

# **Topologies on the Planar Orthogonal Grid**

Reinhard Klette<sup>1</sup>

## **Abstract**

This paper discusses different topologies on the planar orthogonal grid and shows homeomorphy between cellular models. It also points out that graph-theoretical topologies exist defined by planar extensions of the 4-adjacency graph. All these topologies are potential models for image carriers.

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# Topologies on the Planar Orthogonal Grid

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## Abstract

*This paper discusses different topologies on the planar orthogonal grid and shows homeomorphy between cellular models. It also discusses graph-theoretical options defined by planar extensions of the 4-adjacency graph.*

**Keywords:** digital topology, geometric complexes, abstract complexes, Khalimsky-Kovalevsky plane, Wyse topology.

## 1. Introduction

The topology of digital images, and topological problems related to image analysis have been studied over the last thirty years. This article shows that cellular models in 2D, commonly discussed as possible options for image carriers, are topologically equivalent, and that the previously known topology of the 4-adjacency graph also offers ways of defining image topologies for planar adjacency graphs.

A *digital image*  $I$  is a function defined on a set  $\mathbb{C}$ , which is called a *carrier* of the image, and its elements are called *points*. The range of a (scalar) digital image is  $\{0, \dots, G_{max}\}$  with  $G_{max} \geq 1$ . The range of a *binary image* is  $\{0, 1\}$ , i.e.  $G_{max} = 1$ . An image carrier in this article is a subset of the two-dimensional Euclidean space, e.g. of the orthogonal grid defined by *grid points*.

Let  $\mathbb{C}$  be an arbitrary set of points, where a non-negative number  $dim(p)$  is assigned for each  $p \in \mathbb{C}$ .

An *abstract complex*  $[\mathbb{C}, \leq, dim]$  satisfies two axioms:  $\leq$  is a partial order on  $\mathbb{C}$ , and if  $p \leq q$  and  $p \neq q$  then  $dim(p) < dim(q)$ .

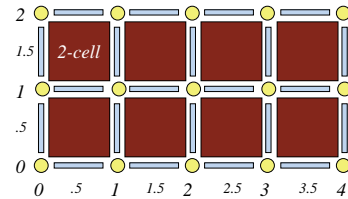
A definition identical by contents may be found in [2] (page 125), and both volumes [2, 3] provide a broad coverage of definitions and results on abstract complexes. The elements in  $\mathbb{C}$  are named *cells* of the complex. If  $dim(p) = n$  then  $n$  is the *dimension* of  $p$ , and  $p$  is called an *n-cell*. 0-cells are named *vertices*. An *n-dimensional complex*  $[\mathbb{C}, \leq, dim]$  is characterized by  $dim(p) \leq n$ , for all  $p \in \mathbb{C}$ , and there is at least one  $p \in \mathbb{C}$  with  $dim(p) = n$ .

Let  $[\mathbb{C}, \leq, dim]$  be an abstract complex. If  $p \leq q$  and  $p \neq q$  then we say that  $p$  is a *proper side* of  $q$ . If  $dim(p) = m$  then  $p$  is an *m-side* of  $q$ . Two cells are *incident* iff  $p \leq q$  or  $q \leq p$ .

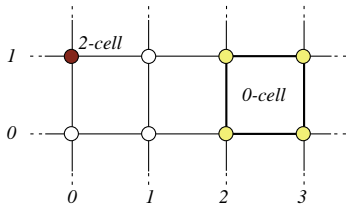
Examples or *models* of abstract complexes are simply called *complexes*. In image analysis, we prefer the homogeneous orthogonal planar grid as a homogeneous image carrier, and we discuss two models of abstract complexes which are normally used as image carrier.

We may identify 2-cells with open grid squares of the homogeneous orthogonal grid, 1-cells with grid edges (without their endpoints), and 0-cells with grid points, see Fig. 1. This defines a partition of  $\mathbb{R}^2$  into pairwise disjoint sets. Because we are interested in topological characterizations of complexes, we may also identify 2-cells with closed grid squares, 1-cells with closed grid edges (i.e. with both end points), and 0-cells with grid points. This is not a partition into pairwise disjoint sets anymore, but defines a Euclidean complex, and it is a topologically equivalent model of the same abstract complex. We decide for the Euclidean complex, and let  $\mathbb{C}_{E2}$  be the set of all these (closed in the Euclidean topology) 2-, 1- and 0-cells of the homogeneous orthogonal grid in the Euclidean plane. For  $p, q \in \mathbb{C}_{E2}$  let  $p \leq_{E2} q$  iff  $p \subseteq q$ .  $[\mathbb{C}_{E2}, \leq_{E2}, dim]$  is a two-dimensional complex.

