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Multigrid Convergence of Geometric Features

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Abstract

Jordan, Peano and others introduced digitizations of sets in the plane and in the 3D space for the purpose of feature measurements. Features measured for digitized sets, such as perimeter, contents etc., should converge (for increasing grid resolution) towards the corresponding features of the given sets before digitization. This type of multigrid convergence is one option for performance evaluation of feature measurement in image analysis with respect to correctness.

The paper reviews work in multigrid convergence in the context of digital image analysis. In 2D, problems of area estimations and lower-order moment estimations do have "classical" solutions (Gauss, Dirichlet, Landau et al.). Estimates of moments of arbitrary order are converging with speed $f(r)=r^{-15/11}$. The linearity of convergence is known for three techniques for curve length estimation based on regular grids and polygonal approximations.

Piecewise Lagrange interpolants of sampled curves allow faster convergence speed. A first algorithmic solution for convergent length estimation for digital curves in 3D has been suggested quite recently. In 3D, for problems of volume estimations and lower-order moment estimations solutions are known for about one-hundred years (Minkowski, Scherrer et al.). But the problem of multigrid surface contents measurement is still a challenge, and there is recent progress in this field.

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The paper reviews work in multigrid convergence in the context of digital image analysis. In 2D, problems of area estimations and lower-order moment estimations do have "classical" solutions (Gauss, Dirichlet, Landau et al.). Estimates of moments of arbitrary order are converging with speed $\kappa(r) = r^{-15/11}$. The linearity of convergence is known for three techniques for curve length estimation based on regular grids and polygonal approximations. Piecewise Lagrange interpolants of sampled curves allow faster convergence speed. A first algorithmic solution for convergent length estimation for digital curves in 3D has been suggested quite recently. In 3D, for problems of volume estimations and lower-order moment estimations solutions have been known for about one-hundred years (Minkowski, Scherrer et al.). But the problem of multigrid surface contents measurement is still a challenge, and there is recent progress in this field.

1 Introduction

Geometric image analysis approaches are normally motivated by concepts in Euclidean geometry. A common strategy is: approximate picture subsets in 2D by *polygons* or in 3D by *polyhedrons* and use Euclidean geometry from that moment on for any further object analysis or manipulation step. A theoretical motivation is given by the fact that rectifiable curves and measurable surfaces can be approximated by polygonal curves or polyhedral surfaces up to any desired accuracy. This means that if we consider grid resolution as a potentially improvable parameter, then polygonal or polyhedral approximations appear to converge (for a set-theoretic metric) to the original preimage of the given object. The important question arises: does a convergence toward the true value also hold for calculated properties? For example, if we measure the length of a digital curve then the calculated value should converge to the correct length of a preimage in Euclidean space digitized with increasing grid resolution.

In ancient mathematics, Archimedes and Liu Hui [39] estimated the length $\mathcal{L}(\gamma)$ of a circular curve γ. Liu Hui used regular n-gon approximations, with $n = 3, 6, 12, 24, 48, 96, ...$, see left of Fig. 1. In case of $n = 6$ it follows $3 \cdot d < \mathcal{L}(\gamma) < 3.46 \cdot d$ for diameter d, and for $n = 96$ it follows that

$$223/71 < \pi < 220/70 \ , \quad \text{i.e.} \quad \pi \approx 3.14 \ .$$

The used method is mathematically correct because the perimeters of inner and outer regular n-gons converge towards the circle's perimeter for $n \to \infty$. For example, for the inner 3×2^n-gons, having perimeters

$$p_{2n} = 2n \cdot r\sqrt{2r^2 - r\sqrt{4r^2 - p_n^2}} \ ,$$

it follows

$$\kappa(n) = |p_n - 2\pi r| \approx 2\pi r/n \ , \quad \text{for} \quad n \geq 6 \ .$$

The function $\kappa(n)$ defines the *speed of convergence*, which is linear in this case.

The perimeters of digitized circles have been calculated in image analysis using sometimes *graph-theoretical concepts* such as the length of a 4-curve, or of an 8-curve where diagonal steps are weighted by $\sqrt{2}$, see, for example, [13, 27]. Such graph-theoretical concepts of path-length measurements are not related to digitized Euclidean geometry. Grid-intersection digitizations of line segments having a slope of 45° (for 4-paths) or of 22.5° (for 8-paths) provide simple examples for illustrating this. Convergence of digital curves toward a preimage with respect to the Hausdorff-Chebyshev-distance does not imply convergence of length calculated for these digital curves, toward the true length, but a proper preprocessing step (e.g. polygonal approximation of digital curves) may ensure such a desirable property as will be shown below.

We recall another historic example [35] cited in [18]. Assume that the lateral face $\mathbf{L}$ of a straight circular cylinder of radius ρ and of height h is cut by $(k-1)$ planes, $k \geq 2$, which are parallel to the base circles and which segment the cylinder into k congruent parts. Furthermore assume a regular n-gon, $n \geq 3$, in every cross section including both base circles, see Fig. 2 for $k = 4$ and $n = 6$.

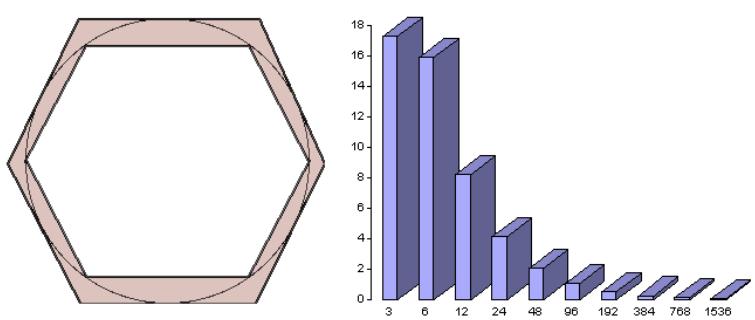

Fig. 1. Inner and outer hexagon approximating a circle (left), and percentage errors for perimeter estimation using inner n-gons.

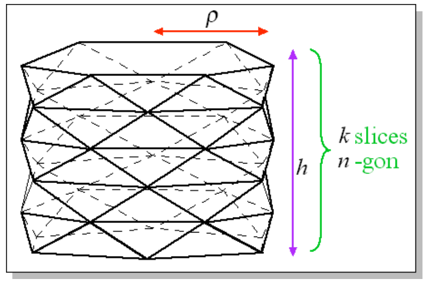

Fig. 2. Triangulation of the lateral face of a straight circular cylinder [35].

