

A New Algorithm for Gradient Field Integration

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Abstract

This paper proposes two improvements for reflectance based shape recovery. First, it is shown that albedo-independent photometric stereo allows albedo computation. This computation is based on photometric equations that relate surface normals to triplets of the image irradiances. Second, the paper also presents as the main result a new algorithm for depth recovery from surface normals. In order to improve the accuracy and robustness and to strengthen the relation between the depth map and surface normals, two new constraints are added into the associated cost function. They constrain the behavior of high-order change rate between the variables. Therefore, the changes of depth maps will be more regular. The Frankot-Chellappa-algorithm is a special case of our algorithm in the sense that it uses a subset of constraints only.

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Abstract

This paper proposes two improvements for reflectance based shape recovery. First, it is shown that albedo-independent photometric stereo allows albedo computation. This computation is based on photometric equations that relate surface normals to triplets of the image irradiances. Second, the paper also presents as the main result a new algorithm for depth recovery from surface normals. In order to improve the accuracy and robustness and to strengthen the relation between the depth map and surface normals, two new constraints are added into the associated cost function. They constrain the behavior of high-order change rate between the variables. Therefore, the changes of depth maps will be more regular. The Frankot-Chellappa-algorithm is a special case of our algorithm in the sense that it uses a subset of constraints only.

1 Introduction

Measured image irradiances of a 3-D object depend upon its shape, its reflectance properties, and the illumination. Accordingly, there are three possible ways of analyzing measured image irradiances: recovering the surface shape from the surface reflectance and illumination of the scene (i.e. reflectance-based shape recovery), recovering the illumination from the surface shape and the surface reflectance of the scene (i.e. reflectance-based light source modeling or calibration), and recovering the surface reflectance from the surface shape and illumination of the scene (i.e. shape-based reflectance recovery). The first and second tasks have been very actively researched in computer vision, e.g. for shape recovery see Horn [5], Klette et. al [7, 8], Wei and Klette [9, 11], and for light source modeling see Zheng and Chellappa [13]. However, there has been not much activity so far on the reflectance recovery, which is of importance for MPEG4 related surface modeling.

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In this paper, we analyze one (basically previously known) algorithm for reflectance recovery from surface normals and present one new algorithm for depth recovery from surface normals. The basic steps in our reflectance recovery approach are as follows: **(i)** first, the surface normals are determined using albedo-independent analysis (Klette et. al [7, 8]), e.g. photometric stereo method with three light sources (3S PSM). Then **(ii)** the surface normals (i.e. the discrete gradient vector field) are integrated, as normally in shading-based shape recovery. The obtained surface data are then used **(iii)** to achieve better gradient fields via discrete differentiation. Finally, **(iv)** according to the *image irradiance equation*, the computation of albedo is carried out by using these improved surface normals.

We will use images captured under constrained environment where background is surrounded with black cloth. The scene object will only illuminated with controllable light with known intensity and direction. We also assume that there are no interreflections and shadows on the surface of the scene object, that is, the scene object do not act as secondary light sources.

The organization of the rest of the paper is as follows. In Section 2 we analyze the computation of the albedo from the surface normals. In Section 3 we present our new algorithm for depth recovery from surface normals. The experimental results and conclusions are given in Section 4 and 5, respectively.

2 Surface normals and albedo computation

In this section, we describe steps **(i)** and **(iii)**, i.e. the computation of surface normals and albedo. We assume that the surface function $Z(x, y)$ of the scene object is formed by an orthographic (parallel) projection of the surface into the xy -image plane, and defined in the image plane over a compact domain Ω . $\rho(x, y)$ is the albedo of the surface material at that surface point which is projected into image point $(x, y) \in \Omega$. In the following, it will assumed that the albedo $\rho(x, y)$

stays unknown, and it varies across the object surface with $0 \leq \rho(x, y) \leq 1$, where 1 means the object is fully reflective or bright and 0 represents that the object is black at this surface point. Denote the gradient of the surface by $(p, q) = (p(x, y), q(x, y))$ with

$$p(x, y) = \frac{\partial Z(x, y)}{\partial x} = Z_x,$$

and

$$q(x, y) = \frac{\partial Z(x, y)}{\partial y} = Z_y.$$

Then the normal of the surface is written as $\mathbf{n} = \mathbf{n}(x, y) = (p, q, -1)^T$.

