects of those actions, helps extend the range of acts and tactics upon which to draw; picking up ideas of ways to act from other people and from writing and thinking, making a note of these. Identifying and labelling typical situations can help recognition occur in the future.</td>
<td><img src="image2" alt="Diagram" /></td>
</tr>
<tr>
<td><strong>Preparing and Noticing</strong></td>
<td>Developing sensitivities through vividly imagining oneself carrying out chosen acts in order to make it more likely they come to mind in the moment in the future; setting oneself to notice specific events or acts.</td>
<td><img src="image3" alt="Diagram" /></td>
</tr>
<tr>
<td><strong>Validating with others</strong></td>
<td>Selecting and honing descriptions which others instantly recognise; refining task-exercises which highlights fruitful issues or sensitivities.</td>
<td><img src="image4" alt="Diagram" /></td>
</tr>
</tbody>
</table>

These actions are not a sequence of routines to be followed in order, but rather, they are practices of attuning oneself to the possibility to act differently in-the-moment, that is to respond, rather than to react instinctively. According to Mason (2002), “labelling” these choices or alternative responses can become “triggers” to notice possible future instances to act. These practices are not new, and can be found in differing
combinations across different productive practices to enhance reasoning. For example, the idea of systematic reflection and recognising choices are similar to the practice of reflecting about student reasoning (See Section 2.2.3), or activities during a Lesson Study (Lewis, Friedkin, Baker, & Perry, 2011), or during the design of a mathematical task (See Section 2.2.1); while the idea of imagining oneself to act in some desired manner forms the basis of lesson play discussed in Section 2.2.2.2.

However, most researchers (Goldsmith & Seago, 2011; Jacobs, Lamb, et al., 2011; Kazemi et al., 2011; M. G. Sherin, Russ, et al., 2011; Star et al., 2011; van Es, 2011) examine noticing, more broadly, as processes of attending to and making sense of instructional moments. While some researchers look at noticing by focusing only on the objects that teachers attend to (M. G. Sherin, Russ, et al., 2011; Star et al., 2011), others focus on both the processes of attending and making sense (Goldsmith & Seago, 2011; Jacobs, Lamb, et al., 2011; Kazemi et al., 2011; van Es, 2011). Noticing can also be extended to include “consideration of teachers’ instructional responses” (Jacobs et al., 2010; van Es, 2011).

Most researchers are in agreement with regard to the process of attending as seeing or paying attention to worthwhile instructional details (Barnhart & van Es, 2015; Erickson, 2011; Star et al., 2011). Nevertheless, there are differences among researchers with regard to what constitutes the processes of making sense. For example, van Es (2011) examined teachers’ noticing in terms of interpreting students’ mathematical thinking and using evidence from observations to justify key assertions about teaching and learning of mathematics. In particular, she developed a framework that looked at what and how teachers notice by assessing the level of specificity and elaboration of teachers’ interpretation (van Es, 2011). In contrast, Jacobs, Lamb, et al. (2011) went beyond interpreting to include “deciding how to respond on the basis of children’s understandings” (p. 99). They argue for its inclusion into the notion of noticing by highlighting how deciding to respond is intricately connected to the other two elements of mathematical noticing—attending to children’s strategies and interpreting their mathematical thinking—and that the three elements occur together for “purposeful responding” to take place (p. 100).
Despite the slight differences in the notion of noticing, researchers are generally in agreement about the crucial role that noticing plays in “sizing up” and “building on” students’ mathematical thinking by mathematics teachers (Goldsmith & Seago, 2011; Jacobs, Lamb, et al., 2011, p. 97; Schifter, 2011; van Es, 2011). The question, then, is how teachers can harness and develop the component skills of mathematical noticing, or hone their abilities to engage in the practices that constitute noticing. As Mason (2002) put it, “noticing is an act of attention, and as such is not something you can decide to do all of a sudden. It has to happen to you, through the exercise of some internal or external impulse or trigger” (p. 61). Therefore, it is important to examine how a teacher’s noticing can be supported.

Informed by Mason’s (2002) practices of noticing and the triad view of noticing: seeing, interpreting, and responding, this study, therefore, examines teachers’ mathematical noticing when they make instructional decisions for teaching reasoning. The key assumption in this study is that there are certain characteristics associated with what teachers see, and how they interpret instructional details, when they make productive decisions during teaching. Understanding these characteristics can support teachers to be more sensitised to what, and how they notice, and to make their noticing more visible in order to bring about the productive practices for teaching mathematical reasoning.

2.4.2. What teachers notice

Not all noticing is productive with regard to teaching for mathematical thinking. Erickson (2011) highlights that teacher noticing is very selective in its focus, and while different teachers can notice various aspects of the classrooms, what they notice may not always be helpful, and at times, direct students’ attention away from the mathematical issues. It is common for teachers, for example, to see students’ active participation in the tasks, or their enthusiastic raising of hands to answer questions, as indicators of students’ understanding (Erickson, 2011; Star et al., 2011; Star & Strickland, 2008). Furthermore, what teachers notice depends on their knowledge (Kazemi et al., 2011; Schifter, 2011), and beliefs or philosophical stance towards teaching (Erickson, 2011; Schoenfeld, 2011b). This diversity of knowledge and orientations has the potential for both “insight” and “misperception” in noticing.
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(Erickson, 2011, p. 32; Miller, 2011). Moreover, the amount of information that a teacher encounters in real time teaching is enormous, and so there is a need to focus that teacher's attention on some aspects of teaching. B. Sherin and Star (2011), for instance, point out the three main approaches of narrowing the focus for noticing: focus on certain events; focus on a subset of teaching practice; and focus on emergent properties of a teacher’s action and thoughts. However, each of these approaches has its own merits and faults (B. Sherin & Star, 2011). Therefore, whether an explicit focus for noticing is useful for encouraging more productive noticing, and if so, what this focus should be, remains a non-trivial question (Erickson, 2011; Star et al., 2011).

Without providing an explicit focus, M. G. Sherin, Russ, et al. (2011) employed the use of a wearable camera for teachers to capture phenomena that caught their attention. They found that teachers attended to a variety of episodes, ranging from those that involved some interesting mathematical ideas, to those related to student participation. These aspects, which teachers notice, are similar to those found in other studies (Borko, Jacobs, Eiteljorg, & Pittman, 2008; Star et al., 2011; Star & Strickland, 2008). Similarly, Star and Strickland (2008) focus on developing teachers’ ability to attend to a wide range of classroom features—classroom environment; classroom management; tasks; mathematical content; communication; and mathematical thinking—without specifying what teachers should notice. From a pre-assessment and post-assessment written instrument administered during a teaching methods course that used video clips as prompts, they found that teachers show significant improvements in attending to aspects of classroom environment and tasks; but a more modest improvement in terms of their abilities to notice the mathematical content of the lesson (Star & Strickland, 2008). A replication study (Star et al., 2011) was done, but there was no similar gain in the noticing of mathematical content. Neither of the two studies tested whether it is better to have an explicit focus, but both suggest that noticing important mathematical details is not easy, especially for novice teachers (Star et al., 2011; Star & Strickland, 2008).

On the other hand, several researchers provide analytic frames for teachers to focus their attention on certain aspects of mathematics teaching (Ceneida Fernandez, Llinares, & Valls, 2012; Goldsmith & Seago, 2011; Jacobs et al., 2010; Santagata, 2011; Stockero, 2014; Vondrová & Žalská, 2013; Yang & Ricks, 2012). For example, Jacobs et
al. (2010) focus on students’ mathematical thinking to investigate how, and the extent to which teachers notice student thinking during a series of professional development activities. Other possible ways to narrow the focus of noticing include the use of a lesson analysis framework when teaching and reflecting about a lesson (Santagata, 2011; Santagata & Angelici, 2010), the idea of mathematically significant moments to direct discussion (Stockero, 2014; van Es, 2011), and explicit use of artefacts related to mathematically important moments (Goldsmith & Seago, 2011, 2013; Vondrová & Žalská, 2013).

Even with a focus on thinking, Jacobs et al. (2010) found that a significant number of pre-service teachers and novice teachers struggled with attending to children’s specific strategies. This highlights the challenge of maintaining a focused noticing in order to make productive decisions during a lesson. This difficulty of noticing significant details has also been observed during the reflection on a lesson (Santagata, 2011). Nevertheless, Levin, Hammer, and Coffey (2009) push for framing as a way to influence what teachers notice, and call for the use of tools and guidelines to focus a teacher’s attention on relevant details. These guidelines or tools have to be more specific in order to provide adequate support for teachers to notice better (Santagata, 2011). Therefore, a more structured approach, such as the Three Points (Yang & Ricks, 2012) mentioned above or the Video Analysis Support Tool (van Es & Sherin, 2002), to highlight the specific aspects to focus on may be useful for enhancing teacher noticing.

It is possible that providing a more detailed or structured guideline may blind the teachers to notice other useful aspects of a lesson. For example, in an experiment conducted by Simons and Chabris (1999), participants were asked to watch a video clip involving students, dressed in black or white, playing with two basketballs; and count the number of times a student in white passes the ball to another student in white. Almost half of the 192 participants did not notice that a person wearing a gorilla suit had come into the midst of the students, beat upon his chest, and left the scene somewhere in the middle of the clip. This finding suggests two opposing implications: On one hand, getting teachers to focus on particular details may result in inattentional blindness to other unexpected events (Simons & Chabris, 1999); and on the other hand, it suggests that teachers may have a higher chance of focusing on the mathematically relevant aspects if directed to do so (Levin et al., 2009). Notwithstanding the limitations
of a structured approach to noticing, Mason’s (2002) idea of validating what one notices with the three worlds of experience may be helpful in mitigating the effects of this blindness, within the context of a learning community. Moreover, as Miller (2011) argues, learning to mark instructional details and respond to what is critical as well as discerning and ignoring observations that are not productive, is crucial in the development of noticing expertise. What this focus can be was explored in this study.

2.4.3. How teachers notice?

Besides the focus on what teachers notice, how they notice is also critical. Jacobs et al. (2010) suggest to examine growth in noticing expertise through the shifts in how teachers notice. In particular, the criterion of specificity, based on Jacobs et al. (2010, p. 196), can be used as an indicator of these shifts in teachers’ expertise as they begin to move from:

- Giving a general description to giving one that contains important mathematical details;
- Making general pedagogical comments to making those that addresses student thinking explicitly;
- Making hasty generalisations about student thinking to carefully linking one’s analysis of observations to specific classroom details;
- Analysing and deciding on instructional decisions by making general references to curriculum, to deciding based on one’s reasoned consideration about students’ current understanding and future strategies;
- Considering the whole class to thinking about individual students in terms of what they understand; and
- Designing tasks without consideration of student thinking to thinking about the mathematical details of the task with regard to their thinking.

In all these shifts, the emphasis is to get teachers to notice the specifics, and not just the general aspects of the observations. Most research focuses on the specificity of what teachers notice as a measure of their noticing expertise (Goldsmith & Seago, 2011; Jacobs et al., 2010; Jacobs, Lamb, et al., 2011; Stockero, 2014; van Es, 2011), but specificity is not sufficient for noticing to be productive. In a study involving seven prospective secondary school mathematics teachers, Ceneida Fernandez et al. (2012)
found that most were unable to relate the strategies used by students to the characteristics of the problem, even though they were all able to describe the specific strategies at the beginning of the study. This suggests the need to examine more closely the sense-making processes of noticing, particularly the skill of explaining the observations, and predicting the possible learning trajectories of the students involved (Berliner, 2001; E. A. Davis, 2006; Mason, 2002; Seidel & Stürmer, 2014; Star & Strickland, 2008; van Es, 2011; van Es & Sherin, 2002).

According to Seidel and Stürmer (2014), the critical element in explaining the observations lies in connecting the observations to the related professional knowledge of teaching (e.g., MKT, see Section 2.3.1), and using what is known about the context to reason about the situation. The main objective is not to seek the best explanation, but more importantly, to make the connections between practice and theory (Jacobs, Lamb, et al., 2011; van Es, 2011; Yang & Ricks, 2012). By drawing upon the principles of learning and teaching in relation to the specific classroom situation, a teacher can then predict possible learning outcomes (Seidel & Stürmer, 2014), and propose possible strategies or suggestions (van Es, 2011). As several researchers emphasise, the key characteristic of good predicting skills, is the use of evidence to make instructional decisions (Hiebert et al., 2007; Jacobs, Lamb, et al., 2011; Santagata & Angelici, 2010; Seidel & Stürmer, 2014).

This attention to the specificity of descriptions, as well as the connectivity of the explanations and predictions, is also captured by the framework for learning to notice student mathematical thinking, developed by van Es (2011). The framework was developed from her analysis of seven elementary school teachers (Grade 4 and 5) who took part in ten video-club meetings. Her framework articulates not only what teachers notice, and how teachers notice, but also presents a possible “developmental trajectory” through four levels: “baseline, mixed, focused and extended” (van Es, 2011, p. 139). Table 2.4 shows the details of this framework.

Even though the framework was developed from one particular video-club, and thus may not reflect the trajectories of other groups of teachers, the descriptions of what and how teachers may notice agree largely with what other researchers have suggested (Jacobs, Lamb, et al., 2011; Santagata, 2011; Seidel & Stürmer, 2014; van Es, 2011).
However, it may not reflect the types of student thinking, the teaching moves in the context of other forms professional development activities (van Es, 2011).

### Table 2.4: Framework for learning to notice student mathematical thinking

<table>
<thead>
<tr>
<th>Level</th>
<th>What Teachers Notice</th>
<th>How Teachers Notice</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Level 1</strong>&lt;br&gt;Baseline</td>
<td>Attend to whole class environment, behaviour, and learning and to teacher pedagogy.</td>
<td>Form general impressions of what occurred; Provide descriptive and evaluative comments; Provide little or no evidence to support analysis.</td>
</tr>
<tr>
<td><strong>Level 2</strong>&lt;br&gt;Mixed</td>
<td>Primarily attend to teacher pedagogy; Begin to attend to particular students' mathematical thinking and behaviours.</td>
<td>Form general impressions and highlight noteworthy events; Provide primarily evaluative with some interpretive comments; Begin to refer to specific events and interactions as evidence.</td>
</tr>
<tr>
<td><strong>Level 3</strong>&lt;br&gt;Focused</td>
<td>Attend to particular students' mathematical thinking.</td>
<td>Provide interpretive comments; Refer to specific events and interactions as evidence; Elaborate on events and interactions.</td>
</tr>
<tr>
<td><strong>Level 4</strong>&lt;br&gt;Extended</td>
<td>Attend to the relationship between particular students' mathematical thinking and between teaching strategies and student mathematical thinking.</td>
<td>Provide interpretive comments; Refer to specific events and interactions as evidence; Elaborate on events and interactions; Make connections between events and principles of teaching and learning; On the basis of interpretations, propose alternative pedagogical solutions.</td>
</tr>
</tbody>
</table>

Many researchers often conceptualise noticing as a connected process integrating the three elements: attending, interpreting, and responding (Barnhart & van Es, 2015; Jacobs et al., 2010; Star & Strickland, 2008; van Es & Sherin, 2002, 2008). For example, Jacobs et al. (2010) see noticing as a set of three inter-related skills that are embedded, as seen in Figure 2.8. This nested relationship suggests that responding to student thinking can occur only if teachers analyse student thinking, and these analyses can only be made if the teachers attend to the details of instances of student reasoning. Moreover, as Jacobs et al. (2010) point out, this nested nature of the three component skills of noticing does not imply that the skills are to be developed in silos, but rather
these skills are to be developed in an integrated manner. However, the connections between the three skills are still unclear.

![Diagram](image.png)

**Figure 2.8. Nested relationship of the three noticing component skills.**

More recently, Barnhart and van Es (2015) explored the relationship between these three skills by investigating whether a video-based course supports a cohort of 16 science prospective teachers in their development of noticing skills. The results were compared to another cohort of 8 teachers who did not participate in the course. Their findings (Barnhart & van Es, 2015) reveal a more complex relationship between the three skills, than previously thought (Jacobs et al., 2010) in four ways.

First, the ability to see or attend to details of student reasoning is crucial for interpreting and responding; and is the “cornerstone” of noticing expertise (Barnhart & van Es, 2015). Developing the skill of seeing the “invisible” (Schifter, 2001) is challenging, and may require teachers to assess students’ ideas in terms of the reasoning and explanations, rather than the accuracy (Hiebert et al., 2007). However, a teacher who is highly competent in attending may still not build on these observations in their analysis of, and response to, the observations (Barnhart & van Es, 2015). This is similar to the “mixed level” of noticing as detailed by van Es (2011) in Table 2.4. Nevertheless, Barnhart and van Es’s (2015) findings seem to indicate that attending, interpreting, and responding to student reasoning are “successively difficult and complicated skills” (p. 91), and that analysing “may be the bridging skill”. This highlights the central role of analysing or interpreting in using the evidence from observations to make instructional decisions.
2.4.4. Key research questions for this study

After examining the complex interactions between the three component skills of noticing and the possible influence an explicit focus for noticing may have on making productive instructional decisions, the three refined research questions that guided this study are stated as follow:

1. Is an explicit focus for noticing useful to encourage more productive noticing, and if so, what kind of foci can be used?

Given that careful attending is the pre-requisite to both interpreting observations, and responding with a decision productive for enhancing student reasoning (Barnhart & van Es, 2015), it was critical for this study to explore this question. An additional challenge for this present study was to document what teachers attend to when they plan, teach, and reflect about their lessons. Most of the professional development centres on the use of retrospective recall or analysis of what teachers notice about video prompts of classroom teaching (Barnhart & van Es, 2015; Jacobs, Lamb, et al., 2011; Kazemi et al., 2011; M. G. Sherin, Russ, et al., 2011; Stockero, 2014; van Es, 2011), classroom artefacts (Goldsmith & Seago, 2011), and analysis of lessons conducted (Santagata, 2011; Santagata & Angelici, 2010). While the use of wearable cameras seems to be a potentially useful method of capturing what teachers attend to, it does not capture teachers’ sense-making processes and is still dependent on the use of interviews to provide a complete picture of what teachers notice (M. G. Sherin, Russ, et al., 2011).

Therefore, this study aimed to demonstrate another way to study noticing in the context of a full lesson cycle—from the planning to the teaching and reflection. An investigation into possible representations of practice to make noticing more visible may be a productive direction of research, and can provide a means to access and assess teachers’ noticing. Through this study, the researcher hoped to develop ways to make teachers’ noticing more explicit and observable so that it not only provides researchers with a means to study mathematical noticing but more pragmatically, a way for teachers to self-assess their noticing.

Next, evidence from research suggests the critical role of interpreting instructional details in noticing during the reflection phase of a lesson. How this process of making sense contributes to the planning and teaching of a lesson is less clear. Hence, an
investigation of teachers' analysing skill in noticing was guided by the following question:

2. How do teachers interpret and make sense of instructional details that lead to decisions that are productive with respect to enhancing student mathematical reasoning?

As highlighted by Barnhart and van Ees (2015), teachers' sense-making of the instructional details may be the link between seeing and responding. In the context of the productive practices (See Section 2.2), which promote student reasoning, teachers' analyses of their students’ thinking can occur during the planning of the task, while orchestrating a whole-class discussions and at the post-lesson reflection. It is, therefore, important that this study captured teachers' pedagogical reasoning, in order to gain insight into how they transform (Shulman, 1987) their knowledge in the process of teaching.

Of particular interest in this study were the characteristics of teachers’ noticing that accompany fruitful instructional decisions. While this researcher expected specificity to be a key component, it remained to be seen whether other factors would feature in cases of productive noticing.

Finally, noticing is hypothesised as a possible means to overcome the challenges of teaching for mathematical reasoning (See Section 2.3.5). This study, therefore, explored the relationships between a teacher's ROG clusters and productive noticing.

3. What are, if any, the changes in teachers’ resources (mainly knowledge), orientations, and goals with respect to teaching for mathematical reasoning when they begin to notice more productively?

Schoenfeld (2011b) suggests that noticing matters a lot. Framing noticing as a goal-oriented decision making process, he suggests that what teachers notice is influenced by their resources, orientations, and goals. On the other hand, noticing has been seen as a “high-leverage” practice (Grossman et al., 2009) that has the potential to improve teaching practices, which are dependent on a teacher’s ROG clusters. While Schoenfeld (2011a) hypothesises that a teacher's growth trajectory may proceed along the three planes depending on the changes to the ROG clusters, what is less clear is the
mechanism by which this may occur. This study also explored this possible link to gain some insight into why teachers notice the way they do, and how their noticing might influence their own ROG clusters.

2.5. Theoretical Framework for this Study

This chapter reviews the complexity of teaching for mathematical reasoning, and proposes the centrality of noticing, as a critical construct in supporting teachers to do this work. It highlights the productive practices associated with promoting student reasoning in Section 2.2 in order to examine the characteristics of productive noticing—mathematical noticing that results in teachers responding with instructional decisions that support student reasoning. The underlying assumption for this study is that productive noticing is manifested when teachers respond to student thinking through these practices during task design, teaching, and reflection. The aim of this study was to investigate what teachers notice, and how they notice, when they make these decisions that promote student reasoning. This study sought to gain an insight as to why teachers notice the way they do by examining their resources, orientations, and goals with respect to mathematics teaching; and highlight how a teacher’s ROG clusters may change as they seek to develop their noticing skills. The design of the study, and the analytical stance taken is guided by the theoretical framework developed from, and informed by the literature reviewed in this chapter. Figure 2.9 shows an overview of this theoretical framework, which serves as a starting point for this study.
Figure 2.9. Theoretical framework for this study.
Chapter 3. Methodological Considerations

Given the complex interactions that exist between teachers, students, and the mathematical content, there were two important methodological considerations for the design of this study. First, the methodological paradigm adopted should allow this researcher to develop a theoretical perspective of productive noticing, as well as a practical means to support teachers in their noticing. Second, the methodology should facilitate the documentation of teachers’ thought processes through all the stages of a lesson cycle: planning, teaching, and reviewing, instead of focusing only on the reflection after the lesson. This chapter, thus, describes how these concerns with regard to the development of the framework and the recording of teachers’ thinking were addressed during the research.

This chapter begins with a discussion of the design research paradigm, which framed the methodology of this research. It addresses how a systematic investigation of a multi-tiered teaching experiment supported the aims of this research. The choice of using Lesson Study as a primary means to investigate teacher noticing is justified, before presenting how the ethical issues for this study were addressed by the researcher in Section 3.2. This is followed by a detailed description of the research context and data collection procedures in the following two sections. Lastly, the approaches to data analysis and representation are presented in Section 3.5.

3.1. The Design Research Paradigm

The design research paradigm addresses the twin challenges of theoretical development and practical application (Zawojewski, Chamberlin, Hjalmarson, & Lewis, 2008). It uses an iterative and highly interventionist approach (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) to generate usable knowledge (Design-Based Research Collective, 2003) that is grounded in complex real-world settings (McKenney & Reeves, 2012). Its methods have been commonly used to study students’ learning in response to interventions in the form of teaching innovations (Anderson & Shattuck, 2012; Brown, 1992); whereas this study’s application of design-based research to study how teachers learn and develop in their practice is less common (Cobb, Zhao, & Dean, 2009; Zawojewski et al., 2008).
Since teacher learning is highly contextual (Lampert, 2009), any attempt to research teacher noticing should be conducted within the authentic setting of a teaching environment. Furthermore, effective teacher professional development should involve “approximations of practice” (Grossman et al., 2009, p. 283) and job-embedded collaborative teacher inquiry (Darling-Hammond & Richardson, 2009; Timperley et al., 2007). It is then useful to design environments or tools that encourage teachers to hone their mathematical noticing. Moreover, it was challenging to identify aspects of their noticing that led to productive instructional decisions. One way to overcome this methodological hurdle is to engage teachers in the development of products, which reveal their thinking, both individually and collectively when they work together to design an artefact (Zawojewski et al., 2008). In the context of this study, the main artefact of interest was the mathematical task used in the lesson; but the key processes associated with the task—the design, implementation, and reviewing—were also important components for the researcher to examine. Hence, design research, which is aimed at “designing learning environments and developing theories of learning” (Design-Based Research Collective, 2003, p. 5), is a suitable methodological paradigm to adopt for this present study.

3.1.1. Multi-tiered teaching experiment

Educational research dealing with teacher learning has “multiple interacting layers of designers”: teachers design educational products such as lesson plans; facilitators design professional development sessions; and researchers design theories (Zawojewski et al., 2008, p. 221). Thus, this created another layer of complexity because of the role of facilitators, in addition to the existing milieu of mathematics, students, and teachers. A way that can capture this multi-layered interaction is through a multi-tiered teaching experiment (Lesh & Kelly, 1997; Lesh, Kelly, & Yoon, 2008). Lesh and Kelly (1997) describe how this method examines learning at three levels: students developing their concepts of mathematics; teachers developing a model to understand students’ thinking; and researchers developing theories about one-to-one tutoring. Similarly, the students, in this study, were encouraged to reason mathematically through the tasks designed by their teachers, while the researcher developed a framework to describe and analyse the teacher noticing, which resulted in these
productive instructional decisions. Hence, the context of this present study warranted the use of a methodology similar to that used by Lesh and Kelly (1997).

Another important consideration for this study is the role of the facilitators (Goldsmith & Seago, 2011; Kazemi et al., 2011). Zawojewski et al. (2008) adds a facilitator tier to the three-tiered model to examine teachers’ learning in the context of professional development. The three tiers of teachers, facilitators and researchers are interdependent, and each affects the development of the other two. This study, which is aimed at developing both theory and practical applications of noticing, hence fit nicely within the framework by Zawojewski et al. (2008). Moreover, a key feature of design-based research, which involves professional development of teachers, is that “teachers externalise their thinking explicitly through the educational objects they design” (Zawojewski et al., 2008, p. 223). Thus this paradigm supported the researcher’s goal of describing teacher noticing, as well as developing, testing and revising his theory of productive mathematical noticing.

The development of a theory to describe productive noticing depends on developing “thick descriptions” (p. 88) of the research settings (Cobb & Gravemeijer, 2008), and there is a need to delineate clearly aspects of thinking of all those involved in the study: the students, the teachers, the facilitators and the researcher (Design-Based Research Collective, 2003; Zawojewski et al., 2008). For this study, it was critical to be explicit about how teachers’ thinking was revealed through the products of a design experiment. Figure 3.1 illustrates the ways in which the thinking of each group of participants was revealed in this study. Although all four tiers of participants contributed and interacted in the classroom, the focus of the researcher, in this study, was to examine closely the teacher tier in order to develop a theory to analyse teachers’ mathematical noticing. Therefore, how teachers plan the lessons, and how they react to students’ thinking during the lesson was documented carefully to achieve the aims of the research.
3.1.2. Systematic investigation of teaching

The second consideration for this study was to develop ways that could capture teachers’ noticing through iterative cycles of planning, teaching and reviewing. Studying teacher noticing often involves the use of video clips of teaching or video case studies. For example, teachers are shown clips of classroom teaching and asked to notice certain features of the instruction (Seidel, Stürmer, Blomberg, Kobarg, & Schwindt, 2011; Star et al., 2011; van Es, 2011). These approaches tend to focus largely on noticing instructional details after lessons are conducted. Other methods attempted to capture teachers’ in-the-moment noticing using wearable cameras (Colestock, 2009; M. G. Sherin, Russ, et al., 2011), or examine how teachers improved the lesson after they had analysed the video recording of their lessons (Santagata, 2011). However, these techniques are mainly retrospective, and do not focus on teachers’ preparation to notice. Thus these methods may not be adequate for investigating teacher noticing through the whole lesson cycle. Therefore, there was a need to look beyond video case studies to explore the teachers’ noticing, particularly during lesson planning.

Going beyond video case studies, the researcher adopted a systematic investigation of teaching for this study to address the need for a more comprehensive approach, which can be applied throughout the lesson cycle. In many ways, the notion of treating
“lessons as experiments” parallels the design research paradigm (Hiebert, Morris, et al., 2003, p. 206):

When teachers treat lessons as experiments, they engage in many of the practices critical for conducting design-based research or design experiments. For example, the goals include both the actual improvement of classroom environments and the generation of shareable knowledge about such environments. The process plays out through continuing cycles of planning, enactment, analysis, and revision, and hypotheses about connections between teaching and learning are used to drive each cycle of the process.

The term experiment does not necessarily mean randomised controlled trials, but is instead taken to highlight that teachers develop the competences to teach well after multiple observations of practice over a period of time. The focus is for teachers to be systematic and rigorous about studying their own practice in order to learn from these experiences (Hiebert, Morris, et al., 2003). This systematic investigation of practice not only generates a product, but also hones teachers’ processes of planning and examining the effectiveness of lessons. The emphasis on the process of learning from practice, as well as the product, was also important for this present study on teacher noticing.

In particular, the four processes involved in this systematic investigation—clarifying the research question, designing the experiment, gathering data, interpreting data and drawing conclusions (Hiebert, Morris, et al., 2003)—helped to frame the data collection methods for this study. First, clarifying the research questions involves articulating the hypotheses that connect the teaching activities with the learning goals explicitly. Next, designing the experiment requires teachers to engage in reasoning about their choice of instructional strategies, and specify how these activities can help change students’ thinking. This shift from “spontaneous” decision making to one in which, teachers plan, and consider possibilities is the essence of the discipline of noticing—“to be methodical without being mechanical” in order to be more sensitised to notice in the moment (Mason, 2002, p. 61). This leads to the collection of information related to students’ thinking, which can help to inform future revisions to the lesson design. To do this, teachers would have to interpret the data collected, and draw conclusions, which focus on student learning in relation to the task design. This set up the experimental cycle again for teachers to clarify their thinking about students’ learning.
Therefore, the *experimental view of teaching*, proposed by Hiebert, Morris, et al. (2003), was useful for this study because it provided a theoretical justification for, and an operationalisation of the design study methodology adopted in this study. This view of teaching, therefore, situates the study of noticing within a cycle of activities to make teachers’ thinking visible. The experimental view thus maintains a focus on improving teaching, instead of improving teachers. This shift is best effected through collaboration with other teachers (Hiebert, Morris, et al., 2003, p. 212):

Shifting from a vision of effective teachers to effective teaching also requires a new set of obligations. Rather than considering only what one is learning from one’s own experience, teachers must ensure that others can learn from their experience, and that they are disposed to learn from others’ experiences. Planning teaching so that others can learn is different from planning teaching so that you can learn. Considering what others can learn from your experience requires collaboration with other teachers who share the same learning goals for students.

Lesson study, which encapsulates the essence of this systematic investigation and the design-research paradigm, was then adopted for this study because it provided a lens, both to examine the mathematical noticing of groups of teachers, and to zoom in to a single teacher.

### 3.1.3. Lesson Study: A design experiment to study teacher noticing

Lesson study is a collaborative inquiry-based teacher professional development approach credited for transforming the teaching of mathematics in Japan (Clea Fernandez et al., 2003; Murata, 2011; Stigler & Hiebert, 1999; Yoshida, 2005). Despite the different adaptations implemented by various countries, there are five essential tasks (See Figure 3.2)—developing a research theme; working, discussing and anticipating student thinking through mathematics tasks; developing a shared lesson plan; collecting data during observation of research lesson; and conducting a post-lesson discussion (Lewis et al., 2011). Sometimes, the protocol includes an additional iteration of observation of research lesson followed by another post-lesson discussion (Murata, 2011; Yoshida, 2005).

Lesson Study is a problem solving process that is designed to help teachers learn about the specific problems of practice. Guided by a research theme, teachers are not only
involved in discussing and solving mathematical tasks, but are also engaged in anticipating student thinking while developing the lesson plan. Lesson study goes beyond teachers coming together for a lesson observation, but is instead focused on getting teachers to collaborate in their investigation of problems of practice. This is done through careful observations; data collection; and discussion of findings based on observable evidence, collected during the lesson. Therefore, in many ways, the Lesson Study protocol mirrors the *experimental model of teaching* as proposed by Hiebert, Morris, et al. (2003). More importantly, the Lesson Study protocol can provide snapshots of a teacher's noticing. With recent calls to improve learning from Lesson Study through a more explicit focus on mathematics (Yoshida, 2012), it is timely to examine how noticing is crucial for teachers to learn from the Lesson Study.

![Figure 3.2. The five key tasks of Lesson Study.](image)

**Figure 3.2. The five key tasks of Lesson Study.**

### 3.1.3.1. The five key tasks of Lesson Study

Similar to the experimental model of teaching (Hiebert, Morris, et al., 2003), Lesson Study begins with the development of a research theme or goal (Lewis et al., 2011; Murata, 2011; Yoshida, 2005). During this stage, teachers consider the gap between their students' current state of learning and their aspirations for students based on deliberate observations of their students or data collected from artefacts, such as samples of student work (Lewis et al., 2011; Yoshida, 2005). The research themes or goals are usually long-term developmental objectives in the learning of mathematics, though they can also be specific to a mathematical topic (Stigler & Hiebert, 1999). The intent of crafting these themes is to guide teachers in their inquiry during Lesson Study.
With these goals in mind, teachers can then begin to design mathematical tasks that may support students to achieve these objectives (Lewis et al., 2011). At times, teachers may also consider how they can use, and test certain strategies or teaching approaches that have been used in other classrooms (Murata, 2011). Whatever the case, it is important that the main tasks designed should be aligned to research themes identified earlier. They would then attempt to solve these tasks; discuss the different solutions; and consider possible student responses to these tasks. In doing so, teachers get to anticipate students’ thinking and see these mathematical tasks through a student’s lens. The solving of tasks; the sharing of solutions; and the anticipating of students’ responses can help hone teachers’ knowledge for practice (Lewis et al., 2011, p. 165):

As teachers discuss their approaches, they make their mathematical thinking visible to colleagues, and teachers may expand their knowledge of solution methods in this way. These conversations may also surface difficulties or misunderstandings related to the subject matter, making problematic ideas available for discussion and revision. By solving tasks, sharing solutions, and anticipating student solution methods, teachers can build their own understanding of both mathematics and student thinking.

Based on these discussions, teachers would then refine the task, and use it to develop a shared research lesson plan. Before they focus on creating the detailed lesson plan, teachers would first study how the topic fits within the curriculum or mathematics; how the unit for this topic can be mapped out in terms of the sequence of lessons; and how the research lesson (within this unit of the topic) can move students towards the goals decided earlier (Stepanek, Appel, Leong, Mangan, & Mitchell, 2007).

The detailed research lesson plan functions like a test-bed for the research theme identified by the teachers. It is a physical manifestation of what the goals look like in the classroom, and provides a way to find out whether, and if so why, the lesson works (Stigler & Hiebert, 1999; Yoshida, 2005). It captures the teachers’ thinking, and serves as a record, with which the teachers can re-examine the lesson to highlight where their original ideas may have changed (Lewis et al., 2011). As a form of design-based research methodology, the lesson plan provides the documentary trail necessary for this study.

The discussion and co-development of the plan also enable teachers to “be clear about where they are going with the lesson they are preparing” (Yoshida, 2005, p. 7) and
provide a platform for teachers to share their diverse perspectives and experiences. In a way, teachers’ thinking about mathematics learning and teaching are “made visible” through this shared process (Lewis et al., 2011, p. 171). Teachers can then integrate their learning into the development of a collective lesson plan that focuses on making the mathematical content visible to their students. Teachers’ discussion and careful study of the lesson plan is thus the mechanism to effect any learning from an experimental perspective.

Lastly, the lesson plan is a “research proposal” developed by the team of teachers to investigate the problems of practice as stated in their research themes (Yoshida, 2005). The teachers can use the lesson plan to help them to observe the lesson more critically, and as a lens to direct data collection during the research lesson. This involves articulating explicitly the evidence of student thinking and learning in the lesson plan, in order to remind teachers of what to look out for during the research lesson.

As the “centerpiece” of Lesson Study, a research lesson is implemented by a research teacher based on the agreed lesson plan (Murata, 2011; Stigler & Hiebert, 1999). The other teachers will observe, and try to notice various aspects of the lesson, which include the interactions between research teacher, students and the content (Murata, 2011). Experienced observers may attend to particular facets of lessons that are invisible to novices, and this expert knowledge may surface when teachers analyse the effectiveness of the lesson during post-lesson discussion.

In order to better understand how their lesson plan may help or hinder their students’ learning, it is important for teachers to pay attention to students’ reasoning in the following way (Yoshida, 2005):

1. How they interpret the concepts;
2. How students indicate their confusion about a concept; and
3. What students do when they are struggling to make sense of the mathematics.

In other words, they have to examine the classroom interactions through a researcher’s lens as non-participatory observers (Chokshi & Fernandez, 2004; Clea Fernandez & Yoshida, 2004; Lewis et al., 2011; Murata, 2011; Stepanek et al., 2007; Yoshida, 2005). Teachers then use their notes taken during the observations to support a post-lesson discussion.
Chapter 3 – Methodological Considerations

The post-lesson discussion provides a forum for teachers to discuss the strengths and weaknesses of their lesson design based on specific observations of the research lesson (Lewis et al., 2011). The post-lesson discussion, facilitated by one of the teachers in the team, usually begins with the research lesson teacher (the one who taught the lesson) reflecting about his or her views on students’ thinking and learning difficulties encountered during the teaching. He or she will also share the thinking behind any decisions made that were different from the lesson plan (Lewis et al., 2011; Yoshida, 2005). The other teachers will follow up with their own thoughts from the notes taken during the observations. The facilitator may then moderate the discussion to focus on a few interesting points raised from observations about students’ thinking without evaluating the teacher’s teaching. At the end of the discussion, the teachers summarise the key learning points and refine the lesson plan based on the points raised during the review. If the logistics permit, the refined lesson can be taught to a different class followed by another discussion.

3.1.3.2. A novel approach to study noticing

The five key tasks of Lesson Study helped operationalise what it means for teachers to participate in a community of practice, and learn from their practice (Jaworski, 2006; Lave, 1993, 1996). In essence, Stigler and Hiebert (1999) position a lesson as an important mechanism to improve teaching because the teachers’ learning is situated in the context of an actual classroom:

The premise behind Lesson Study is simple: If you want to improve teaching, the most effective place to do so is in the context of a classroom lesson. If you start with lessons, the problem of how to apply research findings in the classroom disappears. The improvements are devised within the classroom in the first place. The challenge now becomes that of identifying the kinds of changes that will improve student learning in the classroom, and once the changes are identified, of sharing this knowledge with other teachers who face similar problems, or share similar goals, in the classroom (p. 111).

However, the learning and application of teacher knowledge in mathematics classrooms is not automatic (Clea Fernandez et al., 2003; Clea Fernandez & Chokshi, 2005; Yang & Ricks, 2013). What teachers see, and how they make sense of instructional details encountered during Lesson Study are likely to impact their learning from practice. This
study, therefore, addressed the role of noticing in the context of a Lesson Study, and examined the characteristics of noticing that follow instructional decisions that are productive for enhancing students' reasoning. Expanding the scope of current research on teacher noticing, this study took a novel approach by including lesson planning in addition to video studies of lesson, the method most commonly taken by researchers in this field. Hence, the findings from this present study may potentially advance a more comprehensive perspective of teacher noticing.

3.2. Ethical Considerations

The key guiding principle of educational research is to “do no harm” (British Educational Research Association, 2011; Miles, Huberman, & Saldana, 2013, p. 56; New Zealand Association for Research in Education, 2010). Of critical importance to this project is the need to consider the possible harm to the teachers and students as the researcher interacts with the teachers in the context of a professional learning community. Approval was granted for the study to be conducted in Singapore by the University of Auckland Human Participants Ethics Committee (Ethics approval no. 8218). See Appendix 1 for a sample of the Participant Information Sheet (PIS) for teacher participants. The following sections outline the key considerations regarding informed consent, confidentiality and anonymity.

3.2.1. Informed consent

Three groups of direct and indirect participants—teachers, principals and students—were identified in this study. The primary participants were teachers and they were given an overview of the study, what the study might involve, and how it might be reported during the initial meeting. The researcher’s role was also made very explicit to the teachers and the teachers were made aware that they had the right to withdraw from the study at any time (within a reasonable timeframe) without the need to give any reasons (See Teacher Participant PIS in Appendix 1). The teachers were protected from any potential adverse appraisals as participants of this study through an undertaking by the principal of the school. The researcher on the study briefed the principal of each school, and teachers were assured that their participation and non-participation in the study would not affect their annual appraisal in any way. The study focused on the teachers as the key participants, and their interactions with the students.
While students’ work was not analyzed by the researcher, it was made clear to the parents of the students involved that the researcher would analyze teachers’ responses to students’ work.

3.2.2. Anonymity and Confidentiality

With the use of digital voice recorders and video recorders, anonymity and confidentiality might be difficult to ensure (Miles et al., 2013). However, the researcher explained to the participants how the recordings would be transcribed under a Confidentiality Agreement to ensure that the information given by the teachers would not be linked to them. Access to the raw data was restricted and secured, and data would be destroyed after six years. In addition to the allocation of pseudonyms to all participants, teachers had the control to decide whether their photos or videos could be published in articles or used in presentations. Furthermore, the possibility of identification was made clear to the participants if they chose to allow their photos or video snapshots to be used. As students were not the focus of this study, all students’ faces would be digitally obscured in the unlikely event of using video snapshots involving students.

3.3. The Research Context

This research began when I met up with the three principals, in Singapore, who responded to my advertisement to participate in the study. All three schools had processes in place to support learning communities within and between schools. The teachers in this study had used Lesson Study as one of the professional development activities, and they were familiar with the Lesson Study activities. This section presents a brief description of each of the three schools and the teacher participants to set the context for the study. In addition to the regular Lesson Study sessions, a workshop on mathematical thinking was conducted for each of the three schools before the main part of the study in 2013. The details of the workshop and other forms of support given to the teachers are also described in this section.

3.3.1. Greenhill Primary School

Greenhill Primary School is a government, co-educational single session primary school situated in the eastern part of Singapore. The school enrolment is around 700, which is
smaller than the average size of a regular primary school in Singapore. As with all other primary schools, the school roll comprises of students from the surrounding neighbourhood with multicultural backgrounds. Most of the students come from lower and middle-income families. The students enrolled in the school have a wide spectrum of achievement levels, ranging from low to very high levels.

For Phase 1 of the study, seven mathematics teachers and a school leader, Jaslyn, formed the Lesson Study group that explored the teaching of fractions for Primary Two students (age 8). Four of the teachers have more than 10 years of experience and the others have at least three years. Most of the teachers are not trained in Mathematics. Table 3.1 shows the educational background and teaching experience of the teachers involved in Phase 1 of the study.

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Educational Background/ Professional Qualifications</th>
<th>Years of Experience</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice</td>
<td>Obtained a Diploma in Education, and currently pursuing a degree in English Language and Sociology.</td>
<td>14 years</td>
</tr>
<tr>
<td>Hannah</td>
<td>Obtained a Bachelor of Science in Mathematics and a Diploma in Education.</td>
<td>16 years</td>
</tr>
<tr>
<td>Heather</td>
<td>Obtained a Bachelor of Arts in Psychology and a Post-graduate Diploma in Education.</td>
<td>4 years</td>
</tr>
<tr>
<td>Heidi</td>
<td>Obtained a Bachelor of Arts in English Language with a Diploma in Education.</td>
<td>5 years</td>
</tr>
<tr>
<td>Jacinda</td>
<td>Obtained a Bachelor of Science (Mathematics) and a Diploma in Education.</td>
<td>12 years</td>
</tr>
<tr>
<td>Sherry</td>
<td>Obtained a Bachelor in Business Studies and a Post-Graduate Diploma in Education.</td>
<td>3 years</td>
</tr>
<tr>
<td>Zelina</td>
<td>Obtained a Certificate of Education (equivalent to a Diploma in Education).</td>
<td>25 years</td>
</tr>
</tbody>
</table>

For the second and third phase of the study, another six mathematics teachers formed a Lesson Study group that explored the teaching of ‘Fractions of a set’ and ‘Conversion of Fractions to Decimals’ for Primary Four students (aged 10). All teachers had at least five years of teaching experience and their educational background is shown in Table 3.2. The teachers in Phases 2 and 3 were different from those in Phase 1 because of the school’s deployment.
### Table 3.2: Profile of Greenhill teachers involved in Phases 2 and 3

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Educational Background/Professional Qualifications</th>
<th>Years of Experience</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cindy</td>
<td>Obtained an Advanced Diploma in Teaching Mathematics.</td>
<td>30 years</td>
</tr>
<tr>
<td>Flora</td>
<td>Obtained a Diploma in Marketing and a Diploma in Education (Specialisation in Mathematics and Science).</td>
<td>7 years</td>
</tr>
<tr>
<td>James</td>
<td>Obtained a Bachelor of Engineering (Civil Engineering) and a Post-Graduate Diploma in Education.</td>
<td>5 years</td>
</tr>
<tr>
<td>Kirsty</td>
<td>Obtained a Bachelor of Arts with Diploma in Education, majoring in English Language.</td>
<td>21 years</td>
</tr>
<tr>
<td>Anthony</td>
<td>Obtained a Bachelor of Science in Mathematics, a Post-Graduate Degree in Business Administration.</td>
<td>15 years (varied experience)</td>
</tr>
<tr>
<td>Rani</td>
<td>Obtained a Diploma in Education and an Advanced Diploma in Teaching Mathematics.</td>
<td>23 years</td>
</tr>
</tbody>
</table>

### 3.3.2. Springside Secondary School

Springside Secondary School is a government co-educational secondary school located in the Northeast region of Singapore. In Singapore, students apply for their secondary schools and get posted to their secondary schools based on the t-scores obtained from a single seating of the Primary School Leaving Examination (PSLE). There are three courses offered at Springside Secondary, namely Express, Normal (Academic) and Normal (Technical). The median entry PSLE t-scores for the three courses are 206, 167 and 142 respectively\(^1\). The school enrolment is around 1450 students. Based on the median PSLE t-score requirements to be enrolled in the school, the school can be considered as a typical secondary school in Singapore with students from the average achievement bands.