The axis of the cylinder and any vertex of such an n-gon defines a halfplane, which bisects an edge of the n-gon in the neighboring cross section or base circle. Now we connect for two neighboring n-gons each edge in one n-gon with those vertex of the other n-gon closest to this edge. This results into a triangulation $\mathbf{T}_{k,n}$ (i.e. a specific polyhedrization) of the lateral face $\mathbf{L}$ of the cylinder into $2kn$ congruent triangles having a surface area equal to

$$\mathcal{A}(\mathbf{T}_{k,n}) \;=\; 2\pi\rho \cdot \frac{\sin(\pi/n)}{\pi/n} \sqrt{\frac{1}{4}\pi^4\rho^2 \left(\frac{\sin(\pi/2n)}{\pi/2n}\right)^4 \left(\frac{k}{n^2}\right)^2 + h^2} \;.$$

If k and n go to infinity then the length of the edges of the triangular faces of $\mathbf{T}_{k,n}$ converges to zero. However, the surface area of $\mathbf{T}_{k,n}$ does not necessarily converge towards the surface area $\mathcal{A}(\mathbf{L}) \;=\; 2\pi\rho h$ of the lateral face! This is only true if k and n go to infinity such that k/n^2 converges to zero. If k/n^2 converges to $g > 0$ then $\mathcal{A}(\mathbf{T}_{k,n})$ converges to

$$2\pi\rho \cdot \frac{\sin(\pi/n)}{\pi/n} \sqrt{\frac{1}{4}\pi^4\rho^2 g^2 + h^2} \;.$$

It may even happen that k/n^2 goes to infinity, e.g. $k = n^3$, and then it follows that $\mathcal{A}(\mathbf{T}_{k,n})$ goes to infinity as well! Note that this example is based on sampling of surface points which cannot be assumed in image analysis. Digitization of sets provides an even less accurate input for subsequent steps of feature measurement.

The paper specifies the concept of multigrid convergence and reviews related results for measurements of moments, the length of curves in 2D or 3D, and surface area.

2 Multigrid Convergence

First we recall three digitization models frequently used in image analysis: Gauss digitization, grid-intersection digitization (for 2D only), and inner or outer Jordan digitization.

Let $r > 0$ be a real number called *grid resolution*. The *dilation* of a set $S \subset \Re^n$ by factor r is defined to be

$$r \cdot S \;=\; \{(r \cdot x_1, \ldots, r \cdot x_n) \;:\; (x_1, \ldots, x_n) \in S\} \;,$$

for $n \geq 1$. Following [17], this is a dilation with respect to the origin $(0, \ldots, 0)$, and other points in the Euclidean space $\Re^n$ could be chosen to be the fixpoint as well.

In studies on multigrid convergence sometimes it may be more appropriate to consider sets of the form $r \cdot S$ (the approach preferred, e.g., by Jordan and Minkowski) digitized in the orthogonal grid with unit grid length, instead of sets S digitized in r-grids with grid length $1/r$. The study of $r \to \infty$ corresponds to the increase in grid resolution, and this may be either a study of repeatedly dilated sets $r \cdot S$ in the grid with unit grid length, or of a given set S in repeatedly refined grids. This is a general *duality principle for multigrid studies* [25]. We choose the repeatedly refined grid approach for this paper which is of common use in numerics. An *r-grid point* $g^r_{i_1,\ldots,i_n} = (i_1/r, \ldots, i_n/r)$ is defined by integers $i_1, \ldots, i_n$.

Definition 1. *For a set $S \subset \Re^n$, $n \geq 1$, its* Gauss digitization $G_r(S)$ *is defined to be the set of all r-grid points contained in S. When $r = 1$ the Gauss digitization is denoted by $G(S)$.*

For example, consider $\Re^2$ and all r-grid points as centers of isothetic squares with edge length $1/r$. Then the set $\mathbf{G}_r(S)$ is defined to be the union of all those squares having their center points in $G_r(S)$.

If the given set is a curve γ in the plane then the grid-intersection model [7, 12] is of common use in digital geometry. Of course, this scheme can be adapted to r-grid points for any value of $r > 0$, and the resulting sequence of r-grid points is the *grid intersection digitization* $I_r(\gamma)$, which can be characterized by a start point and a chain code (i.e. a sequence of directional codes). A *digital straight line* $I_r(\gamma)$ is an 8-curve of r-grid points resulting from the grid-intersection digitization of the straight line γ in the Euclidean plane, excluding the straight lines $y = x + i/2$, where i is an integer. A *digital straight line segment* (DSS) is a finite 8-connected subsequence of a digital straight line.

The important problem of *volume estimation* was studied in [17] based on gridding techniques. Any grid point (i, j, k) in the Euclidean space $\Re^3$ is assumed to be the center point of a cube with faces parallel to the coordinate planes and with edges of length 1. The boundary is part of this cube (i.e. it is a closed set). Let S be a set contained in the union of finitely many such cubes. Dilate the set S with respect to an arbitrary point $p \in \Re^3$ in the ratio $r : 1$. This transforms S into S^p_r. Let $l^p_r(S)$ be the number of cubes completely contained in the interior of S^p_r, and let $u^p_r(S)$ be the number of cubes having a non-empty intersection with S^p_r. In [17] it is shown that $r^{-3} \cdot l^p_r(S)$ and $r^{-3} \cdot u^p_r(S)$ always converge to limit values $L(S)$ and $U(S)$, respectively, for $r \to \infty$, independently of the chosen point p. Jordan called $L(S)$ the *inner volume* and $U(S)$ the *outer volume* of set S, or the *volume* $\mathcal{V}(S)$ of S if $L(S) = U(S)$. The volume definition based on gridding techniques was further studied, e.g., in [29, 34].

The following definition is about an n-dimensional situation. For $n = 3$, a *regular Euclidean cell complex* consists of (topologically closed) r-cubes, r-squares, r-edges and r-vertices (see [20] for a review on cell complexes), and generalizations to higher dimensions, as well as a restriction to the two-dimensional case are straightforward.

Definition 2. *For a set* $S \subset \Re^n$, $n \geq 1$, *its* Jordan digitizations $J_r^-(S)$ *and* $J_r^+(S)$ *are defined as follows: the set* $J_r^-(S)$ *(also called the* inner digitization*) contains all* n*-dimensional cells completely contained in the interior of set* S*, and the set* $J_r^+(S)$ *(also called the* outer digitization*) contains all* n*-dimensional cells having a non-empty intersection with set* S*.*

The unions of all cells contained in $J_r^-(S)$ or $J_r^+(S)$ are isothetic polyhedra $\mathbf{J}_r^-(S)$ or $\mathbf{J}_r^+(S)$, respectively. The Hausdorff-Chebyshev distance, generated by the d_∞ metric, between the polyhedral boundaries $\partial\mathbf{J}_r^-(S)$ and $\partial\mathbf{J}_r^+(S)$ is greater than or equal to $1/r$ for any non-empty closed set S, and it holds that

$$\mathbf{J}_r^-(S) \subset S \subseteq \mathbf{J}_r^+(S)$$

in this case.