Suppose that $E_1(x, y)$, $E_2(x, y)$ and $E_3(x, y)$ are the three images of the scene object, which are measured with respect to three known illumination directions \mathbf{s}_1 , \mathbf{s}_2 and \mathbf{s}_3 . E_{01} , E_{02} and E_{03} are the three known illumination intensities, respectively. According to Lambert's cosine law, we have three *image irradiance equations*:

$$E_1(x, y) = E_{01}\rho(x, y) \frac{\mathbf{n}^T \mathbf{s}_1}{\|\mathbf{n}\| \|\mathbf{s}_1\|}, \quad (1)$$

$$E_2(x, y) = E_{02}\rho(x, y) \frac{\mathbf{n}^T \mathbf{s}_2}{\|\mathbf{n}\| \|\mathbf{s}_2\|}, \quad (2)$$

$$E_3(x, y) = E_{03}\rho(x, y) \frac{\mathbf{n}^T \mathbf{s}_3}{\|\mathbf{n}\| \|\mathbf{s}_3\|}, \quad (3)$$

where $\|\cdot\|$ represents the Euclidean norm. From the associated irradiance equations for the image pairs $E_1(x, y)$ and $E_2(x, y)$, $E_1(x, y)$ and $E_3(x, y)$, $E_2(x, y)$ and $E_3(x, y)$, we can deduce the following relations, respectively,

$$\rho(x, y) \mathbf{n}^T (E_{01}E_2\|\mathbf{s}_2\|\mathbf{s}_1 - E_{02}E_1\|\mathbf{s}_1\|\mathbf{s}_2) = 0,$$

$$\rho(x, y) \mathbf{n}^T (E_{01}E_3\|\mathbf{s}_3\|\mathbf{s}_1 - E_{03}E_1\|\mathbf{s}_1\|\mathbf{s}_3) = 0,$$

$$\rho(x, y) \mathbf{n}^T (E_{02}E_3\|\mathbf{s}_3\|\mathbf{s}_2 - E_{03}E_2\|\mathbf{s}_2\|\mathbf{s}_3) = 0.$$

If the albedo $\rho(x, y) \neq 0$ at any image point $(x, y) \in \Omega$, then $\rho(x, y)$ can be eliminated from the above equations, and it follows,

$$\mathbf{n}^T \mathbf{s}_{12} = 0, \quad \mathbf{n}^T \mathbf{s}_{13} = 0, \quad \text{and} \quad \mathbf{n}^T \mathbf{s}_{23} = 0, \quad (4)$$

where

$$\mathbf{s}_{ab} = E_{0a}E_b\|\mathbf{s}_b\|\mathbf{s}_a - E_{0b}E_a\|\mathbf{s}_a\|\mathbf{s}_b \quad (5)$$

for $a, b \in \{1, 2, 3\}$, which consists of the linear combination of the known quantities of the light sources and lies in the symmetry plane defined by the illumination directions \mathbf{s}_a and \mathbf{s}_b . Equation (4) means that the surface normal \mathbf{n} is orthogonal to the vectors \mathbf{s}_{12} , \mathbf{s}_{13} and \mathbf{s}_{23} . In other words, the surface normal \mathbf{n} is collinear

to the cross product of any two of the vectors \mathbf{s}_{12} , \mathbf{s}_{13} and \mathbf{s}_{23} . Therefore, $\mathbf{n} = \sigma \mathbf{v}(\mathbf{x}, \mathbf{y})$, and

$$\begin{aligned} \mathbf{v}(x, y) &= \mathbf{s}_{12} \times \mathbf{s}_{13} \\ &= (E_{01}E_2\|\mathbf{s}_2\|\mathbf{s}_1 - E_{02}E_1\|\mathbf{s}_1\|\mathbf{s}_2) \\ &\quad \times (E_{01}E_3\|\mathbf{s}_3\|\mathbf{s}_1 - E_{03}E_1\|\mathbf{s}_1\|\mathbf{s}_3), \end{aligned}$$

where \times represents the cross product of the vectors. The scaling factor σ must have such a sign that normal vectors \mathbf{n} have a negative z -component. The albedo $\rho(x, y)$ at image point $(x, y) \in \Omega$ can be found by substituting the surface normal \mathbf{n} into the image irradiance equations. The steps involved are summarized as follows