As an alternative model of a two-dimensional abstract complex, we may identify 2-cells with a grid point of the homogeneous orthogonal grid, 1-cells with an undirected



**Figure 1. Two-dimensional Euclidean complex of the homogeneous orthogonal grid.**



**Figure 2. Two-dimensional graph complex of the homogeneous orthogonal grid.**

subgraph consisting of two grid points and one edge forming a grid edge, and 0-cells with an undirected subgraph consisting of four grid points and four edges forming a grid square, see Fig. 2. Let  $\mathbb{C}_{G_2}$  be the set of all of these cells. For  $p, q \in \mathbb{C}_{G_2}$  let  $p \leq_{G_2} q$  iff  $q$  is a subgraph of  $p$ . For example, a grid point  $x$  is a subgraph of an undirected edge  $e = \{x, y\}$ , i.e.  $e \leq_{G_2} x$ . It follows that  $[\mathbb{C}_{G_2}, \leq_{G_2}, \dim]$  is a two-dimensional complex.

**Theorem 1** Complexes  $[\mathbb{C}_{G_2}, \leq_{G_2}, \dim]$  and  $[\mathbb{C}_{E_2}, \leq_{E_2}, \dim]$  are isomorphic.

**Proof:** Let  $\Phi$  be a mapping of  $\mathbb{C}_{G_2}$  into  $\mathbb{C}_{E_2}$  such that grid point  $(i, j)$  is mapped onto a grid square having  $(i, j)$  as its lower-left corner, a graph connecting grid points  $(i, j)$  and  $(i, j + 1)$  is mapped onto a grid edge connecting grid points  $(i, j + 1)$  and  $(i + 1, j + 1)$ , a graph connecting grid points  $(i, j)$  and  $(i + 1, j)$  is mapped onto a grid edge connecting grid points  $(i + 1, j)$  and  $(i + 1, j + 1)$ , and a graph consisting of four grid points  $(i, j), (i + 1, j), (i, j + 1), (i + 1, j + 1)$  and connecting grid edges is mapped onto the single grid point  $(i + 1, j + 1)$ . Then it holds that  $\Phi$  is bijective from  $\mathbb{C}_{G_2}$  onto  $\mathbb{C}_{E_2}$  such that for any  $p, q \in \mathbb{C}_{G_2}$  we have  $p \leq_{G_2} q$  iff  $\Phi(p) \leq_{E_2} \Phi(q)$ . Q.E.D.

This isomorphism shows a general *duality* of grid-point related (graph-theoretical) concepts and of cellular concepts. Models of abstract complexes may be homogeneous geometric complexes such as  $[\mathbb{C}_{G_2}, \leq_{G_2}, \dim]$  or  $[\mathbb{C}_{E_2}, \leq_{E_2}, \dim]$ , or inhomogeneous image carriers.

## 2 Topological Spaces

A topology of an image carrier may be defined via a specification of a locally finite basis. A *poset* is a partially ordered set. The *Aleksandrov topology* of a poset  $[\mathbb{C}, \leq]$  is defined as follows: a set  $M \subseteq \mathbb{C}$  is open iff  $p \in M$  and  $p \leq q$  imply  $q \in M$ , for all  $p, q \in \mathbb{C}$ .

**Example 1**  $[\{\{i\} : i \in \mathbb{Z}\} \cup \{\{i, i + 1\} : i \in \mathbb{Z}\}, \subseteq]$  is a poset. The sets  $\{\{i\}, \{i, i + 1\}, \{i, i - 1\}\}$  and  $\{\{i, i + 1\}\}$ , for  $i \in \mathbb{Z}$ , are a basis of the Aleksandrov topology [9].

For topologies on abstract complexes see, for example, the definition and study of *open* and *closed subcomplexes* in [2, 12, 13]. A subset  $M \subseteq \mathbb{C}$  of an abstract complex  $K$  is *open* iff  $p \in M$  and  $p \leq q$  then  $q \in M$ , for all  $p, q \in \mathbb{C}$ .