A total of eight teachers were involved in the study. During the exploratory study, a team of five teachers was involved in a Lesson Study group looking at a lesson on mathematical problem solving. During the main study, two of the teachers left and three other teachers joined the team to look at ‘Mensuration’ and ‘Gradients of Straight Lines’ for Secondary Two Express students (age 14). Table 3.3 shows the educational and professional profile of the teachers involved in the study.

---

\(^1\) The national median PSLE t-score is around 200.


### Table 3.3: Profile of Springside teachers involved in the study

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Educational Background/Professional Qualifications</th>
<th>Years of Experience</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>Don</em></td>
<td>Obtained a Bachelor of Science (Hons) in Mathematics and a Post-Graduate Diploma in Education.</td>
<td>8 years</td>
</tr>
<tr>
<td><em>Helen</em></td>
<td>Obtained a Bachelor of Science in Mathematics and a Post-Graduate Diploma in Education.</td>
<td>3 years</td>
</tr>
<tr>
<td><em>Nick</em></td>
<td>Obtained a Bachelor of Engineering, a Masters of Science (Business and IT) and a Post-Graduate Diploma in Education.</td>
<td>6 years</td>
</tr>
<tr>
<td><em>Sally</em></td>
<td>Obtained a Bachelor of Science in Mathematics and a Post-Graduate Diploma in Education.</td>
<td>4 years</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(Two years teaching Mathematics)</td>
<td></td>
</tr>
<tr>
<td><em>Winston</em></td>
<td>Obtained a Bachelor of Engineering (Mechanical Engineering) and a Post-Graduate Diploma in Education.</td>
<td>2 years</td>
</tr>
<tr>
<td><em>Anita</em></td>
<td>Obtained a Bachelor of Science and Post-Graduate Diploma in Education.</td>
<td>12 years</td>
</tr>
<tr>
<td><em>Teresa</em></td>
<td>Obtained a Bachelor of Science in Mathematics and a Post-Graduate Diploma in Education.</td>
<td>Beginning Teacher</td>
</tr>
<tr>
<td><em>Kent</em></td>
<td>Obtained a Bachelor of Engineering (Electrical and Electronic Engineering)(Honours) and a Post-Graduate Diploma in Education.</td>
<td>Beginning Teacher</td>
</tr>
<tr>
<td><em>Eddie</em></td>
<td>Obtained a Bachelor of Engineering and a Post-Graduate Diploma in Education.</td>
<td>10 years</td>
</tr>
</tbody>
</table>

#### 3.3.3. Trafford Secondary School

Trafford Secondary School, on the other hand, is an autonomous government secondary school that takes in students with a median PSLE t-score of 238. The school offers only Express/Special course and can be considered as a high achievement school in Singapore. The students can be described as above average in their academic achievements.

A total of 15 teachers were involved in the study: Six took part in the exploratory study and one of them left during the main study in 2013. Nine other teachers then joined the study to form two Lesson Study teams. One team examined ‘Straight Line Graphs’ and ‘Set Theory and Notation’ for Secondary Two (age 14) and the other team looked at ‘Exponential and Logarithmic Functions’ and ‘Trigonometric Equations’ for Secondary Three (age 15). Table 3.4 shows the educational and professional profile of the teachers involved in the study.
Taken together, the teachers in these three schools represent a wide spectrum of teacher knowledge, experience and settings. This purposive sample of teacher participants can be viewed as a representative sample of Singapore schools. Even though the researcher acknowledges that the settings are unique in each school, the lessons from this study can inform efforts to support teacher noticing in other settings.

**Table 3.4: Profile of Trafford teachers involved in the study**

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Educational Background/Professional Qualifications</th>
<th>Years of Experience</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angeline</td>
<td>Obtained a Bachelor of Engineering (Biological and Chemical Engineering)(Honours) and a Post-Graduate Diploma in Education.</td>
<td>2 months</td>
</tr>
<tr>
<td>Albert</td>
<td>Obtained a Bachelor of Engineering (Aeronautical), Masters in Business Administration, and a Post-Graduate Diploma in Education (Physical Education).</td>
<td>1 year</td>
</tr>
<tr>
<td>Cayden</td>
<td>Obtained a Bachelor of Engineering (Civil Engineering) and a Post-Graduate Diploma in Education.</td>
<td>3 years</td>
</tr>
<tr>
<td>Lindy</td>
<td>Obtained a Bachelor of Science and a Diploma in Education.</td>
<td>15 years</td>
</tr>
<tr>
<td>Tim</td>
<td>Obtained a Bachelor of Engineering (Electrical and Electronic Engineering)(Honours) and a Post-Graduate Diploma in Education.</td>
<td>5 years</td>
</tr>
<tr>
<td>Wesley</td>
<td>Obtained a PhD in Mechanical Engineering (No professional qualification in teaching yet).</td>
<td>6 months</td>
</tr>
<tr>
<td>Adam</td>
<td>Obtained a Bachelor of Science with Diploma in Education (Mathematics).</td>
<td>2 years</td>
</tr>
<tr>
<td>Aniyah</td>
<td>Obtained a Bachelor of Engineering (Mechanical Engineering) and a Post-Graduate Diploma in Education.</td>
<td>3 years</td>
</tr>
<tr>
<td>Hayley</td>
<td>Obtained a Bachelor of Science (Mathematics) and a Post-Graduate Diploma in Education.</td>
<td>2 years</td>
</tr>
<tr>
<td>Hazeline</td>
<td>Obtained a Bachelor of Engineering (Electrical and Electronic Engineering) and a Post-Graduate Diploma in Education.</td>
<td>10 years</td>
</tr>
<tr>
<td>Ivy</td>
<td>Obtained a Bachelor of Science (Applied Mathematics) and a Post-Graduate Diploma in Education.</td>
<td>6 years</td>
</tr>
<tr>
<td>Linus</td>
<td>Obtained a Bachelor of Engineering (Mechanical Engineering)(Honours), Masters in Education (Teaching and Curriculum), and a Post-Graduate Diploma in Education.</td>
<td>7 years</td>
</tr>
<tr>
<td>Keaton</td>
<td>Obtained a Bachelor of Science (Mathematics) and a Post-Graduate Diploma in Education (equivalent).</td>
<td>23 years</td>
</tr>
<tr>
<td>Peggy</td>
<td>Obtained a Bachelor of Science (Applied Mathematics) and a Post-Graduate Diploma in Education.</td>
<td>7 years</td>
</tr>
<tr>
<td>Wendy</td>
<td>Obtained a Bachelor of Science with Diploma in Education (Physics) and Masters in Education (Mathematics Specialisation).</td>
<td>13 years</td>
</tr>
</tbody>
</table>
3.3.4. Workshop on mathematical reasoning and other support

A two-hour workshop held for each of the three schools before the commencement of the main study fulfilled three purposes. First, it was an entry point for the researcher to re-establish contact with all the teachers involved in this study. This was important because there was a five-month gap between Phase 1 in 2012 and the other two phases in 2013. Moreover, some of the teachers who were involved in Phase 1 had left, and there were some new teachers who joined in 2013. These staff movements were context specific and beyond the control of the researcher. The workshop was conducted for all the mathematics teachers in the school, and provided a means for the researcher to get to know the specific context of each school. In a way, the workshop became a platform for teachers to articulate their thinking, and this helped the researcher to document the teachers’ “starting point” in noticing mathematical reasoning (Cobb & Gravemeijer, 2008, p. 69). Next, the researcher, through the workshop, introduced the envisioned notion of mathematical thinking, and how it might look like in a classroom context. This helped to establish a common operating definition of mathematical reasoning amongst the teachers. Lastly, the workshop also introduced basic notions of noticing and their role in teaching for mathematical thinking.

The workshop comprised three main activities. The introductory activity was designed to find out about teachers’ current notions of mathematical thinking. It was a ‘Think-Pair-Share’ activity, in which teachers articulated what they thought about mathematical thinking through a description of what they might hear or see in a classroom that promotes thinking. The second activity consisted of two mathematical tasks designed to promote mathematical thinking. The purpose was to allow teachers to experience what it was like to “do mathematics” (Smith & Stein, 1998; 2011, p. 16). A number of tasks were adapted or designed but only two were used for each school. One of the tasks was more exploratory and might not be directly related to the curriculum. Examples of such exploratory tasks include Euclid’s Game, Regions in a Circle and Coins and Paper Clips. Figure 3.3 shows the ‘Coins and Paper Clips’ game used with the Primary School teachers.

The second task was more directly linked to the curriculum in terms of the mathematical content required to complete the tasks. These are possible examples of “Learning Experiences”, which are “opportunities where students discover...
mathematical results on their own” by working collaboratively to “present their ideas using appropriate mathematical language and methods” (Ministry of Education-Singapore, 2013, p. 20). The second task was included in the workshop to create buy-in amongst teachers by demonstrating how mathematical tasks that promote mathematical thinking are aligned with the latest curriculum documents. An example of such a task used with Secondary School teachers was ‘The Best Circle’ task\(^2\) as shown in Figure 3.4.

\textbf{Coins and Paper Clips}

This game is played by two players. Player A will place coins on any two squares. Player B will place paper clips on the board so that each clip lies on two squares that share a common side. The clips may not overlap each other. To win, player B has to place seven paper clips in this way so that they lie on the fourteen squares not occupied by the coins. If player B cannot do this, player A wins. Is there a way to win this game? Can you prove it?

![Coins and Paper Clip Game](image1)

\textbf{Figure 3.3. The ‘Coins and Paper Clip Game’.

Best Circle?}

Four guys—Andrew (top left), Chris (top right), Nathan (bottom left) and Timon (bottom right)—had a circle drawing competition. But they couldn’t decide the winner. Your task is to create a method to rank circles from the least circular to the most circular. You must use your method to rank the circles you have been given. In addition, your method should also work for any circle given to you. Explain how and why your method works.

![Best Circle Task](image2)

\textbf{Figure 3.4. The ‘Best Circle Task’.

\(^2\) Adapted from Dan Meyer’s Three Act Math task found at http://threeacts.mrmeyer.com/bestcircle/}
The last section of the workshop introduced some principles in task design based on the ideas discussed in Section 2.2.1, and the teachers worked on applying the principles to modify an existing mathematical task. Notions of mathematical noticing (See Section 2.4.1) and how they featured in the study were shared with the teachers before the Questions and Answers segment of the workshop.

The effectiveness of such a one-off workshop might be questioned because it takes time for teachers to develop an understanding of how mathematical reasoning can be fostered in the classrooms. Thus, the researcher revisited some of these ideas about mathematical thinking, student reasoning, and noticing throughout the study. For instance, the researcher used the first 30 minutes to revisit some of these concepts with the teachers during the Lesson Study discussions. He also clarified the use of the tools featured in the modified Lesson Study protocol. These on-the-job sessions, together with ad-hoc mathematics study sessions to share some points related to mathematical content, formed part of the support given to teachers as they worked on designing mathematics tasks during the Lesson Study sessions.

3.4. Data Collection

This section explains how the design research framework was applied in the form of a modified Lesson Study protocol during this research. The study took place in three phases over a period of nine months. Table 3.5 and Table 3.6 outline the timeline for Phase 1, Phases 2 and 3 respectively.

3.4.1. Phase 1: Observing teacher mathematical noticing in Lesson Study

The purpose of Phase 1 of this study was to investigate mathematics teacher noticing during Lesson Study sessions in order to inform the design of the modified Lesson Study protocol for next two phases. Specifically, this phase served to highlight what, and how, teachers notice during Lesson Study sessions. The focus was to uncover what teachers attend to during planning, teaching and reviewing of lessons.

Data collection for the first phase of the study took place in each of the three schools from July 2012 to August 2012. The researcher employed participant observation to gain insight into what, and how, teachers discussed during the Lesson Study sessions. Every session was voice-recorded and lessons were video-recorded as part of the
Lesson Study protocol practised by the schools. The use of participant observation allowed the researcher to gain access to a wider range of data such as lesson plans and lesson materials, to have a better understanding of the context in which the mathematical content was discussed, and to understand possible challenges in data collection during the later iterations (Guest, Namey, & Mitchell, 2013; Scott & Usher, 2010). The researcher, however, chose to adopt a “highly observational” stance and shifted between “more and less visible researcher role” in the participant observation continuum (Guest et al., 2013, p. 89), or what some researchers term “reactive observation” (Angrosino, 2012, p. 166). In doing so, the researcher did not have any input into how teachers designed the learning activities, and on the quality of observations made by teachers during lesson observations and post-lesson discussions. On a few occasions the teachers asked the researcher questions pertaining to mathematical content, and the researcher attended to these questions without making any evaluation of teachers’ mastery of content.

While it may be the case that the presence of the researcher, and the use of a recording device during the Lesson Study discussions might affect the behaviour of the teachers (Williams, 2008), this issue was mitigated by a few factors: First, school teachers in Singapore are used to having visitors and observers in classrooms, and in other professional development settings because many schools are part of professional learning communities (Chua, 2009). The researcher had also taken care to introduce himself as an observer-researcher to all teachers and emphasised his primary role as an observer. Furthermore, the first meeting with the teachers was mainly used to establish rapport and a common understanding of the research purpose. He also had an extra meeting with the facilitators of the Lesson Study sessions from the project schools to reiterate his observer stance. Next, the teachers involved in the study were familiar with Lesson Study, and the facilitators involved were experienced teachers. Therefore, they were comfortable with the idea of having a “knowledgeable other” (Watanabe & Wang-Iverson, 2005, p. 85) during the discussion, which is an usual feature of Lesson Study. However, instead of positioning himself as a “knowledgeable other”, the researcher placed himself as a listener willing to learn from the discussions and reassured the teachers that the study would be non-evaluative with regard to their teaching competence throughout the study. Last but not least, the researcher used to be a
teacher, a Head of Department in a school, and a curriculum policy officer with the Ministry of Education (Singapore). These varied experiences had enabled him to understand the settings of the research field and the norms of working with schoolteachers.

A digital voice recorder was used to record all the discussions, and the teachers arranged for their own teachers to videorecord the lesson observations as part of their usual practice. Copies of the lesson plans and lesson materials were obtained with permission from the teachers at the end of the Lesson Study cycles. Field notes were taken and reviewed to mark out any interesting episodes in order to facilitate data analysis.

### Table 3.5: Timeline for Phase 1 of the study

<table>
<thead>
<tr>
<th>Phase 1: (Term 3 2012)</th>
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<tbody>
<tr>
<td><strong>Late June to July</strong></td>
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<tr>
<td><em>(Term 3 week 1)</em></td>
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<tr>
<td>Establish initial contact with the three schools.</td>
</tr>
<tr>
<td>Meeting with the principals and senior management teams of the three schools. Information sheets and consent forms given.</td>
</tr>
<tr>
<td>Introductory meeting with the teacher participants in the Phase 1 of the study to discuss the aims of the research. Information sheets and consent forms given.</td>
</tr>
<tr>
<td><strong>July to August</strong></td>
</tr>
<tr>
<td><em>(Term 3 week 2 to 9)</em></td>
</tr>
<tr>
<td>Lesson study sessions including lesson observations with Greenhill Primary School (7 sessions).</td>
</tr>
<tr>
<td>Lesson study sessions including lesson observations with Springside Secondary School (9 sessions).</td>
</tr>
<tr>
<td>Lesson study sessions including lesson observations with Trafford Secondary School (8 sessions).</td>
</tr>
<tr>
<td>Interviews with teacher participants in Phase 1 of the study (18 teachers).</td>
</tr>
<tr>
<td>Meeting with principals and senior management teams to discuss proposed research plans for the iteration phases in 2013.</td>
</tr>
</tbody>
</table>

### 3.4.2. Phase 2 and 3: A more participatory approach to studying teacher noticing

The iterative nature of design experiments is one of the defining features of design research (Anderson & Shattuck, 2012; Cobb et al., 2003; Design-Based Research Collective, 2003; Zawojewski et al., 2008). These multiple iterations can occur within and between studies (Zawojewski et al., 2008). In this study, the iterations occurred within each school, and between schools throughout the second and third phases of the study. Different modifications, tools and aspects of the modified Lesson Study protocol
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(See Section 3.4.3) were implemented for different schools depending on the context of the school. This is consistent with the notion that design research is highly contextual. In this section, I describe the data collection methods and the modified Lesson Study protocol used during the next two phases of this study.

Table 3.6: Timeline for Phases 2 and 3 of the study

<table>
<thead>
<tr>
<th>Phase 2: (Term 2 2013)</th>
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<tbody>
<tr>
<td><strong>Late February to March</strong></td>
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<tr>
<td><strong>March to May</strong></td>
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<tr>
<td><strong>March to May (Term 2 week 1 to week 10)</strong></td>
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</table>

<table>
<thead>
<tr>
<th>Phase 3: (Term 3 2013)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Late June</strong></td>
</tr>
<tr>
<td><strong>July to early September</strong></td>
</tr>
<tr>
<td><strong>(Term 3 week 1 to week 10)</strong></td>
</tr>
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Data collection for Phases 2 and 3 began in each of the three schools from the end of February 2013 to the end of August 2013. Unlike the observational stance taken during Phase 1, the researcher took on a more participatory role (Guest et al., 2013) and co-facilitated the Lesson Study sessions with the school facilitators. For each Lesson Study session, the objective of the session was highlighted and teachers were guided in their use of the modified protocols. The respective school facilitators would then take over and facilitate the sessions thereafter. The planning processes were steered mainly by the school facilitators, with the researcher acting mainly as a resource person most of the time. Throughout these two phases, the researcher’s participation moved along the twin continua of “participation and observation” (Guest et al., 2013, p. 88), as depicted in Figure 3.5.

In contrast to the first phase, the researcher asked questions to direct teachers’ attention to the modified Lesson Study protocol, and gave comments and suggestions for the task design. The researcher was considered to be part of the Lesson Study team by the teachers as he tried to encourage teachers to be more explicit in their noticing through the use of the various tools in the modified Lesson Study protocol. The stance was viewed as more collaborative, and a documentation trail was produced through the voice recordings of the sessions and the artefacts, such as lesson plans and lesson materials, generated by the teachers as they worked through the modified Lesson Study protocol.

Care was taken to maintain a balance between the researcher designing experiences to prompt teachers’ noticing, and teachers taking ownership of the task designs. In all the sessions, the researcher deferred to respective school facilitators to make the final decisions. The researcher also provided information regarding mathematical content when necessary during the sessions. The school teams and the researcher video-recorded the lessons separately using two separate cameras. During the last session of each cycle, the researcher would summarise the key learning points and invite feedback from the teachers to fine-tune the protocols and tools.

Throughout the study, the researcher took on different roles such as participant-observer and teacher-researcher at different times. While this might be problematic or conflicting in typical experimental research (Ainley, 1999), taking on multiple roles is
consistent in the context of a multi-tiered design experiment like Lesson Study (Lesh et al., 2008).

![Diagram of researcher participation continua]

**Figure 3.5. Continua of researcher participation.**

### 3.4.3. The modified Lesson Study protocol

The modified Lesson Study protocol used in this study was based on the experimental model for teaching (Hiebert, Morris, et al., 2003) and the five key Lesson Study tasks (Lewis et al., 2011). The tasks outlined in Figure 3.6 fit the purpose of this study because they provide teachers with a useful frame to focus their attention on the processes involved in designing a lesson and examining the effectiveness of the design (Hiebert, Morris, et al., 2003).

![Diagram of tasks in the modified Lesson Study protocol]

**Figure 3.6. Tasks in the modified Lesson Study protocol.**

The modified Lesson Study protocol, used in Phases 2 and 3, incorporated tools to document teachers’ practices of noticing—“systematic reflection, recognising of choices
and alternatives, preparing and noticing possibilities and validating ideas with other teachers”—as they plan the lessons (Mason, 2002, p. 95). More specifically, the conceptual framework for this study, as developed and discussed in chapter 2, helped shape the protocol. Based on the findings from Phase 1, the language of the Three Point Framework (Yang & Ricks, 2012) was used to elicit what teachers notice. The modified Lesson Study protocol also integrated lesson play (Zazkis et al., 2009) to provide documentation of teachers’ pedagogical thinking and decisions. Other tools such as the ‘Task Analysis Guide’ (Smith & Stein, 2011) and the ‘Questions Prompts for Mathematical Thinking’ (Watson & Mason, 1998) were given to teachers as resources as they planned to teach for mathematical thinking (See Table 2.1 and Table 2.2 respectively).

There were three main stages in the modified Lesson Study protocol used in this study. The protocol began with clarifying the research question that motivated teacher inquiry in each of the schools. Using the language afforded by Three-point Framework, teachers in the study considered the key mathematical idea (Key Point) of the lesson, discussed the cognitive difficulties that students might face (Difficult Point), and suggested how this difficulty might be bridged (Critical Point). The protocol was designed to prompt teachers to be more explicit in identifying the Three Points and teachers’ thinking was captured through the use of a template (Appendix 2).

Using input gathered from the template, the teachers focused on the design of the lesson by addressing the difficulty of learning the concept through the use of a hypothesised approach or Critical Point. Testing the effectiveness of the proposed Critical Point provided teachers opportunities to “study systematically” and “reason rigorously” about their practice in order to improve their teaching (Hiebert, Morris, et al., 2003, p. 207). The teachers then developed a Key Task that used the suggested approach (Critical Point) to overcome the Difficult Point. The purpose of a Key Task is to initiate some mathematical activity (Mason & Johnston-Wilder, 2006) so that teachers can provide students opportunities to overcome the Difficult Point and learn the Key Point. Teachers were given the Task Analysis Guide (See Table 2.1) to support them in selecting and designing appropriate mathematical tasks, and a set of task design principles developed from Section 2.2.1 was also given to teachers for their consideration.
Another important part of this stage was to anticipate students’ solutions or responses to the Key Task. The opportunity to anticipate students’ solutions or responses (both right and wrong) not only provided a platform to develop teachers’ mathematical knowledge for teaching (Lewis et al., 2011), but also provided a means for the researcher to examine what these teachers attended to. To document teachers’ thinking more explicitly, lesson play (Zazkis, Sinclair, & Liljedahl, 2013) was incorporated into the lesson plan template (See Appendix 3). Crafting a lesson play requires teachers to focus on specific teaching incidents; it directs their attention to relevant mathematical details, and offers them opportunities to consider other alternatives to respond to students’ reasoning (Zazkis et al., 2013). This supported the practices of noticing (Mason, 2002), which the study aimed to develop in teachers, and enabled the researcher to gain insight into teachers’ noticing. More importantly, as argued by Zazkis et al. (2013), lesson plays “provide a window into imagined trajectories of ‘good’ teaching, from which we can infer about teachers’ knowledge of mathematics, their knowledge of mathematics for teaching, as well as their pedagogical inclinations” (p. 30).

In Stage Two, the research teacher from each school conducted the lesson and the rest of the teachers in the respective teams observed the lesson without interacting with the students directly. The observers were told not to teach the students nor initiate side discussions with the students. The protocol was designed to facilitate a teacher’s focus on students’ reasoning by directing their attention to what they see without “accounting for” their observations (Mason, 2002, p. 62). Observation sheets (Appendix 4) were given to teachers for them to record instances of students’ thinking during the lesson, and their thinking about what they observed after the lesson, providing a documentation trail. On the other hand, the teacher who was teaching the class would attempt to “monitor” the students’ discussions, “select” particular students to present their work, “sequence” the presentations and “connect” the different solutions to bring out the mathematical ideas of the lesson (Smith & Stein, 2011, p. 8). An optional ‘Monitoring Chart’ (Appendix 5) was given to the teachers teaching to facilitate the monitoring, selection and sequencing of the presentations. The ‘Monitoring Chart’ was made optional because some teachers found it hard to use when they were teaching.
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The last stage of the modified Lesson Study protocol involved a post-lesson discussion. These discussions provided the opportunity for the teachers to examine the effectiveness of the lesson design. Specific instructions were given to teachers to focus on the ‘Three Points’ and support their comments using specific evidence from the lesson observations. The discussions also highlighted any students’ difficulty that was not previously discussed. The team would then revise the lesson design based on the points raised during the post-lesson discussion. A teacher, who might be different from the original one, then taught the revised lesson and the whole team went through Stage 3 and 4 again.

To summarise, the modified Lesson Study protocol used in the study consist of four stages: Clarifying the ‘Three Points’; designing the lesson; teaching and observing the lesson; and discussing the effectiveness of the design and suggesting refinements after the lesson. Stages Three and Four were usually repeated once more if time and logistics permitted the teams to do so.

3.4.4. Using video technology to record lessons

Using video technology to record lessons is a very useful technique because it can help to capture both planned and unplanned situations (McKenney & Reeves, 2012). It can be used to prompt discussions about lessons and has been used extensively in studying teacher mathematical noticing (Jacobs, Lamb, et al., 2011; Kazemi et al., 2011; Miller, 2011; Seidel et al., 2011; Star et al., 2011; van Es & Sherin, 2008). Besides live observation of lessons, video cameras are also sometimes used in Lesson Study to collect data (Lewis et al., 2009). In Phases Two and Three of this study, the researcher’s video camera was used to capture teachers’ interaction with students during lesson observations, in addition to the usual video cameras as deployed by the respective schools during Phase 1. As described in Section 3.1, this study is a multi-tiered design experiment. Teachers, in the context of their schools’ learning communities, were designing lessons to encourage students’ mathematical thinking; while the researcher was designing tools to promote teachers’ mathematical noticing. Therefore, the teachers’ video captured mainly how students responded to the tasks in order to facilitate the post-lesson discussion, whereas the researcher’s video camera focused mainly on the teacher teaching the lesson in order to study what teachers noticed.
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The researcher considered the use of a wearable camera, such as Camwear 100®, to record the lesson from the teacher’s perspective (M. G. Sherin, Russ, et al., 2011). The Camwear 100 records a 30-second clip each time the camera is activated. Even though M. G. Sherin, Russ, et al. (2011) had used Camwear 100 to access and study teachers’ in-the-moment noticing with some success, they also acknowledged that teachers’ attention and thinking could only be better accounted for during the interviews after lessons. Since this study was situated in the context of a Lesson Study, where post-lesson discussions are central to the protocol, the use of a wearable camera might not have been advantageous. Furthermore, recording teachers’ discourse with the students and whole class was essential for the researcher to gain insight into their noticing. Hence, the camera needed to have the capability to record more than 30 seconds of footage each time. Therefore, a Swivl ® Cameraman system was used during the study because the system enables the video camera to follow the teacher, and records the discourse through a wireless connection with a digital microphone. Consequently, the teachers’ whole-class discourse and their discussions with individual students could be captured during lesson observations to supplement and triangulate the data from the post-lesson discussions.

3.4.5. Semi-structured interviews

A semi-structured interview was conducted with each teacher in the Lesson Study teams to find out more about the teacher’s educational and teaching background, their ideas and beliefs about mathematics and the teaching of it. The interview protocol (Appendix 6) consisted of two parts: Firstly, a structured questionnaire was used to collect basic information about the educational and teaching background of the teachers. This part also served as a way for the researcher to understand the possible perspectives of the interviewee; Secondly, a semi-structured interview sought to provide a description of the teachers’ own teaching and learning experiences with mathematics and their beliefs about it. More specifically, the questions were designed to be “open-ended” to encourage teachers’ sharing of their own “narrative and stories” (Rubin & Rubin, 2012, p. 29). Moreover, the researcher exercised flexibility in the order of the questions asked as he tried to respond and interact better with the teachers, and asked new questions to follow up on the teachers’ responses (Rubin & Rubin, 2012).
The interviews were then transcribed and the demographic details summarised. Teachers’ responses to the open-ended sections were marked and coded to describe teachers’ notions of mathematics, mathematical thinking and mathematics teaching. Using the field notes taken by the researcher during the interview, notable quotes and any other insights relevant to the study were also captured (Rubin & Rubin, 2012). A picture of each teacher was constructed based on the data analysed and detailed in the Contact Summary Form (Miles et al., 2013). These descriptions of teachers’ resources, orientations and goals related to mathematics and mathematics teaching provided a source of data that was used to explain or triangulate findings from the Lesson Study discussions.

3.5. Data Analysis

Consistent with the design research methodology, ongoing data analysis during data collection was done to support teachers’ participation and learning while retrospective analysis of the data was conducted after the data collection phase to examine and define the characteristics of productive mathematical noticing (Cobb & Gravemeijer, 2008; Zawojewski et al., 2008). Data analysis went through iterative cycles of data condensation, data display, drawing and verifying conclusions (Miles et al., 2013) during two distinct phases: one in 2012 (Phase 1), and the other during Phases 2 and 3 in 2013. The first phase of data analysis provided insights into the teachers’ noticing during Lesson Study and generated input for the design of the other two phases. The findings helped the researcher to think about the design of the tools for the modified Lesson Study protocol, and formulated a tentative framework for productive noticing that would be tested and refined throughout the study. The second phase of data analysis focused on analysing teachers’ noticing to derive any characteristics of their noticing that resulted in productive outcomes or decisions during the Lesson Study cycle. Changes in teachers’ resources and thinking were inferred from analyses of discourses amongst teachers during Lesson Study sessions, and classroom discourses between teachers and students during lesson observations.

3.5.1. Data condensation

The first challenge in data analysis was to deal with the huge amount of data generated from the voice recordings (Total of 79 with an average duration of 1.5 hours) and video
recordings (Total of 29 with an average duration of 1 hour). This excludes the interview recordings (Total of 29 with an average duration of 40 min). In order to condense the data to a level which is manageable for the scope of this study, the following procedure was followed for all three phases:

1. All recordings were reviewed with the field notes taken;
2. The voice recordings were marked for discussion segments that dealt with the five key tasks of Lesson Study (See Table 3.7). Segments that were focused on logistical issues (e.g., “Who is doing the video?”), administrative matters (e.g., “Remember to clock in the hours for this Lesson Study”), and other unrelated incidents (e.g., “John is giving me problems again!”) were not marked for further analysis;
3. These selected segments were then reviewed again, and classified initially using the framework for noticing (van Es, 2011). Refer to Table 2.4;
4. Noteworthy segments, which provide a good contrast within the same Lesson Study session, were then selected for transcription. Care was taken to ensure a wide spread of segments ranging from baseline noticing to extended noticing.

The classification of noticing segments, and the selection of noteworthy segments were potentially biased and problematic, because the process depended on the perspectives one holds about the teaching and learning of mathematics (Clarke, 2001). This was negotiated partially through the use of the five key tasks in the Lesson Study (Lewis et al., 2011), and the aims related to enhancing student reasoning (Hiebert, Morris, et al., 2003). Moreover, the selection of noteworthy segments was informed by the literature on productive practices that enhance student reasoning (See Section 2.2). These frameworks guided the researcher in determining the more productive segments as well as those segments, which could have been productive. The selected segments were of mathematical or pedagogical interest, and were characterised mainly by discussions surrounding mathematical concepts; students’ learning difficulties; or teaching approaches (Brousseau, 1997; Cohen et al., 2003; Yang & Ricks, 2012). The decision to “ignore” segments that focused on non-mathematical aspects during the discussion was taken in view of the goals of this study: To study teacher noticing that accompany teachers’ instructional decisions, which are more productive for enhancing student mathematical reasoning.
### Table 3.7: Criteria for marking discussion segments

<table>
<thead>
<tr>
<th><strong>Tasks during Lesson Study</strong></th>
<th><strong>Possible indicators for productive decisions/outcomes</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Planning</strong></td>
<td></td>
</tr>
<tr>
<td>Developing a research theme</td>
<td>Identify the learning goals or key mathematical ideas.</td>
</tr>
<tr>
<td>Solve and discuss mathematics tasks; anticipate students’ thinking</td>
<td>Recognise the main learning difficulty/cognitive obstacle that students face.</td>
</tr>
<tr>
<td>Develop shared teaching-learning plan</td>
<td>Highlight and analyse possible ways to address students’ difficulties through the design of tasks and lessons.</td>
</tr>
<tr>
<td></td>
<td>Design a higher-level demand task aimed at revealing student thinking.</td>
</tr>
<tr>
<td></td>
<td>Generates new understanding of key mathematical ideas and student thinking.</td>
</tr>
<tr>
<td></td>
<td>Anticipate how students may reason about the mathematics in the task.</td>
</tr>
<tr>
<td><strong>Teaching</strong></td>
<td></td>
</tr>
<tr>
<td>Collect data during research lesson</td>
<td><em>For research teacher:</em></td>
</tr>
<tr>
<td></td>
<td>Listen to and solicit ideas from students about their solutions.</td>
</tr>
<tr>
<td></td>
<td>Respond to students’ ideas/questions about or responses to tasks in a way that reveals students’ thinking.</td>
</tr>
<tr>
<td></td>
<td><em>For observers:</em></td>
</tr>
<tr>
<td></td>
<td>Listen to, and record particular instances of students’ thinking.</td>
</tr>
<tr>
<td><strong>Reviewing</strong></td>
<td></td>
</tr>
<tr>
<td>Conduct post-lesson discussion</td>
<td>Describe particular instances of students’ thinking.</td>
</tr>
<tr>
<td></td>
<td>Analyse these instances of students’ thinking.</td>
</tr>
<tr>
<td></td>
<td>Use these instances to refine ideas about the lesson design.</td>
</tr>
<tr>
<td></td>
<td>Refine thinking about students’ reasoning based on evidence drawn from these instances.</td>
</tr>
</tbody>
</table>

### 3.5.2. Transcription procedures

After the process of “selecting, focusing, simplifying” the huge amount data (Miles et al., 2013, p. 12) as described in Section 3.5.1, the researcher began the process of transcribing the selected segments to facilitate further analysis. For the selected episodes, the researcher transcribed word for word, including pauses (...), and ungrammatical or colloquial language, which were not edited. The researcher, however, did not transcribe changes in pitch, intake of breath or any other speech features. This is
because the primary aim of the researcher was to understand what teachers attended to and discussed instead of how they expressed their ideas. Words added into the transcript to enhance clarity were indicated using square parentheses, and actions, if any, were indicated within round parentheses. This transcription protocol was followed through all the phases of the study.

3.5.3. Data analysis during Phase 1

At the beginning of the exploratory study, it was not clear whether providing a focus, such as students' representations, was helpful for teachers to enhance their noticing (Star et al., 2011). It was also unclear what teachers notice about students' thinking during Lesson Study sessions. Therefore, for Phase 1 of the study, the researcher analysed what teachers noticed about mathematics and students' thinking during the Lesson Study, focusing particularly on the selected segments. To provide a frame to aid in analysis, the Three-Point Framework (Yang & Ricks, 2012) and the processes of noticing (Jacobs et al., 2010) were used to code instances in the selected episodes. A “thematic approach” proposed by Bryman (2012, p. 578) was used to develop patterns within the instances of these selected episodes.

In particular, the researcher attended to themes around the 'Three Points' and noticing for which noticing results in a productive or non-productive decision during the discussion. The thematic approach was useful for this exploratory study because it helped to summarise the key features of a large body of data and generate ‘thick description’ of the data set necessary in design research (Braun & Clarke, 2006; Cobb & Gravemeijer, 2008). A matrix containing fragments of transcripts from the selected episodes was constructed to display the theme in relation to the 'Three Points' and the processes of noticing. See Table 3.8 for such an example of such a matrix.

The themes were then reviewed in relation to the coded instances within the same episode and across other selected episodes that formed part of the data set after data condensation. Ongoing analysis was then carried out to generate specific characteristics, as well as a definition and a name for each theme. The systematic thematic approach taken helped to ensure the coherence and consistency of the themes generated (Braun & Clarke, 2006). The themes generated from Phase 1 of the study were used to design
the tools for the modified Lesson Study protocol described in Section 3.4.3 and the tentative framework for productive noticing.

Table 3.8: An example of a matrix for thematic analysis

<table>
<thead>
<tr>
<th>Theme: Lack of specificity</th>
<th>Attending to</th>
<th>Making Sense of</th>
<th>Deciding to</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Key Point</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Relative sizes of unit fractions.</td>
<td>“… particularly the basic skills of ordering fractions...”</td>
<td></td>
<td>“… we then decided to focus on fractions as our area of concern.”</td>
</tr>
<tr>
<td><strong>Difficult Point</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>To see the relationship between the denominators and the size of the unit fraction.</td>
<td>“… still not good in fractions...”</td>
<td>“… particularly the basic skills of ordering fractions... also they are having some problems.”</td>
<td></td>
</tr>
<tr>
<td><strong>Critical Point</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Use of questioning techniques.</td>
<td></td>
<td></td>
<td>“… how we could actually sharpen our questioning techniques to actually help children to learn fractions...”</td>
</tr>
</tbody>
</table>

3.5.4. Data analysis during Phases 2 and 3

Data analysis occurred at two distinct levels during Phases 2 and 3 of the study: An ongoing data analysis that fed into the design cycles, and the retrospective data analysis at the end of the data collection phase. The themes generated, and the tentative theoretical framework for productive noticing in Phase 1, provided ways to identify patterns and relationships during the design cycles. These themes and the tentative framework were revised and tested concurrently throughout the main study as the researcher worked with teacher participants to enact designs for the lessons. This level of data analysis was consistent with the design research paradigm in which the study was situated (Cobb & Gravemeijer, 2008).

Retrospective analysis was conducted after the data collection phase using processes similar to those taken in Phase 1. As in Phase 1, data condensation was carried out through a selection of episodes that were of mathematical and pedagogical interest. Each episode was first coded for the Three Points, and processes of noticing before the themes that emerged from the exploratory study were tested, refuted or refined using the systematic thematic approach outlined in the preceding section. To confirm or
refute the themes, the related transcripts and video clips were read and viewed repeatedly to highlight evidence from the transcripts and video clips that support the researcher’s analysis. Analytic memos were created to help the researcher to generate and crystallise concepts related to the notion of productive noticing. These characteristics of productive noticing then became the object of analyses to examine the relationships between the various proposed constructs of productive noticing. The theoretical framework, constructed through these iterations of testing and refutation, was used as a model to analyse the episodes to check for interpretative consistency. Criteria for using the key constructs of this proposed framework were made explicit, tested and confirmed. The analytic approach taken in this phase drew on the method of Constant Comparison (Cobb & Whitenack, 1996; Glaser, 1965) used by other researchers in design study (Cobb & Gravemeijer, 2008; Zawojewski et al., 2008).

3.5.5. Analysing discourse through a commognitive lens

One of the challenges in studying noticing, and documenting the growth of teachers’ ability to notice, is how to document the sense-making processes of teachers (M. G. Sherin, Russ, et al., 2011). The underlying assumption in this study is that teachers’ mathematical and pedagogical thinking is made visible by their communication during lesson planning, teaching and discussion of lessons. As discussed in Section 2.1.5.4, Sfard (2012) views mathematical thinking as mathematical discourse, and mathematical discourses are characterised by four features: “keywords, visual mediators, routines and endorsed narratives” (p. 2). It is reasonable to assume that there is a close relationship between thinking and communicating about mathematics and mathematics teaching. The commognitive perspective of mathematical thinking views “learning as participation” and “learning as change in discourse” when one participates in a community (Gückler, 2013, p. 440; Sfard, 2008). Since the study was situated in a professional learning community, the commognitive framework is a good way to document changes in teachers’ learning as they begin to notice more productively. Hence, this study adapted a commognitive approach (Sfard, 2008; Sfard & Kieran, 2001) to analyse discourse between the teacher and students during lesson observations; and amongst teachers during other Lesson Study sessions.
### Table 3.9: Description and Definitions of Codes for Commognitive Analysis

<table>
<thead>
<tr>
<th>Codes</th>
<th>Definitions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Student-Teacher discourse</strong> (Newton, 2012)</td>
<td><strong>Teacher-Teacher discourse</strong> (Developed for this study)</td>
</tr>
<tr>
<td><strong>Key Words</strong></td>
<td>Words that signify quantities, shapes and mathematical concepts. E.g., fractions, functions, limits etc.</td>
</tr>
<tr>
<td></td>
<td>Words that signify mathematical concepts.</td>
</tr>
<tr>
<td></td>
<td>Words that signify ideas related to mathematics teaching.</td>
</tr>
<tr>
<td></td>
<td>E.g., practice, explanation, examples, etc.</td>
</tr>
<tr>
<td><strong>Visual Mediators</strong></td>
<td>Visible objects that are operated upon as part of communication. E.g., numerals, symbols, graphs etc.</td>
</tr>
<tr>
<td></td>
<td>Visible objects that are operated upon as part of communication in mathematics learning and teaching.</td>
</tr>
<tr>
<td></td>
<td>E.g., numerals, symbols, graphs, a student’s work sample, or other student artefacts.</td>
</tr>
<tr>
<td><strong>Routines</strong></td>
<td>Repetitive or patterned ways in which mathematical tasks are being performed. E.g., comparing, solving equations, conjecturing etc.</td>
</tr>
<tr>
<td></td>
<td>Repetitive or patterned ways in which mathematical tasks are being performed.</td>
</tr>
<tr>
<td></td>
<td>Repetitive actions used in teaching.</td>
</tr>
<tr>
<td></td>
<td>E.g., revoicing, calling on students, giving explanations, circulating the class, orchestrating a discussion etc.</td>
</tr>
<tr>
<td><strong>Endorsed Narratives</strong></td>
<td>Sequence of utterances framed as description of objects, of relationships between objects, or of processes with or by objects, that is subject to endorsement or rejection with the help of discourse specific substantiation procedures.</td>
</tr>
<tr>
<td></td>
<td>Sequence of utterances framed as description of teaching instances, relationships between these instances and instructional decisions that are subject to endorsement or rejection with the help of discourse specific to reasoning processes.</td>
</tr>
</tbody>
</table>

While the approach has been used in studying discourses between teachers and students (Güçler, 2013; Nachlieli & Tabach, 2012; Sfard, 2008), student discourses (M. B. Wood & Kalinec, 2012), and analysing written and enacted curricula (Newton, 2012) this study extends the use of the commognitive approach to analysis of mathematical pedagogical discourses between teachers. Using similar approaches adopted by other researchers, the discourses between the teacher and students during lesson observations were coded for these four features and analysed. However, modifications had to be made in order to extend its use to analyse mathematical pedagogical discourse between teachers. The teachers’ discourses from the selected episodes were first coded for the same four features: keywords, visual mediators, routines and endorsed narratives. These new codes related to the commognitive features of 
mathematical pedagogical discourses were then used to analyse the discourses of teachers to investigate the changes in teachers’ thinking during the study.

Table 3.9 shows the definitions of codes related to the commognitive framework used for the discourse analysis. The extended commognitive framework thus provided the study with a means to analyse, and a language to describe, teachers’ thinking during the project.

3.5.6. Data Representation

The data were presented at two levels: a wide-angle view and a close up view. The wide-angle perspective gave an overview of the school’s development in noticing as they work through the Lesson Study tasks during each session in each phase. This provided a rough-grained analysis of each of the three schools. Instead of assigning an overall level to each session, I characterised the different levels of noticing observed in each session according to the amount of evidence that supported the ratings. Following similar methods used by Jacobs, Lamb, et al. (2011), the amount of evidence ranged from limited to robust. This characterisation provided a more comprehensive view of the overall development of the Lesson Study teams in each school. In addition, the main points of discussion were tracked in the selected segments in order to relate levels of noticing to the key tasks of Lesson Study (Lewis et al., 2011), and the productive practices associated with teaching for reasoning. To reflect the differences between the schools, these segments were characterised in terms of the evidence base for these productive practices.

The summary for teacher development was presented by overlaying the evidence base for the levels of noticing with the key tasks of Lesson Study, across each session, for each school. Using a representation similar to lesson signatures (Hiebert, Gallimore, et al., 2003, p. 123), a session signature was created for each school to show the co-occurrence and evidence base for the two variables: level of noticing, and components of the key tasks that reflect the productive practices (outlined in Section 2.2), during each session. The notion of a session signature provided a representation of what happened for each session to allow for a closer investigation into the tasks discussed, and the associated levels of noticing. Comparing the session signatures for each school provided insights into the noticing expertise development of teachers, and mapped out
the overall learning trajectory of the teachers. Figure 3.7 shows a sample session signature.

