

---

# Multiobjective (Combinatorial) Optimisation – Some Thoughts on Applications

Matthias Ehrgott<sup>1</sup>

Department of Engineering Science, The University of Auckland, Private Bag  
92019, Auckland Mail Centre, Auckland 1143, New Zealand  
`m.ehrgott@auckland.ac.nz`

## 1 Mathematical Formulation

A multiobjective optimisation problem is the following mathematical programme

$$\begin{aligned} \min f(x) &= (f_1(x), \dots, f_p(x)) \\ \text{subject to } g(x) &\leq 0, \end{aligned}$$

where  $f : \mathbb{R}^n \rightarrow \mathbb{R}^p$  is a vector-valued objective function and  $g(x) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ .

In the case of integer programmes, that we are mainly concerned with in this paper, we further assume that  $f$  and  $g$  are linear functions. Thus a multiobjective integer programme is

$$\begin{aligned} \min f(x) &= Cx \\ \text{subject to } Ax &\leq b \\ x &\in \{0, 1\}^n. \end{aligned}$$

We denote by  $X = \{x \in \mathbb{R}^n \text{ or } X = \{x \in \{0, 1\}^n : Ax = b\}$  the feasible set in decision space and by  $Y = f(X) = \{f(x) : x \in X\}$  the feasible set in objective space. We understand solving a multiobjective integer programme as finding a complete set of efficient solutions  $X_E$ , according to the definition of Hansen (1979). A feasible solution  $\hat{x}$  is efficient if there is no  $x \in X$  with  $f(x) \leq f(\hat{x})$  and  $f(x) \neq f(\hat{x})$ . The image of the efficient set in objective space is the set of non-dominated points  $Y_N := f(X_E)$ .

It is of course impossible to give a comprehensive survey of applications of multiobjective optimisation in the space of this paper. I have therefore made a very subjective selection of problems that I am familiar with. They are nevertheless drawn from widely different application areas. In each of the examples I emphasise why I find it instructive and what lessons can be learned. The applications I consider are the portfolio selection problem in finance (Section

2), train timetable information and airline crew scheduling from the transportation field (Sections 3 and 4), radiotherapy treatment design in medicine (Section 5) and the telecommunication application of routing in IP networks (Section 6).

## 2 Finance: Portfolio Optimisation

The first problem I want to discuss is the portfolio optimisation problem of deciding on an investment of a certain sum of money, for example at a stock exchange, so as to maximise the return and minimise the associated risk. If the number of websites is an indication of importance, this is a very important problem: A Google search for “Risk Return Portfolio Stock Exchange” produces about 10 million hits, among those <http://www.ise.ie/intuition.asp?type=SUCCESS> of the Irish stock exchange, where we can read that “In this section of the Exchange’s e-learning tool you can learn more about the trade off between risk and return.”

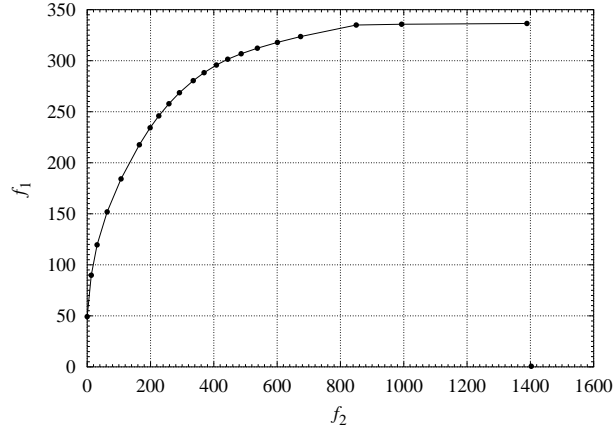
As the phrase “trade off” indicates, portfolio optimisation is a classical bicriteria optimisation problem. It is arguably the first one that has been intensively studied since the seminal work of Markowitz (1952) appeared. The original single objective formulation employed for its solution is nothing but the  $\varepsilon$ -constraint scalarisation of the problem

$$\begin{aligned} \max f_1(x) &= \mu^T x \\ \min f_2(x) &= x^T \sigma x \\ \text{subject to } e^T x &= 1 \\ x &\geq 0 \end{aligned} \tag{1}$$

where  $\mu$  is the expected return,  $\sigma$  is the covariance matrix of the returns, and  $e$  is a vector of ones.

As a biobjective programme with linear and quadratic objectives and linear constraints, the non-dominated and efficient sets are relatively easy to compute. Figure 1 shows the non-dominated set of a portfolio optimisation problem with  $n = 40$  assets from Ehrgott *et al.* (2004).