As a consequence, a subset  $M$  of an abstract complex  $K$  is *closed* iff  $p \in M$  and  $q \leq p$  then  $q \in M$ , for all  $p, q \in \mathbb{C}$ . Note that  $\leq$  is a partial ordering, i.e. the definition of an Aleksandrov topology of a poset  $[\mathbb{C}, \leq]$  is a generalization of the (historically earlier) Tucker topology of an abstract complex.

**Example 2** Consider the two-dimensional Euclidean complex  $[\mathbb{C}_{E_2}, \leq_{E_2}, \dim]$  or graph complex  $[\mathbb{C}_{G_2}, \leq_{G_2}, \dim]$  of the homogeneous orthogonal grid. The following is formulated for the Euclidean complex, and the graph complex may be discussed analogously.

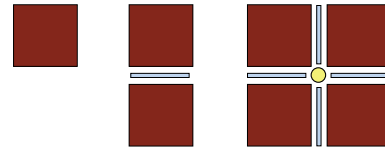
Let  $p$  be a 2-cell. Then  $\{p\}$  is open in the Tucker or Aleksandrov topology: there is no  $q \in \mathbb{C}_{E_2}$  with  $p \neq q$  and  $p \leq_{E_2} q$ . Let  $p$  be a 1-cell. Then there are exactly two 2-cells  $q_1$  and  $q_2$  with  $p \leq_{E_2} q_1$  and  $p \leq_{E_2} q_2$ , see Fig. 3, i.e. the set  $\{p, q_1, q_2\}$  is open. Figure 3 also illustrates (on the right) the case when we start with a 0-cell  $p$ .

Let  $[\mathbb{C}, \leq, \dim]$  be an abstract complex. For  $p \in \mathbb{C}$  let  $U(p) = \{q : q \in \mathbb{C} \wedge p \leq q\}$  be the *smallest neighborhood* of  $p$  in this abstract complex. This smallest neighborhood may be understood as being the  $\varepsilon$ -neighborhood with  $\varepsilon = 1$ , where a distance is defined with respect to the partial ordering  $\leq$ . Figure 4 illustrates the smallest neighborhoods in the graph complex: a grid point (2-cell); a subgraph defined by two grid points and one grid edge (1-cell) and both of its grid points (2-cells); and a subgraph of a 0-cell which is a proper side of four 1-cells and of four 2-cells.

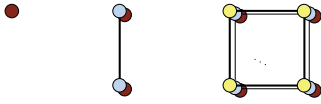
The application of topological spaces of homogeneous Euclidean complexes for image analysis has been proposed in [8], and for more general situations in [10].

## 3 Wyse Topology and a Non-Existence Theorem

A mapping  $\Phi$  of a topological space  $\mathbb{C}_1$  onto a topological space  $\mathbb{C}_2$  is a *homeomorphism* or a *topological map*.



**Figure 3. The smallest neighborhoods of single cells in the two-dimensional Euclidean complex of the homogeneous orthogonal grid.**



**Figure 4.** The smallest neighborhoods of single cells in the two-dimensional graph complex of the homogeneous orthogonal grid.

ping iff it is one-one (i.e. a *bijection*), continuous (i.e.  $\Phi^{-1}(M) = \{p \in \mathbb{C}_1 : \Phi(p) \in M\}$  is open in  $\mathbb{C}_1$ , for any open subset  $M$  of  $\mathbb{C}_2$ ), and  $\Phi^{-1}$  is continuous as well. The cell or graph complex of the homogeneous two-dimensional grid provides one topological space because both models are homeomorphic (see Theorem 1).

