Length Estimation for Curves with Different Samplings

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Summary. This paper* looks at the problem of approximating the length of the unknown parametric curve $\gamma : [0,1] \to \mathbb{R}^n$ from points $q_i = \gamma(t_i)$, where the parameters t_i are not given. When the t_i are uniformly distributed Lagrange interpolation by piecewise polynomials provides efficient length estimates, but in other cases this method can behave very badly [15]. In the present paper we apply this simple algorithm when the t_i are sampled in what we call an ε-*uniform* fashion, where $0 \le \varepsilon \le 1$. Convergence of length estimates using Lagrange interpolants is not as rapid as for uniform sampling, but better than for some of the examples of [15]. As a side-issue we also consider the task of approximating γ up to parameterization, and numerical experiments are carried out to investigate sharpness of our theoretical results. The results may be of interest in computer vision, computer graphics, approximation and complexity theory, digital and computational geometry, and digital image analysis.

1 Introduction

Recent research in digital and computational geometry and image analysis concerns estimation of lengths of digitized curves; indeed the analysis of digitized curves in $\mathbb{R}^2$ or $\mathbb{R}^3$ is one of the most intensively studied subjects in image data analysis. This paper contributes to this topic by showing that there are possible improvements in convergence speed compared to all known methods in digital geometry, however, based on sampling of curves (as common in approximation theory) compared to digitization (as common in digital geometry).

A digitized curve is the result of a process (such as contour tracing or 2D thinning extraction) which maps a curve γ (such as the boundary of a region) onto a computer-representable curve. An analytical description of $\gamma : [0,1] \to \mathbb{R}^2$ is not given, and numerical measurements of points on γ are

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corrupted by a process of *digitization*: γ is digitized within an orthogonal grid of points $(\frac{i}{m}, \frac{j}{m})$, where i, j are permitted to range over integer values, and m is a fixed positive integer called *the grid resolution*.

Depending on the digitization model [9], γ is mapped onto a digital curve and approximated by a polygonal curve $\hat{\gamma}_m$ whose length is an estimator for $d(\gamma)$. This is a standard approach for approximating a digital curve with respect to geometric analysis tasks. However, different smooth approximations (e.g. snake model) are used in image analysis as well, where the convergence analysis with respect to geometric figures is omitted. Approximating polygons $\hat{\gamma}_m$ based on local configurations of digital curves do not ensure multigrid length convergence, but global approximation techniques yield *linearly* convergent estimates, namely $d(\gamma) - d(\hat{\gamma}_m) = O(\frac{1}{m})$ [1], [11], [12] or [20]. Recently, experimentally based results reported in [4] and [10] confirm a similar rate of convergence for $\gamma \subset \mathbb{R}^3$. In the special case of discrete straight line segments in $\mathbb{R}^2$ a stronger result is proved, for example, [6], where $O(\frac{1}{m^{1.5}})$ order of asymptotic length estimates are given. In Theorems 1 and 2 presented in this paper the convergence is of order at least $O(\frac{1}{m^{r+1}})$ and $O(\frac{1}{m^{4\varepsilon}})$ when $0 < \varepsilon \leq 1$, respectively.

For $k \geq 1$, consider the problem of estimating the length $d(\gamma)$ of a C^k regular parametric curve $\gamma : [0,1] \to \mathbb{R}^n$ from $m+1$-tuples $\mathcal{Q} = (q_0, q_1, \ldots, q_m)$ of points $q_i = \gamma(t_i)$ on the curve γ. The parameters t_i are not assumed to be given, but some assumptions are needed to make our problem solvable. For example, if none of the t_i lie in $(0, \frac{1}{2})$ the task becomes intractable. The problem is easiest when the t_i are chosen in a perfectly uniform manner, namely $t_i = \frac{i}{m}$ (e.g. see also [14] or [21]). In such a case it seems natural to estimate γ by a curve $\tilde{\gamma}$ that is piecewise polynomial of degree $r \geq 1$. We prove first in this paper:

Theorem 1. *Let γ be C^{r+2} and let t_i be sampled perfectly uniformly. Then there exists piecewise-r-degree polynomial $\tilde{\gamma}$, determined by $\mathcal{Q}$ such that $d(\tilde{\gamma}) = d(\gamma) + O(\frac{1}{m^{r+p}})$, where p is 1 or 2 according as r is odd or even.*

As usual, $O(g(m))$ means a quantity whose absolute value is bounded by some multiple of $g(m)$ as $m \to \infty$. We are principally concerned with non-uniform sampling. More precisely

