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Digital Topology for Image Analysis
- PART I-
Basics and Planar Image Carriers

Reinhard Klette[1]

Abstract

This report is about basic material on digital topology within the context of image analysis. This first part informs about basics in topology, and different options for two-dimensional homogeneous image carriers. It discusses adjacency graphs more in detail as a possible unifying approach for modelling and applying homogeneous or inhomogeneous planar image carriers. The second part of the report will focus on three-dimensional image carriers with a special treatment of complexes.

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- PART I -
Basics and Planar Image Carriers

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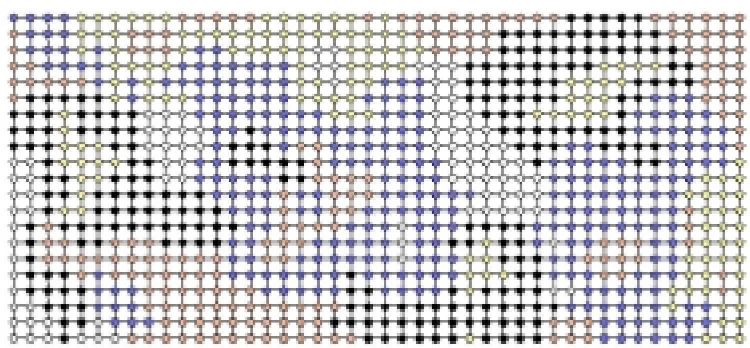
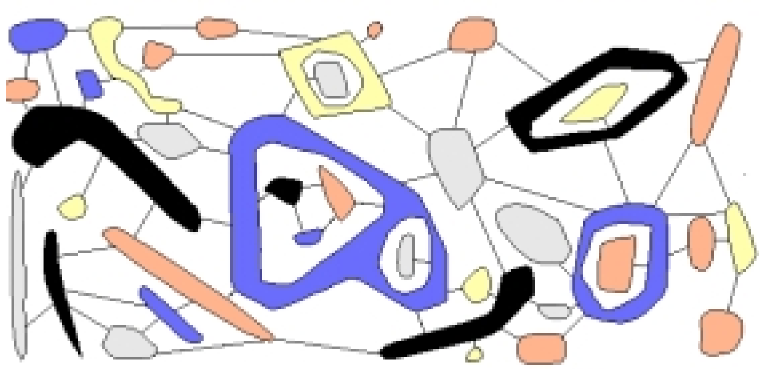




1 Introduction

The topology of digital images, and topological problems related to image analysis have been studied over the last thirty years. Pioneering papers from around 1970 stand at the beginning of this very vivid research process, and we recall such early contributions in Section 3, as a start into topological approaches for image analysis. But before proceeding to these subjects, this introductory section provides a few basic definitions and comments on digital images, and Section 2 is a compressed guide to fundamentals in topology.

Definition 1. *A* digital image I *is a function defined on a set* $\mathbb{C}$*, which is called a* carrier *of the image, and its elements are called* points*. The range of a (scalar) digital image is* $\{0, \ldots, G_{max}\}$ *with* $G_{max} \geq 1$*. The range of a* binary image *is* $\{0, 1\}$*, i.e.* $G_{max} = 1$*.*

Any image I defines different equivalence classes $\mathbb{C}_u$ of points $p \in \mathbb{C}$ by its values $I(p) = u$, $0 \leq u \leq G_{max}$: points p and q are *I-equivalent* iff (read: 'if and only if') $I(p) = I(q)$ (we recall: an equivalence relation is reflexive, transitive and symmetric), i.e. all points $p \in \mathbb{C}$ with $I(p) = u$ are in the same *I-equivalence class* $\mathbb{C}_u$, for any fixed value u.

An image carrier is normally a subset of the two- or three-dimensional Euclidean space, e.g. of the orthogonal grid defined by *grid points*[1] $p \in \mathbb{Z}^n$, *grid edges* bounded by two grid points, *grid squares* bounded by four grid edges, and (if $n = 3$) *grid cubes* bounded by six grid squares. Time-series of digital images may be described in higher dimensions, but the study of dynamic objects is outside of the scope of this report.

An n-dimensional image carrier is *homogeneous*, $n \geq 1$, iff it is either of infinite extent and there exist n linearly independent translations all transforming it into itself again, or it is a connected substructure of finite extent (defined by finite intervals of non-zero length) of such an homogeneous image carrier of infinite extent. An important aspect of topological problems in image analysis is the fact that the homogeneous carrier of input images,

(i) typically the orthogonal grid endowed with a topological structure, for example (in case of the planar grid): two grid points p and q are *adjacent* iff $p \neq q$ and both are endpoints of one grid edge (the so-called *4-adjacency* [69]);

is mapped into *inhomogeneous carriers* at more advanced layers of image analysis approaches: for example,

(ii) segmentation of images leads to carriers where an (abstract) point p represents a *region* (i.e. a component of an I-equivalence class of grid points) of

[1] Some publications call the geometric element $p \in \mathbb{Z}^n$ already a 'pixel' (if $n = 2$) or a 'voxel' (if $n = 3$). In this report it is preferred that such geometric elements do not need to be renamed, and a picture element (*pixel*) is a geometric element p labeled by an image value $I(p)$ defined at this element.

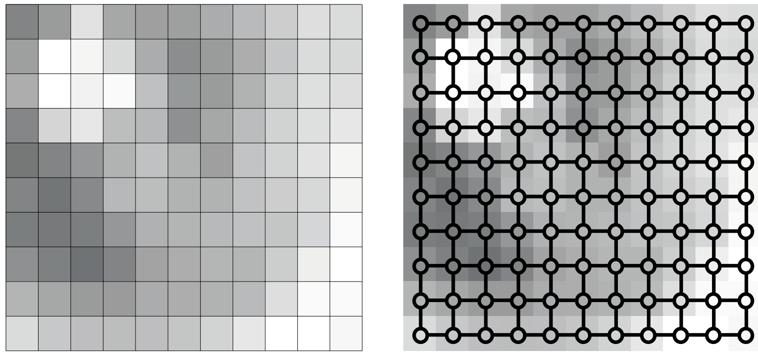


Fig. 1. Magnified image without (left) and with (right) drawing of assumed adjacencies.

the original image and the value $I(p)$ stands for the label of this region p (see Fig. 2, left); two regions are *adjacent* iff they contain grid points being on the same grid edge;

(iii) a set of grid points $\mathbb{C} = \{p_1, \ldots, p_n\}$ in the original image leads to a carrier where a point p_i represents a *Voronoi cell*

$$V(p_i) = \{q \in \mathbb{R}^2 : d_2(p_i, q) \leq d_2(p_j, q), \text{ for } 1 \leq j \leq n\};$$

two points p_i and p_k are *adjacent* iff $p_i \neq p_k$ and $V(p_i) \cap V(p_k)$ is a straight segment (of finite or infinite extent); all the edges of all Voronoi cells form the *Voronoi diagram* or *Voronoi tessellation*,

(iv) an approximation of image data may lead to two-dimensional polygonal or three-dimensional polyhedral sets; a subdivision of such a polygonal or polyhedral set into *simplexes* (which are points, edges, triangles, or tetrahedra) is *regular* iff two non-identical simplexes of this subdivision are either disjoint, or *adjacent*, i.e. they share either a triangular face, an edge or a vertex; the union of all simplexes is the originally given polygonal or polyhedral set;

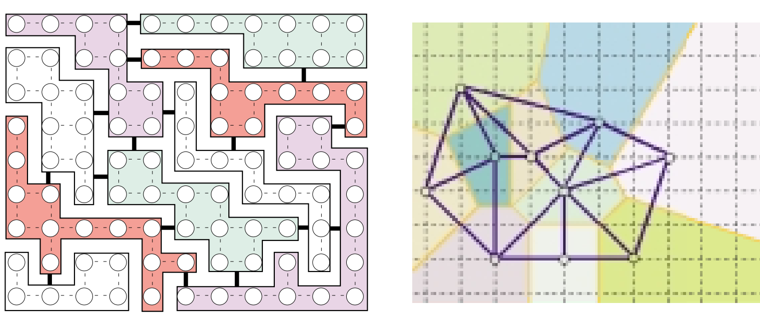


Fig. 2. Left: a region adjacency graph resulting from a four-valued image. Right: a Voronoi adjacency graph defining an inhomogeneous image carrier.

and a point $p \in \mathbb{C}$ represents a simplex where $I(p)$ stands for a category of simplexes

- just to cite a few occurrences of inhomogeneous image carriers. The notions *point* p, *carrier* $\mathbb{C}$, and *image* I will be used for all types of approaches.

Definition 2. *An* adjacency relation A *on* $\mathbb{C}$ *is irreflexive (i.e. never* pAp*) and symmetric (i.e. if* pAq *then* qAp*). A* neighborhood relation N *on* $\mathbb{C}$ *is reflexive (i.e. always* pNp*, also denoted by* $(p,p) \in N$*).*

We say, p *is adjacent to* q iff pAq. For a symmetric relation such as A we also write $\{p,q\} \in A$ instead of pAq. The examples above specify different options for defining such an adjacency relation. We say q *is a neighbor of* p iff qNp. In case of the *degenerated neighborhood relation* $\{(p,p) : p \in \mathbb{C}\}$ any point is only a neighbor of itself. Let $N(p) = \{q : qNp\}$ be the *neighborhood* of point p, always containing point p itself, and $A(p) = \{q : qAp\}$ be the *adjacency set* of point p, never containing point p itself. We have:

A_to_N: Assume an adjacency relation A on $\mathbb{C}$. Let $q \in N_A(p)$ iff $p = q$ or pAq. Then N_A is a symmetric neighborhood relation on $\mathbb{C}$.

N_to_A: Assume a neighborhood relation N on $\mathbb{C}$. Let $\{p,q\} \in A_N$ iff $p \neq q$ and $q \in N(p)$ or $p \in N(q)$. Then A_N is an adjacency relation on $\mathbb{C}$.

Note that $A_N = \emptyset$ if N is the degenerated neighborhood relation. Adjacency and neighborhood relations are not always dual approaches for introducing an algebraic structure on $\mathbb{C}$ (see Fig. 3): assume that a neighborhood relation N is not symmetric on $\mathbb{C}$; the induced adjacency relation A_N induces a symmetric neighborhood relation N_{A_N} not equal to the asymmetric relation N. We call N_{A_N} the (smallest) *symmetric closure* of relation N.

An algebraic structure on $\mathbb{C}$ is crucial for defining image analysis procedures such as tracing of borders. The notation $[\mathbb{C}, N]$ specifies a base set $\mathbb{C}$ and a (not necessarily symmetric) neighborhood relation defined on $\mathbb{C}$. From now on we will always assume that an *image carrier* is such a pair of a base set and a neighborhood relation defined on this base set. Following the axiomatic approach in [36] we will specify further constraints for characterizing image carriers throughout this report (axioms *C1* ... *C5*). We start with:

Axiom C1: $N(p)$ is finite for any point $p \in \mathbb{C}$ of an image carrier $[\mathbb{C}, N]$.

These algebraic concepts of neighborhood or adjacency relation allow to introduce an algebraic definition of connectedness. Let A_N be an adjacency relation on $\mathbb{C}$ induced by a neighborhood relation N. An (algebraic) *path* g *in* $M \subseteq \mathbb{C}$ (defined by relation A_N) is a sequence $g = \langle p_1, p_2, \ldots, p_n \rangle$ of points $p_i \in M$ with $n \geq 1$ and $\{p_i, p_{i+1}\} \in A_N$ for $i = 1, 2, \ldots, n-1$. We omit A_N or N in general if the context allows.

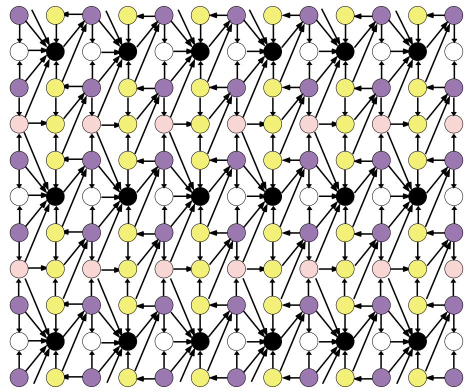


Fig. 3. A periodic neighborhood relation N defined on the infinite orthogonal grid shown as directed graph: if all directed edges are changed into undirected edges then the resulting undirected graph represents the homogeneous adjacency relation A_N and the smallest symmetric closure $N_{A_N}(p)$ of N. Sets N_{A_N} have cardinalities 4, 5, 6, 7 or 9.

Definition 3. *Two points $p, q \in \mathbb{C}$ are (algebraically)* connected *with respect to $M \subseteq \mathbb{C}$ and relation N iff there is a path $\langle p_1, p_2, \ldots, p_n \rangle$ (defined by relation A_N) which is either completely in M or completely in $\overline{M} = \mathbb{C} \setminus M$, with $p_1 = p$ and $p_n = q$. A set $M \subseteq \mathbb{C}$ is (algebraically)* connected *with respect to N iff M is not empty and all points in M are pairwise connected with respect to set M and relation N.*

We say that p and q are in relation Γ_M iff they are connected with respect to $M \subseteq \mathbb{C}$. This relation $\Gamma_M \subseteq \mathbb{C} \times \mathbb{C}$ satisfies $\Gamma_M = \Gamma_{\overline{M}}$, for any set $M \subseteq \mathbb{C}$. Γ_M is an equivalence relation on $\mathbb{C}$ because it is reflexive, symmetric, and transitive. Therefore it defines equivalence classes $\Gamma_M(p) = \{q : q \in \mathbb{C} \land \{p, q\} \in \Gamma_M\}$, for a *representative* $p \in \mathbb{C}$ and $M \subseteq \mathbb{C}$. We have $\{p, q\} \in \Gamma_M$ iff $\Gamma_M(p) = \Gamma_M(q)$. It is $p \in \Gamma_M(p)$, for any $p \in \mathbb{C}$.

It follows that the equivalence classes $\Gamma_M(p)$ are connected and pairwise disjoint subsets of M or of $\overline{M}$, called (algebraic) *components* of M or (algebraic) *complementary components* of M, respectively. $\Gamma_M(p)$ is *the* component of point $p \in \mathbb{C}$. Note that the previous does not necessarily imply that $\mathbb{C}$ is connected itself. We claim:

> *Axiom C2:* An image carrier $[\mathbb{C}, N]$ is (algebraically) connected with respect to relation N.

Later on we will also discuss the topological concept of connectedness based on definitions of topological smallest neighborhoods. We will be interested in characterizing those adjacencies where the defined algebraic concept of connectedness coincides with a topological concept of connectedness.

Adjacencies defined for a ('basic') homogeneous image carrier have direct impact on the design of programs for image analysis on such an image carrier. Important topological problems include

(i) the *separation problem* (closely related to the problem of defining curves [60] or surfaces [2]), e.g. how to define frontiers of subsets M of $\mathbb{C}$ separating the topologically defined interior of M from the topological exterior of M, and how to turn that into efficient algorithms tracing these frontiers, and

(ii) the *shape equivalence problem*, e.g. characterize transformations of subsets of $\mathbb{C}$ which do not change the topology of these subsets, e.g. for the purpose of 'shape simplification' or 'shape matching', and provide efficient algorithms for performing these transformations.

Terms such as 'interior', 'frontier', 'topology of a set of grid points' etc. need to be defined for discussing the separation problem or the shape modification problem.

A calculation of qualitative properties or topological invariants (characterizing the topology of a subset of an image carrier) is a very desirable result in many applications of digital image analysis. Images are typically corrupted by noise, and an expectation of capturing exactly identical 'shapes' (say, silhouettes of 'flat objects' in the two-dimensional case) is unrealistic. Geometry provides (for example) the concept of *congruent sets*, corresponding to such identical shapes, which is a quantitative approach, reflecting features such as size or diameter. Topological invariants are more general: congruent sets are topologically equivalent (*homeomorphic*, see definition below), but congruency cannot be expected for topologically equivalent sets.

Topology has more than 150 years of history, and it is of course of interest to look for approaches and methods provided there, allowing to derive qualitative properties based on images defined on homogeneous or inhomogeneous image carriers. We will see that combinatorial topology is very relevant, but models of point-set topology may be useful as well, actually leading to a combined use of point-set and combinatorial topology models.

EXERCISES

1.1. We consider the Euclidean space $\mathbb{E}^n = [\mathbb{R}^n, d_2]$, for $n \geq 1$, characterized by the *Euclidean metric*

$$d_2(p,q) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}$$

for points $p = (x_1, x_2, \ldots, x_n)$ and $q = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$. Let

$$U_\varepsilon(p) = \{q \in \mathbb{R}^n : d_2(p,q) < \varepsilon\}$$

be the *ε-neighborhood* of $p \in \mathbb{R}^n$, for $\varepsilon > 0$. For $\varepsilon > 0$ assume a set of ε-grid points

$$\mathbb{C}_\varepsilon = \{(i_1 \cdot \sqrt{2} \cdot \varepsilon, i_2 \cdot \sqrt{2} \cdot \varepsilon, \ldots, i_n \cdot \sqrt{2} \cdot \varepsilon) : (i_1, i_2, \ldots, i_n) \in \mathbb{Z}^n\}.$$

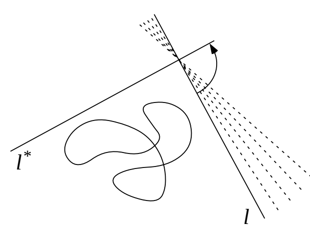
Now define that $pN_\varepsilon q$ iff $U_\varepsilon(p) \cap U_\varepsilon(q) \neq \emptyset$, for points $p, q \in \mathbb{C}_\varepsilon$. What is the cardinality of neighborhoods $N_\varepsilon(p)$ in $\mathbb{C}_\varepsilon$, for $p \in \mathbb{C}_\varepsilon$?

1.2. A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* at $z \in \mathbb{R}$ iff f is defined on an open interval containing z, $f(x)$ tends to a limit as x tends to z, and that limit is equal to $f(z)$. Function f is continuous on $\mathbb{R}$ iff it is continuous for all $z \in \mathbb{R}$. In [4] it is shown that

> any closed curve γ in the Euclidean plane may be circumscribed by a square (i.e. such that any of the four sides of the square has a non-empty intersection with the curve)

using the following property of a function f which is continuous on $\mathbb{R}$: if there are values x and y with $f(x) > 0$ and $f(y) < 0$ then there is a value z (a *zero-crossing*) between x and y with $f(z) = 0$:

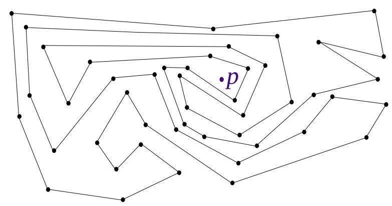
We start with one straight line l which does not intersect γ. Consider a parallel straight line l' such that γ is in the stripe between l and l'. Move both lines by parallel translation towards γ until they intersect γ for the first time. The resulting straight lines are two *lines of support* of γ. Consider two additional lines of support which are orthogonal to l. All four lines of support define a *circumscribing rectangle* for the given curve γ, having side lengths $a(l)$ and $b(l)$. The rectangle is a square iff $a(l) - b(l) = 0$. Let $l^\star$ be a straight line orthogonal



to l which does not intersect γ. We also obtain a circumscribing rectangle, with $a(l^\star) = b(l)$ and $b(l^\star) = a(l)$. Now rotate straight line l until it coincides with straight line $l^\star$. The resulting circumscribing rectangle is changing its shape continuously, and the difference $a(l) - b(l)$ is a continuous function on $\mathbb{R}$ with respect to the slope of line l. This function changes its sign between l and $l^\star$, i.e. there is a line producing the value 0, i.e. producing a circumscribing square.

Let $M \subset \mathbb{E}^n$. Assume that the *diameter* of M, i.e. the upper limit of the Euclidean distance between any two points in M, is less or equal to d. Case $n = 2$: show that there exists a square with side length d which contains M. Case $n = 3$: show that there is a regular (i.e. all edges of constant length) octahedron which contains M where the distance between opposite planar faces of the octahedron is equal to d.

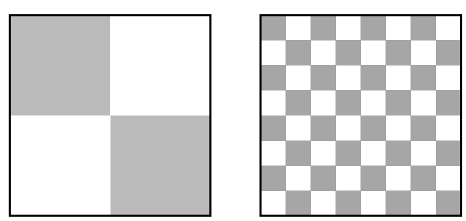
1.3. Let $n \geq 3$. A (finite, connected) *polygonal chain* in the Euclidean plane $\mathbb{E}^2$ is a finite sequence of points $\langle p_1, p_2, ..., p_n \rangle$ forming $n-1$ line segments $p_i p_{i+1}$, $i = 1, 2, ..., n-1$. The polygonal chain forms a *circuit* if we also take the nth line segment $p_n p_1$. The points p_i are called the *vertices* of the chain and the line segments are termed its *edges*. A *simple polygon* P is defined by a finite, connected polygonal chain, forming a circuit, where no point of the plane belongs to more than two edges of the chain, and the only points of the plane that belong to precisely two edges are the vertices of the chain; and P consists of all points on this chain as well as of all points in the interior of this chain. Specify an



algorithm for deciding whether a given point p of the Euclidean plane is in P based on a sequence of points $\langle p_1, p_2, ..., p_n \rangle$ defining the frontier of P.

1.4. Specify the class of all geometric transforms which map any set in the Euclidean plane into a congruent set in this plane. A property of a set in the Euclidean plane is an *invariant with respect to congruency* iff a value of this property is not changed applying one of these transforms. Consider the following properties of simple polygons: the contents of the polygon, the contents of the smallest circumscribing rectangle of the polygon, and the contents of the convex hull of the polygon. Show that these three properties are invariant with respect to congruency.

1.5. A set M in the Euclidean plane is *polygonally connected* iff for any two points $p, q \in M$ there is a finite, connected polygonal chain $\langle p_1, p_2, ..., p_n \rangle$, with $p = p_1$ and $q = q_n$, such that all edges of this chain are contained in M. (i) Show that any simple polygon is polygonally connected. (ii) Consider the following four squares inside of the large (black) square (picture on the left):



Specify a condition that either both shaded squares, or both white squares are polygonally connected. Can you specify a condition that all shaded squares in the upper n rows of the chessboard pattern (right) are polygonally connected as well as all white squares in the lower m rows, for $4 \leq m, n \leq 8$?

1.6. A set M in the Euclidean plane $\mathbb{E}^2$ is *continuously connected* iff for any two points $p, q \in M$ there is a continuous function $f : [0,1] \to \mathbb{E}^2$ such that $f(0) = p$, $f(1) = q$, and $f(x) \in M$ for any real x in the closed interval $[0,1]$. Specify a set $M \subseteq \mathbb{E}^2$ which is continuously connected, but not polygonally connected.

1.7. Consider the neighborhood relation N as defined in Fig. 3. Specify the symmetric closure $N^* = N_{A_N}$ of N by listing all elements in sets $N^*(p)$, for $p = (i,j)$, with respect to coordinates i and j.

1.8. Sets $N_{A_N}(p)$ in Fig. 3 have cardinalities 4,5,6,7 or 9, and there is an upper Euclidean distance bound of $\sqrt{8}$ for grid points being in relation N. The maximal possible cardinality would be 25 for this upper bound. In general let us consider symmetric homogeneous neighborhood relations N_m on $\mathbb{Z}^2$, for $m \geq 3$ and m odd, such that (1) points being in relation N_m are in Euclidean distance less or equal $\sqrt{(m^2 - 2m + 1)/2}$, and (2) the resulting adjacency graph is planar and connected. It follows that $card(N_m(p)) \leq m^2$. Let $\{n_1, \ldots, n_k\}$ be the set of all cardinalities of sets $N_m(p)$. We say that two neighborhood relations $N_m^{\{1\}}$ and $N_m^{\{1\}}$ are *identical* iff they have the same sets of cardinalities of sets $N_m(p)$. Specify all equivalence classes of neighborhood relations N_m, for (i) $m = 3$ and (ii) $m = 5$. For example, $\{5\}$ represents the equivalence class of 4-adjacency for $m = 3$, and $\{4, 5, 6, 7, 9\}$ defines one equivalence class for $m = 5$.

2 Combinatorial and Point-Set Topology

J. B. Listing was the first to use the word *topology* since 1837 in his correspondence.[2] In [52], page 109, he writes that

> "*topologische* Eigenschaften (solche sind), die sich nicht auf die Quantität und das Maass der Ausdehnung, sondern auf den Modus der Anordnung und Lage beziehen." (*translation*: Topological properties are those which are not related to quantity or contents, but on the mode of spatial order and position.)

Being first a student and then a close friend of C. F. Gauss [7] it seems not unlikely that his research followed the advice or example of Gauss himself. It is interesting to note that his work remained often unnoticed. For example, the historic review in [1] cites Listing's more unimportant first note [51], but not [52] which contains the *Listing band* (see below) and his important contributions, e.g. on geometric complexes, or on skeletons of a set.

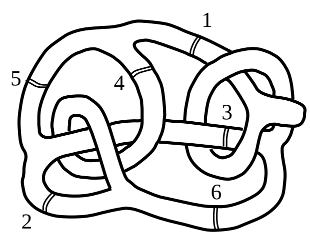


Fig. 4. Figure 15 in [52]: Assume that we cut the shown solid at the labeled six positions. There are 720 different orders of such cuts. In most cases, the resulting solid is simply connected after three cuts. In 24 cases (start sequences 136, 145, 235, 246 and their permutations), the third cut separates the solid into two parts where one is not yet simply connected. The genus of this set is 3.

The term topology (replacing Leibniz's 'geometria situs' or 'analysis situs') was introduced to distinguish qualitative geometry from those geometric studies focussing on quantitative relations. For example, the (topological) *genus* of a set, defined in [52], is the minimum number of cuts to transform this set into a (topologically) *simply connected set*[3]; see Fig. 4. The genus is such a qualitative property characterizing the topological *degree of connectedness* of a given set; see Fig. 5 for more examples.

[2] He is also well known in physiological optics (*Listings's Law*), see [75].

[3] This will be defined later as a set which is topologically connected and where its fundamental group is trivial. Informally speaking, a simply connected set is connected and does not have any (proper) holes.

Fig. 5. Three sets of genus 0, 1 or 2.

2.1 Curves

A *planar Jordan curve* γ as defined by C. Jordan in 1893 is a subset of the Euclidean plane where there exists (note: not 'where we know') a continuous function $\Phi : [a, b] \rightarrow R^2$, with $a \neq b$, $\Phi(a) = \Phi(b)$, where

$$\gamma = \{(x, y) \ : \ \Phi(t) = (x, y) \ \wedge \ a \leq t \leq b\} \, ,$$

with $\Phi(s) \neq \Phi(t)$, for all s, t with $a \leq s < t < b$ (i.e. Φ is a bijection on the open interval (a, b)), and the inverse function Φ^{-1} is continuous as well. Such a parametric and continuity-related curve definition is appropriate for image analysis situations where parametric shape descriptors are used, e.g. for smooth approximation of frontiers. A planar Jordan curve is topologically equivalent (in the Euclidean topology) to the unit circumference. Topological equivalence will be specified later on as 'homeomorphy'.

Non-parametric curve characterizations based on topological connectedness approaches correspond to local (neighborhood based) approaches of curve tracing or curve detection. Digital topology is often concerned about this case. G. Cantor was the first who suggested a topological definition of curves [60], but which had to be revised, and P. Urysohn [81] (also, independently, K. Menger [54]) provided a solution. An *Urysohn-Menger curve* is a one-dimensional topologically connected compact set. In case of planar curves in the Euclidean plane, both definitions, Jordan curves and simple Urysohn-Menger curves, specify the same class of objects. We will discuss the topological approach towards curve definitions in detail. Note that a Urysohn-Menger curve is more general than a Jordan curve: it may be a *simple* (i.e. forming a circuit, without any self-intersection) curve, but also a union of finitely many arcs, each of finite extent. The topological branching point definition (see below) of Urysohn-Menger curves is also of relevance for image analysis situations where line patterns (e.g. skeletons) need to be processed.

Urysohn-Menger Curves: For the exact definition of these curves we recall a few basic notions defined for sets in a Euclidean space $\mathbb{E}^n = [\mathbb{R}^n, d_2]$: a set $M \subseteq \mathbb{R}^n$ is *of finite extent* iff there is a real number $r > 0$ such that M is completely contained in a disk of radius r. A point $p \in \mathbb{R}^n$ is a *frontier point* of $M \subseteq \mathbb{R}^n$ iff any ε-neighborhood $U_\varepsilon(p)$ of p contains points of M as well as points of $\overline{M} = \mathbb{R}^n \setminus M$, for $\varepsilon > 0$. The *frontier* of $M \subseteq \mathbb{R}^n$ consists of all frontier

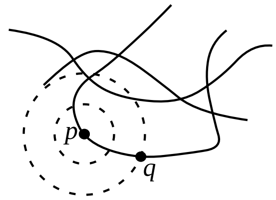


Fig. 6. A reduction of ε allows to analyze the situations at point p.

points of M. A set $M \subseteq \mathbb{R}^n$ is *closed* iff M contains all of its frontier points. A set $M \subseteq \mathbb{R}^n$ is a *compactum* iff it is closed and of finite extent.[4]

A point $p \in \mathbb{R}^n$ is a *point of accumulation* of $M \subseteq \mathbb{R}^n$ iff any ε-neighborhood $U_\varepsilon(p)$ of p contains a point $q \neq p$, $q \in M$, for $\varepsilon > 0$. A set $M \subseteq \mathbb{R}^n$ is *topologically connected in the Euclidean space* iff for any partition of M into two disjoint subsets A and B (i.e. $M = A \cup B$) there is at least one point in one of these two subsets which is a point of accumulation of the other subset. Finally, a *continuum* is a non-empty subset of $\mathbb{R}^n$ which is compact and topologically connected in the Euclidean space.

