Approximately Solving Multiobjective Linear Programmes in Objective Space and an Application in Radiotherapy Treatment Planning

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Abstract

In this paper we propose a modification of Benson's algorithm for solving multiobjective linear programmes in objective space in order to approximate the true nondominated set. We first summarize Benson's original algorithm and propose some small changes to improve computational performance. We then introduce our approximation version of the algorithm, which computes an inner and an outer approximation of the nondominated set. We prove that the inner approximation provides a set of ε-nondominated points. This work is motivated by an application, the beam intensity optimization problem of radiotherapy treatment planning. This problem can be formulated as a multiobjective linear programme with three objectives. The constraint matrix of the problem relies on the calculation of dose deposited in tissue. Since this calculation is always imprecise solving the MOLP exactly is not necessary in practice. With our algorithm we solve the problem approximately within a specified accuracy in objective space. We present results on four clinical cancer cases that clearly illustrate the advantages of our method.

Keywords: Multiobjective linear programming, radiotherapy treatment planning, ε-efficient solution.

1 Outline

The paper is organized as follows. In Section 2 we state the multiobjective linear programming problem and summarize Benson's outer approximation algorithm to solve it in objective space. We illustrate the algorithm by an example and describe some modifications to improve computational performance of the algorithm. In Section 3 we describe the approximation version of Benson's algorithm and prove that it finds a set of weakly ε-nondominated points in the feasible set in objective space. The rest of the paper is dedicated to the application in radiotherapy treatment planning. In Section 4 we review the beam intensity optimization problem, provide a formulation as multiobjective linear programme, and motivate the use of an approximation algorithm by clinical considerations. Finally, we provide results on four clinical cases in Section 5 and draw some conclusions in Section 6.

2 Multiobjective Linear Programming and Benson's Algorithm

In this paper we consider multiple objective linear programming problems of the form

$$\min\{Cx : x \in X\}. \tag{1}$$

We assume that X in (1) is a nonempty, compact feasible set X in decision space $\mathbb{R}^n$ defined by $X = \{x \in \mathbb{R}^n : Ax \geqq b\}$. We have $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. $C \in \mathbb{R}^{p\times n}$ is the $p \times n$ matrix, the rows $c_k, k = 1, \dots, p$, of which are the coefficients of p linear functions $< c_k, x >$, $k = 1, \dots, p$.

The feasible set Y in objective space $\mathbb{R}^p$ is defined by

$$Y = \{Cx : x \in X\}. \tag{2}$$

It is well known that the image Y of a nonempty, compact polyhedron X under a linear map C is also a nonempty, compact polyhedron of dimension $\dim Y \leqq p$ (Rockafellar, 1970).

In this paper we use the notation $y^1 \leq y^2$ to indicate $y^1 \leqq y^2$ but $y^1 \neq y^2$ for $y^1, y^2 \in \mathbb{R}^p$ whereas $y^1 < y^2$ means $y^1_k < y^2_k$ for all $k = 1, \dots, p$.

Definition 1 *A feasible solution $\hat{x} \in X$ is an efficient solution of problem (1) if there exists no $x \in X$ such that $Cx \leq C\hat{x}$. The set of all efficient solutions of problem (1) will be denoted by X_E and called the efficient set in decision space. Correspondingly, $\hat{y} = C\hat{x}$ is called a nondominated point and $Y_N = \{Cx : x \in X_E\}$ is the nondominated set in objective space of problem (1).*

Definition 2 *A feasible solution $\hat{x} \in X$ is called weakly efficient if there is no $x \in X$ such that $Cx < C\hat{x}$. The set of all weakly efficient solutions of problem (1) will be denoted by X_{WE} and called the weakly efficient set in decision space. Correspondingly, the point $\hat{y} = C\hat{x}$ is called weakly nondominated point and $Y_{WN} = \{Cx : x \in X_{WE}\}$ is the weakly nondominated set in objective space of problem (1).*

Researchers have developed a variety of methods for generating all or at least part of the efficient set X_E, such as multiobjective simplex methods and interior point methods, see the references in Ehrgott and Wiecek (2005) for more information. Although some of these approaches have had some success in aiding the Decision Maker (DM) to solve the problem, i.e., to identify a most preferred solution, this success has been relatively limited due to the heavy computational requirements and the near-impossibility to study the overwhelming set of efficient solutions X_E.

For an MOLP problem $Y_N \subseteq \mathbb{R}^p$ and $X_E \subseteq \mathbb{R}^n$ with p typically much smaller than n and many points in X_E are mapped to a single point in Y_N. For these reasons Benson (1998a) argues that generating Y_N should require less computation than generating X_E. Moreover, it is reasonable to assume that a decision maker (DM) will choose a solution based on the objective values rather than variable values. Therefore, finding Y_N instead of X_E is more important for the DM. Benson has proposed an algorithm to solve an MOLP in objective space in (Benson, 1998a,b). Below, we summarize his outer approximation algorithm.

2.1 Benson's Outer Approximation Algorithm

For the MOLP problem (1) let

$$Y' = \{y \in \mathbb{R}^p : Cx \leqq y \leqq \hat{y} \text{ for some } x \in X\}, \tag{3}$$

where $\hat{y} \in \mathbb{R}^p$ is chosen to satisfy $\hat{y} > y^{AI}$. The vector $y^{AI} \in \mathbb{R}^p$ is called the anti ideal point for the problem (1) and is defined as

$$y_k^{AI} = \max\{y_k : y \in Y\}. \tag{4}$$

Theorem 1 (Benson (1998a,b))

1. *The set $Y' \subset \mathbb{R}^p$ is a nonempty, bounded polyhedron of dimension p.*
2. $Y_N = {Y'}_N$.

Theorem 1 is the basis of the outer approximation algorithm. It works on Y' to find all nondominated extreme points of Y. In the course of the algorithm supporting hyperplanes of Y' are constructed. The following primal dual pair P(y) and D(y) of linear programmes depending on $y \in \mathbb{R}^p$ is needed for that purpose.

$$\begin{array}{ll} \mathrm{P}(y) & \min\{z : Ax \geqq b, Cx - ez \leqq y\}, \\ \mathrm{D}(y) & \max\{b^T u - y^T w : A^T u - C^T w = 0, e^T w = 1, u, w \geqq 0\}. \end{array}$$

Theorem 2 (Benson (1998b))

1. *Let* $\bar{p} \in \mathrm{int}Y'$ *and suppose that* $y^k \leq \hat{y}$ *and* $y^k \notin Y'$. *Let* q^k *denote the unique point on the boundary of* Y' *that belongs to the line segment connecting* y^k *and* $\bar{p}$. *Then* $q^k \in Y'_{WN}$.