<table>
<thead>
<tr>
<th>Levels of noticing (van Es, 2011)</th>
<th>Extended</th>
<th>Focused</th>
<th>Mixed</th>
<th>Baseline</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conduct post lesson discussion</td>
<td>Refine thinking about student reasoning based on evidences drawn from instances</td>
<td></td>
<td>Use this instances to refine ideas about lesson design</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Analyse these instances of student thinking</td>
<td></td>
<td>Describe particular instances of student thinking</td>
<td></td>
</tr>
<tr>
<td>Collect data during research lesson</td>
<td>(Observers) Listen to, and record particular instances of students' thinking</td>
<td></td>
<td>(Teacher) Respond to students' ideas/questions; students' responses to task, in a way that reveal their thinking</td>
<td></td>
</tr>
<tr>
<td>Key Take During Lesson Study</td>
<td>(Teacher) Listen to and solicit ideas from their students about their thinking</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Develop a research theme</td>
<td>Anticipate how students may reason about the task</td>
<td></td>
<td>Generates new understanding of key ideas and student thinking</td>
<td></td>
</tr>
<tr>
<td>Solve and discuss mathematics</td>
<td>Design a higher-level cognitive demand task aimed at revealing student thinking</td>
<td></td>
<td>Highlight and analyse possible ways to address students' difficulties through the design of tasks and lessons</td>
<td></td>
</tr>
<tr>
<td>Anticipate students' thinking</td>
<td>Recognise the main learning difficulty/cognitive obstacle that students face</td>
<td></td>
<td>Identify the learning goals or key mathematics</td>
<td></td>
</tr>
<tr>
<td>Develop shared teaching-learning plan</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Session</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
</table>

**Figure 3.7. A sample session signature.**

The close-up view presents the analyses of what teachers noticed through vignettes developed from the episodes. A teacher's noticing processes during these vignettes were then represented using the theoretical models of noticing developed from the FOCUS framework—the theoretical product of the design experiment outlined in Sections 3.1. The development of the FOCUS framework will be outlined in the next chapter.
3.6. Summary

This chapter began with an outline of the key research questions that guided inquiry for this study, and detailed how a design research paradigm together with systematic investigation of teaching fitted the purpose of this study: Revealing what teachers were thinking and noticing as they participated in planning, teaching and reviewing a lesson. The choice of Lesson Study as a means for data collection followed naturally from these two perspectives. Particular attention was given to illustrating how the chosen methods of data collection could provide the necessary documentary trails to ensure trustworthiness and dependability of the findings obtained from this study.

Data collection timeline and methods were provided to give a comprehensive view of how teachers’ attention, sense making and decision making was captured for analysis. More specifically, the modified Lesson Study protocol was described as an intervention designed to create an environment that would promote more educationally productive noticing. Consistent with the design research paradigm, design and data analysis were intertwined and inter-dependent. Methods of retrospective data analysis were described for both the exploratory study and the main study to develop themes and characteristics that might contribute to the constructs of productive noticing. The use of the commognitive perspective of mathematical thinking also provided a means to analyse and document teachers’ thinking from the discourses recorded.
Chapter 4. The FOCUS Framework for Productive Noticing

This chapter presents the theoretical product of the design experiment—the FOCUS framework—for describing, analysing, and modelling teacher noticing. Drawing on a photography metaphor, the framework highlights what teachers attend to (focus), and how they notice (focusing), when they notice productively. The development of the framework is presented through two different lenses: a wide-angle lens for zooming out to view the overall teacher noticing development; and a close-up lens for zooming in to see the noticing processes underlying potentially productive episodes during the planning, teaching, and reflection of the lesson (Hiebert, Gallimore, et al., 2003; Lampert, 2001). The wide-angle view summarises the development of teacher noticing for the three schools, and set the context for the close-up views of teacher noticing described later in this chapter, and the following one. It also presents an overview of the development of the FOCUS framework. The close up view, presented primarily through a case study of Greenhill Primary School, demonstrates the two key components of the FOCUS framework. This school was chosen as the main case because the instances that occurred there provide a rich portrayal of what teachers noticed. Finally, the FOCUS framework will be presented in Section 4.4 to provide a means to create theoretical models of teacher noticing during the lesson cycle.

4.1. Wide-angle view: Summary of Findings

From a wide-angle perspective, most teachers started out with a baseline level of noticing (van Es, 2011), in which they attended to the more generic instructional details with little or no evidence of how they reasoned about the decisions taken during the discussion sessions. The only exception was Trafford Secondary School, where the teachers started the project with a mixed level of noticing. The levels of teacher noticing, however, were not static, and they changed from between, and within segments of the same discussion session over the three phases. Even though the trajectory of noticing is non-linear (van Es, 2011), there is a positive growth in terms of teachers’ noticing expertise at the end of the study. The development of teacher noticing in each of the three schools through the three phases is presented using a session signature (See
Section 3.5.6). The session signatures for the three schools are shown in Figures 4.1, 4.2, 4.3 and 4.4.

4.1.1. Development of teacher noticing in Greenhill Primary School

As seen in Figure 4.1, teachers from Greenhill Primary School started noticing at the baseline level, with a few instances of higher level (focused) noticing when no explicit focus was given in Phase 1 (See Figure 4.1 Phase 1 sessions 1, 4 and 7). These few instances of focused noticing are detailed in Section 4.2 because these episodes are illustrative of the contrast between productive and non-productive noticing.

In Phase 2, where an explicit focus was given, teachers at Greenhill Primary School engaged productively in the lesson study tasks (See Figure 4.1 Phase 2). The improved engagement was evident from teachers’ discussion in terms of their focus on mathematics (Key Point), and what their students find difficult about the concepts (Difficult Point). Teachers also began to analyse how the task might address the students’ difficulties in learning about fraction of a set (Phase 2 Session 5); gave more descriptions of particular students’ reasoning (Phase 2 Session 7); and examined more closely the task in relation to student learning (Phase 2 Session 8). Thus the teachers were mainly noticing at the mixed level, which indicates a move towards attending to particular students’ thinking as opposed to more generic observations in Phase 1. There were also a few instances of extended noticing in Sessions 2 and 5, and these will be discussed in Section 4.3.

Although all three schools continued their development trajectory and reached a higher level of noticing at the end of Phase 3, Greenhill Primary showed the widest range of development from baseline noticing to more focused noticing. Hence, the teachers demonstrated the most progress in terms of their noticing expertise. Furthermore, a few teachers such as Kirsty and Cindy demonstrated extended noticing in a few episodes (See Figure 4.1 Phase 3 Sessions 4, 7, 8 and 9). Cindy, who had demonstrated a consistent pattern of very focused noticing in Phase 3, will be featured in Section 5.2.
Figure 4.1. Session signature for Greenhill Primary School.

<table>
<thead>
<tr>
<th>Levels of noticing (van Es, 2013)</th>
<th>Extended</th>
<th>Focused</th>
<th>Mixed</th>
<th>Baseline</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Conduct post lesson discussion</strong></td>
<td>Refine thinking about student reasoning based on evidences drawn from instances</td>
<td>Use this instances to refine ideas about lesson design</td>
<td>Analyse these instances of student thinking</td>
<td>Describe particular instances of student thinking</td>
</tr>
<tr>
<td><strong>Collect data during research lesson</strong></td>
<td>(Observers) Listen to, and record particular instances of students' thinking</td>
<td>(Teacher) Respond to students' ideas/questions; students' responses to task, in a way that reveal their thinking</td>
<td>(Teacher) Listen to and solicit ideas from their students about their thinking</td>
<td></td>
</tr>
<tr>
<td><strong>Develop a research theme</strong></td>
<td>Anticipate how students may reason about the task</td>
<td>Generates new understanding of key ideas and student thinking</td>
<td>Design a higher level cognitive demand task aimed at revealing student thinking</td>
<td>Highlight and analyse possible ways to address students' difficulties through the design of tasks and lessons</td>
</tr>
<tr>
<td><strong>Solve and discuss mathematics</strong></td>
<td>Recognise the main learning difficulty/cognitive obstacle that students face</td>
<td>Identify the learning goals or key mathematics</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Session</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Phase 1</strong></td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td><strong>Phase 2</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td><strong>Phase 3</strong></td>
<td></td>
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</tr>
</tbody>
</table>
4.1.2. Development of teacher noticing in Springside Secondary School

Similar to Greenhill Primary, the teachers in Springside Secondary School started noticing at the baseline level. They could identify the key mathematical concepts for the lesson, but were less able to recognise students’ confusion and highlight how the task could address students’ learning difficulties. For example, the teachers at Springside (Phase 1) generated a “doing mathematics” task (Smith & Stein, 1998), which required students to estimate the time taken to count from one to one million. However, they neither discussed nor worked out the solutions initially. Thus the teachers did not think about how students would have approached the problem, and only discussed the generic aspects of the lesson without analysing particular instances to refine the task (See Figure 4.2 Phase 1 Session 7).

The teachers at Springside Secondary School, unlike those at Greenhill, seemed to reflect a stable baseline level of noticing characterised by comparatively limited evidence for the productive practices during the Lesson Study sessions in Phase 2 (See Figure 4.2 Phase 2 Sessions 1 to 5). They tended to give students direct instructions, and their task centred on the procedures of determining the height of a pyramid set in a fictitious real-life context. However, the teachers demonstrated a more advanced noticing during the post-lesson discussions (Phase 2 Sessions 7 and 9), and realised the importance of modifying the task to reveal what the students were thinking.

In Phase 3, the teachers’ level of noticing remained relatively varied between mixed noticing and focused noticing, even though there were indications of a slower development trajectory (Compare Figure 4.2 Phases 1, 2 and 3). Of particular interest is Anita, the research teacher for Phase 3, who maintained a less productive noticing stance for most part of Phase 3. However, she began to show some evidence of improved noticing towards the end of the study (See Figure 4.2 Phase 2 Sessions 8 and 9). Anita’s case will be examined more closely in Section 5.1 to exemplify case of non-productive noticing.
### Figure 4.2: Session signature for Springside Secondary School

<table>
<thead>
<tr>
<th>Levels of noticing (van Es, 2013)</th>
<th>Extended</th>
<th>Focused</th>
<th>Mixed</th>
<th>Baseline</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Conduct post lesson discussion</strong>&lt;br&gt; (&lt;br&gt;Observers) Listen to, and record particular instances of students' thinking&lt;br&gt; (Teacher) Respond to students' ideas/questions; students' responses to task in a way that reveal their thinking&lt;br&gt; (Teacher) Listen to and solicit ideas from their students about their thinking</td>
<td>![Limited evidence]</td>
<td>![Some evidence]</td>
<td>![Robust evidence]</td>
<td>![Robust evidence]</td>
</tr>
<tr>
<td><strong>Collect data during research lesson</strong>&lt;br&gt; (&lt;br&gt;Observers) Listen to, and record particular instances of students' thinking&lt;br&gt; (Teacher) Respond to students' ideas/questions; students' responses to task in a way that reveal their thinking&lt;br&gt; (Teacher) Listen to and solicit ideas from their students about their thinking</td>
<td>![Limited evidence]</td>
<td>![Some evidence]</td>
<td>![Robust evidence]</td>
<td>![Robust evidence]</td>
</tr>
<tr>
<td><strong>Key Task: Develop teaching-learning plan</strong>&lt;br&gt; Anticipate how students may reason about the task&lt;br&gt; Generates new understanding of key ideas and student thinking&lt;br&gt; Design a higher-level cognitive demand task aimed at revealing student thinking&lt;br&gt; Highlight and analyze possible ways to address students' difficulties through the design of tasks and lessons&lt;br&gt; Recognize the main learning difficulty/cognitive obstacle that students face&lt;br&gt; Identify the learning goals or key mathematics</td>
<td>![Limited evidence]</td>
<td>![Some evidence]</td>
<td>![Robust evidence]</td>
<td>![Robust evidence]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Session</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
</table>
4.1.3. Development of teacher noticing in Trafford Secondary School

In contrast to the other two schools, the teachers at Trafford Secondary started off with a higher level of noticing (See Figure 4.3 Phase 1), characterised by the features of mixed noticing (See Table 2.4). There is some evidence to suggest several instances of focused noticing (See Figure 4.3 Phase 1 Sessions 1, 5, 6 and 8) because of a distinctively strong emphasis on learning mathematics for teaching amongst the teachers. For example, they generated new understanding of angles in a polygon (See Figure 4.3 Phase 1 Session 1) when they discussed the lesson plan. A noteworthy observation can be found in Session 5, in which Lindy, the research teacher, listened and responded to students’ ideas in a more discursive way. Lindy’s teaching episode is highlighted in Section 4.2.4.1 to provide a contrast to the more IRE teaching style demonstrated at Greenhill Primary during Phase 1.

In Phases 2 and 3, there were two Lesson Study teams in Trafford Secondary School: comprising of teachers from the Secondary Two level (See Figure 4.3 Phases 2 and 3), and Secondary Three level respectively (See Figure 4.4 Phases 2 and 3). As in Phase 1, the teachers started from a higher level, and mainly maintained a focused level of noticing throughout the study. Similar to Phase 1, both teams focused on the mathematical concepts and possible students’ confusion associated with these concepts. The discussions were often shaped by teachers’ sharing about their students’ specific difficulties and mistakes from their teaching experiences. What set them apart from the other two schools was not only the frequency of such segments, but also the quality of their instructional decisions. For example, the teachers from the Secondary Two focused on the different ways to think about gradient of a straight line (Moore-Russo, Conner, & Rugg, 2011), considered the numerical values of coefficients used in examples and questions they posed, and thought about more subtle points including language used by the teacher in his explanation (e.g., numerical value, value, absolute value etc.).

A closer look reveals that the teachers at the Secondary Two level might be less proficient at orchestrating a discussion, because both lesson observations centred about the teacher giving clear explanations (See Phase 2 Sessions 6 and 8). Nevertheless, the teachers showed evidence of attending to student reasoning through their detailed descriptions of instances of student thinking during post-lesson discussions (See Phase 2 Sessions 7 and 9). In addition, the other team focused mainly on a lecture-style of
teaching involving around 100 students, and hence there were fewer opportunities to engage students in a whole-class discussion. Despite the choice of lecture as a mode of instruction, the teachers were still able to reveal what, and how, students think by highlighting students’ work during the lecture, and engaging students in a short discussion around these pieces of work.

Lastly, the teachers at Trafford Secondary continued their relatively focused noticing in Phase 3 (See Figures 4.3 and 4.4). While there were some variations in the level of noticing for the Secondary Two teachers, their noticing still led on to generally productive practices. This can be seen especially during the post-lesson discussion (See Figure 4.4 Phase 3 Session 6), where the teachers analysed extensively students’ thinking about set theory using their observations.

The noticing levels in Trafford Secondary remained relatively higher throughout the study, and the cases from this school were used as a comparison especially in Phase 1 (See Lindy’s classroom interaction in Section 4.2.4.1). However, it is possible that the school was noticing at a higher level because of other factors beyond the scope of this study. For example, the school was designated as a model school for professional learning, which was beyond the control of the researcher. Moreover, Keaton, who is a very experienced Head of Department, had exceptionally good facilitation skills. Therefore, despite the generally higher frequency of productive noticing at Trafford Secondary, a close-up view of Greenhill Primary School will be presented instead because their trajectory of growth can better illustrate the differences between productive and non-productive noticing.
<table>
<thead>
<tr>
<th>Levels of noticing (van Es, 2011)</th>
<th>Extended</th>
<th>Focused</th>
<th>Mixed</th>
<th>Baseline</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conduct post lesson discussion</td>
<td>Refine thinking about student reasoning based on evidence drawn from instances</td>
<td>Use this instance to refine ideas about lesson design</td>
<td>Analyse these instances of student thinking</td>
<td>Describe particular instances of student thinking</td>
</tr>
<tr>
<td>Collect data during research lesson</td>
<td>(Observers) Listen to, and record particular instances of students' thinking</td>
<td>(Teacher) Respond to students' ideas/questions; students' responses to task, in a way that reveal their thinking</td>
<td>(Teacher) Listen to and select ideas from their students about their thinking</td>
<td></td>
</tr>
<tr>
<td>Key tasks during lesson study</td>
<td>Develop a research theme</td>
<td>Solve and discuss mathematics</td>
<td>Anticipate students' thinking</td>
<td>Develop a shared teaching-learning plan</td>
</tr>
<tr>
<td></td>
<td>Anticipate how students may reason about the task</td>
<td>Generates new understanding of key ideas and student thinking</td>
<td>Design a higher-level cognitive demand task aimed at revealing student thinking</td>
<td>Highlight and analyse possible ways to address students' difficulties through the design of tasks and lessons</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Recognise the main learning difficulty/cognitive obstacle that students face</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Identify the learning goals or key mathematics</td>
</tr>
</tbody>
</table>

Figure 4.3. Session signature for Trafford Secondary School (Sec 2).
4.1.4. Focus and Focusing: Two key aspects that characterise productive noticing

The findings from the study highlight two important characteristics of noticing that can most likely lead to making productive instructional decisions: An explicit focus for noticing, and the central role of pedagogical reasoning to focus a teacher’s noticing. The overview of the development of the FOCUS framework is shown in Figure 4.5. The initial framework builds on the nested relationship of noticing processes (See Figure 2.8) and incorporates the practices that occur during task design, teaching, and reflection, which are productive for enhancing student mathematical reasoning (See Section 2.2).
Figure 4.5. Overview of the development of the FOCUS Framework.
Chapter 4 – The FOCUS Framework for Productive Noticing

The results from Phase 1 of this study suggest that the Three Points, as an explicit focus, is useful for teachers to guide their noticing more productively. However, this study goes beyond the usefulness of framing teacher noticing (Levin et al., 2009; Loughran, 2002) to include the alignment of the Three Points as an additional necessary condition for productive noticing. This alignment refers to whether the teacher’s course of action (Critical Point) targets students’ confusion (Difficult Point) about the concept (Key Point). The explicit focus for noticing in terms of the Three Points and their alignment is represented by the first iteration of the Framework (See Figure 4.5). This will be further elaborated, and developed through a case study of Greenhill Primary School as detailed in Section 4.2.

Next, the results from Phases 2 and 3 suggest that attending to and making sense of this alignment between the Three Points is not trivial, even for experienced teachers. Nevertheless, teachers can respond with an instructional decision that aligns with the Key Point and Difficult Point through a sound and purposeful teacher reasoning about the instructional decisions they want to take. By making sense of the Three Points and justifying their instructional decisions using teaching principles or observations, teachers are more likely to respond with a teaching decision that is productive with regard to enhancing student reasoning. This finding positions the process of pedagogical reasoning (Shulman, 1987) at the centre of noticing expertise, which is represented by the second iteration and final version of the framework. The critical role played by the pedagogical reasoning process will be discussed through another case study in Section 4.3.

Throughout the development of the framework, particularly in Phases 2 and 3, the language of the Three Points were used by the teachers. However, in the context of a lesson cycle, the teachers’ decisions to respond may not correspond exactly to Yang and Rick's (2012) notion of Critical Point, which seems to indicate the general approach used in a lesson. To accommodate a wider range of teachers’ responses, the term course of action was used instead of Critical Point in the final version of the FOCUS framework. Consequently, the Key Point and Difficult Point, have also been renamed as concept and confusion respectively. In the following analyses of teachers’ noticing, these terms will be used interchangeably.
A photography metaphor is helpful here to explain the framework. Lampert (2001) uses the analogy of a lens with adjustable focal lengths to examine her teaching practices. Similarly, what teachers notice can be likened to taking a snapshot or photograph of what they see, how they think, and how they decide to respond. However, even a simple snapshot of a classroom situation can contain too much information for a teacher to notice productively (B. Sherin & Star, 2011). So, an explicit selective focal point can support a teacher to notice the essential features of the instructional details related to teaching for reasoning.

In a way, this selective focus (or blurring out of background “noise”) is what photographers term bokeh. A bokeh is a pleasing, but out-of-focus blur in a photograph, which helps to highlight the subject of the photograph. It keeps distracting elements blurred in the background of the photograph, and maintains a sharp focus on the essential subjects. To achieve a bokeh, the photographer, besides using a fast lens, has to adjust the distance between the camera and the subject, and a desired focus on the subject of interest is usually attained by focusing the lens manually. Likewise, for teachers to notice productively the instructional details (take a good snapshot with bokeh) amongst the “noise” in the classroom, it is more fruitful for them to select a clear focus like the Three Points (subject) before they think about, and respond to classroom situations. The thinking or pedagogical reasoning process, which connects the seeing to the responding component of noticing, is then similar to how a photographer tries to attain a bokeh by manual focusing. Together, the explicit focus (the Three Points) and the pedagogical reasoning (focusing), not only provide a way to examine what makes noticing productive on the whole; but more importantly, they can capture a close up view of what happens during the planning, teaching, and reflection of a lesson. These views can be combined to build a model of the noticing process, which describes and decomposes noticing at a more fine-grained level, as demonstrated in Chapter 5.

4.2. A close-up view: Explicit focus for noticing

This section elaborates on the FOCUS framework by highlighting its first component: the need for an explicit focus in noticing. This component will be developed and illustrated through a case study of Greenhill Primary School, which demonstrated a wide range of noticing expertise. In this case study, a group of seven teachers: Hannah
(facilitator); Alice; Heather; Heidi; Jacinda; Sherry; and Zelina (research teacher), worked together to plan a lesson on ordering Unit Fractions for Primary Two students (age 8). Three vignettes are presented and discussed: The first focuses on a few discussion episodes that happened during the planning (See Figure 4.1 Phase 1 Sessions 1 and 4); the second highlights what Zelina noticed in-the-moment during the lesson (See Figure 4.1 Phase 1 Session 6); and the third recounts what teachers noticed during the post-lesson discussion (See Figure 4.1 Phase 1 Session 7).

4.2.1. Vignette 1: The one-quarter incident

4.2.1.1. The need for specificity

During the first session of the Lesson Study at Greenhill Primary School, Hannah started the discussion by sharing the goals of the lesson study on ‘Unit Fractions’ with the teachers, and suggested them to sharpen their questioning techniques:

We have picked fractions as the main cause of concern because of the data that we have collected from last year’s P2 [Primary 2] cohort teachers saying that the children are still not good in fractions and particularly the basic skills of ordering fractions... also they are having some problems. Because of the data we have collected from item analysis, we then decided to focus on fractions as our area of concern. And also... we also talked about questioning techniques that we have gone through as a school... how we could actually sharpen our questioning techniques to actually help children to learn fractions...

Although Hannah made reference to the concept and the confusion targeted in the lesson, she did not elaborate clearly what she meant. Table 4.1 shows the analysis of what Hannah noticed with regard to the Three Points by decomposing her noticing into three processes: Attending; interpreting; and deciding to respond. The evidence for what she saw, interpreted and how she decided to respond are highlighted in the corresponding cells of the table. Hence, a blank cell is used to indicate a lack of evidence for the noticing process associated with the Three Points. As highlighted in Table 4.1, Hannah presented the ordering of unit fractions as the Key Point in the lesson. However, she did not articulate the aspect of ordering fractions that was critical for teachers to consider. Instead, she pointed vaguely to “focus on fractions” as the “area of concern”. Even though Hannah mentioned that the students were “still not good in fractions” based on “evidence” from item analysis, she did not specify what these findings were.
Chapter 4 – The FOCUS Framework for Productive Noticing

These findings would have been useful for teachers to understand students’ difficulties with the concept, which could have led to a better design of the lesson.

Table 4.1: Analysis of Hannah’s Noticing in Session 1 of Phase 1

<table>
<thead>
<tr>
<th>What was noticed</th>
<th>Processes of noticing</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Attending to</td>
</tr>
<tr>
<td><strong>Key Point</strong></td>
<td></td>
</tr>
<tr>
<td>Relative sizes of unit fractions.</td>
<td>“… particularly the basic skills of ordering fractions...”</td>
</tr>
<tr>
<td><strong>Difficult Point</strong></td>
<td></td>
</tr>
<tr>
<td>To see the relationship between the denominators and the size of the unit fraction.</td>
<td>“… still not good in fractions...”</td>
</tr>
<tr>
<td><strong>Critical Point</strong></td>
<td></td>
</tr>
<tr>
<td>Use of questioning techniques</td>
<td>“… particularly the basic skills of ordering fractions... also they are having some problems.”</td>
</tr>
</tbody>
</table>

Moreover, Hannah went on to suggest that teachers should focus on their questioning techniques, but she did not link this suggested course of action (Critical Point) to students’ confusion about the topic. Therefore, although Hannah referred to the Three Points, the lack of specific details prevented teachers from pinpointing students’ confusion about ordering unit fractions, which could have led to a more targeted approach. For example, a discussion on the type of questions given to students the previous year and samples of their work might have revealed students’ conflicting images of fractions (Gould, 2011).

Directing teachers’ attention to these details could have moved the discussions in new directions. Given that Hannah is an experienced Lesson Study facilitator, this lapse in specificity of the Three Points attests to the difficulty faced by Lesson Study practitioners in taking on the *Researcher’s lens* (Clea Fernandez et al., 2003). The teachers’ noticing is thus classified as *baseline* because they provided generic comments without any evidence to support their analysis (van Es, 2011). Their noticing remained largely at baseline level, and little progress was made towards the design for the task for the next three sessions.
4.2.1.2. Focusing on the specifics of mathematical concepts

The first instances of productive noticing in Greenhill Primary School occurred during the fourth session when teachers focused on very specific details of the mathematical concept beyond its surface features. Hannah was able to attend to a subtle point that was missed by the other teachers in the group during a discussion about the use of examples and non-examples to help students recap the fractional notation $a/b$. In order to highlight the role of equal partitioning in the fractional notation, the research teacher Zelina wanted to demonstrate physically an example and a non-example of $1/4$. Zelina showed two rectangles—one was divided into four equal parts and the other was not—to demonstrate what she intended to do during the lesson. (See Figure 4.6.)

To highlight the importance of equal partitioning in the fractional notation $1/4$, Zelina used a detachable piece of the shaded part to show the meaning of $1/4$. She removed the first shaded part and compared it to the rest of the parts of the first rectangle to show that they were equal, and hence demonstrating that the shaded part was $1/4$.

![Figure 4.6. Zelina's representation of an example and non-example of 1/4.](image)

She then took another detachable piece (of the same area) in the second whole, and said that it was not $1/4$ of the second whole because the second whole was not divided equally. Hannah then raised a point of clarification:

1. Hannah: If you take the same piece, the same piece is still $1/4$ of that whole.
2. Jaslyn: This is still $1/4$ of the whole... this one is not, but no... it's still $1/4$ of the whole?
3. Hannah: Yes. You must take the small one or the big one. It's still $1/4$. Because it's equivalent fraction, you can subdivide that...
4. Zelina: I don't know... make up your mind. Take or don't take?
5. Hannah: It is still $[1/4$ of the whole]... you must take something that is not equal to $1/4$ Because that is still $1/4$ of the whole.
In this episode, Hannah attended specifically to Zelina’s statement that the second detachable piece is “not 1/4 because the second whole was not divided equally”. This challenged teachers’ notion of equivalent fractions (Lines 2, 4, 6), and generated a useful point with regard to the choice of example (“Take something that is not equal to 1/4”). Table 4.2 shows the analysis of what Hannah noticed with regard to the concept (Key Point) and connects what she saw and how she interpreted Zelina’s statement to her response (Lines 3, 5, 8). Therefore, Hannah’s noticing helped highlight to the teachers, a new understanding about the concept of fraction, which can be characterised as a productive outcome (See Table 3.7) during the planning of the lesson. In this episode, the teachers became more aware of the subtlety of their own conceptions of fractions, and were more able to see why students might have difficulty with fractions, given that the teachers themselves may also sometimes struggle with the notion.

Consequently, Hannah’s noticing highlighted Zelina’s subtle error to the teachers for discussion, and they were alerted to a possible misconception that might arise as a consequence of over-emphasising the notion of equi-partitioning. Jessie tried to make sense of what Hannah said by physically manipulating the detachable fractional piece (Lines 2, 10), and she struggled with the concept for a brief moment (“...this one is not, but no...”) before she came to same conclusion as Hannah (Lines 2, 10) that “this piece is still 1/4 of this whole” (Line 10). The error involved is not trivial—that the process of dividing a whole into four equal parts gives rise to an object that is 1/4 of the whole and that object can have many different pictorial representations, and it remains 1/4 regardless of any division of the same whole. The error could have occurred because of the misconception that fractions can only involve equal parts. Unequal partitions can be challenging for students (Schoenfeld & Kilpatrick, 2008) and can be difficult even for
some teachers, as suggested in this case. Therefore, as Schoenfeld and Kilpatrick (2008) have emphasised, it is important that teachers are aware of this difficulty and be fluent with the use of different representations of fractions. Hence, Hannah’s noticing of Zelina’s explanation can be classified as more focused, and can be also considered to be productive because it led the whole team to reinforce their understanding of fractions, and to be aware of a possible student misconception.

Table 4.2: Analysis of Hannah’s Noticing in Phase 1 Session 4

<table>
<thead>
<tr>
<th>What was noticed</th>
<th>Processes of noticing</th>
<th>Making Sense of</th>
<th>Deciding to respond</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Key Point</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>The notation (a/b) represents (a) out (b) equal parts of a whole. However, it does not mean that fractions can only be represented by area models with equal partitions.</td>
<td>It can be inferred that she noted that the “same piece” is (1/4) regardless of the division of the “same whole” (1, 3, 8). She saw Zelina’s mistake in her demonstration: “it was not (1/4) of the second whole because the second whole was not divided equally” (1).</td>
<td>Her utterance of “equivalent fraction”, “subdivide” (3), and “shifted it in a way” (8) suggests she had related what Zelina said with the concept of fractions.</td>
<td>She highlighted that Zelina should take the “bigger or smaller piece” (3, 8), or “take something that is not equal to (1/4)” (5).</td>
</tr>
</tbody>
</table>

In contrast to Hannah, Zelina’s noticing is characterised initially as non-productive and an analysis of her noticing is shown in Table 4.3. She was caught by Hannah’s point at first and seemed unsure about the concept of fraction (Lines 4, 7, 9). Zelina’s noticing was not productive at first because her initial attention to the routines of teaching (“take or don’t take”), and the over-emphasis on keywords such as “equal parts”, seemed to have distracted her from thinking about the notion of fractions. It was not until Jessie showed the equivalence physically (Line 10) by comparing the same piece to the parts of the two equivalent wholes that Zelina’s attention was directed at the concept. Thus, Zelina’s baseline noticing of generic teaching actions (“take or don’t take”) shifted to a more mixed level of noticing, in which she began to attend to the mathematical concept in her explanation.

This shift in attention to the mathematical concepts provided an opportunity for Zelina, and other teachers, to make sense of what Hannah was trying to say. The focus on the concept (Key Point) also led teachers to surface a possible confusion (fractions only
involve equal parts) associated with the key idea of the lesson that was not previously highlighted. Thus, the teachers became aware of a new potential misconception, and this episode of noticing began to sensitise them to the different (and sometimes wrong) conceptions of fractions students might have as a result of different models used to represent them.

### Table 4.3: Analysis of Zelina’s noticing in Phase 1 Session 4

<table>
<thead>
<tr>
<th>What was noticed</th>
<th>Processes of noticing</th>
<th>Attending to</th>
<th>Making Sense of</th>
<th>Deciding to respond</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Routines or procedures in teaching</strong></td>
<td>Zelina thought that the issue was about whether to take or not take the piece (4, 7).</td>
<td>Zelina thought that the issue was about whether to take or not take the piece (4, 7).</td>
<td>She did not realise initially that she had made a mistake (9).</td>
<td>Zelina, together with the rest of the group, decided to use a smaller or bigger piece.</td>
</tr>
<tr>
<td><strong>Key Point</strong></td>
<td>She saw Jaslyn’s physical comparison of the piece in the two wholes (10).</td>
<td>She did not realise initially that she had made a mistake (9).</td>
<td>It can be inferred that Zelina must have realised that the fraction represented is 1/4 because it is still the same piece (10, 11).</td>
<td>Zelina, together with the rest of the group, decided to use a smaller or bigger piece.</td>
</tr>
</tbody>
</table>

### 4.2.2. Vignette 2: One-sixth, one-seventh, and one-eighth

#### 4.2.2.1. Focusing on getting students to say what she wanted to hear

This episode took place during the research lesson (See Figure 4.1 Phase 1 Session 6), after Zelina guided her students to compare two fractions 1/6 and 1/8 using the fraction discs. As highlighted during one of the planning discussions, she wanted students to argue why 1/6 is bigger without using the physical discs, because they may just focus on the physical size of the discs instead of the values of fractions being compared. In the exchange below, Zelina guided students’ reasoning using a series of directed questions during the actual lesson, even though according to the lesson plan, the students were supposed to articulate their reasoning. (The word ‘Students’ below refers to the whole class, S1 and S2 are unidentified individual students.)

1. Zelina: Let’s look at the whole part. How many 1/6 do I need to make a whole?
2. Students: (In chorus) Six parts.
3. Zelina: How many 1/8 do I need to make a whole?
5. Zelina: So, 1/8 needs 8 parts and 1/6 need?
7. Zelina: So, the more parts you need, what happen to the size of the fraction?
8. Students: (Silence)
9. Zelina: The more parts you need?
10. S1: (Some hesitation) 1/8?
11. Zelina: The more parts you need? What happen to the size of the fraction?
12. S1: (After a period of silence) Smaller?
13. Zelina: Smaller! Yeah! When there are more parts, the fraction becomes smaller. If there are fewer parts, the fraction is?
14. Students: Greater

Zelina’s exchange with the students can be classified as a typical *Initiate-Response-Evaluate (IRE)* sequence (Mehan, 1979). Zelina tried to encourage students to see the relationship between the number of parts needed to form a whole and the denominator of the unit fraction of interest (Lines 1 to 3). Using the grammar provided by the *commognitive* framework (Sfard, 2008), we could argue that Zelina was trying to guide students to adopt her *endorsed narrative* about the relative sizes of unit fractions: the greater the number of parts (or pieces) needed to form a whole, the smaller the unit fraction. However, students might not have attended to this *metarule* of the narrative, but instead focused on the *key word*. There is a possibility that students simply attended to the idea that the denominator corresponds to the number of parts, and responded simply to the key words “eighth” with “eight” and “sixth” with “six”. This is plausible because the students could not follow Zelina’s argument on the size of the fraction (Lines 7 to 12). There is also no evidence that students realised that the parts are supposed to be equal in size, corresponding to the *visual mediators* afforded by the fraction discs used earlier. So, students might not connect the denominators to the relative sizes of the unit fractions. This possibility is also hinted at in Lines 10 and 12 by two indications: First, the lack of chorus responses seemed to suggest that some students are unsure, and there was some hesitation by Student S1 even though she gave
the correct answer. Moreover, giving a right answer to the question does not necessarily equate to student understanding. Hence, Zelina might not have listened for the “sense in students’ mathematical ideas” even though their understanding was incomplete (Schifter, 2001, p. 126).

4.2.2.2. Listen-to-evaluate instead of listen-to-reason

This possibility that students are attending to the key words is further strengthened by the exchange that followed immediately when Zelina switched to another example, 1/7 and 1/8. This time, Zelina used the same IRE pattern but was less successful:

15. Zelina: The more parts that you need, what happens to the fraction?
16. Students: (In chorus) Become one whole.
17. Zelina: Become one whole. So the size of the fraction become...
18. S1, S2, S3: One whole!
20. Zelina: Does it become bigger or smaller?
22. Zelina: More parts means?
24. Zelina: Not many pieces you know. Size. So, this is 1/8 and this is 1/7 (points to the fraction discs).
25. Students: (Silence)

It seemed that the students tried to articulate the correct key words (“One whole”, “1/8” and “Bigger”) instead of reasoning as Zelina intended (Lines 16, 18, 19, 21, 23). Particularly, the students’ response of “Become one whole” did not correspond to Zelina’s question “What happens to the fraction” (Line 16). When Zelina attempted to prompt for the key word “smaller”, the students responded “one whole” and “1/8” instead (Line 17 to 19). These responses indicate that students did not understand the endorsed narratives corresponding to the metarule that Zelina had in mind. Student responses (Lines 21 and 23) also suggest that they were looking at “bigger” in terms of the number of pieces needed to form a whole. Zelina picked that up (Line 24) and tried to direct their attention to the “size of fractions” by pointing to the fractions discs.
(visual mediator). She then reattempted to guide students to see the reasoning by initiating another IRE sequence:

26. Zelina: One more time. How many eighths do I need to make a whole? How many sevenths do I need to make a whole? Which is more? Eight parts or seven parts?

27. Students: (In chorus) Seven!

28. Zelina: Eight parts... Eight pieces or seven pieces?

29. Students: Eight pieces!

30. Zelina: Ah... Eight pieces, isn't it? If you have more pieces, what happens to the fraction?

31. Students: ... Become one whole.

32. Zelina: What happened to the size of the fraction?

33. Students: Same.

34. Zelina: This is seven pieces... maybe I use another example.

It is unclear which of Zelina’s questions students were responding to when they uttered “Seven” (Line 27). Zelina assumed that they were responding to whether a whole divided into eight equal parts would result in more pieces, because she was expecting “Eight pieces or parts” as the answer (Lines 28, 30). Moreover, Zelina used “pieces” instead of “parts” to try and get students to see her message that “more pieces means smaller fraction” (Lines 28 and 30). Students did not catch her intended rule and instead responded with what they thought was the key words (“becomes one whole” in Line 31) that Zelina was looking for. It looked like Zelina and the students were referring to different aspects even though they might be looking at the same objects, and this may have resulted in a commognitive conflict, which signals a lack of understanding. Without clarifying students’ response of “seven” in Line 27, it would be difficult for Zelina to see what they were referring to.

Zelina’s in-the-moment noticing during this episode is classified as non-productive because she did not listen to solicit ideas from the students, in order to understand what they were thinking. Instead, she reverted to her IRE patterns of discourse to get students to utter the key words. Zelina did not respond to students in a way that could reveal their thinking as discussed prior to the lesson, but instead tried to guide students to see the metarule implicit in her questions. In a way, she failed to “respond” based on
what was deliberated upon during the discussions and seemed to have “reacted” instinctively to students’ responses (Mason, 2002, p. 87). An analysis of Zelina’s noticing is summarised in Table 4.4.

<table>
<thead>
<tr>
<th>What was noticed</th>
<th>Processes of noticing</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Key Point</strong></td>
<td><strong>Attending to</strong></td>
<td><strong>Making Sense of</strong></td>
</tr>
<tr>
<td>The greater the number of equal &quot;pieces&quot;, the smaller the &quot;pieces&quot;.</td>
<td>She attended to students’ response of &quot;smaller&quot; (12) in light of their earlier correct responses (2, 4, 6), which corresponded to the metarule that she was trying to teach.</td>
<td>She interpreted students’ response of &quot;smaller&quot; (12) as indication that students understood the reasoning (13). Zelina interpreted students’ answer of &quot;greater&quot; (14) as an indication that they understood the reasoning.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Difficult Point</strong></th>
<th><strong>Attending to</strong></th>
<th><strong>Making Sense of</strong></th>
<th><strong>Deciding to respond</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Students might not see the correspondence between the number of equal pieces and the denominator, and hence the relationship between the number of equal pieces and the sizes of unit fractions.</td>
<td>She noted that students had difficulty with 1/7 and 1/8 (16, 18, 19, 21, 23). She noted her students’ response “seven” (27). She realised that students did not seem to get the reasoning from their responses (31, 33).</td>
<td>It can be inferred that she thought that students were thinking about “more pieces” instead of “size” (24). She assumed that students tried to answer her question, “How many sevenths do I need to make a whole? Which is more?” (26, 28, 30).</td>
<td>She responded to students’ difficulty by emphasising the word “size” (24) and pointing to the fraction discs. To help her students see the reasoning of her metarule (30), she used the term “pieces” (28) and tried to get students to see that “more pieces mean smaller fraction” (30). Zelina decided to use a simpler example, which was discussed earlier during the Lesson Study meetings.</td>
</tr>
</tbody>
</table>

Referring to Table 4.4, the Key Point of this segment of lesson was to get students to understand the size of unit fractions in relation to the fractional notation. The corresponding Difficult Point was that students might not deduce the relative sizes of unit fractions from the use of physical fraction discs alone without making the connections between the physical or pictorial models of representing fractions and the symbolic representation of fractions. These instructional details were discussed during the Lesson Study sessions.
However, it seems that Zelina was unable to break away from her usual IRE sequence, even though she might have noticed her students’ difficulty in following the reasoning. Her response to students’ incorrect responses, as well as the dubious response of “seven”, did not correspond to the confusion students faced. Zelina could have allowed students to represent fractions in their own way and to explain their reasoning when comparing the size of fractions. That would have helped to reveal students’ thinking and provided some clues to what the students were perplexed about. Furthermore, Zelina was looking for the keywords such as “greater”, “smaller” and “seven pieces” instead of the reasons for students’ answers. She did not consider the possibility that students might have caught the patterns of speech (“eighth with eight pieces”) when they happened to give the “correct answers”. Although she might have noticed students’ difficulty with 1/7 and 1/8, she inferred that the problem was due to the words “pieces” and “size”, and tried to emphasise the word “size”. As highlighted in Table 4.4, Zelina decided to continue with her pattern of questioning and move on with the next example because she interpreted students’ “correct answers” as an indication of their understanding.

The failure to respond differently during the lesson indicates a less productive noticing by Zelina because she did not use what the other teachers had highlighted prior to the lesson. For example, Jessie highlighted the importance of asking students to give the reasoning on several occasions, such as:

- Whether correct or not, we want to know how they argue? What is the reason?
- What makes them believe that that’s the right answer? Even if they said it’s the half piece, we should ask them how do you know?

- After that you will be asking, for those who said this is a bigger piece, can you tell me why this is a bigger piece? Ask the child to show...

Despite being aware of the suggested approach, Zelina could not bring to her mind questions that she could have used during the actual lesson. Instead, she reacted to students’ incorrect responses (Lines 17, 20, 22, 24) by directing students’ attention to the keywords such as “bigger or smaller”. This did not work during the lesson (Lines 18, 19, 21, 23) and Zelina did not respond differently (Lines 26 to 34). Hence, her noticing can also be characterised as baseline because she attended mainly to her own thinking instead of her students’.
Nevertheless, prompted by an earlier discussion during a lesson study session, Zelina then switched to a more contrasting example after this exchange. She used the fractions 1/8 and 1/4 to support students’ reasoning and it seemed to work. While her decision to use a more contrasting example could be seen as a productive decision, she did not go back to the “1/8 and 1/7” example after that, and proceeded with the group work instead. Thus, Zelina missed an opportunity to determine whether her students could indeed reason about the sizes of unit fractions to overcome their confusion about the relationship between denominators and the sizes of unit fractions.

4.2.3. Vignette 3: "I like the song"

4.2.3.1. Lack of specificity and no focus

During the post-lesson discussion, Zelina's first comments were about the clarity of her instructions to the task, and were not focused on student thinking. She was pleased that most students were clear about the key task of making comparison statements about fractions except for a few who picked up two equal pieces representing a tenth:

> What I saw was… my instructions were clear enough. I said, ‘take out one tenth’. But when I was going around, I realised that some of them took two “tenths” instead of one. Instead of one unit fraction, they took a few more. I think they still have difficulty grasping the greater denominators and smaller fractions. They have some inkling but have not touched down yet… it’s not easy… to make the whole.

Even though she gave some detailed descriptions in her observations ("... took two tenths instead of one"), she did not seem to attend to details related to the mathematical concept (comparing fractions), the students’ confusion about the concept (inappropriate ideas related to sizes of numbers), nor how students responded to the lesson approach (the need to reason about the size of fractions). Zelina did not seem to distinguish between what was mathematically relevant and what was not with regard to the lesson, and made general or vague statements about students' thinking. Zelina seemed to be aware that the students might not have fully understood the use of denominators to compare unit fractions ("They have some inkling..."), and students might have difficulties seeing the relationship between denominators and relative sizes of unit fractions (“they still have difficulty grasping the greater denominators and
smaller fractions.”). However, she did not give further details on how she came to that conclusion and why that was so. Therefore, while there was evidence that she attended to some aspects of her students’ thinking, the lack of detailed connections between what she observed and the ‘Three Points’ did not help to refine ideas about the student’ thinking nor the design of the tasks. Hence, her noticing can be characterised as less productive.

4.2.3.2. The Song Episode: Focusing on less relevant aspects

When the other teachers shared their observations, almost every one referred to an incident that happened during the lesson. They highlighted an episode where Zelina tried to help her students recall the meaning of numerator and denominator through the use of a song that she composed. Zelina taught two songs in previous lessons to help students remember the definitions of key words such as fractions, numerators and denominators. Even though the song was never discussed during the meetings, the teachers seemed to be impressed by the use of the song as a mnemonic. See the following transcript of what happened during the lesson.