The following definition of curves has been proposed and studied in [54, 80, 81]. Let $M \subseteq A \subseteq \mathbb{R}^n$. A point $p \in \mathbb{R}^n$ is an *A-frontier point* of $M \subseteq \mathbb{R}^n$ iff any ε-neighborhood $U_\varepsilon(p)$ of p contains points of M as well as points of $A \setminus M$, for $\varepsilon > 0$. The *A-frontier* of $M \subseteq \mathbb{R}^n$ consists of all A-frontier points of M. A continuum $M \subseteq \mathbb{R}^n$ *is one-dimensional at point* $p \in M$ iff there is a value $\varepsilon > 0$ such that any continuum C contained in the M-frontier of $U_\varepsilon(p) \cap M$ is a single-elemented set $C = \{q\}$, see Fig. 6. A continuum M is *one-dimensional* iff it is one-dimensional at any of its points $p \in M$.

Definition 4. (P. Urysohn, 1923, K. Menger, 1932) *A* curve $\gamma \subseteq \mathbb{R}^n$ *is a one-dimensional continuum.*

For example, an isolated point satisfies this definition. It defines a component on its own and does not need any further discussions. S. Mazurkiewicz [55] defined *local topological connectedness*, and a curve is locally topologically connected. P. S. Alexandroff proved that curves defined this way may also be characterized by polygonal chains [60]:

Theorem 1. (P. S. Alexandroff, year unknown) *A compactum $\gamma \subseteq \mathbb{R}^n$ is a curve iff for arbitrarily small $\varepsilon > 0$ there is a mapping Φ of γ onto a polygonal chain such that $d_2(p, \Phi(p)) < \varepsilon$, for any $p \in \gamma$.*

In other words, the *Hausdorff distance*[5] between γ and a polygonal chain is less than ε. Note that this approximation of a curve by polygonal chains with respect

[4] A more general definition, not just for the Euclidean topology, is: A topological space M is *compact* iff every open covering of M contains a finite open covering of M.

[5] The *Hausdorff distance* for sets of points A, B,

$$d_2(A,B) = \max\left\{\sup_{p\in A}\inf_{q\in B} d_2(p,q), \sup_{p\in B}\inf_{q\in A} d_2(p,q)\right\},$$

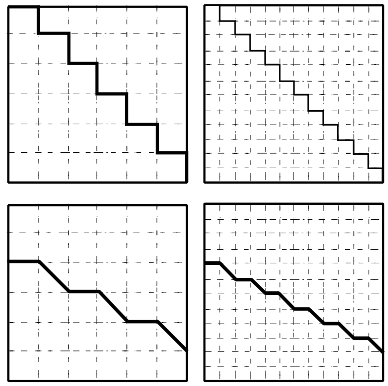


Fig. 7. Upper row: polygonal chains converging towards the diagonal of the square. Lower row: convergence towards a straight segment having a slope of 22.5°.

to the Hausdorff metric (see Fig. 7) does not automatically imply that the length of these polygonal chains converges to the length of curve γ assuming that γ is a measurable curve [41].

Simple Curves and Arcs: A curve γ has *branching index* $m \geq 0$ at point $p \in \gamma$ iff for any $r > 0$ there is a positive real $\varepsilon < r$ such that the cardinality of the γ-frontier of $U_\varepsilon(p) \cap \gamma$ is less or equal to m, and for a sufficiently small real $r > 0$ it follows that for any positive real $\varepsilon < r$ the cardinality of the γ-frontier of $U_\varepsilon(p) \cap \gamma$ is greater or equal to m. Note that the definition of curves allows [60] that m may be equal to infinity (to be precise, to countable infinity $\aleph_0$).

Definition 5. (P. Urysohn, 1923, K. Menger, 1932) *A* simple curve *is a curve where every point p on this curve has branching index 2. A* simple arc *is either a curve where every point p on this curve has branching index 2 with the only exception of two* endpoints *having branching index 1, or a simple curve with one point on this curve dividing it into a simple arc with one endpoint only.*

See Fig. 8, left, for an elementary curve which is a union of simple arcs. A *regular point* has branching index 2 and is not an endpoint. A *branch point* has a branching index greater or equal to 3. A *singular point* is either an endpoint or a branch point.

An *elementary curve* [2] is the union of a finite number of simple arcs having by pairs at most a finite number of points in common. It consists of a finite number of singular points and *regular components*, the latter are either simple curves or simple arcs: every regular point $p \in \gamma$ is on a uniquely determined subcurve $\gamma_p \subseteq \gamma$ which is either a simple curve (*the* component of p in γ), or a simple arc having one endpoint only, or a simple arc having two endpoints.

generalizes the Euclidean distance d_2 between points to a metric between sets of points.

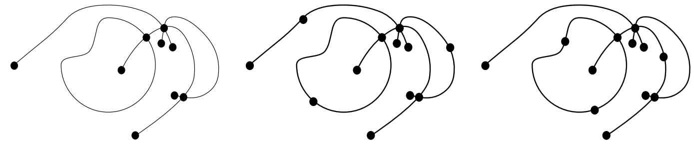


Fig. 8. Left: an elementary curve with Euler characteristic zero, having nine singular points and nine regular components. Right: two different partitionings of the same curve into one-dimensional geometric complexes (both with Euler characteristic zero).

The *Euler characteristic* of a simple curve is defined (see, e.g., [2]) to be zero, and of an elementary curve equal to the difference between the number of all singular points minus the number of all regular components which contain at least one singular point. For example, a simple arc having only one end point, has Euler characteristic zero.

Separation Theorems: The *Jordan-Veblen curve theorem* of Euclidean topology says that not only the (unit) circumference decomposes the Euclidean plane into two disjoint sets, but also any set which is topologically equivalent (i.e. homeomorphic) to the unit circumference:

Theorem 2. (C. Jordan, 1887, O. Veblen, 1905) *Let γ be a Jordan curve in the Euclidean plane $\mathbb{E}^2$. The complementary open set $\mathbb{R}^2 \setminus \gamma$ consists of two disjoint topologically connected open sets whose common frontier is γ.*

This theorem was first stated in [31]. However, the proof in [31] is completely wrong (it attempts to use a sequence of polygons converging towards the given parametric planar curve), and a first correct proof of the Jordan curve theorem was given by O. Veblen in 1905 [82] based on the parametric characterization of the planar Jordan curve itself. This proof left open the question of whether the inside and outside of such a curve is always topologically equivalent to the inside and outside of the unit circumference in $\mathbb{R}^2$ (or any other circumference of non-zero radius). The stronger *Schönflies-Brouwer Curve Theorem* says that

Theorem 3. (A. Schönflies, 1906, L. E. J. Brouwer, 1910) *For any planar Jordan curve γ, there is a bijective mapping Φ of the Euclidean plane into itself, where Φ and Φ^{-1} are continuous functions, such that $\Phi(\gamma)$ is the unit circumference.*

The proof by A. Schönflies in 1906 contains some errors which were fixed in [8, 9]. Note that planar simple (Urysohn-Menger) curves are exactly the same sets as planar Jordan curves, i.e. both theorems also apply (in $\mathbb{E}^2$) to these topologically defined simple curves.

2.2 Early Work in Combinatorial Topology

As far as we know, L. Euler was the first mathematician able to think about polyhedra without limiting his studies to measurements. This step towards abstraction allowed him to build up fundamentals of topology by combinatorial studies

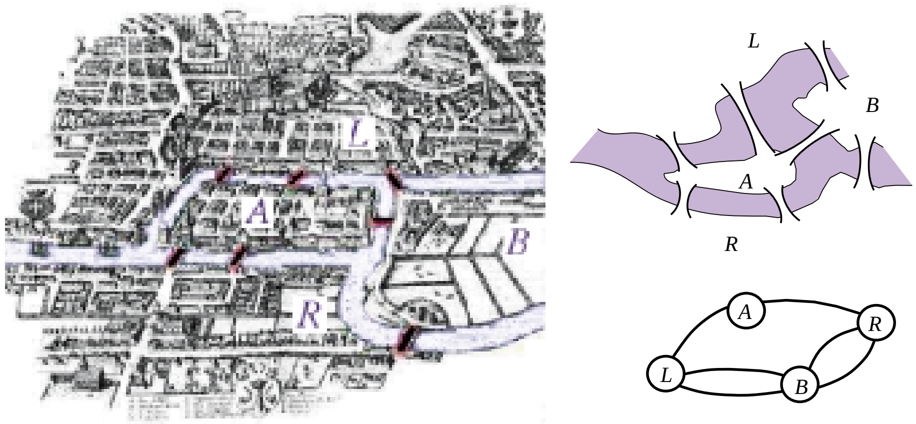


Fig. 9. Three different representations of the Königsberg (left: city at the time of Euler) bridge situation.

of geometric objects. His consideration of the bridge situation in Königsberg (see Fig. 9) is one example: would it be possible to cross all bridges just once during a walk through this city? Today a finite undirected graph is called *Eulerian* iff there exists a *path* (i.e. a sequence of consecutive edges) through this graph, starting and ending at the same vertex, and containing all edges just once. Such a path is also called *Eulerian* if it exists. Undirected graphs are often representations of geometric one-dimensional complexes. Geometric complexes have been introduced into the mathematical literature by J. B. Listing [51, 52]; see also the review [40] on the history of complexes. Complexes are the subject of *combinatorial (or algebraic) topology.*

One-dimensional Geometric Complexes: For analyzing elementary curves γ we may partition them into *one-dimensional geometric complexes* $\mathbb{C}$ which consist of mutually non-intersecting *open arcs* (i.e. obtained by deleting all endpoints from a simple arc) and their endpoints (called isolated points). Note that any elementary curve has a finite number of branch points having only finite branching indices. We obtain a finite set $\mathbb{C}$ containing open arcs (also called *1-cells*) and isolated points (also called *0-cells*).

For a one-dimensional geometric complex $\mathbb{C}$, let α_0 be the number of all 0-cells in $\mathbb{C}$, and α_1 be the number of all 1-cells of $\mathbb{C}$ which contain at least one 0-cell. The difference $\chi = \alpha_0 - \alpha_1$ is the *Euler characteristic* of the given one-dimensional geometric complex [2]. It follows that this number is equal to the Euler characteristic of the (original) elementary curve, for any one-dimensional geometric complex which is a partition of the same elementary curve (see Fig. 8).

Finite undirected graphs are an adequate representation of one-dimensional geometric complexes. See Fig. 10 for an example, where we have $\alpha_0 = 4$ vertices, $\alpha_1 = 6$ edges, and $\chi = -2$. A simple arc with only one endpoint would be

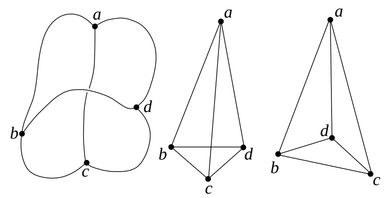


Fig. 10. Figures 26-28 in [52]: the elementary curve in the three-dimensional Euclidean space on the left is topologically equivalent to both graph representations.

represented by a *circuit* in such a finite undirected graph.

Let β_0 be the number of components of a complex. The *connectivity* β_1 of a one-dimensional geometric complex is defined as follows:

$$\beta_1 = -\beta_0 + \alpha_0 - \alpha_1 \ ,$$

and it is equal to the number of *inner cycles* (i.e. sets of regular components whose union is a simple curve) of the complex. Figure 11 shows an example with four 0-cells, six 1-cells, two components, and four inner cycles.

Two-dimensional Geometric Complexes: *Two-dimensional geometric complexes* [52] contain a finite number of subsets of $\mathbb{E}^2$ which may be faces (of finite extent), curves, arcs (of finite extent), or isolated points (e.g. endpoints of arcs). The complement of the union of all of these sets is the *exterior of infinite extent.* A two-dimensional geometric complex is topologically equivalent to surfaces in $\mathbb{E}^3$. For example, a single face may be represented by a simple polygonal area, and a two-dimensional complex by faces, edges and vertices of a *simple polyhedron* (which is topologically equivalent to the surface of a sphere), where one face of the polyhedron corresponds to the exterior of infinite extent of the originally given two-dimensional complex.

L. Euler studied *convex polyhedra* (i.e. simple polyhedra where for any two points in such a polyhedron also the line segment having these two points as endpoints is contained in the polyhedron) and proved (in the second paper in [18]) the formula

$$\alpha_0 - \alpha_1 + \alpha_2 = 2 \ , \qquad (1)$$

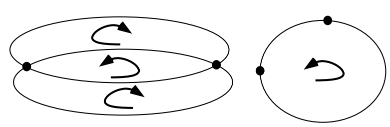


Fig. 11. Example of a one-dimensional geometric complex with four inner cycles.

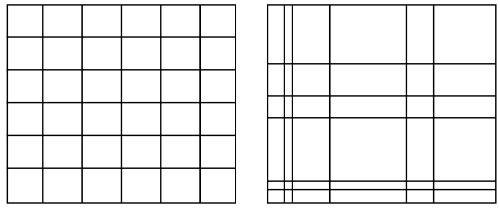


Fig. 12. Two topologically equivalent rectangular gratings formed by a finite number of segments across a rectangle, parallel to its sides.

where α_0 denotes the number of vertices, α_1 the number of edges, and α_2 the number of faces.[6]

Example 1. (Newman 1939) Two-dimensional geometric complexes in the form of (subsets of) rectangular gratings, see Fig. 12 (it is $\alpha_2 = 37$ also counting the face of infinite extent), are considered in [58]. These complexes define a partition of a given rectangle into disjoint sets of vertices, edges without endpoints, and open (in the Euclidean topology) rectangles. The grating indicates that the specific geometric size of the cells is unimportant for topological studies of these complexes in the Euclidean plane.

Two-dimensional geometric complexes are used for modelling a partition of a surface of a three-dimensional set (see part 2 of this report).

Three-dimensional Geometric Complexes: A. Cauchy [10] generalized in 1813 the Descartes-Euler polyhedron theorem (1) by introducing intercellular faces into the given simple polyhedron which replaces 2 by $D + 1$,

$$\alpha_0 - \alpha_1 + \alpha_2 = D + 1 \tag{2}$$

where D is the number of polyhedral cells, see Fig. 13, left. L. Euler and A. Cauchy considered convex polyhedra only.

A.-J. Lhuilier [50] suggested a generalization allowing 'tunnels' and 'bubbles'. He claimed that

$$\alpha_0 - \alpha_1 + \alpha_2 = 2(b - t + 1) + p \tag{3}$$

where b denotes the number of 'bubbles' within a given simple polyhedron, t denotes the number of 'tunnels', and p is the number of polygons ('exits of tunnels') within faces of the given simple polyhedron. However, his (simplifying) induction about the number of 'tunnels' does not cover the full range of possible topological complexity. For example, Fig. 16 shows a simple polyhedron and illustrates the problem of defining a 'tunnel'.

Three-dimensional geometric complexes are used for modelling a partition of a three-dimensional set (see part 2 of this report).

[6] This formula appeared already in an earlier, but unpublished fragment by R. Descartes, see [1] where it is called the *Descartes-Euler polyhedron theorem*.

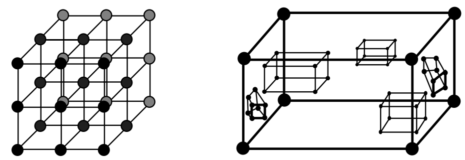


Fig. 13. Left: a cube partitioned into eight subcubes has $\alpha_0 = 27$ vertices, $\alpha_1 = 54$ edges, $\alpha_2 = 36$ faces and $D = 8$ subpolyhedra. Right: a cube with $b = 3$ 'bubbles' (all of the shape of a cube), $t = 2$ 'tunnels', $p = 4$ polygons within the original cube's faces, and it holds $\alpha_0 = 48$, $\alpha_1 = 72$, and $\alpha_2 = 32$.

Euclidean Complexes: Euclidean complexes are a generalization of geometric complexes [66], having convex sets as its elements. A set $C \subset \mathbb{E}^n$ is *convex* iff for any pair of points $a, b \in C$ we have that the straight segment ab is completely contained in C.

The notion of *dimension* played an important role in the examples before. The dimension of a set allows to separate isolated points from straight segments, or curves from faces. Later on we will provide a topological definition of the dimension $dim(M)$ of a set M. In Euclidean space, the dimension of a set coincides with the maximum number of linearly independent directional vectors contained in this set.

Let $C \subset \mathbb{E}^n$ be a convex set, and let P be a *hyperplane* (i.e. any m-dimensional subspace, $m < n$) in $\mathbb{E}^n$. If $dim(P \cap C) = n - 1$ then $P \cap C$ is an $(n-1)$-*side* of C, which is also a convex set. The intersection of finitely many $(n-1)$-sides of an n-dimensional convex set C is, if not empty, a *proper side* of C. If a proper side has dimension m it is also called an m-*side* of C. The 0-sides are the vertices of C. The set C itself is an *improper side* of itself, i.e. also a side of C. It follows that every $(n-2)$-side of an n-dimensional convex set C is a side of exactly two $(n-1)$-sides of C. A *convex cell* is a convex set of finite extent.

Definition 6. *An* Euclidean complex $\mathbb{C}$ *is defined to be a nonempty, at most enumerable family of convex cells of a Euclidean space* $\mathbb{E}^n$*, satisfying the following axioms:*
Axiom E1: If $p \in \mathbb{C}$ *and* q *is a side of* p *then* $q \in \mathbb{C}$.
Axiom E2: The intersection of two cells of $\mathbb{C}$ *is either empty or a joint side of both cells.*
Axiom E3: Each point in a cell of $\mathbb{C}$ *has an* ε*-neighborhood which has points in only a finite number of cells of* $\mathbb{C}$.

See Fig. 14 for an illustration of this definition. Different cells are not assumed to be disjoint; however, this is just a formal aspect and could be resolved if desired. A Euclidean complex is *simplicial* if all of its cells are simplexes (i.e. points, edges, triangles, or tetrahedra in case of $\mathbb{E}^3$). Rinow introduced Euclidean complexes for the discussion of polyhedral complexes (analogous to [64]), and they are called

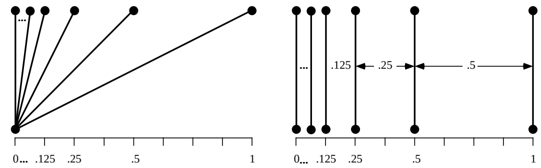


Fig. 14. Rinow [66]: Both sketched sets, consisting of infinitely many vertices and edges, are not Euclidean complexes. Axioms *E1* and *E2* are satisfied but *E3* is not. The set on the right would satisfy *E3* if the limit edge (on the left) and its vertices are omitted.

convex complexes in [22]. Note that the union of all cells of an Euclidean complex needs not to be topologically connected with respect to the Euclidean topology. A subset of an Euclidean complex $\mathbb{C}$ is not necessarily an Euclidean complex again.

Two results from [66]: Each point of the Euclidean space $\mathbb{E}^n$ is contained in only a finite number of cells of such an Euclidean complex $\mathbb{C}$. Each cell of an Euclidean complex $\mathbb{C}$ is a side of finitely many cells of $\mathbb{C}$.

Following [2] we define a *triangulation* $\mathbb{C}$ as a finite family of triangles (i.e. the union of the interior of a triangle and its frontier), line segments (i.e. line segments with endpoints) and individual points in a Euclidean space $\mathbb{E}^n$, satisfying axioms *E1* and *E2*.

Definition 7. *A* polyhedron *is the union of all sets in a triangulation* $\mathbb{C}$*.*

See Fig. 15 for three examples of triangulations. The triangulation on the right allows the construction of surfaces by identification of vertices: identify points P_5, P_8, P_{11} and P_{14} (all four corners of the large square) as being just one point (say, by moving all four corners into one position on top of the plane), where as a result the directed side from P_5 to P_{14} is identified with ('clued to') the directed side from P_8 to P_{11}, and the directed side from P_5 to P_8 is identified with the directed side from P_{14} to P_{11}:

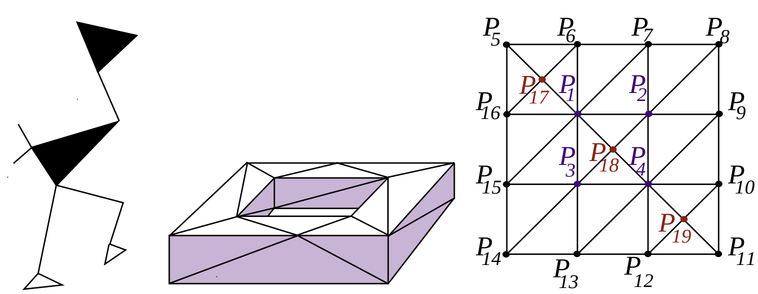


Fig. 15. Figures 19, 22 and 29 in [2]: Example of a triangulation in the plane (left), a polyhedron homeomorphic to the surface of a torus (middle), and triangulation of a rectangle (right).

(i) then proceed with identifying P_6 and P_{13}, then P_7 with P_{12}, then P_9 with P_{16}, and finally P_{10} with P_{15}; all the remaining vertices (i.e. 1, 2, 3, 4, 17, 18 and 19) remain separate; the resulting triangulation is homeomorphic to the surface of a torus; or

(ii) proceed with identifying P_6 and P_{12}, then P_7 with P_{13}, then P_9 with P_{16}, and finally P_{10} with P_{15}; all the remaining vertices (i.e. 1, 2, 3, 4, 17, 18 and 19) remain separate; the resulting triangulation is homeomorphic to the surface of a *Klein bottle*.

An identification of vertices $\{1, 2, 3, 4, 17, 18, 19\}, \{5, 11\}, \{6, 12\}, \{7, 13\}, \{8, 14\}, \{9, 15\}$, and $\{10, 16\}$ defines a triangulation of the *projective plane* [2].

It follows from Definition 7 that any polyhedron is a compactum, i.e. a closed subset of $\mathbb{R}^n$ of finite extent. A simple polygon (as defined in Exercise 1.3) is an example of a union of a one-dimensional triangulation, i.e. also a polyhedron following this definition.

The maximum dimension of elements in a triangulation $\mathbb{C}$ is the *dimension of the triangulation*. A two-dimensional triangulation $\mathbb{C}$ is *pure* iff every point or line segment in $\mathbb{C}$ precedes some triangle in $\mathbb{C}$ with respect to the side-of relation. A simple polyhedron is an example of a union of a pure two-dimensional triangulation.

A *chain* of sets $\langle M_1, M_2, \ldots, M_n \rangle$ is a finite sequence of sets such that $M_i \cap M_{i+1} \neq \emptyset$, for $i = 1, \ldots, n-1$. Such a chain *connects* set M_1 with set M_n. A polygonal chain is a chain of line segments. A pure two-dimensional triangulation $\mathbb{C}$ is *strongly connected* iff every two triangles T_1 and T_2 in $\mathbb{C}$ can be connected by a chain of triangles, where all these triangles are in $\mathbb{C}$. Simple polyhedra are defined by strongly connected triangulations. In image analysis we are interested in analyzing simple polygons and simple polyhedra, and this may be done via partitionings into triangulations.

Abstract Complexes: Studies on collections of polyhedral cells stimulated a development of a general theory of *abstract complexes*. A *poset* is a partially ordered set. We recall that a partial order is reflexive, transitive and anti-symmetric (i.e. if $p \leq q$ and $q \leq p$ then $p = q$).

Let $\mathbb{C}$ be an arbitrary set of points, where a non-negative number $dim(p)$ is assigned for each $p \in \mathbb{C}$. The history of the following definition, which goes back on axiomatic definitions of geometric complexes in [74] and topological spaces of abstract complexes in [77], has been discussed in [40].

Definition 8. *An* abstract complex $[\mathbb{C}, \leq, dim]$ *satisfies two axioms:*
Axiom A1: $\leq$ *is a partial order on* $\mathbb{C}$ *and*
Axiom A2: if $p \leq q$ *and* $p \neq q$ *then* $dim(p) < dim(q)$.

A definition identical by contents may be found in [2] (page 125), and both volumes [2,3] provide a broad coverage of definitions and results on abstract complexes. [66] contains a more recent but shorter discussion of abstract complexes - just to cite a few related texts in combinatorial topology. The elements

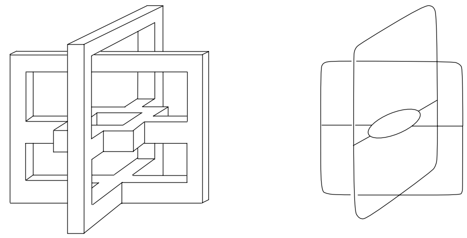


Fig. 16. Figures 59 and 60 in [52]: a three-dimensional complex (left) and its linear skeleton.

in $\mathbb{C}$ are named *cells* of the complex. If $dim(p) = n$ then n is the *dimension* of p, and p is called an *n-cell*. 0-cells are named *vertices*. An *n-dimensional complex* $[\mathbb{C}, \leq, dim]$ is characterized by $dim(p) \leq n$, for all $p \in \mathbb{C}$, and there is at least one $p \in \mathbb{C}$ with $dim(p) = n$. Note that studies of abstract complexes may proceed without interpreting cells by geometric objects such as shown in Fig. 16: this figure shows one (geometric) three-dimensional complex with 88 vertices (0-cells), 132 edges (1-cells), 36 faces (2-cells) and 2 volumes (3-cells, where one volume is a simple polyhedron, and the other one is its exterior of infinite extent). The *linear skeleton* of a set $M \subseteq \mathbb{E}^n$ is defined by continuous contractions on points or lines.[7] For example, the linear skeleton of a topologically simply connected set is a point (and not a set defined by local maxima of a distance transform [61], or a set [28] defined by connectivity-preserving thinning and further constraints) and that of a torus is a simple closed curve. Figure 16 may be sufficient to indicate the potential topological complexity of abstract complexes.

Let $[\mathbb{C}, \leq, dim]$ be an abstract complex. If $p \leq q$ and $p \neq q$ then we say that p is a *proper side* of q. If $dim(p) = m$ then p is an *m-side* of q.

Two cells are *incident* iff $p \leq q$ or $q \leq p$. This *incidence relation* is reflexive and symmetric, and thus a symmetric neighborhood relation N^*. Let $(p, q) = 1$ iff p and q are incident. This notation has been used in combinatorial formulas on abstract complexes, e.g. in generalizations of the Descartes-Euler polyhedron theorem, starting with [74].

Examples or *models* of abstract complexes are simply called *complexes*. Euclidean complexes are models of abstract complexes, and the grating discussed in Example 1 is another example (of a finite two-dimensional complex). In image analysis, we prefer the homogeneous orthogonal planar grid as a homogeneous image carrier, and we discuss two models of abstract complexes which are normally used as image carrier.

[7] Listing introduced the linear skeleton under the name of *cyclomatic diagram* (in German: cyclomatisches Diagramm) of M, see page 116 in [52]. Listing's work on linear skeletons is, for example, briefly discussed in [76]. Due to the fact that the notion 'skeleton' became very popular in image analysis in the context of distance transforms, thinning operations etc. it might be useful to return to Listing's original notion of a cyclomatic diagram for denoting the topological concept of a linear skeleton.

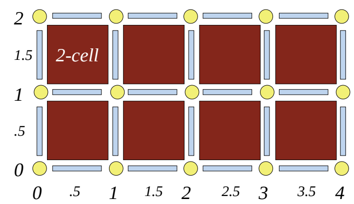


Fig. 17. Two-dimensional Euclidean complex of the homogeneous orthogonal grid.