2. *Assume that* $q^k \in Y'_{WN}$, *and let* (u^T, w^T) *denote any optimal solution to the dual linear programme D(*q^k*). Then* q^k *belongs to the weakly nondominated face* $F(u,w)$ *of* Y' *given by* $F(u,w) = \{y \in Y' : \langle w, y \rangle = \langle b, u \rangle\}$.

If $q^k \in Y'_{WN}$, then P(q^k) has the optimal value $z = 0$, and D(q^k) also has the optimal value $b^T u - {q^k}^T w = 0$. The dual optimal solution (u^T, w^T) is used to construct the supporting hyperplane of Y', $H(u,w) = \{y \in R^p : \langle w, y \rangle = \langle b, u \rangle\}$.

Benson's outer approximation algorithm is shown in Algorithm 1. For details, the reader is referred to (Benson, 1998a,b).

Algorithm 1 (Benson's outer approximation algorithm)

Initialization:	*Compute a point* $\bar{p} \in \operatorname{int}Y'$ *and construct a* p*-dimensional simplex* S^0 *containing* Y'*. Store both the vertex set* $V(S^0)$ *of* S^0 *and the inequality representation of* S^0*. Set* $k = 0$ *and go to iteration* k*.*
Iteration k.	
Step $k1$.	*If, for each* $y \in V(S^k), y \in Y'$ *is satisfied, then go to Step* $k5$*:* $Y' = S^k$*. Otherwise, choose any* $y^k \in V(S^k)$ *such that* $y^k \notin Y'$ *and continue.*
Step $k2$.	*Find the unique value* λ_k *of* λ*,* $0 < \lambda < 1$*, such that* $\lambda y^k + (1-\lambda)\bar{p}$ *belongs to the boundary of* Y'*, and set* $q^k = \lambda_k y^k + (1-\lambda_k)\bar{p}$*.*
Step $k3$.	*Set* $S^{k+1} = S^k \cap \{y \in \mathbb{R}^p : \langle w^k, y \rangle \geqq \langle b, u^k \rangle\}$*, where* $({u^k}^T, {w^k}^T)$ *can be found by solving LP* $D(q^k)$*.*
Step $k4$.	*Using* $V(S^k)$ *and the definition of* S^{k+1} *given in Step* $k3$*, determine* $V(S^{k+1})$*. Set* $k = k+1$ *and go to iteration* k*.*
Step $k5$.	*Let the total number of iterations be* $K = k$*. The nondominated extreme points of* Y' *are* $Y'_{NE} = \{y \in V(S^K) : y < \hat{y}\}$*.* $Y_{NE} = Y'_{NE}$ *is the set of all nondominated extreme points of* Y*. Stop.*

For each $k \geqq 0$, the hyperplane given by $\left\langle w^k, y \right\rangle = \left\langle b, u^k \right\rangle$ is constructed so that it cuts off a portion of S^k containing y^k, thus $S^k \supset S^{k+1} \supset Y'$. This is the reason for the name "outer approximation" algorithm, although at termination, the MOLP is solved exactly in objective space. Theorem 3 proves that Benson's algorithm is finite and it terminates with finding all the nondominated extreme points of Y in Step $k5$.

Theorem 3 (Benson (1998a,b)) . *Algorithm 1 is finite and at termination* $S^K = Y'$*. Let* $Y'_{NE} = \{y \in V(S^K) : y < \hat{y}\}$*. Then* Y'_{NE} *is identical to the set of all nondominated extreme points of* Y*, i.e.,* $Y'_{NE} = Y_{NE}$*.*

The general idea of Benson's algorithm can be explained as follows. First, a simplex cover S^0 that contains $Y'$is constructed. S^0 is given by axes parallel hyperplanes defined by the entries of $\hat{y}$ and a supporting hyperplane of Y' with normal $e = (1, \ldots, 1) \in \mathbb{R}^p$. An interior point $\bar{p}$ of Y' is found. Then, for each vertex y^k of the cover, it is checked whether y^k is in Y' or not. If not, $\bar{p}$ and y^k are connected by a line segment that contains a unique boundary point q^k of Y'. A cut (new supporting hyperplane) containing q^k is constructed and the cover S^k is updated. The procedure repeats until all the vertices of the cover are in Y'. Then the vertices of the cover are the extreme points of Y' and the nondominated extreme points of Y' are $Y'_{NE} = \{y \in V(S^k) : y < \hat{y}\}$.

We give an example to illustrate Benson's algorithm.

Example 1 *Consider the MOLP* $\min\{Cx : Ax \geqq b\}$*, where*

$$C = \begin{pmatrix} 3 & 1 \\ -1 & -2 \end{pmatrix}, A = \begin{pmatrix} 0 & -1 \\ -3 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, b = \begin{pmatrix} -3 \\ -6 \\ 0 \\ 0 \end{pmatrix}.$$

The feasible set Y *in objective space is shown in Fig. 1. Choosing* $\hat{y} = (13, 1)$ *we define* Y' *as* $Y' = \{y \in \mathbb{R}^2 : Cx \leqq y \leqq \hat{y},\ Ax \geqq b\}$*. Fig. 2 shows* Y, Y'*,* S^0 *and the interior point* $\bar{p}$*.*

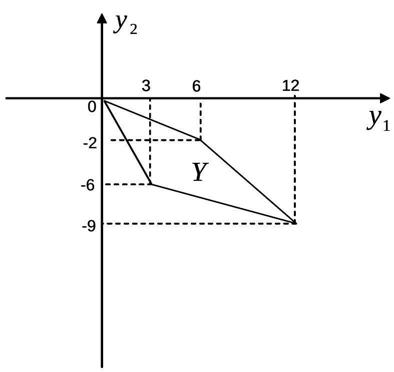

Figure 1: Objective space Y.