1. Zelina: So, what are fractions?
2. Students: (Singing.) Fractions are parts of a whole, parts of a whole, parts of a whole... (Fading.)
3. Zelina: Come, let's sing it again.
4. Students: (Singing.) Fractions are parts of a whole... parts of a whole... parts of whole... fractions are parts of a whole... yes... we learn!
5. Zelina: Good. Now there's something special about fractions. Here is an example of a fraction. (Writes 1/7 on the board.) What do you call this digit here (Points to the numerator 1)?
6. Students: One!
7. Zelina: Yes... it is the digit 1, number 1, what do you call? What is the name we give [to it]?
8. S1: First!
9. S2: One!
10. Zelina: Friends of fraction?
11. Students: Who are...
12. S3: Numerator.
14. Zelina: So, this is the numer...
15. Students: ...rator!
16. Zelina: Yes. What does it tell us?
17. S5: It tells us the total!
18. Zelina: It tells us the total? Anybody else? Perhaps you want to sing the song?
19. Students: (Singing.) Friends of fraction... (Clap a rhythm.) Who are they? Nu- me-ra-tor (Point upwards.)...De-no-mi-na-tor (Point downwards.)... Nu-me-ra-tor (Point upwards.) number of parts... De-no-mi-na-tor (Point downwards.) total parts.
20. Zelina: Good. Again. What does the numerator tell you?
21. S4: Total...
22. S3: Number of parts!
23. Zelina: Number of parts! Yes... How about the second digit (Points to the denominator 7.)?
24. S2: Seven!
25. Zelina: Yes... What do you call this...?
26. Students: Denominator!
27. Zelina: What does the denominator tell us?
28. S3: The total of parts...
29. Students: Total parts!

Here, Zelina used cues (Line 1 and 10) to remind students of the two songs, whose purpose is to help students remember the key words. The lyrics contain the definitions of fractions (Line 2 and 4), numerators, and denominators (Line 19). Zelina also used visual mediators in the form of gestures ("pointing up" corresponds to the numerator and "pointing down" corresponds to the denominator) to accompany the song. A closer look at the “song episode” suggests that some students did not understand what they were singing (S4 in Line 21 and S2 in Line 24). However, Zelina ignored the wrong answer and responded to the right answer (Line 23). She then repeated the same pattern of questioning (Lines 7 and 25; Lines 16 and 27) in a bid to help students say the keywords she was looking for. Although the students were able to utter the expected response, it was not clear whether they understood the intended endorsed narrative.

However, when discussing this episode the teachers focused mainly on the songs and concluded that the students remembered the definitions. For example, Heidi liked how
Zelina used the songs to help them recall the definition without providing further evidence:

Actually, I like how she get them to recall... the numerator and denominator... using a simple song. It’s very interesting.

Jacinda also commented that the lesson was good and liked the use of the song to “reinforce” the definitions:

Overall, I think that her lesson was very good because I can see that her children, even though they are lower ability, they managed to get the concept very well. Like Heidi, I also like the use of the song to reinforce the fractions, the numerators and denominators...

Sherry reiterated the use of the songs as well:

I also like the song because the songs helped them to remember certain concepts...

Only Heather highlighted the need to emphasise the notion of “equal parts” in the definition of a fraction:

Like everybody else, I like the song. It’s very interesting and it’s very catchy. The children really liked it, and it really helped them to remember what numerator and denominator is. I thought, maybe it will be good to repeat what the pupils say, with emphasis on equal parts. They can tell you that fractions are parts of a whole, but maybe the emphasis should be on equal parts so that they can grasp it better.

Even though the use of the song might have counted as an instructional strategy, the teachers mostly attended to how the song was “interesting” and “catchy”. All the teachers highlighted that the song helped the students to remember the terms, but they did not provide any further substantiation. Heather was the only teacher who commented on the key notion of “equal parts” and claimed that students would “grasp it better”. However, no further reason was given to support her claim. What the teachers noticed about the use of the songs is summarised in Table 4.5, and is largely at the baseline level.

Furthermore, as shown in Table 4.5, the teachers’ comments were very general and they did not really relate the use of the song to the mathematical concept targeted in the
lesson. None of the teachers attended to the wrong answers given by students during the episode (Lines 6, 8, 9, 17, 21 and 24 in the song episode), which signalled a possible gap in student understanding of fractions. Like Zelina, they focused on the correct responses given in chorus by the students as a means to justify their claims that students remembered the definitions.

**Table 4.5: Analysis of teachers’ noticing about the song episode**

<table>
<thead>
<tr>
<th>What was noticed</th>
<th>Processes of noticing</th>
<th>Making Sense of</th>
<th>Deciding to respond</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Critical Point</strong>&lt;br&gt;Using a song to help students recall the definitions of numerator and denominator.</td>
<td>Heidi (&quot;interesting&quot;).&lt;br&gt;Heather (&quot;very interesting... catchy.&quot;).</td>
<td>Jacinda (&quot;reinforce the fractions&quot; etc.)&lt;br&gt;Sherry (&quot;helped them to remember certain concepts...&quot;)&lt;br&gt;Heather (&quot;... helped them to remember what numerator and denominator is.&quot;)&lt;br&gt;Heather (&quot;... maybe it will be good to repeat what the pupils say, with emphasis on equal parts... so that they can grasp it better.&quot;).</td>
<td>While Heather suggested the need to emphasise on &quot;equal parts&quot;, there was no further suggestion. There was no comment on how the song might have helped students to understand the notion of fractions beyond remembering the key words.</td>
</tr>
</tbody>
</table>

Besides the song, Heidi commented that Zelina walked the students through the reasoning “step by step” and this “systematic approach” helped students to see the reasoning. However, Heidi did not really draw on the observations of her students’ reasoning, and relate them to how she thought that the approach helped the students. Similarly, Sherry mentioned about the use of paper cut-outs to illustrate how a cake was indeed cut into two equal parts during the lesson, but she also did not link that comment to any observation of her students’ thinking. Hence, most of the teachers in this discussion did not use their observations to refine their views about student thinking or the lesson plan. Therefore, their noticing is generally classified as mixed noticing.

**4.2.3.3. Focusing on the student’ interactions with mathematics**

An episode of productive noticing occurred when Hannah generated useful pedagogical considerations from her detailed observations, which go beyond the classroom
incidents observed (Yang & Ricks, 2012). By describing how two students struggled with a question, she highlighted that these two students were still thinking about fractions physically rather than symbolically because they used the aids to help them:

... [the question] 1/7 is smaller than... he put 1/8. I said look again... then he look and looked. Although he put there 1/7, they still take the 1/7 fraction disc and put it on top of the representation 1/7. They want to see it ... so obviously they are looking at the size, the physical size. So, they put there 1/7 and then put there 1/8, and they put it again ... is it smaller, oh, it’s swapped. But you can’t swap it because it's already written there 1/7. Because it's not an open-ended... 1/7 is written... then they said, 'Oh no, cannot erase...' and then they panicked already... so what to do... Then later, a few minutes later... what can you do ... then swapped, swapped, swapped back, but when it's swapped back, it’s wrong, wrong, then stack, yeah, it's smaller... then how... then finally [Student S1] said, 'take another fraction!'

Hannah’s noticing contrasted with the other teachers in terms of the level of details she gave, and more importantly, how she linked her interpretations to specific instances and combined her understanding to generate a useful principle. Her comments contradicted the other teachers’ remarks that the students’ thinking was well scaffolded by Zelina through a series of guided questions. Instead, Hannah felt that not all the students understood, and saw beyond the students’ seemingly correct answers during the classroom discussion in her relatively detailed description of a particular student’s thinking. She contended that students might not have seen fractions as a representation of a part-whole relationship without the physical manipulative. Moreover, Hannah also noted that the students might have had problems seeing how the number of equal pieces needed to make up the whole could have been related to the size of the pieces even though they could have performed the task, or answered Zelina’s questions correctly:

They are able to do but may not be able to relate it back to the whole. Like why is the whole... I think it’s logic and we assume that they know... that for the same whole, this one has many pieces and this one has lesser pieces, then this should be a smaller piece. Maybe this logic must come in at another platform... However, the children need some wait time, some thinking time, some verbalisation and articulation among themselves... You might want to hear... are they saying it?
It seemed probable that Hannah did not consider “chorus answers” to be indicative of students’ ability to reason about the relative sizes of the unit fractions. An analysis of Hannah’s noticing is shown in Table 4.6.

Table 4.6: Analysis of Hannah’s noticing during post-lesson discussion

<table>
<thead>
<tr>
<th>What was noticed</th>
<th>Processes of noticing</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Key Point</strong></td>
<td>Attending to</td>
</tr>
<tr>
<td>How the denominators of unit fractions relate to the relative sizes of unit fractions.</td>
<td>Attended to students’ understanding of the sizes of unit fractions. (Students struggling with 1/7 and 1/8.)</td>
</tr>
<tr>
<td><strong>Difficult Point</strong></td>
<td>She noted that students are still looking at the physical sizes of the fraction discs. (“Although he put there 1/7, they still take the 1/7 fraction disc and put it on top of the representation 1/7.”)</td>
</tr>
<tr>
<td><strong>Critical Point</strong></td>
<td>Saw that students are not able to reason how the fractions are related to the whole. (“They are able to do but may not be able to relate it back to the whole.”)</td>
</tr>
</tbody>
</table>

Reflecting a more focused noticing, Hannah’s reflections proposed the possibility that the students may not understand the key idea of the lesson even though they had responded correctly to Zelina’s questions. Using what she observed about the two
students, she analysed their thinking, and suggested that students need more opportunities to reason amongst themselves. Hence, Hannah reminded the teachers of an important principle necessary for examining student thinking—listen to what students are saying. As a result, the teachers became more aware of the need to listen to their students’ reasoning and Hannah’s noticing led to a possible refinement of the teachers’ thinking about their students' thinking. Hence, her noticing can be said to be productive because it generated new insight into student reasoning.

4.2.4. Discussion of the three vignettes: Explicit focus for more productive noticing?

The three vignettes, together, represent the typical episodes that occurred in all three schools during Phase 1 of the study. By extending the use of van Es’ (2011) framework, the teachers' noticing can be categorised according to the different levels, ranging from baseline to extended. However, a key distinction in the characterisation of productive noticing undertaken in this study, as compared to the different levels of noticing, lies in the course of action taken by the teachers. When a teacher’s course of action reflects the productive practices, highlighted in Section 2.2, the noticing that drives the decision making is said to be productive for encouraging student reasoning (See Hannah’s noticing in Section 4.2.1.2 and 4.2.3.3). On the contrary, when the course of action taken by a teacher does not reflect these practices, then the noticing is seen to be non-productive (See Zelina’s noticing in Section 4.2.2.1, 4.2.2.2 and 4.2.3.1). The three vignettes not only highlight the importance of noticing during reflection, but more critically, suggest a central role for noticing in the planning of a lesson, which is generally under-represented in studies related to teacher noticing.

However, given the wide spectrum of things to observe, it is not surprising that teachers may focus on aspects that do little to enhance their understanding of student thinking. Without an explicit guiding focus, teachers noticed a wide variety of events and details, both relevant and irrelevant to the tasks of Lesson Study (Star et al., 2011; Star & Strickland, 2008). For instance, Hannah’s noticing in the first discussion session (See Section 4.2.1.1) was not productive because she did not attend to the specifics of the mathematical concepts, and the students’ confusion about these concepts. Without discussing these concepts, it is unlikely for a task to be useful for enhancing student reasoning (Mason & Johnston-Wilder, 2006; Smith & Stein, 1998). So, even if Hannah’s
suggestion to improve questioning is appropriate, the lack of connection between her suggestion and the students’ difficulty did not help the teachers to propose the necessary features for the task to work.

Likewise, teachers did not generate useful input in terms of knowledge, or refinements to the tasks that could potentially enhance student reasoning when they did not focus on the mathematically worthwhile aspects of the lesson. The teachers’ noticing of aspects related to the song, instead of how the use of the song relates to student understanding during the post-lesson discussion in Vignette 3 is a good illustration of the challenges faced by teachers in reflecting about their lessons. This finding is not surprising given that it may not be natural for teachers to focus on the things that might enable them to put on the lens of a researcher, curriculum developer and student (Clea Fernandez et al., 2003). Therefore, Phase 1 of the study extends the findings by Star et al. (2011) to indicate that in-service teachers, both experienced and beginning teachers, are also not necessarily effective observers of mathematics lessons.

Analyses of the vignettes seem to suggest that productive noticing occurs when teachers begin to focus on issues related to the concepts, the students’ confusion, and the teachers’ corresponding course of action. The focus on these three aspects of teaching seemed particularly important during the planning and reviewing of lessons. (See Vignettes 1 and 3). Similar to other studies on noticing, Phase 1 also suggests that the ability to give specific details by referring to particular events or students is an indicator of noticing expertise (van Es, 2011, p. 139). Particularly, from the three vignettes, teachers’ noticing seems to be more productive when they are able to refer to specific aspects of the Three Points (See Sections 4.2.1.2 and 4.2.3.3), and not a vague passing comment about generic aspects of a lesson (See Section 4.2.1.1). Hence, the evidence suggests that the Three-Point Framework may support teachers in thinking about the concept (Key Point), students’ confusion (Difficult Point) and teaching strategies (Critical Point) at the end of Phase 1.

Moreover, teachers also seemed better at responding to student thinking in the midst of teaching when they were able to see and interpret the Key Point and Difficult Point from the students’ in-class interactions. For instance, Zelina’s decision to switch to another example of “1/4 versus 1/8” highlights how she responded by focusing on her students’
confusion. However, as illustrated by her subsequent failure to return to the more difficult example of “1/6 and 1/8”, it is challenging to maintain a clear focus on the mathematical aspects of classroom interactions in-the-moment.

Nevertheless, as Mason (2002) suggests, advance preparation to noticing can support a teacher’s in-the-moment noticing. To illustrate this, a contrasting example depicted by Lindy, from Trafford Secondary, will be presented next. Unlike Zelina, Lindy was able to bring to her mind an example that was discussed during a Lesson Study session in response to her students’ vague ideas about the definition of an interior angle in a polygon (See Figure 4.3 Phase 1 Session 5).

4.2.4.1. A productive instance of in-the-moment noticing

In this segment, Lindy tried to listen to her students’ definition of interior angles before moving on to the main task of investigating the interior angles of a polygon. During the planning, the teachers realised that it was important to be precise about definitions, and produced a few examples that test the boundaries of weak definitions such as “angles inside the polygon”. As anticipated by the teachers, most students were not precise enough in their definition, and the exchange presented here describes what Lindy did in class to help her students understand the notion of interior angles of a polygon:

1. Lindy: Let me ask you, anybody knows what is an interior angle of a polygon? Who wants to try? Because I remember that last lesson someone mentioned about the interior angle of a polygon. Who wants to offer a definition? What do you think is an interior angle of a polygon?
2. S1: They are angles inside the polygon...
3. Lindy: Angles inside a polygon. [S2], do you agree? It’s angle inside the polygon?
4. S2: ...
5. Students: Say no because it’s [S1] (laughter.)
6. Lindy: No? Because it’s [S1]?
7. S2: Kind of.
8. Lindy: Kind of. [S3], what do you think? Angles inside a polygon. Good enough? Do you think that his definition is good enough?
9. S3: Nooo... (Laughter from classmates.)
10. Lindy: No? Then, can you suggest a better definition?
11. S3: Don’t know ... (Laughter from classmates.)
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12. Lindy: You don’t know? Ai ya... (Laughter from classmates) He said it’s angle inside the polygon. Why not we get [S1] to draw what does he meant by that? What does it look like? Just draw for me any polygon you like, and then mark for me the interior angles.

13. S1: (Comes out and draws a triangle.)

14. Lindy: Ok. He draws a triangle and marks an interior angle. (Points to the other two interior angles one at a time.). Is this also an interior angle?

15. Students: (Chorus answer.) Yes...

16. Lindy: Yes? This one?

17. Students: (Chorus answer.) Yes...

In contrast to Zelina’s IRE pattern, Lindy’s discourse was more discursive. Instead of focusing on getting the students to say the key words, Lindy tried to model how mathematicians refine (routine) mathematical definitions (endorsed narrative). She did that by eliciting her students’ responses (Lines 1, 3, 8) without revealing whether the proposed definition (Line 2), and other responses to Student S1’s definition were correct (Lines 3, 8, 10). When Student S3 could not offer a better definition (Line 11), Lindy responded by getting Student S1 to draw what he meant by an interior angle (Line 12). In doing so, Lindy skilfully used a visual mediator (geometric diagram) to orchestrate a discussion that followed (Lines 18 to 35). In getting Student S1 to represent his notion of an interior angle, Lindy was able to ascertain whether the students had a common understanding of what Student S1 meant by “angles inside a polygon” (Lines 13 and 14). In order to direct her students’ attention to the lack of precision in Student S1’s proposed definition, Lindy drew a second shape besides the triangle, as proposed during the Lesson Study discussion:

18. Lindy: Let’s look at this. (Draws the following shape on the board).
This one? (Pointing to the angle created by two line segments within the polygon.) Do you call this an interior angle of a polygon?

19. Students: Yes...
20. Students: No...
21. S4: Not a polygon, what?
22. S5: It’s part of a polygon...
23. Students: (Discussion amongst the students. Impossible to transcribe.)
24. Lindy: You said angle inside the polygon? This one, my polygon. Then this one angle inside the polygon?
25. Students: This one not inside?
26. S6: But there’s still a polygon there...
27. Lindy: There’s another polygon there?
28. S7: Two polygons?
29. Lindy: So, that means just now, the definition offered by [S1], can we help him tighten a bit...
30. Students: (Laughter.)
31. Lindy: (Smiles.) I mean tighten means try to improve his definition. He said angles inside a polygon. (Writes on the board.) Then you realise that if I draw my polygon like that, this angle is inside my polygon but it may not be an interior angle of a polygon, right? So, how can I improve on his definition? This angle is formed by what? This interior angle is formed by what?
32. S8: From the two line segments of a polygon
33. Lindy: The two line segments of a?
34. S8: Polygon
35. Lindy: Two line segments... or maybe we can call this... make it easier... two sides... of a polygon. (Writes on the board.)

In this segment of interaction between Lindy and her students, she drew a polygon and constructed an angle formed by two line segments within the polygon. Through the use of this visual mediator, Lindy wanted to direct the students to see that their metarule—interior angles are angles inside the polygon—is not precise enough. It seemed that her
students were unsure whether the angle constructed is an interior angle of a polygon (Lines 19 and 20). Furthermore, the students were testing the definition on their own (Lines 21 to 23) before Lindy brought to their attention the notion of “angle inside the polygon” (Line 24). From the key words used by the students (Lines 25, 26 and 28), it can be inferred that they were trying to make sense (routine) of the definition using the diagram (visual mediator) drawn by Lindy. In order to “tighten” Student S1’s definition, Lindy focused on getting her students to describe the location of the angle more precisely (Line 31). Student S8 provided the key words to complete the definition of interior angles. Lindy then proceeded to use Student S8’s input to build on Student S1’s definition.

Therefore, in contrast to Zelina’s, Lindy’s in-the-moment noticing can be classified as productive because she not only solicited and listened to her students’ ideas (Lines 2, 3, 8, 10, 18 and 24) about the definition, but also responded in a way that invited the students to reveal their thinking (Lines 12 and 18). More importantly, she was able to respond to the student definition of interior angles she anticipated by constructing an illustrative example for them to compare against their initial definition. In a way, the teachers’ preparation during Lesson Study had helped Lindy to heighten her in-the-moment noticing. During the post-lesson discussion that followed, she highlighted how the preparation to notice had helped her respond:

> I can feel the difference after the discussion, and coming up with the solutions, and then being more confident in class, expecting some of the students’ responses actually helped a lot. If we didn’t go into such details, I think, it depends on the level of experience of the teacher to see students’ responses and know how to react.

As a result of the preparation for noticing, Lindy’s response was to create a good example (Critical Point) for her students to discuss instead of telling them the correct definition of interior angles. Her response targeted at helping the students to test out their own definition so that they can understand the necessity of rigorous definitions in mathematics.
4.2.4.2. Another focus: Alignment of the Three Points

Lindy’s focus on the mathematical concepts (Key Point), and her students’ understanding of the concept (Difficult Point) was instrumental to her orchestration of a productive classroom discussion as depicted in Vignette 4. Teachers might revert to their habitual reaction instead of responding appropriately when they could not bring to mind the ideas discussed prior to the lesson. The Three Points appear to provide a useful frame for examining what teachers see and interpret, and how they use what they notice to make instructional decisions. Besides, the use of a focal point may serve to encourage extended noticing, which examines the relationship between the students’ thinking and teaching strategies. Therefore, as a result of the findings from Phase 1, the use of the ‘Three Point Framework’ to direct teachers’ noticing was used as a means to test the usefulness of providing an explicit focus in Phase 2 of the study.

In addition, Lindy’s interaction with her students suggests another important aspect of the focus: the alignment between the Three Points. While Lindy’s attention on the Three Points was useful for her to coordinate a discussion on the notion of interior angle, it was her focus on responding to the students in a way, which addressed their confusion (Difficult Point) related to the concept, that made a difference. First, she challenged her students’ imprecise notion of interior angles by creating an example that corresponded to their proposed definition of angles in a polygon. Second, she probed her students’ thinking by asking them to explain or show their thinking. And finally, she built on their understanding through a more discursive style of teaching. Her courses of action taken, in this case, were directed or aligned with at the issue at hand. Likewise, when Zelina switched to a more contrasting example, the course of action taken was targeted at the students’ difficulty to understand the concept. But unlike Lindy, Zelina could not maintain the alignment through the interaction.

The importance of alignment also featured in Hannah’s attention to how the use of the song, and the chorus answers given by students. She was able to point out that the song, which targeted the learning of the terms, may not address her students’ understanding of the sizes of unit fractions. Likewise, Hannah’s assertion that teachers need to “hear the students say it” is directly related to the gap in the logic of students’ thinking. This contrasted to the other teachers’ noticing of the song, which was not related to the confusion faced by the students in relation to the size of the fractions.
Therefore, these two aspects—the Three Point framework and the alignment between the Three Points—are essential components of the proposed focus for productive noticing. The Three Points provide a focal point for teachers to consider mathematically worthwhile aspects of instructional details; while the alignment between the Three Points helps to ensure that the course of action is directly related or linked to the observations attended to by the teachers on the basis of their interpretation of these events. Figure 4.7 shows a tentative schematic representation of productive noticing. It builds on the nested model suggested by Jacobs et al. (2010) as shown in Figure 2.8, and incorporates the Three Points, represented by a green, red, and blue circle respectively, within the process of attending as an explicit focus. In addition, the lines connecting the three circles, representing the Three Points, signify the importance of the alignment between the three points: The pedagogical approach (Critical Point) should target the difficulty faced by students (Difficult Point) associated with the concept (Key Point). This tentative framework formed the basis for investigating teacher noticing in Phase 2 of the study.

![Figure 4.7. Tentative framework for productive noticing.](image-url)
4.3. Another close-up view: Focusing noticing

This section presents the second key element of the FOCUS framework: the process of focusing attention in order to bring the Three Points into alignment. It highlights the role of pedagogical reasoning as the mechanism to connect the process of attending to the process of responding. This notion is presented through another case study of Greenhill Primary School, developed from the episodes in Phase 2 of the study, in which the Three Points were used explicitly as a focus. Six teachers were involved in this case study: Kirsty (facilitator); Cindy; Flora; Anthony; Rani; and James (research teacher). They collaborated to design a lesson on *Fraction of a Set* for Primary Four students (age 10). Three vignettes are presented and discussed: The first focuses on a few discussion episodes that happened during the planning (See Figure 4.1 Phase 2 Sessions 1, 2 and 3); the second highlights what James noticed in-the-moment during the lesson (See Figure 4.1 Phase 2 Session 6); and the third recounts what teachers noticed during the post-lesson discussion (See Figure 4.1 Phase 2 Session 7).

4.3.1. Vignette 4: “Fraction as part of a whole”

4.3.1.1. The challenge of aligning the Three Points

During the first session, Anthony identified a few specific difficulties that his students might have when they worked with operations involving fractions. In this episode, Anthony discussed the common error of adding the numerator to the numerator, and denominator to denominator when adding two fractions:

1. Researcher: So, usually, how do you show 2/3 + 1/4?
2. Anthony: So, if we follow exactly the textbook, what we are using now. The textbook tries to draw a complete whole, and cut into parts. They didn’t explain, but just give the LCM [Lowest Common Multiple]. Students start to be confused, why give 12 parts and not 24 or 18 parts?

I (Researcher) asked Anthony to show the teachers how the textbook presents the explanation for 2/3 + 1/4. He drew the following diagrams on the board.

![Diagram](image-url)
Chapter 4 – The FOCUS Framework for Productive Noticing

3. Anthony: So the children will ask, why do you give me 12 equal parts? Why didn’t you give me 6 or 18 equal parts? So, Ah… we look at the multiples of 3, 6, 9, and so on... at the end, we have 4, 8, 12... Coincidentally, we find just the lowest common multiple, so we have to use is 12. So, to me, this has to come back to 2/3 of a set of 12, and this is 1/4 of a set of 12. So, this goes back to the same problem [referring to the fraction of a set] and we are turning around the same problem.

4. Researcher: So how do you help them to bridge the gap?

5. Anthony: No choice... Because they are not in the same family, we want them to do some transaction, or you want to mix them together, we need to make them look alike. Like I told them, you come from Thailand, you come from China, you speak your own language, you cannot stay in Singapore. Then what to do? Follow our language lah. Speak English. English is our common language. In this case, 12 is our common language, so we use 12.

6. Researcher: So diagrammatically, because the problem is that we start with the 12 units. If we don’t start off with the 12 units, how do we explain that it is 12?

7. Anthony: They will tell me this... My teacher tell me... this one, you multiply me and I multiply you. [Laughter] So, if the question is not that big, some times they are given 5/6 and then 4/9. They start to multiply 9 with 6 and 6 with 9 [yeah! That’s right?] and the numbers get bigger and bigger? Then they don’t know how to do.

8. Researcher: Did they know why they have to do it this way?


11. Anthony: They will say, because you told me, they are not in the same family, so I had to make them alike. So, if I have denominators of the same kind, it is easier for us to do. So, they forget about the lowest common multiple and then they just make the numbers bigger and bigger. There are cases like 12 and 9, they multiply the numbers together to make 108. In the end, they do not know how to reduce to smaller fractions.

12. Kirsty: Actually, I agree with him. This problem is very common.

13. Anthony: This problem is really very, common. Even the best children also end up with this problem.

14. Kirsty: Because they fail to understand the factors and multiples well. They don’t know the least common [multiple]...

15. Rani: They don’t know how to list and find the lowest.

In the excerpt above, Anthony described the Three Points with specific details when he highlighted how the textbook presented the explanation (Critical Point) for the addition of fractions (Lines 2 and 3). He pinpointed the lack of explanation for the use of the lowest common multiple (LCM), and saw this as a potential problem (Difficult Point) for
his students (Line 2). Moreover, Anthony could see the Difficult Point from the students’ perspective (e.g., “So the children will ask, why do you give me 12 equal parts? Why didn’t you give me 6 or 18 equal parts?”). His attention to this Difficult Point was further demonstrated by his elaborated students’ responses (Line 11). Therefore, Anthony could focus explicitly on the Three Points, and highlighted them for the teachers’ consideration.

However, Anthony appeared to have only considered the seemingly arbitrary division of the whole into 12 equal parts, and did not really connect this to the original issue he raised—that the students might add $\frac{2}{3}$ and $\frac{1}{4}$ to give $\frac{3}{7}$—when he used the area models of fractions to explain the operation. He did not explain how students’ failure to see the 12 parts might have caused some confusion in students when they operate with fractions. While the other teachers also resonated with him on the prevalence of this difficulty amongst the students (Lines 9 to 14; Kirsty - Line 12 and 14; James – Line 10), they seemed to focus mainly on the procedural aspects, and did not consider how the use of the area model could have led to the students’ erroneous answer shown in Figure 4.8.

![Figure 4.8. A possible student error on the use of area model.](image)

Hence, Anthony and the other teachers did not consider the possibility that the students might still make the same mistake, even though they could have changed the two fractions to the same denominator. In this case, they did not examine how the area model (a visual mediator) might fail to represent the reasoning behind the procedure (endorsed narrative). From what Anthony attended to, which is the routine of getting the common multiple, it can be inferred that he might not have noted this possibility even though Anthony had the necessary subject expertise (a degree in Mathematics).
Therefore, the teachers missed an opportunity to gain new understanding that could have helped them to think more carefully about the representations they would be using. In addition, Anthony responded with the use of an analogy (“because they are not in the same family…”) to highlight the requirement to change the denominators of both fractions to the same number when he was asked about his approach to help the students understand the operation. Even if the analogy was helpful as a memory aid to remind students of the need to change denominators (routine), it lacked the mathematical connection to explain the procedure (endorsed narrative). So, in this case, Anthony’s suggested course of action did not align with the Difficult Point and the Key Point. An analysis of Anthony’s noticing is shown in Table 4.7.

<table>
<thead>
<tr>
<th>What was noticed</th>
<th>Processes of noticing</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Attending to noticing</strong></td>
<td><strong>Making Sense of</strong></td>
<td><strong>Deciding to respond</strong></td>
</tr>
<tr>
<td><strong>Key Point</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Using equivalent fractions to add two fractions with different denominators.</td>
<td>He highlighted how the textbook presents the explanation of the procedure (2, 17). In particular, he noted that the textbook did not explain why the LCM is used to obtain the equivalent fractions (2).</td>
<td>He seemed to focus on why the LCM is 12 rather than the reasoning behind the procedure (5).</td>
</tr>
<tr>
<td><strong>Difficult Point</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Students do not understand the reason behind the procedure. In particular, they may misinterpret the area model of fractions to justify their incorrect procedures.</td>
<td>Anthony highlighted that his students did not understand the reason for the use of the LCM to convert the fractions to the same denominator (3, 7).</td>
<td>It seemed that he interpreted students’ difficulty in terms of seeing why the LCM is used (3, 7, 11). He interpreted this problem as similar to the topic at hand – “1/4 of a set of 12” (3).</td>
</tr>
<tr>
<td><strong>Critical Point</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Use area models to represent the procedure but also highlight the relationship of each part to the whole unit.</td>
<td>He noted that the textbook uses an area model that is already divided into 12 parts without explanation (2).</td>
<td>Anthony did not consider how students might misinterpret the area model. (See Figure 1.4.)</td>
</tr>
</tbody>
</table>

Once again, this episode highlights the two aspects of the explicit focus for productive noticing: First, the specifics of the Three Points; and next, the alignment between the Three Points. As highlighted in Table 4.7, even though Anthony was rather specific and
Chapter 4 – The FOCUS Framework for Productive Noticing

detailed in his attending to the 'Three Points', the lack of alignment between his decision to use an analogy and the possible Difficult Point had rendered the discussion less productive towards enhancing the students' thinking. The alignment between the Three Points could have been enhanced if teachers had considered different alternatives or explanations (Mason, 2002), and analysed these alternatives to justify their effectiveness.

4.3.1.2. Aligning the Three Points in task design

The role of analysing and justifying in aligning the Three Points can be seen in examining how another teacher, James, who explained how a met-before (McGowen & Tall, 2010) of “a fraction as part of a whole” may hinder his students’ understanding of ‘fraction of a set’ during the same session:

16. James: I think the objective for fraction of a set is for students to see, to interpret fraction as part of a set of objects. Previously, the fraction [concept] they learnt is more of part of a whole. They are very used to thinking about part out of a whole. Now that we give them a lot of whole things, they cannot link that actually these fractional parts can refer to a set of whole things also. So I think, to me I feel that the connection that is missing, is that, how this fraction concept—which is part of one whole which they have learnt so far—can be linked to whole things. For example, previously we used to teach fractions as parts of a cake or pizza. From that, how can it be that we have many pizzas, we don’t cut out the pizza, there is a fraction of the pizzas. I think they cannot make a link there. If they can see that, they should be to compute a fraction of a set of items using the unitary method, for P4 lah, and using multiplication and division as well. So, er… is this part of interpreting a bar model of a fraction as a set, divided into equal subsets. That means, how... it’s more of model drawing already.

17. Teachers: [Inaudible discussion.]

18. James: For me, the main difficulty is to relate part of a whole into items that are “whole” but you take a fraction out of it. So, I think that’s where the confusion comes. [After a while, James gave a more concrete example to illustrate what he meant.]

19. James: For example, if you say 3/4 of the cats are... [Imitating the students] Ah... you cut the cat into three quarters? (Laughter.) Cut each cat into four parts. So, yeah, but based on what they learnt so far, that may be the first thought they might have. To them, fraction could still be cutting up into parts. Whereas, fractions of a set, we leave the things as a whole entity but we look it as a collection of things. So out of these five things, how many are blue etc... For me, that would be the main difficulty.

In this episode, James not only described specific details about the Key Point (“... to interpret a fraction as part of a set of objects.”) and the Difficult Point (“They are very
used to thinking about part out of a whole.”), but he was also able to relate these aspects to his knowledge and experience. James suggested that students may only possess an image of a fraction as “part of a whole” (See Figure 4.9); and highlighted how the type of examples used by teachers to teach fractions (“...previously we used to teach fractions as parts of a cake or pizza…”) may have been stuck in the students’ minds. And according to James, students’ notion of a fraction as “part of a whole” might have conflicted with the notion of fractions as part of a set of objects.

Figure 4.9. An example of students’ image of a fraction as "part of a whole".

The link between students’ image of a fraction as “part of a whole” and their difficulty to grasp the idea of a ‘fraction of a set’ was further elaborated by James with the use of two examples—the pizza and the cat (Lines 16 and 19). Particularly, he drew teachers’ attention to the students’ way of thinking about fractions with his vivid example of “cutting up the cat” to illustrate how students might be thinking of fractions as “cutting up into parts” (Line 19). James reiterated the same idea when Kirsty asked whether students could identify what is 1/4 of a set of eight circles later in Session 2.

20. Kirsty: What I was thinking is... let’s say, I have 8 circles right... I leave all of them blank, and I tell them to shade to show 1/4 of the circles is shaded red...

21. James: Oh... My first thought is...

22. Kirsty: Would they be able to know that actually 2 will be shaded out of the 8?

23. James: They will cut out every circle and circle 1/4...

24. Kirsty: Oh... they will?

25. James: That’s my first thought that the students will do...
In this short exchange, James reiterated the Difficult Point by anticipating how students might respond to the question posed by Kirsty. This (Line 23) and the other examples (Lines 16 and 19), possibly drawn from what he previously noticed, provided an explicit link between the Difficult Point and the Key Point. James’s noticing of his students’ possible confusion thus had productive potential for enhancing student thinking because it helped other teachers to focus their attention on the Key Point and Difficult Point when designing the task.

In the next session, the teachers (particularly James) attended to the same point (Lines 26 and 34), highlighted an example from the textbook (Line 36), and generated a possible approach (Critical Point) to overcome this cognitive difficulty (Line 36):

26. James: I think that the difficulty is putting the things into the sets. And imagining that each of this set is one part. The textbook makes it look like a very good way to teach this, they arrange the items very neatly into visible lines like this, for example, like this one, 2 fifths of the circles are yellow. It is very clear and you can see two sections. But without the pictures, the children cannot imagine neatly like that.

27. Flora: Pictorial to abstract.

28. James: So, when we are doing this, they can get it. Because it is very clear, because visually, yellow colour two sides, purple colour three sides, so, total you have five rows and all that. So, very clearly they can tell you, but once without pictorial, they get lost already. So I think it’s the transition from the pictorial to abstract.

29. Cindy: Actually, we usually use the colour cubes as manipulative…

30. James: Here, we start with concrete and then moved to pictorial. From concrete to pictorial, they are still okay. Once we move to a non-picture, they cannot imagine already.

31. Cindy: Yes…

32. James: Especially when some of the numbers, we are talking about something like 36 pupils in the class, 1/3 of them, too many for them to imagine in their mind, how to arrange into something like that. So, I think that’s the biggest difficulty. So, if we just talk about this part, they can understand, even if bigger numbers, they can still see and can tell… oh ¼ of the cups, how many are there? Oh… 4… With the colours everything there. Once, there are no more pictures, they cannot make it.

33. Flora: So, when you give them concrete right? Did you tell them how to
arrange it or they themselves will arrange it? And then, maybe we could ask them to articulate why they arrange it that way. You know like, they then probably say, if there are 22 cubes, then they know they can use their multiplication tables and they group it into groups of 2 or... you know. Maybe we find out what they are thinking... you know... and bring it to the abstract part. So, they have the items already, but they have the numbers, then they can visualise and link it later?

By maintaining a focus on the Key Point and Difficult Point, James was able to direct the teachers’ attention to how their textbook presented the concept. He stressed the diagrams might have made it obvious for the students to see the partition, and the students might possibly find it difficult if the diagrams were removed (Lines 28 and 32). James's noticing might have prompted Flora to suggest the possibility of getting the students to “arrange” the items into the partitions and explained why they arranged it that way (Line 33). James then suggested how the problem can be dealt with:

34. James: I think the confusion part also comes when... for example... this example here... we tell that ... 1/4 of the cups are yellow and then the answer is 4 cups. Huh...1/4 and then why got 4 in the 1/4? They cannot link between the... the 1/4 in their mind is still 1/4 of a whole... and then there is this four cups, four whole things... and so they cannot link.

35. Researcher: So, what is the key thing that can help them to link these two ideas?

36. James: I was thinking whether we can put it into... something more familiar because... eh... they have learnt models, how to represent questions in model also, so, I was just looking at this... instead of just doing this, could we box the whole thing up instead. And to them, they are familiar with the part-whole model... a whole box is a whole... so while keeping the items inside and we draw the box... and... and... yes... we tell them that this looks familiar, and it looks like the model as a whole, right? These lines can be the partitioning of the whole model. While doing that... they can still see that the 4 items are still inside the parts. I don't know whether that can help them to make the connection that if this 1 box is 1/4 of the whole, inside that box, I have four things. And this is where the 4 came from?

37. Flora: Would they have... the experience of drawing the model for this? Because the whole and the fraction, they don’t get the connection, right?

38. James: If we tell them to draw us a model of 1/4, I think they can at least can draw the model and partition it 1/4 one part. Because there's still that concept of whole. But it's the... when you put in all the items, then they don’t know what to do with it already.

Here, James tried to use the part-whole model method (familiar to all students in
Singapore) to direct the students’ attention to the possibility that one partition of the whole (a fraction) can contain “whole objects” (Line 36). This suggestion was directly linked to the student’s image of a fraction as “part of a whole” and thus provided a bridge for the students to extend their notion of fractions by emphasising a fraction as a means to express the relationship between a part and its whole (See Figure 4.10).

The use of the part-whole model can thus be classified as a Critical Point that teachers could use to help their students overcome the Difficult Point. By directing the students’ attention to the number of discrete items in a partition of the whole, James hoped to create a way for the students to see that fractions can be used to refer to “whole things” (Line 1). Hence, what James attended to and analysed provided some design considerations for the task. Therefore, James’s noticing would be characterised as productive in this case. An analysis of James’s noticing is shown in Table 4.8.

As seen from Table 4.8, what distinguished James’s noticing from Anthony’s was not the workability of the approach suggested, but rather the justification that reinforces the alignment between the ‘Three Points’. For example, James highlighted that the use of the model method could serve as a means to represent the unitising and partitioning processes of understanding fractions. This reasoning serves as a warrant to link his claim about the students’ concept images of a fraction to the examples his students’ responses. This kind of justification or pedagogical reasoning that is built on prior experience and supported by what was noticed helped the other teachers maintain their attention on relevant aspects of task design. Furthermore, it also lessened the likelihood of generating a response that does not provide opportunities to enhance student reasoning. Hence, unlike Anthony’s approach, which did not match the Difficult Point,
the approach of using the bar model to represent the whole collection of objects is targeted at James's identified Difficult Point.

Table 4.8: Analysis of James's Noticing in Vignette 4

<table>
<thead>
<tr>
<th>What was noticed</th>
<th>Processes of noticing</th>
<th>Making Sense of</th>
<th>Deciding to respond</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Key Point</strong></td>
<td>He identified the objective as seeing and interpreting a fraction &quot;as part of a set&quot; (1, 3).</td>
<td>It could be inferred that he broke down the thinking processes into two parts: unitising the set or seeing the set of all the objects as a whole (1, 3, 13, 21), and taking part of it (3, 11, 13, 21).</td>
<td>He suggested using a box (model) to represent the whole, and show that there are &quot;whole&quot; objects inside each partition (21) created by the students (23).</td>
</tr>
<tr>
<td><strong>Difficult Point</strong></td>
<td>He highlighted that the students previously encountered a fraction as &quot;part of a whole&quot; (1) and could not how a fraction could refer to a set of discrete items (1, 3, 8, 19). He hinted that the way the textbook presented the notion of a fraction of a set (&quot;it seems like a good way...&quot;) might not be a &quot;good way&quot; (11).</td>
<td>He was able to relate his student's limited &quot;part of a whole&quot; notion to specific instances he encountered, e.g., pizza cutting (1), cat cutting (4), shading 1/4 of 8 circles (8), and 1/4 refers to 4 cups (21). He was able to give an example of student response when students only relied on the circular model of fractions (8).</td>
<td>James suggested using the part-whole model to explicitly highlight the possibility that a fraction can refer to &quot;whole objects&quot; (21) instead of &quot;part of a whole [object]&quot;. He mentioned that the partitioning of the model was to be done by the students (23).</td>
</tr>
</tbody>
</table>

4.3.2. Vignette 5: Partitioning the cubes

4.3.2.1. Noticing student reasoning through student responses

During the lesson, James started with a simple ‘warm-up’ task in which he got the students to make simple fraction statements about a set of items with two distinct subsets (e.g., four buttons were shown: one orange triangular button and the others were grey round buttons.). The objects were created digitally using a piece of software that allows for the shapes and lines to be added and moved.

1. James:  I’m going to start with a very quick recap of what we did yesterday. So far, the fractions that we learn, we have always talked about 1 whole thing, right. But yesterday, you saw that when we talked about fractions of a set, it doesn’t mean that I cut up one thing. For example, I don’t cut the apple into 4 parts anymore, right? We talk about many apples. For example, now we
look at the screen... ok...

There are some buttons of different shapes. Can you tell me... what fraction of the buttons is orange? Yes, [S1].


3. James: 1/4. Do you agree?

4. Students: (Chorus) Yes...

5. James: Yesterday, what we have been talking about is... [?] 1 part out of ... 4. That's the simple one that we started, right? Now I have more buttons, can someone tell me what fraction of the buttons is orange again? Yes, [S2]?

6. S2: ... 3 twelfths

7. James: 3 twelfths. [S2] said 3/12. So, [S2], can I ask you why did you say it's 3/12?

8. S2: There are 3 orange buttons...

9. James: Ok. There are 3 orange buttons. So, where did you get the 12?

10. S2: From the...

11. James: From?

12. S2: The total...

13. James: All of... the total number of buttons... good. And simply, 3 buttons out of 12 buttons that are orange, right? Ok. Yes. [S3]... you have something to add?


15. James: Ok. [S3] said 1/4 of the buttons are orange... do you agree with that?

16. Students: Yes...

17. James: So, [S3], why did you say it's 1/4 then?

18. S3: Because... because it's like divide...

19. James: You divide 12 by...


21. James: 4... why did you do that? What do you call that? Yes... [S4]?

22. S4: Grouping...
As discussed during the Lesson Study sessions, James tried to probe his students’ reasoning for the answers given (Lines 7, 9, 17 and 21). It seemed that James attempted to structure the discourse to minimise direct instruction by repeating his students’ responses (Lines 7 and 9), revoicing (Lines 13 and 15), and asking students to justify their answers (Lines 7, 9, 17 and 21). Particularly, in Lines 9 and 13, James reinforced the endorsed narrative of a fraction as part of a set by focussing on the number of orange buttons out of the total number of buttons in the set. It seemed that S2 knew the reasoning for his answer (Lines 8 and 12) because he also focussed on the same metarule: Find the number of objects of interest, the total number of objects in the set, and express it as a fraction:

$$\frac{\text{Number of objects of interest}}{\text{Total number of objects in the set}}$$

S3’s response of 1/4 in Line 14 opened another line of discussion with the possibility of attending to the other metarule: Equal partitioning of objects in the set. James changed his use of key words from “3 out of 12 buttons” to “1/4 of the buttons” to indicate a possible alternative to S2’s response. When James asked S3 what he meant by “divide” (Line 19), James added in the number “12” in his question (Line 19) to hint at equal partitioning of the 12 marbles. It is not clear what S3 meant by “…it’s like divide…” in Line 18: whether he meant to divide both the numerator and denominator by 3 or divide the set of 12 marbles into four equal groups. However, S3 responded to James’s question (Line 19) with the answer “4”, which indicated that he understood that James was asking about the metarule of equal partitioning.