Theorem 1 (Albedo Computation) *Let $\rho(x, y) \neq 0$ be the albedo at any image point $(x, y) \in \Omega$. If $\mathbf{n} = \sigma \mathbf{v}(x, y)$ is the surface normal, where σ is a scaling factor, and $\mathbf{v}(x, y)$ is the cross product of the vectors \mathbf{s}_{12} , \mathbf{s}_{13} and \mathbf{s}_{23} , then the albedo $\rho(x, y)$ can be calculated by any one of the three image irradiance equations (1), (2) and (3).*

The main advantage of the above algorithm is that the surface normals can be found without knowledge of the surface reflectance property. So far this is the ideal case in theory. When $\mathbf{v}(x, y)$ is divided by one of the three irradiances of the light sources, for example, by E_{01} , then the direction of the calculated vector

$$\begin{aligned} \mathbf{v}^*(x, y) &= (E_2\|\mathbf{s}_2\|\mathbf{s}_1 - \frac{E_{02}}{E_{01}}E_1\|\mathbf{s}_1\|\mathbf{s}_2) \\ &\quad \times (E_3\|\mathbf{s}_3\|\mathbf{s}_1 - \frac{E_{03}}{E_{01}}E_1\|\mathbf{s}_1\|\mathbf{s}_3) \end{aligned}$$

does not change with respect to $\mathbf{v}(x, y)$. This property means that only the ratios of the irradiances of the light sources have to be known, but this leads to a different scaling of the albedo value.

3 Depth recovery from surface normals

In this section, we present a new algorithm for depth recovery from surface gradients in order to obtain improved surface normals. The gradient values of this surface at discrete points $(x, y) \in \Omega$ are only available as input data. Essentially there are two main classes of integration techniques for finding depth $Z(x, y)$ from gradients $p(x, y)$ and $q(x, y)$: *local integration techniques* and *global integration techniques* (for a review, see Klette and Schlüns [6]).

Local integration methods (Coleman and Jain [1], Healey and Jain [3], Wu and Li [12]) are conceptually

simple and based on the following curve integrals:

$$Z(x, y) = Z(x_0, y_0) + \int_{\gamma} p(x, y)dx + q(x, y)dy.$$

where γ is an arbitrarily specified integration path from (x_0, y_0) to $(x, y) \in \Omega$. Starting with initial height values, the methods propagate height values according to a local approximation rule (e.g., based on the 4-neighborhood) using the given gradient data. Such a calculation of relative height values can be repeated by using different scan algorithms. Finally, resulting height values can be determined by averaging operations. However, initial depth values have to be provided. The locality of the computations propagates errors, i.e. this approach strongly depends on data accuracy. Therefore, local integration techniques perform badly when the data are noisy.

Global integration techniques for gradients vector fields (Horn and Brooks [4], Frankot and Chellappa [2], Horn [5], Wei and Klette [10]) are based on the optimization process minimizing the following functional (cost function):

$$W = \iint_{\Omega} [|Z_x - p|^2 + |Z_y - q|^2] dxdy, \quad (6)$$

where p and q denote the given gradient field components. Z_x and Z_y denote the unknown gradient field components which have to be reconstructed. Comparing with the local integration methods (previously known in 1998), the *Frankot-Chellappa algorithm*, based on the results of the paper [2] and presented in Klette et. al [7], leads to better results for the task of calculating depth from gradients. Nevertheless, errors at locations of very low albedo result in reconstruction errors. Also, the algorithm is very sensitive to the abrupt changes in orientation, i.e. there are large errors at the object boundary.