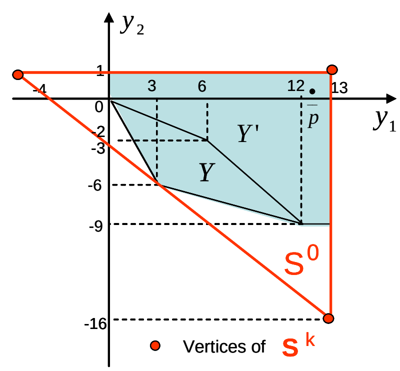

Figure 2: Y', S^0 and interior point $\bar{p}$.

Figs. 3, 4, 5, and 6 show the first, second, third and fourth hyperplane, respectively. The first hyperplane cuts off vertex $(-4, 1)$*, the second cuts off vertex* $(13, -16)$*, the third cuts off vertex* $(0, -3)$*, the fourth cuts off vertex* $(6, -9)$*. We can see the change of* S^k *after each cut. After the fourth cut, we have* $S^4 = Y'$*. Therefore, the vertices of* S^4 *are the extreme points of* Y'*. We obtain all nondominated extreme points by* $Y'_{NE} = \{y \in V(S^4) : y < \hat{y}\}$*. In this example we obtain the three nondominated extreme points* $(12, -9), (3, -6)$ *and* $(0, 0)$*.*

2.2 Improvements to Benson's Algorithm

In **Step** $k2$ of Algorithm 1 it is necessary to find the unique λ $(0 < \lambda < 1)$ which determines the boundary point $q^k = \lambda y^k + (1 - \lambda)\bar{p}$ of Y'. Benson

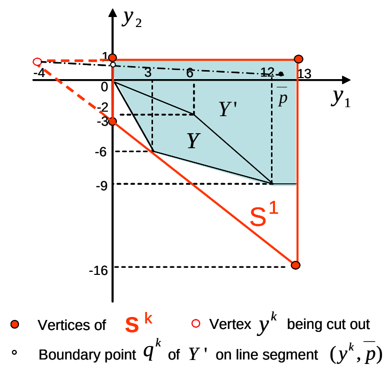

Figure 3: After the first cut.

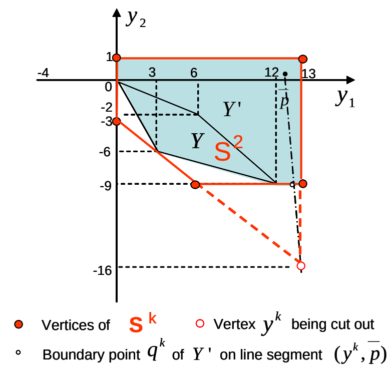

Figure 4: After the second cut.

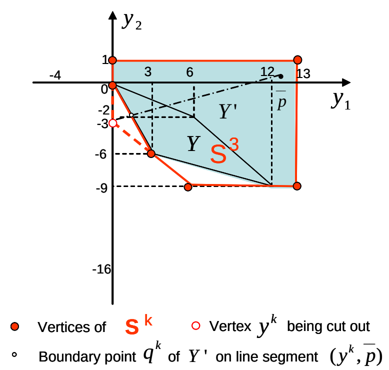

Figure 5: After the third cut.

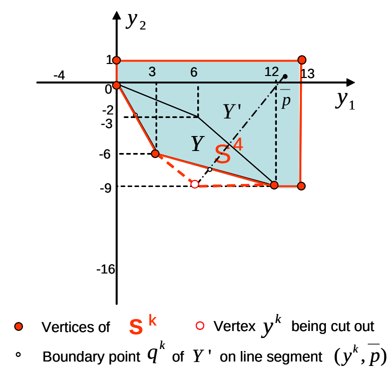

Figure 6: After the fourth cut.

(1998b) suggests using a bisection method. However, this requires the solution of many LPs. We show that it is possible to find the appropriate λ solving just a single LP.

Proposition 1 *Let y^k be a vertex of S^k and $y^k \notin Y'$. Let $\bar{p}$ ∈intY'* and let $\mu = y^k - \bar{p}$. Then there must exist some $i \in \{1, \ldots, p\}$ such that $\mu_i < 0$.

Proof: Suppose, to the contrary, that $\mu_i \geqq 0$ for all $i = 1, \ldots, p$. According to Theorem 2 there is a unique λ $(0 < \lambda < 1)$ such that $q^k = \lambda y^k + (1 - \lambda)\bar{p} \in Y'_{WN}$. Then $q^k - \bar{p} = \lambda y^k + (1 - \lambda)\bar{p} - \bar{p} = \lambda(y^k - \bar{p}) = \lambda\mu \geqq 0$. For a minimization problem this means that q^k is dominated by $\bar{p}$ which contradicts $q^k \in Y'_{WN}$. Therefore, there must exist i with $\mu_i < 0$. □

By Proposition 1 it is possible to choose $l \in \{1, \ldots, p\}$ with $\mu_l = y^k_l - \bar{p}_l < 0$. We choose that $\mu_l < 0$ which has the biggest absolute value among the negative μ_i, i.e., $|\mu_l| \geqq |\mu_i|$ for all $i = 1, \ldots, p$ with $\mu_i < 0$. Due to the convexity of Y', among all points of Y' on the line segment connecting points y^k and $\bar{p}$, the boundary point q of Y' attains the smallest value of q_l.

Therefore the unique λ for determining q^k can be found by solving the following LP.

$$\begin{array}{lrcl} \min & \lambda y_l^k + (1-\lambda)\bar{p}_l & & \\ \text{s.t.} & \lambda y^k + (1-\lambda)\bar{p} & \leqq & \hat{y} \\ & \lambda y^k + (1-\lambda)\bar{p} & \geqq & Cx \\ & Ax & \geqq & b \\ & \lambda & \leqq & 1 \\ & \lambda & \geqq & 0, \end{array} \tag{5}$$

where λ and x are the variables. Note that, since $y^k \notin Y'$ and $\bar{p} \in \text{int}Y$, λ cannot be 0 or 1 in an optimal solution of (5).

This modification dramatically improves the computation time.