It can be argued that James’s in-the-moment noticing in this short exchange could be classified as productive because he was able to attend to his students’ reasoning and responded in a way that probed their thinking. In some ways, James can be seen to engage students by listening or attending to their answers in an interpretive way. According to B. Davis and Renert (2014, p. 87), “interpretive listening” is a stance in teaching in which the teacher tries to listen attentively to interpret or make sense of student thinking to figure out what students understand. It can thus be inferred that James tried to maintain a focus on the Key Point of the lesson, which is to interpret a fraction as part of a set, from the way he directed the discussion. In the example he used, there are at least two possible approaches to interpret the situation, which
corresponded to the two metarules in the preceding paragraph, and so, James’s responses to S2’s and S3’s answers provided opportunities for students to attend to the reasoning behind the answers. An analysis of James’s noticing of the Key Point is shown in Table 4.9.

<table>
<thead>
<tr>
<th>What was noticed</th>
<th>Processes of noticing</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Key Point</strong></td>
<td><strong>Attending to</strong></td>
</tr>
<tr>
<td>To see a collection of objects as a whole and to interpret a subset of the collection as a fraction of that set.</td>
<td>He attended to S3’s answer of 1/4 as another way to think about the same fraction (15, 17, 19).</td>
</tr>
</tbody>
</table>

### 4.3.2.2. Responding and reacting to student interactions during a critical incident

After the initial warm-up activity, James then went on to explain the task using 12 physical cubes. He used a colour configuration of 2 green, 4 blue, 3 red and 3 yellow. In the following interaction, James engaged Student S5, who was seen to be competent in mathematics, in an interesting conversation.

1. James: What fraction of my cubes is green? OK, [S5]?
3. James: 1 out of 6... 1 sixth. Let me shift it up a bit (James shifts the cubes on the table so that everyone can see on the projector). Anybody disagree with [S5]? He said it’s 1/6. Hey... [S6]? No? Do you agree or disagree with [S5]?
5. James: Don’t agree. Then what would be your answer then?
7. James: Ok. We have two answers here. 2 out of 12 and S5 said 1 out of 6. (Writes the fractions on the white board) Do you think they are related?
8. Students: [Chorus] Yes...
9. James: Ok. First, [S5]. Can you come and show us how you got 1 part out of 6 when there are so many cubes here. (S5 comes out and...
arranges the cubes.)

Ok. [S5], stay there... stay there. Where's your six parts? (S5 points to the cubes and counts 1, 2, 3, 4, 5, 6...)

And the green is what? 1 out of 6, is it?

10. S5: Yeah.
11. James: Then what about the remaining cubes?
12. S5: Still the same.
13. James: Still the same, ok? If I put it this way? (James puts the two groups of cubes together.)

Would you all be able to see the six parts?

14. Students: [Chorus] Yes...
15. James: Yes... So, [S5], where are the six parts? (S5 points to the cubes again, and shrugs his shoulders.) Ok. Can you imagine the imaginary lines between the cubes? OK. How can you have put this better? (S5 rearranges the cubes.)

How many parts can you see now? Anybody wants to give Gerald a hand?

Yes, [S7]. Ok. Thank you, [S5]. (S7 comes out to do another arrangement.) Mmm .. Something different from what [S5] did. (S7 rearranges the cubes to be 6 groups of 2. See Figure 5.)

Ok. Let's shift this a bit. Ok. Do you see 6 parts now?

16. Students: [Chorus] Yes...
17. James: A bit clearer?
18. Students: [Chorus] Yes...
19. James: Thank you, [S7]. I was asking for the fraction of...
20. Students: One out of six...
21. James: Green cubes right? So, it's one part out of...
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22. Students: Six.
23. James: Six parts. Same thing, yeah? Has my number of cubes changed?
24. Students: No...
25. James: So, actually, is [S6] right to say that it's actually 2 parts of 12 also?
27. James: Actually, he's correct also? But how did I get from 2/12 to 1/6?
28. Students: Divide... Simplify...
29. James: Yes... we could have simplified it, right? They are equivalent fractions, right?

In this episode, James attended to S5’s use of the visual mediator to reveal how S5 thought about the partitioning. (See Figure 4.11.)

Figure 4.11. S5’s first arrangement of cubes to illustrate 1/6.

He realised that S5’s idea of partition was different from what he had in mind. (See Figure 4.12.)

Figure 4.12. James's intended arrangement of 1/6.

James then tried to ask S5 some questions to understand what S5 was thinking with regard to the six groups (Lines 9 and 11). S5 seemed to have understood about the “six parts” and counted each cube (Line 9) in one of the rows he created in Figure 4.11. James could see that S5 understood that 1/6 of the total number of cubes in the first row is green (“And the green is what? 1 out of? 6, is it?”). S5’s answer of “still the same” in Line 12 indicated that he perceived the grouping as two equal groups of 6 cubes, with one green cube in each group or possibly a different partition. James’s intended metarule—that the two green cubes form one out of the six equal partitions—was thus different from S5’s. Therefore, James tried to get S5 to see the intended metarule by putting the two rows of cubes together (See Figure 4.13).
James’s question in Line 13 and 15 indicated he was trying to get the students to see the intended arrangement of the cubes. (See Figure 4.12.) His use of the cubes as a visual mediator to hint at the intended arrangement did not seem to convince S5 (Line 13). S5’s hesitation pointed to a possible confusion and that he did not attend to the same features (imaginary lines) as his teacher. This was evident from S5’s arrangement of the cubes (visual mediator) that did not show the six partitions clearly (See Figure 4.14).

Sensing that S5 might not have caught his intended metarule, James then asked another student S7 to do the arrangement. It appears that James noted and interpreted specifically what S5 was thinking with regard to the partitioning, but his response was limited in revealing explicitly what S5 was thinking. James tried to direct S5 to see the intended arrangement through a series of questions to funnel his thinking. This approach did not seem to work and S5 was confused at the end of this episode. It might have been better for James to ask S5 to explain his own reasoning for his arrangement, so that James could then make sense of what S5 was thinking (Burns, 2005; B. Davis & Renert, 2014). Hence, James did not respond in a way that could have enhanced students’ reasoning. His response during the interaction (Critical Point) did not help S5 to overcome his difficulty of seeing the partition, and James missed an opportunity to find out what S5 was thinking. Therefore, his noticing would be classified non-productive even though his attention was focused and interpretation might have been accurate. It appears that James did not have, at his disposal, other ways of responding when students did not use the key words in their explanation. A summary of James’s noticing during this Vignette is shown in Table 4.10.
### Table 4.10: Analysis of James’s noticing of students’ thinking in Section 4.3.2.2

<table>
<thead>
<tr>
<th>What was noticed</th>
<th>Processes of noticing</th>
<th>Deciding to respond</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Attending to</strong></td>
<td><strong>Making Sense of</strong></td>
<td><strong>To see a collection of objects as a whole (unitising) and to interpret a subset of the collection as a fraction of that set (partitioning).</strong></td>
</tr>
<tr>
<td><strong>Key Point</strong></td>
<td>James figured that S5 might have got 1/6 from other methods because S5’s arrangement represents 1/6 but not the partitioning that he was looking for (9, 10, 11, 12).</td>
<td>He tried to direct S5’s attention to the “six parts” in order to make the partitioning clearer (9).</td>
</tr>
<tr>
<td><strong>Difficult Point</strong></td>
<td>He sensed that S5 might not be able to see the “six parts” (15).</td>
<td>He asked S5 to rearrange the cubes after he asked Gerald to imagine the lines between the cubes (15).</td>
</tr>
<tr>
<td>Students cannot extend the notion of a fraction beyond “part of a whole” and may not be able to see a subset of discrete objects as a partition of the whole.</td>
<td>S5 gave an arrangement that did not show the partitioning clearly (15).</td>
<td>He asked S7 to show the partitioning (15).</td>
</tr>
</tbody>
</table>

Moreover, James did not use the opportunity to discuss the different partitions after S7 showed the intended arrangement. Instead, he launched an IRE sequence to direct his students’ attention on simplifying the fraction 2/12 to 1/6 (Line 25 to 27). Students seemed to catch the routine (counting, writing it as a fraction and then simplify the fraction) correctly (Line 28). This placed emphasis on students getting the correct answer, and might have contributed to his students’ tendencies to stick to a routine or procedure without understanding. Therefore, James shifted from a more interpretative stance of listening to a more evaluative stance, and did not attend to student reasoning during this segment of interaction with students. Subsequent observations revealed that students counted the cubes of interest, and then expressed it as a fraction of the total before simplifying it to the lowest terms. As James did not go through the reasoning of the partitioning with the students, they might not see the connection between the symbolic representation and the grouping of the cubes. Furthermore, the use of the model method discussed during the Lesson Study sessions was not used here to help students make the connection (The model method connection was used the day before). As a result, students did not pay attention to the partitioning and thus, it was unclear whether they really understood how the notion of a fraction was extended to express the relationship between a subset and the total collection of discrete objects.
This episode was considered a critical or crucial incident during the lesson (Goodell, 2006; Yang & Ricks, 2012) because it highlighted opportunities to explore connections in mathematics, student learning and teacher pedagogy. Such incidents have potential to deepen teachers’ understanding of students’ mathematical thinking (Goodell, 2006; Yang & Ricks, 2012). These events usually involve students’ unexpected responses to teachers’ questions (Yang & Ricks, 2012); episodes that raise questions about teaching approaches or students’ understanding (Goodell, 2006); or interactions that change the direction of the lesson from what was planned (Clea Fernandez et al., 2003).

As illustrated in Vignette 5, the ability to see and interpret these events in-the-moment can impact how teachers decide to respond to these events. It seemed that the key to respond productively to enhance student reasoning lies in the ability of the teacher to adopt a more interpretive stance in listening, and allow students’ responses or answers to modify the flow of the lesson (B. Davis & Renert, 2014). Moreover, the teacher has to think-on-the-spot to attend selectively to the myriad of responses from the students. The ‘Three Point Framework’, particularly the Key Point and Difficult Point, continues to be a useful frame to see and understand student responses. The findings from these two vignettes and other similar episodes in the three schools suggest a need to see and interpret student interactions, in the form of their questions and answers, through the ‘Three Points’. By being more sensitive to students’ Difficult Point, and maintaining a focus on the Key Point, teachers might be able to raise their own awareness of how they listen to students’ responses that make aspects of their thinking visible. In so doing, they might have a better chance at generating a Critical Point or a decision to respond that enhances students’ mathematical thinking.

4.3.3. Vignette 6: “Simplifying over partitioning”

4.3.3.1. Maintaining the focus on the Three Points

Using the ‘Three-Point Framework’ in the modified Lesson Study protocol, James identified the most crucial learning point, which dominated the post-lesson discussion:

Maybe I share what I observed from my point of view? The glaring thing that I noticed about my pupils is that too many of them, they didn’t get their fraction by partitioning .. they got it more by counting and then simplifying... so that was the easy option to them. Which was why later when I got them to explain
“How did you get this fraction for example?”... “one sixth of the cubes were red” or something like that. Some of them were not able to show the six parts or to group the objects into six parts. So they were a bit lost. Because how they did it was, count the number of red cubes over the total number of cubes, then simplify. When they cannot put it in parts, right ... it was very clear what their thought process was – simplify ...

James highlighted that the students “didn’t get their fraction by partitioning”, but instead by “counting and then simplifying”. He explained how that prompted him to try asking students to reason how they arrived at the fraction (See Vignette 5). James was able to give very specific details about students’ difficulty in showing the partitioning of the cubes (“they were not able to show the six parts...”), and interpreted that as a manifestation of their “thought processes”. James’s noticing was not only specific and focused on the ‘Three Points’, but also more importantly, it set the stage for the teachers to learn about another possible Difficult Point not previously discussed. Moreover, James supported his claim about students’ thinking with reasoning based on the observations he noted during the lesson. This is a consistent pattern that seems to accompany instances of productive noticing throughout the post-lesson discussion. For instance, James went on to describe how a girl did not see the number of cubes in reference to the whole collection:

One girl was saying something about the red cubes can be halves or something like that. So what she meant was she can put the red cubes, let’s say into two groups and they look like halves – but it was not in reference to the whole set of cubes. There was a bit of confusion or the link wasn’t there lah. To me it was a group activity, which meant that the rest of the group agreed, that’s why they wrote down that statement. Whatever idea it was, the whole group seemed to agree. So whatever explanation they had, we assume that the whole group accepted it. So when they explained it... it wasn’t in reference to the whole set – it was just with reference to that colour – for example there were eight red cubes. So they just took the eight red cubes and divided them in two groups – they see halves. That’s what they saw and that’s what they wrote.

Here, James made a claim that some students might not make fraction statements “in reference to the whole set of cubes”, and substantiated his claim with a specific observation that is linked to one of the Key Points. This justification helps to strengthen the “validity” of what was noticed. According to Mason (2002), noticing is validated
when what was noticed makes an impression with one’s experience, and is interpreted in light of this experience to effect changes in the future. Raising awareness to act differently and recognising similar instances by others are indicators of validity in noticing. Therefore, James’s reasoning based on what he saw gave other teachers some insight into what he noticed during the lesson itself. This insight has the potential to heighten other teachers’ awareness of what James noticed, and may lead teachers to think of other possible responses to students’ questions and answers.

In contrast to Zelina’s noticing during the post-lesson discussion (See Section 4.2.3.1), James’s noticing was more focused and the connections between what he noted in relation to the Three Points were more explicit. He was also able to identify the key incidents during the lesson and reasoned about them. James pointed out the essence of these critical incidents (See Section 4.2.3.1) and summarised the important observations with regard to the Three Points.

Some did not simplify, but they got the idea of taking a certain number of objects out of the whole set. At least I think that’s the idea that we wanted to see, not so much the simplifying. Because once they simplify, it’s actually the answer that we want from partitioning. They didn’t do partitioning... actually if they simplify... it doesn’t show the partitioning part of our lesson... unitising, partitioning and that a group of objects can be many things. So, I feel that my class, most of them, they don’t .... The concept of partitioning was not very strong for them. They are more in a sense... technical ... abstract... abstract, they saw the numbers first and simplifying instead of ... visual and all that... they see the parts. They went straight into the fraction itself.

The Key Point of the lesson was to extend the notion of fractions to a collection of discrete objects, and this was linked to two concepts: A collection of discrete objects can be considered as a whole (unitising); and the relationship between two sets can be expressed as a fraction using the idea of equal partitioning. Clearly, James realised that most students (except for “one girl” and her group) were able to see a collection of discrete objects as a whole (“They got the idea of taking a certain number of objects out of a whole set...” and “a group of objects can be many things”), but were less adept at the link between equal partitioning and fractions.
In order for the post-lesson discussion to be fruitful, the points raised should help teachers to refine ideas about students’ thinking or lesson design (See Section 2.2.3). They should go beyond vague or broad statements to focus on supporting or refuting claims made by teachers on students’ learning. Moreover, the discussions can move towards a more generative position when these claims are supported or refuted based on teachers’ observations of specific instances. As can be seen from the rest of the discussion, teachers are generally focused on the ‘Three Points’ related to this critical incident because James’s noticing had set the stage for a productive discussion. A summary of James’s noticing is shown in Table 4.11.

Particularly, James’s noticing of the Difficult Point in Table 4.11 showed that he was aware of student thinking during the critical incident (See Section 4.2.3.1), even though he might not have responded in a way that promoted student reasoning. During the post-lesson discussion, James attended specifically to what his students could and could not do. He claimed that students did not understand the partitioning aspect of the task and hypothesised possible reasons for his students’ difficulty. His noticing can be classified as productive, in this case, because he used what he observed to support his claim. By highlighting the specific incidents he was referring to, James was able to direct other teachers’ noticing on similar issues.
4.3.3.2. Analysing the observations

As a result of James’s productive noticing, the other teachers also seemed to focus their attention more on the students’ difficulty in demonstrating the partitioning of the set according to the fraction. An interesting conversation then revolved around whether the partitioning process is essential for understanding fractions:

1. James Mmmm... I think it’s good to learn the partitioning part. But the most direct way actually to them is... I think... I take the number and put it over the total objects, which is the idea of a set and get the answer. To them they don’t see there’s a need to partition it. I can just simplify. I just take a number, a big number and simplify... I still get the same answer. Perhaps they didn’t see the need to actually physically partition it... even though the previous lesson we did that. Because this class... they are more practical in a sense, so “if I can get answer like that, then why do I need to do that?” You know. So what I feel is that it’s good they learn the partitioning because certain questions require them to see partitions. I don’t know...

2. Flora Actually, what you said got me thinking... because many of the kids in that class I realise, we were forcing them to go into reverse because they were already at that simplifying. We were forcing them to reverse, go backwards, and play with the cubes and see them in parts. So I kind of question myself: is it necessary to bring them backwards, you know, to really see the cubes and partition them when they already know what it means. They already have, like for example, one group... didn't need the whole set of cubes all the time – they knew 24... that was the total and that was it. And so red, they just counted the red, they don't even bother looking at the rest, they just simplify it immediately. Is it necessary to get them to go a couple steps back that far, when they're already there? They know it is a fraction of a set. It's like we are forcing them backwards. So, I don't know.

[After some discussion, James added:]

3. James: I think ...towards the end of the activity, most of them were not playing with the cubes anyway, they are just sitting there thinking about what else they can do with the numbers ...instead of the moving around part. They were making other things... nothing to do with finding a fraction, but they are playing a fool with the cubes, making shapes and all that... Actually, all they needed was a table and that they could make a fraction out of the table. So, what I’m thinking is... Would this simplifying skill and idea be a more advanced idea than the partitioning part? If so, then we are actually moving them back? They already had the idea of the simplifying part from the previous lesson.

4. Flora Some kids even got confused. When you put it in front, right, they are like... what am I going to do what? With the cubes?

5. Rani I agree with what Flora said, because the group I observed... the only thing they had to use the cubes, to handle the cubes is... because they have to fill in the table... they just count and then they put them aside – oh I can simplify simplify... rather than actually put together and do the partitioning. And even the kid
knows. One of them actually is so bored he just put his head on the table. He already got the answer.

6. James  
He already got the idea... I already learned what I need to learn.

7. Cindy  
For the group that I observed. [S8]'s group... he was the one doing all the manipulatives. The two girls had no chance to play with it – but they could actually put them in groups. Very fast, they put the colours in separate groups. On their own, they worked at their own table separately. S8 was the only that was fiddling around. He was the only one going deeper into it. 'How can I put the groups of six... oh, divide by eight, multiples of eight', then he wanted to do it his way. There was some friction in that sense. But like I said they actually knew how to simplify – they didn't refer to the cubes and all. He was playing his own little game.

8. Flora  
They have the total in mind already... My question is – is it necessary to, you know?

9. James  
So the idea of the unitising part is there already. That's the number I need to use. In this sense, the progression of the lesson, right?

10. Me  
What do you all think?

11. James  
Because I think our idea of this part of the lesson was to help them to understand it better, right? The fraction of a set... unitising... partitioning. Perhaps for this class when we did this activity, most of them already had the idea of the unitising, and then the simplifying, so the partitioning part wasn't that important to them anymore. Because they were able to get answer via... without doing partitioning.

12. Cindy  
They are doing the procedural part... in other words, rather than actually [incomplete sentence]...

13. Kirsty  
Actually, I... I don't know... What I saw was something – the group that I observed, all right, to me they didn't get actually really the concept of unitising, partitioning, fraction of the set, really to be really honest: I was attached to a group that had set B. Why I said that is because one boy went even so far as because there was four colours given, he said one quarter of the cubes are blue, when there was eight blue cubes given... where there was 8 blue cubes given... Set B has 8 blue cubes given and He said one quarter are blue. It was later on when his friends actually corrected him – that's when he realised... all he did was partition by colour.

Which I found that, to some extent, I could say that... okay... maybe the majority of your class could possibly have understood... you know... this concept of unitising and partitioning, but I wouldn't say all did. But I wouldn't say that all of them – let's say if they are told to apply it... whether they'll be able to do so very convincingly and very successfully. Because to me... not everyone got the concept. That's the feeling that I got. I did a general walk around, I mean more towards where I was sitting. I think the danger is that they really went by colour. And they couldn't see that this whole 24 is... actually that whole collection we were talking about. That's the way I see it, they really went according to just by... basically... a lot by colours.
In this episode, James while thinking from the perspective of a student (Lines 1 and 3), realised that students could bypass the partitioning and still get the required answers (Lines 1 and 11). His observation prompted Flora to wonder whether it was necessary for the students to show the partition (Line 2). Flora referred to a particular group that she observed (Line 2), and noted that the students just counted and simplified because they knew that there were 24 cubes. However, she pointed out students were confused when asked to display their partitioning. Similarly, Rani suggested that the students just needed the cubes to figure out the total number of cubes and the number of cubes of each colour (Line 5) and substantiated her claim by referring to a particular student who was bored. Cindy also highlighted how she noticed that the girls were able to work out the answers from “their own tables” without the cubes because they knew how to simplify (Line 7). This, as she argued, might have indicated that students were just focused on “doing the procedural part” (Line 12).

### Table 4.12: Analysis of teachers’ noticing during post lesson discussion

<table>
<thead>
<tr>
<th>Difficult Point</th>
<th>Processes of noticing</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Attending to</strong></td>
<td><strong>Making Sense of</strong></td>
</tr>
<tr>
<td>Students cannot extend the notion of a fraction beyond “part of a whole” and may not be able to see a subset of discrete objects as a partition of the whole.</td>
<td>Teachers realised that students did not use partitioning to obtain their answers (4, 7, 9).</td>
</tr>
<tr>
<td>Flora noted that students were confused about partitioning (6).</td>
<td>Cindy thought that students were just focusing on the procedural aspect (14).</td>
</tr>
<tr>
<td>Kirsty saw students partitioning by the number of colours (15).</td>
<td>Kirsty thought students might look at number of colours as the whole (15).</td>
</tr>
</tbody>
</table>

Whereas most of the other teachers focused mainly on the partitioning part of the task, Kirsty also noted that some students might not have got the “concept of unitising and partitioning”. She highlighted that students might be looking at cubes of the same colour as the “whole”. Kirsty, like the other teachers, was able to focus on the details related to the ‘Three Points’. In particular, she was able to link what she saw (“...one boy went even so far as because there was four colours given, he said one quarter of the cubes are...” with...
blue.”) to the possibility that the boy was “partitioning by colour”. Based on her observation and interpretation, she expressed her doubts that all students had understood the Key Point, even the possibility that students might not see the whole collection of 24 cubes as the whole. Instead, students might be thinking that the number of colours formed the whole (“I think the danger is that they really went by colour. And they couldn't see that this whole 24 is ... actually that whole collection we were talking about.”). Kirsty’s points were more convincing because she linked her claims directly to what she noticed. A summary of what teachers noticed about students’ thinking is given in Table 4.12.

Even though the teachers did not decide how to respond as seen in Table 4.12, they suggested different possible interpretations, which had the potential to generate new understanding of how students think. The teachers attended to specific instances, and made connections between their observations, and that of others to their own knowledge and experience. The process of detailed interpretation further encouraged teachers to look at these observations at a deeper level. For example, Kirsty and Rani then linked a seemingly insignificant observation that some students hesitated with the fraction $\frac{5}{24}$ to the possibility that they relied primarily on simplification to get the answers:

14. Kirsty ... But really, you know... because like the five green... you notice? The five green wasn’t used in tandem. Because they didn’t know how to use the five green – I mean the one the set I had, you noticed the odd number... they didn’t know how to use it. They never used it actually. Some did I would think and some didn’t.

15. Rani They didn’t. For some reason, they did not want to put 5 out of 24 right? (Kirsty agrees.) They did not what to use the denominator 24. They wanted to simplify.

16. Kirsty They just simply ... just go and simplify

When teachers were able to connect what they saw to the ‘Three Points’, there was a higher likelihood of generating appropriate modifications to the lesson design. For example, when Kirsty focused on students’ difficulty in unitising and partitioning, she highlighted that it might be good to reinforce students’ understanding by the consistent use of the model method discussed throughout the lesson design. The model method (See Section 4.3.1.2) was used in the previous lesson but was largely left out during the lesson that teachers observed. While the teachers might have overlooked the
importance of maintaining the use of the models throughout the unit, they were able to think about the possibility of emphasising the connection with the model for future iterations of the lesson.

4.3.3.3. Analysing observations by linking to experiences

Productive noticing can also be initiated based on recalling or accounting of (Mason, 2002) specific instances from past experience. For example, Anthony highlighted that it is important for students to know partitioning by drawing from what he observed about students’ understanding (Key Point and Difficult Point) in the past:

17. Anthony Another thing... I didn’t go through the worksheet. I’m thinking of, if...from P4, P5 and P6 maybe when you give them 24, anything about 24, if I put it into 24 colours – can I group them in one single or I can further put it in one pack of two cubes... so must make some planning so in the end when they have red, blue, green or yellow... they have maybe this one two... this one four... this one eight.... At the end, they can say... if I put them separately, they have 2... if I put into what they called... cards... 1... so they can have 1 cards compared to 2 cards... But actually the number of cards and the number of equal parts doesn’t mean the actual quantity. It might helpful when they go to ratio. When they go from A to B. It doesn’t mean that A has one unit. One equal part represents many other quantities. So if you can build a strong base in P4 ... when they go further... it will help.

18. James I think that will be where the partitioning will come in...

19. Anthony If I give them 24... I just plan – red four, blue six and the other one about 8 or 10. Afterward I can do some even number of packing. Maybe I can give them some example of their science assessment... when you collect leaves or flowers you put them together. Two in the cards or three in the cards. When we talk about how many equal parts, 3 leaves in the ... card ... But it doesn’t mean that one card means exactly one leaf. It sometimes means three or sometimes four.

20. James That one was done in the previous lesson

21. Anthony It may be help them to have a further think about when I do a fraction, I’m talking about equal parts... make equal parts. So further emphasise unitising and partitioning skill – it will be better. Now when we go to P5 ratio, some children they say... teacher – 2 : 3... why 2 unit, I can give them 12 kg? We need to have a long discussion. Two units means they have same equal part. 2 units doesn’t mean 2 kilogram...may be one unit is 5kg. So when they are in the same quantity, they can compare.

22. James So a problem that this class might have with P5 ratio when they cannot partition equally... Uh... cannot simplify already...

In this excerpt, Anthony argued that it is important for students to know how to unitise
and partition. Although he did not attend the lesson observation, Anthony tried to connect what he heard during the post-lesson discussion with his own teaching experiences (Line 21). He reiterated that it is common for students to have the misconception that a fractional unit corresponds to only one item or part or quantity (Line 21). This was the Difficult Point identified during the planning discussion in Session 1 (See Section 4.3.1.2). Anthony underlined the important notion that a fractional unit can refer to more than one discrete object (Lines 19 and 21). In a way, Anthony highlighted that it is critical for students to link the idea of partitioning with fractions. His comment prompted James to realise that students who relied solely on simplification to get the correct answers might experience difficulties dealing with ratio in Primary Five (P5). Therefore, because of Anthony’s comments, the teachers could come to realise that students might not have understood the concept even if they could give the correct answers. Hence, students’ inability to show the partitioning could indicate a gap in their reasoning about fractions, and it is thus important to listen to their reasoning. This realisation stimulated a series of recommendations for modifying the lesson design:

23. Kirsty: Besides crafting out the sentences, would getting them to explain help? how they actually .. I mean not too tedious of course, but that would be important in actually capturing their understanding. Because It’s like I said, if you craft any of their sentences they are all perfectly correctly, but it’s the thinking behind the sentences. Whether they have...

24. Researcher: So that their explanation is maybe more critical than the answer itself? (the team agrees.)

25. Flora: Like how do you view two-thirds? Why two-thirds? What does that two-thirds represent? Then we can just give them a blank space and get them to draw it.

26. James: We can’t even ask them how they get two-thirds. They will answer… I simplify … it has to be more directed question … It needs to be a more specific questions – getting them to explain why is it three parts: two parts out of three parts.

27. Flora: Getting them to illustrate it. Maybe some children will actually draw a model to illustrate it. Some will draw all the cubes and circle it. I don’t know… I guess… I seen it in some math journals… how how kids express and explain – you get to actually see their thought process, and what they understand about 2/3.

28. James I was just thinking the danger of – during the design of this lesson, we didn’t see that maybe they may skip the partitioning part of it… they didn’t show how the answer is found. It is something we need to recognise. It is good that we now know that if they missed the partitioning part… this may cause a problem later. Missing the
partitioning part will be fine until we show them they have a problem. Even though they can do a fraction of a set, and they can solve fraction of a set problems – it will pose learning problems in future when they move on.

29. Flora  It may seem trivial.

30. James  Mmm. Just now you asked whether it is important that they learn partitioning. At this point, no. Because they have what they need to get answers, but in terms of further learning and forming the foundation for further learning, I think we need to look at it more carefully.

Based on Anthony’s interpretation, Kirsty and Flora suggested that it is important to “capture their understanding” (Lines 23 and 27); and James became cognisant of the need to ask specific questions that encouraged students to articulate their thinking through their explanations (Line 26). Moreover, these were indications that the teachers were beginning to understand the need for mathematical reasoning, and become more aware of how the concepts ought to be built up over the different levels. James stressed that even though “students could solve fraction-of-a-set problems”, the bypassing of the partitioning could pose problems in the future, and articulated the need to look at this issue more carefully (Line 30).

Through these discussions, teachers came to understand more deeply a Difficult Point not previously discussed (Line 28) during the design of the lesson, and suggested possible ways to deal with the issue. Therefore, in some sense, this noticing had triggered “resonance” with teachers’ collective experiences during the Lesson Study, and enhanced their own “sensitivities to notice” in order to “inform” and improve their practices (Mason, 2002, p. 62).

4.3.4. Discussion of Phase 2 vignettes: The central role of pedagogical reasoning

The findings from Phase 2 of the study strengthen the case for using an explicit focus, such as the Three Points, to guide teachers in examining mathematically worthwhile aspects during lesson planning, lesson observation and lesson reviewing. As illustrated in the vignettes, the two aspects of an explicit focus—the Three Points and the alignment between the Three Points—are both critical for teacher noticing to be productive. For example, Anthony attended to the Three points specifically, but his chosen course of action (the use of an analogy) did not align with the confusion he raised (See Section 4.3.1.1). Similarly, while James attended to his students’ thinking
during the lesson to some extent (See Section 4.3.2 and his reflection in Section 4.3.3.1), he could not respond with a course of action that was aligned to the difficulty faced by students. On the other hand, when teachers attended to both the Three Points, and the alignment between the Three Points, they are more likely to generate a productive instructional decision. James’s noticing during lesson planning is a good example of how this dual focus enabled the teachers to think about the design of the task (See Section 4.3.3). However, this alignment is not automatic even when teachers can focus on the specific details of Three Points.

The aligning of the Three Points depends on how teachers connect their responses (instructional decisions or Critical Point) to what they see or attend to. The findings from Phases 1 and 2 indicates that teacher pedagogical reasoning or analysing is a critical component skill in noticing, and may be a possible mechanism behind this alignment. For example, when James suggested that students’ image of $1/4$ may hinder their understanding of a fraction of a set (See Section 4.3.1.2), it was his justification that linked students’ limited notion of a fraction to what he observed about their thinking. Moreover, his proposed use of the model method (See Section 4.3.1.2) was also motivated by his reasoning.

This reasoning process was made explicit when teachers based their instructional decisions on the interpretation of what they attended to. A good example of this can be found in Vignette 6 (See Section 4.3.3.2) where the teachers discussed the role of unitising and partitioning in the teaching of fractions. Following James’s observation and analysis of how students might have bypassed partitioning, each teacher in the Lesson Study group was able to add to the discussion by giving specific examples of what they observed, and linked that observation to what they hypothesised about student thinking. They focused their attention on the concept (Fraction of a set, the notion of a whole, and equal partitioning), and the mistakes committed by the students (e.g., taking the number of colours as the whole etc.), in order to make some claims about student reasoning or understanding of the concept. As Anthony had demonstrated, this reasoning can also be based on prior experiences evoked as a result of the ongoing discussions (See Section 4.3.3.3). It is through the process of validating (Mason, 2002) one another’s observations and claims that the teachers were able to
come to a consensus on the role of partitioning. This consensus came about as a result of examining the evidence and various interpretations of what was observed.

Thinking about different interpretations is important for promoting the feasibility of a teacher’s course of action. For example, Anthony’s suggested use of analogy could have been avoided if he had considered other possible explanations for students’ mistakes beyond the need to change fractions to common denominators. If he had thought about the possibility of students making a similar error as a result of misinterpreting the area model (See Figure 4.8), the teachers would have had a better understanding of the Difficult Point, and perhaps suggested another possible course of action.

Besides aligning the ‘Three Points’, it is teachers’ reasoning that makes their thinking or sense-making processes of noticing visible. Although it was difficult to know directly what James saw, and how he interpreted classroom events during the lesson; his thinking was revealed when he began to reason about his decisions and what he observed about the Three Points. Therefore, James’s less productive responses to Gerald could then be explained in terms of his ability to make sense of Gerald’s thinking in-the-moment (See Section 4.3.2.2). This is a challenging task, and requires a heightened sense of situation awareness (Miller, 2011). To respond productively to student reasoning, a teacher has to perceive mathematically worthwhile aspects in a student’s response; make sense of them; and draw upon his resources to come up with an appropriate response. This insight, a hallmark of giftedness, is dependent on the processes of “selective encoding, selective comparison and selective combination” (Sternberg & Davidson, 1983, pp. 53-54). Similar to how gifted professionals such as doctors and lawyers work, an effective teacher who wants to orchestrate a productive discussion has to sift through a huge amount of information to differentiate the relevant from the irrelevant; compare and relate this relevant information with prior knowledge or experience; and combine this information in a meaningful way to make a response to student thinking. This rapid reasoning process can be promoted with the use of an explicit focus, and advance preparation (Mason, 2002).

Drawing on the photographic metaphor introduced in Section 4.2.4.2, this reasoning process undertaken by a teacher can be likened to a photographer who actively adjusts his perspective, and distance between the camera and the subject, in order to bring the
desired focal points in alignment to achieve the bokeh (selective focus). The reasoning process of a teacher during teaching can thus be enhanced if the teacher actively keeps in mind the issue of alignment as he focuses on the Three Points. As Lindy had demonstrated, her active sense-making of the students’ thinking when she orchestrated the discussion was enhanced by keeping in mind the different examples created during the planning (See Section 4.2.4.1).

Building on the tentative framework developed in Section 4.2.4.2, the framework has been revised to emphasise the central role of reasoning by embedding the Three Points and its alignment within the making-sense component of noticing. Moreover, the revised framework highlights the importance of the focal points and their alignment during the interpretation process of noticing. This revised framework, as shown in Figure 4.15, was used in Phase 3 of the study.
4.4. The FOCUS Framework for Productive Noticing

The two characteristics of productive noticing—the focus and focusing—continue to feature prominently and are further strengthened by the findings in Phase 3. A close-up view of teacher noticing during the lesson cycle reveals two essential characteristics of productive noticing, which form the basis of the FOCUS framework for productive noticing:

1. An explicit focus: The Three Points, and its alignment;
2. Focusing: The active process of pedagogical reasoning that aligns the instructional decisions to the observations made.

Despite the teachers’ positive reception towards the Three Points during the study, the language of the Three Points may present a few challenges in describing and analysing
teacher noticing. For example, the teacher may decide to ask students questions to reveal their thinking, which is considered a productive decision. However, this may not fit well into the definition of Critical Point, which has more to do with the overall approach of a lesson (Yang & Ricks, 2012). To better reflect the responding component of noticing during planning, teaching or reviewing, the term 'Course of Action' was used in place of the Critical Point. This change highlighted the different types of responses taken by teachers during the different tasks of Lesson Study. Therefore, the course of action could now be directly related to the productive practices listed in Section 2.2 (See Figure 4.16).

In addition, the Key Point was replaced by the term Concept, which refers to the mathematical ideas discussed in the segment or teaching episode. This change highlighted the possibility that there can be more than one key idea in a given episode, and that the Key Point may not remain the same throughout the lesson, as in the case of how the Key Point was used by Yang and Ricks (2012). Finally, the Difficult Point was changed to Confusion to highlight the wider range of cognitive difficulties faced by students. Despite the change in terminology, the alignment between these three points continued to be the focus for productive noticing. That is, the course of action, or instructional decisions taken by the teacher should be directly related to the students’ confusion about the concept encountered during the planning, teaching, and reviewing of the lesson. In summary, these three crucial focal points for productive noticing are as follow:

1. Concept: The mathematical idea(s), themes, or construct that is(are) of interest in the discussion or teaching episode;
2. Confusion: The mathematical difficulty, cognitive obstacles, errors, misconceptions, or uncertainties demonstrated by students;
3. Course of action: The instructional decision or response made by the teachers during the planning, teaching, and reviewing of the lesson.

A schematic representation of the FOCUS Framework for productive noticing is given in Figure 4.16.
4.4.1. Theoretical models of productive noticing

The FOCUS Framework for productive noticing relates to what, and how, teachers notice when they make productive instructional decisions. Drawing mainly from the analyses of the productive episodes of discussion, I identify particular elements in what teachers see, think, and respond, to develop a model of their noticing for each of the three main productive practices (See Section 2.2, and Figure 4.16) using the FOCUS Framework. The overall theoretical model for the three main stages of learning from practice (planning, teaching, and reviewing) is shown in Figure 4.17.
Chapter 4 – The FOCUS Framework for Productive Noticing

Figure 4.17. A theoretical model to describe productive teacher noticing.
This theoretical model from the FOCUS Framework describes what, and how, a teacher can notice productively when learning from practice. It maps a teacher's noticing processes (attending, making sense, and responding) through three stages of learning from practice (planning, teaching, and reviewing) to the three key productive practices for mathematical reasoning (designing lesson to reveal thinking; listening and responding to student thinking; and analysing student thinking). In other words, the model describes a theoretical process of productive noticing, which highlights explicitly the three crucial focal points, and how the alignment between these three points can be achieved. For example, referring to the planning portion of Figure 4.17, a teacher, who notices productively, can step through the following during the planning of a lesson:

1. Identify specifics of the mathematical concept(s) for the lesson;
2. Recognise what students find difficult or confusing about the concept;
3. Analyse why students find the concept difficult or confusing;
4. Analyse possible ways to address students' confusion when learning the concept;
5. Develop and implement a high-level cognitive demand task (Smith & Stein, 1998) to target students' confusion about a concept.

Steps 1 and 2 can broadly be subsumed under the attending component of noticing; steps 3 and 4 under making sense; and step 5 under responding. Therefore, this model provides a wide-angle view of a teacher's noticing or a group of teachers' noticing through the lesson cycle, and a close-up view of noticing at each of the stages. To illustrate how a teacher's noticing can be modelled, I will present James's noticing during the planning of a lesson (See Section 4.3.1.2) using the model.

As seen in Section 4.3.1.2, James identified the concept as a fraction of a set, and highlighted the extension of the notion of a whole to include a collection or set of objects. He realised that students may have a limited conception of a fraction as “part of a whole” (See Section 4.3.1.2) in terms of cutting a whole object into equal parts, which hinder this extension to include a whole as a collection of “whole objects”. James suggested that it would be important to use objects that cannot be cut, and provide opportunities for students to do the partitioning. The idea for the lesson was to emphasise the two processes: unitising (taking a collection of objects as a whole) and partitioning (dividing the collection into equal parts). He suggested that this can be
done using the model method representation, and a task that required students to form fraction statements with explanation of how they partition a set of coloured cubes. Figure 4.18 shows how James’s noticing can be modelled.

Figure 4.18. A model of James’s noticing in Vignette 4.

Referring to Figure 4.18, brief details of what and how James noticed are captured in the boxes corresponding to the three component skills of noticing. Moreover, some key words and ideas are coded in green, red, or blue, according to how these words are
related to the concept, confusion, and course of action respectively during the analysed episode. Figure 4.18, which models James’s noticing, thus reflects a model of productive noticing because of all three crucial focal points, and their alignment were noticed by James. This is represented by the unbroken arrows that connect all the boxes depicting details of James’s noticing.

On the other hand, the lack of attention to the three focal points, and the misalignment between the three points can also be represented using the model when noticing is not productive. This can be represented by the broken-lined boxes; broken arrows; and the missing colours (green, red, or blue) to signify a lapse in terms of the focus and alignment of productive noticing. For example, the teachers’ non-productive noticing during the song episode, as highlighted in Section 4.2.3.2, is represented in Figure 4.19.

4.4.2. The FOCUS Framework: A central result of this research

The aim of this chapter is to offer the FOCUS framework for productive noticing as a product to guide and study noticing. The framework highlights the two key dimensions to promote productive noticing: the need for an explicit focus (the three focal points and its alignment), and the central role of reasoning to ensure the alignment between these three focal points. These two dimensions were developed from the analyses of the potentially productive segments during the discussions (See Section 4.1); and presented through two case studies involving teachers from Greenhill Primary School in Sections 4.2 and 4.3.

Furthermore, a theoretical model, which describes the processes of productive noticing, was developed from the FOCUS framework to describe and analyse teachers’ noticing from two perspectives: A wide-angle view and a close-up view. The wide-angle view provides a portrayal of teachers’ noticing through the three stages of learning from practice, in relation to the productive practices for mathematical reasoning; whereas a close-up view can provide a lens to examine teacher noticing at each stage of learning from practice. The FOCUS framework, together with the model for productive noticing, therefore positions productive noticing as a critical process in supporting teachers in the systematic investigation of their own practice.
Figure 4.19. A model of teachers' noticing for the song episode in Vignette 3.
Chapter 5. Two Portraits of Teacher Noticing: Anita and Cindy

This chapter develops the *portraits of noticing* of two contrasting teachers, Anita and Cindy, using the FOCUS framework for productive noticing. Portraits of noticing, which consist of a series of snapshots depicting *what* teachers see, and *how* they interpret these observations to *make* instructional decisions, present a new perspective on teacher noticing. This new notion is different from the levels of noticing expertise developed by van Es (2011), and does not give a static score to assess teachers’ ability to notice (Jacobs, Lamb, et al., 2011). Instead, these snapshots capture the flow of teachers’ noticing processes as they learn from their observations of teaching through each stage of the lesson cycle. When put together, these snapshots form a portrait of noticing, which reflect a teacher’s (or a group of teachers) underlying resources, orientations, and goals, to illustrate the developmental trajectories of their noticing expertise. Examining different portraits of noticing can thus provide a more dynamic and comprehensive way to characterise teacher noticing.

I present two case studies detailing Anita and Cindy’s noticing, respectively. The case studies begin by describing Anita and Cindy in terms of their resources, orientations, and goals for teaching mathematics. Analyses of representative vignettes developed from Phase 3 of the study are presented from two perspectives. Firstly, a close-up snapshot of what, and how, a teacher noticed at each stage of the lesson cycle; and a wide-angle view of the teacher’s noticing through the whole lesson cycle. The chapter then concludes with a comparison of Anita’s and Cindy’s noticing to demonstrate how the framework might be used to characterise differences in teacher noticing.

5.1. Anita: A portrayal of non-productive noticing

Anita’s portrait of noticing is presented through a vignette developed from Phase 3 of the study. The vignette details how Anita’s attention on the three focal points, and how a lack of focusing on her noticing had often led to a misalignment between these three points.
5.1.1. Who is Anita? Her resources (R), orientations (O) and goals (G)

Anita revealed her orientations (O) in terms of teaching, mathematics, and students during an interview conducted at the beginning of the study (all quotations below were taken from Anita’s interview). An important belief, which crossed over these three areas, was her perceived dichotomy between “high and low capability students”. Two distinct orientations were:

O1: “High capability” students can be given more opportunities to explore Mathematics:

It will depend on the capability of the students. Of course for the better capability students, they can do more things. For the less capability students, of course they can do less [sic] things. For the more capable ones, I guess you can use GSP [Geometer’s Sketchpad], you can use certain things; they can go and explore to find the formula themselves.

O2: “Less capability” students should focus more on the basics and do simpler tasks:

But for the general students, like my school, they can go back to the basic things... basic things like the simple stuff... Simple tasks to get their attention... or something that can make them feel that they have achieved something... then you just touch on the basics. To be frank, O Level [a national examination] is not very difficult.

Anita’s goals (G) for teaching mathematics were closely linked to these two major orientations (O1 and O2). For example, O2 seemed to suggest that she aimed to teach only the basics and was willing to forgo certain parts of the syllabus (G1). She also placed a high priority on studying to excel in examinations (G2).

G1: Make “low capability” students “feel that they have achieved something” by focusing on skill development and foregoing some parts of the topic.