Example 2. We may identify 2-cells with open grid squares of the homogeneous orthogonal grid, 1-cells with grid edges (without their endpoints), and 0-cells with grid points, see Fig. 17. This defines a partition of $\mathbb{R}^2$ into pairwise disjoint sets. Because we are interested in topological characterizations of complexes, we may also identify 2-cells with closed grid squares, 1-cells with closed grid edges (i.e. with both end points), and 0-cells with grid points. This is not a partition into pairwise disjoint sets anymore, but defines a Euclidean complex, and it is a topologically equivalent model of the same abstract complex. We decide for the Euclidean complex, and let $\mathbb{C}_{E2}$ be the set of all these (closed in the Euclidean topology) 2-, 1- and 0-cells of the homogeneous orthogonal grid in the Euclidean plane. For $p, q \in \mathbb{C}_{E2}$ let $p \leq_{E2} q$ iff $p \subseteq q$. Let N^*_{E2} be the symmetric neighborhood relation defined by cell incidence. $[\mathbb{C}_{E2}, \leq_{E2}, dim]$ is a two-dimensional complex, and $[\mathbb{C}_{E2}, N^*_{E2}]$ satisfies axioms *C1* and *C2* as formulated in the Introduction.[8]

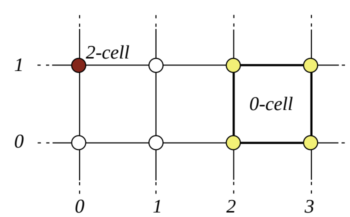


Fig. 18. Two-dimensional graph complex of the homogeneous orthogonal grid.

As an alternative model of a two-dimensional abstract complex, we may identify 2-cells with a grid point of the homogeneous orthogonal grid, 1-cells with an undirected subgraph consisting of two grid points and one edge forming a grid edge, and 0-cells with an undirected subgraph consisting of four grid points and four edges forming a grid square, see Fig. 18. Let $\mathbb{C}_{G2}$ be the set of all of

[8] The three-dimensional case (3-cells are open or closed grid cubes in the Euclidean topology of $\mathbb{E}^2$) is discussed in [26, 27]. Generalizations to orthogonal grids in arbitrary dimensions, or applications of cell complex models for image analysis are the subject in [16, 34, 35]. The application of regular, but not necessarily homogeneous cell complexes for image analysis is proposed in [45].

these cells. For $p, q \in \mathbb{C}_{G2}$ let $p \leq_{G2} q$ iff q is a subgraph of p. For example, a grid point x is a subgraph of an undirected edge $e = \{x, y\}$, i.e. $e \leq_{G2} x$. Let N^*_{G2} be the symmetric neighborhood relation defined by cell incidence. It follows that $[\mathbb{C}_{G2}, \leq_{G2}, dim]$ is a two-dimensional complex, and $[\mathbb{C}_{G2}, N^*_{G2}]$ satisfies axioms *C1* and *C2* as formulated in the introduction.[9] □

Theorem 4. *Complexes* $[\mathbb{C}_{G2}, \leq_{G2}, dim]$ *and* $[\mathbb{C}_{E2}, \leq_{E2}, dim]$ *are isomorphic.*

Proof. Let Φ be a mapping of $\mathbb{C}_{G2}$ into $\mathbb{C}_{E2}$ such that grid point (i, j) is mapped onto a grid square having (i, j) as its lower-left corner, a graph connecting grid points (i, j) and $(i, j + 1)$ is mapped onto a grid edge connecting grid points $(i, j + 1)$ and $(i + 1, j + 1)$, a graph connecting grid points (i, j) and $(i + 1, j)$ is mapped onto a grid edge connecting grid points $(i + 1, j)$ and $(i + 1, j + 1)$, and a graph consisting of four grid points $(i, j), (i + 1, j), (i, j + 1), (i + 1, j + 1)$ and connecting grid edges is mapped onto the single grid point $(i + 1, j + 1)$. Then it holds that Φ is bijective from $\mathbb{C}_{G2}$ onto $\mathbb{C}_{E2}$ such that for any $p, q \in \mathbb{C}_{G2}$ we have $p \leq_{G2} q$ iff $\Phi(p) \leq_{C2} \Phi(q)$. □

This isomorphism shows a general *duality* of grid-point related (graph-theoretical) concepts and of cellular concepts, and this isomorphism generalizes to arbitrary dimensions, see [34, 35], or page 48 in [88] for the three-dimensional case.

Models of abstract complexes may be homogeneous geometric complexes such as $[\mathbb{C}_{G2}, \leq_{G2}, dim]$ or $[\mathbb{C}_{E2}, \leq_{E2}, dim]$, or inhomogeneous image carriers (e.g. the cells of the adjacency graphs shown in Fig. 2).

2.3 Point-set Topology

In the last third of the 19th century, H. Poincaré [63] and others (see [59]) established topology as a branch of modern mathematics. *Point-set topology* studies topological spaces. The base set $\mathbb{C}$ used to be a Euclidean space in early topological publications (e.g. in [52]), but modern topology considers (abstract) sets of points.

Topological Spaces: A system of neighborhoods (*a topology*) is the preferred way for defining topological connectedness between sets of points in $\mathbb{C}$, and it is a family of subsets of points satisfying just three axioms:

Definition 9. $[\mathbb{C}, \mathcal{Z}]$ *is a* topological space *iff* $\mathcal{Z}$ *is a family of subsets of set* $\mathbb{C}$ *satisfying*
Axiom T1: $\{\emptyset, \mathbb{C}\} \subseteq \mathcal{Z}$,
Axiom T2: $\mathcal{Z}$ *is closed under arbitrary unions and*
Axiom T3: $\mathcal{Z}$ *is closed under finite intersections.*

[9] For generalizations to orthogonal grids in arbitrary dimensions see [34, 35].

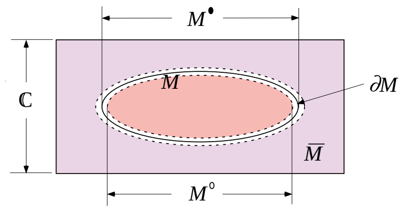


Fig. 19. Illustration of basic topological notations as used in this report.

The family $\mathcal{Z}$ of sets is a *topology* on $\mathbb{C}$. If the context allows we also say that $\mathbb{C}$ is a (topological) space. The elements of $\mathcal{Z}$ are *open sets*. A subset $M \subseteq \mathbb{C}$ is *closed* iff the complementary subset $\overline{M} = \mathbb{C} \setminus M$ is open. It follows that the family of closed subsets of $\mathbb{C}$ is closed under finite unions and under arbitrary intersections.

A topological space $\mathbb{C}$ is called an *Aleksandrov space* iff the intersection of any number of open subsets of $\mathbb{C}$ is open. A finite space $\mathbb{C}$ is always Aleksandrov. The *discrete topology* is given if every subset of $\mathbb{C}$ is open.

For $M \subseteq \mathbb{C}$, the *interior* M° of M in $\mathbb{C}$ is the union of all subsets of M which are open in $\mathbb{C}$. The *closure* $M^\bullet$ of M in $\mathbb{C}$ is the intersection of all closed subsets of $\mathbb{C}$ which contain M. The *frontier* ∂M of M in $\mathbb{C}$ is $M^\bullet \cap (\mathbb{C} \setminus M)^\bullet$. See the graphical sketch of these three important topological notations in Fig. 19.

Example 3. Assume that d is a metric on set $\mathbb{C}$, such as the Euclidean metric d_2 on $\mathbb{R}^n$, $n \geq 1$. The pair $[\mathbb{C}, d]$ is called a *metric space*. Any metric space defines a topology as follows, called the topology *induced* by the metric space:

> for $M \subseteq \mathbb{C}$ and $p \in \mathbb{C}$, let $d(p, M)$ be the greatest lower bound of all nonnegative numbers $d(p, q)$, with $q \in M$. The value $d(p, M)$ defines the *distance between point* p *and set* M. The set of all points $p \in \mathbb{C}$ with $d(p, M) = 0$ defines the closure of set M, and a set is closed iff it coincides with its closure. A set N is open iff there is a closed set M such that $N = \mathbb{C} \setminus M$.

The Euclidean metric d_2 induces the *Euclidean topology* on $\mathbb{R}^n$, $n \geq 1$. For example, in case $n = 1$ we have that the closure of an *open interval* $(x, y) = \{z \in \mathbb{R} : x < z < y\}$, $x, y \in \mathbb{R}$ and $x < y$, is equal to the *closed interval* $[x, y] = \{z \in \mathbb{R} : x \leq z \leq y\}$. □

Definition 10. *A collection of open sets* $\mathcal{Z}' \subseteq \mathcal{Z}$ *is called a* basis *of the space* $[\mathbb{C}, \mathcal{Z}]$ *iff any nonempty open set in* $\mathcal{Z}$ *is an arbitrary union of sets in* $\mathcal{Z}'$.

A basis uniquely specifies the topology $\mathcal{Z}$ on $\mathbb{C}$. For example, the set of all open intervals on $\mathbb{R}$ is a basis of the Euclidean topology $\mathbb{E}^1$. A topological space has a *countable basis* iff it has a basis of cardinality less or equal $\aleph_0$, where $\aleph_0$ is the cardinality of the set $\mathbb{N}$ of natural numbers. For example, the set of all open intervals with rational endpoints is a countable basis of $\mathbb{E}^1$. A topology of an image carrier may be defined via a specification of a locally finite basis.

Example 4. We recall that a poset is a partially ordered set. The *Aleksandrov*[10] *topology of a poset* $[\mathbb{C}, \leq]$ is defined as follows:

a set $M \subseteq \mathbb{C}$ is open iff $p \in M$ and $p \leq q$ imply $q \in M$, for all $p, q \in \mathbb{C}$.

For example, $[\{\{i\} : i \in \mathbb{Z}\} \cup \{\{i, i+1\} : i \in \mathbb{Z}\}, \subseteq]$ is a poset, and a basis of the Aleksandrov topology are the sets $\{\{i\}, \{i, i+1\}, \{i, i-1\}\}$ and $\{\{i, i+1\}\}$, for $i \in \mathbb{Z}$ [44]. For an early example of an Aleksandrov topology, specified in 1935, see exercise 4^0, page 26, in [1][11]. □

Let $\mathbb{C}$ be a triangulation of a surface. For two sets $T_1, T_2 \in \mathbb{C}$ let $T_1 \leq T_2$ iff T_1 is a side of T_2 or $T_1 = T_2$. This defines a partial ordering on $\mathbb{C}$. Consider the Aleksandrov topology of a poset on $\mathbb{C}$. A space $\mathbb{C}$ was defined to be topologically connected iff it is not the union of two nonempty disjoint closed subcomplexes. In [2] it is shown that a triangulation $\mathbb{C}$ is connected iff for any pair of vertices in $\mathbb{C}$ there is a chain of simple arcs connecting this pair of vertices in $\mathbb{C}$.

For topologies on abstract complexes see, for example, the definition and study of *open* and *closed subcomplexes* in [2, 64, 77]:

Definition 11. (A. W. Tucker 1933) *A subset $M \subseteq \mathbb{C}$ of an abstract complex K is* open *iff $p \in M$ and $p \leq q$ then $q \in M$, for all $p, q \in \mathbb{C}$.*

As a consequence, a subset M of an abstract complex K is *closed* iff $p \in M$ and $q \leq p$ then $q \in M$, for all $p, q \in \mathbb{C}$. Note that $\leq$ is a partial ordering, i.e. the definition of an Aleksandrov topology of a poset $[\mathbb{C}, \leq]$ is a generalization of the (historically earlier) Tucker topology of an abstract complex.

Example 5. Consider the two-dimensional Euclidean complex $[\mathbb{C}_{E2}, \leq_{E2}, dim]$ or graph complex $[\mathbb{C}_{G2}, \leq_{G2}, dim]$ of the homogeneous orthogonal grid. The following is formulated for the Euclidean complex, and the graph complex may be discussed analogously.

Let p be a 2-cell. Then $\{p\}$ is open in the Tucker or Aleksandrov topology: there is no $q \in \mathbb{C}_{E2}$ with $p \neq q$ and $p \leq_{E2} q$. Let p be a 1-cell. Then there are

[10] Different transcriptions of his name: 'P. S. Alexandroff' [1] in German, and 'P. S. Aleksandrov' [2] in English.

[11] "Wir unterziehen die Ebene einer Einteilung in kongruente achsenparallele Quadrate (etwa von der Seitenlänge 1). Die Elemente der Menge R seien: a) die Quadrate dieser Einteilung, b) diejenigen geradlinigen Strecken, die als Seiten dieser Quadrate auftreten, c) die Punkte, die als Eckpunkte dieser Quadrate auftreten. Es sei p ein Element von R. Ist p ein Quadrat, so soll die abgeschlossene Hülle $\overline{p}$ der aus dem einzigen Element p bestehenden Teilmenge von R aus neun Elementen bestehen: aus p selbst und as den vier Seiten und den vier Eckpunkten des Quadrates p; ist p eine Strecke, so bestehe $\overline{p}$ aus drei Elementen: aus p selbst und den beiden Endpunkten von p; ist schließlich p ein Eckpunkt, so sei $\overline{p} = p$ gesetzt. Des weiteren definieren wir für eine beliebige Teilmenge $M = (p_1, p_2, \ldots)$ von R

$$\overline{M} = \sum \overline{p}_i. \text{ ''}$$

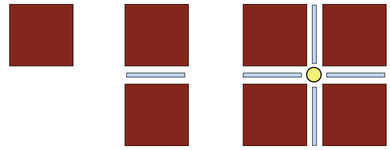


Fig. 20. The smallest neighborhoods of single cells in the two-dimensional Euclidean complex of the homogeneous orthogonal grid.

exactly two 2-cells q_1 and q_2 with $p \leq_{E2} q_1$ and $p \leq_{E2} q_2$, see Fig. 20, i.e. the set $\{p, q_1, q_2\}$ is open. Figure 20 also illustrates (on the right) the case when we start with a 0-cell p. Every cell defines its *smallest neighborhood*. The application of topological spaces of homogeneous Euclidean complexes for image analysis has been proposed in [34], and for more general situations in [45].

Figure 21 illustrates the smallest neighborhoods in the graph complex: a grid point (2-cell); a subgraph defined by two grid points and one grid edge (1-cell) and both of its grid points (2-cells); and a subgraph of a 0-cell which is a proper side of four 1-cells and of four 2-cells. □

Definition 12. (A. W. Tucker 1933) *Let* $[\mathbb{C}, \leq, dim]$ *be an abstract complex. For* $p \in \mathbb{C}$ *let* $U(p) = \{q : q \in \mathbb{C} \wedge p \leq q\}$ *be the* smallest neighborhood *of* p *in this abstract complex.*

This smallest neighborhood may be understood as being the ε-neighborhood with $\varepsilon = 1$, where a distance is defined with respect to the partial ordering $\leq$. Because $\leq$ is reflexive we have that U is a neighborhood relation as defined in Definition 2, which induces an adjacency relation A_U on $\mathbb{C}$. An abstract complex is a potential option for modelling an image carrier $[\mathbb{C}, U]$.

Topological Connectedness, Neighborhoods, and Dimensions: Let M be a subset of a topological space $\mathbb{C}$. The family $\{A \cap M : A \in \mathcal{Z}\}$ of subsets of M is the *inherited topology* on M, and M is a *topological subspace* of $\mathbb{C}$.

Definition 13. *A topological space (or subspace) is said to be* topologically connected *iff it is not the union of two disjoint nonempty closed sets.*

Topological components of a subset M of a topological space are maximum connected subsets of M. In Axiom *C2* we have claimed that an image carrier is (algebraically) connected. With respect to topological concepts we claim:

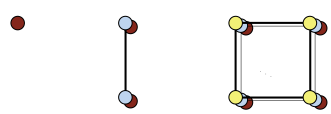


Fig. 21. The smallest neighborhoods of single cells in the two-dimensional graph complex of the homogeneous orthogonal grid.

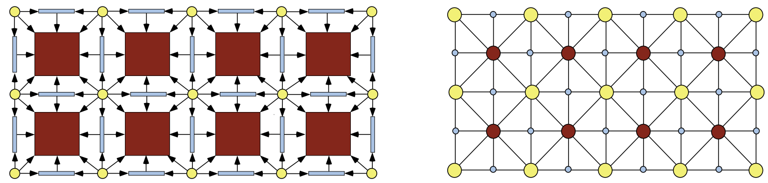


Fig. 22. Diagrams for relations U (directed edges on the left) and A_U in the two-dimensional homogeneous orthogonal Euclidean complex defined in Example 2.

> *Axiom C3:* A topological image carrier $[\mathbb{C}, N]$ allows the introduction of a topology on $\mathbb{C}$ such that algebraic connectedness coincides with topological connectedness;

i.e. any $M \subseteq \mathbb{C}$ should be connected w.r.t. relation N iff it is connected in the corresponding topology. This axiom formalizes our claim that an image carrier supports topological concepts. $[\mathbb{C}_{E2}, N^*_{E2}]$ and $[\mathbb{C}_{G2}, N^*_{G2}]$ satisfy axiom *C3*.

Definition 14. *A* topological neighborhood *of a point p in a topological space $\mathbb{C}$ is any set containing an open subset of $\mathbb{C}$ which contains p.*

For example, a closed interval $[x, y]$ is a topological neighborhood of $z \in [x, y]$ in the Euclidean topology $\mathbb{R}$ iff $z \neq x$ and $z \neq y$. Aleksandrov spaces allow to define the intersection of all topological neighborhoods of a point p, which is the *smallest topological neighborhood* $U(p)$ of p. It follows that $U(p)$ is an open set. As in the case of algebraic neighborhoods and adjacencies we define:

Definition 15. *Two points p and q of an Aleksandrov space $\mathbb{C}$ are* topologically adjacent *iff $p \neq q$, and $p \in U(q)$ or $q \in U(p)$.*

Adjacency in the two-dimensional homogeneous orthogonal complexes in Example 2 coincides with incidence, see Fig. 22. The discrete topology is Aleksandrov with $U(p) = \{p\}$, for all points, and does not support any adjacency. The smallest topological neighborhood of a 0-cell in a two-dimensional complex, endowed with the Tucker topology, is the set of all 1- and 2-cells incident with the given 0-cell.[12]

A topological space $\mathbb{C}$ is a *Kolmogorov space* (see [5]), also called a T_0*-space*, iff for any two distinct points of $\mathbb{C}$, at least one of them has a topological neighborhood not containing the other point.[13] The discrete topology is a T_0-space.

[12] Such a smallest topological neighborhood has been earlier called a *triangle star* in [83], and the 0-cell is the *center* of the triangle star, resembling a focus on simplicial complexes.

[13] For the sake of completeness, let us also mention that a topological space $\mathbb{C}$ is a T_1*-space* iff for any two distinct points of $\mathbb{C}$, each of the points has a topological neighborhood not containing the other point. This implies that $N(p) = \{p\}$, and such a space would not be of value for defining topological adjacencies. A topological space $\mathbb{C}$ is a T_2*-space* or *Hausdorff space* iff any two distinct points of $\mathbb{C}$ have disjoint topological neighborhoods. For example, the Euclidean plane is a Hausdorff space.

The important notion of the *dimension* $\dim(M)$ of a subset M of an Aleksandrov space $[\mathbb{C}, \mathbb{Z}]$ may be answered using the given (see Definition 15) adjacency relation A. For $p \in \mathbb{C}$ let $N(p) = \{q \in \mathbb{C} : p = q \vee pAq\}$. The relation N is the smallest symmetric extension of relation U.

Definition 16. *Let M be a subset of an Aleksandrov space $[\mathbb{C}, \mathbb{Z}]$. The* dimension $\dim(M)$ *of set M is defined as follows:*

(i) $\dim(M) = -1$, *if $M = \emptyset$;*
(ii) $\dim(M) = 0$, *if M is a non-empty* topologically totally disconnected set *(i.e. no pair of adjacent points in M),*
(iii) $\dim(M) = 1$, *if $card((N(p) \setminus \{p\}) \cap M) \leq 2$ for all $p \in M$, and there is at least one $p \in M$ with $card((N(p) \setminus \{p\}) \cap M) > 0$, and*
(iv) $\dim(M) = \max_{p \in M} \dim((N(p) \setminus \{p\}) \cap M) + 1$ *otherwise.*

The dimension of a set is a topological invariant. The relevance of this definition for image analysis has already been discussed in [57]. Axiom *C3* ensures that the dimension of M at a point $p \in M$ may be specified by considering the symmetric closure of its (algebraic) neighborhood, because this coincides then with neighborhoods N as used in Definition 16.

2.4 Topological Equivalence and Invariants

Now we arrive at the central notion of topology which is topological equivalence or homeomorphy. This will be the main tool for discussing topological properties of sets.

Homeomorphy, Isotopy and Homotopy: Let Φ be a mapping of a topological space $\mathbb{C}_1$ into a topological space $\mathbb{C}_2$. The mapping Φ is *continuous* iff the set $\Phi^{-1}(M) = \{p \in \mathbb{C}_1 : \Phi(p) \in M\}$ is open in $\mathbb{C}_1$, for any open subset M of $\mathbb{C}_2$.

Definition 17. (H. Poincaré, 1895) *A mapping Φ of a topological space $\mathbb{C}_1$ in a topological space $\mathbb{C}_2$ is a* homeomorphism *or a* topological mapping *iff it is one-one (i.e. an* injection*), onto $\mathbb{C}_2$ (i.e. even a* bijection*), continuous, and Φ^{-1} is continuous as well.*

Two topological spaces are *homeomorphic* iff one of them can be mapped by a homeomorphism onto the other. Two subsets M and N of a topological space $\mathbb{C}$ are considered to be identical with respect to the topological point of view (*topologically equivalent*) iff they are homeomorphic, i.e. if there exists a homeomorphism from M onto N.

The Euclidean plane $\mathbb{R}^2$ is homeomorphic to an open halfsphere. The gnomonic azimuthal projection (perspective projection from the center onto a plane tangential to the surface) of the open halfsphere defines a homeomorphism onto the Euclidean plane, see left of Fig. 23. A triangular line is homeomorphic to

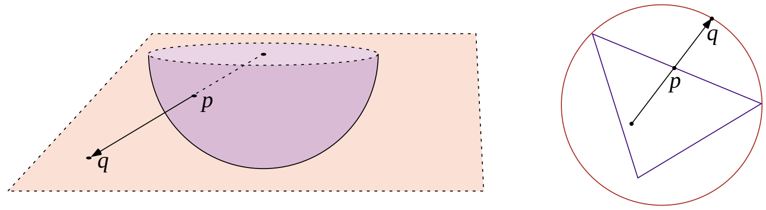


Fig. 23. Left: gnomonic azimuthal projection of an open halfsphere (point p) onto the Euclidean plane (point q). Right: projection of a triangle onto a circumference.

a circumference, see right of Fig. 23. Homeomorphisms are one way to prove topological equivalence between subsets of topological spaces. The surfaces of a sphere, a cube, and a cylinder are pairwise homeomorphic, but they are not homeomorphic to the surface of a torus. Non-homeomorphy may be shown by comparing topological invariants (see Definition 18): if two sets possess different instances of a topological invariant (for example, different genus values) then they cannot be homeomorphic.

An isomorphism between two posets defines a homeomorphism between the Aleksandrov topologies of these posets. From Theorem 4 it follows:

Corollary 1. *The cell and the graph complex of the homogeneous two-dimensional grid are topologically equivalent.*

But note that the graph complex is more than just the set of grid points. Theorem 4 may be generalized to arbitrary dimensions, see [34, 35], and this corollary as well. The corollary points out that we may either prefer the cellular approach (e.g. in [46]) or the graph-theoretical approach (e.g. in [88]) in image analysis, but both are topologically identical, and algorithmic concepts may be 'translated' accordingly from one model to the other.

Isotopy denotes the identity of positions of two sets M, N within one topological space $\mathbb{C}$. Isotopic sets not only have to be homeomorphic, there has to be a homeomorphism Φ with $\Phi(\mathbb{C}) = \mathbb{C}$ such that $\Phi(M) = N$ and $\Phi(N) = M$. For example, if we have 05 (as a non-connected elementary curve) and 70 (also as an elementary curve) in the plane then both sets are isotopic, but the set 92 is not isotopic to these two sets because 9 (as an elementary curve) is already not homeomorphic to simple curve 0. The two bands on the left of Fig. 24 are homeomorphic subsets of the Euclidean space $\mathbb{E}^3$, but they are not isotopic in $\mathbb{E}^3$. The two curves g_1 (a meridian) and g_2 (a parallel of latitude) on the surface of a torus (right of Fig. 24) are isotopic in the surface of the torus. Curves g_1 and g_3 are isotopic in E^3, but not in the surface of the torus.

Any two simple Urysohn-Menger (or Jordan) curves in $\mathbb{E}^2$ are isotopic (see Schönflies-Brouwer theorem above), and this is a stronger result than the Jordan-Veblen theorem.

We conclude this subsection with a brief introduction into the important notion of the *fundamental group* of a subset M of a topological space $[\mathbb{C}, \mathcal{Z}]$, which allows a precise definition of *simply connected sets*. Let $\phi : [0,1] \rightarrow M$ be a continuous function with $\phi(0) = p$ and $\phi(1) = q$. This function describes a *topological path* γ from p to q in M. Two topological paths γ_1, γ_2 in M with identical endpoints are called *homotopic* iff γ_1 may be continuously transformed into γ_2 within M. To be precise, let γ_1, γ_2 be two paths defined by functions $\phi_1 : [0,1] \rightarrow M$ and $\phi_2 : [0,1] \rightarrow M$. A continuous transformation of γ_1 into γ_2 means that there exists a continuous function $\psi : [0,1] \times [0,1] \rightarrow M$ with $\psi(x,0) = \phi_1(x)$ and $\psi(x,1) = \phi_2(x)$, for any real x in $[0,1]$. If γ_2 is a single point then we speak about a *contraction of* γ_1 *in* M *into a single point*.

Homotopy defines an equivalence relation on the class of all paths in set M. Figure 25 shows three curves: γ_1 and γ_2 are homotopic, but γ_3 is not homotopic to these two curves.

Now consider two paths γ_1, γ_2 in M which both start and end at the same point $p_0 \in M$. The *product* $\gamma_1 \otimes \gamma_2$ of these two paths is their concatenation, i.e. both functions $\varphi_1 : [0,1] \rightarrow M$ and $\varphi_2 : [0,1] \rightarrow M$ for these two paths are combined into one function

$$\varphi(x) = \begin{cases} \varphi_1(2x), & \text{if} \quad 0 \leq x < .5 \\ \varphi_2(2x-1), & \text{if} \quad .5 \leq x \leq 1 . \end{cases}$$

Let $[\gamma]$ be the class of all paths homotopic in M to a path γ in M (with respect to point p_0). Let $\pi(M)$ be the set of all of these classes (with respect to p_0).

The set of all homotopy classes $\pi(M)$ and the operation $\otimes$ define a (in general non-abelian, i.e. non-commutative) algebraic group: *zero-homotopic* are all paths which are contractable in M into a single point, i.e. into point $p_0 \in M$ which was chosen for defining the homotopy classes. The set of all of these paths defines the *unit* ϵ in $\pi(M)$, i.e. for any class $\xi \in \pi(M)$ we have $\xi \otimes \epsilon = \epsilon \otimes \xi = \xi$. Furthermore, for a curve γ defining a class $[\gamma] \in \pi(M)$ and defined by a function $\phi : [0,1] \rightarrow M$, let $\psi(x) = \phi(1-x)$, for any real x in $[0,1]$. This function ψ defines a curve γ^{-1} such that $[\gamma] \otimes [\gamma^{-1}] = [\gamma^{-1}] \otimes [\gamma] = \epsilon$. The operation $\otimes$ is also associative on $\pi(M)$.

Example 6. The fundamental group of a circumference is the *free cyclic group*: consider a curve which encircles the circumference $n \geq 0$ times in clockwise orientation. This curve defines a homotopy class α^n. If the curve encircles the circumference $m \geq 0$ times in counter-clockwise orientation than it defines a homotopy class α^{-m}. The class α^0 is the unit ϵ. □

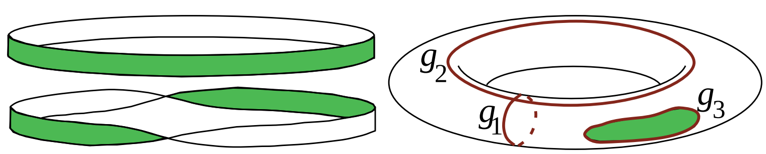


Fig. 24. Left: two homeomorphic bands, the one below is twisted twice. Right: three curves on a torus.

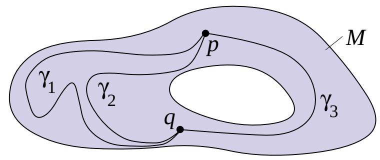


Fig. 25. Three paths in a set M.

If points p_0 and p_1 may be connected by a path in M then both fundamental groups, defined with respect to p_0 or with respect to p_1, are isomorphic. It follows that for any topologically connected set M the fundamental group is uniquely defined, independent upon point p_0. For example, the surface of a torus has a fundamental group which contains the unit, classes of meridians, classes of parallels, and classes defined by products of these classes. The fundamental group is the most important topological invariant in homotopy theory.

Theorem 5. (M. Dehn, P. Heegard, 1907) *Two connected, closed sets in the Euclidean topology of the plane are homeomorphic iff they have isomorphic fundamental groups.*

This theorem [14] is based on work by C. Jordan (1866, introduction of homotopy) and by H. Poincaré (1892, definition of the fundamental group).