So why should I talk about this problem? The reason becomes apparent when we compare theory with reality. As Konno (1990) observes, most investors do not actually buy efficient portfolios, but rather those behind the non-dominated frontier. Can this behaviour be explained? The assumption underlying the Markowitz model is that investors are “after the money” and therefore only interested in return and risk. We might call such investors “average” or “standard” investors. However, individual investors might not act according to the Markowitz assumption, and consider other, additional, objectives. Such *multiobjective* models have been proposed, e.g., by Steuer *et al.* (2006) and Ehrgott *et al.* (2004). The latter uses formulation (2).



**Fig. 1.** The non-dominated frontier of a portfolio optimisation problem.

$$\begin{aligned}
 \max f_1(x) &= \mu_1^T x \\
 \min f_2(x) &= x^T \sigma x \\
 \max f_3(x) &= \mu_3^T x \\
 \max f_4(x) &= d^T x \\
 \max f_5(x) &= s^t x \\
 \text{subject to } e^T x &= 1 \\
 x &\geq 0,
 \end{aligned} \tag{2}$$

where  $\mu_1$  and  $\mu_3$  are the one and three year expected returns,  $d$  is the dividend and  $s$  is the Standard and Poor's star ranking. Steuer *et al.* (2006) call investors that use such non-standard criteria "suitable portfolio investors." Investors may also like to have control over the number of assets in the portfolio and the fraction of investment in a single asset. This can be incorporated by using additional binary variables as in (Chang *et al.*, 2000) to yield model (3).

$$\begin{aligned}
\max f_1(x) &= \mu_1^T x \\
\min f_2(x) &= x^T \sigma x \\
\max f_3(x) &= \mu_3^T x \\
\max f_4(x) &= d^T x \\
\max f_5(x) &= s^t x \\
\text{subject to } e^T x &= 1 \\
x_i &\leq u_i y_i \\
x_i &\geq l_i y_i \\
e^T y &= k \\
y_i &\in \{0, 1\},
\end{aligned} \tag{3}$$

where  $k$  is the number of assets, and  $l_i$  and  $u_i$  are lower and upper bounds on the fraction of capital invested in asset  $i$ .

By now, the continuous, convex, linear-quadratic biobjective Markowitz model has become a true multiobjective and mixed integer problem, which is certainly worthy of further study. But besides showing that (multiobjective) portfolio optimisation remains an interesting topic more than 50 years after its first appearance we can learn another important lesson. Conventional portfolio theory cannot predict behaviour of individual investors. However, the introduction of additional objectives provides a rather plausible explanation of this phenomenon. The model of “suitable portfolio investors” opens possibilities for further research. Assuming that investors make rational (optimal) decisions, how many and which objectives are needed to explain a particular solution as efficient? Furthermore, the importance of multicriteria decision aid increases, as criteria need to be made explicit and decision support is necessary to find an investors’ most preferred portfolio.

Much more information on this topic can be found in Chapter 20 in Figueira *et al.* (2005) and references therein.

### 3 Transportation: Train Timetable Information

At the time of planning my trip to Europe, including attendance at the MOPGP conference, I considered using the train from Pirmasens, Germany, to Tours. The online timetable information of Deutsche Bahn (see <http://www.reiseauskunft.bahn.de>) provided two possible connections shown in Table 1.

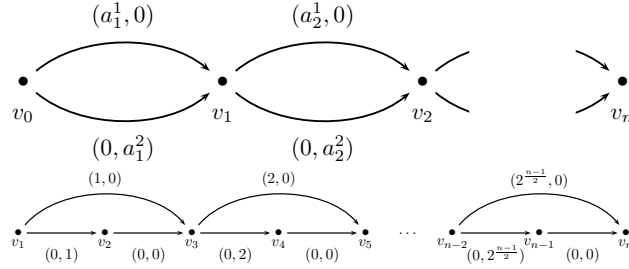
So I had a choice between shorter travel time or fewer train changes. Obviously this is a multiobjective shortest path problem (a third objective, fare, is not available due to the international connection).

The multiobjective shortest path problem is a well studied multiobjective combinatorial optimisation (MOCO) problem. In particular, we know that

**Table 1.** Two train connections between Pirmasens and Tours.

Station	Date	Time	Duration	Changes
Pirmasens	Su, 11.06.06	09:32	8:49	4
Tours	Su, 11.06.06	18:21		
Pirmasens	Su, 11.06.06	09:32	10:05	3
Tours	Su, 11.06.06	19:37		

already the biobjective version is NP-hard because the digraph at the top of Figure 2 can be used to demonstrate a reduction from the knapsack problem. Moreover, the graph at the bottom of Figure 2 shows that it is intractable, i.e., there can be exponentially many efficient paths and nondominated paths, see, e.g., Ehrgott (2005) for proofs.