In this paper, we apply the Fourier transform theory to derive a new algorithm for solving the depth from gradients. In order to improve the accuracy and robustness, and to strengthen the relation between the estimated surface and the original image, we introduce two new constraints as

$$Z_{xx} = \frac{\partial^2 Z}{\partial x^2} = p_x,$$

and

$$Z_{yy} = \frac{\partial^2 Z}{\partial y^2} = q_y.$$

The two new constraints model the behavior of the second-order derivatives change rate between the variables. Therefore, the changes of depth maps will be

more regular. Having the new constraints, the cost function can be redefined as

$$W = \iint_{\Omega} [|Z_x - p|^2 + |Z_y - q|^2] dxdy + \lambda \iint_{\Omega} [|Z_{xx} - p_x|^2 + |Z_{yy} - q_y|^2] dxdy, \quad (7)$$

where the non-negative parameter λ establishes a trade-off between the constraints, i.e. it is used to adjust the weighting between them. The above new cost function reflects the relations among $Z(x, y)$, $p(x, y)$ and $q(x, y)$ more effectively, and make the best use of the information provided by the surface normals. The following objective is to solve the unknown $Z(x, y)$ subject to an optimization process which minimizes the cost function W . Instead of using the calculus of variations to derive the Euler equations for the solution to the cost function (7), we use the Fourier transform theory.

Suppose that the Fourier transform of the surface function $Z(x, y)$ is

$$Z_F(u, v) = \iint_{\Omega} Z(x, y) e^{-j(ux+vy)} dxdy, \quad (8)$$

and the inverse Fourier transform is

$$Z(x, y) = \frac{1}{2\pi} \iint_{\Omega} Z_F(u, v) e^{j(ux+vy)} dudv. \quad (9)$$

According to the differentiation properties of the Fourier transform, we have

$$\begin{aligned} Z_x(x, y) &\leftrightarrow juZ_F(u, v), \\ Z_y(x, y) &\leftrightarrow jvZ_F(u, v), \\ Z_{xx}(x, y) &\leftrightarrow -u^2 Z_F(u, v), \\ Z_{yy}(x, y) &\leftrightarrow -v^2 Z_F(u, v). \end{aligned}$$

Let $P(u, v)$ and $Q(u, v)$ be the Fourier transforms of gradients $p(x, y)$ and $q(x, y)$, respectively. Taking the Fourier transform in the functional (7), and using the above differentiation properties and the following Parseval's formula

$$\iint_{\Omega} |Z(x, y)|^2 dxdy = \frac{1}{2\pi} \iint_{\Omega} |Z_F(u, v)|^2 dudv,$$

we obtain

$$\begin{aligned} &\frac{1}{2\pi} \iint_{\Omega} [|juZ_F(u, v) - P(u, v)|^2 + \\ &+ |jvZ_F(u, v) - Q(u, v)|^2] dudv + \\ &+ \frac{\lambda}{2\pi} \iint_{\Omega} [| -u^2 Z_F(u, v) - juP(u, v)|^2 + \\ &+ | -v^2 Z_F(u, v) - jvQ(u, v)|^2] dudv \\ &\rightarrow \text{minimum}, \end{aligned}$$

The left side of the above expression can be expanded as

$$\begin{aligned} & \frac{1}{2\pi} \iint_{\Omega} [u^2 Z_F Z_F^* - ju Z_F P^* + ju Z_F^* P + \\ & + P P^* + v^2 Z_F Z_F^* - jv Z_F Q^* + jv Z_F^* Q + \\ & + Q Q^*] dudv + \frac{\lambda}{2\pi} \iint_{\Omega} [u^4 Z_F Z_F^* - ju^3 Z_F P^* \\ & + ju^3 Z_F^* P + u^2 P P^* + v^4 Z_F Z_F^* - \\ & - jv^3 Z_F Q^* + jv^3 Z_F^* Q + v^2 Q Q^*] dudv, \end{aligned}$$