Moreover, to calculate the vertices of $S^{k+1} = S^k \cap \{y \in \mathbb{R}^p : \langle w^k, y\rangle \geqq \langle b, u^k\rangle\}$ in **Step** $k4$, Benson proposes the method of Horst *et al.* (1988) in (Benson, 1998b) and the simplicial partitioning technique in (Benson, 1998a). We use the on-line vertex enumeration algorithm of Chen and Hansen (1991). This algorithm is based on the algorithm of Horst *et al.* (1988) but its complexity is smaller, as shown in Chen and Hansen (1991). The principle of the on-line vertex enumeration algorithm is to find the vertex sets of S^k on both sides of the cutting plane $H(u, w) = \{y \in \mathbb{R}^p : \langle w, y\rangle = \langle b, u\rangle\}$ and then to use adjacency lists of extreme points to identify all edges of S^k intersecting $H(u, w)$. The corresponding intersection points are computed and the adjacency lists updated. We found that the on-line vertex enumeration method leads to an improvement in computation speed compared to the simplicial partitioning technique.

3 Approximation Version of Benson's Algorithm

Let us first define ε-efficient solutions.

Definition 3 (Loridan (1984)) *Consider the MOLP (1) and let* $\mathbb{R}^p_{\geqq} = \{y \in \mathbb{R}^p : y \geqq 0\}$ *and* $\varepsilon \in \mathbb{R}^p_{\geqq}$.

1. *A feasible solution* $\hat{x} \in X$ *is called an* ε*-efficient solution of (1) if there does not exist* $x \in X$ *such that* $Cx \leq C\hat{x} - \varepsilon$. *Correspondingly,* $\hat{y} = C\hat{x}$ *is called an* ε*-nondominated point in objective space;*

2. *A feasible solution* $\hat{x} \in X$ *is called a weakly* ε*-efficient solution if there does not exist* $x \in X$ *such that* $Cx < C\hat{x} - \varepsilon$. *Correspondingly,* $\hat{y} = C\hat{x}$ *is called a weakly* ε*-nondominated point in objective space.*

We modify Algorithm 1 in order to find weakly ε-nondominated points of Y'. In addition to the vertex set $V(S^k)$ we introduce sets O and I (initially empty) of points used for the construction of an inner and an outer approximation of Y'. In the algorithm, if an extreme point y^k of $V(S^k)$ is close to Y', i.e., has a distance less than $\epsilon > 0$ from the boundary point q^k we omit construction of the hyperplane in **Step** $k3$ but remember both y^k and q^k to construct the inner and outer approximation of Y'.

Our approximation version of Benson's algorithm is identical to algorithm 1 except for **Step** $k1$, **Step** $k3$, and **Step** $k5$. Let $\epsilon \in \mathbb{R} > 0$ be a tolerance and let d denote the Euclidean distance, then the changes are as follows.

Step $k1$.	If, for each $y \in V(S^k), y \in Y'$ or $y \in O$ is satisfied, then go to Step $k5$. Otherwise, choose any $y^k \in V(S^k) \setminus O$ such that $y^k \notin Y'$ and continue.
Step $k3$.	If the distance $d(y^k, q^k)$ from y^k to the boundary point q^k of Y' is at most ϵ, then add y^k to O and add q^k to I. Go to Step $k1$. Otherwise set $S^{k+1} = S^k \cap \{y \in \mathbb{R}^p : \langle w^k, y\rangle \geqq \langle b, u^k\rangle\}$, where (u^k, w^k) can be found by solving LP D(q^k).
Step $k5$.	Let the total number of iterations be $K = k$. Define the set of points of the outer approximation $V_o(S^K) = V(S^K)$ and define the set of points of the inner approximation $V_i(S^K) = (V(S^K) \setminus O) \cup I$. The convex hull Y'^i of $V_i(S^K)$ represents the inner approximation of Y'. The convex hull Y'^o of $V_o(S^K)$ represents the outer approximation of Y'. Stop.

We apply the modified algorithm to Example 1.

Example 2 *In Example 1, set* $\bar{p} = (12.5, 0.5)$ *and* $\epsilon = 2.0$. *After two cuts there are two points* $y^1 = (0, -3)$ *and* $y^2 = (6, -9)$ *outside* Y'*, see Fig. 7. The boundary points corresponding to* y^1 *and* y^2 *are* $q^1 = (1\frac{6}{19}, -2\frac{12}{19}) \approx (1.316, -2.632)$ *and* $q^2 = (7\frac{4}{35}, -7\frac{13}{35}) \approx (7.114, -7.371)$*, respectively. The distances between the infeasible points and the boundary points are* $d(y^1, q^1) \approx 1.366$ *and* $d(y^2, q^2) \approx 1.973$. *We accept these two infeasible points for the outer approximation due to the distances to their corresponding boundary points being less than* ϵ. *When the algorithm terminates, the total number of iterations* K *is equal to* 2, $V_o(S^2) = (13, 1), (0, 1), (0, -3), (6, -9), (13, -9)$ *and* $V_i(S^2) = (13, 1), (0, 1), (1.316, -2.632), (7.114, -7.371), (13, -9)$. *In Fig.*

8, we show the outer approximation Y'^o and the inner approximation Y'^i of Y' and their corresponding sets of points.

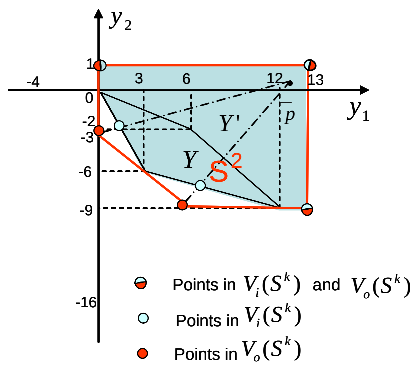

Figure 7: Accepted infeasible points for approximation.

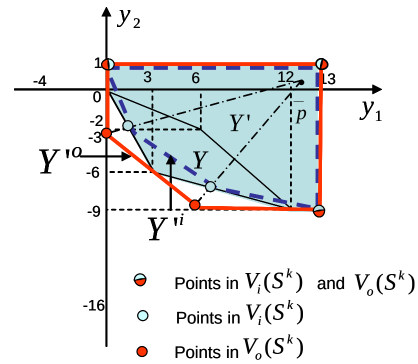

Figure 8: Inner and outer approximation.