**Fig. 2.** The multiobjective shortest path problem is NP-hard (top) and intractable (bottom).

This is pretty bad news for a problem which is so easy in its single objective version. Table 2 shows computation times and the number of efficient paths for relatively big networks from Raith and Ehrgott (2006).

Apparently, the NP-hardness and intractability are not an issue in these examples (and all others tested in Raith and Ehrgott (2006)). It is particularly striking that the large road networks have very few efficient paths. Can this discrepancy be explained?

Indeed it can. Müller-Hannemann and Weihe (2006) have investigated properties of networks with two objectives that allow better estimates of the number of efficient paths. Using the ratio between the first and second objective on the arcs they prove Theorem 1.

**Theorem 1.** *1. Even if the ratio between first and second length of an arc assumes only 2 values there can be exponentially many efficient paths.*

**Table 2.** Number of efficient paths and CPU times for biobjective shortest path problems of different types.

Type	Nodes	Edges	Efficient Paths	CPU Time
Grid	20,002	79,600	247	960.51
Grid	10,002	39,600	390	71.08
NetMaker	10,000	155,157	10	1.05
NetMaker	3,000	8,333	27	0.11
Road	9,559	39,377	6	0.63
Road	53,658	192,084	17	4.41

2. If there are  $k$  different ratios between first and second length of an arc there are at most  $O(n^{2k-2})$  efficient bitonic paths. A bitonic path is a path where the sequence of ratios switches only once from increasing and decreasing.

Müller-Hannemann and Weihe (2006) have conducted experiments on the train graph of the Deutsche Bahn rail network, which has 1.4 million nodes, 2.3 million arcs and found that 84% of efficient paths are bitonic. Moreover, the number of efficient paths using different combinations of objectives is very small. For distance versus time on average 2 and most 8 paths are efficient. For fare versus time the numbers are 3 and 22 and for the three objectives distance, time, and train changes they are 10 and 96.

Again, we learn a number of lessons from this. Firstly, the concepts of NP-hardness may not be too relevant in multiobjective optimisation. Since almost all MOCO problems are NP-hard and intractable, there is virtually no distinction among problems by these criteria. Moreover, worst case estimates may simply not apply in a particular application, even if problem instances become very large. It is therefore always worse studying the circumstances of the application. That will be beneficial for the application and it might lead to interesting mathematical results.

## 4 Transportation: Airline Crew Scheduling

BBC News of Sunday, 4 August, 2002, had an item that serves well to explain a problem in airline crew scheduling.

Passengers with low-cost airline Easyjet are suffering delays after 19 flights in and out of Britain were cancelled. The company blamed the move – which comes a week after passengers staged a protest sit-in at Nice airport – on crewing problems stemming from technical

hitches with aircraft. Crews caught up in the delays worked up to their maximum hours and then had to be allowed home to rest. Mobilising replacement crews has been a problem as it takes time to bring people to airports from home. Standby crews were already being used and other staff are on holiday.

But first, we need to understand the standard integer programming model of the airline crew scheduling problem. The goal is to partition the scheduled flights into a set of pairings each of which can be operated by a crew member to minimise cost. Let

$$a_{ij} = \begin{cases} 1 & \text{pairing } j \text{ includes flight } i \\ 0 & \text{otherwise.} \end{cases}$$

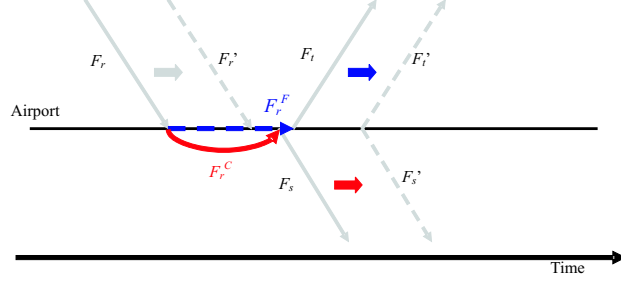
The problem can then be formulated as a generalised set partitioning problem

$$\begin{aligned} \min &= c^T x \\ \text{subject to } & Ax = e \\ & Mx = b \\ & x \in \{0, 1\}^n. \end{aligned} \tag{4}$$

This particular type of set partitioning problem can be solved using column generation and constrained branching strategies. Software to solve (4) (optimally or heuristically) is in use by all airlines. In fact, airline crew scheduling has been one of the biggest successes of Operations Research.