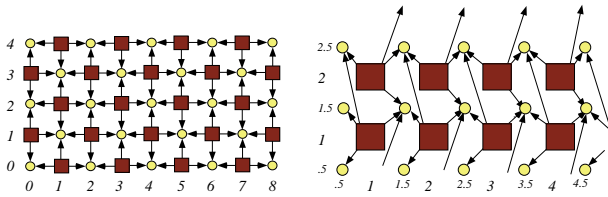
There are further topological spaces defined on  $\mathbb{Z}^2$ . Topologies on the two-dimensional homogeneous orthogonal grid may be defined by specifying a basis (as noted earlier for the general case):

**Example 3** Let  $A_4(p) = \{q \in \mathbb{Z}^2 : d_1(p, q) = 1\}$  be the set of 4-adjacent points of grid point  $p = (i, j)$ . Let

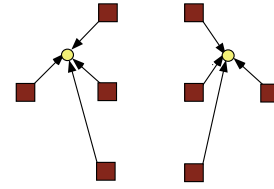
$$U_4(p) = \begin{cases} \{p\}, & \text{if } i+j \text{ is odd} \\ \{p\} \cup A_4(p), & \text{if } i+j \text{ is even} \end{cases}.$$

The family of all of these sets  $U_4(p)$ ,  $p \in \mathbb{Z}^2$ , defines a topological basis on  $\mathbb{Z}^2$ , and a set of grid points is connected in the resulting 4-topology iff it is 4-connected, see [14]. Set  $U_4(p)$  is the smallest topological neighborhood of point  $p$ . The neighborhood relation  $U_4$  is asymmetric. See Fig. 5.

Due to the correspondence of topological connectedness and 4-connectedness it is obvious that the 4-topology does not add further ‘structure’ to the concept of 4-adjacency. However, it is possible to discuss open and closed sets in this 4-topology, or the closure of a set (see Fig. 6). For example, any set containing an *even* grid point  $p = (i, j)$ ,



**Figure 5.** Left: a directed graph visualizing the asymmetric neighborhood relation  $U_4$ . Right: Indication of one possible mapping of this graph such that exactly all odd grid points (shown as squares) are in grid point positions.



**Figure 6.** The smallest neighborhoods of single even grid points for the drawing shown in Fig. 5 on the right.

with  $i + j$  even, but only at most three of its 4-neighbors (which are *odd* grid points), is not open, and sets containing only even grid points are closed.

The following theorem states that 8-adjacency does not allow such a topological model.

**Theorem 2** [5] Let  $\mathbb{C}$  be  $\mathbb{Z}^2$ , or a finite subset of  $\mathbb{Z}^2$  containing a translation of the set below. Then there exists no



topology on  $\mathbb{C}$  in which connectivity would be the same as 8-connectivity.

The geometric location of this set of grid points is unimportant.

The 8-adjacency graph is non-planar. Now consider a planar adjacency graph  $S$  on  $\mathbb{Z}^2$ , e.g. an extension of the 4-adjacency graph. The homogeneous grid used may suggest a straightforward adaptation of the Wyse topology for such an extension: let  $A_S(p) = \{q \in \mathbb{Z}^2 : d_S(p, q) = 1\}$  be the set of all grid points being in distance 1, where  $d_S$  is defined by graph  $S$ , for grid point  $p = (i, j)$ . Let

$$U_S(p) = \begin{cases} \{p\}, & \text{if } i+j \text{ is odd} \\ \{p\} \cup A_S(p), & \text{if } i+j \text{ is even} \end{cases}.$$

However, the family of these sets  $U_S(p)$ ,  $p \in \mathbb{Z}^2$ , does not define a topological basis on  $\mathbb{Z}^2$ : the 6-adjacency is a special example of an  $S$ -adjacency. Assume two ‘diagonally adjacent’ and even grid points  $p$  and  $q$ . Then the intersection of  $U_S(p)$  with  $U_S(q)$  contains exactly four grid points, and this is not one of the defined sets. A further intersection with another set  $U_S(r)$ , where  $r$  is another even ‘diagonally adjacent’ point to  $p$ , allows to produce the singleton  $\{p\}$ . It follows that all subsets of  $\mathbb{Z}^2$  are open, i.e. only singletons are connected.

This example only shows that a straightforward adaptation of the Wyse topology fails in case of  $S$ -adjacency. A more general discussion is required to analyze the existence or non-existence of a topology on  $\mathbb{Z}^2$  corresponding to  $S$ -connectivity.