By definition of the approximation version of Benson's algorithm, we have the following observations.

Theorem 4

1. *The number of points in $V_o(S^K)$ is equal to the number of points in $V_i(S^K)$.*

2. *All points in $V_i(S^K)$ are on the boundary of Y'. Some points in $V_o(S^K)$ are outside Y', while the others are on the boundary of Y'. Moreover, $y \in V_o(S^K)$ is not on the boundary of Y' if and only if $y \notin V_i(S^K)$.*

3. *If y_{ov} is a point in $V_o(S^K)$, there exists a point y_{iv} in $V_i(S^K)$ corresponding to y_{ov} with $d(y_{ov}, y_{iv}) \leqq \epsilon$ and vice versa.*

4. *If $Y_N'^i$ is the nondominated set of the inner approximation Y'^i and $Y_N'^o$ is the nondominated set of the outer approximation Y'^o, then we have $Y_N'^i + \mathbb{R}^p_{\geqq} \subseteq Y_N' + \mathbb{R}^p_{\geqq} \subseteq Y_N'^o + \mathbb{R}^p_{\geqq}$.*

Point 4 in Theorem 4 means that $Y_N'^i$ and $Y_N'^o$ are upper and lower bound sets for Y_N as defined by Ehrgott and Gandibleux (2006). We would also like to note that the approximation depends on the choice of the interior point $\bar{p}$. Of course, if $\epsilon = 0$ the algorithm is Benson's original algorithm. We proceed to show that $Y_N'^i$ is a set of weakly ε-nondominated points for Y'.

Proposition 2 *If y_o is a weakly nondominated point of the outer approximation set Y'^o, then there exists a weakly nondominated point y_i of the inner approximation set Y'^i such that $d(y_o, y_i) \leqq \epsilon$.*

Proof: Let y_o be a point on F_o, a weakly nondominated face of the outer approximation set with vertices $y_{ov}^1, y_{ov}^2, \ldots, y_{ov}^l \in V_o(S^K)$. Then y_o can be expressed as a convex combination of the vertices, i.e., $y_o = \sum_{j=1}^l \lambda_j y_{ov}^j$ with $\sum_{j=1}^l \lambda_j = 1$ and $\lambda_j \geqq 0$, $j = 1, \ldots l$.

Let $y_{iv}^1, y_{iv}^2, \ldots, y_{iv}^l \in V_i(S^K)$ be the corresponding points to $y_{ov}^1, y_{ov}^2, \ldots, y_{ov}^l$ on the inner approximation. Then $d(y_{ov}^j, y_{iv}^j) \leq \epsilon$, for $j = 1, \ldots, l$. Let $y_i = \sum_{j=1}^l \lambda_j y_{iv}^j$, then $d(y_o, y_i) = || \sum_{j=1}^l \lambda_j y_{ov}^j - \sum_{j=1}^l \lambda_j y_{iv}^j)|| \leqq \sum_{j=1}^l \lambda_j ||y_{ov}^j - y_{iv}^j|| \leq \sum_{j=1}^l \lambda_j \epsilon = \epsilon$. If y_i is a weakly nondominated point of the inner approximation we are done. Otherwise, choose the intersection point $\tilde{y}$ of the line connecting y_i and y_o with the boundary of Y'^i. Clearly $d(\tilde{y}, y_o) \leq d(y_i, y_o) \leqq \epsilon$. □

Combining Proposition 2 with Definition 3 we obtain our main result.

Theorem 5 *Let $\varepsilon = \epsilon e$, where $e = (1, \ldots, 1) \in \mathbb{R}^p$. Then $Y_N'^i$ is a set of weakly ε-nondominated points for Y'.*

Proof: Let $y_i \in Y_N'^i$ and suppose there is $y \in Y'$ such that $y < y_i - \varepsilon$. Thus $y_i - y > \varepsilon$ and $d(y, y_i) > ||\varepsilon|| > \epsilon$. By Theorem 4 we have that $Y' \subseteq Y_N'^o + \mathbb{R}^p_{\geqq}$, i.e., there is $y_o \in Y_N'^o$ such that $y_o \leqq y$. Now observe that the intersection of the hypercube defined by y and y_i with Y'^i contains the single point y_i because $Y'^i + \mathbb{R}^p_{\geqq}$ is convex. The hypercube has edge length at least ϵ. Thus we have that $d(y_o, \bar{y}_i) \geqq d(y, \bar{y}_i) \geqq d(y, y_i) > \epsilon$ for any $\bar{y}_i \in Y_N'^i$, contradicting Proposition 2. □

Theorem 5 shows that the approximation version of Benson's algorithm allows a guaranteed approximation quality for the weakly nondominated set of Y'. Because $Y_N \subset Y'_{WN}$ it is valid for Y_N as well. But $Y_N'^i$ may contain weakly nondominated points of Y and even points of $Y' \setminus Y$, see Fig 8.

To approximate the nondominated set of Y and avoid weakly nondominated points and points of $Y' \setminus Y$, we define $Y_{NE}^i = \{y \in V_i(S^K) : y < \hat{y}\}$ and $Y_{NE}^o = \{y \in V_o(S^K) : y < \hat{y}\}$. We construct faces using the points in Y_{NE}^o on the same cutting plane (found during the algorithm) and let Y_N^o be the union of the faces. Similarly, we can construct Y_N^i. Then the true nondominated set Y_N can be approximated from outside by Y_N^o and from inside by Y_N^i.

Example 3 *For Example 1, the points in Y_{NE}^o are $(0, -3)$ and $(6, -9)$, while the points in Y_{NE}^i are $(1.316, -2.632)$ and $(7.114, -7.371)$. The set Y_N^o is the* *line segment from point* $(0,-3)$ *to point* $(6,-9)$ *and the set* Y_N^i *is the line segment from point* $(1.316,-2.632)$ *to point* $(7.114,-7.371)$*, see Fig. 8. Note that there might exist* $y \in Y_N$ *which are farther than* ϵ *from any point in* Y_N^o *and* Y_N^i*.*

4 Application: Radiotherapy Treatment Planning

The aim of radiation therapy is to destroy the DNA of tumour cells thus preventing them from reproducing. Since radiation harms healthy cells, too, albeit to a lesser extent, it is necessary to protect tissue surrounding the tumour and critical organs at risk from this damaging effect of radiation. In order to achieve these goals with current treatment modalities, notably intensity modulated radiotherapy, optimization based inverse planning systems are necessary. Given the number of beams and beam directions, the beam intensity optimization consists in calculating beam intensity profiles (also called fluence maps) for all beams that yield the best achievable dose distribution under consideration of clinical and physical constraints. For background on radiotherapy we refer to Schlegel and Mahr (2002). In the following sections we review the beam intensity optimization problem and introduce the MOLP model we use.