However, as can be seen from the news item above, things do not always go as planned and delays are common occurrences in operation. Optimal crew schedules according to (4) often operate with minimal “sit times” between flights, that is without buffer time between flights to be operated by the same crew member. In addition, aircraft are also kept operating with minimal “turn time” between flights. In such a scenario consider Figure 3. If arriving flight  $F_r$  is late the next flight operated by the same aircraft is inevitably late, too. Moreover, if two crew members  $C$  and  $F$  arriving on flight  $F_r$  are scheduled to operate flights  $F_s$  and  $F_t$ , these will also be delayed. It is easy to imagine that this propagation of delays through the schedule can cause major and very expensive disruptions.

Thus, dealing with delay has become a focus of research in recent years. I shall explain two approaches. The first one is based on the stochastic nature of delays and incorporates the cost of delay into the problem formulation resulting in a stochastic programme with recourse (Yen and Birge, 2006). The formulation is



**Fig. 3.** Delay propagation due to aircraft changes.

$$\begin{aligned}
 \min \quad & c^T x + Q(x) \\
 \text{s.t.} \quad & A_1 x = e \\
 & A_2 x = b \\
 & x \in \{0, 1\}^n,
 \end{aligned} \tag{5}$$

where  $Q(x) = \sum_{\omega \in \Omega} p(\omega) Q(x, \omega)$  and  $Q(x, \omega)$  is the cost of delay under schedule  $x$  in scenario  $\omega$ . Details of the solution algorithm – a branch and bound algorithm, which requires a set partitioning problem to be solved in every node – can be found in Yen and Birge (2006).

The second approach is based on the conflicting goals of minimising cost and minimising delay caused by aircraft changes, i.e., it is a biobjective programme (Ehrgott and Ryan, 2002):

$$\begin{aligned}
 \min \quad & r^T x \\
 \min \quad & c^T x \\
 \text{s.t.} \quad & A_1 x = e \\
 & A_2 x = b \\
 & x \in \{0, 1\}^n,
 \end{aligned} \tag{6}$$

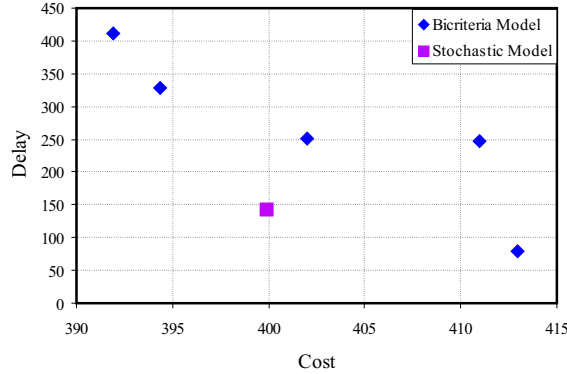
where  $r_j$  is a penalty for short ground time that does not allow recovery from previous delays. It uses the 95%-quantile of the delay distribution as a parameter. The biobjective set partitioning models were solved using the method of elastic constraints



$$\begin{aligned}
& \min r^T x + ps \\
& \text{s.t. } A_1 x = e \\
& \quad A_2 x = b \\
& \quad c^T x - s = \varepsilon \\
& \quad x \in \{0, 1\}^n \\
& \quad s \geq 0,
\end{aligned} \tag{7}$$

a variant of the  $\varepsilon$ -constraint scalarisation which allows the  $\varepsilon$ -constraints to be violated but penalises the violation. It is known (Ehrgott and Ryan, 2003) that all solutions found are weakly efficient, that all efficient solutions can be found. But most importantly it turned out to be computationally superior to the  $\varepsilon$ -constraint method. In fact, an instance of (7) could be solved in approximately the same time as (4), whereas solving the  $\varepsilon$ -constraint scalarisation often exceeded the node limit of 1000 and ran more than 10 times longer.

Both approaches have been implemented using the same crew scheduling software and the same schedule. The optimal solution of the stochastic programme and efficient solution of the biobjective programme were simulated using 100 delay scenarios. Figure 4 shows the average costs and delays.



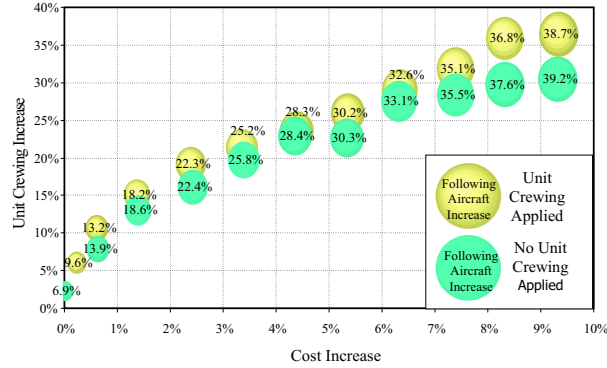
**Fig. 4.** Cost versus delay for schedules obtained with the stochastic and biobjective programmes.