A subset M of a topological space is *topologically simply connected* iff $\pi(M) = \{\epsilon\}$. In other words, any simple curve in M is contractible in M into a single point.

Topological Invariants: In the remainder of this subsection we illustrate topological concepts using solely the Euclidean space for examples. Starting with Section 3 we will also have topologies related to ('popular') image carriers at hand, and the question arises how topological concepts apply to these topologies on image carriers. The provided definition of topological equivalence may also be applied to these topologies (to some extent: the constraint of asking for a one-one mapping is not appropriate anymore for finite sets of discrete points).

Definition 18. (H. Poincaré, 1895) *A property of a subset M of a topological space $\mathbb{C}$ is a* topological invariant *iff the same property is also valid for set $\Phi(M)$, for any homeomorphism Φ.*

For example, being the empty set is a topological invariant as well as being a non-empty set (for any topological space $\mathbb{C}$). The genus and the dimension (both defined above) are non-trivial examples of topological invariants. The specification of topological invariants is *the* core problem in combinatorial and point-set topology. In digital topology it is the *calculation* of topological invariants for the purpose of image analysis. Of, course, this at first requires that invariants are well defined.

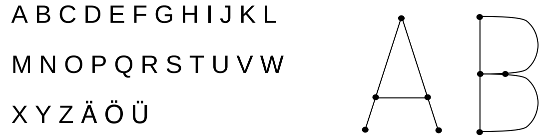


Fig. 26. Capitals of the German alphabet: assume that all letters are given as elementary curves having endpoints or branch points as indicated on the right for letters A and B.

We recall: components of a subset M of a topological space are maximum connected subsets of M. The number of components is a topological invariant: assume set M has n components and Φ is a homeomorphism of M onto set $N = \Phi(M)$; then the homeomorphic set N has n components as well.

Figure 26 shows the capitals of the German alphabet (example discussed in [4]), where letters are assumed to be elementary curves (Note: a rectangle of non-zero width, i.e. a rectangular line and its interior, is not homeomorphic to a line segment.). The three (pairwise non-homeomorphic!) letters Ä, Ö and Ü have three components compared to only one component for all of the remaining letters, i.e. these three letters cannot be homeomorphic to any of the other letters.

The homeomorphy of letters in Fig. 26 depends upon whether we consider a letter as a set being not one-dimensional at any of its points, or as an elementary curve. Figure 27 shows both options for two letters. The elementary curve of letter E has one *decomposition vertex* (a deletion of this point from the given set causes a partition into more than just two connected segments), and the elementary curve of letter M has no decomposition vertex. The number of decomposition vertices is another topological invariant: if Φ is a homeomorphism of M onto N and p is a decomposition vertex in M then $\Phi(p)$ is a decomposition vertex in $N = \Phi(M)$.

Branching indices have been introduced for curves and partitions of these curves into one-dimensional geometric complexes. The list of branching indices of an elementary curve, or of one-dimensional geometric complexes representing such a curve, is another example of a topological invariant; see Fig. 28 for an example.

Fig. 27. Both sets on the left are homeomorphic, but both elementary curves on the right (which are not linear skeletons of the sets on the left, but which might be considered to be 'skeletons' in an image-analysis context) are not.

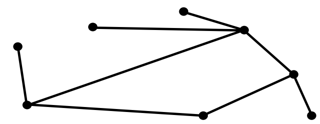


Fig. 28. The index list is (0,4,1,2,4): no 0-cell with index 0, four 0-cells with index 1 etc.

2.5 Surfaces

We conclude our short tour through basics in topology with a topological definition of the important notion of a surface, and a few results related to surfaces. Part II of the report deals with three-dimensional sets, and there will be more material on surfaces in this second part of the report.

***n*-Manifolds:** Consider a topological space $[\mathbb{C}, \mathcal{Z}]$. A subset $M \subseteq \mathbb{C}$ is *locally compact* iff every point $p \in M$ has a topological neighborhood in M whose closure in M is compact. The following definition introduces one of the basic objects in topology:

Definition 19. *The set M is an* n-manifold *iff it is locally compact, has a countable basis, and each point $p \in M$ has a topological neighborhood in M which is homeomorphic to the* open n-sphere *(i.e. the interior of a unit sphere in $\mathbb{E}^n$), for $n \geq 1$.*

Note that we could also ask for homeomorphy to the whole $\mathbb{R}^n$ because $\mathbb{R}^n$ and the open n-sphere are homeomorphic sets, for any $n \geq 1$. Note that 2-manifolds may be of finite or infinite extent, e.g. the set $\mathbb{R}^2$ is a 2-manifold. A 2-manifold of finite extent in $\mathbb{E}^2$ is an open set.

The Euclidean space has a countable basis. Let $M \subseteq \mathbb{R}^m$. The claim that M is locally compact and has a countable basis avoids topologically extreme situations. Basically we say that set M is an n-manifold iff for any point $p \in M$ there is an $\varepsilon > 0$ such that $U_\varepsilon(p) \cap M$ is homeomorphic to an open n-sphere, for $1 \leq n \leq m$. A 1-manifold M is a simple (Urysohn-Menger arc) without its endpoints, defined by homeomorphy of $U_\varepsilon(p) \cap M$ to an open line segment, for all points $p \in M$.

In three-dimensional image analysis we are interested in analyzing 2-manifolds defined by homeomorphisms with the open disc. If this set $U_\varepsilon(p) \cap M$ is homeomorphic to three open halfcircular areas as shown in Fig.29 then p is called a *bifurcation point* of set M. Of course, a 2-manifold cannot have such a bifurcation point.

An n-manifold is *closed* iff it is compact, i.e. of finite extent and topologically closed. For example, the surface of a sphere is a closed 2-manifold, and the surface of a torus as well. We have [21]:

Theorem 6. (I. Gawehn, 1927) *Every closed 2-manifold is homeomorphic to some polyhedron.*

This allows to introduce and study triangulations of closed 2-manifolds (see [2]): for a given 2-manifold M assume a homeomorphism Φ on such a homeomorphic polyhedron $\Phi(M)$. Consider a triangulation $\mathbb{C}$ defining this polyhedron as the union of all sets in $\mathbb{C}$. All sets $\Phi^{-1}(T)$, $T \in \mathbb{C}$, define a *triangulation on the 2-manifold* M, consisting of curvilinear triangles, their sides (which are simple arcs) and vertices (points).

Closed Surfaces and Surfaces with Frontiers: The following definition is one possible option for defining surfaces, following [2]. A *Jordan surface* [31] is defined by a parametrization, establishing a homeomorphism to the unit sphere. In image analysis we are normally not interested in calculating such a parametrization of a surface. The following topological approach, analogously to the topological definition by Urysohn and Menger of a curve, seems to be better suited for our purposes:

Definition 20. *A* closed surface *is a closed 2-manifold. A* surface with frontiers *is a compactum S homeomorphic to a polyhedron, where S is partitioned into two non-empty subsets S° and ϑS: all points in S° have a neighborhood in S which is homeomorphic to the open disc (or the Euclidean plane), and all points $p \in \vartheta S$ have a neighborhood in S which is homeomorphic to the union of the interior of a triangle and one of its sides (without both endpoints) where p is mapped onto this side of the triangle. Points in S° are* interior points *of S, and points in ϑS are* frontier points *of S.*

A surface without frontiers (i.e. $\vartheta S = \emptyset$) is a closed surface. A *surface* is either a closed surface or a surface with frontiers. A *simple closed surface* is a closed surface which is homeomorphic to the surface of a sphere, i.e. it is a Jordan surface.

The *frontier* ϑS of S is the set of all frontier points, the *interior* S° of S is the set of all interior points. It is $S = S^\bullet = S^\circ \cup \vartheta S$, where $S^\bullet$ denotes the closure of set S.

The frontier ϑS of a surface S with frontiers is an elementary curve which is the union of pairwise disjoint simple curves. Figure 30 shows two examples: the surface of a sphere without a few circular areas, and the surface of a torus without one circular area (called a *handle*). Every frontier point of a surface with frontiers is an interior point relative to some simple curve contained in ϑS. The

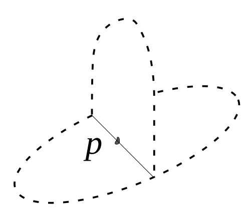


Fig. 29. A bifurcation point.

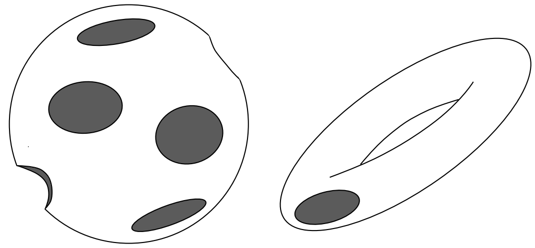


Fig. 30. The surface of a sphere without a few circular areas (left) and the surface of a torus without one circular area (right), called a *handle*.

surface of a sphere with r circular holes, whose frontiers are pairwise disjoint, is a *normal simple surface with r contours*. A *simple surface with r contours* is a surface with frontiers which is homeomorphic to a normal simple surface with r contours.

Above we already discussed triangulations of closed 2-manifolds, containing points, simple arcs or curvilinear triangles. In the sequel we also assume that any surface with frontiers may be defined by a triangulation, i.e. it is equal to a union of finitely many points, simple arcs and curvilinear triangles. As a generalization (by forming faces as unions of finitely many elements of a triangulation) we define that a finite or infinite connected graph drawn on a surface S defines a *tiling* of this surface iff every edge of this graph is on a circuit (encircling a face), there is a vertex at any intersection point of edges, and ϑS is contained in the point set defined by the union of all edges and faces of this graph. We identify a tiling with its set of faces (2-cells), edges (1-cells) and vertices (0-cells). The side-of relation specifies a partial ordering on this set of all 0-, 1- and 2-cells of a tiling. It follows that

Corollary 2. *A tiling of a surface and the side-of relation between its cells defines a complex (i.e. a model of an abstract complex).*

Triangulations are examples of tilings. A finite or infinite connected graph drawn on a surface defines a *homogeneous tiling* of this surface iff every face of this tiling is an *n-gon* (i.e. a simple polygon having $n \geq 1$ vertices on its frontier), and every vertex is incident with exactly $k \geq 1$ faces. Topologically homogeneous tilings define complexes which are of special interest for specifying homogeneous image carriers.

Example 7. Assume a homogeneous finite tiling of a Jordan surface, defined by $n \geq 1$ and $k \geq 1$. Following Theorem 6 we know that a Jordan surface is homeomorphic to some polyhedron, and we may actually assume that faces and vertices of this polyhedron are defined by faces and vertices of the given tiling, i.e. we may apply the Descartes-Euler polyhedron theorem $\alpha_0 - \alpha_1 + \alpha_2 = 2$, where α_0 denotes the number of vertices, α_1 the number of edges, and α_2 the number of faces. It follows that $2 \cdot \alpha_1 = k \cdot \alpha_0$ (because every vertex is incident with exactly k edges, and an edge is defined by two vertices) and $2 \cdot \alpha_1 = n \cdot \alpha_2$ (because every edge is incident with two faces, and each face has exactly n edges), i.e. we obtain a Diophantine equation

$$\frac{1}{k} + \frac{1}{n} = \frac{2 + \alpha_1}{2\alpha_1}$$

for positive integers k,n and α_1. However, solutions are only given by the five regular polyhedra, i.e. there are no positive integers k and n solving this equation for $\alpha_1 > 30$, i.e. there is no homogeneous finite tiling on a Jordan surface which might be of interest for defining an image carrier. Of course, there are (for example) triangulations of a Jordan surface (i.e. $n = 3$) for $\alpha_1 > 30$, but there is no constant value k for the degree of vertices of such a triangulation. There are homogeneous finite tilings on the surface of a torus, which is another example of a closed 2-manifold, without limitations on α_1, see [87], i.e. finite and non-trivial (i.e. $\alpha_1 > 30$) homogeneous image carriers may be defined on the surface of a torus, but not on a sphere.

Let $\mathbb{C}$ be a triangulation of a closed surface, or of a surface with frontiers. With the relation $\leq$, defined via the side-of relationship, we have a poset with its Aleksandrov topology. We recall: for $T \in \mathbb{C}$ we have that $U_{\mathbb{C}}(T) = \{S :\in \mathbb{C} \wedge T \leq S\}$ is the smallest neighborhood of T in $\mathbb{C}$. Let $P \in \mathbb{C}$ be a point. The smallest neighborhood $U_{\mathbb{C}}(P)$ of P in $\mathbb{C}$ contains P, and may also contain curvilinear triangles PR_1R_2 and simple arcs PQ. The set of all of the vertices Q of these simple arcs, and of all of the simple arcs R_1R_2 of these triangles specifies a subcomplex $F_{\mathbb{C}}(P)$, called the *outer frontier* of $U_{\mathbb{C}}(P)$. The smallest neighborhood $U_{\mathbb{C}}(P)$ is *cyclic* iff its outer frontier $F_{\mathbb{C}}(P)$ only contains simple arcs whose union is a simple curve. It follows that $U_{\mathbb{C}}(P)$ is homeomorphic to a closed circular area if it is cyclic.

In [2] there are several theorems about surface triangulations:

(i) The triangulation $\mathbb{C}$ is a triangulation of a closed surface iff $\mathbb{C}$ is connected and for every point $P \in \mathbb{C}$ it follows that $U_{\mathbb{C}}(P)$ is cyclic.
(ii) Every simple arc in a triangulation $\mathbb{C}$ of a closed surface is a side of exactly two triangles in $\mathbb{C}$.
(iii) Any triangulation of a surface is strongly connected[14].

Note that statement **(i)** provides a local criteria for testing a global property of a given triangulation.

Oriented Surfaces: A simple arc with endpoints p, q is denoted by $[p, q]$, and the corresponding open arc by (p, q). A simple arc is homeomorphic to the

[14] We recall that a pure two-dimensional triangulation $\mathbb{C}$ is strongly connected iff every two triangles T_1 and T_2 in $\mathbb{C}$ can be connected by a chain of triangles, where all these triangles are in $\mathbb{C}$.

closed interval $[0,1]$. The *direction* of a simple arc is defined by a homeomorphism $\Phi : [0,1] \to [p,q]$ with $\Phi(0) = p$ and $\Phi(1) = q$: for $r_1, r_2 \in [p,q]$ let

$$r_1 \lhd r_2 \quad \text{iff} \quad r_1 = \Phi(x_1) \ \wedge \ r_2 = \Phi(x_2) \ \wedge \ x_1 < x_2 \ .$$

The relation $\lhd$ defines an order on the simple arc $[p,q]$, and it can be shown that this order is independent upon the chosen homeomorphism Φ with $\Phi(0) = p$ and $\Phi(1) = q$. Analogously we can define a direction for a simple curve. For example, the frontier of a triangle in $\mathbb{R}^2$ is a simple curve. A direction on a simple curve induces a direction for simple arcs contained in this curve.

An *oriented triangle* is a triangle with a prescribed direction (called the *orientation* of this triangle) of describing its frontier, say, 'either clockwise or counter-clockwise'. The orientation of a triangle induces orientations for its sides. Two adjacent triangles (i.e. with a joint edge) of one triangulation are *coherently oriented* if they induce opposite orientations on their common side.

Definition 21. (P. S. Aleksandrov, 1956) *A triangulation of a surface is* orientable *iff it is possible to orient all the triangles in such a way that every two adjacent triangles are coherently oriented (the orientations of all these triangles specify an* orientation of the triangulation*. Otherwise the triangulation is called* nonorientable*.*

If $\mathbb{C}$ is an orientable, strongly connected triangulation then the orientation of one triangle specifies already the orientation of the (whole) triangulation. This shows that any orientable, strongly connected triangulation has exactly two orientations.

Theorem 7. (P. S. Aleksandrov, 1956) *If $\mathbb{C}_1$ and $\mathbb{C}_2$ are two different triangulations of the same surface, then $\mathbb{C}_1$ is orientable iff $\mathbb{C}_2$ is orientable.*

This invariance of orientability with respect to the chosen triangulation of a surface allows us to define that a surface is *orientable* iff any triangulation of the surface is orientable. Theorem 7 implies that orientability of a surface is a topological invariant.

Example 8. (J. B. Listing, 1861, A. F. Möbius, 1865) A famous example of a non-orientable surface is the *Listing Band* originally described by J. B. Listing in [52], see Fig. 31. Traditionally this band has been called the 'Möbius Band' after A. F. Möbius who discovered it independently of Listing a few years later, see [56]. The Listing band is a surface with frontiers. Its frontier is exactly one simple curve, homeomorphic to a circumference. By identifying this frontier with one of the frontiers of the surface of a sphere without circular areas (Fig. 30) we are able to cover one of these holes, and by doing this for all holes we may transform the surface with frontiers into a closed surface which is not homeomorphic to the surface of a unit sphere, i.e. it is not a Jordan surface. □

The example may act as a preliminary illustration of the topological complexity of surfaces, which will be further discussed in part II of this report. Of

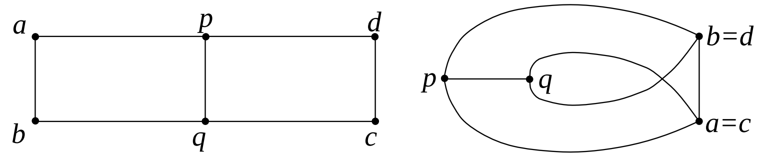


Fig. 31. Listing Band as drawn in [83]: the rectangle (left) is transformed into the Listing Band (right) if a corresponds to c and b to d.

course, we could also cover one or all of these holes with copies of the handle (torus with one hole). This process of *glueing* one surface to the other requires that both frontiers used in the process are homeomorphic, and the result may also depend on the chosen orientations of both frontiers. For example, we may glue two handles together and obtain a closed surface which is homeomorphic to the result of glueing two handles to a normal simple surface with two contours.

Connectivity and Euler Characteristics: Above we already briefly discussed the Descartes-Euler polyhedron theorem and Euler characteristics of elementary curves. Let α_0, α_1, α_2 be the number of points, simple arcs and curvilinear triangles, respectively, of a triangulation of a surface. The *Euler characteristic* of a triangulation $\mathbb{C}$ of a surface is equal to $\chi(\mathbb{C}) = \alpha_0 - \alpha_1 + \alpha_2$.

Theorem 8. (P. S. Aleksandrov, 1956) *If $\mathbb{C}_1$ and $\mathbb{C}_2$ are two different triangulations of the same surface, or of two homeomorphic surfaces, then their Euler characteristics are equal.*

The theorem allows that we speak about the Euler characteristic $\chi(S)$ of a surface S. It also follows that the Euler characteristic of any finite tiling of a surface S (for example, also of a homogeneous tiling), where α_0, α_1, α_2 be the number of vertices, edges and faces, respectively, of this tiling, is equal to $\chi(S)$.

For example, every face of a cube may be triangulated by two triangles. This results in 8 points, 18 edges and 12 triangles, i.e. the Euler characteristic is equal to 2. The surface of a sphere may be (e.g.) subdivided into 4 curvilinear triangles with 4 points and 6 simple arcs, i.e. Euler characteristic 2 again (as stated in the theorem, because surface of the cube and surface of a sphere are homeomorphic).

We say that a one-dimensional closed subcomplex M of a triangulation $\mathbb{C}$ of a surface *does not separate* $\mathbb{C}$ iff the open subcomplex $\mathbb{C} \setminus M$ is (still) strongly connected. The *connectivity* $q(\mathbb{C})$ of a triangulation $\mathbb{C}$ is the maximum number n for which there exists a closed one-dimensional geometric subcomplex of connectivity $\beta_1 = n$ which does not separate $\mathbb{C}$. The *Euler theorem* says that

$$\chi(\mathbb{C}) = 2 - q(\mathbb{C}) \ ,$$

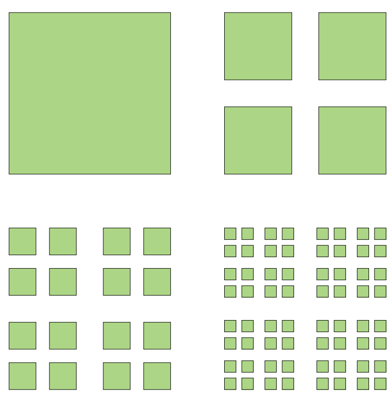
for any triangulation $\mathbb{C}$ of a surface. Together with Theorem 8 it follows that triangulations or finite tilings of homeomorphic surfaces have identical connectivity. Therefore we can speak about *the connectivity $q(S)$ of a surface S*, and this is a topological invariant. For example, the Euler characteristic of a simple surface with r contours is $2-r$, see [2]. A proof is as follows: in case of a simple surface (i.e. $r=0$) we have the Euler characteristic 2 as shown above using the surface of a cube or of a sphere; and every deletion of one triangle from a triangulation of a simple surface (such that its vertices and edges remain in the triangulation) decreases the Euler characteristic by one.

The *genus* $p(S)$ of a surface is one half of its connectivity if the surface is orientable, and it is connectivity minus 1 if the surface is nonorientable. For example [2], covering all r holes of a normal simple surface with r contours with r copies of the Listing band yields a nonorientable surface of genus r.

EXERCISES

2.1. Classify all elementary curves ('letters') in Fig. 26 with respect to topological equivalence.

2.2. The following figure shows a square S_0 with contents 1 in the upper left corner. Assume that we delete an 'open cross' with contents $\frac{1}{4}$ as shown in the upper right corner, resulting in four squares forming a (disconnected) set S_1. We proceed with this for any resulting square: next we obtain 16 squares forming set S_2, then 64 squares forming set S_3 etc. The remaining set S_n is always a closed (in the Euclidean topology) set of a finite number of squares. Assume we are continuing ad infinitum. The family of closed sets is closed with



respect to arbitrary intersections. What is the contents of the resulting *limiting set* $S_0 \cap S_1 \cap S_2 \ldots$?

2.3. Let $\mathbb{C}$ be the set of all grid squares of the homogeneous orthogonal grid $\mathbb{Z}^2$. We define that p is a neighbor of q (we write pNq) iff the intersection of p and q is a grid edge, for any $p, q \in \mathbb{C}$. Is $[\mathbb{C}, N]$ a topological image carrier satisfying axioms *C1*, *C2* and *C3*?

2.4. A *tree* is a connected elementary curve without any circuit. Show that

$$\alpha_1 - \alpha_0 = 1 \,,$$

for any tree (i.e. the Euler characteristic of any tree is equal to 1).

2.5. Let γ be a simple (Urysohn-Menger) curve in the surface $S \subseteq \mathbb{E}^3$ of a sphere. Show that the complementary set $S \setminus \gamma$ consists of two open subsets of S whose common frontier is γ.

2.6. Assume $n > 0$ connected finite polygonal chains in $\mathbb{E}^2$, all starting at point p and all ending at point $q \neq p$. Assume that any two of these polygonal chains only intersect in points p and q. Show that these chains separate the Euclidean plane into n disconnected sets.

2.7. Show that the property of a set $S \subseteq \mathbb{R}^2$ of being a (Urysohn-Menger) curve is a topological invariant in the Euclidean plane.

2.8. Let S_1 and S_2 be two surfaces with frontiers. Assume that we glue both together by identifying one simple curve in ϑS_1 and one simple curve in ϑS_2. Show that the resulting surface has Euler characteristic $\chi(S_1) + \chi(S_2)$.

2.9. What are the Euler characteristics and the genus of the surface of a sphere, the surface of a torus, the handle and the Listing band?

2.10. Show that the handle allows a tiling defined by a graph with only one vertex, three edges and one face.

2.11. Show that the following equation results for a homogeneous tiling of the surface of a sphere [4]:

$$\frac{1}{n} + \frac{1}{k} = \frac{1}{2} + \frac{1}{\alpha_1} \,,$$

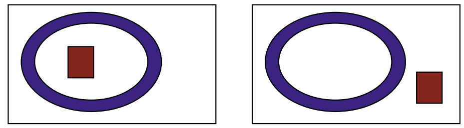
where α_1 denotes the number of edges.

2.12. A connected graph as defined in the previous exercise is an example of an elementary curve. Show that the graph of a homogeneous tiling of the surface of a sphere is either homeomorphic to one of the five *Platonic graphs* (Tetrahedron, Octahedron, Icosahedron, Hexahedron, or Dodecahedron) or to a graph defined by either $n = 2$ and $\alpha_1 = k \geq 2$, or $n = \alpha_1 \geq 2$ and $k = 2$.

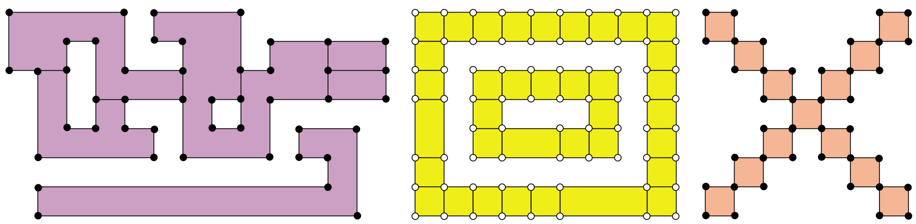
2.13. Show that the Euclidean plane only allows homogeneous tilings (with $\alpha_1 = \aleph_0$) for either $n = k = 4$, $n = 3$ and $k = 6$, or $n = 6$ and $k = 3$.

2.14. Explain that $[\mathbb{C}_{E2}, N^*_{E2}]$ and $[\mathbb{C}_{G2}, N^*_{G2}]$ satisfy axioms *C1*, *C2* and *C3*.

2.15. Characterize the topological difference between the sets shown on the left and on the right in the following figure:



2.16. Calculate topological invariants for the following three tilings (i.e. complexes):



2.17. Show that any homogeneous tiling of the Euclidean plane is topologically equivalent to such a homogeneous tiling of the plane where the set of vertices coincides with $\mathbb{Z}^2$.

2.18. Show that the surface of a torus only allows homogeneous tilings (with $\alpha_1 < \aleph_0$) for either $n = k = 4$, $n = 3$ and $k = 6$, or $n = 6$ and $k = 3$.

2.19. [4] Assume that there exists a homogeneous tiling on a closed surface with $n = 5$ and $k = 4$. Show that this surface is not orientable if the number of faces of this tiling is not a multiple of 8.

2.20. Show that any convex set in a Euclidean space (such as a single point, a straight segment, a sphere, a convex polyhedron etc.) is simply connected, i.e. its fundamental group only contains the unit ϵ.

2.21. Show that the free cyclic group is the fundamental group of the annulus and also of $\mathbb{R}^2 \setminus \{p\}$, where p is any point in $\mathbb{R}^2$.

2.22. Consider a surface with frontiers. Show that this surface is not simply connected if the frontier consists of more than one simple curve.

2.23. Consider a subcomplex $[\mathbb{C}_{m,n}, \leq_{G2}, dim]$ of complex $[\mathbb{C}_{G2}, \leq_{G2}, dim]$ where $\mathbb{C}_{m,n}$ contains all grid points within a rectangle of size $m \times n$, and all subgraphs defined by grid edges or grid squares having their vertices in this rectangle. Specify the closure of the following subsets of $\mathbb{C}_{m,n}$: (i) the set of all grid points in $\mathbb{C}_{m,n}$, (ii) a set containing three consecutive collinear grid edges, (iii), a set containing two grid squares which share one edge, and (iv) the set of all grid squares in $\mathbb{C}_{m,n}$.

2.24. We specified several finite and countable models of abstract complexes. The following shows that there are also non-countable models: let $\mathbb{C} = \{s_a : a \in [0,1]\}$ be a non-countable set of (abstract) cells defined for the real index interval $[0,1]$, with $s_a \neq s_b$ iff $a \neq b$. For $a \in [0,1]$ let $a = 0.a_1a_2a_3\ldots$ with $a_i \in \{0,1,\ldots,9\}$. Let $s_a \leq s_b$ iff $a = b$ or $a_1 < b_1$ and $a_i = b_i$ for all $i \geq 11$. Let $dim(s_a) = a_1$. Show that $[\mathbb{C}, \leq, dim]$ is an abstract cell complex.

3 Models in Image Analysis

As already mentioned at the beginning of this report, pioneering papers from around 1970 stand at the beginning of research on topological approaches for image analysis. A. Rosenfeld introduced in [68, 69] connected subsets in the orthogonal grid. The carrier of a binary image was assumed to be a finite, rectangular set $\mathbb{C}_{m,n} \subset \mathbb{Z}^2$ of grid points. Different algebraic adjacencies, possibly in combination, have been suggested and studied for the analysis of binary images.

3.1 Adjacencies in Binary Images

A grid point $p_1 \in \mathbb{C}_{m,n}$ is *4-adjacent to* $p_2 \in \mathbb{C}_{m,n}$ iff $d_1(p_1, p_2) = 1$, and *8-adjacent to* $p_2 \in \mathbb{C}_{m,n}$ iff $d_\infty(p_1, p_2) = 1$.[15] In the sense of Definition 3 we have that a *4-path* from $p \in \mathbb{C}$ to $q \in \mathbb{C}$ of length n is a sequence $g = \langle p_0, p_1, \ldots, p_n \rangle$, with $p_0 = p$ and $p_n = q$ of grid points such that p_i is 4-adjacent to p_{i-1}, $1 \leq i \leq n$. An *8-path* g is defined based on 8-adjacency.

