

Angle Counts for Isothetic Polygons and Polyhedra

Ben Yip¹ and Reinhard Klette²

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Abstract. In the case of isothetic simple polyhedra there are only six different types of 3D angles. This article states and proves a formula about counts of these angles. This complements formulas in combinatorial topology such as Euler’s polyhedron formula, or the previously known formula on angle counts for isothetic polygons. The latter formula and the shown equality for angle counts of isothetic simple polyhedra are useful formulas for analyzing isothetic boundaries in 2D digital images (e.g. classification into inner (boundary of a hole) or outer boundaries, see [5]) and isothetic surfaces in 3D digital images (e.g. necessary condition for a complete surface scan).

Keywords: Isothetic polyhedra, isothetic polygons, angle counts, combinatorial topology.

1 Introduction

In many geometric applications the objects under consideration are line segments, rectangles, polyhedra etc. which are parallel to the axes of the coordinate system. Exploiting this fact often leads to more efficient and simpler algorithms, see, e.g. [6]. They also allow combinatorial studies, see, e.g. [1] on visibility of isothetic rectangles, and they correspond to boundaries in cell complex approaches in image analysis [4]. Isothetic polygons and polyhedra also have applications to problems of VLSI layout synthesis, database design, computational morphology, and stock cutting. Often they are studied with respect to partitioning problems [2, 3]. An *isothetic* polygon has all edges aligned to one of the axes of a Cartesian coordinate system, and an isothetic polyhedron has all faces parallel to one of the coordinate planes. We recall that a simple polygon is homeomorphic to a (closed) disk, and a simple polyhedron is homeomorphic to a (closed) sphere.

This paper presents and proves a formula on angles of isothetic simple polyhedra. First we recall a basic result on isothetic simple polygons. This result for

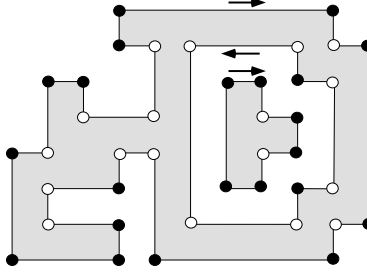


Fig. 1. Three examples of isothetic simple polygonal chains.

the two-dimensional (2D) case, see [5], is included here due to the obvious analogy to our 3D case, and we provide a new proof for the 2D case for illustrating analogies to our proof of the formula for isothetic simple polyhedra. This result for isothetic simple polygons allows in image analysis to discriminate *inner* and *outer boundaries*, what identifies it as a very practical tool for 2D image pattern analysis, see Theorem 2.3.-1 in [5].

Let P be a simple polygon, defined by a closed polygonal chain $v_1v_2, v_2v_3, \dots, v_kv_{k+1}$, $k \geq 2$, of its vertices. We assume that the chain circumscribes P in clockwise orientation. This defines the *outer boundary* of P , see Fig. 1 for two examples of outer boundaries. A closed polygonal chain with counter-clockwise orientation is an *inner boundary* of a bounded or unbounded planar region, see Fig. 1 for one example of an inner boundary.

We exclude that three consecutive vertices of a polygonal chain may be collinear. A chain $v_1v_2, v_2v_3, \dots, v_kv_{k+1}$, $k \geq 2$ of edges of P is *concave* if the inner angles of the vertices v_2, v_3, \dots, v_k are of more than π radians. The angle of a vertex v_2 is *concave* iff the sequence v_1v_2, v_2v_3 is concave, otherwise it is *convex*. Note that there is a one-to-one correspondence between concave angles of isothetic simple polygons representing inner boundaries and convex angles of the inscribed isothetic polygon, and between convex angles of the inner boundary and concave angles of the inscribed polygon. We only discuss the case of outer boundaries in the following theorem.

Theorem 1. *Let P be an isothetic simple polygon. Let P_V and P_C denote the numbers of convex and concave angles of P , respectively. We have*

$$P_V - P_C = 4.$$

Proof. Any isothetic simple polygon can be partitioned into a finite number of isothetic rectangles. Consequently, it can be constructed by joining, step by step, an isothetic rectangle at a time with the previous isothetic simple polygon, starting with a single isothetic rectangle satisfying the formula. There are four options for these joints, shown in Fig. 2.

In case (1) we have that P_V and P_C remain unchanged, in case (2) both P_V and P_C increase by 1, and in cases (3) and (4) both P_V and P_C increase by 2. \square

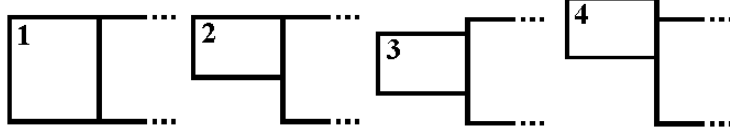


Fig. 2. Four possible joining steps.