I come to a point that I just want to teach a bit lesser. At least they get the basic point. As I mentioned, we must hit at least 70 to 80% of the pupils to understand the topic, rather than just finish up the topic. So, we may have to forego certain parts in the topic... It is just like a computer... they [the lower achieving students] do not have the memory yet.

G2: Prepare her students to “study and excel”, in order to perform well in examinations:
Chapter 5 – Two Portraits of Teacher Noticing: Anita and Cindy

If you talk about mathematics in the academic sense, it is just a subject that you need to study and excel in Singapore.

However, Anita felt that it was important to show the “relevance of mathematics” (O3) to make teaching more interesting, though she found that difficult at the secondary school level:

It is important to show the relevance of mathematics in order to understand how Maths is applied in real life. They have to see the application so that they will find that it’s interesting. It’s valued. ... That’s why I find Physics is a more interesting subject to teach, because Physics and Science, they can see the relevance. But sadly secondary school maths is a bit harder.

And at the same time, she viewed mathematics as a highly logical subject (O4) even though the relevance of the subject may not be clear at times:

But one thing you cannot deny is I think Mathematics is the study of logic. And yes, it’s true. For the majority of the topics, you don’t get to see the relevance. But you cannot deny that when you do maths, you are applying your logic and developing your analytical thinking.

Hence, her orientations about “higher and lower capability students” (O1 and O2), which influenced her goals (G1 and G2), seemed to take priority over her belief that it is important to show the relevance of the subject:

I always find it very sad. Ever since I started teaching, my intention was to see how maths can be applied in real life application, but that part is lacking. Even now, we tried to make it link, but sometimes at the higher level, say A Level, is easier. But at a lower level, they try but I feel that certain topics are still about skills development.

Possibly directed by her goal to prepare students to excel in examinations (G1), Anita saw the Singapore Education Ministry’s attempts to make school mathematics relevant to real life as being too contrived (O5). She argued that students can only explore real life applications at the tertiary level (GCE ‘A’ Level), and not at the secondary level (O6). Furthermore, she found the current mathematics syllabus “too taxing” and “packed” to allow for any meaningful exploration or connection to real life (O7). These orientations could also explain Anita’s preference for worksheet-driven (O8) and teacher-directed kinds of teaching style (O9), when she described her ideal classroom as one, where
“students are able to listen carefully to instructions, and work on their task worksheets”. These preferences (O8 and O9) were also demonstrated when Anita emphasised the importance of giving good instructions and explanations in her classrooms.

Anita also hoped that her students would learn the values of “integrity”, and “honesty” through the study of the subject, so that they would be willing to learn and be “less selfish” with regard to sharing their knowledge (G3). To achieve her goals (G1, G2 and G3), Anita drew on her resources (R) about mathematics, teaching, and students. She obtained a degree in Mathematics, and had the necessary content knowledge needed for secondary school mathematics (R1). In addition, Anita could be considered as an experienced teacher, having taught similar levels of students in different countries. Hence, she could draw on prior experiences when planning for the lesson (R2). More importantly, Anita was aware that her students were “lower-band achievement” students who came from less privileged backgrounds with little monitoring by their parents (R3).

As a result of the interactions amongst G1, G2, O2, O8, O9 and R3, Anita focused on teaching procedures, and was less likely to respond to students’ thinking during lessons. This was seen when she explained why she did not take time to explore the concept of gradient during a post-lesson discussion:

Not wrong [to explore]. But... time constraints. If the objective is just to find the value mathematically (Refers to the answers.)... but now to fulfil MOE [Ministry of Education] criteria to get the results, must get the answer. (Laughs.) I’m serious. If I get the results, I get the results.

Therefore, Anita’s goal of preparing for examinations (G1) and her belief about her “lower capability” students (O2) dominate her teaching decisions: she was less likely to consider other teaching approaches, and instead, insisted on her usual approach of focusing mainly on procedures.

5.1.2. Context of Anita’s vignettes

The vignettes focus on what, and how, Anita noticed when she worked with five other teachers from Springside Secondary School—Eddie, Winston, Teresa, Don, and Kent—during Phase 3 of the study. An overview of the sessions in which the teachers planned,
implemented, and reviewed a lesson on \textit{Gradients of Straight Lines Graphs} for Secondary One students (age 13) is presented in Figure 4.2 (Phase 3). The notion of gradient is defined as the ratio of the vertical change to the horizontal change, and students are expected to find gradients of straight lines set in the Cartesian Coordinate system (1 cm represents 1 unit). However, the formula for computing the gradient of a straight line using coordinates is not required.

Referring to Figure 4.2 (Phase 3), Anita provided the first draft of the task, which took the form of a worksheet on finding gradients. The teachers, including Anita, then discussed and modified the task over five discussions (Sessions 1 to 5) before Anita taught the first research lesson (Session 6). After reviewing the first lesson, the teachers suggested a few instructional strategies for Anita to consider (Session 7). Anita then taught the second research lesson (Session 8), which was followed by the final review discussion with the teachers (Session 9).

5.1.3. Vignette 7: How Anita noticed students’ difficulty of seeing “rise” and “run”

5.1.3.1. Recognising the Key Idea and Difficulty related to “rise” and “run”

During the first discussion, Anita identified the notion of gradient, which she introduced as “rise over run”, as the main concept for the lesson. She also noted that her students might have had difficulties when the coordinate axes were introduced.

![Figure 5.1. A possible student error of calculating “run”.

Anita was very specific about students’ errors, and was able to provide possible examples of them. For instance, Anita highlighted that students might determine the “run” wrongly by taking reference to the origin (See Figure 5.1). This possible error was
also highlighted by Winston, who gave a similar but more specific example to illustrate his point (See Figure 5.2):

The very common problem is... when the line is here (Draws Figure 5.2 and points to the x-coordinate “4” on the diagram.)... y-axis [vertical distance] is not the problem, but it's always the x-axis [horizontal distance] that is having the problem. The line will stop at four... they say that the horizontal distance... the run is four. Ok? Yeah... this is the only problem. Somehow, I don't know, students have no problems with y-axis... don't know why they have problems seeing the run. They will put four.

Figure 5.2. Winston's example of student error.

Hence, the teachers had identified that their students may be confused about how the distances in the rise and run were calculated.

5.1.3.2. Trivialising the Difficulty: “It's obvious! They can see!”

When Eddie asked whether her students could see the “rise” and “run”, Anita highlighted that her students should not have problems identifying them:

1. Eddie: So can they see? Can they see the rise over run? (He laughs.) So... we want to make sure that...
2. Anita: Can see one... I don't think they cannot see... It's [obvious.] Can see...
3. Eddie: So, the students know that for the "rise", they know how to count the squares?
4. Researcher: Is this the first time they encounter this?
5. Anita: Yeah. This is the first time. They are Secondary One [students].
6. Eddie: So, do they know this one? (Points to the formula “rise over run”.)
7. Anita: Sorry, I have been doing this for a long time already. At the beginning, the teacher will tell them that gradient is the measure of steepness or slope, so they will know that gradient is rise over
run... then tell them that to measure the slope, it’s rise over run. Just start with [calculating gradient.]

8. Researcher: So, you define for them?

9. Anita: Yes. Just tell them. It’s just as simple as that. Gradient is just “rise over run”. Some students, the better ones, can even tell you if they read the textbooks.

10. Eddie: No... no... Is this “rise over run” idea introduced to them before this?

11. Anita: No.

12. Eddie: So, this is very new to them?

13. Anita: Yes. This is very new to them.

14. Eddie: So, the rise is referring to what rise? Let’s say I am a student. What is rise?

15. Anita: Eddie, this one they will have no problems. They know what’s the meaning of rise. I can tell you they know. (Eddie laughs.) They know that rise is the straight one [perpendicular]. This is not a problem to lower secondary students.

From the above exchange, Anita seemed confident that students could understand gradient simply by telling them (See Lines 7 and 9). Anita’s confidence might be rooted in her knowledge about the students she was teaching (R3), and her orientation about mathematics as a “logical subject” (O3). She expected that her students would see “rise” and “run” in the same way (Lines 7 and 15) and catch the same ‘logic’. The questions she posed in the worksheet also presupposed students’ ability to see the different ‘distances’ under different coordinate systems. (See Figure 5.3).

![Figure 5.3. Different types of questions in the worksheet.](image)

Referring to Figure 5.3, the worksheet moved from four questions asking for gradients of lines set in unmarked gridlines (Type A), to six questions that involved the use of the
standard Cartesian coordinates (Type B), before moving on to another six, which were situated in a non-homogeneous coordinate system (Type C). Hence, what counts as distance is different in each of the three cases, and this can pose problems for students trying to find the “rise” and “run” when they are learning about coordinate systems for the first time. Given Anita’s knowledge of mathematics (R1) and experience (R2), she might have overlooked this possible source of confusion because of her perceived rush to complete the syllabus in a “taxing” curriculum (O7), so as to do well in the examinations (G1). Anita did not analyse the possible confusion about the different distances when the various coordinates axes were introduced. As a result, she did not consider any modification for the design of the task because she insisted that the error identified was “trivial”, and could be overcome with sufficient practice. Hence, Anita did not seem to see the difficulty from the perspective of a student.

Eddie tried to get Anita to see the possibility that a student might not see the “rise and run” in the same way by adopting the perspective of a student (“Let’s say I am a student...” in Line 14). Following the cue by Eddie (Lines 14 and 16), the researcher tried to probe Anita’s thinking about “rise and run”:

16. Eddie: So, what is rise? What is run?
17. Anita: The height is the rise, then the run...
17. Researcher: The height from where?
19. Anita: The height from the bottom to the top of the mountain...
20. Researcher: So, where’s the bottom? So, for this line (points to a line on the worksheet), where is the bottom?
21. Anita: Perpendicular? I mean, usually when I teach this topic, I will tell them that it is the line perpendicular to the straight line. I do not think they have a problem with that. I only intend to spend two to three minutes on this. I will tell them, during the introduction, that the bottom is the perpendicular line to the height.
22. Researcher: So, where’s the base for this line?
23. Anita: The base is where the person is standing... (Points to the boy in Figure 5.3 Type A)
24. Researcher: (Points to the left hand corner of the boy.) So, is the base here?
25. Anita: I trivialise all these because so far, I have not seen any of my students having problems counting the rise and the run.
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Anita’s responses further revealed her confidence in using the explanations she had in mind to help students overcome their confusion about the distances. Anita tried to support her confidence by asserting that she had not seen anyone having problems with “counting rise and run” (Line 25). However, this contradicted the difficulty she had earlier identified (See Figure 5.1). Another possible point that may confuse some students, which she overlooked, was her reference to the “where the person is standing” as an indication of the base (Line 23). Her method of using the figure of a boy to indicate the “base” for the run (Line 23) might be problematic because this “boy” is not a single point on the line, and instead, covers a few squares beyond the line. Therefore, her explanation might not work well for students when the coordinate axes are introduced.

In contrast, Eddie then highlighted the importance of making explicit the meanings of “rise” and “run” with reference to the points on the line:

The rise over run... you need to identify two points, right? You need to identify two points... for example, identify this point and this point... with these two points... you draw a triangle. They need not identify these two points, they can even identify this point and this point, also can draw a triangle, right? This is important, because, if you say they know they know, but when they come to here, they won’t be able to transfer the learning to these questions.

In his explanation, Eddie expanded the concept of gradient targeted in the lesson to include two additional aspects: The gradient of a line is calculated by considering the gradient of a line segment; and different line segments of the same line give the same gradient. While these notions may seem trivial, they are critical for students to understand the concept of gradient (Rowland, 2010). While Eddie proposed getting students to test the latter empirically by examining different cases, Anita insisted that she would “just tell them to plug into the formula”, and would not dwell on the reasoning. Instead, she decided to tell students that “this was decided by mathematicians”. Despite Anita’s belief that these two ideas should be “obvious” to the students, students’ failure to understand these two ideas could have led to their error suggested previously (See Figure 5.1).

Anita’s insistence on simply telling students the definition of a gradient did not seem to target the specific error she had identified earlier. This suggests that she might have under-estimated the cognitive obstacles posed by the concept of a gradient. In spite of
the reasons given by Eddie, Anita was not convinced about importance of these key ideas and how students might be confused by these different distances. Even though she may have attended to the three focal points suggested by the FOCUS framework, Anita did not focus her noticing to bring these three points into alignment. Without analysing how the different aspects of the concept relate to the questions she had set in her worksheet, she did not notice the lack of alignment between her course of action (“spend two to three minutes” on teaching about rise and run) and students’ possible confusion about gradients. Therefore, her noticing is non-productive with regard to the design of the task as represented in Figure 5.4.

As seen from Figure 5.4, Anita did not make any reference to students’ confusion about the concept in her choice of teaching strategy. There is a gap, therefore, between what she might have noticed from the discussion and her response in planning the lesson. This is most likely influenced by strong beliefs about her “low capacity” students (O2), and her goal to prepare them for the examinations (G1) in a “taxing” curriculum (O7). Possibly influenced by her orientation that lower-achievement students require “simpler” and “straight-forward” tasks (O2), Anita expected students to abstract the essential features of gradients of straight lines from the task through her explanations, instead of exploring the concept through a discussion. The next section portrays how Anita’s preference to tell the students (O9) might have influenced her interaction with the students.
Figure 5.4. Anita’s noticing during lesson planning (Vignette 7).
5.1.3.3. Dealing with students’ confusion: What’s the “rise”? How to count?

Contrary to Anita’s expectation, students had difficulty with “counting the run” and an unanticipated confusion about the “rise”. For instance, Anita realised that students mistook height as the gradient during the first lesson (See Lines 30 to 33 below). Although she started off with a potentially illustrative example, Anita did not take the opportunity to initiate a discussion, and instead reverted to telling her students that slope is rise over the run (See Line 36):

26. Anita: Today, we are going to move on to a sub-topic, gradient. If you look at the notes, the meaning of gradient is what?

27. Students: … (No answer)

28. Anita: (Almost immediately) If I got a line…

29. Students: (Barely audible.) \( y = … \) (not audible)

30. Anita: The purpose to find gradient today the lesson right? Previous lesson, we have been drawing lines. (Draws four lines) But lines can be this way… something like that or something like that.

So we notice, they have different? Steepness. Another word for steepness is? Slope. Today, we are going to learn, basically, just on gradients. And talking about the steepness … how is it applicable in our daily lives? How can you see the link? If you notice, when you climb up a mountain, or go up the staircase… Here’s a mountain… (Draws a mountain.) If you are climbing up the mountain, the bigger the number, what do you notice about the slope?


32. Anita: Higher? You mean higher mountain when the number is bigger?

33. Students: (Various answers. Cannot be transcribed.)

34. Anita: So, if you mean that if a mountain that is higher, and this mountain is higher, the gradient is even higher? Remember? The key word here is the steepness. So, let’s say I have two mountains… same height. The two mountains are of the same height. One mountain is like that… the other mountain … I’m drawing here.
This is the base... the land. They have the same land right? And this is the top. When we are learning gradient, right? What do you think we are learning? So, we are learning about how high is the mountain?

35. Students: How steep...

36. Anita: How steep is the mountain? It’s not a matter of the height. It’s about how steep. So, we are talking more about the slope, or what we call steepness. This is what we are addressing today. Clear? In order to find the gradient, which is the steepness... in order to find how steep, we are going to use? The? Height over the? Horizontal. So, we are looking at the steepness here, right? I need to use the reference from the height, which in this case is the? Rise. Because we are going to address how steep is the slope right, we will look at how high it is. So, today, imagine that you are going to climb up a mountain, to know how steep is the height, we will look at the perpendicular length that you come down over the run, in order to find out how steep is this slope that you are running up.

So, you can have 2 slopes. This one, obviously, if you go up from here, this is the height. Imagine both of them are of the same height, but this one within a very short distance, you are running up to the top. This one, over a longer distance, you are running up to the top. Obviously, this one is steeper. And that one is less steep, which we also use another word, gentle. So this is about steepness, but we also have direction, which I will talk about later. Let’s go straight to the examples. We can’t always draw a mountain... let’s look at a straight line. (Refers students to the example on the worksheet.)

In this excerpt, Anita recognised students’ misconception that “a slope that is steeper is higher” (Line 34) from students’ responses (Line 31). This prompted her to draw two mountains to illustrate the key idea of steepness (Line 34). Anita’s decision demonstrated her awareness of the misconception, which indicated that she had drawn
on her knowledge of mathematics (R1) and experience (R2) to make sense of the situation. Anita’s choice of using two “mountains” with the same height but different gradients indicates that she had analysed her students’ confusion in-the-moment, because this scenario was not discussed during the planning sessions. While the analogy of mountains may not correspond directly to straight line graphs, Anita possibly assessed that the use of these two figures would draw students’ attention to the relationship between steepness, gradient, height (rise) and the run. Her choice might also have been driven by a belief in the importance of linking mathematics to real life (O5) to show the relevance of the subject (O4).

However, Anita was more focused on getting students to know the important terms and calculations than orchestrating a discussion to listen to their thinking. This can be seen in her decision to launch into an explanation that involves calculating gradient as “the height over the horizontal” (Line 36). Her explanation only revolved around finding “height” or “rise”, and she left out the relationship between the “rise” and “run”. Even when Anita explained the difference between a steep slope and a gentle slope, she assumed that students could see the convention of “running” from left to right, and that a steeper slope corresponded to a “bigger number”. Furthermore, her questioning was more evaluative, requiring students to give a closed answer (“How steep.” In Line 35). Therefore, evidence suggests that while Anita noticed student thinking, she might be motivated by her orientations about her students and the curriculum (O2, O7), as well as the goal of examination preparation (G1), to use a more teacher-directed mode of instruction (Lines 34 and 36).

Although Anita was able to bring to her mind a counterexample to illustrate her point about steepness and height; her preference for telling instead of exploring seemed to hinder a more discursive mode of teaching that would have provided some opportunities for students to reveal their understanding. Her noticing is modelled in Figure 5.5 as shown.
Figure 5.5. Anita’s noticing processes during teaching (Vignette 7).
This preference for “telling” students during teaching was also clearly seen when Anita interacted with the students on a one-to-one basis. For example, when Student S4 asked Anita about the determination of “rise”, she reacted by “telling” without listening to find out what Student S4 was thinking about:

37. Anita: The same thing for the next one, you try out. (Moves around the class to look at students’ answers)

38. S4: Ms Anita, what’s the rise? How to count?

39. Anita: The rise is the height. You find the perpendicular, and count how many units are there here.

40. S4: So, you count by the?

41. Anita: Yes. You count by the boxes. How many boxes are there? (Moves to the front of the classroom to explain how to count the “rise”).

Anita’s definition (Line 39) requires Student S4 to see the same right-angled triangle as hers. The “most natural” triangle in the problem of interest is the biggest triangle formed with the end points of the line segment. Anita did not ask what Student S4 was specifically puzzled about with regard to the “rise”. Instead, Anita’s emphasis on “counting” might cause Student S4 to associate the “counting of boxes” with the measurement of “rise” and “run”. It was not clear whether Student S4 understood the key idea, that what matters is the ratio of the “vertical change” to the “horizontal change” at this point. Figure 5.6 represents her in-the-moment noticing during her interaction with Student S4.
Figure 5.6. Anita’s noticing processes during Teaching (Vignette 7 Lines 37-41)
In spite of Anita’s explanations, student S4 encountered difficulties with Question 1b (See Figure 5.7) when coordinates were introduced a few minutes later. She was confused about how the “rise” and “run” were counted when the coordinates involved were negative and asked Anita about the counting for Question 1(b):

42. S4: Is it start from here or here?
43. Anita: You must start at the base, right? Which is the right angle triangle, right?
44. S4: But I thought we must start from zero? Then zero, should be minus three?
45. Anita: So, if today we don’t have all these things (referring to the grid), then there’s no more zero. So we look at the line. Remember just now there’s no zero at all, we just try to form a right angle triangle? Ok. Don’t bother about the zero first, just try to get the right angle triangle…
46. S4: But...
47. Anita: Never mind. I will show you again at the board.

Student S4, like several other students, mistook the run as “negative three” because they took reference from the origin (Line 44). Despite Anita’s beliefs that students should be able to catch the “rule” she had in mind, it turned out that a number of students, like Student S4, had problems working out the answers. Instead of thinking about a different response or finding out what students were thinking, Anita continued to focus on “forming the right-angle triangle” and “counting the rise and run”. She asked Student S4 to ignore the zero and emphasised the drawing of the right-angled triangle (Line 45) despite knowing that students might have problems when coordinate were introduced. Anita also did not make explicit to the students during the lesson how distance was measured or calculated in a coordinate system.
In these episodes, Anita’s perception of the difficulty as trivial might have constrained her noticing in-the-moment during the lesson. Despite being made aware of the difficulty about ‘slope as height’, and the confusion when the coordinates were introduced, Anita preferred a “telling” mode of instruction that dominated the rest of the lesson (Lines 41 and 47). Her noticing during this episode may be summarised as follows:

1. Anita attended to an unexpected difficulty of confusing gradient with height; and the difficulty of finding the rise;
2. Even though she might have analysed the difficulty as evidenced by the use of the mountain example, Anita still decided to tell the students;
3. Without preparing to listen to her students’ reasoning, Anita’s instructional strategy remained consistent (telling mode) through the lesson, which might not have targeted the difficulty that surfaced; and
4. Again, her decisions were seen to be influenced by her orientations about students and curriculum (O2 and O7) towards achieving examination-based outcomes (G1).

5.1.3.4. Insisting on doing the same thing: Just tell them ‘gradient is rise over run’

Anita continued to see her students’ difficulty with finding “rise” and “run” as non-critical during the post-lesson review. Instead of analysing the students’ responses in relation to the content and difficulty, she attributed the issues highlighted by other teachers to factors such as time constraints. For example, during the review of her first lesson, Anita expressed surprise at her students’ mistake of seeing “gradient as the height of the mountain” although this is a well-documented error (Leinhardt, Zaslavsky, & Stein, 1990; Zaslavsky, 2010):

Like [Question] 1c, I expected some problems, but I did not manage to go around the class. [...] But from the response of the class, I think most of them could get concept of the height and the run. I linked back to the starting, because at the start, I did not foresee them telling me that gradient is the height of the mountain. It was something that I did not expect... I thought it was very straightforward.
While she repeatedly acknowledged throughout the reviewing of the lesson that she was surprised by the students’ responses, Anita did not think beyond her initial surprise to analyse what went on during the lesson. Moreover, she thought that “most of them could get the concept of *height* and the run”, whereas other teachers had observed a number of misconceptions demonstrated by students during the lesson. Anita did not explain why she thought her students had understood, but instead, attributed their lack of understanding to time constraints:

48. Anita: I actually miscalculated the time because in front spent a bit more time, I was trying to target to finish at least until page 4, but [I could not.] In front, hopefully, those 1e, 2e, edge questions, those that they are supposed to count until the edge... So, hopefully in the next lesson in question 3, they can look at those that they need to use convenient points. I can also let them think about the different triangles.

49. Researcher: So, do you think that students got the idea of the convenient point?

50. Anita: Not too bad, I would say. I think not too bad. A few of my weaker ones, I think, though they had queries, but after I explained, I think they helped each other... Because even like, question 1f, or 2f, the line crossed and they were still looking at the last point. It was deliberate, and I sort of expected [that]. Actually I was quite happy that they made mistakes, but I just hope that they would check with others to get the correct answers.

51. Researcher: Do you think students had problems with the scale?

52. Anita: They did have, but I’m... but actually all these things, we expected them to make errors, and through all these, they will learn. But this class is actually more bothered academically... the other class is a bit more passive, passive as in, if they don’t know, they will just move on. But somehow I am going to use the same notes and so I am thinking whether I should modify the worksheet. Or maybe I just don’t modify. Instead, I cut short my intro, just tell them that left hand side is negative, right hand side is positive, tell them about the story about good person point to right hand side. Straight away go to the problems and don’t short-change them.

53. Eddie: So, when you say you cut short your intro, which part of the intro?

54. Anita: Maybe I will just tell them that we are just talking about the gradient of the line. Basically it’s just about the slope. I will tell them about the mountain. So, when the mountain is gentler, the number is smaller, steeper, the number is higher. Just use the first two examples.

In this exchange, Anita was able to recognise what students found difficult: Use of ‘convenient points’ to determine gradient, and confusion caused by the introduction of the coordinate axes (Lines 48, 50 and 52). However, Anita viewed these “mistakes” as “expected” (Lines 50 and 52) and she thought that students would learn from their
mistakes ("they will learn" in Line 52) without any intervention on her part. Hence, there was no evidence to suggest that she had analysed the concept to understand why students had problems during the lesson. Anita hoped that "students would check with one another to get the correct answers", instead of relating her students’ responses to analyses of their confusion to determine whether her explanations were useful. This contrasted with Eddie’s claim that the introduction of the lesson was too brief for students to make sense of the concept that they encountered for the first time.

55. Eddie: My perception [from observing you] is that you had already introduced to them the concepts of gradients before this lesson.

56. Anita: No. I did not. I was just trying to tell them that using this notes that gradient is actually steepness of a slope. I brought in the example of steepness.

57. Winston: You did this during the last lesson?

58. Anita: No, no. I didn’t. Today is really the first lesson on gradient. The last lesson, I just focused on the plotting of linear graphs.

59. Eddie: When you dismissed the class, I actually picked a few students and asked them what they understood by the word "gradient". They told me that gradient is "rise over run". I told them that I don’t want the formula; I wanted them to tell me what they understood by the word "gradient". One of them told me that it is the steepness. When I asked further, he could not tell me what he understood by steepness. He did not know what is steepness. The rest just applied formula “rise over run”. That’s why one of the questions I asked them is whether they really understand what is ‘gradient’. Because my perception is that when you flashed this on the screen and drew a few lines on the whiteboard, it was as if that they had heard of this word gradient, or you had introduced to them the idea of the gradient.

60. Anita: I think that could be because of the fact that our students’ English is not very good. That’s why maybe… I wonder whether they understand the meaning of steepness, or maybe they do not know how to explain.

61. Eddie: Yeah. So, the guy who answered me steepness. I asked him what he understood by steepness. He said "steepness, [is] steepness". But only that guy told me about steepness. The rest just referred to gradient as "rise over run". When I asked them whether they had heard of the phrase "gentle slope" or "steep slope". A few said "yes". I asked them what is a "gentle slope". I asked them to elaborate but they could not. Maybe the first part, we need to spend a bit more time so that they could understand the notion of gradient.

62. Anita: Not wrong. But time constraint. If the objective is just to find the value mathematically… but now to fulfil MOE criteria to get the results, must get the answer. (Laughs.) I’m serious. If I get the result, I get the results.
Both Eddie (Line 55) and Winston (Line 57) initially thought Anita had introduced the concept to students the day before because she only spent a brief time introducing the notion of gradient during the lesson. Eddie observed that students could articulate gradient as “rise over run” but they were unable to say more about what they understood about gradient. His response was different from Anita’s because Eddie tried to offer some student-centred evidence to support his view (Lines 59 and 61): Eddie thought that more time could be spent on helping them understand the concept better (Line 61). However, Anita suggested that the students’ inability to explain was due to their language barriers (Line 60), and she was more concerned about the “time constraint” (Line 62) than understanding. She also indicated that her goal was to “get the results” and “fulfil MOE (Ministry of Education) criteria”, which indicates her focus on getting the correct answers (G1).

Besides the notion of gradient, Eddie also described how students had problems counting the “rise” and “run”, and highlighted that students might be confused about the definitions and conventions involved in understanding gradients. This corroborated with Kent’s observations towards the end of Session 7:

The first page was very clear. Most of them could identify the sign of the gradient. Once we include the axes, they were confused. I liked the second page, where the numbers are the same but the directions are different. I asked one of the students sitting at the corner, who got it wrong. She did not put in the negative sign. I asked her whether there was any difference. It seemed that she was a bit confused about the convention from left to right, pointing left, pointing right, highest point etc. There are a lot of conventions…. They are also unable to identify the convenient points, as well as the drawing the triangles and reading of the scales.

Kent not only highlighted the possibility that students might be confused by the different conventions (“There are a lot of conventions...”), but he also raised a new point about the calculation of the “rise” and “run” when the scales of the axes were different (“...reading of scales”). This observation is important because in both homogeneous (same scale but not the standard 1 cm to represent 1 unit) and non-homogeneous (different scale on different axes) graphs, distance is not preserved. The confusion between geometric and analytic distance has been known to cause a number of cognitive difficulties for understanding gradients (Zaslavsky, Sela, & Leron, 2002).
These cognitive obstacles were not trivial and the researcher took some time to discuss these differences for the teachers to consider.

Despite the different perspectives offered, Anita decided to “spend more time going through the examples” so that students would be “exposed to more questions”. She believed that this increased exposure would help students to understand. Anita did not consider the balance between being explicit in her explanations and leaving everything for the student to figure out (Mason, 2000). Since Anita’s dominant mode of instruction was centred on her explanations and questions, considering this delicate balance between scaffolding and fading (Mason, 2000) would have been productive with regard to hearing students’ reasoning. The discussion could have moved the teachers towards understanding the need for students to make their thinking visible to the teacher, while providing enough support for students to make sense of the mathematics. Furthermore, Anita did not provide any evidence from her observations to support her views, and took cursory notice of the perspectives offered. Therefore, there was little connection between what Anita attended to, what happened during the lesson, and the general principles of teaching and learning.
Figure 5.8. Anita’s noticing during reviewing of lesson (Vignette 7).
Figure 5.8 shows a representation of Anita’s noticing during the post-lesson review, and it may be summarised as follow:

1. Anita was able to describe some of her students’ responses that emerged during the lesson;
2. She was able to identify and recognise the mathematical concepts involved, and the difficulties;
3. However, she did not analyse the difficulties in order to relate them to her explanation;
4. Thus, there was no indication that her decision to “expose students to more questions” was based on her noticing; and
5. Therefore, her noticing processes were fragmented (as seen in Figure 5.8), and influenced largely by the goal to “get results” (G1) and perceptions about her students (O2).

5.1.4. A portrait of Anita’s noticing in Vignette 7

In looking at snapshots of Anita’s noticing through Vignette 7, some regularities emerged with regard to what and how she noticed, and why she noticed the way she did. Together, these snapshots present a portrait of Anita’s noticing as shown in Figure 5.9, which highlights three aspects of her non-productive noticing throughout the whole lesson cycle.

First, even though Anita had demonstrated consistently that she could attend to specific details related to the gradient concept (See Section 5.1.3.1), her course of action was often misaligned with the other two focal points. For example, she did not factor the difficulty of finding distances in various coordinate systems into the design of the questions she used in the worksheet (See Figure 5.3). By ‘mixing’ the three types of questions, it was not surprising that students encountered many difficulties when the scales were different (See Kent’s comments in Section 5.1.3.4). Although Anita was aware of these difficulties, she suggested “cutting the introduction” as one of the main modifications to the lesson. This suggestion had little connections to student thinking, and was contrary to what Eddie suggested (Line 61 in Section 5.1.3.4), which was focused on asking students questions to reveal their thinking (Mason & Johnston-Wilder, 2006). Instead of considering explanations that target students’ difficulties,
Anita used analogies such as the “boy” as the “base” (See Section 5.1.3.2) to help students remember the procedures. These rules might be useful for students; however, they are not connected to any mathematical reasoning, and may confuse some students (Teuscher, Reys, Evitts, & Heinz, 2010).

As seen in Figure 5.9, this lack of alignment between the focal points was also demonstrated during Anita’s teaching. She had the tendency of telling students the right answers, or the correct procedures; instead of building on their thinking to help them understand when students gave unanticipated responses during discussions (See 5.1.3.3). A shift from evaluative listening to hermeneutic listening (B. Davis & Renert, 2014) during her lesson would have given Anita opportunities to get a sense of what her students were confused about. This would have been beneficial in helping her to decide whether the explanations were effective, and to vary them, if necessary, to target their confusion. Furthermore, without focusing on the alignment, Anita ascribed other teachers’ comments about students’ confusion to other less important factors such as time constraints and language barriers during the post-lesson discussion (Section 5.1.3.4).

Next, the focusing dimension of productive reasoning was generally absent as presented in Figure 5.9. This is demonstrated by her lack of analyses of student thinking throughout the cycle from the planning to the reviewing of the lesson. For instance, Anita did not analyse why students had difficulty finding the “run” although she was able to identify clearly their difficulty in determining the “run” when the axes were introduced (See Figure 5.4). It appears that Anita persistently trivialised the difficulty that students had (Sections 5.1.3.2 and 5.1.3.3). She underestimated the issues associated with the introduction of different coordinate axes (homogeneous and non-homogeneous), and did not seem to understand the difficulty, even when other teachers highlighted these students’ errors from their observations.

Even in cases where she was able to attend to, and make sense of, student thinking; Anita eventually did not respond to her students’ questions in order to build on their understanding. For instance, when Anita recognised and interpreted the students’ confusion about height and steepness (See Figure 5.5), her initial response to use an illustrative example was potentially pedagogically productive. However, she went on to
tell them the rule instead of orchestrating a discussion to find out what they were thinking, which would have provided her opportunities to tailor her explanations.

Finally, her non-productive noticing is evident from the design of the lesson. The main task, which consisted of questions seeking the gradients of straight lines, did not lend itself for Anita to see the reasoning and thinking of her students. The focus of the task was to get the right answers, and this might have hindered her attempts to respond to students in ways that reveal their thinking. For example, in her interaction with Student S4 (See Section 5.1.3.3), Anita emphasised her own solution instead of listening to Student S4, in order to find out what was confusing to the student. Moreover, the analysis of 'critical incidents' such as her interaction with Student S4 also did not result in gaining new understanding of how students think. Instead, Anita decided to do more of the same thing, without considering other alternatives (See Section 5.1.3.4).

Moreover, despite having discussed the mathematical aspects of analytic and geometric distance in different coordinate systems, Anita did not show evidence of using that knowledge to modify the questions or approach to teaching. Hence, there was no evidence to indicate any expansion of her ROG clusters. Instead, a common dominance of her preferences for worksheets (O8) and teacher-directed instruction (O9), together with her strong perception of ‘lower capability students’ (O2), emerged from the analyses. Driven by her goal to prepare students well for examinations (G1), Anita might have experienced conflicts between her views of mathematics, as a logical and sense-making subject (O3), and her perception of the “taxing” curriculum demands (O7). This could have restricted her choice of alternative teaching strategies, and discouraged her from choosing more discursive mode of instruction in view of the “limited” curriculum time.

Therefore, Anita’s noticing, which often fails to take the alignment of the three focal points into account through her pedagogical reasoning, is largely characterised as non-productive according to the FOCUS framework.
Chapter 5 – Two Portraits of Teacher Noticing: Anita and Cindy

Figure 5.9. Portrait of Anita’s noticing in Vignette 7.
5.2. Cindy: A portrayal of productive noticing

In contrast to Anita’s, Cindy’s portrait of noticing presents she attended to alignment of the three focal points through an active focusing of her noticing processes. The snapshots are presented through a vignette developed from Phase 3 of the study, before a portrayal of her productive noticing is depicted at the end of the chapter.

5.2.1. Who is Cindy? Her resources (R), orientations (O) and goals (G)

Cindy revealed her goals related to teaching mathematics during an interview conducted at the beginning of the study. Two overarching goals stood out during the interview:

G1: Teach mathematics in a way that “relates to everyday situations” so that students would learn to love the subject.

   I always try to use real life experiences, relate to everyday experiences, every day situation which they can relate to... The first thing I want them to learn from me is to love mathematics.

G2: Teach mathematics in a way that will “cater to the different ability groups” in her class.

   I’m trying to cater to three different groups: The high end like these Indian internationals; the middle, who are ok but need more pushing; the lower end, who need to be with you all the time, but I can’t afford the time... Now that I have a very challenging class this year, I really think I have to think of different ways to engage my three main ability groups. To get them more engaged.

Cindy’s desire to teach mathematics using real life situations (G1) can be seen in her description of several lessons she had conducted in the past. For example, she described how she had brought students around the school to learn ‘Area and Perimeter’, and had set up a “shopping centre” in her classroom to teach ‘Money’. In many ways, Cindy’s view of mathematics as a subject that is closely related to real-life application (O1) seems to influence her teaching:

   Maths is everywhere. You need maths in everything you do in daily life. So, you have to apply what you know, and the clues you know, to the questions. I suppose everyday you have to use it; it’s part and parcel of our daily living.
right? Students have to know the basic maths concepts in order to get on with life, simple calculations, or estimation...

Therefore, she believed that a “hands-on” approach (O2) would make the subject come alive for her students so that they could see the role of mathematics in their daily lives (G1). Her other perceptions related to the subject included:

O3: Mathematics is a highly connected subject, where one needs to apply different concepts (“apply the clues”) to solve problems.

O4: Mathematical thinking is the ability to analyse the information given in the problem to see how “different mathematical ideas fit into the solution of the problem”.

Cindy, therefore, believed that “every step in the solution was motivated by some reasoning” (O5), and aimed to examine students’ mistakes in their written work, and understand why the students had these difficulties (G3).

When I get to see their written answers, I can see how they think... I think it’s important for them to tell me how they get [the answers]. I do ask them, how you get the 7? Where did the 7 come from? They look me like that [shows a blank face], ok, let us go back, see what clues are given in the story sum.

To achieve her goals of teaching mathematics, Cindy drew on her resources (R) related to mathematics, teaching and students. She obtained a ‘GCE A Level’ certificate; before she went on to obtain an Advanced Diploma in Primary Mathematics Teaching 15 years ago. Hence, Cindy has sufficient mathematical knowledge to teach the current syllabus in Singapore (R1). Moreover, having taught for more than 30 years, she had a wealth of experience in teaching mathematics at all levels (R2) except for Primary Six (age 12, the highest level in the primary school system). Lastly, Cindy knew that her students had problems in grasping difficult topics, such as ‘Fractions and Decimals’, and solving more challenging questions. They were considered to be in the “low-end” achievement bands, but had good learning attitudes (R3).

They enjoy the hands-on activities, but when it comes to applying what they learned, it’s question mark, question mark. ... for example, in model drawing, you are supposed to draw the blocks and apply them to part part whole... when it comes to actually drawing the model, they can’t ... like today, when we are
doing squares and rectangles... they know the properties, but when it comes to finding missing sides, they can't apply like opposite sides are equal...

I think there are too many concepts. Like decimals has been brought down to Primary 4, which I think is very hard... they can't cope with fractions, what more decimals?

However, Cindy lamented that her students lacked procedural fluency, and felt that mastery of computational skills should form the basis for mathematical understanding (O6). This belief (O6), coupled with the knowledge of her students (R3), seemed to constrain her goals to emphasise reasoning and thinking (G2 and G3). She sometimes felt that she should concentrate on developing her students’ competence in routine skills than understanding the reasons behind the procedures (O7).

To me, there is too much information. They are not mature enough to grasp the concept. Why are we doing away with the procedure? I know, it's partly procedure, and we are trying to move away from that... But still, the kids still have to know it [sic], certain things have to have the procedures... It’s quite tough to reconcile the two, get the kids to try to understand [the procedures]. That is the trouble we are facing. At the end of the day, they sit for exams, and if they do not know what’s the answer to that, that’s it. I think a bit of procedure, and a bit of ... at that age, they are too young to understand.

As a result, Cindy considered that the “reasoning behind the procedures” might be too “high-level” for her students to understand (O8), and hypothesised that her students might be too “young” to understand. As a consequence of these conflicting beliefs about mathematics (O3, O4, and O5) and teaching (O6, O7 and O8), Cindy expressed a sense of frustration in her struggle to “achieve the balance between procedural and conceptual understanding”. She articulated that it was important for students to have more opportunities to experience mathematics in different ways. However, she believed that the current curriculum was too packed (O9); and this, according to her, contributed to many teachers choosing to “touch and go with the mathematical connections”.

So, I think too many concepts have been cramped into our syllabus... In order for them to grasp the concepts, you need to have more experiences, hands on and stuff like that, instead of touch and go touch and go... like 3 weeks for this, 3 weeks for that, in the end they learn like 12 topics, but how good are they?
Nevertheless, Cindy could see how the syllabus was targeted at developing both skills and mathematical thinking (O10), and understood the need to strike a balance between procedural understanding and conceptual understanding. She embraced different ways of teaching so that she could engage all her students, and was open to changing her current practice (O11) to make mathematics more accessible to all students (G4).

5.2.2. Context of Vignette 8

The vignettes focus on what, and how, Cindy noticed when she worked with five other teachers from Greenhill Primary School—Kirsty, James, Flora, Rani, and Anthony—during Phase 3 of this study. These noteworthy discussions occurred as the teachers planned, implemented, and reviewed a lesson on *Conversion of common fractions to decimal fractions* for Primary Four students (age 10). This lesson is part of a unit on *Decimals*, which is the first formal encounter with decimal representations in the curriculum for the students. In this lesson, the teachers focused on the conversion of common fractions with denominators that are factors of powers of 10. Prior to this lesson, students would have been taught place values of the decimal system, and conversion of common fractions with denominators 10, 100 and 1000.

The teachers, including Cindy, developed the task over five discussions (See Sections 5.2.3.1, 5.2.3.2 and Figure 4.1 Phase 3 Sessions 1 to 5) before James taught the first research lesson. After reviewing the first lesson, the teachers made a few suggestions about instructional strategies for Cindy to consider in her iteration of the lesson. She then taught the second research lesson (See Section 5.2.3.3 and Figure 4.1 Phase 3 Session 8), which was followed by the final review discussion with the teachers (See Section 5.2.3.4 and Figure 4.1 Phase 3 Session 9).

5.2.3. Vignette 8: Cindy's productive noticing of students' thinking about decimals

5.2.3.1. Recognising the Key Point and Difficult Point

In the first discussion, the teachers discussed some of the difficulties that students faced when learning decimals. Kirsty highlighted a key idea—decimals and fractions are different representations of the same number—that students find difficult to understand. In particular, Cindy and Kirsty pointed out a specific common problem encountered by their students:
The problem highlighted by Kirsty (Line 2) focused on the students’ confusion about the different symbolic representations: they did not understand why the common fraction looked so different in an equivalent decimal fraction. For example, students could see where the digits “1” and “5” came from when “1/5” was used to denote one out of five equal parts of a whole; whereas, they were perplexed as to where the digit “2” comes from, when 1/5 is written as a decimal fraction “0.2” (Lines 11 and 12). All the teachers (Line 4), particularly Kirsty and Cindy (Line 6), were able to give specific examples of this error. Based on her understanding of her students (R3), Cindy subsequently highlighted a few additional common errors, such as “4/25” as “4.25” and “1/25” as “1.25”, which were committed by her students.
Cindy analysed these errors and highlighted that students might have made this mistake when they tried to make sense of decimal fractions using knowledge of whole numbers (Line 17). This may be a significant cognitive obstacle, which students have to overcome in order to extend their understanding of the number system (Callahan & Hiebert, 1987; Desmet, Grégoire, & Mussolin, 2010; Roche, 2005; Steinle & Stacey, 2011). Cindy’s comments prompted the teachers to realise that the students’ error patterns were rather systematic in the numbers used. They began to think about how a lack of understanding of place value might have led to these errors. Cindy then commented further on the error by highlighting a related problem:

18. Cindy: I guess it is the place value...
19. Kirsty: That’s what I think also. But if it’s the place value, then how does it have bearing when we’re teaching...
20. Researcher: So you all mentioned a few things. So the prerequisite to achieve the key point... So maybe it is good to think about the prerequisite to achieve the key point... So the prerequisite to understand the key point... would be? Understanding the place value. Right? Okay... and by this you mean?
21. Kirsty: Being able to distinguish tens and tenths. And also being able to distinguish tenths, hundredths and thousandths.
22. Cindy: Because our children view both tenth the same as ten...
23. Flora: Actually if they know this one (Points to the tenths), there’s no way they will put one over 5 as 1.5.
24. All: (Agreeing) Yes...
25. Cindy: [Understanding that] One tenth is 0.1 (Zero point one).
26. Researcher: So understanding place value seems to be an important key.
27. Cindy: Yeah, it’s an important key.
28. Researcher: Any other important key?
29. Cindy: Maybe also the language of tenth, “th”, the “t-e-n-t-h”... you have to know it’s actually a fractional part, which most of them don’t know. The vocab[ulary]...

Here, Cindy understood that the students’ reasoning about the notion of place value would play an important role in understanding the concept of fraction-decimal equivalence (Line 18). She highlighted that the difference between “ten” and “tenth”, while obvious for the teachers, might be confusing to students (Lines 22 and 29). Cindy’s comments also concurred with Kirsty’s (Lines 19 and 21), and invoked a sense of agreement or validation among the rest of the teachers (Line 24). This connection
between common fractions and decimal fractions, in terms of the place values (Lines 18, 22, 25 and 29), was a crucial factor to consider in the design of the task. This was important because teaching approaches that reinforce this connection can help the students see the meaning behind the procedures (Wang & Siegler, 2013). The discussion on the students’ confusion about the concepts of place value had the effect of shifting the teachers’ view of decimal-fraction conversion from being purely procedural to a more conceptual one. The discussion emphasised the possibility that a procedural approach to teaching conversion, which the teachers had used in the past, might not be effective as previously thought.