As noted in Section 1, an adjacency relation A defines a symmetric neighborhood relation N with $p \in N(p)$. Here we have two adjacency relations, $A_4(p) = \{q \in \mathbb{Z}^2 : d_1(p, q) = 1\}$ and $A_8(p) = \{q \in \mathbb{Z}^2 : d_\infty(p, q) = 1\}$, for grid point p, defining symmetric neighborhood relations N_4 and N_8, respectively.

Let $M \subseteq \mathbb{C}_{m,n}$. Two grid points $p, q \in M$ are *4-connected in M* iff there exists a 4-path from p to q consisting entirely of grid points in M. This equivalence relation defines *4-components* of M. A set $M \subseteq \mathbb{C}$ is *4-connected* iff for any $p, q \in M$ we have that p and q are 4-connected in M. Similarly, *8-connectedness* and *8-components* are defined using 8-paths.

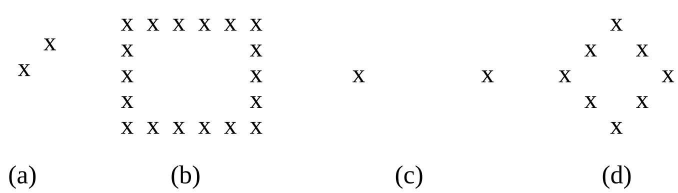


Fig. 32. Examples of connected and nonconnected sets [68].

Without explicitly using the terms '8-neighborhood' or '4-neighborhood', the pioneering paper [68] deals with these neighborhoods. Figure 1a (an object containing two grid points) is 8-connected, as is its complementary set, which also defines an '8-diagonal cut through both object points'. Figure 1b is both 4- and 8-connected, and its complement is neither 4- nor 8-connected. Figure 1c is not 4- or 8-connected but its complement is both 4- and 8-connected, and [68]

[15] We use the metrics $d_1(p, q) = |x_1 - y_1| + \ldots + |x_n - y_n|$ and $d_\infty(p, q) = max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$, for points $p = (x_1, \ldots, x_n)$ and $q = (y_1, \ldots, y_n)$ in the n-dimensional Euclidean space $\mathbb{E}^n$, $n \geq 1$.

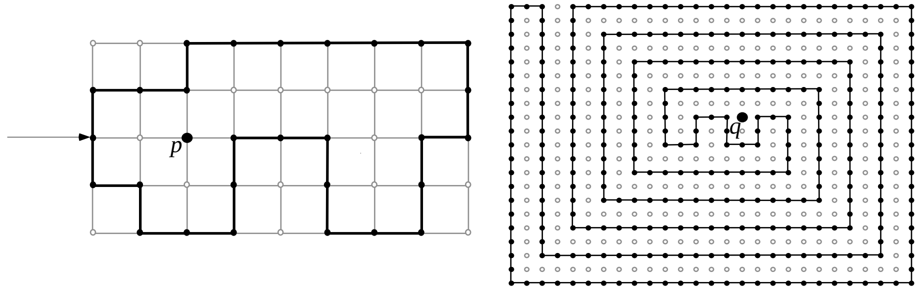


Fig. 33. Left: A grid point p inside of a simple 4-curve. Right: A simple 4-curve where decisions about inside or outside are more difficult (is point q inside or outside?).

> "the 'paradox' of Fig. 1d can be (expressed) as follows: If the 'curve' of shaded points is connected ('gapless'), it does not disconnect its interior from its exterior; if it is totally disconnected it *does* disconnect them. This is of course not a mathematical paradox[16], but it is unsatisfying intuitively; nevertheless, connectivity is still a useful concept. It should be noted that if a digitized picture is defined as an array of hexagonal, rather than square, elements, the paradox disappears".

The first case assumes 8-connectedness for 'curve' and 'background points', and the second case assumes 4-connectedness for both. The first case occurs if a non-planar adjacency graph (here: grid points as vertices with horizontal, vertical and diagonal edges) is used.

Curves in digital topology are often defined based on local connectedness properties following the topological approach of Cantor, Menger and Urysohn, instead of the parametric approach of Jordan:

Definition 22. (A. Rosenfeld 1970) *A 4-path is a* simple 4-curve *iff its length n is greater or equal 4, it consists of $n+1$ different grid points $p_0, p_1, \ldots, p_n$, and p_i is 4-adjacent to p_k iff $i \equiv k \pm 1$ (modulo $n+1$). An 8-path is a* simple 8-curve *iff its length n is greater or equal 4, it consists of $n+1$ different grid points $p_0, p_1, \ldots, p_n$, and p_i is 8-adjacent to p_k iff $i \equiv k \pm 1$ (modulo $n+1$).*

A *simple 4-arc* is a 4-connected proper subset of a simple 4-curve, and a *simple 8-arc* is an 8-connected proper subset of a simple 8-curve.

A grid point $p = (i, j)$ is *inside* of a simple 4-curve iff p is not on that 4-curve, and the grid line j has an odd number of crossings with that simple 4-curve to the left, and an odd number of crossings with that simple 4-curve to the right

[16] A mathematical *paradox* (antinomy) is characterized by a deduction of a contradiction within one theory [6]. A deduction of statements in digital topology which do not resemble statements in Euclidean topology is not a mathematical paradox. For example, for a natural number $n \in \mathbb{N}$ we do not have a natural number $m \in \mathbb{N}$ such that $n + m = 0$, as we have for $n \in \mathbb{Z}$ with $m = -n$, but this is of course not a paradox.

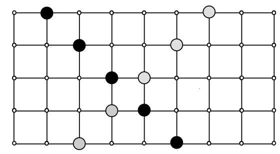


Fig. 34. Two non-intersecting 8-paths.

of p. For a more rigorous definition see Fig. 33 on the left: the crossing of the simple 4-curve by grid line j to the left of p does not need a further explanation, the touching of the curve on the right of p is defined by 'coming and going' of the 4-curve from/in the same halfplane, and the 4-curve comes from/ goes in different halfplanes at the final crossing. We refrain from a formal definition. A grid point p is *outside* of a simple 4-curve iff p is not on that 4-curve, and it is also not inside of this 4-curve. Inside or outside of a simple 8-curve is defined analogously. Note that all grid points inside of a simple 4-curve do not need to be 4-connected, see left of Fig. 33.

Let $G(g)$ be the set of all grid points listed in the sequence g. A 4- or 8-path g_1 *intersects* another 4- or 8-path g_2 iff $G(g_1) \cap G(g_2) \neq \emptyset$. Figure 34 shows two non-intersecting 8-paths. A 4- or 8-path g *8-separates* (*4-separates*) two grid points p and q iff any 8-path (4-path) from p to q intersects path g.

Theorem 9. (A. Rosenfeld 1970) *Any simple 4-curve (8-curve) g 8-separates (4-separates) all grid points inside of g from all grid points outside of g.*

In [69] it has been mentioned that studies towards this specific separation theorem were motivated by the Jordan-Veblen curve theorem of Euclidean topology.

The *border* of the rectangular carrier $\mathbb{C}_{m,n}$ is the set of all grid points in $\mathbb{C}_{m,n}$ having one 4-adjacent grid point in $\mathbb{Z}^2 \setminus \mathbb{C}_{m,n}$. Let $M \subseteq \mathbb{C}_{m,n}$ and assume that one 4-component of $\overline{M} = \mathbb{C}_{m,n} \setminus M$ contains the border of $\mathbb{C}_{m,n}$. This is called the *background component of infinite extent* of $\overline{M}$. All other 4-components of $\overline{M}$, if any, are *4-holes* in M. If M has no 4-holes, it is called *simply 4-connected*. An *8-hole* in M is an 8-component of $\overline{M}$ excluding the background component of infinite extent, and M is *simply 8-connected* if there is no 8-hole in M. We consider 4- and 8-curves being such sets $M \subseteq \mathbb{C}$. Theorem 9 implies:

Corollary 3. (A. Rosenfeld 1970) *Any simple 4-curve 8-separates its 8-holes from the background. Any simple 8-curve 4-separates its 4-holes from the background.*

This establishes an approach of solving the separation problem in binary image analysis by using two different adjacency or neighborhood definitions (called a *good pair* in [19]) on the same non-planar adjacency graph of 8-adjacency:

Definition 23. *(a, b) is a* good pair *iff any simple a-curve b-separates its b-holes from the background, and any simple b-curve a-separates its a-holes from the background.*

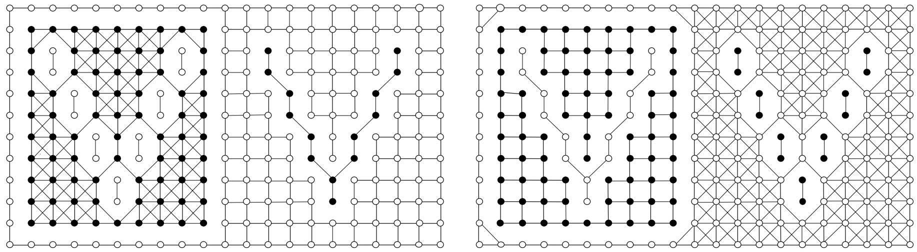


Fig. 35. Left: good pair (8,4). Right: good pair (4,8) (this binary image example has been discussed in [46]). There are 'cuts' of the V-shape in both cases established by 8-adjacencies.

Assume a binary image I defined on a rectangular set $\mathbb{C}_{m,n}$ of grid points. $(4,8)$ and $(8,4)$ are good pairs, and this suggests for $M = I^{-1}(1)$ to use 4-connectedness for M, and to use 8-connectedness for $\overline{M} = I^{-1}(0)$; or vice-versa for $(8,4)$. See Fig. 35 for an illustration of these two good pairs, where all object points are shown as filled dots and all background points are shown as hollow dots.

Definition 24. Valid adjacencies *are between adjacent grid points which are labeled by identical image values.*

Valid adjacencies are shown by connecting line segments. *Invalid adjacencies* are between points in different I-equivalence classes defined by different values of image I.

$(4,4)$ or $(8,8)$ are not good pairs, but $(6,6)$ is a good pair. The good-pair neighborhood approach is successfully applied in binary image analysis: normally (4,8) or (8,4) adjacencies in $\mathbb{Z}^2$, and (6,26), (26,6), (6,18) or (18,6) adjacencies in $\mathbb{Z}^3$ [19, 42]. A drawback of two different adjacency definitions combined into one good pair is that M and $\overline{M}$ then do have different connectedness definitions.

The left half of the binary image (an example from [46]) in Fig. 35 shows a 'background V', the right half shows an 'object V'. There is only one connected 'V' in both copies of this image, for good pair (8,4) and for good pair (4,8). The second 'V' is (already) disconnected by 8-connected pixels, i.e. it may happen that subsequent image analysis procedures have to disconnect these pixels again.

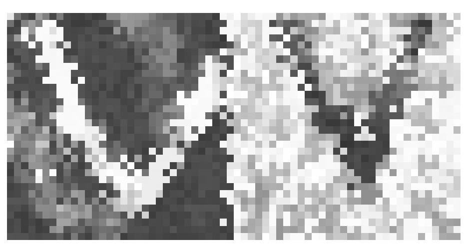


Fig. 36. Multi-level input image as normally given in image analysis.

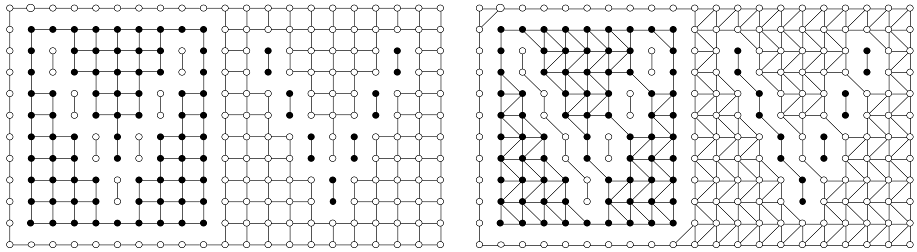


Fig. 37. Pair (4,4) and good pair (6,6): there are still a few 'cuts' as in Fig. 35 for (6,6). Missing connections may be obtained by subsequent image analysis approaches.

3.2 Open Questions for Multi-Level Images

Image analysis normally deals with multi-level input images (actually even with multi-channel images in an increasing number of applications), i.e. we have $G_{max} > 1$. See Fig. 36 for an example; pixel values are shown as shaded squares. The concept of good pairs cannot be extended to such multi-level images I where a similar consistency of different neighborhood definitions for $I^{-1}(u)$, for $0 \leq u \leq G_{max}$, is impossible if $G_{max} > 1$.

Alternative orientations of diagonals defining *6-adjacencies* introduce a (systematic) directional bias into the resulting 6-components. Figure 37 illustrates the pair (4,4) and the good pair (6,6). Both, $(4,4)$ and $(6,6)$ are planar graph structures. The connectedness approach defined by the pair $(4,4)$ is used in several major commercial image processing systems sold worldwide, see Fig. 38: pixels are shown again as squares; an 8-curve 4-separates one interior 4-component from one exterior 4-component, but the 8-curve itself is not connected according to the system.

A 'duality of separation and connectedness' becomes increasingly important for higher-level image analysis, where data are often defined with respect to inhomogeneous carriers (e.g. for image segments, approximated simplicial object surfaces, or polyhedral approximations of scene objects). Cases of inhomogeneous

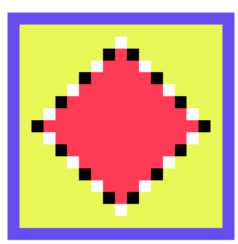


Fig. 38. 4-components as produced by a major commercial image processing system: the 8-curve is a disconnected set of isolated points, but separates interior and exterior components.

carriers (see Fig. 2 for two examples) cannot be discussed using homogeneous 4-, 6- or 8-neighborhood approaches.

The Euclidean and graph complexes of the homogeneous orthogonal grid, $[\mathbb{C}_{E2}, \leq_{E2}, dim]$ and $[\mathbb{C}_{G2}, \leq_{G2}, dim]$, provide alternative models compared to base sets $\mathbb{C}_{m,n}$ containing only grid points. Abstract complexes provide a uniform topological approach for all the different carriers in image analysis, which simplifies concepts and implementations. Planar graph structures also provide a (methodologically simpler) alternative for dealing with input images defined on the orthogonal planar grid. We briefly discuss a general way to introduce and implement such planar graph structures.

3.3 Switches

Let us continue with using a graph-theoretical model and the orthogonal grid for discussing possible options for specifying a topological approach for homogeneous grids. Undirected grid edges represent a symmetric and irreflexive adjacency relation.

We use all isothetic grid edges representing 4-adjacency, plus selected diagonals in grid squares specified in the following definition.

Definition 25. *Take the lower left corner of a grid square as the reference point for a* switch *which is a grid diagonal being either in an on-, or in an off-state, see Fig. 39. The state of a switch needs to be such that the grid diagonal connects grid points being in the same equivalence class (i.e. having identical image values) if there is such a pair of diagonal points in the given grid square; if both diagonals connect grid points in identical equivalence classes then a state may be chosen.*

Note that we only allow one grid diagonal per grid square. The resulting (inhomogeneous in general) planar graphs are examples of adjacency graphs as studied in [36], examples of two-dimensional strongly normal digital picture spaces in the sense of [43], and also examples of planar generic axiomatized digital surface-structures (GADSs) as discussed in [19],

Figure 39 shows on the right all possible 2×2 image value configurations: filled dots illustrate pixels $(p, I(p))$ belonging to one equivalence class C, and hollow dots illustrate pixels belonging to different equivalence classes (not necessarily

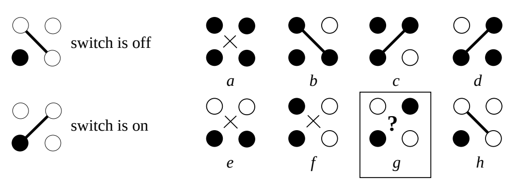
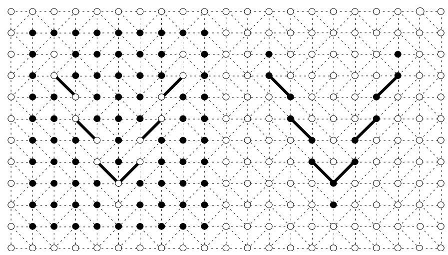


Fig. 39. The reference point is at the lower left corner: states of a switch (left), and for all possible image value assignments on a grid square, there is only flip-flop case (g) where the position of the switch needs to be decided. The cross stands for a don't-care-situation.



-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
1	0	1	1	0	0	1	0	1	0	0	1	0	0	0	1	0	1	0	0	-
0	1	0	1	1	0	1	1	0	0	1	1	0	1	1	0	0	1	0	0	-
1	1	1	0	1	0	0	1	0	1	0	1	0	1	1	0	1	0	1	0	-
1	0	0	0	0	0	1	1	1	0	0	0	0	0	1	1	1	1	1	1	-
0	1	0	1	1	0	1	1	0	0	1	1	0	0	0	0	1	1	1	0	-
0	0	0	0	0	1	1	1	0	0	1	0	0	0	0	1	1	1	0	1	-
0	0	0	0	1	1	1	0	1	1	0	0	1	0	0	0	1	1	1	0	-
0	1	0	0	0	1	1	0	1	0	1	0	1	0	0	1	1	1	1	1	-
1	1	1	0	0	1	0	0	1	0	0	1	0	0	0	1	1	0	1	1	-
0	0	1	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	0	-
0	0	1	0	1	1	0	1	0	1	0	1	1	0	1	0	0	0	0	0	-

Fig. 40. States of switches are uniquely defined, or can be chosen randomly in most of the cases. Just a few flip-flop switches (12 in this example) are decided by a procedure *SetSwitches* defined by the local templates shown in Fig. 41. The binary matrix **S** on the right encodes the states of all switches.

just to one category different to C). The state of the switch is unimportant in cases (a) (both diagonals connect points in class C), (e) and (f) (both diagonal pairs are points in different classes). The state of the switch is uniquely defined in cases (b), (c) and (d) because there is just one diagonal pair of points which are in the same class. In situation (h) we choose the off-state because the connected diagonal pair might be in the same class.

Switch State Matrix: The only remaining problem is the *flip-flop case* (g) (in fact absolutely analogously to the Euclidean plane when two curves intersect at one point, and the assignment of the intersection point decides how these two curves subdivide the Euclidean plane!): if both diagonal points shown by hollow dots are in different classes then the switch will be in on-state. Otherwise we call a procedure *SetSwitches* to chose either the on- or the off-state. Important is that the state can't be changed again during one topological operation on a picture after it has been set.

The procedure *SetSwitches* may, for example, analyze larger neighborhoods of the reference point for defining the state of its switch. See Fig. 40 for a possible specification of switches, where a procedure *SetSwitches* has been used in a bottom-up, left-to-right fashion: assigning switches randomly in don't-care situations, and using the local templates shown in Fig. 41 for the flip-flop case (g). These templates are such that the state of the switch in the grid square below is simply copied as the new state of the switch in the recent grid square. The example shows that it is possible to assign switches such that both V-shapes remain connected. Of course, more advanced procedures *SetSwitches* may be

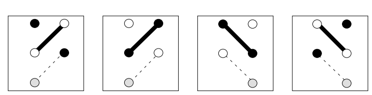


Fig. 41. Set of simple templates for defining flip-flop switches.

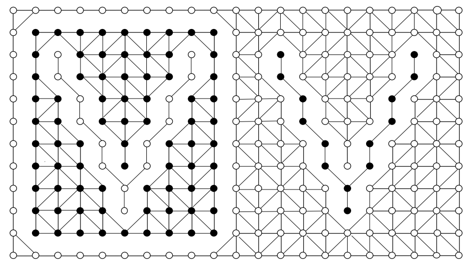


Fig. 42. Valid **S**-adjacencies for the planar adjacency graph shown in Fig. 40.

designed, using larger neighborhoods for enforced control about switch settings. In an implementation of the approach a binary image may be used with value 0 at point p iff the switch at reference point p is in off-state, and value 1 otherwise (see left of Fig. 40). The topology of a digital image is then specified by such an accompanying binary *switch state matrix* **S**, which defines an **S**-adjacency between grid points and, subsequently, **S**-connectedness. Figure 42 shows the resulting subgraph of valid adjacencies between **S**-adjacent grid points p, q having identical image values, i.e. $I(p) = I(q)$, where **S** as shown in Fig. 40.

Practical Aspects: The discussed switches ensure that the resulting **S**-adjacency graph is always planar. Any image processing step, e.g. a simple local filter, contrast enhancement, or an interactive modification of single image values, will create a need to update the switch state matrix **S** of a given image. Of course, matrix **S** is only needed if a topological operation is called, and its calculation is very simple. The *switch-approach* can be summarized as follows: every grid square contains one grid diagonal only, as in case of the good pair (6,6). However, to avoid a directional bias, or to reflect the given image data, the diagonals (switches) may be either set randomly or based on rules as discussed above. For image processing this means that a switch state matrix **S** needs to be available at the time of a topological operation in an image such as contour tracing or thinning. The matrix **S** ensures that only planar adjacency graphs are used, and it can be

(i) always the same switch state matrix (look-up table $\mathbf{S}_{m,n}$), just defined by the size $m \times n$ of the image and calculated by using a random number generator,
(ii) a function which produces a binary pseudo-random number based on the coordinates of the reference point $p = (i, j)$, allowing that actually no switch state matrix is needed, just a local calculation of the pseudo-random switch state (e.g., if the size of the images varies frequently),
(iii) an updated switch state matrix **S** using the rules as discussed for Fig. 39 and an image data-dependent procedure for dealing with the flip-flop cases (which, in fact, appear very rarely in captured images, see Fig. 43, i.e. this option might be of interest for cases of very high-precision image capturing only).

Due to a certain degree of randomness in captured image data (due to sensor noise, uncertainties in image data, illumination changes etc., see Fig. 44) and the low percentage of locally (i.e. in a 2×2 window) undecidable flip-flop states (note: typically less than 0.5% for grayscale images, see Fig. 43, and less than 0.2% for color images) it is normally appropriate to use one of the first two options. A data-dependency of adjacencies is also discussed in [67] using hypergraphs, allowing even incorporations of local image data variances into the adjacency definition.

The switch-approach has shown that context-dependent connectivities may be achieved by adding more structural elements (namely grid diagonals) to the 4-adjacency graph. The *SetSwitches* procedure may be designed such that curve-like patterns are preferably connected, for example using larger neighborhoods than in the simple set of templates shown in Fig. 41. The switch-approach is not designed for inhomogeneous image carriers. It is also not designed for extreme cases such as a 'chessboard-like' binary image segments.

Image analysis approaches depend on calculated components provided at the base layer of processes in the homogeneous carrier, but are flexible in dealing

Fig. 43. Upper left: this 2014×1426 picture, i.e. 2,872,964 pixels, with $G_{max} = 255$, possesses 14,359 flip-flop cases, i.e. 0.50% of all grid points. Upper right: 0.38%. Lower left: 0.38%. Lower right: 0.22%.

129	61
61	129

128	61
61	129

129	61
62	129

Fig. 44. Only one of these three labeled grid squares is a flip-flop situation.

with them: for example, the pair (4,4) (see left of Fig. 37 and Fig. 38) provides subsegments which may be clustered into lines, frames of windows, yucca trees, cars or other meaningful image segments based on context-dependent image analysis approaches. This even leads to a good degree of justification that the pair (4,4) may be considered as being 'sufficient', leaving establishments of missing connections to higher levels of an image analysis approach. Besides the rare case of purely binary image analysis and the justified use (see Theorem 9) of either good pair (4,8) or (8,4) in such a binary situation, any use of 8-adjacency in other situations causes conflicts as already sufficiently discussed in the pioneering paper [68], see Fig. 32.

3.4 Wyse Topology and a Non-Existence Theorem

The question arises what topologies and topological concepts may be defined and used for topological image carriers as claimed in Axiom *C3*. The cell or graph complex of the homogeneous two-dimensional grid provides one topological space because both models are homeomorphic (see Corollary 1). Topologies on the two-dimensional homogeneous orthogonal grid may be defined by specifying a basis (as noted earlier for the general case):

Example 9. (F. Wyse et al. 1970) We defined $A_4(p) = \{q \in \mathbb{Z}^2 : d_1(p,q) = 1\}$ as set of 4-adjacent points of grid point $p = (i,j)$. Let

$$U_4(p) = \begin{cases} \{p\}\,, & \text{if } i+j \text{ is odd} \\ \{p\} \cup A_4(p)\,, & \text{if } i+j \text{ is even}\,. \end{cases}$$

The family of all of these sets $U_4(p)$, $p \in \mathbb{Z}^2$, defines a topological basis on $\mathbb{Z}^2$, and a set of grid points is connected in the resulting *4-topology* iff it is 4-connected, see [89].[17] Set $U_4(p)$ is the smallest topological neighborhood of point p. The neighborhood relation U_4 is asymmetric. It generates (see Section 1) a symmetric adjacency relation which coincides with A_4, and A_4 generates the symmetric (algebraic) neighborhood relation N_4. See Fig. 45. □

Corollary 4. *$[\mathbb{Z}^2, N_4]$ is a topological image carrier, i.e. it satisfies the image carrier axioms C1, C2 and C3.*

Due to the correspondence of topological connectedness and 4-connectedness it is obvious that the 4-topology does not add further 'structure' to the concept

[17] The note [89] actually defined an Aleksandrov topology on $\mathbb{Z}^n$, $n \geq 1$, and we only cite case $n = 2$.

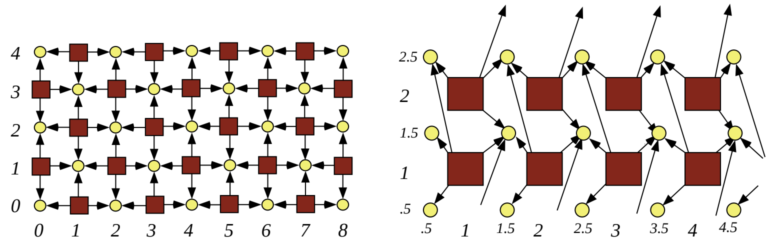


Fig. 45. Left: a directed graph visualizing the asymmetric neighborhood relation U_4. Right: Indication of one possible mapping of this graph such that exactly all odd grid points (shown as squares) are in grid point positions.

of 4-adjacency. However, it is possible to discuss open and closed sets in this 4-topology, or the closure of a set (see Fig. 46). For example, any set containing an *even grid point* $p = (i, j)$, with $i + j$ even, but only at most three of its 4-neighbors (which are *odd grid points*), is not open, and sets containing only even grid points are closed.

The following theorem states that 8-adjacency does not satisfy the topological image carrier axiom *C3*.

Theorem 10. (Chassery 1979) *Let* $\mathbb{C}$ *be* $\mathbb{Z}^2$*, or a finite subset of* $\mathbb{Z}^2$ *containing a translation of the set* G_0*. Then there exists no topology on* $\mathbb{C}$ *in which connectivity would be the same as 8-connectivity.*

The theorem refers to set G_0 which is shown in Fig. 47; the geometric location of this set in the Euclidean plane is unimportant. The proof in [11] for this theorem consists of a series of case discussions.

The homogeneous grid used in the switch-approach suggests a straightforward adaptation of Example 9 for this model:

Example 10. Let $A_{\mathbf{S}}(p) = \{q \in \mathbb{Z}^2 : d_{\mathbf{S}}(p, q) = 1\}$ be the set of all grid points being in distance 1, where $d_{\mathbf{S}}$ is defined by a switch state matrix $\mathbf{S}$, for grid point

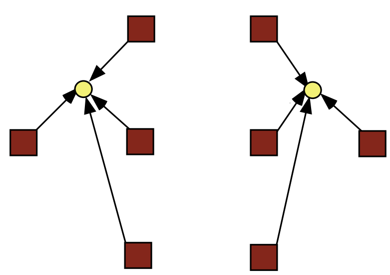


Fig. 46. The smallest neighborhoods of single even grid points for the drawing shown in Fig. 45 on the right.

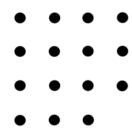


Fig. 47. Set G_0 referenced in Theorem 10.

$p = (i, j)$. Let

$$U_{\mathbf{S}}(p) = \begin{cases} \{p\}, & \text{if } i + j \text{ is odd} \\ \{p\} \cup A_{\mathbf{S}}(p), & \text{if } i + j \text{ is even} \end{cases} .$$

However, the family of these sets $U_{\mathbf{S}}(p)$, $p \in \mathbb{Z}^2$, does not define a topological basis on $\mathbb{Z}^2$: the 6-adjacency is a special example of an **S**-adjacency. Assume two 'diagonally adjacent' and even grid points p and q. Then the intersection of $U_{\mathbf{S}}(p)$ with $U_{\mathbf{S}}(q)$ contains exactly four grid points, and this is not one of the defined sets. A further intersection with another set $U_{\mathbf{S}}(r)$, where r is another even 'diagonally adjacent' point to p, allows to produce the singleton $\{p\}$. It follows that all subsets of $\mathbb{Z}^2$ are open, i.e. only singletons are connected.