Using either the stochastic or biobjective approach to robust crew scheduling we may solve the problem of delays caused by aircraft changes. This does not address the problem of an arriving crew splitting up to operate different flights. Thus, the issue is only partially resolved: What is the use of having robust solutions for pilots if cabin crew do something different?

Robust crew scheduling should also address unit crewing, i.e., the problem of keeping crew together for a sequence of flights for as long as possible. Thus, we want to solve the pairings problem for several crew groups simultaneously so as to minimise cost and maximise unit crewing. The corresponding problem formulation is

$$\begin{aligned}
& \min && c_1^T x_1 + c_2^T x_2 \\
& \min && e^T s_1 + e^T s_2 \\
& \text{subject to} && A_1 x_1 &= e \\
& && M_1 x_1 &= b_1 \\
& && A_2 x_2 &= e \\
& && M_2 x_2 &= b_2 \\
& && U_1 x_1 - U_2 x_2 - s_1 + s_2 &= 0 \\
& && x_1 & x_2 & s_1 & s_2 \in \{0, 1\}^n.
\end{aligned} \tag{8}$$

Using new branching strategies and the elastic constraint method as in (7), (Tam, 2004) has obtained results that show that unit crewing, crew changing aircraft, and robustness of crew schedules are closely related, as shown in Figure 5.



**Fig. 5.** Unit crewing versus number of aircraft changes and cost.

We see here, that working on a particular application necessitates the development of new solution techniques. Such developments drive multiobjective programming, eventually making it applicable in other real word situations, where multiobjective models haven't been considered yet or haven't been solved before. We can also see that biobjective models may be an alternative to stochastic programming, if the recourse objective can be captured by means of a deterministic objective. This is especially relevant for integer models, where the stochastic programme is particularly hard to solve and computationally expensive. Finally, multiobjective modelling can lead to a more

comprehensive view of an application. From the biobjective model for robust scheduling it is only a small step towards the simultaneous consideration of several crew groups in the unit crewing problem. In fact, the next question is immediate: Why not include aircraft routing and consider assignment of aircraft to flights simultaneously with crew scheduling? This is an unsolved problem that is currently under investigation (Weide, 2005).

## 5 Medicine: Radiotherapy Treatment Design

External beam radiotherapy is one of the major forms of cancer treatment. About 50% of cancer patients receive radiotherapy for curative or palliative purposes. Beams of electrons or high energy photons generated by a linear accelerator are focused on the tumour from several directions. An oncologist prescribes a dose distribution to be achieved by the treatment, that is a radiation dose to be delivered to the tumour that achieves the curative or palliative intent of the treatment but avoids damage to healthy tissues.

Given the beam directions, the purpose of radiotherapy treatment design is to find beam intensity (or fluence) maps for each beam that realise the desired dose distribution. Here I consider the treatment design problem for intensity modulated radiotherapy (IMRT), where beam intensity can vary across a beam. The advantage of IMRT is described on <http://www.cancernews.com/data/Article/259.asp> as follows

IMRT represents an advance in the means that radiation is delivered to the target, and it is believed that IMRT offers an improvement over conventional and conformal radiation in its ability to provide higher dose irradiation of tumour mass, while exposing the surrounding normal tissue to less radiation. <http://www.cancernews.com/data/Article/259.asp>

Many optimisation models, both linear and nonlinear, are available for this problem. The most popular optimisation model is based on an oncologist's prescription of a goal dose  $TG$  to the target and upper bounds  $CG$  and  $NG$  on the dose to critical structures and normal tissue. It consists of the minimisation of a norm of the (nonnegative) deviation of delivered and goal dose:

$$\min_{x \geq 0} \omega_T \|A_T x - TG\| + \omega_C \|(A_C x - CG)_+\| + \omega_N \|(A_N x - NG)_+\|, \quad (9)$$

where  $(\cdot)_+ = \max\{\cdot, 0\}$ .  $A_T, A_C$  and  $A_N$  are matrices. In practice the Euclidean norm is most often used and the most popular solution technique is simulated annealing. The result of the optimisation depends crucially on the values of  $\omega_T, \omega_C, \omega_N$ , which are often deemed indispensable for effective treatment planning. A trial and error process is usually needed to find values that result in a good quality treatment.

The words “higher” and “lower” in the above quotation indicate that treatment planning is about conflicting goals. And to anyone familiar with multi-objective optimisation it is obvious that the standard dose based model (9) is the weighted sum scalarisation of the multiobjective programme

$$\min_{x \geq 0} (\|D_{\mathcal{T}}x - TG\|, \|(D_{\mathcal{C}}x - CG)_+\|, \|(D_{\mathcal{N}}x - NG)_+\|) \quad (10)$$

However, this model has only been used in the form (9), with a set of pre-selected weights to produce several efficient plans (Cotrutz *et al.*, 2001; Lahanas *et al.*, 2003). The first non-scalarised multiobjective LP model has been proposed by Küfer and Hamacher (2000).