Gauss and Jordan digitizations have been used in gridding studies in mathematics (geometry of numbers, number theory, analysis). The model of grid-intersection digitization has been introduced for computer images, and it may be applied to planar curves.

A general scheme for comparing results obtained for picture subsets with the true quantities defined by the corresponding operation on the preimage in Euclidean space has been formalized in [36]. The following definition [18] specifies a measure for the speed of convergence toward the true quantity.

Definition 3. *Let* F *be a family of sets* S *in* $\Re^n$*, and* $dig_r(S)$ *a digital image of set* S*, defined by a digitization mapping* dig_r*. Assume that a quantitative property* $\mathcal{P}$*, such as area, perimeter, or a moment, is defined for all sets in family* F*. An estimator* $E_\mathcal{P}$ *is* multigrid convergent *for this family* F *and this digitization model* dig_r *iff there is a grid resolution* $r_S > 0$ *for any set* $S \in F$ *such that the estimator value* $E_\mathcal{P}(dig_r(S))$ *is defined for any grid resolution* $r \geq r_S$*, and*

$$|E_\mathcal{P}(dig_r(S)) \; - \; \mathcal{P}(S)| \leq \kappa(r)$$

for a function κ *defined for real numbers, having positive real values only, and converging toward 0 if* $r \to \infty$*. The function* κ *specifies the* convergence speed*.*

Gauss and Dirichlet knew already that the number of grid points inside a planar convex curve γ estimates the area of the set bounded by the curve within an order of $O(\mathcal{L}(\gamma))$, where $\mathcal{L}(\gamma)$ is the length of γ.

Theorem 1. (Gauss/Dirichlet ca. 1820) *For the family of planar convex sets, the number of* r*-grid points contained in a set approximates the true area with at least linear convergence speed, i.e.* $\kappa(r) = r^{-1}$*.*

Today we know that the convergence speed of this estimator is actually at least $r^{-1.3636}$ [15] for planar, bounded, *3-smooth* (i.e. continuous 3rd derivatives with positive curvature at all boundary points except a finite number of arc endpoints) convex sets, and it cannot be better than $r^{-1.5}$, which is a trivial lower bound.

Theorem 2. (Huxley 1990) *For the family of planar, bounded, 3-smooth convex sets, the number of r-grid points contained in a set approximates the true area with a convergence speed of* $\kappa(r) = r^\alpha$, *for* $-1.5 \leq \alpha < -1.3636$.

Closing the gap is an open problem which is a famous subject in number theory [28], and is closely related to digital geometry [24].

3 Moments and Moment-Based Features in 2D

In this section we cite worst-case error bounds from [25] in estimating real moments (and related features) of sets $S \subset \Re^2$ from corresponding discrete moments. Note that in case of 3-smooth sets the claimed positive curvature excludes straight boundary segments. Throughout this section we assume that S is a planar convex set whose boundary consists of a finite number of C^3 arcs, also allowing straight line segments if not otherwise stated. The (p, q)-*moments* of set S are defined by

$$\mathcal{M}_{p,q}(S) = \iint_S x^p y^q \; dx \; dy \; ,$$

for integers $p, q \geq 0$. The moment $\mathcal{M}_{p,q}(S)$ has the *order* $p+q$. In image analysis, the exact values of moments $\mathcal{M}_{p,q}(S)$ remain unknown. They are estimated by *discrete moments* $\mu_{p,q}(S)$ where

$$\mu_{p,q}(S) = \sum_{(i,j)\in G(S)} i^p \cdot j^q$$

which can be calculated from the corresponding digitized set $G(S)$ of set S. The grid resolution r has to be used as scaling factor if the approach involves repeatedly refined grids. The moment-concept has been introduced into image analysis in [14].

The *contents* or *area* $\mathcal{A}(S)$ of a planar set S, i.e. the moment $\mathcal{M}_{0,0}(S)$ of order zero, is estimated by the number of grid points in $G(S)$, i.e. by the discrete moment $\mu_{0,0}(S)$. For the *center of gravity* of a set S,

$$\left(\frac{\mathcal{M}_{1,0}(S)}{\mathcal{M}_{0,0}(S)}, \frac{\mathcal{M}_{0,1}(S)}{\mathcal{M}_{0,0}(S)} \right)$$

the estimate

$$\left(\frac{\mu_{1,0}(S)}{\mu_{0,0}(S)}, \frac{\mu_{0,1}(S)}{\mu_{0,0}(S)} \right)$$

is calculated from its digital set $G(S)$. The orientation of a set S can be described by its axis of the least second moment. That is the line for which the integral of the squares of the distances to points of S is a minimum. That integral is

$$I(S, \varphi, \rho) = \iint_S r^2(x, y, \varphi, \rho) dx dy \; ,$$

where $r(x, y, \varphi, \rho)$ is the perpendicular distance from the point (x, y) to the line given in the form

$$x \cdot \cos\varphi - y \cdot \sin\varphi = \rho \ .$$

We are looking for the value of φ for which $I(S, \varphi, \rho)$ takes its minimum, and by this angle we define the *orientation* of the set S. This φ-value will be denoted by $\mathcal{D}(S)$, i.e.

$$\min_{\varphi,\rho} I(S, \varphi, \rho) \ = \ I(S, \mathcal{D}(S), \overline{\rho}), \text{ for some value of } \overline{\rho} \ .$$

Again, this feature is estimated by replacing integration and set S by a discrete addition and a digital set $G(S)$, respectively. With respect to applications note that this feature requires sets with "a main orientation", i.e. $\mathcal{M}_{2,0}(S) \neq \mathcal{M}_{0,2}(S)$. Finally, we also mention the *elongation* of S (see [16, 42]) in direction φ which is the ratio of maximum and minimum values of $I(S, \varphi, \rho)$, i.e.

$$\mathcal{E}(S) \ = \ \frac{\max\limits_{\varphi,\ \rho} I(S, \varphi, \rho)}{\min\limits_{\varphi,\ \rho} I(S, \varphi, \rho)} \ .$$

It may be estimated by digital approximations of the I-function values as in case of the orientation of set S.

The curvature of the boundary of the considered set plays an important role. It makes an essential difference whether at least one straight section on the boundary is allowed or not.

Theorem 3. (Klette/Žunić 2000) *Let S a convex set whose boundary consists of a finite number of C^3 arcs, then $\mathcal{M}_{p,q}(S)$ can be estimated by $r^{-(p+q+2)} \cdot \mu_{p,q}(r \cdot S)$ within an error of $O(r^{-1})$, and this error term is the best possible.*

However, if S is 3-smooth and convex, i.e. the boundary does not possess any straight segment, then the application of *Huxley*'s theorem leads to a reduced upper error bound.