This example only shows that a straightforward adaptation of the Wyse topology fails in case of **S**-adjacency. A more general discussion is required to analyze the existence or non-existence of a topology on $\mathbb{Z}^2$ corresponding to **S**-connectivity.

3.5 Cellular Model

A more flexible approach than using just grid points as possible elements of consideration consists in considering subsets of Euclidean spaces (towards Euclidean complexes, see Euclidean complex $\mathbb{C}_{E2}$) or in specifying topologies based on subgraphs (towards graph complexes, see graph complex $\mathbb{C}_{G2}$). We start with a simple example defined in an exercise in [5]:

Example 11. (N. Bourbaki 1961) The family $\{[x, +\infty) : x \in \mathbb{R}\}$ is a basis of a topology on $\mathbb{R}$, called the *right topology* on $\mathbb{R}$. It follows (for example) that intervals $(-\infty, x)$ are closed. Analogously, the family $\{(-\infty, x] : x \in \mathbb{R}\}$ is also a basis of a topology on $\mathbb{R}$, called the *left topology* on $\mathbb{R}$. Note that sets $[x, +\infty)$, open in the right topology, are not open in the Euclidean topology on $\mathbb{R}$. □

We recall that a subset of a topological space induces a topological subspace. The set $\mathbb{Z} \subset \mathbb{R}$ defines an inherited Euclidean, right or left topology, all topological subspaces of $\mathbb{R}$, depending upon whether $\mathbb{R}$ is considered as being endowed with either the Euclidean (see Example 3), or the right or left topology as specified in Example 11. In case of the Euclidean topology we induce a *discrete topology* on $\mathbb{Z}$, where any subset of $\mathbb{Z}$ will be open and closed as well, and this trivial topological space is not connected: any two nonempty and complementary subsets of $\mathbb{Z}$ define a partition of $\mathbb{Z}$ into two closed subsets. In case of the right or left topology we obtain connected subspaces.

Example 12. (C. O. Kiselman 2000) For a different approach for inducing a topology assume a connected topological space $\mathbb{C}$ and a surjection $f : \mathbb{C} \to M$ into a set M. Equip M with the finest topology such that f is continuous. [18] Then M is a connected topological space.

Now consider the Euclidean topology on $\mathbb{R}$ and a surjection $f : \mathbb{R} \to \mathbb{Z}$, i.e. the set $f^{-1}(i)$ defines a subset of $\mathbb{R}$, for $i \in \mathbb{Z}$: let $f(x)$ be the nearest integer to x, and, if x is a half-integer $x = i + \frac{1}{2}$ then let $f(x) = i$. It follows $f^{-1}(i) = (i - \frac{1}{2}, i + \frac{1}{2}]$, for $i \in \mathbb{Z}$, i.e. $f^{-1}(i)$ is neither open or closed in the Euclidean line $\mathbb{R}$. The same may be stated if we taken $f(i + \frac{1}{2}) = i + 1$ instead. As a result, no proper subset of $\mathbb{Z}$ may be open or closed, i.e. the induced (finest) topology is the trivial topology which only has the empty set $\emptyset$ and $\mathbb{Z}$ itself as open and closed sets. □

This example has been discussed in [33] for illustrating the basic idea underlying the introduction of the following alternating topology:

Example 13. (E. Khalimsky 1986) Consider the function f as in Example 12, but choose the nearest even integer as the best approximation of a half-integer $i + \frac{1}{2}$ this time. This function induces an *alternating topology* on $\mathbb{Z}$: $f^{-1}(2i)$ is a closed subset of $\mathbb{R}$ in the Euclidean topology, and $f^{-1}(2i+1)$ is an open subset, i.e. $\{2i\}$ is a closed subset of $\mathbb{Z}$ and $\{2i+1\}$ is an open subset of $\mathbb{Z}$. In general, a subset M of $\mathbb{Z}$ is open iff $f^{-1}(M)$ is open in the Euclidean topology on $\mathbb{R}$. □

This alternating topology on $\mathbb{Z}$ combines the basic ideas of Example 9, i.e. it uses the properties odd or even for alternations, and of Example 11, i.e. it defines a topology on $\mathbb{R}$ by intervals. For image analysis we are interested in topologies on $\mathbb{Z}^n$, with $n \geq 2$:

Definition 26. *Let $\mathbb{C}_1$ and $\mathbb{C}_2$ be topological spaces; their* product $\mathbb{C}_1 \times \mathbb{C}_2$ *is the set of ordered pairs (p_1, p_2) such that $p_1 \in \mathbb{C}_1$ and $p_2 \in \mathbb{C}_2$, endowed with the* product topology [1]; *namely, $M \subseteq \mathbb{C}_1 \times \mathbb{C}_2$ is open iff for each $(p_1, p_2) \in M$ there are open sets M_1 in $\mathbb{C}_1$ and M_2 in $\mathbb{C}_2$ such that $(p_1, p_2) \in M_1 \times M_2 \subseteq M$.*

Examples 11, 12 and 13 introduced topologies on $\mathbb{R}$ or $\mathbb{Z}$ which can be used to form topologies on $\mathbb{R}^n$ or $\mathbb{Z}^n$, $n \geq 2$.

Figure 48 illustrates the product of two alternating topologies on $\mathbb{Z}$, resulting into the *Khalimsky plane* on $\mathbb{Z}^2$ (left), or a scaled version of it defined on $\{i/2 : i \in \mathbb{Z}\}^2$ (right). A subset $M \subseteq \mathbb{Z}^2$ of the Khalimsky plane is open iff the set

$$S_M = \bigcup_{(i,j) \in M} f^{-1}(i) \times f^{-1}(j) \tag{4}$$

is open in the Euclidean plane.

The (infinite) Khalimsky plane is Aleksandrov because an arbitrary intersection of open sets S_M as specified in formula (4) is open. For example, the smallest

[18] Given two topologies $\mathbb{Z}_1$ and $\mathbb{Z}_2$ on N. $\mathbb{Z}_1$ is *finer* than $\mathbb{Z}_2$ iff every subset of M which is open in $\mathbb{Z}_2$ is also open in $\mathbb{Z}_1$. The use of function f allows to avoid the definition of the topological concept of a quotient space.

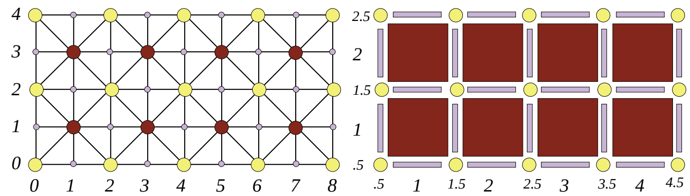


Fig. 48. Left: shaded (yellow) large dots show closed sets $\{(2i, 2j)\}$, filled (red) large dots indicate open sets $\{(2i + 1, 2j + 1)\}$, and small dots show sets $\{(2i, 2j + 1)\}$ or $\{(2i + 1, 2j)\}$ which are neither open nor closed. Right: use of different scaling and different symbols for indicating the same alternating topology.

topological neighborhood of an open set $\{(2i+1, 2j+1)\}$ in the Khalimsky plane is this set itself, and that of a closed set $\{(2i, 2j)\}$ is the set containing grid point $p = (2i, 2j)$ as well as all of its eight 8-neighbors.

The Khalimsky plane is a T_0-space. For example, $N(p) = \{p\}$ for point $p = (2i + 1, 2j + 1)$ does not contain any of its eight 8-neighbors.

Figures 17 and 48 indicate a bijection between the base set $\mathbb{C}_{E2}$ of the two-dimensional Euclidean complex of the orthogonal grid, and the base set $\{i/2 : \ i \in \mathbb{Z}\}^2$ of the scaled alternating topology. Earlier we already realized that the two-dimensional complexes $[\mathbb{C}_{E2}, \leq_{E2}, dim]$ and $[\mathbb{C}_{G2}, \leq_{G2}, dim]$ are homeomorphic (see Corollary 1), and that this topology on the two-dimensional homogeneous grid is an example of an Aleksandrov topology of a poset. The Euclidean complex $[\mathbb{C}_{E2}, \leq_{E2}, dim]$ endowed with the Tucker topology has been popularized by [46] in image analysis as an option of a homogeneous image carrier, also known as *Kovalevsky plane.*

Theorem 11. *Khalimsky and Kovalevsky plane are homeomorphic.*

Proof. We define a bijection Φ as indicated by Figs. 17 and 48: 0-cells (grid points) at (i, j) are mapped onto points $(2i, 2j)$, 2-cells (grid cells) with reference point (i, j) (which is assumed to be the lower left corner of the grid square) are mapped onto points $(2i+1, 2j+1)$, 1-cells (grid edges) between (i, j) and $(i+1, j)$ are mapped onto points $(2i + 1, 2j)$, and 1-cells (grid edges) between (i, j) and $(i, j + 1)$ are mapped onto points $(2i, 2j + 1)$.

Φ is continuous: let $M \subseteq \mathbb{Z}^2$ be an open set in the alternating topology, i.e. set S_M (see Equ. 4) is an open set in $\mathbb{R}^2$, and assume that there exists a pair of points $p \in \Phi^{-1}(M)$ and $q \notin \Phi^{-1}(M)$ with $p \leq q$, i.e. $\Phi^{-1}(M)$ is not open in the Tucker topology.

q is an n-cell, $0 \leq n \leq 2$, and p is one of its m-sides, $0 \leq m \leq n$. Case $n = m$ is impossible because this implies $p = q$, i.e. it would be $p \in \Phi^{-1}(M)$ and $p \notin \Phi^{-1}(M)$. Let $n = 2$ and $m = 1$. Then q is an open grid square s, which can be represented as $s = f^{-1}(i) \times f^{-1}(j)$, and p is a grid edge e (without both of its endpoints) which can be represented as $e = f^{-1}(k) \times f^{-1}(l)$. Because e bounds s we have that (k, l) is an 8-neighbor of (i, j). $p \in \Phi^{-1}(M)$ implies $(k, l) \in M$, and $e \subseteq S$. $q \notin \Phi^{-1}(M)$ implies $(i, j) \notin M$, and $s \nsubseteq S$. This means that S cannot be open in contradiction to our assumption, i.e. the assumed pair of points p, q cannot exist. - Cases $n = 2$ and $m = 0$, and $n = 1$ and $m = 0$ may be treated analogously.

Φ^{-1} is continuous: let M be an open subset of the Kovalevsky plane. Assume that $\Phi(M)$ is not open in the alternating topology, i.e.

$$S_{\Phi(M)} = \bigcup_{(i,j)\in\Phi(M)} f^{-1}(i) \times f^{-1}(j)$$

is not open in the Euclidean topology, i.e. there is one set $S_0 = f^{-1}(i) \times f^{-1}(j)$ such that at least one of its frontier subsets $S_1 = f^{-1}(k) \times f^{-1}(l)$ is contained in $S_{\Phi(M)}$, but S_0 is not. Let S_0 be an open square q and S_1 be an edge p of this square (without both of its endpoints). $S_1 \subseteq S_{\Phi(M)}$ implies $(k, l) \in \Phi(M)$, and $p = \Phi^{-1} \in M$. $S_0 \not\subseteq S_{\Phi(M)}$ implies $(i, j) \notin \Phi(M)$, and $q = \Phi^{-1} \notin M$ with $p \leq q$. It follows that M is not an open set. This contradicts our assumption on M, and $\Phi(M)$ needs to be open in the alternating topology. - The remaining cases (S_1 is a vertex, or S_0 is an edge and S_1 is a vertex) follow by using analogous arguments. □

Due to this theorem we may speak from *the Khalimsky-Kovalevsky plane* if we like to refer to this special example, e.g. in the form $[\mathbb{C}_{E2}, \leq_{E2}, dim]$ or in the form $[\mathbb{C}_{G2}, \leq_{G2}, dim]$, of a Tucker or Aleksandrov topology. Historically the first definition of this plane is actually Example 4 on page 26 in [1] (see citation above in Example 4), i.e. it might also be called the *Alexandroff-Hopf plane*.

Note that any scaling operation, such as on $\{i/2 : \ i \in \mathbb{Z}\}^2$ in Fig. 48, may be incorporated into the definition of the homeomorphism given in the proof of the Theorem. In [44] it was also pointed out that Theorem 11 may be obtained as a corollary of a more general theorem, saying that the product of the Aleksandrov topologies of any two posets is the Aleksandrov topology of the product of those posets.

3.6 Discussion

Let us return to the 'double-V' image example from [46] discussed in Section 3. As illustrated in [46], the Khalimsky-Kovalevsky or Alexandroff-Hopf plane allows an image-data dependent specification of a subset of $\mathbb{C}_{E2}$, the set of all 0-, 1- and 2- cells in the plane, such that this set corresponds exactly to both V-patterns. Pixel locations are identified with 2-cells, and 0- or 1-cells may be assigned to different 2-cells. Note that the switch-approach only requires specifications of a binary state of one switch, for every pixel.

Altogether we have two different topological spaces for the orthogonal planar grid:

(i) 4-adjacency and the corresponding 4-topology as defined in Example 9, and
(ii) the graph complex $[\mathbb{C}_{G2}, \leq_{G2}, dim]$ or Euclidean complex $[\mathbb{C}_{E2}, \leq_{E2}, dim]$, in image analysis literature also known as Khalimsky-Kovalevsky plane defined by one of two equivalent topologies, the Tucker topology on $\mathbb{C}_{E2}$ of

abstract complexes, or the product topology of two alternating topologies defined in Example 13.

Both provide alternative options for discussing topological problems at the lowest (initial) layer of image analysis approaches.

Option (i) is the *graph-theoretical model* for image carriers, compared to option (ii) which is the *the cellular model*

$$[\mathbb{C}^{\{2\}}, \leq^{\{2\}}, dim] \; .$$

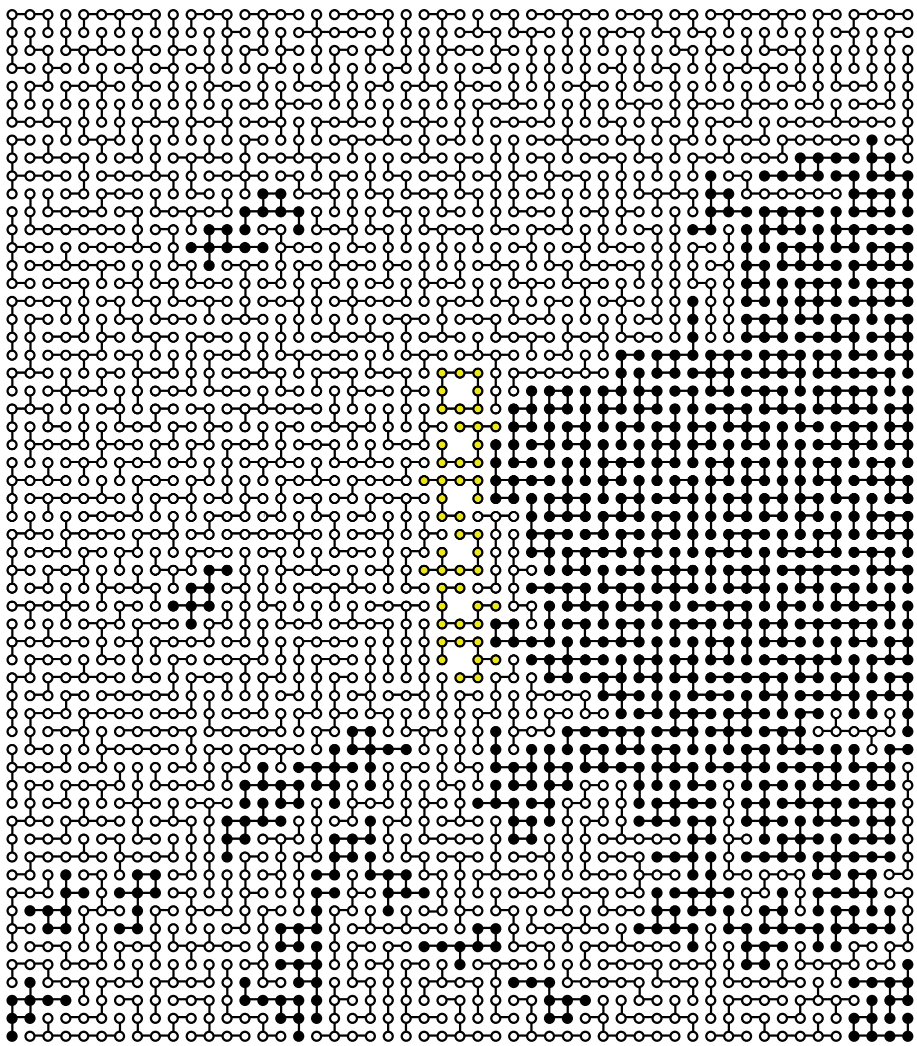


Fig. 49. All 369 4-connected sets of grid points of cardinality eight [23].

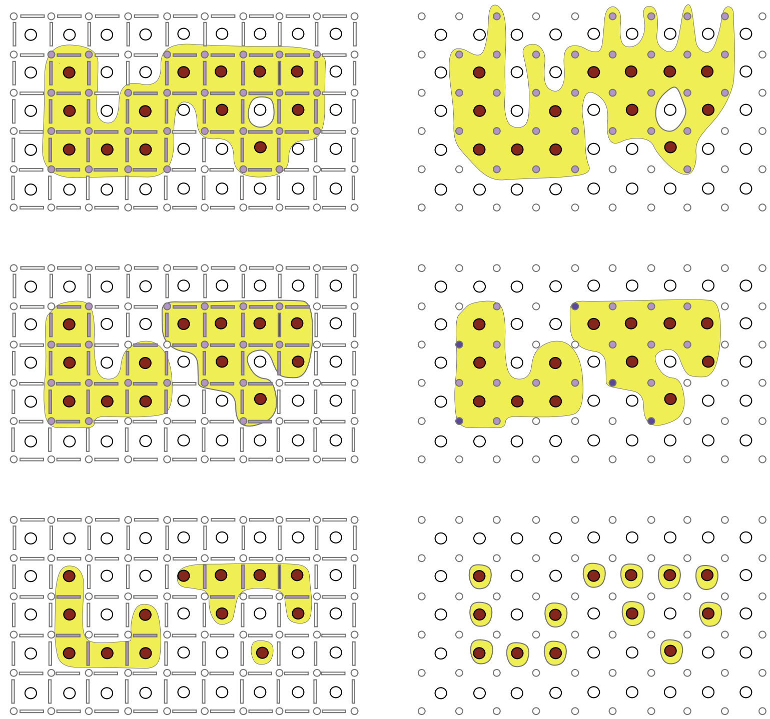


Fig. 50. Cellular model (left column) and 4-topology (right column): closed sets (upper row), 'partially open' (middle row), and open (bottom row).

See Fig. 50 for an example of a binary input image (filled dots) where the assumed additional elements of the cellular model (right column) and of the 4-topology (left column, using the mapping into $\mathbb{Z}^2$ as indicated on the right of Fig. 45) are also shown. It is a matter of the specification in the program whether a given set of pixels (here indicated by filled dots) is considered as being closed in the assumed topology (upper row), 'partially open' (middle row), or open (bottom row). Any assumption implicates a consistent assignment of model elements to the complementary set, e.g. if the given set of pixels is assumed to be closed, then the complementary set should be open.

Because 8-adjacency fails to provide a topological model for an image carrier we will not mention this concept anymore. The main difference between the graph-theoretical model and a cellular model consists in the fact that topological neighborhoods defined in cellular models are asymmetric, i.e. they establish directed relations between cells. Adjacency (in the graph-theoretical model) is symmetric. Smallest topological neighborhoods (of cellular models) induce symmetric algebraic neighborhoods via the incidence relation, and this defines a simplified view on the previously asymmetric topological neighborhood.

Topologies on image carriers allow to use (many of) the general topological concepts discussed in Section 2; and this report can only illustrate a few minor aspects of this large potential. For example, the definition of dimension (Definition 16) is based on the smallest symmetric extension of smallest topological neighborhoods $U(p)$. Figure 49 from [23] shows all 369 4-connected sets of grid points of cardinality eight: all two-dimensional sets (w.r.t. the 4-topology, i.e. using algebraic neighborhood relation N_4 resulting from the topological relation U_4) are drawn with filled dots, and the six sets having a (proper or improper) *hole* (i.e. a 4-connected component 'separated by the set' (will be specified later) from the 4-component of infinite extent of the background) are positioned in the center of the drawing. In case of 4-adjacency it follows that a set is two-dimensional iff it contains at least one 2×2 square of grid points. Note that the definition of dimension depends upon the used connectedness definition.

From the practical point of view (design of common commercial image analysis systems) it may appear that a choice besides 4-adjacency is more of academic interest than of practical relevance, e.g. due to the facts that planar 6-adjacency is not supported by hexagonal CCD-arrays or 'hexagonal frame grabbers', the small percentage of flip-flop cases (see examples in Section 3; of course, every flip-flop case may have crucial impact on analysis results for components), or that image segmentation problems will not be solved at basic topological levels of image carriers (signal-theoretical approaches, image texture, larger neighborhoods, image acquisition models etc. are keywords for image segmentation). This observation is supported by the fact that major commercial image analysis systems actually use 4-adjacency (in the non-anti-aliased mode), and nothing else. The switch-approach seems to be straightforward for implementations, and it has obviously benefits with respect to solutions of the separation problem compared to 4-adjacency. It remains an open problem here to show whether 6-adjacency may satisfy the topological axiom *C3* or not (or **S**-adjacency in general).

The full advantage of cellular models, such as $[\mathbb{C}_2, \leq_2, dim]$ for the plane, becomes visible if we deal with 'advanced' image carriers where directed relations between cells are important to be modelled; for example in case of polyhedral structures for geometric modelling [53]. We will return to complexes in part II of this report. Assuming that a cellular model approach is implemented for higher levels of image analysis it is also important to state that the same approach may also be used for homogeneous image carriers (which is not the case for the graph-theoretical models). The slightly higher complexity of the 'asymmetric' cellular model may be acceptable due to this option of unifying used methodologies for homogeneous and inhomogeneous 'complicated' image carriers.

The following adjacency graph discussions provide a uniform framework for graph-theoretical models of ('symmetric representations of') image carriers, e.g. also for adjacencies induced by topological neighborhoods of cellular models.

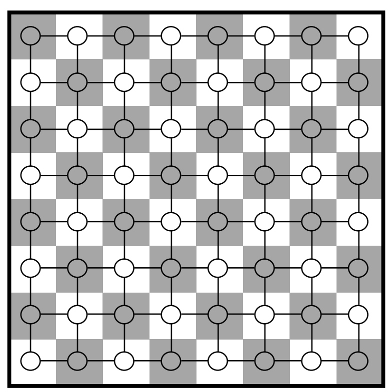
EXERCISES

3.1. B. Russell published in 1903 the following paradox of the set of all sets: *Naive set theory* or *Cantor set theory* allows to define a set by specifying a mathematically precise property for all the elements in this set. Let $\mathcal{S}$ be the family of all sets of naive set theory. For example, $A = \{x : x \text{ is a citizen of Kaitaia}\}$ is not precise because the used relation for specifying this set depends on time, and there may be even several cities worldwide called *Kaitaia*, i.e. this is not an acceptable way of defining a set in naive set theory. But $B = \{x : x \in \mathbb{R} \wedge x^2 = x\}$ is a precise definition in naive set theory, and we have $B \in \mathcal{S}$. This shows that $\mathcal{S}$ is not empty. Russell's set is $R = \{x : x \in \mathcal{S} \wedge x \notin x\}$, and it follows that $R \in R$ iff $R \notin R$, what contradicts elementary logics. This means that naive set theory and elementary logics leads to a contradiction, which is known as *Russell's paradox* in this form. The mathematical conclusion is that we cannot define sets in the way as assumed in naive set theory, based on the knowledge that elementary logics is free of contradictions. - G. Cantor, the founder of naive set theory, arrived already[19] at a paradox in studying power sets of sets: he showed that in his naive set theory every power set $\mathcal{P}(S)$ of a set S is of larger cardinality than set S. Thinking about the family $\mathcal{S}$ of all sets he arrived at his paradox. Explain.

3.2. Define a set of 4×4 templates for defining flip-flop switches (see Fig. 41) and discuss your motivation.

3.3. Show that $(6, 6)$ is a good pair for $\mathbb{Z}^2$.

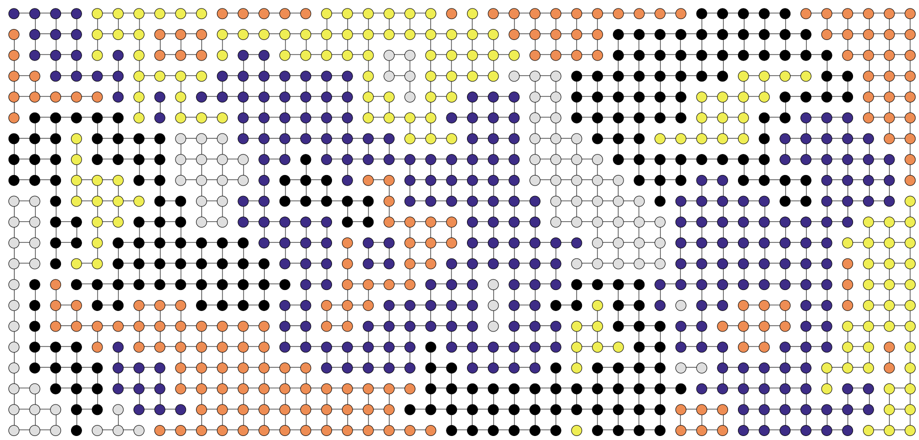
3.4. Assume that the chessboard pattern from Exercise 1.5 (ii) has been digitized into an 8×8 binary image: What are the resulting components assuming



good pairs (4,8), (8,4) or (6,6)? How do these results compare to your discussion of Exercise 1.5 (ii)? Can you specify a switch state matrix **S** such that all 'dark pixels' in the upper 4 rows are **S**-connected as well as all 'white pixels' in the lower four rows?

[19] In 1899 or between 1895 and 1897.

3.5. The deletion of all invalid edges in the image shown on the title page leads to a non-connected undirected graph. Consider that each grid point is the



centroid of an isothetic square of side length 1, and take the union of all squares of grid points being in the same 4-component. The frontiers of these resulting sets are isothetic simple curves, and all these curves define an elementary curve. What is the connectivity β_0 of this elementary curve?

3.6. Let $\mathbb{C}_G$ be the subcomplex of $\mathbb{C}_{G2}$ containing all grid points in $\mathbb{Z}^2$ and all subgraphs of the homogeneous orthogonal grid defined by grid edges, but no subgraph of any grid square, i.e. we only keep all the 2- and 1-cells of $\mathbb{C}_{G2}$. Now consider the smallest neighborhoods $U(p)$, $p \in \mathbb{C}_G$, in this subcomplex and the adjacency relation A_U induced by this neighborhood relation in $[\mathbb{C}_G, \leq_{G2}, dim]$. Define A_4 using A_U.

3.7. Consider subcomplex $[\mathbb{C}_{m,n}, \leq_{E2}, dim]$ of Euclidean complex $[\mathbb{C}_{E2}, \leq_{E2}, dim]$ where $\mathbb{C}_{m,n}$ contains all grid points (0-cells) within a rectangle of dimension $m \times n$, and all grid edges (1-cells) and all grid squares (2-cells) having their vertices in this rectangle. Specify the closure $G(g)^\bullet$ in the poset topology of $[\mathbb{C}_{m,n}, \leq_{E2}, dim]$, for the grid point set $G(g)$ of 4-arcs g or 8-arcs g. Specify the closure of the set (union of grid point sets of both paths) as shown in Fig. 34.

3.8. Consider the Aleksandrov topology of the poset $[\mathbb{R}, \leq]$. Identify a topology (discussed in this section) which is homeomorphic to this topology.

3.9. Discuss Definition 16 (dimension) for the Khalimsky-Kovalevski plane assuming that this model is mapped onto the homogeneous orthogonal grid of given image data as follows: identify all pixel positions with 2-cells, and assume that all 0- or 1-cells are virtual cells in-between.

3.10. Consider the neighborhood graph in Fig. 3 (see also Exercise 1.7) which induces an adjacency A on $\mathbb{Z}^2$. Is $[\mathbb{Z}^2, A]$ a dense graph, i.e. does it possess an equivalent topology such that both connectedness definitions (graph-theoretical and topological) coincide?

4 Adjacency Graphs

In this final section of part I of this report we focus on planar adjacency graphs as models for (not necessarily topological) two-dimensional homogeneous or inhomogeneous image carriers. We discussed 4-, 6- and **S**-adjacencies as examples, and the graphs shown in Fig. 48, left (Khalimsky plane), and Fig. 2 are planar as well. Graph-theoretical models or induced adjacencies of cellular models may be represented by adjacency graphs.