In the context of the multitude of objective functions used in radiotherapy treatment planning models a theorem stated in Romeijn *et al.* (2004) becomes important.

**Theorem 2.** *The two multiobjective problems  $\min\{(f_1(x), \dots, f_p(x)) : x \in X\}$  and  $\min\{(h_1(f_1(x)), \dots, h_p(f_p(x))) : x \in X\}$  with strictly increasing functions  $h_1, \dots, h_p$  are equivalent.*

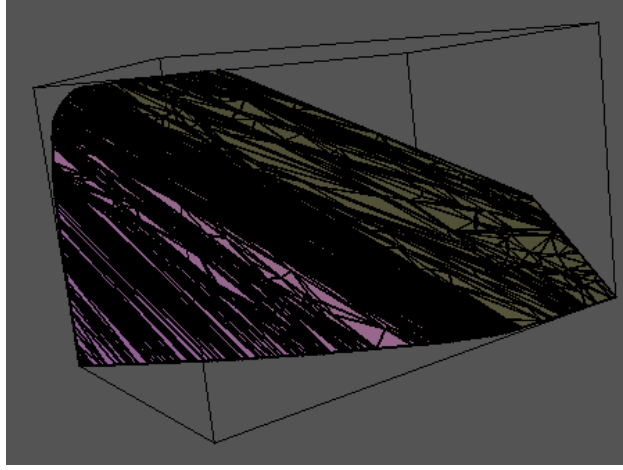
Theorem 2 is not really surprising, but it is important as it illustrates that much of the discussion about the right model is void. We present here a linear model with three objectives derived from (the scalar) LP in Holder (2003):

$$\begin{aligned} & \min(z_{\mathcal{T}}, z_{\mathcal{S}}, z_{\mathcal{N}}) \\ & \text{subject to } A_{\mathcal{T}}x + z_{\mathcal{T}}e \geq l_{\mathcal{T}} \\ & \quad A_{\mathcal{T}}x \leq u_{\mathcal{T}} \\ & \quad A_{\mathcal{S}}x - z_{\mathcal{S}}e \leq u_{\mathcal{S}} \\ & \quad A_{\mathcal{N}}x - z_{\mathcal{N}}e \leq u_{\mathcal{N}} \\ & \quad z_{\mathcal{N}} \geq 0 \\ & \quad x \geq 0. \end{aligned} \quad (11)$$

This multiobjective linear programme may have thousands of variables and tens of thousands of constraints. Since it has only three objectives it is advantageous to solve it in objective space. Benson’s “outer approximation” algorithm (Benson, 1998) can be used for this purpose. In Shao and Ehrgott (2006) we have solved a simplified version of this problem with 1293 constraints and 821 variables. 3165 non-dominated extreme points have been obtained, shown in Figure 6. The computation took nearly one hour.

From Figure 6 it can already be seen that many of the extreme points differ only very slightly. This result points to another issue: How is a solution to be selected among so many options? And do we want an extreme point solution in the first place?

At this stage it is necessary to reconsider the model. It uses a dose deposition matrix  $A$ , separated by rows into  $A_{\mathcal{T}}$ ,  $A_{\mathcal{C}}$  and  $A_{\mathcal{N}}$  as input. The entries



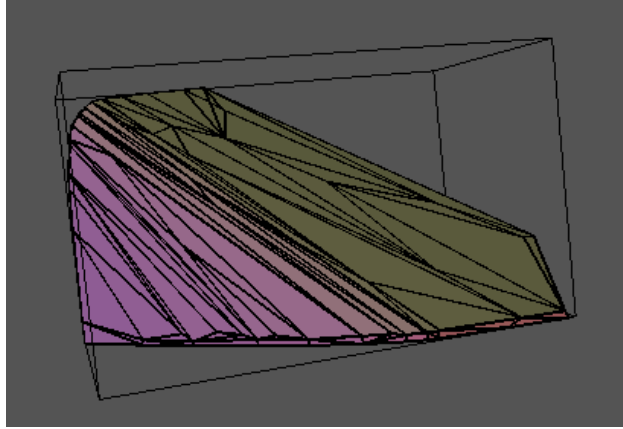
**Fig. 6.** The non-dominated set of (11) for a prostate cancer example.

in  $A$  are the result of a dose calculation model, that calculates the amount of dose deposited at a point in the body at unit intensity of a sub-beam. Even with sophisticated dose models this calculation is imprecise. That means that the data of (11) are imprecise. It turns out that in clinical practice calculating a dose distribution to 0.1 Gy precision is sufficient.