where $*$ denotes the conjugate. Differentiating the above expression with respect to Z_F and Z_F^* , we can deduce the following minimal conditions for the cost function (7)

$$\begin{aligned} & (u^2 + \lambda u^4) Z^* - j(u + \lambda u^3) P^* + \\ & + (v^2 + \lambda v^4) Z^* - j(v + \lambda v^3) Q^* = 0, \end{aligned}$$

and

$$\begin{aligned} & (u^2 + \lambda u^4) Z - j(u + \lambda u^3) P + \\ & + (v^2 + \lambda v^4) Z - j(v + \lambda v^3) Q = 0. \end{aligned}$$

Adding the above two equations together, then subtracting the second one from the first one, this results in the following equations

$$\begin{aligned} & (u^2 + v^2 + \lambda(u^4 + v^4))(Z_F + Z_F^*) + \\ & + j(u + u^3)(P - P^*) + j(v + v^3)(Q - Q^*) = 0, \end{aligned}$$

and

$$\begin{aligned} & (u^2 + v^2 + \lambda(u^4 + v^4))(Z_F - Z_F^*) + \\ & + j(u + u^3)(P + P^*) + j(v + v^3)(Q + Q^*) = 0. \end{aligned}$$

Solving the above equations except for $(u, v) \neq (0, 0)$, we obtain

$$\begin{aligned} Z_F(u, v) = & \frac{-1}{u^2 + v^2 + \lambda(u^4 + v^4)} \times [\\ & j(u + \lambda u^3) P(u, v) + j(v + \lambda v^3) Q(u, v)], \quad (10) \end{aligned}$$

where $(u, v) \neq (0, 0)$. The main result is summarized in the following theorem.

Theorem 2 (Depth Recovery) *The cost function (7) is minimized by taking the Fourier transform of surface $Z(x, y)$ as in the formula (10).*

The Frankot-Chellappa algorithm [2] as formulated in [7], is a special case when parameter $\lambda = 0$ in (7). Therefore, let $\lambda = 0$ in (10), we obtain that the objective functional (6) is minimized by taking the Fourier transform of the surface $Z(x, y)$ as

$$Z_F(u, v) = \frac{-1}{u^2 + v^2} [juP(u, v) + jvQ(u, v)], \quad (11)$$

where $(u, v) \neq (0, 0)$. The solution calculated by the Frankot-Chellappa algorithm is optimal in the sense of the quadratic error function between ideal and given gradient values. It only provides a relative depth function up to an additive constant.

The formula (11) can also be derived using the above process directly. If so, the process deriving (11) is much simpler than the one used by Frankot-Chellappa in [2]. On the other hand, our new algorithm is capable of dealing with additional constraints taking second order derivatives into account.

The following algorithm shows our proposed method for the task of calculating depth from gradients, which use the transformation as specified in Theorem 2 after having the Fourier transforms of the given gradient field. Then an inverse Fourier transform leads to the desired depth map, which allows us to reconstruct object surfaces in 3D space within a subsequent computation step of a general back projection approach.