Assume a graph-theoretical model of an image carrier. Let us delete all invalid adjacencies in the adjacency graph of an image. The situation resembles the 'white and black marbles example' as described in [12] illustrating that the *(joint) boundary* of two adjacent marbles is actually formed by different frontiers being next to one another. We used the word 'boundary' for the first time in this report, and it was reserved for 'something between' two sets. Figure 51 illustrates such a 'double' frontier situation. These isothetic frontiers correspond to a double count (in two directions) of oriented frontiers of unions of 2-cells. Such frontiers are one-dimensional complexes in the orthogonal grid model. Adjacency graphs and related frontier or border definitions have been studied in a sequence of papers, starting with [84] and ending with [38], completely documented in the book [87], briefly reviewed in the paper [39] and cited at some places in the book [88].

This section discusses this *theory of adjacency graphs* which generalizes previously used adjacency definitions. Throughout this section we assume that $[\mathbb{C}, N]$ is a not-necessarily topological image carrier, so far specified by axioms *C1* and *C2*, and we will not make use of axiom *C3* in this section.[20] Due to axiom *C1* we have that all sets $A(p)$ are of finite cardinality, and due to axiom *C2* we deal with algebraically connected image carriers.

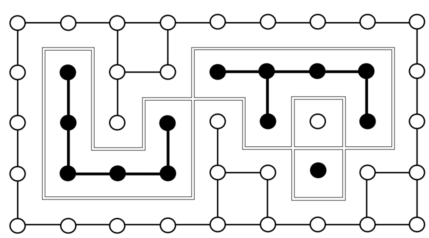


Fig. 51. Frontiers of all components forming parallel line segments of isothetic polygons; see Clifford's white-and-black marble analogy.

[20] Axiom *C3* does not imply planarity: for example, the homogeneous orthogonal grid in three-dimensional space with its so-called 6-adjacency of grid points satisfies axioms *C1*, *C2* and *C3*.

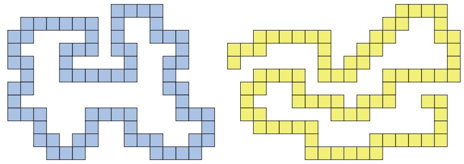


Fig. 52. Sequences of grid squares as discussed in [73], where they have been called *grid continua* resembling the continua definition in Section 3.

4.1 Adjacency Graphs and Basic Morphology

We review [36] (in the context of the previous sections) and start with considering a finite or denumerable set $\mathbb{C}$ of *points* and an irreflexive and symmetric *adjacency relation* $A \subseteq \mathbb{C} \times \mathbb{C}$ which can be represented by an undirected graph $[\mathbb{C}, A]$: points $p \in \mathbb{C}$ specify vertices of the graph, and pairs $\{p, q\} \in A$, with $p \neq q$, define an undirected edge between p and q in this graph, which is called the *adjacency graph* $\mathcal{G}_A = [\mathbb{C}, A]$. Let $A(p) = \{q : q \in \mathbb{C} \wedge \{p, q\} \in A\}$ be the set of adjacent points of $p \in \mathbb{C}$, not containing p itself. Let N be the symmetric neighborhood relation generated by A on $\mathbb{C}$, i.e. $N(p) = \{p\} \cup A(p)$.

We show that this very generic adjacency model already allows to start with analyzing *morphological operations* (i.e. operations concerning the *mathematical shape* of a set of vertices). No embedding into the Euclidean space, and no planarity constraints are needed for doing so. Typically we may imagine that a point p in $\mathbb{C}$ stands for a point or a vertex, positioned on a surface in a Euclidean space. Actually we do not ask for that at this moment. For example, an adjacency graph may model sequences of grid squares in the plane (see Fig. 52), where two squares p and q are *adjacent* iff their intersection is exactly one grid edge; and the induced adjacency graph of a cellular model is another example.

Definition 27. *A point $p \in M \subseteq \mathbb{C}$ is (algebraically)* inner *iff $A(p) \subseteq M$, i.e. iff $N(p) \subseteq M$; and it is (algebraically)* border *otherwise. Any set $M \subseteq \mathbb{C}$ splits into two disjoint sets M^{∇} and δM of inner and border points, respectively, called* inner set *and* border *of set M.*

Set M^{∇} is defined similar to the (topological) interior M°, and set δM is defined similar to the (topological) frontier ϑM. Because of $M = M^{\nabla} \cup \delta M$ it follows that there is no sense in introducing an algebraic closure of set M. Axiom *C3* ensures that a subset M of a topological image carrier possesses border, frontier, interior and inner set.

Example 14. Consider the 4-topology of Z^2 and a set $M = \{p\}$ containing just one grid point $p = (i, j)$. In the adjacency graph $[\mathbb{Z}^2, A_4]$ we have

$$M^{\nabla} = \emptyset \quad \text{and} \quad \delta M = \{p\} .$$

Let p be even, i.e. with $i + j$ even. In the 4-topology it follows that

$$M^{\circ} = \emptyset \, , \quad \text{but} \quad \vartheta M = N_4(p) .$$

In case of an odd grid point p it follows that

$$M^\circ = \{p\} \quad \text{and} \quad \vartheta M = (N_4(p_1) \cup N_4(p_2) \cup N_4(p_3) \cup N_4(p_4)) \setminus \{p\} ,$$

for $A_4(p) = \{p_1, p_2, p_3, p_4\}$. This shows that axiom *C3* is insufficient to guarantee that border coincides with frontier, or inner set with interior.

Now consider the cellular model $[\mathbb{C}^{\{2\}}, \leq^{\{2\}}, dim]$ with the poset topology, say in one of its topologically equivalent forms (i.e. graph $\mathbb{C}_{G2}$, Khalimsky plane, or Euclidean $\mathbb{C}_{E2}$, also known as Kovalevsky plane), inducing an adjacency relation $A^{\{2\}}$ and a neighborhood relation $N^{\{2\}}$ on $\mathbb{C}^{\{2\}}$. Let $p \in \mathbb{C}^{\{2\}}$ be a 2-cell and let $M = \{p\}$. In the adjacency graph $[\mathbb{C}^{\{2\}}, A^{\{2\}}]$ we have again

$$M^\nabla = \emptyset \quad \text{and} \quad \delta M = \{p\} ,$$

but $M^\circ = \{p\}$ and ϑM is the set of all proper sides of p. Again we note that algebraic and topological concepts of inner sets and interior, or border and frontier, differ. Set $M = \{p\}$ is open in the $\mathbb{C}^{\{2\}}$ topology. In case of the closed set $M^\bullet = M^\circ \cup \vartheta M$, what is the set of all sides of p, it follows that

$$(M^\bullet)^\nabla = \{p\} \quad \text{and} \quad \delta M^\bullet = \vartheta M ,$$

as well as

$$(M^\bullet)^\circ = \{p\} \quad \text{and} \quad \vartheta M^\bullet = \vartheta M .$$

The identities $A^\nabla = A^\circ$ and $\delta A = \vartheta A$ are true for any closed set $A \subseteq \mathbb{C}^{\{2\}}$ (Exercise 4.1). □

We only claim that axioms *C1* and *C2* are valid for our adjacency graphs in this section. In general it is $\delta\mathbb{C} = \emptyset$, and $\delta M \neq \emptyset$ for any non-empty proper subset M of $\mathbb{C}$ due to axiom *C2*. It is possible that a set M only contains border points, and no inner points at all. The connectedness relation Γ_M (defined following Definition 3) partitions M^∇ and δM into components, called *inner* and *border components* of M. A connected subset M with $M^\nabla = \emptyset$ consists of one border component. Figure 53 illustrates components of borders and inner sets.

Let $A(M)$ be the *set of all points adjacent to* $M \subseteq \mathbb{C}$, defined as set of all $p \in \overline{M}$ with $A(p) \cap M \neq \emptyset$. The set $A(M)$ contains exactly all border points of $\overline{M}$. We have $A(\mathbb{C}) = A(\emptyset) = \emptyset$. Any non-empty finite set $M \subseteq \mathbb{C}$ has a finite *adjacency set* $A(M)$ due to axiom *C1*.

Let $M \subseteq \mathbb{C}$. For any inner point $p \in M^\nabla$ there is at least one border point $q \in \delta M$ such that p and q are connected with respect to M. There may be sets with border points which are only connected to border points. If $A(M)$ consists of only one component, for $M \subseteq \mathbb{C}$, then $\overline{M}$ as well. It is $A(M) = \overline{M^\nabla} \cap A(\delta M)$.

The operation $\mathbf{E}$ of *erosion* transforms a set $M \subseteq \mathbb{C}$ into M^∇, and the operation $\mathbf{D}$ of *dilation* transforms M into $M \cup A(M)$.

It is $\mathbf{D}M = M^\nabla \cup \mathbf{D}(\delta M)$, and $\mathbf{D}(M \cup L) = \mathbf{D}M \cup \mathbf{D}L$. Any set $M \subseteq \mathbb{C}$ satisfies the inclusion sequence

$$\mathbf{E}M \subseteq \mathbf{DE}M \subseteq M \subseteq \mathbf{ED}M \subseteq \mathbf{D}M . \tag{5}$$

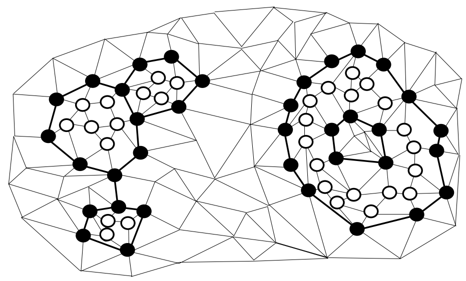


Fig. 53. Two connected sets in a graph: with one border component and three inner components on the left, and with two border components and one inner component on the right [36].

Further operations of mathematical morphology [72], such as opening and closing, can already be studied for adjacency graphs, see [36].

We cite one more axiom from this paper: infinite adjacency graphs as defined so far would allow that a finite set $M \subseteq \mathbb{C}$ may have several infinite complementary components, which is not desirable in an image analysis context. Because this cannot be excluded based on our previous axioms we state this as

Axiom C4: Any finite subset $M \subseteq \mathbb{C}$ of a (not necessarily topological) image carrier $[\mathbb{C}, N]$ possesses at most one infinite complementary component.

This axiom allows a termination criterion for computational problems such as: for a given finite set M in an adjacency graph, calculate all finite complementary components of M. Note that the number of complementary components of M is less or equal to the number of components of $A(M)$. Axioms *C1*, *C2*, *C4* together with our initial claim that $\mathbb{C}$ is finite or denumerable, define the *non-topological theory of adjacency graphs* as introduced in [36]. Axiom *C3* ensures that an adjacency graph allows an introduction of an equivalent (w.r.t. connectedness) topology, and this axiom has not been used in the non-topological theory of adjacency graphs.

Let $M \subseteq \mathbb{C}$ and $A_M(p) = A(p) \cap M$, for all $p \in M$. Then $[M, A_M]$ is the subgraph *induced by subset M*. It is an adjacency graph again if it continues to satisfy axiom *C2*, as $[\mathbb{C}, A]$ did before.

4-adjacency together with a non-empty rectangular set $\mathbb{C}$ of grid points is an example of a model of the (topological or non-topological) theory of adjacency graphs, and 8-adjacency also satisfies axioms *C1*, *C2*, *C4*, but will fail later to satisfy the planarity constraint in the definition of a mesh.

Region adjacency graphs have been discussed by many authors in the image analysis literature. For example, [37] specifies hierarchies of region adjacency graphs within the theory of adjacency graphs. In Section 1 we defined a region as a component of an I-equivalence class of grid points, for a given image I, i.e. a region is a non-empty connected subset of an image carrier. The neighborhood relation N of an image carrier $[\mathbb{C}, N]$ induces an adjacency relation A_N which defines adjacencies $A_N(M) \subseteq \mathbb{C}$, for $M \subseteq \mathbb{C}$.

Definition 28. *Two disjoint subsets M_1, M_2 of an image carrier $[\mathbb{C}, N]$ are* adjacent *(we write $M_1 \mathcal{A}_N M_2$ or $(M_1, M_2) \in \mathcal{A}_N$) iff $A_N(M_1) \cap M_2 \neq \emptyset$.*

Obviously, due to the symmetry of relation A_N we have $A_N(M_1) \cap M_2 \neq \emptyset$ iff $A_N(M_2) \cap M_1 \neq \emptyset$, i.e. the relation $\mathcal{A}_N$ is symmetric. Due to the claimed disjointness of M_1 and M_2 it also follows that $\mathcal{A}_N$ is irreflexive, i.e. an adjacency relation on the powerset of $\mathbb{C}$. Let $\mathcal{R}$ be a *partition* of $\mathbb{C}$ (i.e. all sets in $\mathcal{R}$ are pairwise disjoint, non-empty, and their union is $\mathbb{C}$). It follows that $[\mathcal{R}, \mathcal{A}_N]$ satisfies axioms *C2* and *C4* again. Regarding axiom *C1* we may claim that $\mathcal{R}$ is such that any set in $\mathcal{R}$ is only adjacent to a finite number of sets in $\mathcal{R}$. Regarding axiom *C3* we may introduce a so-called quotient topology on $\mathcal{R}$ specified by the topology on $\mathbb{C}$. The resulting algebraic structure $[\mathcal{R}, \mathcal{A}_N]$ is an *adjacency graph of second order*, and the original graph $[\mathbb{C}, A_N]$ was of first order. By continuing this process [37] we may arrive at adjacency graphs of nth order, forming an irregular *image pyramid* where level n is occupied by the adjacency graph of nth order. However, we will not further discuss this subject in this topology report; hierarchical image data structures are a separate topic.

4.2 Oriented Adjacency Graphs

Oriented manifolds (one-dimensional complexes or graphs, or higher-dimensional complexes) have been studied in combinatorial topology since [63] (see Section 2). Finite adjacency graphs may be drawn as elementary curves. They are one-dimensional complexes (as any undirected graph), and any one-dimensional complex is orientable. We discuss orientations for undirected graphs where also *border cycles* may be specified, and these orientations can be defined for finite or infinite adjacency graphs generalizing the concept of local orders of adjacency sets which is well-known in image analysis since H. Freeman's chain codes [20].

The typical approach for studying oriented manifolds is characterized by specifying circuits in triangulations or tilings, see Section 2. An alternative ap-

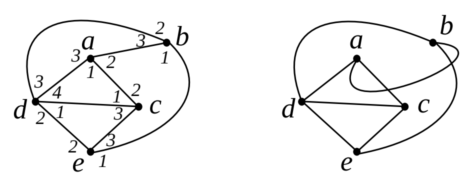


Fig. 54. Two representations of the same oriented graph: cyclic numberings of point neighborhoods on the left, and the clockwise drawing of outgoing edges on the right [85].

proach has been suggested in [85] by starting with cyclic orderings of adjacency sets, which allows to deduce oriented circuits as results of such cyclic orderings.

We review [85] in this subsection. This paper introduced a general study of cyclic orderings for adjacency sets $A(p)$, which introduce an orientation on the given directed graph.

Definition 29. (K. Voss, R. Klette, 1986) *Let $p \in \mathbb{C}$ be an arbitrary vertex of an undirected graph where every vertex is of finite degree. Let $\xi(p) = \langle q_1, \ldots, q_n \rangle$ be a cyclic order of all points in the adjacency set $A(p)$, called* adjacency cycle*. An* oriented graph *$\mathcal{G}_\xi = [\mathbb{C}, A, \xi]$ is defined by an undirected graph $\mathcal{G}_A$ with vertex set $\mathbb{C}$ (where every vertex is of finite degree) and adjacency cycles $\xi(p)$ for all of its vertices $p \in \mathbb{C}$.*

Figure 54 shows an oriented graph $[\{a, b, c, d, e\}, A, \xi]$, with $\xi(a) = \langle c, b, d \rangle$, $\xi(b) = \langle e, d, a \rangle$, $\xi(c) = \langle d, a, e \rangle$, $\xi(d) = \langle c, e, b, a \rangle$, and $\xi(e) = \langle b, d, c \rangle$. In the sequel we will use the clockwise drawing of outgoing edges as used on the right of Fig. 54. Of course, crossings of edges may be unavoidable for this type of representation even for planar graphs. The drawings in Fig. 31 (Listing band) show two oriented graphs: on the left $[\{a, b, c, d, p, q\}, A, \xi]$, with $\xi(a) = \langle p, b \rangle$, $\xi(b) = \langle a, q \rangle$, $\xi(c) = \langle d, q \rangle$, $\xi(d) = \langle c, p \rangle$, $\xi(p) = \langle d, q, a \rangle$, and $\xi(q) = \langle p, c, b \rangle$, and on the right (the Listing graph) $[\{a, b, p, q\}, A, \xi]$, with $\xi(a) = \langle b, p, q \rangle$, $\xi(b) = \langle p, a, q \rangle$, $\xi(p) = \langle b, q, a \rangle$, and $\xi(q) = \langle a, b, p \rangle$.

Cyclic orders may be assigned to any adjacency graph due to the finiteness claim in axiom *C1*. 'Classical' combinatorial topology (see Section 2) introduced orientations on surfaces based on oriented triangulations or tilings, where every orientation of a single ('first' or initial) triangle or face specifies already the orientation of the whole triangulation or tiling, i.e. there are only two different options of orientations. The concept of oriented graphs generalizes this: now we can define a larger diversity of orientations depending on 'local choices' of cyclic orders of adjacency sets. We specify a substructure of an oriented adjacency graph *induced by a subset M of $\mathbb{C}$*:

Definition 30. (K. Voss, R. Klette, 1986) *Let $[\mathbb{C}, A, \xi]$ be an oriented graph, $M \subseteq \mathbb{C}$ and let $\xi_M(p) = \langle r_1, \ldots, r_m \rangle$ be a cyclic order of all points in $A(p) \cap M$ being a subsequence of the cyclic order $\xi(p)$, for $p \in M$. Then ξ_M is a cyclic order on the subgraph $[M, A_M]$ induced by M, and $\mathcal{G}_{\xi_M} = [M, A_M, \xi_M]$ is a* substructure*. We call $\xi_M(p)$ a* reduced adjacency cycle*.*

Note that the subgraph $[M, A_M]$ does not need to be connected, i.e. it is not an adjacency graph if disconnected. Disconnected sets of object points as in Fig. 37 may be described as being substructures. If M is a finite connected subset in the oriented adjacency graph $\mathcal{G}_\xi = [\mathbb{C}, A, \xi]$ then the induced substructure $\mathcal{G}_{\xi_M} = [M, A_M, \xi_M]$ is also an oriented adjacency graph.

Assume an oriented graph $[\mathbb{C}, A, \xi]$, a set $M \subseteq \mathbb{C}$ with points $p, q, r \in M$, and $q, r \in A_M(p)$. The directed edge (q, p) is called *predecessor* of the directed edge (p, r) with respect to M iff q is the cyclic predecessor of r in the reduced adjacency cycle $\xi_M(p)$. Analogously, the directed edge (p, r) is called *successor*

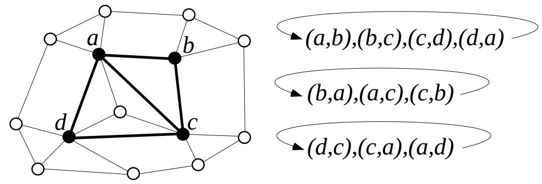


Fig. 55. All circuits of directed edges in an oriented graph [85].

of the directed edge (q,p) with respect to M iff r is the cyclic successor of q in $\xi_M(p)$.

Figure 55 follows our convention to illustrate adjacency cycles by clockwise drawings of outgoing edges. The subset $M = \{a,b,c,d\}$ defines reduced adjacency cycles $\xi_M(a) = \langle b,c,d\rangle$, $\xi_M(b) = \langle a,c\rangle$, $\xi_M(c) = \langle b,d,a\rangle$, and $\xi_M(d) = \langle a,c\rangle$. Let us start with the directed edge (a,b) which is the predecessor of (b,c) according to $\xi_M(b)$. Edge (b,c) is predecessor of (c,d) because b is predecessor of d in $\xi_M(c)$, etc. This specifies an *oriented circuit* of directed 1-cells (i.e. edges) $\langle (a,b),(b,c),(c,d),(d,a)\rangle$, specified by one of its shortest periodic subsequences (which only differ by cyclic permutation). The shown oriented graph has three oriented circuits containing all of its directed edges. [21] A *shorter notation* for these oriented circuits is by the cycle of points instead of the cycle of edges. The given example has cyclic point sequences $\langle a,b,c,d\rangle, \langle a,c,b\rangle, \langle a,d,c\rangle$.

Every directed edge (p,q), with $p,q \in M$, produces an infinite sequence of directed edges or points in an oriented graph, just by proceeding, step by step, to the next successor. Such a sequence is either an *edge sequence generated by* (p,q), or a *path generated by* (p,q), depending upon whether directed edges or vertices are used to specify the sequence.

If M is finite then any directed edge (p,q) generates a periodic (infinite) oriented circuit u^ω of directed edges, and a (finite) shortest periodic subsequence u is used to represent such a circuit. In case of an infinite set $\mathbb{C}$ of points, depending upon the cyclic order defined on adjacency sets of points $p \in \mathbb{C}$, there may be, or may be not acyclic infinite oriented circuits of directed edges generated by an initial directed edge. An infinite oriented circuit of directed edges generated by an directed edge cannot contain any directed edge twice.

Figure 56 illustrates two different cyclic orders of adjacency sets for $[\mathbb{Z}^2, A_4]$, where A_4 denotes 4-adjacency. The cyclic order shown on the left produces only acyclic infinite circuits of directed edges, and the cyclic order shown on the right produces only periodic circuits having shortest periodic subsequences of length four. For oriented image carriers we exclude graphs with aperiodic infinite circuits (such as the one on the left in Fig. 56):

[21] Generalizations to oriented n-circuits will be discussed in part II of the report (with a focus on $n = 2$).

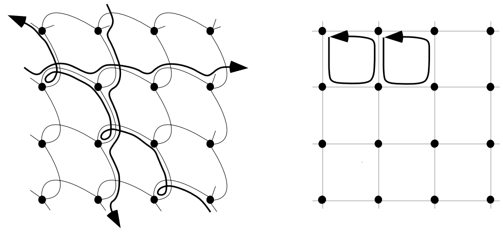


Fig. 56. Two different schemes of oriented circuits for the orthogonal planar grid [85].

> *Axiom C5:* Any directed edge in a (not necessarily topological) oriented image carrier generates a periodic sequence of directed edges.

This axiom specifies a constraint for orientations on adjacency graphs of image carriers. The infinite orthogonal grid, i.e. the 4-adjacency graph on all grid points in $\mathbb{Z}^2$, satisfies axioms *C1*, *C2*, *C3* and *C4*, but Fig. 56 shows that only specific cyclic orders may be chosen according to axiom *C5* for defining an oriented adjacency graph $[\mathbb{Z}^2, A_4, \xi]$.

Definition 31. (K. Voss and R. Klette 1986) *A* cycle *g is a shortest periodic subsequence $\langle p_1, \ldots, p_n \rangle$ of a periodic oriented circuit, generated by an directed edge (p_1, p_2) in an oriented graph $\mathcal{G}_\xi$. The* length $\lambda(g)$ of the cycle *g is n. We say that cycle g* is generated by (p_1, p_2) in $\mathcal{G}_\xi$.

Cycles $g_1 = \langle p_1, \ldots, p_n \rangle$ and $g_2 = \langle q_1, \ldots, q_n \rangle$ are identified with one another iff they only differ by a cyclic permutation. The oriented graph in Fig. 55 possesses three cycles.

4.3 Combinatorial Maps

Oriented adjacency graphs have been proposed and studied in the 1980's independently of an earlier development in graph theory: the theory of *combinatorial maps*, initiated by L. Heffter [25] who published in 1891 his results on *maps*. For defining these maps, first recall that an undirected graph $[\mathbb{C}, A]$ allows different ways of graphical representation, and it is called *planar* iff it allows such a representation in the plane (or on a sphere) that edges only intersect at their vertices. An *embedding* is a representation of an undirected graph $[\mathbb{C}, A]$ on a closed compact surface iff no two edges intersect except at their vertices. Such an embedding is characterized by local circular orders ξ of the edges around the vertices, and a *map* is such a graph together with all of its local circular orders, i.e. it is an oriented adjacency graph $[\mathbb{C}, A, \xi]$.

L. Heffter introduced maps and proved a dual characterization theorem for them. Maps have been used, e.g., by G. Ringel (starting in the 1950s), see [65],

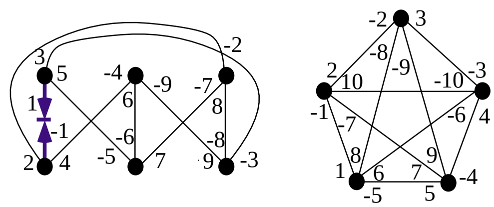


Fig. 57. Combinatorial maps for undirected graphs $K_{3,3}$ and K_5 (see Fig. 58).

and were reinvented in [17]. Finally they became popular in discrete mathematics based on publications such as [13, 24, 29, 30, 78, 79, 90].

Combinatorial studies of maps are of relevance for modelling images and segmented images, see, e.g. [15, 49]. A *planar map* is a map on a Jordan surface, i.e. it is an orientable map of Euler characteristic 2. Of course, such a tiling is an option for a finite representation of a closed compact surface in three-dimensional space.

A *combinatorial map* is defined by a finite undirected graph $[\mathbb{C}, A]$ and two permutations α and σ: first assume that all undirected edges are numbered by $1, 2, ..., m$, and we split each undirected edge i of the graph into two directed edges $+i$ and $-i$, also (sometimes) called *darts* in the theory of directed graphs or combinatorial maps. The permutation α is just a mapping of all darts $+i$ into $-i$, and $-i$ into $+i$, i.e. α is defined by cycles of length 2, and we can write

$$\alpha = \begin{pmatrix} 1 & 2 & \cdots & m & -1 & -2 & \cdots & -m \\ -1 & -2 & \cdots & -m & 1 & 2 & \cdots & m \end{pmatrix}$$
$$= (1, -1)(2, -2) \cdots (m, -m) \ .$$

The permutation σ combines all adjacency cycles $\xi(p)$. Assume we have $\mathbb{C} = \{p_1, p_2, \cdots, p_n\}$. Then it follows that

$$\sigma = \xi(p_{i_1})\xi(p_{i_2}) \cdots \xi(p_{i_n}) \ ,$$

for any of the possible $n!$ permutations

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}$$

of all n vertices.

See Fig. 57 for two examples of graphs in dart representation. Assume anti-clockwise adjacency cycles $\xi(p)$ for the graph $K_{3,3}$ on the left. We obtain[22]

$$\alpha = (1, -1)(2, -2)(3, -3)(4, -4)(5, -5)(6, -6)(7, -7)(8, -8)(9, -9) \ ,$$
$$\sigma = (5, 3, 1)(-4, 6, -9)(-7, 8, -2)(-1, 2, 4)(-6, -5, 7)(-8, 9, -3) \ , \text{ and}$$
$$\phi = \sigma \circ \alpha = (1, 2, -7, -6, -9, -3)(-1, 5, 7, 8, 9, -4)(-2, 4, 6, -5, 3, -8) \ .$$

[22] As a quick reminder for product calculations of permutations: in this example, 1 goes into -1 in α, -1 goes into 2 in σ (we have: 1 goes into 2 in $\sigma \circ \alpha$), 2 goes into -2 in α, -2 goes into -7 in σ (we have: 2 goes into -7 in $\sigma \circ \alpha$), etc.

Note that the cycles in the product $\phi = \sigma \circ \alpha$ are exactly all circuits in the oriented adjacency graph. As another example, assume clockwise adjacency cycles $\xi(p)$ for the graph K_5 on the right. We obtain

$$\alpha = (1,-1)(2,-2)(3,-3)(4,-4)(5,-5)(6,-6)(7,-7)(8,-8)(9,-9)(10,-10)\,,$$
$$\sigma = (2,10,-7,1)(-2,3,-9,-8)(-3,4,-6,-10)(9,-4,5,7)(8,6,-5,1)\,, \text{ and}$$
$$\phi = \sigma \circ \alpha = (-2,10,-3,-9,-4,-6,-5,7,-1,8)(1,2,3,4,5)(6,-10,-7,9,-8)\,,$$

and ϕ lists all three circuits of this oriented adjacency graph. Based on the given definitions of permutations α and σ, this is valid for any product $\phi = \sigma \circ \alpha$, for the case of counter-clockwise as well as for the case of clockwise orientations, as long as we start with a finite connected undirected graph $[\mathbb{C}, A]$, split its undirected edges into directed edges via α, and define adjacency cycles via σ all with respect to the same orientation, either counter-clockwise or clockwise. Of course, the handling of such global permutations of all edges or all adjacency sets becomes unefficient when dealing with high-resolution digital images.