Theorem 4. (Klette/Žunić 2000) *Let a planar bounded 3-smooth convex set S be given. Then $\mathcal{M}_{p,q}(S)$ can be estimated by $r^{-(p+q+2)} \cdot \mu_{p,q}(r \cdot S)$ within an error of $O\left((\log r)^{\frac{47}{22}}) \cdot r^{-\frac{15}{11}}\right) \approx O\left(r^{-1.3636\ldots}\right)$.*

The following theorem specifies how progress in the estimation of the "basic difference" $|\mathcal{M}_{0,0}(r \cdot S) - r^2 \cdot \mathcal{A}(S)|$ by $O(\kappa(r))$ can be used to improve error bounds for higher-order estimates $|\mathcal{M}_{p,q}(S) - r^{-(p+q+2)} \cdot \mu_{p,q}(r \cdot S)|$, for a set S being n-smooth, for some integer $n = 3, 4, \ldots$, including $n = \infty$. For a function $\kappa(r) \geq 0$, for $r \geq 0$, a family $F_{\kappa(r)}$ of classes C of planar sets is defined such that the the following conditions are satisfied:

(i) C is nonempty;
(ii) if $S \in C$ then it satisfies

$$|\mu_{0,0}(r \cdot S) - r^2 \cdot \mathcal{A}(S)| = O\left(\kappa(r)\right) \ ;$$

(iii) if a set S belongs to C then any isometric transformation of S belongs to C as well;
(iv) any set which can be represented by a finite number of unions, intersections and set-differences of sets from C also belongs to C.

This definition allows a formulation of the following theorem which 'translates' possible future progress in number theory into a formulation of related error bounds for moments of arbitrary order. Let F_0 be the smallest family of sets which contains all n-smooth planar convex bounded sets, and which is closed with respect to finite numbers of intersections, unions and set-theoretical differences.

Theorem 5. (Klette/Žunić 2000) *Let S be a planar n-smooth convex set and let $\kappa(r)$ be such that F_0 is contained in the family $F_{\kappa(r)}$. It follows that $\mathcal{M}_{p,q}(S)$ can be estimated by $r^{-(p+q+2)} \cdot \mu_{p,q}(r \cdot S)$ within an error of $O\left(\kappa(r) \cdot r^{-2}\right)$.*

Let S be a set in a class in F_0. The given theorems allow a derivation of the following upper error bounds for feature estimations [25]. An upper error bound for area estimates $\frac{1}{r^2}\,\mu_{0,0}\,(r \cdot S)$ is directly given by *Huxley*'s theorem, i.e.

$$\left|\mathcal{A}\,(S) - \tfrac{1}{r^2} \cdot \mu_{0,0}\,(r \cdot S)\right| \;=\; O\left(r^{-\left(\frac{15}{11}-\varepsilon\right)}\right) \;.$$

The same upper error bound holds for the estimates

$$\frac{1}{r} \cdot \frac{\mu_{1,0}\,(r \cdot S)}{\mu_{0,0}\,(r \cdot S)} \quad \text{and} \quad \frac{1}{r} \cdot \frac{\mu_{0,1}\,(r \cdot S)}{\mu_{0,0}\,(r \cdot S)}$$

of the coordinates

$$\frac{m_{1,0}\,(S)}{m_{0,0}\,(S)} \quad \text{and} \quad \frac{m_{0,1}\,(S)}{m_{0,0}\,(S)}$$

of the center of gravity, respectively. For the estimate of the orientation only sets S with $\mathcal{M}_{2,0}(S) \neq \mathcal{M}_{0,2}(S)$ are relevant. Then S's orientation $\mathcal{D}(S)$ can be recovered within an worst-case error of $\mathcal{O}(r^{-\frac{15}{11}+\varepsilon})$, by using the estimate

$$\tan(2 \cdot \mathcal{D}(S)) \;\approx\; \frac{2 \cdot \overline{\mu}_{1,1}(r \cdot S)}{\overline{\mu}_{2,0}(r \cdot S) - \overline{\mu}_{0,2}(r \cdot S)} \;.$$

The elongation $\mathcal{E}$ of a 3-smooth convex set S can be estimated by

$$\Theta(r \cdot S) = \frac{t_1(r \cdot S) + \sqrt{t_2(r \cdot S)}}{t_1(r \cdot S) - \sqrt{t_2(r \cdot S)}} \;,$$

for

$$t_1(r \cdot S) = \overline{\mu}_{2,0}(r \cdot S) + \overline{\mu}_{0,2}(r \cdot S)$$

and

$$t_2(r \cdot S) = 4 \cdot (\overline{\mu}_{1,1}(r \cdot S))^2 + (\overline{\mu}_{2,0}(r \cdot S) - \overline{\mu}_{0,2}(r \cdot S))^2 \;,$$

for a planar set S. The error in the approximation $\mathcal{E}(S) \approx \Theta(r \cdot S)$ has an upper error bound in $O\left(r^{-\frac{15}{11}+\varepsilon}\right)$.

Of course, Theorems 3, 4, and 5 may also be used to derive error bounds for features defined by moments of higher order than just up to order two.

4 Length of Curves in 2D and 3D

There are provable convergent length estimators in 2D and 3D, where linear multigrid convergence has been shown for planar convex curves. A superlinear convergence $O(r^{-1.5})$ of asymptotic length estimation has been achieved in [10] just for the case of digitized straight lines, and there are superlinear length estimators in 2D if sampling of curves is assumed instead of digitization.

4.1 Polygonal Approximations of Curves in 2D

Boundaries of digitized planar sets, or digitized planar curves, can be approximated by a polygon, using a *digital straight segment* (DSS) procedure to segment the boundary into a sequence of maximal-length DSSs. The resulting polygon depends on the starting point and the orientation of the scan. Besides this DSS-based approach to the approximation of digital curves by polygons, there are other possible approaches using minimum-length polygons; see [5, 6, 38]. We review all three methods with respect to multigrid convergence.

Given a connected region S in the Euclidean plane and a grid resolution r, the *r-frontier* $\partial\mathbf{G}_r(S)$ of S is uniquely determined. Note that an r-frontier may consists of several non-connected curves even in the case of a bounded convex set S. A set S is *r-compact* iff there is a number $r_S > 0$ such that $\partial\mathbf{G}_r(S)$ is just one (connected) curve, for any $r \geq r_0$.