## 4 Cellular Model

We start with a simple example:

**Example 4** [4] *The family  $\{[x, +\infty) : x \in \mathbb{R}\}$  is a basis of a topology on  $\mathbb{R}$ , called the right topology on  $\mathbb{R}$ . It follows (for example) that intervals  $(-\infty, x)$  are closed. Analogously, the family  $\{(-\infty, x] : x \in \mathbb{R}\}$  is also a basis of a topology on  $\mathbb{R}$ , called the left topology on  $\mathbb{R}$ . Note that sets  $[x, +\infty)$ , open in the right topology, are not open in the Euclidean topology on  $\mathbb{R}$ .*

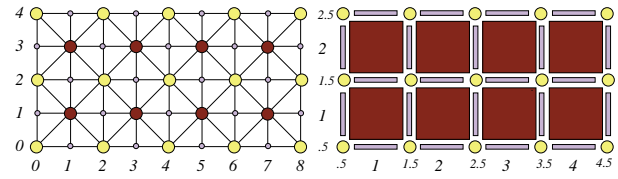
Any subset of a topological space induces a topological subspace. The set  $\mathbb{Z} \subset \mathbb{R}$  defines an inherited Euclidean, right or left topology, all topological subspaces of  $\mathbb{R}$ , depending upon whether  $\mathbb{R}$  is considered as being endowed with either the Euclidean, or the right or left topology as specified in Example 4. In case of the Euclidean topology we induce a *discrete topology* on  $\mathbb{Z}$ , where any subset of  $\mathbb{Z}$  will be open and closed as well, and this trivial topological space is not connected: any two nonempty and complementary subsets of  $\mathbb{Z}$  define a partition of  $\mathbb{Z}$  into two closed subsets. In case of the right or left topology we obtain connected subspaces.

For a different approach for inducing a topology assume a connected topological space  $\mathbb{C}$  and a surjection  $f : \mathbb{C} \rightarrow M$  into a set  $M$ . Equip  $M$  with the finest topology such that  $f$  is continuous. Then  $M$  is a connected topological space.

Now consider the Euclidean topology on  $\mathbb{R}$  and a surjection  $f : \mathbb{R} \rightarrow \mathbb{Z}$ , i.e. the set  $f^{-1}(i)$  defines a subset of  $\mathbb{R}$ , for  $i \in \mathbb{Z}$ : let  $f(x)$  be the nearest integer to  $x$ , and, if  $x$  is a half-integer  $x = i + \frac{1}{2}$  then let  $f(x) = i$ . It follows  $f^{-1}(i) = (i - \frac{1}{2}, i + \frac{1}{2}]$ , for  $i \in \mathbb{Z}$ , i.e.  $f^{-1}(i)$  is neither open or closed in the Euclidean line  $\mathbb{R}$ . The same may be stated if we taken  $f(i + \frac{1}{2}) = i + 1$  instead. As a result, no proper subset of  $\mathbb{Z}$  may be open or closed, i.e. the induced (finest) topology is the trivial topology which only has the empty set  $\emptyset$  and  $\mathbb{Z}$  itself as open and closed sets. This example has been discussed in [7] for illustrating the basic idea underlying the introduction of the following alternating topology:

**Example 5** [6] *Consider the function  $f$  as in the example before, but choose the nearest even integer as the best approximation of a half-integer  $i + \frac{1}{2}$  this time. This function induces an alternating topology on  $\mathbb{Z}$ :  $f^{-1}(2i)$  is a closed subset of  $\mathbb{R}$  in the Euclidean topology, and  $f^{-1}(2i+1)$  is an open subset, i.e.  $\{2i\}$  is a closed subset of  $\mathbb{Z}$  and  $\{2i+1\}$  is an open subset of  $\mathbb{Z}$ . In general, a subset  $M$  of  $\mathbb{Z}$  is open iff  $f^{-1}(M)$  is open in the Euclidean topology on  $\mathbb{R}$ .*

This alternating topology on  $\mathbb{Z}$  combines the basic ideas of Example 3, i.e. it uses the properties odd or even for alternations, and of Example 4, i.e. it defines a topology on  $\mathbb{R}$  by intervals. For image analysis we are interested in



**Figure 7. Left:** shaded (yellow) large dots show closed sets  $\{(2i, 2j)\}$ , filled (red) large dots indicate open sets  $\{(2i+1, 2j+1)\}$ , and small dots show sets  $\{(2i, 2j+1)\}$  or  $\{(2i+1, 2j)\}$  which are neither open nor closed. **Right:** use of different scaling and different symbols for indicating the same alternating topology.