In case of an inner boundary we have a change of the sense of orientation, and the formula goes over into

$$P_V - P_C = -4 .$$

The article generalizes Theorem 1 to the three-dimensional (3D) case of isothetic polyhedra.

2 Angles in Isothetic Simple Polyhedra

There are six kinds of angles in an isothetic simple polyhedron, see Fig. 3. Following this figure these angles will be referenced to as of type A , C , D_1 , D_2 , E or G .

Theorem 2. *Let H be an isothetic simple polyhedron. Let H_A , H_C , H_{D_1} , H_{D_2} , H_E and H_G denote the numbers of A , C , D_1 , D_2 , E and G angles of H , respectively. We have*

$$(H_A + H_G) - (H_C + H_E) - 2(H_{D_1} + H_{D_2}) = 8 .$$

Proof. Consider one coordinate plane and all faces of an isothetic simple polyhedron parallel to this coordinate plane. These faces define a finite number of cuts which separate the given isothetic polyhedron into *layered polyhedra*. We separate these layered polyhedra into simple isothetic layered polyhedra.

Any isothetic simple polyhedron H can be constructed by joining, step by step, two smaller simple isothetic layered polyhedra at a time, such that only one face is the merging face. We may continue with splitting these simple isothetic layered polyhedra into smaller ones such that the following assumption is valid:

H can be obtained by a finite sequence of joints of a simple isothetic polyhedron H_1 with a simple layered isothetic polyhedron H_2 . This joint is defined by a partial overlap of a face P_1 of H_1 with a face P_2 of H_2 such that P_2 is a subset of P_1 , i.e. both faces are coplanar, and the merging face is the only face that H_1 and H_2 joins.

The process may start with two isothetic parallelepipeds, each satisfying the formula with $|H_A| = 8$, and all other values are zero.

Now consider an arbitrary step within the joining process. Note that both polyhedra H_1 and H_2 may only have A and C angles on the merging faces. In the joining process, there are six possibilities for angles A and C , shown in Fig. 4:

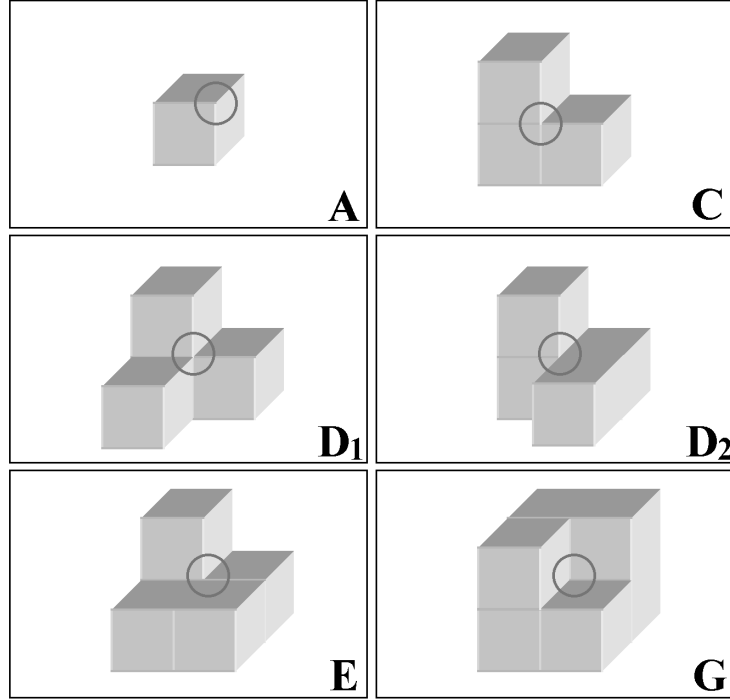


Fig. 3. Six kinds of angles in an isothetic simple polyhedron

- (i) an A -angle on P_2 is joining with an A -angle on P_1 ,
- (ii) an A -angle on P_2 is joining with a C -angle on P_1 ,
- (iii) an A -angle on P_2 is joining with an edge on P_1 ,
- (iv) an A -angle on P_2 is joining with an interior point of P_1 ,
- (v) a C -angle on P_2 is joining with a C -angle on P_1 , and
- (vi) a C -angle on P_2 is joining with an interior point on P_1 .

We assume that Theorem 2 is valid for the isothetic polyhedra H_1 and H_2 . Let H_A denotes the total number of A -angles of H_1 and H_2 , H_C denotes the total number of C -angles of H_1 and H_2 , and so on. Let T be defined as:

$$T = (H_A + H_G) - (H_C + H_E) - 2(H_{D_1} + H_{D_2}) .$$

Before the joining operation, T has a value of 16. We shall prove that T has the value of 8 after the joining operation, and hence satisfies Theorem 2.