We could therefore use an approximation version of Benson's algorithm that is guaranteed to solve the MOLP (11) to within 0.1 precision and achieve a dramatic effect. This algorithm calculates 88 non-dominated extreme points in less than one minute. The approximated non-dominated set  $se$  are shown in Figure 7.

We have so far assumed that the beam directions are given. However, they also have to be chosen. This choice is currently done manually. Mathematically the optimisation of beam directions can easily be incorporated in (11):

$$\begin{aligned}
 & \min(z_T, z_S, z_N) \\
 & \text{subject to } A_T x + z_T e \geq l_T \\
 & \quad A_T x \leq u_T \\
 & \quad A_S x - z_S e \leq u_S \\
 & \quad A_N x - z_N e \leq u_N \\
 & \quad z_N \geq 0 \\
 & \quad x \geq 0 \\
 & \quad x \leq Mye \\
 & \quad e^T y \leq r.
 \end{aligned} \tag{12}$$



**Fig. 7.** The approximate non-dominated set of (11) for the same prostate cancer example.

The solution of the large scale multiobjective mixed integer programme (12) is a challenge for both multiobjective optimisation and radiotherapy treatment design.

I find this application particularly instructive. It shows how hard it can be to convince practitioners of the usefulness of multiobjective optimisation, even if they already use elements of it, albeit unknowingly. It is a reminder that the results of optimisation can never be more precise than the input data, and that it is always worth exploiting features of the application to simplify methods. In addition, applying multiobjective optimisation can lead to improved processes in the application area as implicit benefits. In this example the trial and error search for “optimal” weights can be eliminated. Instead, treatment planners can concentrate on their main task, namely to find a best possible treatment plan for the patient. Because the multiobjective model allows a separation of plan calculation and selection, a speed up of the planning process can be expected. Again, multicriteria decision aid is called upon to provide appropriate decision support systems.

## 6 Telecommunication: Routing in IP Networks

Routing of data packets in computer networks using the internet protocol is usually based on the OSPF protocol (open shortest path first). This protocol applies Dijkstra’s algorithm to minimise the number of hops (the number of intermediate routers) along the path from the origin of the packet to its destination. While other protocols exist that allow aggregation of several objectives, routing is still using a “best effort” rather than “Quality of Service” philosophy.

It does not take much imagination to see that several objectives are relevant in this context. Gandibleux *et al.* (2006) have developed a routing protocol that uses the three objectives

- $\min f_1(p) = \sum_{(i,j) \in p} c^1(i,j)$ , where  $c^1(i,j)$  denotes the delay on link  $(i,j)$ ,
- $\max f_2(p) = \min_{(i,j) \in p} c^2(i,j)$ ,  $c^2(i,j)$  denoting the available bandwidth of link  $(i,j)$ , and
- $\min f_3(p) = |\{(i,j) \in p\}|$ , counting the number of hops

as well as additional constraints. They have implemented a modification of Martin's label correcting algorithm (Martins, 1984) to deal with the constraints and the bandwidth objective, which is of the min max rather than the min sum type.

Considering the delay and bandwidth objectives only there are five efficient paths from node seven to node eleven in the network of Figure 8, an actual IP network (bandwidth and delay are listed along the arcs).

This application shows that even long known algorithms can be useful in today's problems. Once more, as seen in the other examples, multiobjective modelling helps thinking outside the box.

Chapter 22 in Figueira *et al.* (2005) and references therein contain much more on multicriteria decision analysis in telecommunication.

## 7 Conclusion

In this paper I have sketched a number of applications of multiobjective programming. I have tried to show that interacting with practitioners in many areas is mutually beneficial in the sense that real world applications provide opportunities for progress in multiobjective optimisation methodology and theory and that multiobjective models provide insights in applications that conventional models cannot reveal. In particular, multiobjective models help question standard procedures and thus induce the practitioner to think outside a conventional framework. It is easily possible that multiobjective optimisation results in benefits that are not at all part of the model. Last but not least, the real world has many challenges and new application areas in store to motivate established and future researchers to work in this area.

## References

- Benson, H. (1998). An outer approximation algorithm for generating all efficient extreme points in the outcome set of a multiple objective linear programming problem. *Journal of Global Optimization*, **13**(1), 1–24.
- Chang, T., Meade, N., Beasley, J., and Sharaiha, Y. (2000). Heuristics for cardinality constrained portfolio optimisation. *Computers and Operations Research*, **27**, 1271–1302.