Definition 1. *For $0 \leq \varepsilon \leq 1$, the t_i's are said to be ε-uniformly sampled when there is an order-preserving C^k reparameterization $\phi : [0,1] \to [0,1]$ such that $t_i = \phi(\frac{i}{m}) + O(\frac{1}{m^{1+\varepsilon}})$.*

Note that ε-uniform sampling arises from two types of perturbations of uniform sampling: first via a diffeomorphism $\phi : [0,1] \to [0,1]$ combined subsequently with added extra distortion term $O(\frac{1}{m^{1+\varepsilon}})$. In particular, for $\phi = id$ and $\varepsilon = 0$ ($\varepsilon = 1$) the perturbation is *linear* i.e. of uniform sampling order (*quadratic*), which constitutes asymptotically a big (small) distortion of a uniform partition of $[0,1]$. The extension of Definition 1 for $\varepsilon > 1$ could also be considered. This case represents, however, a very small perturbation of uniform sampling (up to a ϕ-shift) which seems to be of less interest in applications.

Lagrange estimates of length can behave badly for 0-uniform sampling (the more elaborate algorithm of [15] is needed for this case), but for $0 < \varepsilon \leq 1$ we prove the following, using piecewise-quadratic Lagrange interpolants $\mathcal{Q}^i$ (see Section 4).

Theorem 2. *Let the t_i be sampled ε-uniformly, where $0 < \varepsilon \leq 1$, and suppose that $k \geq 4$. Then, there is a function*** $\tilde{\gamma}$*, determined by $\mathcal{Q}$, such that $d(\tilde{\gamma}) = d(\gamma) + O(\frac{1}{m^{4\varepsilon}})$.*

Whereas Theorem 1 permits length estimates of arbitrary accuracy (for r sufficiently large), Theorem 2 refers only to piecewise-quadratic estimates, and accuracy is limited accordingly. The interest in this (the main result of the present paper) lies in the non-uniform distribution of the unknown parameters t_i's. The proofs of Theorems 1 and 2 also permit uniform estimates of γ up to reparameterization. Note that the construction of the piecewise-r-degree polynomial interpolant P_r^j including $\mathcal{Q}^i$ (see Sections 3 and 4) requires neither the explicit knowledge of γ nor of the parameters t_i (each P_r^j is constructed over a uniform local grid in $s \in [0, r]$; for $\mathcal{Q}^i$ over uniform grid in $[0, 2]$). The latter are used merely to compare $d(\gamma)$ with $d(\tilde{\gamma})$ and $\tilde{\gamma}$ with γ, respectively. More specifically, in order to prove Theorems 1 and 2 both *global* and *local* t- and s-parameterizations shall be used. On the other hand, the explicit construction of the interpolant P_j^r (or Q^i) approximating γ (and thus $d(\gamma)$) resorts exclusively to the local parameterization.

For these results $\mathcal{Q}$ arises from uniform or ε-uniform samplings as opposed to digitization. So strict comparisons cannot be made. Our results seem relevant to digital and image geometry nonetheless for the following reasons. They provide comparisons with the interpolation and indicate potential problems which might arise in digitization based on non-uniform distribution of t_i. Moreover, they show that using piecewise Lagrange polynomial approach to estimate length of a digiztized curve $\hat{\gamma}$ may not always be appropriate. Finally, as a special case we provide upper bounds for optimal rates of convergence when piecewise polynomials are applied to the digitized curves. Related work can also be found in [2], [3], [7], [8], [17], and [19]. There is also some interesting work on complexity [5], [18], [22], and [23].

2 Sampling and Curves

We are going to discuss different ways of forming ordered samples $0 = t_0 < t_1 < t_2 < \ldots < t_m = 1$ of variable size $m + 1$ from the interval*** $[0, 1]$. The simplest procedure is *uniform sampling*, where $t_i = \frac{i}{m}$ (where $0 \leq i \leq$

** See section 4 for details.

*** In the present context there is no real gain in generality from considering other intervals $[0, T]$.

m). Uniform sampling is not invariant with respect to *reparameterizations*, namely order-preserving C^k diffeomorphisms $\phi : [0,1] \to [0,1]$, where $k \geq 1$. A small perturbation of uniform sampling is no longer uniform, but may approach uniformity in some asymptotic sense, at least after some suitable reparameterization. The ε-uniform sampling in Definition 1 of the previous Section is a possible example of such perturbation. Note that ϕ and the asymptotic constants are chosen independently of $m \geq 1$, and that ε-uniform implies δ-uniform for $0 \leq \delta < \varepsilon$. Uniform sampling is ε-uniform for any $0 \leq \varepsilon \leq 1$. At the other extreme are examples, where sampling increments $t_i - t_{i-1}$ are neither large nor small, considering m, and yet sampling is not ε-uniform for any $0 < \varepsilon \leq 1$:

Example 1. Set t_i to be $\frac{i}{m}$ or $\frac{2i-1}{2m}$ according as i is even or odd. Then $(1/2m) \leq t_i - t_{i-1} \leq (3/2m)$ for all $1 \leq i \leq m$ and all $m \geq 1$. Thus sampling is 0-uniform. To see that sampling is not ε-uniform for $\varepsilon > 0$ assume the opposite. Then, for some C^1 reparameterization $\phi : [0,1] \to [0,1]$, $t_{i+1} - t_i = \frac{1}{2m} = \phi\left(\frac{i+1}{m}\right) - \phi\left(\frac{i}{m}\right) + O(\frac{1}{m^{1+\varepsilon}})$ and $t_{i+2} - t_{i+1} = \frac{3}{2m} = \phi\left(\frac{i+2}{m}\right) - \phi\left(\frac{i+1}{m}\right) + O(\frac{1}{m^{1+\varepsilon}})$. By the Mean Value Theorem

$$\frac{1}{2m} = \frac{\phi'(\xi_{1i}^{(m)})}{m} + O(\frac{1}{m^{1+\varepsilon}}), \qquad \frac{3}{2m} = \frac{\phi'(\xi_{2i}^{(m)})}{m} + O(\frac{1}{m^{1+\varepsilon}}), \tag{1}$$

for some $\xi_{1i}^{(m)} \in (\frac{i}{m}, \frac{i+1}{m})$ and $\xi_{2i}^{(m)} \in (\frac{i+1}{m}, \frac{i+2}{m})$. Fixing i and increasing m, $\phi'(\xi_{1i}^{(m)}) \to \phi'(0)$ and $\phi'(\xi_{2i}^{(m)}) \to \phi'(0)$. On the other hand, by (1), $\phi'(\xi_{1i}^{(m)}) \to 1/2$ and $\phi'(\xi_{2i}^{(m)}) \to 3/2$: a contradiction.

Let $\|\cdot\|$ be the Euclidean norm in $\mathbb{R}^n$, where $n \geq 1$, with $< \cdot, \cdot >$ the corresponding inner product. The *length* $d(\gamma)$ of a C^k parametric curve $(k \geq 1)$ $\gamma : [0,1] \to \mathbb{R}^n$ is defined as $d(\gamma) = \int_0^1 \|\dot{\gamma}(t)\| dt$, where $\dot{\gamma}(t) \in \mathbb{R}^n$ is the derivative of γ at $t \in [0,1]$. The curve γ is said to be *regular* when $\dot{\gamma}(t) \neq \mathbf{0}$, for all $t \in [0,1]$. A *reparameterization* of γ is a parametric curve of the form $\gamma \circ \psi : [0,1] \to \mathbb{R}^n$, where $\psi : [0,1] \to [0,1]$ is a C^k diffeomorphism. The reparameterization $\gamma \circ \psi$ has the same image and length as γ. Let γ be regular: then so is any reparameterization $\gamma \circ \psi$. We say that curve γ is *parameterized proportionally to arc-length* when $\|\dot{\gamma}(t)\|$ is constant for $t \in [0,1]$. We want to estimate $d(\gamma)$ from ordered $m+1$-tuples $\mathcal{Q} = (q_0, q_1, q_2, \ldots, q_m) \in (\mathbb{R}^n)^{m+1}$, where $q_i = \gamma(t_i)$, whose parameter values $t_i \in [0,1]$ are not known but sampled in some reasonably regular way: sampling might be ε-uniform for some $0 \leq \varepsilon \leq 1$. ε-uniform sampling is invariant with respect to C^k reparameterizations $\psi : [0,1] \to [0,1]$. So suppose, without loss of generality, that γ is parameterized proportionally to arc-length.

We close this section with Figure 1 indicating why arbitrary sampling and piecewise-quadratic Lagrange interpolation (see Section 3) in most cases gives poor estimates for $d(\gamma)$ (and indeed for γ). In Figure 1 only the uniform data yields reasonable approximations. In the next sections we show that some kinds of non-uniform sampling also give good approximations.