4.1 Beam Intensity Optimization and Multiple Objectives

In the past, the beam intensity optimization problem has been formulated as a linear or nonlinear optimization problem, see Shao (2005) for a survey of these models. Usually, the conflicting objectives – effective treatment of the tumour and limiting the radiation dose to the surrounding normal tissue and organs at risk – are summed up using a weight or “importance factor” for the tumour, each organ at risk, and the normal tissue. However, selecting weights before optimization is problematic because it leads to a trial-and-error process of getting the “correct” weights.

Recently, multiobjective optimization has been introduced into radiation therapy planning. For example, Hamacher and Küfer (2002) and Küfer *et al.* (2003) formulate the beam intensity optimization problem as a multiobjective linear programming (MOLP) problem, and Cotrutz *et al.* (2001) and Lahanas *et al.* (2003b) formulate it as a multiobjective nonlinear programming (MONP) problem. In their unifying framework Romeijn *et al.* (2004)

show that many of the objectives commonly used in beam intensity optimization are convex and can be transformed into one another by strictly increasing transformations. Thus, for most multiobjective models efficient solutions can be obtained using weighted sum scalarisation. Moreover, the efficient sets of different models coincide.

Hamacher and Küfer (2002) and Küfer *et al.* (2003) describe an idea to generate a subset of the nondominated set based on the concept of neighbour solutions. Cotrutz *et al.* (2001) and Lahanas *et al.* (2003b) solve the MONP using the weighted sum method and they obtain a subset of the nondominated set by choosing a set of weights. Lahanas *et al.* (2003a) use multiobjective evolutionary algorithms to obtain some discrete nondominated points. Craft *et al.* (2005) use the normalized normal constraint (NC) method (Messac *et al.*, 2003) to achieve two dimensional tradeoffs between tumour dose homogeneity and critical organ sparing and Craft *et al.* (2006) propose a method to iteratively choose weights to gradually build up the nondominated set.

However, most of the above methods cannot give us a comprehensive view of the entire nondominated set. They either find a subset of the nondominated set or try to approximate the whole nondominated set using the nondominated points obtained, usually without guaranteed quality of approximation.

It is difficult to choose a set of weights to make the nondominated points evenly distributed. Even if an evenly distributed set of weights is used, it is possible that the points obtained on the nondominated set are not uniformly distributed (Das and Dennis, 1997). The normalized normal constraint method is based on the normal boundary intersection (NBI) method of Das and Dennis (1998). Both of the methods generate a set of equidistant reference points on the convex hull of the individual minima (CHIM). For each reference point, a corresponding nondominated point is found solving a single objective subproblem. These methods can find evenly distributed nondominated points, but they have the limitation that the solution may overlook a portion of the nondominated set if the normal of the CHIM has negative components (which may happen for $p > 2$ objectives). The method of Craft *et al.* (2006) uses the idea of sandwiching the nondominated set between a lower and an upper convex approximation. In each iteration it calculates a new weight and updates the lower and upper approximation. However, the CHIM is taken as the upper approximation initially.

In this paper we use Algorithm 1 to determine the entire nondominated set of an MOLP model for the beam intensity optimization problem. Due to the size of this MOLP model for clinical cases and despite the improvements described in Section 2.2 it turns out that computation times are excessive.

We also observed that for the clinical examples the tradeoffs between the objectives vary widely. Thus the nondominated sets in objective space appear to be "curved" (see the figures in Section 5. This means that very many cutting planes are needed to describe Y'. This explains why Benson's algorithm has computational problems and takes very long to terminate. This is one motivation for using an approximation version of Algorithm 1.

4.2 An MOLP Model for the Beam Intensity Problem

In order to calculate dose in the beam intensity optimization problem, the patient's 3D volume is divided into m small voxels (volume elements), and the beams are discretised into n small bixels (beam elements). Then

$$d = Ax, \tag{6}$$

where $d \in \mathbb{R}^m$ is a dose vector and its elements d_i correspond to the dose deposited in voxel i. Vector $x \in \mathbb{R}^n$ describes the beam intensity, x_j representing the intensity of bixel j. $A \in \mathbb{R}^{m \times n}$ is called dose deposition matrix. The elements a_{ij} of A represent the dose deposited in voxel i due to unit intensity in bixel j. We assume that A is given. A can be partitioned and reordered into sub-matrices $A_T \in \mathbb{R}^{m_T \times n}, A_C \in \mathbb{R}^{m_C \times n}$ and $A_N \in \mathbb{R}^{m_N \times n}$ $(m_T + m_C + m_N = m)$ according to the rows corresponding to tumour, critical organ and normal tissue voxels, respectively.

For treatment planning, the physician needs to specify a "prescription dose" for the tumour, each organ at risk and the normal tissue. The "prescription dose" is used to construct $TLB \in \mathbb{R}^{m_T}, TUB \in \mathbb{R}^{m_T}, CUB \in \mathbb{R}^{m_C}$ and $NUB \in \mathbb{R}^{m_N}$ representing lower bounds on the dose to tumour voxels, upper bounds on the dose to tumour voxels, upper bounds on the dose to critical organ voxels, and upper bounds on dose to normal tissue voxels.