Theorem 6. (Kovalevsky/Fuchs 1992, Klette/Zunic 2000) *Let S be a convex, r-compact polygonal set in $\Re^2$. Then there exists a grid resolution r_0 such that for all $r \geq r_0$, any DSS approximation of the r-frontier $\partial\mathbf{G}_r(S)$ is a connected polygon with perimeter p_r satisfying the inequality*

$$|\mathcal{L}(\partial(S)) - p_r| \leq \tfrac{2\pi}{r}\left(\varepsilon_{DSS}(r) + \frac{1}{\sqrt{2}}\right) .$$

This theorem and its proof can be found in [25]. The proof is based to a large extent on material given in [26]. The value of r_0 depends on the given set, and $\varepsilon_{DSS}(r) \geq 0$ is an algorithm-dependent approximation threshold specifying the maximum Hausdorff-Chebyshev distance (generalizing the Euclidean distance between points to a distance between sets of points) between the r-frontier $\partial\mathbf{G}_r(S)$ and the constructed (not uniquely specified !) DSS approximation polygon. Assuming $\varepsilon_{DSS}(r) = 1/r$, it follows that the upper error bound for DSS approximations is characterized by[1]

$$\frac{2\pi}{r^2} + \frac{2\pi}{r \cdot \sqrt{2}} \approx \frac{4.5}{r} \quad \text{if} \quad r \gg 1 \quad \text{(i.e. } r \text{ is large)} .$$

The grid resolution $1/r$ is assumed in the chord property in [33], where a DSS is defined to be a finite 8-path. In the case of using cell complexes it is appropriate to consider a finite 4-path as a DSS iff its main diagonal width is less than $\sqrt{2}$, see [1, 32].

[1] Let $\kappa(r) = 2\pi/r^2 + 2\pi/r \cdot \sqrt{2}$. Then it follows that $\kappa(r) \to \pi\sqrt{2}$ as $r \to \infty$.

A second approach, see [38], is based on Jordan digitization of sets S in the Euclidean plane. The difference set $\mathbf{O}_r(S) \setminus \mathbf{I}_r(S)$ can be transformed into a subset such that the Hausdorff-Chebyshev distance (generated by the d_∞ metric) between its inner and outer boundary is exactly $1/r$, i.e. the grid constant. The perimeter of S can be estimated by the length of a minimum-length polygon (MLP) contained in this subset, and circumscribing the inner boundary of this subset, which is homeomorphic to an annulus. The subset can be described by a sequence of r-squares, where any r-square has exactly two edge-neighboring r-squares in the sequence. Such a sequence is called a *one-dimensional grid continuum* (1D-GC). Such 1D-GCs are treated in the theory of 2D cell complexes in the plane. This specifies an alternative approach (GC-MLP in short) to the approximation of digital curves; it has been experimentally compared with the DSS method in [23].

For the case of GC-MLP approximations there are several convergence theorems in [38], showing that the perimeter of the GC-MLP approximation is a convergent estimator of the perimeter of a bounded, convex, smooth or polygonal set in the Euclidean plane. The following theorem is basically a quotation from [38]; it specifies the asymptotic constant for GC-MLP perimeter estimates.

Theorem 7. (Sloboda/Zaťko/Stoer 1998) *Let γ be a (closed) convex curve in the Euclidean plane which is contained in a 1D-GC of r-squares, for $r \geq 1$. Then the GC-MLP approximation of this 1D-GC is a connected polygonal curve with length l_r satisfying the inequality*

$$l_r \leq \mathcal{L}(\gamma) < l_r + \frac{8}{r} \, .$$

Finally we sketch a third method, see [5,6], which is also based on minimum-length polygon calculation. Assume an r-frontier of S which can be represented in the form $P = (v_0, v_1, \ldots, v_{n-1})$ where vertices are clockwise ordered and the interior of S lies to the right. For each vertex of P we define forward and backward shifts: The *forward shift* $f(v_i)$ of v_i is the point on the edge (v_i, v_{i+1}) at the distance δ from v_i. The *backward shift* $b(v_i)$ is that on the edge (v_{i-1}, v_i) at the distance δ from v_i.

In the approximation scheme as detailed below we replace an edge (v_i, v_{i+1}) by a line segment $(v_i, f(v_{i+1}))$ interconnecting v_i and the forward shift of v_{i+1}, which is referred to as the *forward approximating segment* and denoted by $L_f(v_i)$. The *backward approximating segment* $(v_i, b(v_{i-1}))$ is defined similarly and denoted by $L_b(v_i)$. Now we have three sets of edges, original edges of the r-frontier, forward and backward approximating segments. Let $0 < \delta \leq 0.5/r$. Based on these edges we define a connected region $A_r^\delta(S)$, which is homeomorphic to the annulus, as follows:

Given a polygonal circuit P describing an r-frontier in clockwise orientation, by reversing P we obtain a polygonal circuit Q in counterclockwise order. In the initialization step of our approximation procedure we consider P and Q as the *external* and *internal* bounding polygons of a polygon P_B homeomorphic to the annulus. It follows that this initial polygon P_B has area contents zero, and as a set of points it coincides with $\partial \mathbf{G}_r(S)$.

Now we 'move' the external polygon P 'away' from $\mathbf{G}_r(S)$, and the internal polygon Q 'into' $\mathbf{G}_r(S)$ as specified below. This process will expand P_B step by step into a final polygon which contains $\partial\mathbf{G}_r(S)$, and where the Hausdorff-Chebyshev distance between P and Q becomes non-zero. For this purpose, we add forward and backward approximating segments to P and Q in order to increase the area contents of the polygon P_B.

To be precise, for any forward or backward approximating segment $L_f(v_i)$ or $L_b(v_i)$ we first remove the part lying in the interior of the current polygon P_B and updating the polygon P_B by adding the remaining part of the segment as a new boundary edge. The direction of the edge is determined so that the interior of P_B lies to the right of it. The resulting polygon P_B^δ is referred to as the *approximating sausage* of the r-frontier and denoted by $A_r^\delta(S)$. The width of such an approximating sausage depends on the value of δ. An *AS-MLP curve* for approximating the boundary of S is defined as being a shortest closed curve $\gamma_r^\delta(S)$ lying entirely in the interior of the approximating sausage $A_r^\delta(S)$, and encircling the internal boundary of $A_r^\delta(S)$. It follows that such an AS-MLP curve $\gamma_r^\delta(S)$ is uniquely defined, and that it is a polygonal curve defined by finitely many straight segments. Note that this curve depends upon the choice of the approximation constant δ.

Theorem 8. (Asano/Kawamura/Klette/Obokkata 2000) *Let S be a bounded, r-compact convex polygonal set. Then, there exists a grid resolution r_0 such that for all $r \geq r_0$ it holds that any AS-MLP approximation of the r-frontier $\partial\mathbf{G}_r(S)$, with $0 < \delta \leq .5/r$, is a connected polygon with a perimeter l_r and*

$$|\mathcal{L}(\partial S) - l_r| \leq (4\sqrt{2} + 8 * 0.0234)/r = 5.844/r. \tag{1}$$

These three Theorems 6, 7 and 8 show that the DSS error bound of $4.5/r$ is smaller than the AS-MLP bound $5.844/r$, and the AS-MLP is smaller than the GC-MLP bound $8/r$. Further theoretical and experimental measures may be used for performance comparisons such as *effectiveness* defined by the product of error and number of generated line segments, or the time efficiency of implemented algorithms. With respect to asymptotic time complexity, a linear-time algorithm is known for any of these three linear approximation (i.e. polygonalization) methods [6, 23].