5.2.3.2. Designing the task to target the Key Point and Difficult Point

Building on what they discussed about the key ideas, and the difficulties encountered by the students, Rani proposed a “Fraction Sorting Task’ to highlight the idea of equivalent representations in the fourth session. Her original suggestion was to give students a few sets of three cards that are equal in magnitude, but different in representation. Rani gave “1/5, 2/10 and 0.2” as an example of a set of three cards. The students would find these sets of three cards from amongst nine given cards, and explain how and why they put these three cards together. The crux of the task was to provide students opportunities to reason about how they put the cards with same value together, in order to abstract the key ideas from their experiences. More importantly, the teachers intended students to see the need to change the denominators to 10, 100 or 1000.

Cindy and the other teachers tried to see the concept and understand the difficulty from the perspective of a student. Teachers’ attention on the three focal points enabled them to refine Rani’s task to target the misunderstanding in decimals. For example, Rani’s anticipation of the students’ responses to the task prompted the teachers to consider knowledge of equivalent fractions as a prerequisite:

30. Anthony: If you ask them to explain why they choose 1/5 equal to 2/10. Do you think they can explain?
31. Rani: Hmm… I think they will somehow say that they are equivalent fractions…
32. Cindy: You actually [can’t be heard] 2/10 equals to 0.2 initially right?
33. Rani: So this set of cards, a set of three… for example, there’s a one fifth, two tenths and 0.2. What I anticipate is that they will just take 2 over 10 matches 0.2. And then, [imitates the students] where does
the one fifth fit in? So, they will somehow... they will come to a point where they realise that the 2/10 is actually the 1/5 because they learned the equivalent fractions before. Perhaps at the end of the lesson, we can sum up... that in order for them to express a decimal, they have to go to a base [denominator] 10 first.

34. Anthony: So, for this approach, they need to revisit the equivalent fractions.
35. Cindy: Yes.
36. Rani: Yes. That should be in the introduction.

Rani (Line 33), and possibly Cindy (Line 32), anticipated that students would see the equivalence between 0.2 and two tenths (2/10), before they thought about the one fifth (1/5). She was able to justify her conjecture using the prior knowledge (R3) of students (“because they learned the equivalent fractions before.”). This focus on how students might think created opportunities for teachers to attend to student reasoning. Later during the same session, Anthony proposed that the connection between the choice of denominators and place value might not be clear to the students:

So at the end, we still have to figure out, they need to concentrate on the denominator 10, 100 and 1000. But then, we have to come back to the concept of place value [...] If they ask you why must I choose denominator 10 or 100, we need to go back to place value. This is what happens at the upper primary. If not, if you give them something like 1/4, they will give you equivalent fractions 2/8; 3/12 and so on. Then they will ask you... [Imitates students] why must you use denominator 10 or 100, why can’t we use other equivalent fractions?

Anthony’s comments about the choice of denominators prompted other teachers to recall teaching incidents related to this connection between the Key Point and Difficult Point. For instance, James shared how his high-achieving students had the same queries when they were asked to fill in an appropriate fraction with denominator 10, 100, 1000 in the following question:

\[ 1.5 = 1 + ( \quad ) \]

Many students answered with the fraction 1/2 instead of 5/10 as intended. James did not mark them wrong but asked them to change the fraction to one involving tenths:
I did not mark them wrong... the fraction is in a sense correct... I put an extra bracket there for them, and ask them... can you write it in a fraction in tenths, and they asked me "why?"

This unexpected response encouraged the teachers to reflect on their students’ understanding of the place value in decimal fractions. It also raised their awareness of what students were thinking when they encountered decimal fractions. James then hypothesised how students might just have latched onto a previously taught procedure without really understanding it:

Because previously when we taught fractions, we always tell them whatever it is, simplify [reduce to primitive fractions]. So, now whatever it is, they simplify. But now, we tell them "don’t simplify"... so there is a need to [give] this extra explanation why certain things we don’t simplify.

These discussions, which focused on student thinking, made explicit to the teachers the need to listen to the students’ reasoning. Rani further reiterated that the task was not meant as a matching activity, but a reasoning activity:

The fact that they actually articulate or write it [reasoning], it will help them to reinforce [the concepts]... The main thing we want to gather from that [task] is that they have to change the base [denominator] to a 10 or a 100 or a 1000. That’s the ultimate, that’s the main thing we want to get from the activity. It’s not just the matching thing.

Based on these discussions, the teachers made several modifications to the task by considering the students’ thinking. They decided that there should be a different number of fractions in each set to eliminate guessing; and they included 0.41 and 0.14 as cards, which do not belong to any set. This decision targeted the students, who might use whatever digits they see in the common fraction “1/4” (Difficult Point) in their decimal representation. Another important aspect of the task was the choice of fractions used. Hence, Cindy demonstrated sensitivity to the students’ thinking, when she suggested 5/20 to be included in a set related to 1/4, which consisted of 1/4, 2/8, 25/100 and 0.25:

37. Cindy: How about 5/20?
38. James: 5/20?
Here, Cindy attended to one of the points raised about different ways of thinking about conversion of common fractions to decimals. She reasoned that students would not be able to convert 5/20 into a common fraction with denominator 10 because of the numerator “5” (Line 39), and so they would have to think about changing it to a denominator of 100. It was plausible that students could not see how 1/4 can be expressed as 25/100 directly, but they could see 1/4, 5/20, and 25/100 as equivalent fractions. This equivalence of fractions would provide students a way to relate 1/4 to 25/100 through 5/20. There is evidence, therefore, to suggest that Cindy attended to the mathematical details discussed during the meetings (Line 43), and took note of the scope from the textbook (Line 41). The teachers firmed up the set of numbers as shown in Table 5.1.

<table>
<thead>
<tr>
<th>Table 5.1: Sets of numbers used in the task</th>
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<tbody>
<tr>
<td>Set</td>
</tr>
<tr>
<td>A</td>
</tr>
<tr>
<td>B</td>
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<tr>
<td>C</td>
</tr>
<tr>
<td>D</td>
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<td>E</td>
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The numbers chosen reflected some thought given to the Difficult Point. For instance, students might think that 0.15 is the representation for 1/5 because of the “1” and “5” in the fractional notation. Together with the emphasis on student explanation of how they group the fractions, this task has the potential to reveal and enhance student thinking. The planning of the task built on what each teacher noticed, and these comments
contributed constructively towards the refinement of the task during the discussions. Moreover, these refinements were based on the teachers’ sharing of what they observed about their own students, and how they reasoned about these observations.

Therefore, Cindy’s and other teachers’ noticing was productive towards the design of a lesson that reveals student thinking. Particularly in the case of Cindy, it can be seen that:

1. She could identify the key concepts and recognise what her students might find difficult;
2. Cindy highlighted the importance of understanding the concept of place value;
3. Together with the other teachers, Cindy considered the connections between the points of confusion about place value to propose refinements to the task;
4. This analysis had enabled teachers to consider how a focus on reasoning about the equivalence of representations can help target the students’ confusion about the concept.

Cindy’s noticing during the planning of the lesson is summarised and represented as shown in Figure 5.10.
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Figure 5.10. Cindy’s noticing during planning of the lesson (Vignette 8).
5.2.3.3. Listening and responding to students’ thinking

During the lesson she taught (See Figure 4.1 Phase 3 Session 8), Cindy tried to engage her students, who struggled with procedural fluency, in whole-class discussions to understand what they were thinking. This was despite her preference for procedural understanding over conceptual understanding (O6, O7, and O8) in a packed curriculum (O9). In so doing, Cindy withheld her own preferences to consider other ways to make mathematics more accessible to her students (G4) by focusing on the reasoning over the correct answers (G2). For example, she tried to probe student thinking when a few students produced the wrong answers to the question on their personal writing board:

44. Cindy: Too easy for you? How about this? (Writes 78/100 on the board.) Ok... (Waits for every student to show their answer on their writing board.) I have this answer and I have this answer. (Writes 0.78 and 7.08 on the board.) Which one?

45. Students: 0.78!

46. Cindy: Who said 7.08? (No student raises his/her hand.) All agrees that it is 0.78? Why? Why is it not 7.08? Anyone can explain? Yes. [Student S1]?

47. Student S1: You did not show the whole. There is no whole at the side.

48. Cindy: There is no whole. What do you mean by “there’s no whole”?

49. Student S1: Because there’s no whole at the side.

50. Cindy: There’s no whole at the side. That means?

51. Student S1: There is no “1”.

52. Cindy: So, in other words? What can you say about the numerator and the denominator?

53. Student S1: ... 

54. Cindy: There’s no whole like the earlier one. The whole number that you are talking about, right? So, in other words, that means, this fraction is? (No response.) Can anyone help out? He said that there is no whole there. Can anyone help out? Yes, [Student S2]?

55. Student S2: The fraction is 0.78.

56. Cindy: Yes. But why is it not 7.08? Yes, [Student S3]?

57. Student S3: Seven is the whole, and there is no seven at the side.

58. Cindy: Yes. Seven is the whole [part]. That’s right. So, there is no seven whole at the front [of the fraction.] That’s what you are trying to say right? (Points to Student S1)

59. Student S1: (Nods.)
As seen from the exchange, Cindy saw a few students with wrong answers (Line 44) and highlighted the two answers she saw (0.78 and 7.08). When no one owned up to the wrong answer (Line 46), she decided to ask her students for their reasoning instead of moving on to the next question. Knowing her students (R3), it can be inferred that Cindy asked for the thinking behind the answers so that these few students (who got it wrong) could understand why (G2), and made mathematics more accessible to them (G4). She wanted to see if the students could see the concept of place value, which seemed to pose problems for a few students, from their responses. Even though Student S1 gave a reasonably correct answer (Line 47), Cindy probed further to find out what Student S1 meant by “no whole” (Line 48) and realised that the student might have some gaps in his understanding (Lines 49, 51, and 53). Instead of telling the correct answer (“7” instead of “1”), she tossed the question to Student S2, and then to Student S3. This short exchange demonstrated Cindy’s willingness to listen to what students think, so that she could vary her explanation if necessary. Her more discursive approach also helped to reveal student thinking to the whole class. Cindy’s in-the-moment noticing is presented in Figure 5.11.

The excerpt presented was representative of Cindy’s teaching throughout her lesson. Possibly motivated by her desire to understand the students’ errors (G3), and her belief that “every step” has a reason (O5), Cindy adopted a more interactive approach even during one-to-one interaction.
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Figure 5.11. Cindy’s noticing during her lesson (Lines 44 to 59).
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During the group work, students needed to sort out the 19 cards given and group those with the same value together. As Cindy circulated the classroom, she noted that Student S1’s group had put 1/5 and 0.15 together:

60. Cindy: Ok. Can you tell why you put 1/5 (one fifth) in this group?
61. Student S1: Because it’s the same number.
62. Cindy: It’s the same number. What do you mean by the same number?
63. Student S1: It’s like... 1 over 5.
64. Student S5: Same digits.
65. Cindy: (Points to 15/100) This is actually what?
66. Student S1: 0.15.
67. Cindy: (Points to 1/5 and 0.15) Are these two the same value?
68. Student S1: Yes.
69. Cindy: Are they equivalent?
70. Student S1: No.
71. Cindy: So, how can you put them [as] the same value? Do you think they are the same value? Is 1/5 (one fifth) the same value as 15 hundredths?
72. Student S1: (After a few seconds of silence) Yes.
73. Cindy: What makes you think so? Why is 1/5 (one fifth) the same value [as 15 hundredths]? How did you get the answer?
74. Student S1: I divide...
75 Cindy: Divide what by what?
76. Student S5: No, it's not [the same.]
77. Student S6: No, it's wrong.

As discussed during the planning sessions (See Section 5.2.3.1), Cindy and the teachers anticipated this error of using the same digits in the decimal representation without reference to the place values (Difficult Point). Instead of telling Student S1 that the answer was wrong, she engaged the students in a discourse focused on their thinking (Lines 60 to 73). Cindy tried to listen to the students’ reasoning as they explained why they had put 1/5 and 0.15 together in the same group (Lines 60 and 62). She realised from Student S1’s answer (Lines 61 and 62), as well as Student S5’s answer (Line 64), that they had made the error anticipated. Cindy tried to find out whether the students could deal with 15/100 (Line 65), and found that they were confident that 15/100 is the
same as 0.15. Cindy repeated her question (Line 69) with a change of wording from “same value” (Line 67) to “equivalent” when Student S1 answered wrongly. In doing so, she realised that Student S1 had a different understanding about “equivalent” and “same value” in this context (Compare Lines 68 and 70). Cindy then directed students to focus on 1/5 (one fifth) and 15/100 (15 hundredths). When Student S1 continued with his wrong answer (Line 72), Cindy continued to probe his understanding (Lines 73 and 75) instead of giving the right answer, until Student S5 and Student S6 realised their mistake.

Here, Cindy was able to identify the students’ understanding from their responses, and she made sense of their thinking through a series of questions that revealed her students’ reasoning. A similar scenario was previously discussed during the planning stage, and the teachers wrote a lesson play (Zazkis et al., 2009) on it (See Figure 5.12).

<table>
<thead>
<tr>
<th>Anticipated Students’ Responses (Key Task)</th>
<th>Lesson Play (for possible teacher’s responses)</th>
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<tbody>
<tr>
<td>Possible students’ responses</td>
<td></td>
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<tr>
<td>2/5 = 0.25</td>
<td>T: Why do you think that 2/5 is equal to 0.25?</td>
</tr>
<tr>
<td></td>
<td>S: The numbers look the same. /</td>
</tr>
<tr>
<td></td>
<td>They have the same numbers.</td>
</tr>
<tr>
<td></td>
<td>T: So if the numbers have the same digits, they</td>
</tr>
<tr>
<td></td>
<td>are equal? Did you make any calculations to</td>
</tr>
<tr>
<td></td>
<td>come up with this conversion?</td>
</tr>
<tr>
<td></td>
<td>S: Not this one. They look the same.</td>
</tr>
<tr>
<td></td>
<td>T: I see. Hmm...What must be done to a fraction</td>
</tr>
<tr>
<td></td>
<td>in order for you to convert it to a decimal?</td>
</tr>
<tr>
<td></td>
<td>S: Change the denominator to a base 10, 100 or</td>
</tr>
<tr>
<td></td>
<td>1000? Oh, right. I must change it first.</td>
</tr>
<tr>
<td></td>
<td>T: Yes. Can you do that to the fraction now,</td>
</tr>
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<td></td>
<td>please?</td>
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**Figure 5.12. Excerpt from teachers’ planned lesson play.**

Although the exact numbers and planned responses were different, Cindy seemed to be able to draw from what was prepared, and responded effectively to students. More importantly, her willingness to listen to students had given them opportunities to reveal their thinking. Cindy’s noticing in this excerpt is modelled as shown in Figure 5.13.
Figure 5.13. Cindy’s noticing during interaction with students.
5.2.3.4. Understanding student thinking about equivalent fractions

Cindy continued to demonstrate her focus on looking at her students' thinking during the review of the lesson (See Figure 4.1 Phase 3 Session 9). She was able to describe specific instances of her students' responses to the task, and highlight a few key insights that led to a better understanding of their thinking about fractions and decimals. For example, Cindy noted that students persistently used a “pairing strategy” to put pairs of cards with the same value together first, and some of them manifested the Difficult Point (See Section 5.2.3.3 Lines 60 to 62):

78. Cindy: Then the other thing that I also noted that they were actually putting in pairs, pairs, pairs, pairs. And they had more than five groups yeah. And they actually put rest all back in the packet, I asked them what happened...why are you putting it back? [...] And then some put the wrong ones, [...], they put one fifth as 0.15. And I asked her why is it together... and she said because the same number.

79. Flora: Dan said the same thing... yeah... say the same thing.

80. Cindy: Evan said the same thing too. (Demonstrates how she ask Evan) So why is 0.15 the same as 51, is 51 the same as 15 and she say no, then why do you put them in the same group. Then she couldn’t answer me. I said you must try and get the equivalent of one fifth, what would it be, equivalent fraction. Then she realised she put it in the wrong group. That was one of the mistakes. And then some of them um actually, group according to digits, same digits, put them all together yeah. So that one maybe I need to actually highlight to them the place value is different.

Besides the pairing strategy, both Cindy and Flora noted a common error that surfaced several times during the lesson: $1/5$ is the same as 0.15. Later in the discussion, Cindy made the connection between this error and her students' understanding of equivalent fractions and place value (Line 80). In response to her point, the researcher asked Cindy whether this error was due to issues related to language ability (Line 82), and she suggested it was more than a language issue:

81. Cindy: It’s a mathematical connection... for example, Justina... if you think about it, I asked her why one fifth equals 0.15 and she couldn’t answer.

82. Researcher: When they couldn’t answer is it because of language or is it because of ...

83. Cindy: Could be partly language also yeah, 'cause she can say that okay 15 and 51 not the same value, so why do you say one fifth which is a fraction, it's the same as 0.15. So I asked her what is 0.15 as
a fraction and she could say 15 over a hundred... then I asked her... is one fifth equals to 15 over a hundred?

Cindy was able to recall specific interactions she had with the student to explain why she believed that the problem was more of a mathematical connection. She argued that since Justina could explain why 15 and 51 are not equal; and was able to say that 0.15 is 15 hundredths; then the issue was not just about language, but the failure to see the “mathematical connection” (Line 81).

This point was then taken up by James when he commented on the lesson. He suggested a possible explanation for the pairing strategy, which seemed to hinder student thinking:

It's like they limit themselves to the pair, in their mind only pairs will match. So they didn't consider, not that they don't know, they didn't consider that there could be more, I think. So those are, they know the equivalent fractions, I think they do, but they didn't, because they only saw it as pair they didn't consider a bigger group of numbers would be equivalent. Maybe because usually we test them or they only show two numbers equivalent, when they do their work also we ask them to change or simply... is always one to one. So maybe that condition their mind to see numbers one equals this, that's it.

James's observations prompted Cindy and the other teachers to think about the role of equivalent fractions in the task, and more importantly, in the conversion of common fractions to decimal fractions. He saw a connection between how they used to teach equivalent fractions (“they only saw it as pair”; “they only show two numbers equivalent”; “is always one to one”) might have reinforced, in students’ minds, that equivalent fractions only come in pairs. Cindy then realised that the tendency to see equivalent fractions only in pairs might have prevented them to see the logic underlying the design of the task: the transitive nature of equivalence, that is, if $2/10 = 0.2$, and $1/5 = 2/10$, then $1/5 = 0.2$. The teachers then realised that students might not see this logic, and could not figure out that if $0.15 = 15/100$, and $1/5$ is not equal to $15/100$, then $1/5$ cannot be equal to $0.15$.

Anthony then added a similar observation, which linked to Cindy's whole class explanation of how “$4/25 = 8/50 = 16/100 = 0.16$”. He highlighted that this persistence
in students looking at only two fractions as equivalent could have caused the students to reject Cindy’s explanations that these fractions are equal:

So in the end they still cannot make it… even when Cindy shared with them the second groups’ answer on that, there’s one 4 out of 25 equal to 8 out of 50 and the children ask her to take out a card [8/50]. (imitates the students)... no... this you cannot use. So even when you tell them 4 out of 25 is the same as 8 out of 50... they are stubborn, they don’t want to put the card there. So if you look at the result [on the task sheet]... they did not put it there.

Anthony then went on to suggest the need to reiterate that there are infinitely many equivalent fractions:

Okay so overall I think maybe when we touch on equal fractions... that’s what we ask this class... we need to start with a simple fraction and multiply it by two, multiple by three, multiple by four and I give them a long list of these fractions, and then they realise oh, they give me a fraction, there are a lot and unlimited number of equivalent fractions. So when it comes to this question they will not just stick with two or three.

Kirsty also highlighted the same observation and suggest the need to ensure that students ought to have experiences dealing with “multiple sets of equivalent fractions”. As seen from this short excerpt, Cindy’s detailed description of the “pairing strategy” led to a productive discussion of how her students’ thinking about equivalent fractions might have hindered their understanding of the conversion procedure. On reflection, Cindy realised she had overlooked the importance of equivalent fractions:

But I overlooked the part of actually bringing back the equivalent, actually I had more practice on that, on hindsight now you are saying, cause they actually didn’t see the equivalent fraction without the cards.

Drawing on these comments, Cindy and Kirsty both highlighted that it would be good to emphasise and revise equivalent fractions, instead of factors of 10, 100, and 1000, as a pre-task activity. As a result of these constructive conversations focusing on the ‘Three Points’, the teachers were able to gain a new understanding about the students’ thinking on equivalent fractions and its role in understanding decimal conversion. Cindy’s noticing in this episode is modelled as shown in Figure 5.14.
Figure 5.14. Cindy’s noticing during the review of the lesson.
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5.2.4. A portrait of Cindy’s noticing in Vignette 8

In looking at snapshots of Cindy's noticing through Vignette 8, some regularities emerged with regard to what and how she noticed, and why she noticed the way she did. Together, these snapshots present a portrait of her noticing as shown in Figure 5.15, which highlights three aspects of her productive noticing throughout the whole lesson cycle.

First, Cindy attended strongly to the alignment between the three focal points. She was able to identify specific aspects of the concept, and understand the common mistakes made by students (See Section 5.2.3.1). For example, Cindy could describe how her students “used whatever digits that appeared in one representation for the other”, and related it to the key idea of place value. Building on Rani’s task, Cindy and the other teachers suggested refinements to the task design that were targeted at the students’ difficulty. Specifically, she showed a keen sense of awareness for the choice of numbers used in the task (See Table 5.1), and gave a justification for her choice to include 5/20. During the planning of the task, Cindy and the other teachers were able to draw on their observations to make sense of the difficulties encountered by their students (See Section 5.2.3.2), and designed a task, which had the potential to draw out students’ reasoning about decimal conversion.

By maintaining a clear focus on the focal points and the alignment, Cindy’s response to the students’ reasoning was always aimed at revealing their thinking. She adopted a more discursive style of instruction, which provided opportunities for her students to articulate what they were thinking. Through her preparation to notice (via the Lesson Play—See Figure 5.12), and careful orchestration of mathematical discourse (see her interaction with Student S1 in Section 5.2.3.3), Cindy was able to support her students’ understanding of the procedure. By adopting a hermeneutic listening approach (B. Davis & Renert, 2014), she was able to adjust her explanations and scaffolding of the task to achieve a better learning effect.

The focus—concept; confusion; course of action; and the alignment between the three—was also evident during the review of the lesson. Cindy was able to describe the students’ responses to the task, and highlighted the mistakes manifested by her students. She linked these observations to the key ideas targeted in the lesson, and was
able to gain new understanding of what her students know about equivalent fractions (See Section 5.2.3.4). Her detailed description of the students was also corroborated by other teachers; and together, Cindy and the teachers were able to suggest a few key modifications to the lesson. More importantly, teachers such as Flora, Anthony and Kirsty, realised the need to introduce equivalent fractions as equivalence classes of fractions (with infinitely many fractions in each class), instead of the usual two-by-two comparison. This created a new awareness of how students think about fractions, and teachers were able to generate new principles of teaching fractions, applicable to other lessons across different school levels.

Next, models of Cindy’s noticing show a consistent emphasis on analysing what she noticed during the planning, teaching and reviewing of the lesson (See Figure 5.10, Figure 5.11, Figure 5.13, and Figure 5.14). For example, she was attentive to, and analysed her students’ thinking during the whole-class discussion, as well as her interaction with Student S1 and his group of friends (See Section 5.2.3.3). Her analysis and preparation to notice prompted her to respond in a way that revealed the students’ reasoning. Cindy was then able to analyse her students’ responses, and tried to orchestrate a more effective discussion by achieving a balance between between scaffolding and fading (Mason, 2000). Cindy’s attention to, and interpretation of, what she observed provided the necessary information for her to decide on how best to draw out her students’ reasoning, which then provided opportunities for further attention and analyses. The inter-connectedness of Cindy’s noticing processes sharpened her pedagogical thinking, and this provided the impetus to generate new understanding for teaching decimals (See Section 5.2.3.4).

Besides generating new understanding, which provided insights for redesigning the tasks for future instruction, the connected noticing processes were instrumental in guiding teachers in the design of the original task. The task was designed to support student thinking about equivalent fractions, and connections to the conversion of fractions involving denominators other than powers of 10. Prompted by Cindy’s detailed description and analysis of student errors (See Section 5.2.3.1), other teachers highlighted related incidents they encountered, which provided useful considerations in the design of the task. For example, both Anthony and James pointed out that the reason for changing the denominator to powers of 10 might not be obvious to the students (See
Section 5.2.3.2). During the lesson, the task revealed the students’ thinking clearly and teachers gained insight into how they think (e.g., the “pairing strategy”), and realised that the students might be hindered by their erroneous thinking that equivalent fractions come in pairs only. The teachers then decided to emphasise equivalent fractions as a pre-task activity as one of the key modifications of the task. More importantly, the new understanding created a new awareness in teachers, such as Flora and Kirsty, to be careful about the way equivalent fractions are first introduced. Hence, unlike the case in Springside Secondary, there is a strong validation by other teachers in Greenhill Primary of what Cindy noticed. As Mason (2002, p. 93) highlights:

Validation of noticing and acting is based not on convincing others through rational argument or through the weight of statistics or of tradition, but rather through whether the other can recognise what is being described or suggested, usually through resonance with their own experience...

The new understanding generated by these teachers indicates an expansion in their resources. Similar to what Ball et al. (2008) describe, when Cindy and the teachers examined the conceptual ideas behind the conversion (Common Content Knowledge); designed a task that involved numbers that reveal the students’ thinking (Specialised Content Knowledge); recognised the students’ difficulty with particular fractions such as 1/5 and 1/4 (Knowledge of Content and Student); and decided to introduce equivalent fractions in “multiple sets” (Knowledge of Content and Teaching); they experienced a growth across these four domains of mathematical knowledge for teaching (MKT). Through the series of constructive discussions and observations, Cindy realised the importance of listening to her students to make sense of their reasoning, even when they belong to the “lower achievement bands”. During her reflection at the end of the study, Cindy expressed her renewed commitment to the goals of understanding her students’ errors (G3), and teaching for conceptual understanding (G2), which indicated her strengthened beliefs about emphasising reasoning in teaching (See Section 6.3).
Figure 5.15. Portrait of Cindy’s noticing in Vignette 8.

PLANNING THE LESSON

- She noted the importance of seeing decimals and fractions as different representations of the same number, and identified the notion of place value as the key to understanding the procedure.
- She recognised that students used any digit in the common fraction representation when they convert fractions to decimals.
- She analysed and realised that students did not understand the notion of place value when they perform fraction-decimal conversions.
- She analysed student mistakes and realised that a focus on procedures without reasoning (e.g., simplifying) may have contributed to students’ confusion over fraction-decimal conversion.
- She built on Ren’s card-sorting task, and suggested the numbers to use in order to target students’ confusion over fraction-decimal conversion. She also realised the need for students to articulate how they think about fractions and decimals.

TEACHING THE LESSON

- She noted that a few students had problems with the conversion of 76/100 to decimal.
- She recognised students might have difficulty understanding ‘place value’ and ‘hundredths’ from their responses of ‘7.08’.
- She considered the possibility that students might not see the role of the decimal point to indicate the unit from the fractional parts.
- She wanted to listen to Student S1’s reasoning to find out what he was thinking, in order to vary her explanation if necessary.
- She probed Student S1’s thinking and asked the question to other students to understand what they were thinking about the concept.
- She concentrated a discussion to reveal student thinking.

TEACHING THE LESSON

- She noted that Student S1 had problems with the conversion of 1/5 to decimal.
- She recognised he thought that 1/5 = 0.15 because ‘they have the same digit’ from his responses.
- She considered the possibility that students might not understand place value, equivalence, and logic underlying the conversion (change to power 10 denominator).
- She might have recalled the lesson plan done during the planning to reveal Student S1’s understanding of fraction-decimal conversion.
- She probed Student S1’s thinking to find out what he knew and understood about the fraction-decimal conversion.
- She orchestrated a discussion to reveal student thinking.

REVIEWING THE LESSON

- She described students’ responses to the task, e.g., pairing strategy etc.
- She realised that students might have problems with understanding that a number can have different representations.
- She recognised students assumed the same digit in all representations and might mix up equivalent fractions as pairs.
- She interpreted from other teachers’ description and analyses that students’ pairwise thinking might have something to do with how they were taught equivalent fractions.
- She realised that students might have abstracted the wrong idea about equivalent fractions as pairs only at the underlying logic in conversion, and its connection to place value may not be obvious.
- She refined the task by incorporating a quick review of equivalent fractions into the task to reveal their pairwise thinking about equivalent fractions.

Design lesson that reveals students’ thinking

Listen and build on students’ thinking to promote mathematical reasoning

Listen and build on students’ ideas during the interaction to support their attention to the errors in their thinking.

Analyze students’ thinking to gain a new understanding of how students think about equivalent fractions, and the possible impacts of the misconception.
5.3. Characterising productive noticing through portraits of noticing

5.3.1. Comparing their instructional decisions

A comparison of Anita’s and Cindy's instructional decisions during the planning, teaching, and reviewing, reveals that Cindy's instructional decisions could be described as more productive than Anita's in terms of enhancing the students’ reasoning. Table 5.2 shows a comparison of the two teachers’ decisions during the last phase of this study.

<table>
<thead>
<tr>
<th>Planning</th>
<th>Anita</th>
<th>Cindy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Design lesson that reveals the students’ thinking</td>
<td>Developed a worksheet, which consisted of a variety of questions on finding gradients as “rise over run”. The task did not consider any opportunities for students to show their reasoning.</td>
<td>Co-developed a task (a card sorting and reasoning task), which provides opportunities for students to show their understanding of equivalent fractions and place value, so as to make connections with decimal conversion.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Teaching</th>
<th>Anita</th>
<th>Cindy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Listen and build on the students’ thinking to overcome Difficult Point</td>
<td>Mainly focused on telling students the rule or procedure, or the use of analogies such as “good boy” and “base” to help students remember the ideas. Did not listen to her students’ thinking most of the time. Did not build on her students’ thinking but decided just to tell or explain without any reference to the difficulty faced by students.</td>
<td>Mainly focused on listening to her students’ explanations, regardless of whether students obtained the correct answers. Tried to build on their understanding to move students towards overcoming the Difficult Point.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Reviewing</th>
<th>Anita</th>
<th>Cindy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analyse the students’ thinking to expand cluster of resources, orientations and goals.</td>
<td>Did not analyse instances of her students’ errors in relation to the Key Point and Difficult Point, but attributed errors to other factors such as lack of time. Did not relate other teachers’ observations to perspectives on student thinking to develop new understanding on teaching gradients.</td>
<td>Able to pick up and analyse specific instances in relation to the Key Point and Difficult Point. Related other teachers’ observations to the Three Points and derived new understanding of the connection between equivalent fractions and conversion of fractions to decimals.</td>
</tr>
</tbody>
</table>

Referring to Table 5.2, Anita’s worksheet did not provide any opportunities for students to articulate their thinking, and was just focused on applying the method of “rise over run”. Although several difficulties, such as the confusion caused by the introduction of coordinates, were raised by other teachers, Anita did not view them as important (See
Section 5.1.3.2 Lines 15 and 25). She did not consider any of these concerns in the design of questions, nor think about other possible instructional strategies for her lesson. She designed the task to focus on getting correct answers with little or no emphasis on understanding. Hence, Anita’s task can be considered as one that focused on procedures without connections (Smith & Stein, 1998). In contrast, the task co-developed by Cindy and the other teachers provided opportunities for students to make sense of the relationships between the different fractions in order to figure out a way to convert decimals. As Rani pointed out during the discussion, reasoning played an important role and students were expected to explain how they came to the answers. According to Smith and Stein (1998), this task is considered to be of a higher cognitive demand, which requires students to be aware of their own thinking about fractions.

In terms of teaching, Anita typically focused on telling instead of listening, as illustrated in Vignette 7. Even though she might have taken notice of her students’ difficulty in understanding the notion of a gradient (See Section 5.1.3.3), Anita did not give students opportunities to explain what they were thinking before she explained to the students. For example, she used an illustrative example to clarify the confusion about steepness as height (See Section 5.1.3.3) by explaining instead of initiating a discussion to listen to students’ reasoning, whereas Cindy tried to find out what her students were thinking (See Section 5.2.3.3) by orchestrating a whole class discussion when the students gave an unexpected response or different answers. In contrast to Anita, Cindy also asked her students to explain what they were thinking during her small-group interactions, before she went on to ask more directed questions to make them aware of their errors (See Section 5.2.3.3 Lines 60 to 77). The key difference between Anita and Cindy is that Cindy always tried to listen and build on her students’ reasoning; while Anita always tried to explain her own reason with the students. Therefore, Cindy’s in-the-moment decisions could be considered to be more productive with regard to enhancing the students’ reasoning.

Lastly, Anita seemed to be more focused on justifying her decisions without analysing the students’ thinking from the observations highlighted by the teachers during post-lesson discussions. For example, she said that the confusion was expected (See Section 5.1.3.4 Lines 48 to 52), and attributed the lack of discussion to “MOE requirements” (See Section 5.1.3.4 Lines 60 to 62). Even when she attempted to look at her students’ errors,
Anita highlighted generic issues such as language ability without any reference to the mathematical ideas or the students’ learning difficulties. She also did not consider the issues related to the non-preservation of distances under different coordinate systems even though these ideas were discussed in depth with the teachers. Cindy, on the other hand, focused on describing and analysing her students’ reasoning processes. For instance, she highlighted her students’ “pairing strategy”, and described how they were confused about “1/5” as being equal to “0.15”. Cindy showed evidence of analysing her students’ errors by offering possible reasons based on the observations. She also listened to other teachers’ comments and made sense of them. Together with the teachers in her group, Cindy realised something that she was unaware of in the past—students may think that equivalent fractions only come in pairs. This realisation provided the teachers with useful information for revising the lesson, and implications for teaching fractions at other grade levels. Therefore, the teachers at Greenhill Primary School had begun to see the general meaning of such teaching incidents (Yang & Ricks, 2012), to generate useful pedagogical principles for future practice (Clea Fernandez et al., 2003; van Es, 2011).

As one can see from the analyses of these episodes, Anita’s instructional decisions are considered to be generally non-productive, while Cindy’s are considered productive in terms of their ability to enhance student reasoning. The noticing underlying these instructional decisions are captured through the snapshots of the two teachers’ noticing depicted in Vignettes 7 and 8. These noticing snapshots demonstrate fine-grained analyses of what and how these two teachers noticed, which build up a more complete picture to form a portrait of noticing for each teacher (See Figure 5.9 and Figure 5.15). A comparison of what, and how, they noticed is presented in the next two sections.

5.3.2. Comparing what Anita and Cindy noticed

As one can see, both Anita and Cindy were able to notice specific details with regard to the key mathematical ideas (concept) and the associated difficulties encountered by students (confusion). Table 5.3 presents examples of what Anita and Cindy noticed. Even though the examples in Table 5.3 are not exhaustive, they give a good representation of what Anita and Cindy noticed, and the level of sophistication of their noticing. Both teachers were able to highlight specific instances of student interactions
or students’ work, and identify students’ understanding of the key ideas from these
descriptions. They were also able to identify specifically the difficulties faced by
students. For instance, Anita highlighted specific students’ errors in identifying the
“run” when the coordinate axes are introduced in Vignette 7; while Cindy was able to
describe students’ mistakes in converting fractions to decimals in detail.

**Table 5.3: Examples of what Anita and Cindy noticed about the KP and DP**

<table>
<thead>
<tr>
<th>Concept</th>
<th>Confusion</th>
<th>Concept</th>
<th>Confusion</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Planning</strong></td>
<td>Gradient as “rise over run”.</td>
<td>Fractions and</td>
<td>Students assumed that the digits in one representation had to appear in</td>
</tr>
<tr>
<td></td>
<td>Students could not find the “run” when coordinate systems were introduced.</td>
<td>decimals are</td>
<td>the other.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>different</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>representations of</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>the same number.</td>
<td></td>
</tr>
<tr>
<td><strong>Teaching</strong></td>
<td>Gradients of any two line-segments on the same line are equal.</td>
<td>Conversion of</td>
<td>Students did not see the connection between conversion and place values.</td>
</tr>
<tr>
<td></td>
<td>Students mistook the hypotenuse as the gradient.</td>
<td>fractions to</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>decimals relies</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>on the notion of</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>place value.</td>
<td></td>
</tr>
<tr>
<td><strong>Reviewing</strong></td>
<td>Direction of the gradients (positive or negative).</td>
<td>Use of equivalent fractions to relate fractions to those with denominators of powers of 10.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Students confused the coordinates with the direction of the gradients in a coordinate system.</td>
<td>Students seemed to think that equivalent fractions only come in pairs.</td>
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</tr>
</tbody>
</table>

What, therefore, set Cindy’s noticing apart from Anita’s relates to what they noticed
about their decisions (courses of action) in response to what they attended to. In Anita’s
case, the approaches she thought of were often not aligned with the key ideas and their
associated difficulties. For example, her choice of analogies, such as “good boy to mean
positive slope”, did not have any mathematical connection with the key idea. Even
though these analogies might be helpful as memory aids, they were not related to
difficulties posed by the introduction of the coordinate systems. In fact, it was pointed
out by Don in the last session of the study, that most of the students did not use the
analogies to help them identify the sign of the slope; and they were still confused about
the directions of gradients. Anita consistently underestimated the cognitive obstacle
posed by the Difficult Point, and regarded the students’ errors as “trivial” (See Section 5.1.3.2 Line 25) or “expected” without seeing them in the context of the concept (See
Section 5.1.3.4 Line 52). Therefore, her non-productive instructional decisions often resulted from a lack of alignment between her response to the issue at hand.

In contrast, Cindy maintained her focus clearly on how the design of the task, and the type of questions she used, could reveal her students’ understanding about the concept. This can be seen in her choice of, and justification for, the numbers to be used in the task (See Section 5.2.3.2 Lines 37 to 43). This careful noticing prompted the other teachers to design the lesson by deliberating on important aspects of the task, such as the choice of numbers, and the number of cards in each set. Cindy was also able to identify the students’ understanding of fraction-decimal conversion, and analysed their confusion to engage them in mathematical discourses that revealed their thinking. Therefore, what she noticed about the three focal points were always specific and aligned, and this focus resulted in the design of a task that reveals student thinking. Hence, a distinguishing feature of these two teachers’ noticing is the alignment, or lack of, between the three focal points, that is, whether the instructional decisions made come from the consideration of an approach that provide possible ways for students to overcome their confusion, in order to learn the concept.

5.3.3. Comparing how Anita and Cindy noticed

The portraits of noticing of Anita and Cindy also suggest important differences in how they notice. As seen in Vignettes 7 and 8, both teachers were competent in attending to the three focal points. They could provide detailed description of what they saw with regard to the mathematical content, the student learning difficulties, and the teaching strategies or approaches. In Vignette 7, Anita attended to the notion of gradient as “rise over run”, and the students’ mistake in identifying the “run” (See Figure 5.1). She also attended to her approach of “telling”, which she claimed to have worked in the past (See Section 5.1.3.2 Line 7). Likewise in Vignette 8, Cindy was able to identify the key mathematical ideas related to fraction-decimal conversion, as well as detailed descriptions of the students’ common error (See Figure 5.10). She was also attentive to the former instructional approach of emphasising the procedural aspects of the conversion.

However, a key distinction in the noticing patterns of Anita and Cindy lies in the processes of making sense and deciding to respond. First, Anita did not examine more
closely the difficulties imposed by the introduction of the coordinate system during the design of the task. Without analysing the students’ difficulties in relation to the different coordinate systems, she trivialised the error, and did not target her instructional explanations (a key part of her approach) to the students’ confusion about gradients. As aligning the explanation with the key ideas and student thinking is an important criterion for effective instructional explanations (Wittwer & Renkl, 2008), Anita missed the opportunities to analyse why students were confused about the distances in order to prepare or improve her explanations beyond explaining the procedure. By contrast, Cindy not only attended to how related procedures involving fractions were taught, but more importantly, analysed them to see how these approaches might have contributed to the errors. For example, Cindy and the other teachers realised that the link between place value, and the need for powers-of-ten denominators might not be clear to the students (See Anthony’s comments in Section 5.2.3.2). Building on these ideas, Cindy went on to suggest important refinements to the list of numbers used for the task (See Section 5.2.3.2 and Table 5.1).

Next, the importance of making sense of what was attended to was also illustrated in the different ways Anita and Cindy responded to the students’ questions. In Vignette 7, Anita’s interaction with Student S4 revealed that Anita was aware of the students’ difficulty of counting the rise and run. However, she did not make sense of what exactly Student S4 was unsure about in “counting the rise” (See Section 5.1.3.3 Lines 37 to 41), or what she was thinking when dealing with negative coordinates (See Section 5.1.3.3 Lines 42 to 47). In both cases, Anita went on to explain in her own way (“You find the perpendicular” and “You count by the boxes”) without seeking to understand the student’s confusion. She assumed that Student S4 saw the problem in the same way, and did not stop to ask and listen to the student’s reasoning. As a result, Anita’s explanations did not seem to be effective, because they were not based on her reasoning of what she noticed about Student S4’s thinking. Contrasting with Anita’s reaction during teaching is Cindy’s response to Student S1 in Vignette 8. Cindy not only orchestrated an instructional dialogue with Student S1 to reveal his thinking, but more critically, built on his understanding to help him and other students (Student S5 and S6) see the error (See Section 5.2.3.3 Lines 60 to 77). Cindy’s questions to Student S1 demonstrated her active
interpretation of what he was thinking, and possibly, a consideration of the lesson play prepared during the discussion (See Figure 5.12).

Lastly, the reasoning component skill of noticing is critical for teachers to generate new understanding about the students’ thinking during the post-lesson discussion. Anita’s insistence on using the same instructional approach; and her decision to “cut back” on the time spent on explaining the concepts, was not based on the analysis of her students’ understanding of the key ideas. In spite of several insightful observations by other teachers, Anita did not seem to think much about the various student errors raised by teachers. When teachers discussed some of the ideas, which were introduced with regard to the possible confusion about different distances, Anita could not see the connection between these ideas and the students’ errors. Instead, she attributed the confusion to lack of practice, and wanted to spend more time on “going through the examples”. This stood in contrast to how Cindy and the other teachers thought about their students’ responses to the task. The teachers at Greenhill Primary School interpreted the students’ responses in relation to the key mathematical ideas, and considered how the students thought about these concepts. They analysed their mistakes and figured out a possible connection to students’ image of equivalent fractions as a pair-wise entity.

Therefore, teachers’ reasoning about what they observe in terms of the three focal points can play a critical role as the mediator between attending and responding with a productive instructional decision. The central role of sense-making in noticing, as suggested by the analyses in this Chapter, supports the hypothesis by Barnhart and van Es (2015) that interpretation serves as a conduit for attending and responding. As consistently demonstrated by Anita, attending to specific details about the focal points without analysing them did not result in productive instructional responses in Vignette 7. Whereas, Vignette 8 not only presents what and how Cindy noticed about the focal points; but more importantly, how she reasoned about these points to bring these three points into alignment. Therefore, it is not just the focus, but also the focusing of noticing that matters in order to make productive instructional decisions to enhance student reasoning.
Chapter 6. Discussion

This thesis extends the research on mathematics teacher noticing to investigate what, and how, teachers notice through the three main stages of a lesson cycle: planning, teaching and reviewing. This study sought to answer one main research question:

*What makes teachers’ mathematical noticing productive for enhancing student mathematical reasoning?*

Specifically, in the context of planning, teaching and reviewing a mathematics lesson, the researcher aimed to answer the following subquestions in this study:

RQ 1: What do mathematics teachers attend to when they productively notice about their students’ reasoning? In particular, is an explicit focus for noticing useful to encourage more productive noticing, and if so, what kind of foci can be used?

RQ 2: How do teachers interpret and make sense of instructional details that lead to decisions that are productive in terms of enhancing student mathematical reasoning?

RQ 3: What are the changes in teachers’ resources (mainly knowledge), orientations, and goals with respect to teaching for mathematical thinking when they begin to notice more productively?

This chapter answers the research questions posed through the constructs of the FOCUS framework, and discusses what teachers notice, how they notice, and when their mathematical noticing is productive. The discussion draws from existing research to highlight the key contributions of this thesis to the study of teacher noticing. The following chapter then presents the theoretical, and practical implications of the FOCUS framework; the limitations of this present study; and suggests some recommendations for future research in this field.