Algorithm 1 New algorithm for depth recovery

```

1: input gradients  $p(x, y)$ ,  $q(x, y)$  and  $\lambda$ 
2: for  $0 \leq x, y \leq N - 1$  do
3:   if  $(|p(x, y)| < \max_{pq} \ \& \ |q(x, y)| < \max_{pq})$  then
4:      $P1(x, y) = p(x, y); \quad P2(x, y) = 0;$ 
5:      $Q1(x, y) = q(x, y); \quad Q2(x, y) = 0;$ 
6:   else
7:      $P1(x, y) = 0; \quad P2(x, y) = 0;$ 
8:      $Q1(x, y) = 0; \quad Q2(x, y) = 0;$ 
9:   end if
10: end for
11: Calculate the Fourier transforms of  $P1(x, y)$  and  $P2(x, y)$ :  $P1(u, v), P2(u, v);$ 
12: Calculate the Fourier transforms of  $Q1(x, y)$  and  $Q2(x, y)$ :  $Q1(u, v), Q2(u, v);$ 
13: for  $0 \leq u, v \leq N - 1$  do
14:   if  $(u \neq 0 \ \& \ v \neq 0)$  then
15:      $\Lambda = u^2 + v^2 + \lambda(u^4 + v^4);$ 
16:      $\Delta 1 = (u + \lambda u^3)P2(u, v) + (v + \lambda v^3)Q2(u, v);$ 
17:      $\Delta 2 = -(u + \lambda u^3)P1(u, v) - (v + \lambda v^3)Q1(u, v);$ 
18:      $H1(u, v) = \Delta 1 / \Lambda;$ 
19:      $H2(u, v) = \Delta 2 / \Lambda;$ 
20:   else
21:      $H1(0, 0) = \text{something}; \quad H2(0, 0) = 0;$ 
22:   end if
23: end for
24: Calculate the inverse Fourier transforms of  $H1(u, v)$  and  $H2(u, v)$ :  $H1(x, y), H2(x, y);$ 
25: for  $0 \leq x, y \leq N - 1$  do
26:    $Z(x, y) = H1(x, y) + \text{BackgroundValue};$ 
   {e.g. LSE optimization }
27: end for

```



Figure 1. Image triplet of Beethoven statue.

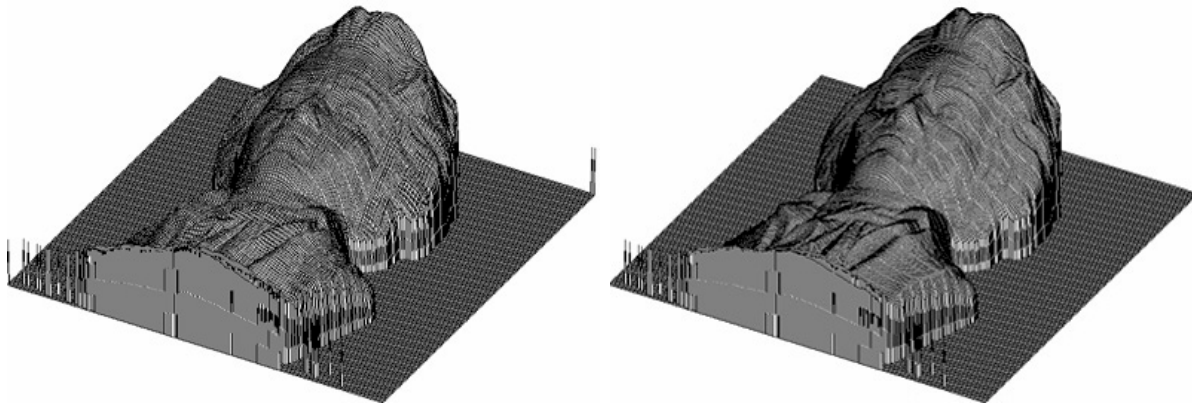


Figure 2. Recovered surfaces: left - using the Frankot-Chellappa integration method, right - using our new method.

If the gradient vectors of any length are used as input to the algorithm, then the reconstructed surface is distorted. To avoid this, the value $max_{pq} = 4$ was used in the experiments that are described in the next section.

4 Experimental results

This short note only allows us a very brief report on experimental results. Figure 1 shows three captured images of a Beethoven plaster statue. The gradients were generated using photometric stereo method with

three light sources (3S PSM). 3S PSM shape recovery results have been discussed in [7] for this statue.

Figure 2 illustrates both recovered surfaces. The left surface was calculated using the Frankot-Chellappa algorithm and the right one was done using our new algorithm with $\lambda = 0.5$. 3D plots such as shown in Figure 2 actually demonstrated that our new integration algorithm improves recovered shapes in the face region.

5 Conclusions

We briefly sketched a way to recover albedo from surface normals, which may be determined using albedo-independent 3S PSM. As an important model in this approach, we designed a new algorithm for depth recovery from surface normals or gradients. The appropriateness of the approach has been illustrated through experiments using real object, and these experiments will be reported in a longer paper.

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