Based on Holder's linear programming formulation (Holder, 2003), we formulate the beam intensity optimization problem as a multiple objective linear programme (MOLP). In this model, we minimize the maximum deviation from tumour lower bounds α, critical organ upper bounds β and normal tissue upper bounds γ at the same time. The model can be described as follows:

$$
\begin{array}{llll}
\min & (\alpha,\beta,\gamma) & & \\
\text{s.t.} & TLB-\alpha e & \leqq A_T x & \leqq TUB \\
& A_C x & \leqq CUB+\beta e & \\
& A_N x & \leqq NUB+\gamma e & \\
& 0 & \leqq \alpha & \leqq \alpha UB \\
& -\min CUB & \leqq \beta & \leqq \beta UB \\
& 0 & \leqq \gamma & \leqq \gamma UB \\
& 0 & \leqq x, &
\end{array}
\tag{7}
$$

where e is the vector in which each entry is 1, $\alpha UB \in \mathbb{R}$, $\beta UB \in \mathbb{R}$, and $\gamma UB \in \mathbb{R}$ are upper bounds for α, β, and γ, respectively. They are specified by the physician and restrict the search to clinically relevant values.

We can see that the three objectives α, β and γ in (7) are limited by upper and lower bounds. The same effect can be achieved by adding upper bounds and lower bounds on the decision variables, i.e., the beam intensity x (Lim *et al.*, 2006). Moreover, we need to point out that due to the non-negativity of the beam intensity, this MOLP problem is always feasible as long as appropriate lower bounds and upper bounds for α, β and γ are set, in particular, if these values are set to ∞ (Holder, 2003).

The constraints of (7) involve the dose deposition matrix A. As mentioned before, a_{ij} describes the dose deposited in voxel i if unit intensity is applied in bixel j. The coefficients a_{ij} are calculated by mathematical models of the physical behaviour of radiation as it travels through the body. While sophisticated techniques are available and in clinical use (Nizin *et al.*, 2001) with the gold standard today being Monte Carlo simulation (Verhaegen, 2003) the results are always imprecise due to the nonuniform composition of the patient body. Thus, solving (7) exactly may give an unwarranted impression of precision, but the result of the optimization can of course not be more precise than the input data. Thus, for clinical purposes it is perfectly acceptable to solve (7) approximately to within a small fraction of a Gy (Gray, the unit of measure for radiation dose). Note that the objectives α, β, γ are commensurate and have the unit Gy and that the tolerance ϵ in the approximation algorithm is absolute, not relative. This is the second motivation for solving the problem by an approximation version of Benson's algorithm.

5 Results

We solve (7) both by Algorithm 1 and our approximation algorithm described in Section 3. Four clinical cases are used, namely an arterial veinous malformation (AVM), an acoustic neuroma (AN), a prostate (P), and a pancreatic lesion (PL). Simplified CT images that show the outline of the tumour and critical organs at risk are shown in Fig. 9.

Figure 9: Pictures from left to right are AVM, AN, P, and PL.

These cases have a voxel size of 5 mm on a single CT slice. For all examples, a total of 72 evenly spaced beams were used at angles $5^o n$, where $n = 0, \ldots, 71$. The number of voxels and bixels used for optimization of each case and the prescription information that defines parameters in (7) is shown in Table 1. The algorithm was implemented in Matlab 7.1 (R14) using CPLEX 10.0 as LP solver and the tests were run on a dual processor CPU with 1.8 GHz and 1 GB RAM.

Table 1: Number of voxels (total = m) and bixels (n). Lower and upper bounds for tumour, critical organs, and normal tissue (in Gy).

Case	AVM	AN	P	PL
Tumour voxels	1	9	22	67
Critical organ voxels	0	47	89	91
Normal tissue voxels	1206	999	1182	986
Bixels	319	594	821	1140
TUB	90.64	87.55	90.64	90.64
TLB	85.36	82.45	85.36	85.36
CUB	—	60/45	60/45	60/45
NUB	0.00	0.00	0.00	0.00
αUB	17.07	16.49	42.68	17.07
βUB	—	12.00	30.00	12.00
γUB	90.64	87.55	100.64	90.64

For the AVM case, both algorithms find two nondominated extreme points. They are $y^1 = (0, 0, 79.31)$ and $y^2 = (17.07, 0, 63.45)$. Here, β is equal to zero because there is no critical organ in this case. The nondominated set is the line segment from point $(0, 79.31)$ to point $(17.07, 63.45)$, see Fig. 10. The clinical meaning of point $(0, 79.31)$ is that there is a solution for which the (single) voxel in the tumour will receive a dose greater than the tumour lower bound and smaller than the upper bound, while some voxel in the normal tissue will receive a dose as high as 79.31 Gy (this is most likely a voxel in immediate proximity of the tumour). The clinical meaning of point $(17.07, 63.45)$ is that there is a solution for which the voxel in the tumour will receive a dose as low as $TLB - 17.07 = 85.36 - 17.07 = 68.29Gy$, while some voxel in the normal tissue will receive a dose as high as 63.45 Gy. We can explain all the other nondominated points in between those two similarly.

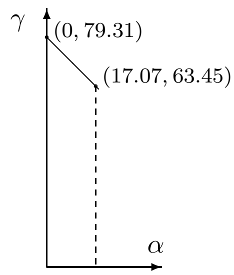

Figure 10: Nondominated set of the AVM case.

For the other cases, it is not possible to list all nondominated extreme points. We show the set Y' obtained by Benson's algorithm for the acoustic neuroma in Fig. 11 and for the prostate in Fig. 13 side by side with the set Y'^o of the outer approximation obtained by the approximation version of the algorithm with $\epsilon = 0.1$. The acoustic neuroma is shown in Fig. 12 and the prostate in Fig. 14.

The pancreatic lesion case could not be solved exactly within 10 hours of computation. Therefore, we show the sets Y'^o obtained by the approximation algorithm for various values of ϵ. Fig. 15 shows the result for $\epsilon = 0.3$, Fig. 16 is for $\epsilon = 0.1$, Fig. 17 is for $\epsilon = 0.05$, and Fig. 18 is for $\epsilon = 0.005$.

Summarizing information comparing the number of nondominated extreme points, the number of cutting planes and the computation time of Benson's algorithm and our approximation version of Benson's algorithm with various values of ϵ is given in Table 2.

Benson's algorithm can solve the first three clinical cases exactly in less than 1.5 hours. For the pancreatic lesion case, Benson's algorithm did not terminate after 10 hours of computation. On the other hand, the approximation version of Benson's algorithm can solve all four problems within 30 minutes within an error of 0.1.