4.2 Higher-Order Approximation for Sampled 2D Curves

Higher-order approximations of curves with the purpose of length estimations have been studied in [30, 31]: for $k \geq 1$, estimate the length $\mathcal{L}(\gamma)$ of a C^k regular parametric curve $\gamma : [0,1] \to \Re^n$ from $(m+1)$-tuples $Q_m = (q_0, q_1, \ldots, q_m)$ of points $q_i = \gamma(t_i)$ positioned on the curve γ. The parameters t_i's are not assumed to be given. Of course, sampling (see both examples in the Introduction) is a different situation compared to digitization. An increase in grid resolution r defining a scale for two dimensions in the plane, corresponds to an increase in the number m of sampling points defining a one-dimensional scale on the curve.

Some assumptions about the distribution of the t_i's are needed to make the sampling problem solvable [30]. The problem is the easiest when the t_i's are chosen in a perfectly uniform manner, namely $t_i = \frac{i}{m}$. In such a case it seems natural to estimate γ by a curve $\widetilde{\gamma}$ that is piecewise polynomial of degree $a \geq 1$. Then we prove

Theorem 9. (Noakes/Kozera/Klette 2001) *Let γ be C^{s+2} and let t_i's be sampled perfectly uniformly. Then $\mathcal{L}(\widetilde{\gamma}) = \mathcal{L}(\gamma) + O(\frac{1}{m^{s+s_0}})$, where s_0 is 1 or 2 according as s is odd or even.*

It is known [31] that Lagrange estimates of length based on a uniform grid do not always converge to $\mathcal{L}(\gamma)$ when the unknown t_i's are non-uniform. In [30] it is shown that there are some approximately uniform samplings of t_i's for which those estimates are well-behaved. More precisely

Definition 4. *For $\varepsilon \geq 0$, the t_i's are (ε, k)-uniformly sampled if there is a C^k reparameterization $\phi : [0,1] \to [0,1]$, for $k \geq 1$, such that $t_i = \phi(\frac{i}{m}) + O(\frac{1}{m^{1+\varepsilon}})$.*

Lagrange estimates of length can behave badly for $(0, k)$-uniform sampling, see [31], but for $0 < \varepsilon \leq 1$ the following theorem holds [30], using piecewise Lagrange interpolants $\widetilde{\gamma}$.

Theorem 10. (Noakes/Kozera/Klette 2001) *Let the t_i's be sampled (ε, k)-uniformly where $0 < \varepsilon \leq 1$ and $k \geq 4$. Then, for piecewise-quadratic Lagrange interpolants $\widetilde{\gamma}$, determined by a sampled $(m+1)$-tuple Q_m and based on a uniform grid, $\mathcal{L}(\widetilde{\gamma}) = \mathcal{L}(\gamma) + O(\frac{1}{m^{4\varepsilon}})$.*

Whereas Theorem 9 permits length estimates of arbitrary accuracy (for s sufficiently large), Theorem 10 refers only to piecewise-quadratic estimates (i.e. $s = 2$), and accuracy is limited accordingly. However, even in the latter case it holds that the quartic convergence speed[2] is three magnitudes faster then the linear convergence speed discussed for DSS, GC-MLP and AS-MLP polygonalizations (with $s = 1$). These sampling-based results encourage further research on higher-order approximations for digitized curves.

4.3 A Polygonal Approximation of Curves in 3D

Consider the length estimation problem for rectifiable curves γ in the three-dimensional Euclidean space. We assume curves γ which lead to simple cube curves for the digitization model $J_r^+(\gamma)$.

A *cube-curve* is a sequence $g = (f_0, c_0, f_1, c_1, ..., f_n, c_n)$ of r-faces f_i and r-cubes c_i in $\Re^3$, for $0 \leq i \leq n$, such that r-faces f_i and f_{i+1} are sides of r-cube c_i, for $0 \leq i \leq n$ and $f_{n+1} = f_0$. Such a cube-curve is *simple* iff $n \geq 4$, and for any two r-cubes c_i, c_k in g with $|i-k| \geq 2 \ (mod\ n+1)$ it holds that if $c_i \cap c_k \neq \emptyset$ then either $|i-k| = 2 \ (mod\ n+1)$ and $c_i \cap c_k$ is an r-edge, or $|i-k| = 3 \ (mod\ n+1)$

[2] Note that m specifies an increase in the order of $\sqrt{m}$ only with respect to a scale for two dimensions in the plane, i.e. the quartic convergence speed in m may be compared with a quadratic convergence speed in r.

and $c_i \cap c_k$ is an r-vertex. A *tube* **g** is the union of all r-cubes contained in a cube-curve g. Such a tube is a polyhedrally-bounded compact set in $\Re^3$, and it is homeomorphic with a torus in case of a simple cube-curve.

A curve is *complete in* **g** iff it has a non-empty intersection with any r-cube contained in g.

Definition 5. *A* minimum-length polygon *(MLP) of a simple cube-curve g is a shortest polygonal simple curve σ which is contained and complete in tube* **g**.

Following [38], the *length* of a simple cube-curve g is defined to be the length $\mathcal{L}(\sigma)$ of an MLP σ of g. Theorem 7 states that this length estimation approach is multigrid convergent to the true value in case of planar convex curves γ as specified in this theorem.

An algorithm for approximating such an MLP in a simple cube-curve has been specified in [8]. It is based on the following theorem [21]. An edge contained in a tube **g** is *critical* iff this edge is the intersection of three cubes contained in the cube-curve g.

Theorem 11. (Buelow/Klette 2000) *Let g be a simple cube-curve. Critical edges are the only possible locations of vertices of a shortest polygonal simple curve contained and complete in tube* **g**.

The algorithmic solution in [8] provides a polygonal approximation of desired MLP's, and thus a length measurement method for simple cube-curves in 3D space. The algorithm possesses a measured time complexity in $O(n)$. However, two open problems remain at this stage: the time complexity might be provable always in $O(n)$, and the convergence might be provable always towards the MLP. For details of the algorithm see [8].

5 Surface Area of Regular Solids

A '3D object' can be modeled by a regular solid, which is defined to be a simply-connected compact set having a measurable surface area [18]. Algorithms for multigrid-convergent surface area estimation are still a research topic. Obviously, increasing the grid resolution in a digitization of a regular solid in the form of a 3D cell complex [4], and measuring the area of the resulting isothetic surface, does not result in convergence to the true value. This might be compared with the fact that 4-path length is not related to the length of a digitized curve in 2D. Marching-cube based polyhedrizations, see Fig. 3 do not support multigrid-convergent surface area estimations toward the true value [19]. This might be compared with the fact that 8-path length (with weighting factor $\sqrt{2}$ for diagonal steps) is not related to the length of a digitized curve in 2D.