**Fig. 8.** Efficient routes in an IP network using bandwidth and delay objectives.

- Cortutz, C., Lahanas, M., Kappas, C., and Baltas, D. (2001). A multiobjective gradient-based dose optimization algorithm for external beam conformal radiotherapy. *Physics in Medicine and Biology*, **46**, 2161–2175.
- Ehrgott, M. (2005). *Multicriteria Optimization*. Springer-Verlag, Berlin, 2nd edition.
- Ehrgott, M. and Ryan, D. (2002). Constructing robust crew schedules with bicriteria optimization. *Journal of Multi-Criteria Decision Analysis*, **11**, 139–150.
- Ehrgott, M. and Ryan, D. (2003). The method of elastic constraints for multiobjective combinatorial optimization and its application in airline crew scheduling. In T. Tanino, T. Tanaka, and M. Inuiguchi, editors, *Multi-Objective Programming and Goal-Programming*, pages 117–122. Springer Verlag, Berlin.



- Ehrgott, M., Klamroth, K., and Schwehm, C. (2004). An MCDM approach to portfolio optimization. *European Journal of Operational Research*, **155**, 752–770.
- Figueira, J., Greco, S., and Ehrgott, M., editors (2005). *Multiple Criteria Decision Analysis: State of the Art Surveys*, volume 78 of *Springer's International Series in Operations Research & Management Science*. Springer Science + Business Media Inc., New York, NY.
- Gandibleux, X., Beugnies, F., and Randriamasy, S. (2006). Martins' algorithm revisited for multiobjective shortest path problems with a maxmin cost function. *JOR*, **4**, 47–59.
- Hansen, P. (1979). Bicriterion path problems. In G. Fandel and T. Gal, editors, *Multiple Criteria Decision Making Theory and Application*, volume 177 of *Lecture Notes in Economics and Mathematical Systems*, pages 109–127. Springer Verlag, Berlin.
- Holder, A. (2003). Designing radiotherapy plans with elastic constraints and interior point methods. *Health Care Management Science*, **6**, 5–16.
- Konno, H. (1990). Piecewise linear risk function and portfolio optimization. *Journal of the Operational Research Society of Japan*, **33**, 139–156.
- Küfer, K.-H. and Hamacher, H. (2000). A multicriteria optimization approach for inverse radiotherapy planning. In W. Schlegel and T. Bortfeld, editors, *XIIIth International Conference on the Use of Computers in Radiation Therapy*, pages 26–28, Heidelberg, Germany. Springer Verlag, Berlin.
- Lahanas, M., Schreibmann, E., and Baltas, D. (2003). Multiobjective inverse planning for intensity modulated radiotherapy with constraint-free gradient-based optimization algorithms. *Physics in Medicine and Biology*, **48**, 2843–2871.
- Markowitz, H. (1952). Portfolio selection. *Journal of Finance*, **7**, 77–91.
- Martins, E. (1984). On a multicriteria shortest path problem. *European Journal of Operational Research*, **16**, 236–245.
- Müller-Hannemann, M. and Weihe, K. (2006). On the cardinality of the Pareto set in bicriteria shortest path problems. *Annals of Operations Research*, **147**, in print.
- Raith, A. and Ehrgott, M. (2006). Comparing optimization strategies for the biobjective shortest path problem. Technical report, Department of Engineering Science, The University of Auckland.
- Romeijn, H., Dempsey, J., and Li, J. (2004). A unifying framework for multicriteria fluence map optimization models. *Physics in Medicine and Biology*, **49**, 1991–2013.
- Shao, L. and Ehrgott, M. (2006). Multiple objective linear programming in radiotherapy treatment planning. Technical report, Department of Engineering Science, The University of Auckland.
- Steuer, R., Qi, Y., and Hirschberger, M. (2006). Suitable-portfolio investors, nondominated frontier sensitivity, the effect of multiple objectives on standard portfolio selection. *Annals of Operations Research*, **??**, in print.

- Tam, B. (2004). Multi-criteria methods for unit crewing in the airline tour-of-duty planning problem. In M. Ehrgott, editor, *Proceedings of the 39th Annual Conference of the Operational Research Society of New Zealand*, pages 174–183. Available online at <http://www.esc.auckland.ac.nz/Organisations/ORSNZ/conf39/Papers/TamB.pdf>.
- Weide, O. (2005). Robust and integrated airline scheduling. In T. Halliburton, editor, *Proceedings of the 40th Annual Conference of the Operational Research Society of New Zealand*, pages 245–254. Available online at <https://secure.orsnz.org.nz/conf40/content/papers/Weide.pdf>.
- Yen, J. and Birge, J. (2006). A stochastic programming approach to the airline crew scheduling problem. *Transportation Science*, **40**(1), 3–14.