In situation (i), one A -angle of H_1 and one A -angle of H_2 are lost, this decreases T by 2.

In situation (ii), one C -angle of H_1 and one A -angle of H_2 are lost, but the union also gains either a D_1 - or a D_2 -angle. This operation decreases T by 2.

In situation (iii), one A -angle of H_2 is lost, but the union also gains a C -angle. This operation decreases T by 2.

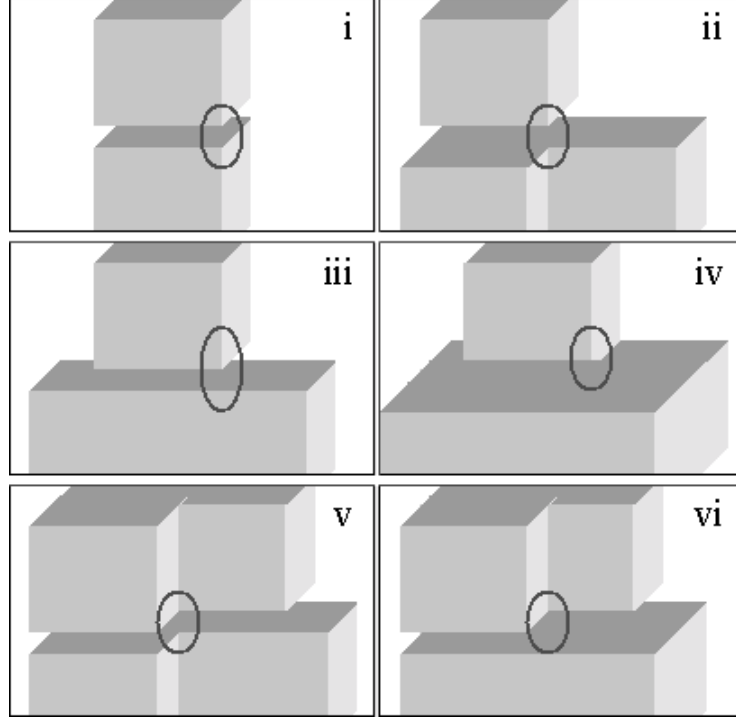


Fig. 4. Six possible joining operations.

In situation (iv), one A -angle of H_2 is lost, but the union also gains an E -angle. This operation decreases T by 2.

In situation (v), one C -angle of H_1 and one C -angle of H_2 are lost, this increases T by 2.

In situation (iv), one C -angle of H_2 is lost, but the union also gains a G -angle. This operation increases T by 2.

All joining operations of A -angles decrease T by 2, while all joining operations of C -angles increase T by 2. The number of A -angles on P_2 is always four more than the number of C -angles, as stated in Theorem 1. This shows that a joining operation decreases the value of T (which was 16 before the joining operation) by 8, which shows that the equality $(H_A + H_G) - (H_C + H_E) - 2(H_{D_1} + H_{D_2}) = 8$ remains true after any of the joining operations. \square

3 Conclusion

The previously known equality $P_V - P_C = 4$ for isothetic simple polygons and the shown equality $(H_A + H_G) - (H_C + H_E) - 2(H_{D_1} + H_{D_2}) = 8$ for isothetic simple polyhedra may be used towards generalizations for isothetic simple polyhedra in arbitrary dimensions.

Both formulas are useful for analyzing isothetic boundaries. In 2D digital images, we classify isothetic boundaries into inner (i.e. boundary of a hole) or outer boundaries (see [5]) depending on whether $P_V - P_C < 0$ or $P_V - P_C > 0$, respectively. This classification scheme is based on the one-to-one mapping (duality) of convex into concave angles, and vice-versa, if changing from clockwise to counterclockwise orientation.

Consider an isothetic cube in 3D. We have $H_A = 8$. If this cube specifies a hole then we have $H_G = 8$. All other angle counts are equal to zero. In general, for isothetic surfaces in 3D digital images there is a duality of angles of type A and G , C and E , D_1 and D_1 and D_2 and D_2 , leaving the formula $(H_A + H_G) - (H_C + H_E) - 2(H_{D_1} + H_{D_2}) = 8$ unchanged independent of whether the isothetic polyhedral surface is considered as being the outer boundary of a simple polyhedron, or an inner boundary of a polyhedral hole. However, the formula is still of interest for pattern analysis purposes by providing a necessary condition for having traced a complete 3D surface of an isothetic polyhedron, or by using angle counts in the specified six categories as shape descriptors. A classification into inner or outer boundary is possible by considering angle counts for a closed isothetic circuit on the given isothetic polyhedral surface, e.g. defined by one of the layered polyhedra.

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