Polyhedrization is a common goal of segmenting the surface of a digitized regular solid, normally given in the form of boundary points of a 3D grid point set (e.g. using 3D Gauss digitization) or in the form of a two-dimensional grid continuum (2D-GC) defined by a difference between the inner and outer Jordan digitizations.

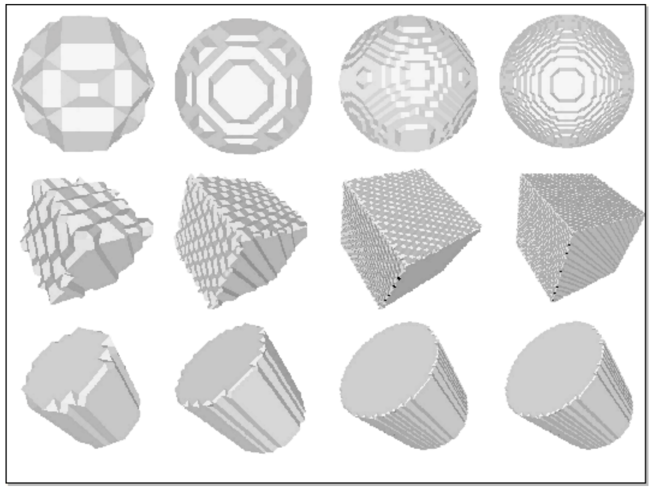

Fig. 3. Three Euclidean sets digitized for increasing grid resolution and approximated by marching-cube polyhedrizations.

5.1 Experimentally Measured Convergence

Expanding the ideas of DSS approximations into 3D leads to a *digital plane segment* (DPS) approach for achieving multigrid-convergent measurement of surface area: the boundary of a Gauss- or Jordan-digitized set is segmented into maximum-size DPSs, and surface areas of related polyhedral faces are added to form a final estimate.

[32] introduced *arithmetic geometry* which allows characterizations of hyperplanes in n-dimensional spaces. [2] proposed a general definition that linked planes and topology, introducing $|a| + |b| + |c|$ thick planes. These planes were further specified and used in [11]. For a generalization to n dimensions see [3]. Digital plane segments can be defined within arithmetic geometry as follows: r-cubes have eight directed diagonals. The *main diagonal* of a Euclidean plane is those directed diagonal (out of these eight) that has the largest dot product (inner product) with the normal of the plane. Note that in general there may be more than one main diagonal for a Euclidean plane; if so, we can choose any of them as the main diagonal. The distance between two parallel Euclidean planes in the main diagonal direction is called the *main diagonal distance* between these two planes.

Now consider a finite set of faces of r-cubes in 3D space. A Euclidean plane is called a *supporting plane* of this set if it is incident with at least three non-collinear vertices of the set of faces, and all the faces of the set are in only one of the (closed) halfspaces defined by the plane. Note that any non-empty finite set of faces has at least one supporting plane. Any supporting plane defines a *tangential plane*, which is the nearest parallel plane to the supporting plane such that all faces of the given set are within the closed slice defined by the supporting

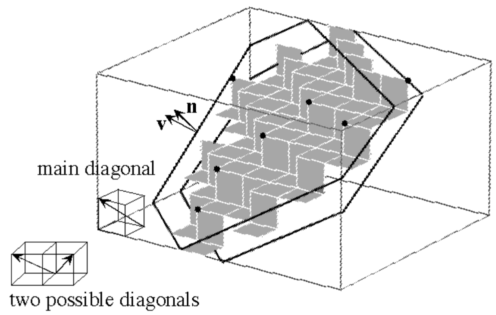

Fig. 4. Illustration of the main diagonal of a DPS.

and tangential planes. Note that a tangential plane may be a supporting plane as well. Figure 4 gives a rough sketch of such a set of faces, where **n** denotes the normal to the two parallel planes, and **v** is the main diagonal.

Definition 6. *A finite, edge-connected set of faces in 3D space is a* digital planar segment *(DPS) iff it has a supporting plane such that the main diagonal distance between this plane and its corresponding tangential plane is less than* $\sqrt{3}/r$ *(i.e. the length of a diagonal of an r-cube).*

Such a supporting plane is called *effective* for the given set of grid faces. Let **v** be a vector in a main diagonal direction with a length of $\sqrt{3}/r$, let **n** be the normal vector to a pair of parallel planes, and let $d = \mathbf{n} \cdot \mathbf{p}_0$ be the equation of one of these planes. According to our definition of a DPS, all the vertices **p** of the faces of a DPS must satisfy the following inequality:

$$0 \leq \mathbf{n} \cdot \mathbf{p} - d < \mathbf{n} \cdot \mathbf{v}$$

Let $\mathbf{n} = (a, b, c)$. Then this inequality becomes

$$0 \leq ax + by + cz - d < |a| + |b| + |c| ,$$

i.e. an DPS is an edge-connected subset of faces in a *standard plane* [11]. A *simply-connected DPS* is such that the union of its faces is topologically equivalent (in Euclidean space) to the unit disk.

The general DPS recognition problem can be stated as follows: Given n vertices $\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n\}$, does there exist a DPS such that each vertex satisfies the inequality system

$$0 \leq \mathbf{n} \cdot \mathbf{p}_i - d < \mathbf{n} \cdot \mathbf{v}, \quad i = 1, \ldots, n ,$$

[40] suggests a method of turning this into a linear inequality system, by eliminating the unknown d as follows:

$$\mathbf{n} \cdot \mathbf{p}_i - \mathbf{n} \cdot \mathbf{p}_j < \mathbf{n} \cdot \mathbf{v}, \quad i, j = 1, \ldots, n ,$$

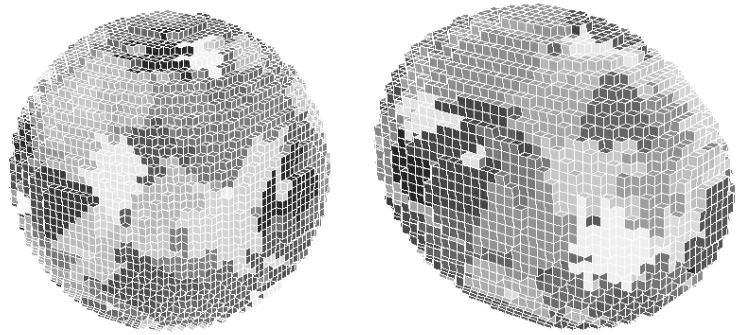

Fig. 5. Agglomeration of faces of a sphere and an ellipsoid into DPSs.