topologies on  $\mathbb{Z}^n$ , with  $n \geq 2$ : let  $\mathbb{C}_1$  and  $\mathbb{C}_2$  be topological spaces; their *product*  $\mathbb{C}_1 \times \mathbb{C}_2$  is the set of ordered pairs  $(p_1, p_2)$  such that  $p_1 \in \mathbb{C}_1$  and  $p_2 \in \mathbb{C}_2$ , endowed with the *product topology* [1]; namely,  $M \subseteq \mathbb{C}_1 \times \mathbb{C}_2$  is open iff for each  $(p_1, p_2) \in M$  there are open sets  $M_1$  in  $\mathbb{C}_1$  and  $M_2$  in  $\mathbb{C}_2$  such that  $(p_1, p_2) \in M_1 \times M_2 \subseteq M$ . Examples 4 and 5 introduced topologies on  $\mathbb{R}$  or  $\mathbb{Z}$  which can be used to form topologies on  $\mathbb{R}^n$  or  $\mathbb{Z}^n$ ,  $n \geq 2$ .

Figure 7 illustrates the product of two alternating topologies on  $\mathbb{Z}$ , resulting into the *Khalimsky plane* on  $\mathbb{Z}^2$  (left), or a scaled version of it defined on  $\{i/2 : i \in \mathbb{Z}\}^2$  (right). A subset  $M \subseteq \mathbb{Z}^2$  of the Khalimsky plane is open iff the set

$$S_M = \bigcup_{(i,j) \in M} f^{-1}(i) \times f^{-1}(j) \quad (1)$$

is open in the Euclidean plane.

The (infinite) Khalimsky plane is Aleksandrov because an arbitrary intersection of open sets  $S_M$  as specified in formula (1) is open. For example, the smallest topological neighborhood of an open set  $\{(2i+1, 2j+1)\}$  in the Khalimsky plane is this set itself, and that of a closed set  $\{(2i, 2j)\}$  is the set containing grid point  $p = (2i, 2j)$  as well as all of its eight 8-neighbors.

The Khalimsky plane is Kolmogorov. For example,  $N(p) = \{p\}$  for point  $p = (2i+1, 2j+1)$  does not contain any of its eight 8-neighbors.

Figures 1 and 7 indicate a bijection between the base set  $\mathbb{C}_{E2}$  of the two-dimensional Euclidean complex of the orthogonal grid, and the base set  $\{i/2 : i \in \mathbb{Z}\}^2$  of the scaled alternating topology. Earlier we already realized that the two-dimensional complexes  $[\mathbb{C}_{E2}, \leq_{E2}, \dim]$  and  $[\mathbb{C}_{G2}, \leq_{G2}, \dim]$  are homeomorphic (see Theorem 1), and that this topology on the two-dimensional homogeneous grid is an example of an Aleksandrov topology of a poset. The Euclidean complex  $[\mathbb{C}_{E2}, \leq_{E2}, \dim]$  endowed with the Tucker topology has been popularized by [11] in image

analysis as an option of a homogeneous image carrier, also known as *Kovalevsky plane*.

**Theorem 3** *Khalimsky and Kovalevsky plane are homeomorphic.*

**Proof:** We define a bijection  $\Phi$  as indicated by Figs. 1 and 7: 0-cells (grid points) at  $(i, j)$  are mapped onto points  $(2i, 2j)$ , 2-cells (grid cells) with reference point  $(i, j)$  (which is assumed to be the lower left corner of the grid square) are mapped onto points  $(2i + 1, 2j + 1)$ , 1-cells (grid edges) between  $(i, j)$  and  $(i + 1, j)$  are mapped onto points  $(2i + 1, 2j)$ , and 1-cells (grid edges) between  $(i, j)$  and  $(i, j + 1)$  are mapped onto points  $(2i, 2j + 1)$ .

$\Phi$  is continuous: let  $M \subseteq \mathbb{Z}^2$  be an open set in the alternating topology, i.e. set  $S_M$  (see Equ. 1) is an open set in  $\mathbb{R}^2$ , and assume that there exists a pair of points  $p \in \Phi^{-1}(M)$  and  $q \notin \Phi^{-1}(M)$  with  $p \leq q$ , i.e.  $\Phi^{-1}(M)$  is not open in the Tucker topology.