For a problem with many nondominated extreme points and a "curved" nondominated surface, such as Fig. 18 suggests, the approximation ver-

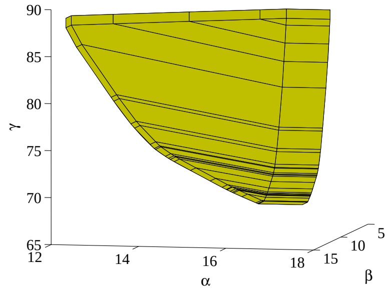

Figure 11: AN: Y'.

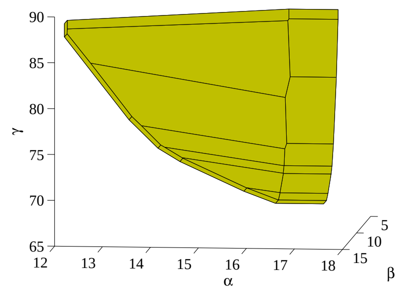

Figure 12: AN: Y'^o with $\epsilon = 0.1$.

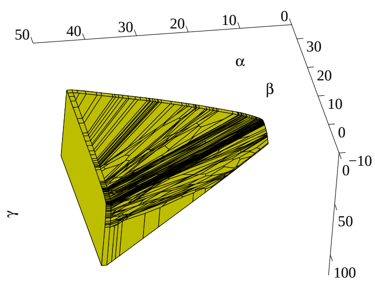

Figure 13: P: Y'.

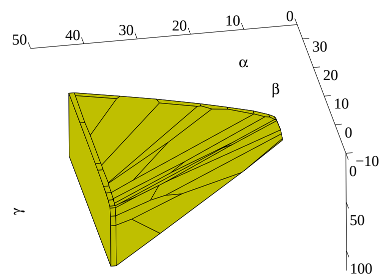

Figure 14: P: Y'^o with $\epsilon = 0.1$.

Table 2: Running time and number of nondominated extreme points and cutting planes for the four cases with different values of ϵ.

Case	ϵ	Time (seconds)	Nondominated extreme points	Cutting planes
AVM	0.1	1.56	2	3
	0	1.56	2	3
AN	0.1	39.19	27	21
	0	178.09	55	85
P	0.1	205.95	56	42
	0	4196.78	3165	3280
PL	0.3	677.15	57	37
	0.1	1256.58	152	90
	0.05	1990.25	278	159
	0.005	30973.51	1989	1041

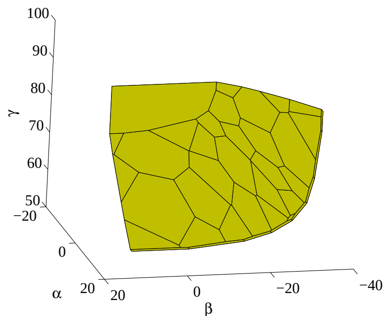

Figure 15: PL: Y'^o with $\epsilon = 0.3$.

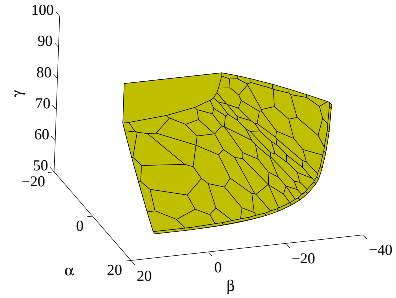

Figure 16: PL: Y'^o with $\varepsilon = 0.1$.

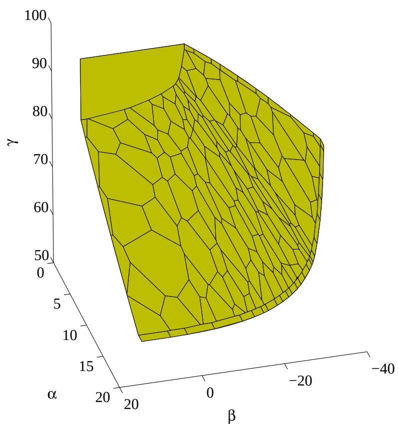

Figure 17: PL: Y'^o with $\epsilon = 0.05$.

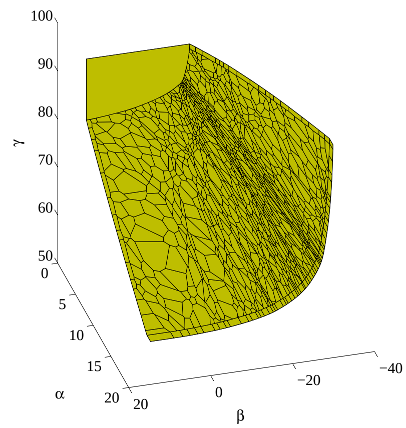

Figure 18: PL: Y'^o with $\varepsilon = 0.005$.

sion of Benson's algorithm generates fewer extreme points and fewer cutting planes compared with Benson's algorithm. For the prostate example, 3165 nondominated extreme points were found with Benson's algorithm, while the approximation version of Benson's algorithm generates only 56 nondominated extreme points when $\epsilon = 0.1$

Table 2 and the figures clearly show the effect of the choice of ϵ. The smaller the error parameter, the more cutting planes and the more nondominated extreme points are generated and the longer the computation time.

6 Conclusion

In this paper we have developed an approximation version of Benson's algorithm to solve MOLPs in objective space. We have shown that the algorithm guarantees to find weakly ε-nondominated points to within a specified accuracy.

The development of the algorithm was motivated by the beam intensity optimisation problem of radiotherapy treatment planning, which can be formulated as an MOLP. The constraint matrix of this model depends on the model of the physical behaviour of radiation. Since this calculation is inaccurate the application of an approximation algorithm is justified in the practical application. In this context the parameter ϵ can be chosen by the physician, based on his knowledge on how accurately the beam model used by the specific treatment planning system calculates dose deposited in the body.

We have used four different clinical cancer cases to test the algorithm, using only a single CT slice and a voxel size of 5 mm. The results suggest that our approach can be used to solve three dimensional cases with clinically relevant 2 mm spaced CT slices and voxel size of 2 mm using computational resources that are available at leading cancer treatment centres.

Our method provides an approximation of the whole nondominated set. Further work is necessary to combine this approach with decision support tools to assist the treatment planner in selecting a treatment plan from this set that is best suited for the individual patient under consideration.

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