This system of n^2 inequalities can be solved in various ways. [11, 40] use a Fourier elimination algorithm. However, this algorithm is not time-efficient even for very small cell complexes. In fact, in [40] a more advanced elimination technique than Fourier-Motzkin was proposed to eliminate unknowns from systems of inequalities. This technique eliminates all variables at once, whereas the Fourier-Motzkin technique eliminates one variable at a time. Eliminating all variables at once leads to an $O(n^4)$ algorithm for recognizing a DPS, which is faster than the algorithm sketched in [11]. [40] included results for hyperplanes of arbitrary dimension. Note that two different definitions are actually used to define digital planes, depending on what kind of connectivity relation is required:

$$0 \leq ax + by + cz - d < |a| + |b| + |c|$$

or

$$0 \leq ax + by + cz - d < max(|a|, |b|, |c|) .$$

The second definition was used in [40], but the results obtained there for the elimination technique are equally valid for the first definition.

An incremental algorithm for DPS recognition has been proposed in [22], based on updating lists of effective supporting planes. This algorithm can be used for segmenting boundaries of digitized 3D sets into maximal-size DPSs, see Fig. 5 for two examples.

Actually, any DPS recognition algorithm could be used for segmenting a surface of a 3D cell complex into maximal-size DPSs. However, the starting point and the search strategy during the process of 'growing' a DPS are critical for the behavior of such an algorithm. Also, after obtaining maximal-size DPSs, it is not straightforward to derive a polyhedron from the resulting segmentation of the surface.

Analytical surface area calculation of an ellipsoid, with all three semi-axes a, b, c allowed to be different, is a complicated task. If two semi-axes coincide, i.e. in the case of an ellipsoid of revolution, the surface area can be analytically

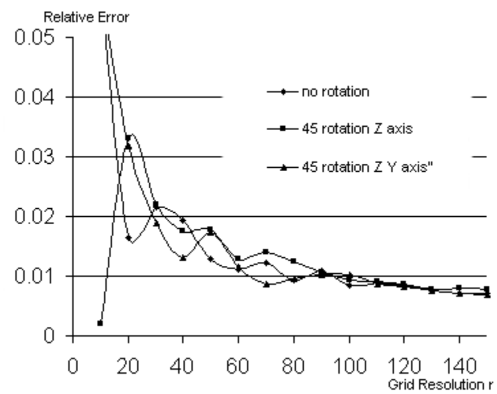

Fig. 6. Relative errors in surface area estimation for an ellipsoid in three orientations for increasing grid resolution. Figure 5 illustrates resolution $r = 40$.

specified in terms of standard functions. The surface area formula in the general case is based on standard elliptic integrals. Example 2 in [18], reporting recent work by G.Tee, specifies an analytical method of computing the surface area of a general ellipsoid. This area can be used in experimental studies as *ground truth* to evaluate the performance of DPS algorithms in surface area estimation. Figure 6 shows the error in the estimated value relative to the true value for an ellipsoid in three different orientations, using a search depth of 10 in region growing (breadth-first search). In general these DPS-based estimates behave 'better' than those based on convex hull (for digitized convex sets) or on marching-cube algorithms. Convex hull and marching-cube methods lead to relative errors of 3.22% and 10.80% for $r = 100$, respectively, while the DPS error is less than 0.8%. The DPS method shows a good tendency to converge, but theoretical work needs to be done to prove this. Altogether, there are working algorithms which appear to provide multigrid-convergent surface area estimations, but there is no related theorem stating this property for some type of 3D sets.

5.2 Multigrid Convergence of Estimated Surface Area

Recently there is actually progress on proving multigrid-convergent behavior of surface area estimation, but so far without an algorithmic solution for the proposed method! [9, 37] introduce the *relative convex hull* $\mathrm{CH}_Q(P)$ of a polyhedral solid P which is completely contained in the interior of another polyhedral solid Q. If the convex hull $\mathrm{CH}(P)$ is contained in Q then $\mathrm{CH}_Q(P) = \mathrm{CH}(P)$; otherwise $\mathrm{CH}_Q(P) \subseteq Q$ is a 'shrunk version' of the convex hull. To be precise, let $\overline{\mathbf{pq}}$ be the (real) straight line segment from point $\mathbf{p}$ to point $\mathbf{q}$ in $\Re^3$, and introduce the following definition[37]:

Definition 7. *A set* $A \subseteq Q \subseteq \Re^3$ *is* Q-convex *iff for all* $\mathbf{p}, \mathbf{q} \in A$ *such that* $\overline{\mathbf{pq}} \subseteq Q$ *we have* $\overline{\mathbf{pq}} \subseteq A$. *Let* $P \subseteq Q$. *The* relative convex hull $CH_Q(P)$ *of* P *with respect to* Q *is the intersection of all* Q*-convex sets containing* P.

For a set $S \subseteq \Re^3$ we defined the inner and outer Jordan digitizations $\mathbf{J}_r^-(S)$ and $\mathbf{J}_r^+(S)$ for grid resolution $r \geq 1$. If S is a regular solid with a defined surface area, let $\mathcal{A}(S)$ be its surface area in the Minkowski sense [29].

Theorem 12. (Sloboda/Zaťko 2000) *Let* $S \subset \Re^3$ *be a compact set bounded by a smooth closed Jordan surface* ϑS. *Then*

$$\lim_{r\to\infty} s\left(CH_{\mathbf{J}_r^+(S)}\left(\mathbf{J}_r^-(S)\right)\right) = \mathcal{A}(S) \ .$$

This theorem, from [37], specifies a method of multigrid convergence which still requires research on algorithmic implementation, theoretical and experimental convergence speed, and performance evaluation in comparison with other methods such as the DPS segmentation method sketched above.

6 Conclusions

Euclidean geometry specifies the ground truth, the correct moment, length or surface area prior to digitization. The concept of multigrid convergence may provide a general methodology for evaluating and comparing different approaches. The measurement of quantitative properties is certainly a main topic in digital geometry. [27] is one of the early publications in this area, and [23] is one of the more recent ones, both focusing on length estimates. Probability-theoretical aspects of digitization errors [41, 43] have only been studied for a few elementary figures and simple geometric problems; further studies should provide answers to open problems such as those listed in [25].

There is still no solution with respect to multigrid convergence for surface area estimation which combines a convergence theorem and an algorithmic implementation. The study of non-polygonal approximations of digitized curves with respect to improvements in convergence speed appears as another important open problem.

For all multigrid-convergence problems, it is important to determine what optimum convergence speed $\kappa(r)$ is actually possible (for example, see open problem defined by Theorem 2). A test set of six curves has been specified in [23] for evaluations of curve length estimations, and general ellipsoids are proposed in [18] for surface area performance evaluations. Evaluation measures might be, e.g., absolute error, efficiency (error times number of generated segments), and computing time. A classification of properties $\mathcal{L}, \mathcal{A}, \mathcal{M}_{p,q}, \mathcal{V}, \mathcal{E}, \mathcal{D}, \ldots$ with respect to families of sets, optimum convergence speed, and optimum algorithmic time complexity might be a long-term project.

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