$q$  is an  $n$ -cell,  $0 \leq n \leq 2$ , and  $p$  is one of its  $m$ -sides,  $0 \leq m \leq n$ . Case  $n = m$  is impossible because this implies  $p = q$ , i.e. it would be  $p \in \Phi^{-1}(M)$  and  $p \notin \Phi^{-1}(M)$ . Let  $n = 2$  and  $m = 1$ . Then  $q$  is an open grid square  $s$ , which can be represented as  $s = f^{-1}(i) \times f^{-1}(j)$ , and  $p$  is a grid edge  $e$  (without both of its endpoints) which can be represented as  $e = f^{-1}(k) \times f^{-1}(l)$ . Because  $e$  bounds  $s$  we have that  $(k, l)$  is an 8-neighbor of  $(i, j)$ .  $p \in \Phi^{-1}(M)$  implies  $(k, l) \in M$ , and  $e \subseteq S$ .  $q \notin \Phi^{-1}(M)$  implies  $(i, j) \notin M$ , and  $s \not\subseteq S$ . This means that  $S$  cannot be open in contradiction to our assumption, i.e. the assumed pair of points  $p, q$  cannot exist. - Cases  $n = 2$  and  $m = 0$ , and  $n = 1$  and  $m = 0$  may be treated analogously.

$\Phi^{-1}$  is continuous: let  $M$  be an open subset of the Kovalevsky plane. Assume that  $\Phi(M)$  is not open in the alternating topology, i.e.

$$S_{\Phi(M)} = \bigcup_{(i,j) \in \Phi(M)} f^{-1}(i) \times f^{-1}(j)$$

is not open in the Euclidean topology, i.e. there is one set  $S_0 = f^{-1}(i) \times f^{-1}(j)$  such that at least one of its frontier subsets  $S_1 = f^{-1}(k) \times f^{-1}(l)$  is contained in  $S_{\Phi(M)}$ , but  $S_0$  is not. Let  $S_0$  be an open square  $q$  and  $S_1$  be an edge  $p$  of this square (without both of its endpoints).  $S_1 \subseteq S_{\Phi(M)}$  implies  $(k, l) \in \Phi(M)$ , and  $p = \Phi^{-1} \in M$ .  $S_0 \not\subseteq S_{\Phi(M)}$  implies  $(i, j) \notin \Phi(M)$ , and  $q = \Phi^{-1} \notin M$  with  $p \leq q$ . It follows that  $M$  is not an open set. This contradicts our assumption on  $M$ , and  $\Phi(M)$  needs to be open in the alternating topology. - The remaining cases ( $S_1$  is a vertex, or  $S_0$  is an edge and  $S_1$  is a vertex) follow by using analogous arguments. Q.E.D.

Due to this theorem we may speak from the *Khalimsky-Kovalevsky plane* if we like to refer to this special example, e.g. in the form  $[\mathbb{C}_{E2}, \leq_{E2}, \dim]$  or in the form  $[\mathbb{C}_{G2}, \leq_{G2}$

,  $\dim]$ , of a Tucker or Aleksandrov topology. Note that any scaling operation, such as on  $\{i/2 : i \in \mathbb{Z}\}^2$  in Fig. 7, may be incorporated into the definition of the homeomorphism given in the proof of the Theorem. In [9] it was also pointed out that Theorem 3 may be obtained as a corollary of a more general theorem, saying that the product of the Aleksandrov topologies of any two posets is the Aleksandrov topology of the product of those posets.

## 4.1 Conclusion

Altogether we have two different topological spaces for the orthogonal planar grid: 4-adjacency and the corresponding 4-topology as defined in Example 3; and the graph complex  $[\mathbb{C}_{G2}, \leq_{G2}, \dim]$  or Euclidean complex  $[\mathbb{C}_{E2}, \leq_{E2}, \dim]$ , in image analysis literature also known as Khalimsky-Kovalevsky plane defined by one of two equivalent topologies, the Tucker topology on  $\mathbb{C}_{E2}$  of abstract complexes, or the product topology of two alternating topologies defined in Example 5.

Both provide alternative options for discussing topological problems at the lowest (initial) layer of image analysis approaches.

6-adjacency or a planar S-adjacency graph in general requires further studies on existence or non-existence of a corresponding topology.

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