As a matter of fact, oriented adjacency graphs may be seen as another reinvention of combinatorial maps, but results of this theory, which will be briefly sketched in the following sections, will demonstrate that the development of this theory in the context of image analysis had impacts on specific intentions and directions, which do not have analogies in the theory of combinatorial maps.

4.4 Euler Characteristic and Separation Theorem

Any finite oriented adjacency graph $\mathcal{G}_\xi = [\mathbb{C}, A, \xi]$ with $\alpha_0 = card(\mathbb{C}) \geq 0$ (or a substructure of $\mathcal{G}_\xi$) possesses a finite number α_2 of cycles.

Let α_1 be the cardinality of undirected edges (any of these edges represents a symmetric adjacency pair). We have $card(A) = 2\alpha_1$. To be more detailed, let $\nu(p) = card(A(p))$, then

$$\sum_{p \in \mathbb{C}} \nu(p) = 2\alpha_1 \tag{6}$$

as well as

$$\sum_{g} \lambda(g) = 2\alpha_1 \tag{7}$$

where the sum is over all cycles of $\mathcal{G}_\xi$. Because of axiom *C2* it follows that there are at least $\alpha_0 - 1$ edges, i.e. $\alpha_1 \geq \alpha_0 - 1$. On the other hand, every vertex may only be connected to any other vertex by one undirected edge at most, i.e. $\alpha_1 \leq \alpha_0(\alpha_0 - 1)/2$. If $\alpha_0 > 1$ then there is at least one cycle. However, the number of cycles will vary with different choices of cyclic orders.

Following the general concept in topology (see Section 2) we define the *Euler characteristic* of $\mathcal{G}_\xi$ to be $\chi = \alpha_0 - \alpha_1 + \alpha_2$. In [85] it has been shown that

Theorem 12. (K. Voss and R. Klette, 1986) *We have* $\chi = \alpha_0 - \alpha_1 + \alpha_2 \leq 2$, *for any finite oriented adjacency graph* $\mathcal{G}_\xi$.

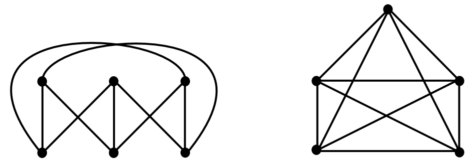


Fig. 58. Graphs $K_{3,3}$ (left) and K_5 (right).

Axiom *C3* is not needed for the proof of this theorem in the non-topological theory of adjacency graphs, which specifies an upper bound for the number of cycles.

A subset $M \subseteq \mathbb{C}$ defines a substructure $\mathcal{G}_{\xi_M} = [M, A_M, \xi_M]$ of $\mathcal{G}_\xi$. Let $\alpha_0^M = card(M)$, $\alpha_1^M = card(A_M)/2$ be the number of undirected edges of the substructure, and α_2^M is its number of cycles. Then the Euler characteristic of the substructure, $\chi_M = \alpha_0^M - \alpha_1^M + \alpha_2^M$, is greater or equal to the Euler characteristic χ of $\mathcal{G}_\xi$. Let τ be the number of components of M in $\mathcal{G}_\xi$. Then it follows that $\chi_M \leq 2\tau$, which defines an upper bound for the Euler characteristic of the substructure.

Definition 32. *A finite oriented graph with Euler characteristic*

$$\chi = \alpha_0 - \alpha_1 + \alpha_2 = 2 \tag{8}$$

is called planar. *An infinite oriented graph is* planar *iff any non-empty finite connected oriented subgraph is planar.*

This definition is consistent with graph theory and combinatorial topology. It follows that substructures of planar finite oriented adjacency graphs consist of components where each defines a planar finite oriented adjacency graph again.

Example 15. (C. Kuratowski, 1930) An undirected graph is planar iff it does not have either $K_{3,3}$ nor K_5 (see Fig. 58) as a homeomorphic subgraph [48]. Note that planarity of graphs is also defined for infinite graphs. □

If cyclic orders are defined for all adjacency sets of $K_{3,3}$ or K_5 then the resulting finite oriented graph has always an Eulerian characteristic of less than 2 (Exercise 4.2).

It follows that any finite undirected graph having $K_{3,3}$ or K_5 as a homeomorphic subgraph has a Euler characteristic less than 2, for any cyclic order defined on it. On the other hand, a planar representation of a finite planar graph specifies a cyclic order with $\chi = 2$. Altogether we have that a finite graph is planar iff there exists at least one cyclic order on it such that the Euler characteristic is two.

Assume a finite subset $M \subseteq \mathbb{C}$. Cycles of the induced substructure $\mathcal{G}_{\xi_M}$ may differ from cycles of the oriented graph $\mathcal{G}_\xi$. Let (p, q) be a directed edge in the induced substructure $\mathcal{G}_{\xi_M}$, let g_1 the cycle generated by (p, q) in $\mathcal{G}_{\xi_M}$, and g_2 the cycle generated by (p, q) in $\mathcal{G}_\xi$.

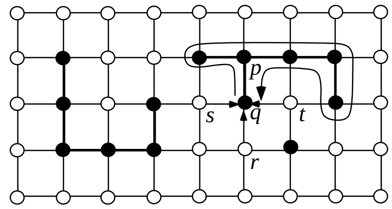


Fig. 59. The directed invalid edges (r, q), (s, q) and (t, q) initiate the same border cycle generated by (q, p) in the substructure defined by filled dots.

Definition 33. (K. Voss and R. Klette 1986) *g_1 is an* original cycle *iff $g_1 = g_2$, and a* border cycle *otherwise.*

The subset $M = \{a, b, c, d\}$ shown in Fig. 55 induces an oriented adjacency graph with one original cycle $\langle b, a, c\rangle$ and two border cycles $\langle a, b, c, d, e\rangle$ and $\langle d, c, a\rangle$. The substructure shown in Fig. 59 does not have any original cycle.

Assume a finite set $M \subseteq \mathbb{C}$. Every border cycle of the induced substructure $\mathcal{G}_{\xi_M}$ contains at least one border point of M, and any border point in δM is incident with at least one border cycle. If M is a proper subset of $\mathbb{C}$ then $\mathcal{G}_{\xi_M}$ possesses at least one border cycle.

Let (r, q) be a directed edge in $[\mathbb{C}, A]$, $M \subseteq \mathbb{C}$, $q \in \delta M$ and $r \in \overline{M}$. We call (r, q) a *directed invalid edge from $\overline{M}$ to M*. A directed invalid edge is one direction of an *undirected invalid edge between $\overline{M}$ and M*.[23]

A directed invalid edge (r, q) *initiates a cycle* in the induced substructure $\mathcal{G}_{\xi_M}$ by starting with that directed edge (q, p) such that p is the first point in M contained in the (original) adjacency cycle $\xi(q)$ following r. See Fig. 59 for an example: for the directed invalid edges (r, q), (s, q) and (t, q), point p is the next (and only one in $\xi_M(q) = \langle p\rangle$) point in M and $\xi(q) = \langle p, t, r, s\rangle$ to be used for generating the shown border cycle. Every directed invalid edge initiates uniquely one border cycle in the substructure $\mathcal{G}_{\xi_M}$. This defines a partition of all (directed or undirected) invalid edges into equivalence classes such that one class is the set of all (directed or undirected) invalid edges *assigned* to one border cycle.

Theorem 13. (K. Voss and R. Klette 1986) *Let $\mathcal{G}_\xi = [\mathbb{C}, A, \xi]$ be a (finite or infinite) planar oriented adjacency graph and M a non-empty finite connected proper subset of $\mathbb{C}$. Then $\mathcal{G}_\xi$ splits into at least two non-connected substructures after deleting all undirected invalid edges assigned to one of the border cycles of M.*

Axiom *C3* is not necessary for proving this separation theorem of the non-topological theory of adjacency graphs, which is again a theorem which may be compared with the Jordan-Veblen curve theorem (see Theorem 2 in Section 2) in the Euclidean plane: let (r, q) be a directed invalid edge from $\overline{M}$ to the connected set M, and $g = \langle q_1, \ldots, q_n\rangle$ be the border cycle in $\mathcal{G}_{\xi_M}$ initiated by (r, q). It

[23] This definition and the related comments differ slightly from the original paper [85].

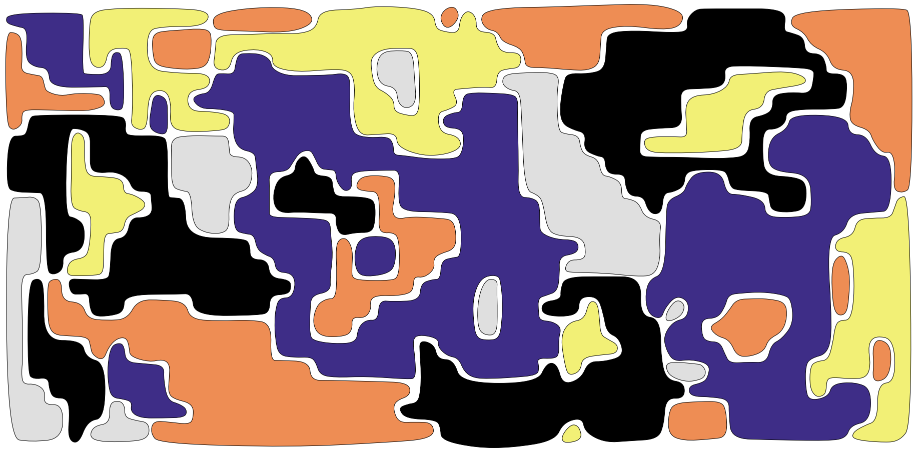


Fig. 60. All border cycles for the image shown on the title page, also used in Exercise 3.5.

follows that for any point $p \in M$ and any path $g = [p, \ldots, r]$ we have $G(g) \cap \{q_1, q_2, \ldots, q_n\} \neq \emptyset$. In other words: border cycles realize separations in planar cyclic adjacency graphs (which are the 'symmetric representations' of image carriers). Exercise 3.10 illustrates the deletion of invalid edges for an image example using 4-adjacency (the original image is on the title page). Figure 60 shows all the border cycles for this example.

The importance of Equations (6), (7) and (8) for the study of planar cyclic neighborhood structures has been studied in [86]. For a more general context see [39].

Definition 34. *A* mesh *is a planar oriented adjacency graph satisfying axioms C1, C2, C4 and C5.*

4-, 6- and **S**-adjacency on $\mathbb{Z}^2$ define meshes just by using their graphical (planar) representations for specifying adjacency cycles, called *4-mesh*, *6-mesh* and **S**-*mesh*. Theorem 13 (note: axiom *C3* has not been used in the proof) says that a mesh is a possible, not necessarily topological image carrier for digital images allowing separations based on 'tracing' of border cycles.

4.5 Meshes

We review a few basic formulas on homogeneous meshes from [86] in this subsection; see also [38, 87, 88]. We reconsider Equations 6 and 7. Let $\mathcal{G}_\xi = [\mathbb{C}, A, \xi]$ be a finite oriented graph with $\alpha_0 = card(\mathbb{C})$ and $\alpha_1 = card(A)/2$. Let

$$\overline{\nu} = \frac{1}{\alpha_0} \sum_{p \in \mathbb{C}} \nu(p) \quad \text{and} \quad \overline{\lambda} = \frac{1}{\alpha_0} \sum_{g} \lambda(g)$$

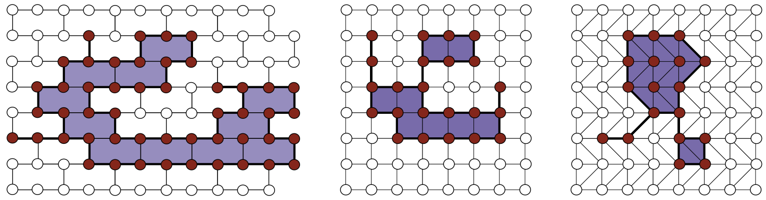


Fig. 61. Figure 4 in [86]: $\nu = 3$, $\lambda = 6$, $\alpha_0 = 49$, $\alpha_1 = 59$, $\alpha_2 = 12$, $l = 52$, $k = 29$ and $f = 11$ (left), $\nu = 4$, $\lambda = 4$, $\alpha_0 = 23$, $\alpha_1 = 30$, $\alpha_2 = 9$, $l = 28$, $k = 32$ and $f = 8$ (middle), and $\nu = 6$, $\lambda = 3$, $\alpha_0 = 18$, $\alpha_1 = 32$, $\alpha_2 = 16$, $l = 19$, $k = 44$ and $f = 15$ (right).

be the mean outdegree of a vertex and the mean length of a cycle, respectively, where the second sum is over all cycles of $\mathcal{G}_\xi$. It follows that

$$\alpha_0/\alpha_1 = 2/\overline{\nu} \quad \text{and} \quad \alpha_2/\alpha_1 = 2/\overline{\lambda} \, ,$$

which results into

$$2/\overline{\nu} \ + \ 2/\overline{\lambda} = 1 \ + \ 2/\alpha_1 \; . \tag{9}$$

This equation is also applicable to infinite oriented graphs if both means are well-defined: 2 α_1 goes to zero for infinite graphs.

Definition 35. *A mesh is* homogeneous *iff* $\nu = \nu(p)$ *is constant, for any* $p \in \mathbb{C}$, *and* $\lambda = \lambda(g)$ *is constant, for any (original or border) cycle of the mesh.*

A homogeneous mesh $\mathbb{C}_{\nu,\lambda}$ specifies a homogeneous tiling (of some surface, see Section 2), and vice-versa. Finite homogeneous tilings were the subject of Exercises 2.12 (on the surface of a sphere), 2.13 and 2.14 (on the Euclidean plane), 2.15 (on the surface of a torus), and 2.16 (on a closed surface). For infinite graphs (as in the Euclidean plane) it results that Equation (9) only allows three integer-valued solutions: $\nu = \lambda = 4$, $\nu = 3$ and $\lambda = 6$, and $\nu = 6$ and $\lambda = 3$. Obviously we may assume that $\mathbb{C} = \mathbb{Z}^2$ for these three infinite homogeneous meshes.

Now assume a finite subgraph in such an infinite homogeneous mesh defined by a finite subset $M \subseteq \mathbb{Z}^2$. First let us consider the case that M has only one border cycle of length l. We call such a set M *simply connected* following Section 2.

Let k be the number of all invalid edges from $\overline{M}$ to M. The number of original cycles is denoted by f. See Figure 61 for an example. Due to the planarity of the mesh and Equations (6) and (7) we obtain in this case:

$$\alpha_0 - \alpha_1 + \alpha_2 = 2 \tag{10}$$

$$\nu\alpha_0 - k = 2\alpha_1 \tag{11}$$

$$\lambda(\alpha_2 - 1) + l = 2\alpha_1 \tag{12}$$

and altogether

$$\nu l - \lambda k + \nu\lambda = (2\nu + 2\lambda - \nu\lambda)\alpha_1 \; . \tag{13}$$

Considering ν and λ as being constants of the given homogeneous mesh it follows that the relationship between l and k is already defined by the number α_1 of undirected edges in the given subgraph defined by the simply connected set M. In case of the three infinite homogeneous meshes it follows that $2\nu+2\lambda-\nu\lambda=0$:

Theorem 14. (K. Voss, 1986) *For a simply connected set M of an infinite homogeneous mesh $\mathbb{C}_{\nu,\lambda}$ it follows that $k=\nu+\nu l/\lambda$.*

In other words, the relation between invalid edges and length of the border cycle only depends on the parameters ν and λ of the infinite homogeneous mesh.

Now assume that M possesses r border cycles, similar to the simple surfaces with r contours discussed in Section 2. Instead of Equations (10-12) in case of a simply connected set we obtain

$$\alpha_0-\alpha_1+\alpha_2 \;=\; 2 \tag{14}$$

$$\nu\alpha_0-\sum_{i=1}^{r}k_i \;=\; \nu\alpha_0-K \;=\; 2\alpha_1 \tag{15}$$

$$\lambda(\alpha_2-r)+\sum_{i=1}^{r}l_i \;=\; \lambda(\alpha_2-r)+L \;=\; 2\alpha_1 \tag{16}$$

for length l_i of border cycle i and k_i invalid edges connecting $\overline{M}$ with points on this border cycle, for $1\le i\le r$. In case of the three infinite homogeneous meshes $(2\nu+2\lambda-\nu\lambda=0)$ it follows that

$$\nu\sum_{i=1}^{r}l_i-\lambda\sum_{i=1}^{r}k_i-(r-2)\nu\lambda \;=\; \nu L-\lambda K-(r-2)\nu\lambda \;=\; 0\,. \tag{17}$$

Theorem 13 implies that any border cycle of the connected set M separates M from at least one complementary component of M in a (not necessarily homogeneous) mesh $\mathbb{C}$. Furthermore [86]:

(i) different border cycles of M separate M from different complementary components of M in a mesh $\mathbb{C}$ (and one border cycle of M may separate M from more than just one complementary component);
(ii) finite complementary components of M possess exactly one border cycle in a mesh $\mathbb{C}$, and,
(ii) if $\mathbb{C}$ is an infinite mesh and M a finite connected subset, then there is only one border cycle of M which separates M from an infinite complementary component, which may be called *the* infinite complementary component of M.

These conclusions of Theorem 13 allow a further specification of border cycles of a finite connected set M of an infinite mesh: the uniquely specified border cycle separating M from its infinite complementary component is called the *outer border cycle*, and all the remaining border cycles are called *inner border cycles*. Assume that the complementary component A of M is separated from M

by border cycle g of M; then A is *assigned to* g. A complementary component of M assigned to an inner border cycle is a *proper hole* of M, and a finite complementary component assigned to the outer border cycle is an *improper hole* of M. Finite complementary components of a finite or infinite connected subset of an infinite mesh do not possess proper or improper holes. Finally, let $C(M)$ be the *cover* of the finite set M defined by the union of M with all of its proper holes. Note that $C(M)$ does not possess inner border cycles anymore, but an outer border cycle which may also separate improper holes from $C(M)$.

Let us return to Equation (17) and the connected set M with r border cycles, being a subset of a homogeneous mesh. The outer border cycle of M coincides with the outer border cycle of its cover $C(M)$. Theorem 14 is valid for $C(M)$. It follows that we always have $\nu l_r - \lambda k_r + \nu\lambda = 0$, where r is assumed to be the index of the outer border cycle of M. Subtracting this equation from Equation (17) it follows that

$$\nu \sum_{i=1}^{r-1} l_i - \lambda \sum_{i=1}^{r-1} k_i - (r-1)\nu\lambda \ = \ \nu(L - l_r) - \lambda(K - k_r) - (r-1)\nu\lambda \ = \ 0 \ . \quad (18)$$

All the $r-1$ inner border cycles of M may be considered as independent events, and this Equation (18) splits into $r-1$ equations $\nu l_i - \lambda k_i + \nu\lambda = 0$, for $1 \leq i \leq r-1$.

Theorem 15. (K. Voss, 1986) *For a connected set M of an infinite homogeneous mesh $\mathbb{C}_{\nu,\lambda}$ and any of its border cycles it follows that $k = \pm\nu + \nu l/\lambda$, where the outer border cycle implies the positive sign, and any inner border cycle the negative sign; k is the number of invalid edges assigned to this border cycle, and l is the length of this border cycle.*

This theorem provides a simple algorithmic rule for deciding whether a traced border cycle is either inner or outer, just by bookkeeping of k and l during border cycle tracing.

From Equation (17) it follows that $r = 2 + L/\lambda - K/\nu$. The total length L of all border cycles, and the total number K of all invalid edges allows to calculate the number r of border cycles, what is a topological invariant of the given finite connected set M. Note that values L and K may be accumulated by passing through all 4-neighborhoods of points in M, i.e. border cycle tracking is not necessary for calculating L and K.

We recall that f is the number of original cycles of set $M \subseteq \mathbb{C}_{\nu,\lambda}$. Combining Theorem 15 and Equations (10-12) leads to

$$\alpha_0 \ = \ \frac{\lambda}{\nu} f \ + \ (\frac{1}{\lambda} \ + \ \frac{1}{\nu}) l \ + \ 1 \ .$$

It is $1/\lambda + 1/\nu = 1/2$ for all infinite homogeneous meshes $\mathbb{C}_{\nu,\lambda}$. This proves:

Theorem 16. (K. Voss, 1986) *It is $\alpha_0 = \lambda f/\nu + l/2 + 1$ for a connected, set M without proper holes, contained in an infinite homogeneous mesh $\mathbb{C}_{\nu,\lambda}$, with $\alpha_0 = card(M)$, f original cycles in M and l is the length of the outer border cycle of M.*

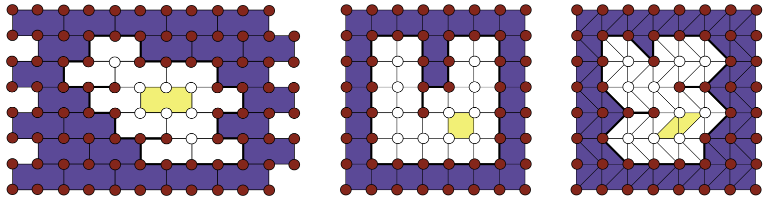


Fig. 62. Inner border cycles with $\nu = 3$, $\lambda = 6$, $\alpha_0 = 8$, $l = 30$, $k = 12$ and $f = 11$ (left), $\nu = 4$, $\lambda = 4$, $\alpha_0 = 11$, $l = 26$, $k = 22$ and $f = 23$ (middle), and $\nu = 6$, $\lambda = 3$, $\alpha_0 = 10$, $l = 22$, $k = 38$ and $f = 40$ (right).

This theorem is a graph-theoretic generalization of a result in [62][24] for the homogeneous orthogonal grid ($\nu = \lambda = 4$) that the contents f of a *simple grid polygon* (i.e. a simple polygon in the Euclidean plane having only grid points as its vertices) satisfies $f = \alpha_0 - l/2 - 1/2$, where α_0 is the number of grid points inside of the polygon (which is a closed simply-connected set in the Euclidean plane) and l is the number of grid points on the frontier of the polygon.

The cover of a finite connected set M is always free of proper holes. Theorem 16 applies for outer border cycles. For an inner border cycle g let α_0 denote the number of points of $\mathbb{C}_{\nu,\lambda} \setminus M$ circumscribed by g ('in the interior of g'), and f be the number of cycles of the mesh defined by these α_0 points and the points on the inner border cycle. Then we have:

Theorem 17. (K. Voss, 1986) *It is* $\alpha_0 = \lambda f/\nu - l/2 + 1$ *for an inner border cycle of a connected subset* M *of an infinite homogeneous mesh* $\mathbb{C}_{\nu,\lambda}$*, where* l *denotes the length of this inner border cycle.*

Note that an inner border cycle may separate several proper holes from set M, and Theorem 17 specifies a result for the union of these proper holes, see Fig. 62, where k be the number of invalid edges assigned to the inner border cycle.

In Fig. 62 we also shaded the original cycles of proper holes assigned to the shown inner border cycles: on the left we have three proper holes, and only one original cycle in one of these proper holes. In the middle we have one proper hole with one original cycle, and on the right we have one proper hole with two original cycles. Let m be the total number of original cycles of all proper holes assigned to the given inner border cycle. The remaining $f - m$ cycles of the mesh, defined by points in the complementary set $\overline{M}$ and on the inner border cycle, are called *boundary cycles*. Note: the word 'boundary' is used for the second time, again in the meaning of specifying the 'space between'. The λm edges of all boundary cycles split into k invalid edges, the length l of the inner border

[24] Georg Pick was professor of mathematics in Prague. He was in close contact with A. Einstein when Einstein worked at Prague university in 1911/12, performing music together, discussing philosophy and establishing the mathematical apparatus of general relativity theory. Pick was assassinated by the Nazi regime in concentration camp Theresienstadt.

cycle of M, and the sum of all lengths l_i of all the outer border cycles of all $n > 0$ proper holes assigned to the given inner border cycle of M:

$$\lambda m = 2k + l + \sum_{i=1} nl_i \ .$$

Together with Theorems 16 and 17 this implies that

$$n = 1 + k - m \ .$$

This is another example how a topological invariant (the number n of proper holes) can be calculated by accumulating local counts of invalid edges and boundary cycles.

In case of of the infinite homogeneous mesh $\mathbb{C}_{6,3}$, i.e. with $\nu = 6$ and $\lambda = 3$, we have [86] that there is exactly one complementary component assigned to any (inner or outer) border cycle of a finite connected set M, As a conclusion, a connected set in $\mathbb{C}_{6,3}$ does not have any improper hole, and a region adjacency graph, generated by any subset of $\mathbb{C}_{6,3}$, is a tree. Any directed invalid edge in this mesh generates exactly one boundary cycle; for any inner border cycle of a connected subset of $\mathbb{C}_{6,3}$.

EXERCISES

4.1. Return to Example 14 and show that the identities $A^{\nabla} = A^{\circ}$ and $\delta A = \vartheta A$ are true for any closed set $A \subseteq \mathbb{C}^{\{2\}}$.

4.2. Show that if cyclic orders are defined for all adjacency sets of $K_{3,3}$ or K_5 (see Fig. 58) then the resulting finite oriented graph has always an Eulerian characteristic of less than 2.

4.3. Calculate all possible Euler characteristics for the Listing graph (for all possible cyclic orders) shown on the right of Fig. 31.

4.4. Consider Fig. 37 as a representation of a subset M (shown by filled dots) in the 4-mesh (left) and in the 6-mesh (right), assuming that both meshes continue outside of the shown rectangle at infinitum. Specify M^{∇} and ϑM in both cases and draw all border cycles.

4.5. Is it possible to have a finite oriented graph with $\alpha_0 = \alpha_1 = \alpha_2$?

4.6. Identify all proper and improper holes in Fig. 3.6, also shown in Fig. 60.

4.7. Consider the infinite homogeneous mesh with $\nu = 6$ and $\lambda = 3$. Show that there is exactly one complementary component assigned to any (inner or outer) border cycle of a finite connected subset M of this mesh.

4.8. A *complete graph* with α_0 vertices has $\alpha_1 = \alpha_0(\alpha_0 - 1)/2$ edges. Show [87] that $\chi \leq -\alpha_0(\alpha_0 - 7)/6$, for any orientation on this graph defined by adjacency cycles, where we assume that $\alpha_0 \geq 3$.

5 Concluding Remarks

This report informed about basics in topology and reviewed a few approaches in digital topology. It also added a new one, the switch-approach, and discussed a new model $[\mathbb{C}_{G2}, \leq_{G2}, dim]$ of the poset topology $[\mathbb{C}^{\{2\}}, \leq^{\{2\}}, dim]$. Oriented adjacency graphs generalize the concept of oriented triangulations or tilings of surfaces (as known in 'classical' combinatorial topology), and the theory of oriented adjacency graphs has been briefly reviewed (also due to the fact that this theory remained widely unknown so far). The books [87] and [88] contain more material (and also all the proofs) about this theory than cited in this report (besides the original journal papers).

Due to the immense volume of publications in digital topology it was only possible to report about a very small fraction of all the available literature. This part I of the report focused on the homogeneous orthogonal planar grid, which is of special interest for the initial layer of image analysis. The material suggests different approaches for dealing with multi-level images defined on such grids:

1. meshes (i.e. planar oriented graphs): these approaches do not necessarily require the material discussed in Section 2. There are different options:
   (a) just 4-adjacency (even allowing an introduction of a 4-topology), and this does not require any mentioning of 8-adjacency or other non-planar adjacencies at all, and Theorem 13 is a theoretical justification of boundary tracing (note: digital straight lines may be modeled using 4-adjacency only, for digitization and self-similarity description, see the review [71]),
   (b) **S**-adjacency and Theorem 13 as theoretical justification, but the practical gain is expected to be minor compared to using solely 4-adjacency because image segmentation problems cannot be solved at pixel level anyway,
   (c) planar adjacencies in general, i.e. Section 4, and 4- or **S**-adjacency as examples (note: 6-adjacency has a directional bias), with the benefit of modelling inhomogeneous adjacencies (Voronoi or Delaunay adjacency, region adjacency etc.) as well, and possible extensions of the graph-theoretical approach to 3D, see [88],
2. cellular models which do have a long history in topology, and the alternating topology may be given as an interesting example for achieving the same Tucker topology (Aleksandrov topology for a poset) for homogeneous two-dimensional complexes but using different approaches for defining this topology, i.e. part of the material on complexes in Section 2. This should be in the context of generalizations to 'asymmetric' image carriers to justify extra model complexity.

Approaches characterized by good pairs may be chosen if solely binary images are the subject, and material as discussed in Section 3 provides a good introduction into the subject.

The section on basics of topology contains material which is important for understanding topological concepts. Even if these subjects are not further 'digitized' (i.e. related discussions for image carriers, e.g. how to define the fundamental group for subsets of an image carrier); basic knowledge on these subjects may support faster access to topological studies relevant to application-dependent problems.

Part II of the report will be on ‘asymmetric’ image carriers (cellular models), also motivated by needs of analyzing three-dimensional digitized sets. The boundary between components will be characterized in the way of *grid continua* as defined in [73].

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