6.1. What makes noticing productive?

The FOCUS framework, developed from detailed analyses of Lesson Study discussions that took place during this study, suggests two critical components for productive
noticing: the focus for noticing; and the focusing of noticing. These two dimensions address RQ1 and RQ2 respectively.

6.1.1. Focus for noticing: Beyond the specifics of an explicit focus

*RQ1: What do mathematics teachers attend to when they productively notice about their students’ reasoning? In particular, is an explicit focus for noticing useful to encourage more productive noticing, and if so, what kind of foci can be used?*

The focus dimension of the framework not only suggests that an explicit focus is useful, but also, more importantly, that the alignment between what is noticed and the course of action is critical. Findings from Phase 1 of the study concur with results from other research on noticing, that teachers notice a wide variety of instructional details when no explicit focus is given (Borko et al., 2008; Erickson, 2011; B. Sherin & Star, 2011; M. G. Sherin, Russ, et al., 2011; Star et al., 2011; Star & Strickland, 2008). For example, many of the teachers from Greenhill Primary School focused on superficial aspects of instructional details during the Song episode (See Section 4.2.3.2), and may have overgeneralised about the learning of the students based on their enthusiasm in singing the song. Likewise, Zelina focused on how she could get her students to respond with the keywords she wanted to hear, instead of what her students were thinking (See Section 4.2.2.1), and concluded that the students had “an inkling” of the key mathematical idea in the lesson (See Section 4.2.3.1). As highlighted by Erickson (2011), it is easier to attend to more generic aspects of classroom instruction than specific students’ mathematical ideas. Whereas many of the studies on teacher noticing (Erickson, 2011; Star et al., 2011; Star & Strickland, 2008; van Es & Sherin, 2002) suggested that noticing can be challenging for novice teachers; this study highlights that noticing specific and relevant mathematical details may not be trivial even for experienced teachers.

By examining what teachers noticed when they made instructional decisions that supported student reasoning, this researcher found that these episodes of noticing were focused on one or more of the following aspects: content; student understanding; or pedagogy. When teachers direct their attention to the mathematical details of the concept, they appear to gain a better understanding of the topic they are teaching. For instance, Hannah demonstrated a keen awareness of the subtlety of Zelina’s comment
that a fourth part of a rectangle was not 1/4 because the whole was not divided equally into four parts (See Section 4.2.1.2). Similarly, Lindy’s attention to the definition of an interior angle, during both the planning and teaching stage of the lesson cycle, enabled her to respond to her students’ thinking in-the-moment during the lesson (See Section 4.2.4.1). This increased, detailed attention to mathematics is similarly observed when a frame (Levin et al., 2009), such as a classroom artefact (Goldsmith & Seago, 2011) or an organisation framework (Santagata, 2011), is used to direct noticing.

For the remaining part of the study, an explicit focus in the form of the Three Point Framework (Yang & Ricks, 2012), which examines aspects related to the content, student learning, and pedagogy of the lesson, was provided for teachers to direct their attention. The teachers’ noticing became more focused, as evident by the improvement to focused or extended noticing (van Es, 2011) demonstrated by all the schools in Phases 2 and 3 (See session signatures in Section 4.1). These findings provide an affirmative answer to the question posed by Star et al. (2011), which seeks to test whether an explicit focus would be useful for enhancing teacher noticing. Analyses of these vignettes suggest that providing teachers a focus, such as the Three Points, may help them to zero in on more mathematically significant events. This emphasis on seeing the critical details, and not just any event, supports the suggestion by Barnhart and van Es (2015) that “learning to see the important details of student thinking... is a cornerstone for more sophisticated analysis and instructional responses” (p. 91). The Three Points thus served as a good frame for directing teachers’ attention to these important details.

Moreover, the findings from this study also establish specificity as an essential characteristic of productive noticing. This agrees with several other studies, which posit the ability to focus selectively on detailed aspects of instruction as a mark of noticing expertise (Ceneida Fernandez et al., 2012; Goldsmith & Seago, 2011; Jacobs, Lamb, et al., 2011; Santagata, 2011; Schifter, 2011; van Es, 2011; Vondrová & Žalská, 2013). Instances of specificity in noticing, which occurred only a few times in Phase 1 (See Hannah’s noticing in Section 4.2.3.3, and Lindy’s noticing in Section 4.2.4.1), were more abundantly evident in Phases 2 and 3. Anthony’s analysis of how textbooks presented fraction addition (See Section 4.3.1.1); James’s description of his students’ confusion regarding fractions (See Section 4.3.1.2); Anita’s and Winston’s analysis of their students’ error in finding the run (See Section 5.1.3.1); and Cindy’s interpretation of her
students’ mistakes in converting fractions to decimals (See Section 5.2.3.1), are just a few of these instances that occurred when teachers planned the lessons for this study. Similar episodes were observed during teaching (See Lindy’s teaching in Section 4.2.4.1; Cindy’s in Section 5.2.3.3), and during the post-lesson discussions as well (See Hannah’s in Section 4.2.3.3; James’s in Section 4.3.3.1; Cindy’s and her other colleagues’ in Section 5.2.3.4).

However, specificity is not a sufficient condition for productive noticing. For instance, Anthony gave a detailed analysis of the way textbooks presented the concepts of adding fractions, and highlighted some possible errors by students when they add fractions (See Section 4.3.1.1). Although he could have explored other possible errors in interpreting the area models (See Figure 4.8), Anthony’s description was nonetheless very specific. Hence, what made his noticing less productive in terms of enhancing student reasoning was the lack of attention to the alignment between his suggested approach and the error. Similarly, Anita was detailed and specific in her identification of the students’ confusion about the run; but she did not examine whether her suggested approach was targeted at the problem students faced (See Section 5.1.3.2).

On the other hand, when teachers attend or make sense of both the Three Points and their alignment, their noticing tends to be more productive. The case study of Cindy and her colleagues demonstrated how a more focused noticing on the alignment of the Three Points can promote instructional decisions, aimed at revealing and understanding student reasoning. As detailed in Section 5.2.3.2, Cindy and the other teachers sustained a discussion on understanding their students’ errors in fraction-decimal conversion (e.g., “1/5 = 0.15’’), which then supported them in developing the fraction-sorting task. The teachers realised that guiding students to see the mathematical connections between the concepts of place value and equivalent fractions is the critical feature of the task. As a result, they designed the task to provide students opportunities for articulating their thinking when they convert fractions to decimals. Similarly, James and his teachers not only focused their attention on specifics of students’ answers, but also on how their answers relate to the concept during the post-lesson discussion. They noticed that their students’ answers, which were obtained by simplifying without partitioning, indicated a possible gap in their knowledge of fractions, and could cause confusion when students work on more advance concepts. In both these cases, the
teachers’ attention to the alignment between the intent of the task, and the students’ thinking from their responses, had enabled them to gain new understanding about how students reason about the topic (See Sections 4.3.3.2 and 4.3.3.3). These, and other examples, suggest that noticing the alignment between the instructional response, and the sources of the students’ confusion about the concept, is critical when thinking about the design of the task during the lesson cycle.

This focus on the alignment between the Three Points, or more generally, the mathematics-learner-teacher milieu (Brousseau, 1997) is related to what researchers like van Es (2011), and Barnhart and van Es (2015), had highlighted about responding with instructional decisions that are based on teachers’ observations. However, these researchers were more concerned about the issue of alignment during the responding component of noticing; whereas this study demonstrates that the alignment of the milieu is crucial even during the attending and making sense part of noticing. Hence, attending to, and analysing whether a course of action (either a teacher’s decision; a textbook’s presentation or example; or task etc.) is directly related or linked to the concept, and its possible points of confusion will be another mathematically worthwhile aspect to focus on for noticing. Therefore, this study’s findings propose that the alignment between the three focal points of the framework (concept; confusion; course of action) can be a new fruitful focus for noticing.

6.1.2. Focusing noticing: Teacher reasoning to align the three focal points

RQ 2: How do teachers interpret and make sense of instructional details that lead to decisions that are productive in terms of enhancing students’ mathematical reasoning?

Another key finding of this research is the central role of teacher reasoning in aligning what teachers see, to their instructional responses. In order to coordinate the instructional decisions with what they observe, it is necessary for teachers to make sense of their students’ difficulties when learning the key mathematical ideas. Based on their interpretation of the students’ errors, teachers can then make a reasoned decision about a potential approach or strategy targeted at the mistakes observed. Hence, there are two aspects of this alignment to be considered for the focusing of teacher noticing. Firstly, to see whether there is an alignment between the three points; and secondly, to ensure that a teacher’s decision to respond is aligned to what he or she has seen and
interpreted with regard to the concept and confusion. For each of these aspects, teacher reasoning is key for focusing one’s noticing to engage in productive practices. Although other similar research suggests the importance of analysing or interpreting instructional details in teacher reflection (Barnhart & van Es, 2015; Berliner, 2001; Timperley et al., 2007), this present study extends their findings by addressing the object of this teacher reasoning process.

Seeing the alignment between the three focal points can be challenging. Even with the explicit focus afforded by the Three Point Framework in Phase 2, teachers still seemed to find it difficult to maintain a focus on the alignment of the three focal points. This was so even for Trafford Secondary, which consistently placed higher in the noticing levels according to van Es’ framework (2011). For instance, during the ninth session of Phase 2 (See Figure 4.3 Phase 2 Session 9), several teachers made many observations about student thinking during the lesson on straight line graphs, and highlighted a number of connections between what was observed and the principles of teaching and learning. But it was Keaton, the head of department and the facilitator, who went one step further to point out that there may be distractors embedded in the design of the task used for the lesson:

I was thinking more about the design of the worksheet. The design is such that we want them to plot points, draw the graph, and make observations. Then plot some more points, draw some more graphs, and make observations. The intent is for them to compare the first and second set of observations, and come up with a general observation. I felt that there are certain distractions in the design. The scale thing is one thing. The plotting… generally most of them could get. But in terms of getting the relationship between $x$ and $y$, especially in activity 1, it stunned a few students. This relationship thing could be a distraction. It wasn’t the key focus for this lesson. There were other observations that the students make. I think they said, all are straight lines, they pass through the same point... these are all correct observations, but they could be possible distractions to the students.

Keaton’s comments highlighted to the teachers the possibility that students may perceive the term relationship differently; and that this distraction was not helpful for students to learn the key idea. Hence it would be important for the teachers to examine closely whether there were any other such elements in the worksheet that might need
some tweaking. Therefore, Keaton noticed specific details of the students’ responses to the worksheet, and attended to whether the task design targeted the students’ confusion about the concept by relating evidence from his observations to his knowledge about mathematics and teaching.

The role of reasoning in ensuring the alignment between the three focal points is also illustrated by Cindy’s reasoning to include 5/20 as part of a set of fractions equivalent to 1/4 (See Section 5.2.3.2). She understood that the critical point was to get students to see the central role of place value in fraction-decimal conversion, in particular, the need to convert fractions to equivalent fractions of denominators of 10, 100 and 1000. Anticipating that students may fail to see the need to convert 1/4 to 25/100, Cindy suggested 5/20 as a transitional fraction for students to relate 1/4 to 25/100. She hypothesised that students would encounter problems if they tried to convert 5/20 to a fraction with denominator 10, and this may prompt them to consider converting to a denominator of 100 instead.

In contrast, Anita, unlike Cindy, did not make sense of the difficulties faced by students when she designed the worksheet for them. The lack of reasoning about the students’ cognitive obstacles in learning the key idea had impacted the type of questions included in the worksheet. As seen in Section 5.1.3.2, she included three different types of coordinate systems (See Figure 5.3) without realising that the way distances were calculated was different in each coordinate system. Given the students’ limited experience with coordinate systems, these questions posed considerable problems. These problems were further augmented because Anita did not think of other ways to support her students’ understanding of these subtle differences. Therefore, the reasoning or analysis needed to ensure the alignment between the Three Points was missing in this case. For this reason, the instructional decisions taken by Anita did not correspond to difficulties faced by her students, and this can be seen when she could not respond effectively to Student S4’s confusion (See Section 5.1.3.3) during the lesson.

Furthermore, Cindy’s reasoning about her students’ mistakes had enabled her to respond more effectively to Student S1’s confusion about 1/5 and 0.15 (See Section 5.2.3.3). Having attended to this error (Section 5.2.3.1 Lines 2 to 17), and anticipated possible ways to engage the students to make sense of their own thinking (See Figure
5.12), Cindy was in a better position to respond to Student S1’s error. Therefore, she could bring to her mind the necessary knowledge for orchestrating a mathematical discourse to find out what Student S1 was thinking during a classroom interaction (See Section 5.2.3.3 Lines 17 to 34).

These snapshots of noticing indicate that the component skill of responding, within the construct of noticing, is dependent on whether, and how, teachers analyse their observations. Hence, this study suggests that a response, which is productive with regard to enhancing student reasoning, has to be mediated through teacher reasoning on what they observe. In their quantitative modelling (using Item Response Theory) of noticing, Seidel and Stürmer (2014) found that the three skills of reasoning: describing, explaining, and predicting, are highly interrelated, and significantly associated with the construct of professional vision (Goodwin, 1994). Thus, their study (Seidel & Stürmer, 2014) supports the centrality of reasoning as suggested by the findings from this present study. The emphasis on teacher reasoning to bring about the alignment of the three focal points also supports the hypothesis, proposed by Barnhart and van Es (2015), that “analysis is the bridging skill between attending and responding” (p. 91). The Three Points thus not only serve as an essential focus, but also as a means to encourage systematic analysis—describing, explaining, and predicting (Seidel & Stürmer, 2014)—by highlighting the alignment between these three aspects of teaching mathematics as a criterion for teachers’ consideration. Finally, teachers’ reasoning about the alignment of the focal points, can serve as a support for teachers to link their knowledge about teaching and learning to their classroom observations—an important skill of learning to teach from a situative perspective (Putnam & Borko, 2000).

To summarise, the FOCUS framework, developed from the analyses of instructional decisions that enhance student reasoning, reflects the following two characteristics of productive mathematical noticing:

1. **The Focus – What to notice:** (a) Specific mathematically significant aspects of learning and teaching, such as the Three Points; mathematics-learner-teacher milieu; or simply the concept, confusion, and course of action. (b) The alignment between the teaching approach and the students’ learning difficulties associated with the mathematical concepts; and
2. *The Focusing – How to notice:* The central role of sense-making or reasoning as a mediator between seeing and responding. It is the analysis of the observations that provide the evidence or justification for making an instructional response that promotes student reasoning.

As introduced in Chapter 4, a photography metaphor of focus and focusing highlights the active role of a photographer to create and capture a beautiful bokeh in the picture. Similarly, the FOCUS framework posits that learning from a systematic investigation of practice (capturing a series of snapshots with bokeh) is not automatic; but rather, it is a moment-by-moment reasoned noticing (focusing) of mathematically worthwhile details (focus) and their alignment (focus), which promotes a teacher’s engagement with productive practices that enhance reasoning. Each snapshot of noticing presents a picture of the teacher’s noticing in terms of what was noticed, and how. Moreover, this metaphor can present opportunities to think of professional development as putting a series of photos together to form a time-lapse movie, which portrays the changes in a teacher’s ROG clusters over time.

6.2. Productive noticing and ROGs

*RQ 3: What are the changes in teachers’ resources (mainly knowledge), orientations, and goals with respect to teaching for mathematical thinking when they begin to notice productively?*

As seen from the snapshots of noticing featured in Chapters 4 and 5, productive noticing is highly consequential. Schoenfeld (2011b) highlights that noticing matters, and proposes studying noticing in the context of teaching as a goal-oriented activity. Examining the teachers’ noticing in this study, with regard to their ROG clusters, reveals a possible relationship between teachers’ ROGs and the way they notice. For example, Anita’s persistence in her “telling” strategy can be explained in terms of her preference for highly structured teaching strategies, such as worksheets, and her goal of preparing the students for examinations. Her knowledge and beliefs about mathematics and her students also play a part in the instructional decisions she made during the study. Likewise, Cindy’s openness to adapt her practice can also be linked to her resources, orientations, and goals. Hence, as Schoenfeld (2011b) has proposed, a teacher’s noticing
can be explained to some extent in terms of one's ROGs (Hannah et al., 2011; Paterson et al., 2011; Schoenfeld, 2008, 2010).

6.2.1. Changes in Mathematical Knowledge for Teaching (MKT)

Another key finding of this study proposes that productive noticing is an important mechanism to effect changes in a teacher's ROG clusters during the lesson cycle. The notion of productive noticing, thus provides a possible means of describing how a teacher's ROG clusters may change during professional development.

When teachers productively notice mathematically significant details that they had missed out in the past, they can gain new insights into *mathematics knowledge for teaching* or *MKT* (Ball et al., 2008). Hannah's observation of Zelina's misconception about $1/4$ (See Section 4.2.1.2), and Lindy's planned example to challenge the anticipated definition of an interior angle as an angle inside a polygon (See Section 4.2.4.1), are examples of how teachers' existing knowledge for teaching mathematics can be expanded. Lindy's example was actually generated during a fruitful session (See Figure 4.3 Phase 1 Session 1), when the teachers discussed whether the angular properties of a convex polygon also extend to a concave one. In examining additional mathematical content related to the lesson, Lindy and her colleagues extended their understanding of interior and exterior angles to all polygons, and explored different ways of proving the formula. Likewise, when Cindy and her teachers discussed the procedure of converting fractions to decimals (See Figure 4.1 Phase 3 Session 2), they noticed a gap between the procedure (converting to fractions with denominator 10, 100 or 1000) taught in Primary Four (age 10) and that taught (using long division) in Primary Five (age 11). Their productive noticing of this gap motivated them to ask the researcher why the two procedures seemed so different. The researcher then highlighted the connection between the two procedures, through a short lecture on rational and irrational numbers, and the connection to the base 10 representation system. That short exposition on real number supported teachers to see the connections between the two procedures, and they became aware of possible implications when they switched to a new procedure without making the connection.

Changes in teachers’ MKT were also documented during several post-lesson discussions. For instance, the teachers at Greenhill Primary noticed the students’
possible concept image of equivalent fractions during the last session in Phase 3 (See Section 5.2.3.4). They realised that students see equivalent fractions as a pairwise entity, and this image may have hindered their understanding of fractions. As a result, the teachers suggested that it is crucial to introduce equivalent fractions as a collection of many fractions, instead of focusing only on pairwise comparisons. This helped to deepen their understanding of fundamental mathematical knowledge.

6.2.2. Changes in orientations and goals

Studies have indicated that a focus on developing student reasoning has a positive effect on teachers’ beliefs and goals about mathematics teaching (Franke et al., 2007; Jacobs et al., 2010; Philipp, 2007; Stipek et al., 2001), though these changes are probably complex and dynamic (Barkatsas & Malone, 2005). The snapshots of noticing presented above not only support the importance of focusing on student reasoning, but also address how this dynamic change in beliefs can be captured.

For example, even though Anita’s noticing was largely characterised as non-productive, the snapshots of her noticing reflect her emphasis on how she reasoned about mathematics: a highly logical subject, which demonstrated in her expert blind spot (Nathan & Petrosino, 2003) that students would see a task or procedure in the same way as she did (See Sections 5.1.3.2, 5.1.3.3, and 5.1.3.4). Her perception was challenged during the last session of Phase 3 (See Figure 4.2 Phase 3 Session 9) when Winston and Don highlighted the possibility that students might not use the same rule (“drawing the person at the base”) she taught:

1. Winston: The little person that you drew, right? It doesn’t work for this class? I don’t know. Sometimes you drew on the left; sometimes you drew on the right…
2. Anita: But it’s always drawn at the base of the line. I told them always look at the person walking up.
3. Winston: Oh? It’s always drawn at the base?
4. Anita: Actually I wanted to tell them a story, but no time.
5. Don: But actually most of them don’t use this. When I asked them right, they looked at me with eyes opened. They did not seem to remember.
7. Anita: Was E3 [the other class] using this?
Chapter 6 – Discussion

8. Winston: They were using the same thing, looking at where the line is pointing.

9. Don: I only saw one boy draw [sic]. He drew the man everywhere. So, he got the directions right.

Anita’s blind spot was pointed out by Don in the discussion that followed, when he referred to another observation:

There’s one thing I realised. Every triangle you drew is the big one, and so the students also drew the big one. And you don’t always tell them why they have to start here or there. They also don’t get a hang of why they have to start here or there. You just drew the triangle and said, the rise is 5... Here, Anita realised that she had only used the biggest possible triangle to find gradient in all her examples; and the reasoning behind her choice of triangle was not made explicit. This revelation prompted Anita to consider the possibility that she may have overlooked this in her teaching. This noticing in the intersection of the three worlds of experience (Mason, 2002) was crucial for Anita to raise awareness of her own blind spot, and was indicative of her changing orientations with regard to teaching.

Likewise, Cindy’s interactions with Student S1 on “1/5 as 0.15” highlighted her attention and making sense of student thinking. It also reflected how she reconciled her conflicting beliefs about mathematics (O3, O4, and O5) and teaching (O6, O7 and O8). There is evidence from the analyses of these discussions that what, and how, Cindy noticed during the planning sessions influenced how she responded in class. The emerging disposition of the teachers to examine closely the reasoning of students (See Section 5.2.3.4) highlights a shift towards genuine mathematical conversations (T. Wood et al., 1991), which values her students’ arguments in order to support their thinking (Kazemi & Stipek, 2001). This change in orientation is perhaps best captured by Cindy in her own reflection at the end of the whole study, when she reflected about her awareness of the importance of listening to student thinking at the end of the study:

I now find myself asking my pupils to explain their answers, when in the past, I used to tell them their mistakes... thank you for ‘opening’ my eyes and making this project a fruitful one, and to appreciate my pupils’ answers/errors...

Cindy’s growing emphasis on seeking her students’ reasoning (Brodie, 2010c, 2010f; Burns, 2005; Watson, 2002), and listening for their reasoning is consistent with the
questioning moves that promote mathematical thinking (Brodie, 2010d; Franke et al., 2009; Isoda & Katagiri, 2012; Smith & Stein, 2011).

Even though, these snapshots of Cindy’s noticing only captured a small slice of her development through Phases 2 and 3, the findings demonstrate the potential of the models to capture her noticing at a finer grain over the entire phase. This can build up a more detailed portrait, which will better reflect the changes of her ROG clusters over time. Nevertheless, these instances not only highlight the possible links between expansion of a teacher’s ROGs and his or her noticing, but also propose noticing as a possible means to effect expansion in teachers’ clusters of ROGs, which may in turn change practices in teaching.

6.3. Summary

The FOCUS framework, developed in this study, positions productive noticing as a high-leverage core practice (Grossman et al., 2009) that enables teachers to work on the components of effective teaching, while honouring its complexity. It captures the regularities or patterns demonstrated by teachers as they work through the planning, teaching and reviewing of a lesson, and provides a way to characterise and analyse their noticing. The two components, focus and focusing, highlight what and how teachers can notice to support their students’ reasoning. The focus on the mathematical concepts, associated students’ confusion, teachers’ course of action, and the alignment between these three focal points directs teachers to notice mathematically worthwhile details through the lesson cycle. The focusing emphasises the role of teacher reasoning to bring these three aspects of the milieu into alignment.

A theoretical model, which describes a teacher’s productive noticing through the lesson cycle, was developed from the FOCUS framework to analyse teachers’ noticing. This model is a step towards the development of a more comprehensive one as proposed by B. Sherin and Star (2011) to advance the field of teacher noticing. Last but not least, the photography metaphor afforded by the Framework, provides a way to represent teacher noticing across the processes of learning from practice over time. Snapshots of a teacher’s noticing can be examined together to provide a portrait of their noticing. These portraits of noticing not only reflect the teacher’s underlying ROGs cluster, but also highlights changes in the ROGs as a result of productive noticing.
Chapter 7. Implications, Limitations and Conclusion

Teaching is highly complex. Lampert (2001) attempts to capture this complexity through the metaphor of a lens. She contends that a teacher has to actively zoom in and zoom out, across time and relationships, to focus on different aspects of teaching, from the big picture of the bigger educational context; to the moment-by-moment interactions with a student; and this may involve more than a single focal point. The FOCUS framework developed in this study, not only provides a lens to view these classroom interactions, but more importantly, proposes how teachers can direct their attention to mathematically worthwhile details. The framework thus complements Lampert’s lens metaphor by highlighting what and how teachers notice in their classrooms. In this concluding chapter, the implications of this study will be presented before a brief discussion of the limitations of this research, which sets the stage for the suggestions for future research.

7.1. Implications of productive noticing

This research is most likely the first study in Singapore that examines teachers’ mathematical noticing during planning, teaching, and reviewing of a lesson. The findings highlight the significance of this study in two aspects: its theoretical and practical implications.

7.1.1. Implications for researching teacher noticing

The FOCUS framework developed in Chapter 4 offers a means to characterise aspects of noticing that result in pedagogically productive outcomes, which provide opportunities to enhance student reasoning. The two dimensions of productive noticing—focus and focusing—can be used to develop snapshots of teacher noticing to analyse what a teacher notices at two levels. At a more fine-grained level, the FOCUS framework is an example of a core practice that enables analysis of the key components of teaching, while preserving the complexities of practice. The snapshots of noticing developed from the theoretical model of productive noticing provide a way to describe what and how teachers notice. At the macro level, these snapshots can be combined to give a portrait of teachers’ interactions with students and mathematics in terms of what and how they
notice. These portraits serve as representations of practice, which can be used to
discuss and analyse teaching (Grossman & McDonald, 2008) and its interactions with
the processes of noticing.

As demonstrated in Chapter 5, the framework serves as a tool to analyse a teacher’s
noticing processes during planning, teaching, and reviewing of a lesson. The snapshots
can be used to conduct a fine-grained analysis, and construct a portrayal of a teacher’s
noticing during each step of the lesson cycle. These models contain representations of a
teacher’s focus in noticing and the mechanisms involved in noticing, which are linked to
outcomes of lesson planning, teaching and reflection. Regularities in terms of what and
how a teacher notices can account for why a productive or non-productive response is
taken. Thus, the snapshots and portraits of noticing can offer comparisons and contrasts
of different teachers’ noticing, which may be useful in giving researchers insights into
the construct of teacher noticing.

Second, the meta-level schematic representation of teacher noticing afforded by the
portraits also allows researchers to examine the interactions between what teachers
notice, how they notice, and the motivations or hindrances to notice productively. In
particular, this meta-level representation highlights the relationship between a teacher’s
resources, orientations, and goals (ROG) with his or her noticing. As argued by
Schoenfeld (2011b) what a teacher notices is a “function” of one's ROGs (p. 231). The
findings from this study suggest that noticing, when productive, can expand a teacher’s
ROGs with regard to “engaging in diagnostic teaching” (Schoenfeld, 2011a, p. 191).
Referring to Figure 7.1, Schoenfeld (2011a) models the growth of teacher activity along
three planes in response to the growth of the ROG clusters held by the teacher.
Productive noticing, as characterised by the dimensions of noticing, builds on this model
of teacher activity, and proposes a possible mechanism by which this expansion can
occur. Therefore, the framework for productive noticing not only tracks the trajectory of
noticing expertise, but can also explain and predict the possible evolution of a teacher’s
clusters of ROGs.
7.1.2. Implications for improving teaching practice

As a practical tool, the framework supports teachers’ noticing by directing their attention to the *mathematically significant* aspects of engaging in diagnostic teaching. The Three Points offer a focus for teachers to attend to, and make sense of, in order to respond with a productive instructional decision. The findings from this study support the use of an explicit focus to frame noticing (Goldsmith & Seago, 2013), instead of not specifying the area to direct teacher noticing to (Star et al., 2011). By emphasising both specificity (Jacobs, Lamb, et al., 2011; Stockero, 2014; van Es, 2011) and alignment, the framework can encourage teachers to reason about the evidence from observations about the Three Points, in order to make instructional decisions that enhance student reasoning.

The model, developed from the FOCUS framework, demonstrates that productive noticing is highly consequential. The empirical evidence from this study suggests that if teachers can:

1. Attend to mathematically worthwhile details related to the concept, the students’ confusion and their courses of action (three focal points);
2. Attend to whether their course of action is targeted at the students’ confusion about the concept (alignment); and
Chapter 7 – Implications, Limitations and Conclusion

3. Make sense of these details and their alignment in order to justify their response to the students’ reasoning (focusing),
then they are more likely to:

1. Design a task that provides students opportunities to express their thinking about the concepts;
2. Plan more effective explanations of concepts to target the students’ confusion about the key ideas;
3. Listen to the students’ reasoning and interpret their explanations mathematically to generate an effective response;
4. Analyse instances of student reasoning to generate new understanding about how students think during post-lesson discussions.

The proposed model (See Figure 4.17) therefore can act as a self-reflection tool for teachers as they examine their own practice to learn from teaching. The model explicates and highlights possible areas for teachers to take note of as they engage in professional learning activities such as Lesson Study. This encourages them to take on the critical lenses needed to learn from Lesson Study (Clea Fernandez et al., 2003). More importantly, it is a means to reflect systematically, to suspend one’s habitual reactions to classroom events, in order to have a different act in mind (Mason, 2002). The framework is a way for teachers to engage in the “practices” of noticing, so as to enhance their “sensitivity to notice opportunities to act freshly in the future” (Mason, 2002, p. 59). The outcomes proposed as indicators of productive noticing provide considerations for teachers in the design of key tasks in any lesson that is aimed at enhancing students’ mathematical reasoning. Besides using the Three Points to design lessons, teachers can apply the Three Points to deliberate on orchestrating classroom discourses that build on student thinking. Lastly, these proposed outcomes provide a means to examine the effectiveness of teachers’ reflection on, and reviewing of, the lesson.

7.2. Limitations

While this thesis aimed to characterise the notion of productive mathematical noticing, it is important to acknowledge that the study was limited to investigating the
mathematical and pedagogical aspects of enhancing students’ reasoning. This study, for example, did not investigate what teachers notice about classroom management (van den Bogert, van Bruggen, Kostons, & Jochems, 2014) even though it may play a role in carrying out mathematical activities to enhance reasoning. Also, the study focused only on teacher noticing, due to the time-consuming nature of qualitative data analysis. However, what students notice mathematically about a task, and how they interact with the task as a result of their noticing, would have an impact on whether the tasks designed had enhanced the students’ reasoning (Lobato, Hohensee, & Rhodehamel, 2013). Therefore, teacher noticing and student noticing are ‘two sides of the same coin’, and it will be a fruitful area for future research to explore the relationships between teacher and student noticing (See Section 7.3).

Another constraint of this study was that the findings were developed from an empirical analysis of a small sample of teachers from three schools in Singapore, and hence, generalisations to other international school contexts may be limited. This limitation was partially mitigated by the wide spectrum of experience and school settings represented by the teachers from the three schools. Collectively, the teachers involved in the study reflected the typical clusters of resources and orientations with respect to mathematics teaching. Moreover, the fine-grained modelling of teacher noticing, at each stage of the Lesson Study, afforded by the framework developed in this study can provide opportunities for other researchers to trial similar studies in other countries and contexts.

Finally, this study is limited by the researcher’s own perspectives of teaching and learning mathematics. Given the complex nature of teaching, what is deemed as productive depends on what one believes about teaching and learning (Clarke, 2001). Although the criteria for productive outcomes of noticing were based on a paradigm of treating lessons as experiments (Hiebert, Morris, et al., 2003) and the key tasks of Lesson Study (Lewis et al., 2011), I acknowledge that my interpretations of the data only constitute one possible emerging narrative about productive teacher noticing. Despite drawing from a wide range of data sources, such as video recordings, audio recordings, teachers’ artefacts, and teacher interviews, to triangulate the findings, researcher bias is still a possibility. However, multiple iterations of analysis and coding using explicit constructs were carried out to reduce the bias of interpretation. The
reliability of similar studies could be improved by involving a team of researchers in the future.

7.3. Suggestions for future research

As discussed in Section 7.1, the framework for productive noticing provides a tool to analyse and support teacher noticing at both micro and macro levels. However, it remains to be seen whether the framework can be applied and tested in other contexts. In future research, I intend to trial and test the framework to see whether it offers a sufficiently robust explanatory model for teacher noticing in other school contexts and instructional settings. One such instructional setting is a technologically driven classroom environment, which is an interesting and potentially fertile site to explore teacher noticing. In particular, it may be useful to examine teachers’ noticing when they design, teach, and review lessons involving the use of technology, such as graphing calculators and other computer-based mathematical tools. Are the Three Points, for example, sufficient to direct teachers’ attention in a more technologically-driven lesson, or are there aspects of Pedagogical Technology Knowledge (Thomas & Hong, 2013), such as technology instrumental genesis, that may warrant teachers’ attention? These questions, and other related ones, may be a useful starting point to extend the framework in future research.

As demonstrated by Anita (See Section 5.1.3.3) and Cindy (See Section 5.2.3.3), noticing is highly consequential in promoting a student-centred mathematics instruction. Because this research only examines teacher noticing, important questions about the interactions between teacher noticing and student noticing remain unanswered. For example, how does teachers’ productive mathematical noticing influence students’ mathematical noticing of the concept targeted in the design of the tasks? How does students’ noticing of mathematics in tasks impact teachers’ noticing of the students’ confusion? What do students notice with regard to mathematical ideas in teachers’ explanations or mathematics discourses? What and how do teachers respond to students’ mathematical noticing? Addressing these questions may advance the research on teacher practices that build on student thinking.

Although the role of mathematical noticing in professional development was not the focus of this research, the vignettes described in Chapters 4 and 5 suggest that
productive noticing contributes to the expansion of a teachers’ ROG clusters. Hence, how productive noticing can be scaffolded or supported in other forms of professional learning activities, besides Lesson Study, would be an interesting area to explore in future studies. For example, Cindy experienced some significant changes in her orientations in terms of listening to her students’ reasoning (See Section 6.2.2), which suggests a productive shift in her noticing stance. Examining the conditions that promote this shift may be a highly consequential subject of future research. This resonates with other similar studies, which also put forth the important role of facilitation in enhancing noticing (Goldsmith & Seago, 2011; Kazemi et al., 2011). The framework developed in this study can thus provide a means to investigate the connections between a facilitator’s noticing and teachers’ noticing in a professional development context.

7.4. Concluding remarks

The intent of this research is to investigate characteristics of teacher noticing that result in instructional responses, which are productive for enhancing students’ mathematical reasoning. This study has extended the notion of teacher noticing to include an explicit focus for teachers to see, analyse, and respond. The focus, adapted from the Three Point Framework (Yang & Ricks, 2012) covers mathematically significant aspects of content, learning and teaching. The findings from this research suggest that such a focus may be helpful to develop teachers’ expertise in noticing. Moreover, this focus also provides two key criteria—specificity and alignment—for examining teacher noticing. Another key finding relates to the importance of the analysing component of noticing, as a bridge between seeing and responding. This thesis also examines teachers’ responses in terms of their instructional decisions and compares them to the outcomes developed from research on enhancing student reasoning. These three components—the object, the process, and the product, of noticing—are encapsulated in the FOCUS framework, which can serve both as a research tool, and as a practical tool for teachers to reflect on their practice. Last but not least, a teacher’s noticing can be captured and represented in a more comprehensive way through the snapshots and portraits of noticing.

Even with a fast lens, a photographer who wants to achieve a beautiful bokeh in a photograph, has to actively adjust his distance to the subject to maintain a sharp focus
on the subject, while blurring out the distracting background. Likewise, teachers make interactions with learners more mathematically worthwhile when they actively attend to, and make sense, of the relationship between their learners with mathematics to maintain a focus on student reasoning while blurring out the “noise” in the classrooms. To do this, one needs to prepare to notice. As Mason and Johnston-Wilder (2006, p. 127) put it: “The heart of teaching is interaction with learners; the rest is preparation to make this interaction useful”.
References


Barnhart, T., & van Es, E. (2015). Studying teacher noticing: Examining the relationship among pre-service science teachers’ ability to attend, analyze and respond to
References


References

(Eds.), *The Cambridge companion to Vygotsky*. Cambridge, UK: Cambridge University Press.


References


272


References


References


References


References


278
References


280
References


Appendices

Appendix 1: Sample of Participant Information Sheet

![Participant Information Sheet](image)

My name is Ban Heng Choy, and I am a doctoral student (Mathematics Education) with the Department of Mathematics at the University of Auckland. I am conducting research into how we can help mathematics teachers improve their practice through disciplined noticing. I will examine what disciplined noticing could look like in practice and develop ways to help teachers grow in their competences of noticing students' mathematical thinking processes.

I am inviting mathematics teachers from schools who are currently or intending to use lesson study as part of their professional development for this study. If you agree to take part in this research, you and other teachers, in the same school who volunteered for the study, will form a lesson study group comprising up to a maximum of four teachers. The research will be conducted as part of the school's existing professional development schedules over a period of 15 months in 2012 and 2013. The Singapore Ministry of Education has also given approval to conduct this study.

The study will take place in three phases. Each phase will involve one complete lesson study cycle lasting up to 6 weeks (2 hours each week). Phase 1 was completed in December 2012. In Phase 2 and 3, teacher participants need to go through one to two sessions of workshops on mathematical thinking and noticing. There will also be an interview (30 min) at the beginning and at the end of the study. Therefore, participants will need to give at least 30 hours (up to maximum of 35 hours) spread over a period of 6 months in 2013. For each lesson study session, you will be observed and audio recorded. Lesson observations will be video recorded and lesson plans and materials that resulted from the lesson study will be recorded and analysed. We will also collect and discuss samples of students' work as part of the lesson study processes.

The data (i.e., video recordings, audio recordings, lesson materials etc.) will be securely stored at the university and will be destroyed after six years. It will not be used for any other purpose outside of this project, and research personnel will have sole access to them. A third party, who will sign a confidentiality agreement, will transcribe the video and audio recordings. The transcripts will be kept securely for six years since it is anticipated that the research will lead to further peer-reviewed publication following completion of the PhD.

Should you feel the need to withdraw from the project, you may do so without question at any time before the data is analysed at the end of each phase. Any data supplied by you in that phase will not be used in the research. The end dates for each phase are as follows: Phase 1 – Sep 2012; Phase 2 – June 2013; Phase 3 – Sep 2013. Your principal has assured that your participation or non-participation in this study will not affect your employment. In addition, your participation in or withdrawal from the project will not affect your performance.
appraisals in any way. The learning points from the study will be shared with you at the end of each phase.

Pseudonyms will be used to label the audio recordings and video recordings, as well as in the transcripts and any publications. If you consent, video snapshots of you may (with your face digitally obscured if you prefer) be published in reports and academic journals. However, this part is optional – you may choose not to allow video images of you to be published or shown in professional presentations, while still being involved in the other aspects of this project.

If you choose not to allow video images of you to be published or shown in professional presentations, any information you provide will be reported/published in a way that does not identify you as its source. If you choose to allow video images of you to be published or shown in professional presentations, your anonymity cannot be guaranteed as you may be recognized from the images.

If you have any questions and would like to receive information about my study, please email me at bcho247@aucklanduni.ac.nz or call me at +64 212 660 081 (New Zealand) or at +65 958 11 958 (Singapore). You can reach me at the following postal address when I am in Singapore.

Contact Address in Singapore: Bk 321, Tah Ching Road, #21-82, Singapore 610321.

Alternatively, you can contact my supervisors at the following email addresses and numbers:

Prof Mike Thomas:
moi.thomas@auckland.ac.nz
+64 9 373 7599 ext 88791

Dr Caroline Yoon:
c.yoon@auckland.ac.nz
+64 9 373 7599 ext 88740

For any queries regarding ethical concerns you may contact the Chair, The University of Auckland Human Participants Ethics Committee, The University of Auckland, Research Office, Private Bag 92019, Auckland 1142. Telephone 09 373-7599 extn. 87830/83761. Email: humanethics@auckland.ac.nz.

APPROVED BY THE UNIVERSITY OF AUCKLAND HUMAN PARTICIPANTS ETHICS COMMITTEE ON 15 June 2012 FOR (3) YEARS. REFERENCE NUMBER K218.
Appendix 2: Three Point Template

KEY POINT

DIFFICULT POINT

APPRAOCH

What you should see / hear in order to know that your students have understood the key point?
Appendix 3: Lesson Plan Template

**KEY TASK DESIGN**

<table>
<thead>
<tr>
<th>Basic Information</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Subject / Unit / Topic</td>
<td></td>
</tr>
<tr>
<td>Level / Class</td>
<td></td>
</tr>
<tr>
<td>Lesson Duration</td>
<td></td>
</tr>
<tr>
<td>Key Idea</td>
<td></td>
</tr>
<tr>
<td>Difficulty</td>
<td></td>
</tr>
<tr>
<td>Approach</td>
<td></td>
</tr>
</tbody>
</table>

**Prior Knowledge / Skills from previous lesson**

In the previous lesson, students should have opportunities to ...

**Key Task:**

**Description**

**Set up**

**Anticipated Students’ Responses (Lesson Play for Key Task)**

<table>
<thead>
<tr>
<th>Possible students' responses</th>
<th>Teacher’s responses (Lesson Play)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Sequence of Lesson**

**Introduction (Duration)**

<table>
<thead>
<tr>
<th>Description</th>
<th>Duration</th>
<th>Comments (if any)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Key Task (Duration)**

<table>
<thead>
<tr>
<th>Description</th>
<th>Duration</th>
<th>Comments (if any)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

See above

**Presentation (Duration)**

<table>
<thead>
<tr>
<th>Description</th>
<th>Duration</th>
<th>Comments (if any)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Conclusion (Duration)**

<table>
<thead>
<tr>
<th>Description</th>
<th>Duration</th>
<th>Comments (if any)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## Appendix 4: Observation Sheet

<table>
<thead>
<tr>
<th>Who</th>
<th>Mathematics</th>
</tr>
</thead>
<tbody>
<tr>
<td>List down (who/group)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>THINK</th>
<th>Analysis (after the lesson)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Did the students make progress toward the key idea?</td>
<td></td>
</tr>
<tr>
<td>How did you know that evidence was missing?</td>
<td></td>
</tr>
<tr>
<td>What difficulties did the students encounter?</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>WONDER</th>
<th>Questions and alternatives (after the lesson)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Were there any new difficulties that students faced?</td>
<td></td>
</tr>
<tr>
<td>What alternative strategies could the teacher use?</td>
<td></td>
</tr>
<tr>
<td>What evidence is missing, how could the teacher collect that evidence?</td>
<td></td>
</tr>
</tbody>
</table>
### Appendix 5: Monitoring Chart

*FOR TEACHER TEACHING THE STUDY LESSON TO DECIDE ORDER OF PRESENTATION*

<table>
<thead>
<tr>
<th>Possible strategies / responses</th>
<th>Who and what?</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Make a list of possible strategies / responses and leave a row or two for unexpected ones. This is taken from the lesson plan (particularly the section on possible students' responses).</td>
<td>List down who (group) and a brief description of what they did</td>
<td>Decide the order of presentation</td>
</tr>
</tbody>
</table>
Appendices

Appendix 6: Interview Protocol

Dear ___________________

Thank you for participating in this study. The interview and video recording will be used only for research purposes, unless you have signed an agreement that states otherwise. The person who will be transcribing this information will sign a confidentiality agreement to protect your privacy. Only pseudonyms will be used for both the school and participants in the dissertation and any subsequent publications resulting from the study.

Thank you for your careful attention to this interview. I appreciate the time you are taking to help me better understand mathematics teaching.

There is no preferred answers to the questions and so answer the questions to the best of your knowledge.

A. YOUR BACKGROUND

Your name: ___________________________  □  Male  □  Female

School’s name: ___________________________  Date: ________________

1. What was the highest level of formal education you have completed?

2. In what subject areas and grade levels are you certified to teach?

3. What was your undergraduate/post-graduate major field of study?

4. Counting this year, how many years in total have you taught mathematics.

5. During the last two years, how many university courses have you taken in mathematics or mathematics education? Briefly describe the content of these courses?

6. During the last two years, have you participated in professional development activities related to mathematics or mathematics education?

7. Briefly describe the students in your mathematics classroom(s)? You may describe them in terms of their ability, performance, attitudes etc.

B. IDEAS ABOUT MATHEMATICS

8. In your opinion, what is mathematics about?

9. How is your view of mathematics similar or different to that of the mathematics taught in schools?

10. What is mathematical thinking? What does it look like in the classrooms?
11. Do you enjoy doing mathematics? Why or why not?

12. How is your view of mathematics similar or different to that espoused by the syllabus documents?

C. IDEAS ABOUT TEACHING AND LEARNING OF MATHEMATICS

13. Describe your own experience in learning mathematics (at the different levels)? To what extent and how have these experiences shape your own teaching?

14. List the three most important things you would like your students to learn from studying mathematics.

15. List the three most important things that shape your teaching of mathematics this year.

16. Describe an ideal mathematics classroom that you feel exemplifies your current ideas about the teaching and learning of mathematics and explain why you think it exemplifies these ideas.

17. Describe your current mathematics classroom practices. How is it similar or different to the ideal that you had described earlier?