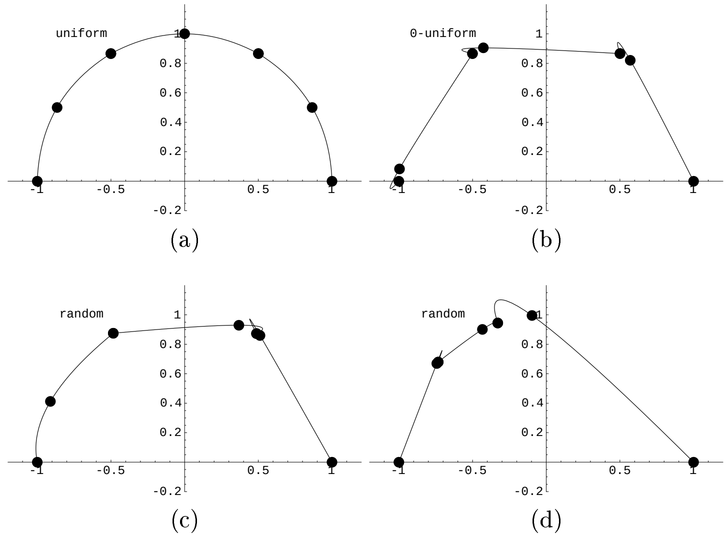

Fig. 1. Absolute errors $E = |\pi - d(\widetilde{\gamma})|$ for a unit semicircle approximated with the piecewise-quadratic interpolant $\widetilde{\gamma}$: (**a**) For perfectly uniform sampling $E = 0.00362662$. (**b**) For 0-uniform sampling (where $t_i = \frac{i}{6}$ for i even and $t_i = \frac{i}{6} - \frac{1}{12}$ for i odd; $0 \leq i \leq 6$) $E = 0.323189$. (**c**) For some random sampling $E = 0.22992$. (**d**) For another random sampling $E = 0.15394$.

3 Uniform Sampling

We first consider length estimates of γ in the easier case, where the t_i's are sampled perfectly uniformly: $t_i = \frac{i}{m}$ (with $0 \leq i \leq m$). Suppose $k = r + 2$, where $r \geq 1$, and (without loss of generality) that m is a multiple of r. Then $\mathcal{Q}$ gives $\frac{m}{r}$ $r+1$-tuples of the form $(q_0, q_1, \ldots, q_r)$, $(q_r, q_{r+1}, \ldots, q_{2r})$, $\ldots$,$(q_{m-r}, q_{m-r+1}, \ldots, q_m)$. The j-th $r+1$-tuple is interpolated by the r-degree Lagrange polynomial $P_r^j : [0, r] \to \mathbb{R}^n$, here $1 \leq j \leq \frac{m}{r}$: $P_r^j(0) = q_{(j-1)r}$, $P_r^j(1) = q_{(j-1)r+1}$,$\ldots$,$P_r^j(r) = q_{jr}$. Note that P_r^j is defined in terms of a local variable $s \in [0, r]$. Recall Lemma 2.1 of Part 1 of [13]:

Lemma 1. *Let $f : [a, b] \to \mathbb{R}^n$ be C^l, where $l \geq 1$ and assume that $f(t_0) = \mathbf{0}$, for some $t_0 \in (a, b)$. Then there exists a C^{l-1} function $g : [a, b] \to \mathbb{R}^n$ such that $f(t) = (t - t_0)h(t)$.*

The proof of Lemma 1 shows that $g = O(\frac{df}{dt})$. If f has multiple zeros $t_0 < t_1 < \ldots < t_k$ then $k + 1$ applications of Lemma 1 give

$$f(t) = (t - t_0)(t - t_1)(t - t_2) \ldots (t - t_k)h(t), \qquad (2)$$

where h is $C^{l-(k+1)}$ and $h = O(\frac{d^{k+1} f}{dt^{k+1}})$.

Assuming that γ is C^{r+2} (i.e. $k = r + 2$) we are now going to prove Theorem 1, where estimation of $d(\gamma)$ is based on piecewise-r-degree polynomial interpolation. For each j-th r-tuple consider the interpolating polynomial P_r^j. Let $\psi : [t_{(j-1)r}, t_{jr}] \to [0,r]$ be the affine mapping given by $\psi(t_{(j-1)r}) = 0$ and $\psi(t_{jr}) = r$, namely $\psi(t) = mt - (j-1)r$. Thus $\dot{\psi}(t)$ is identically m (a diffeomorphism). Note that since both intervals $[t_{(j-1)r}, t_{jr}]$ and $[0,r]$ are *uniformly sampled*, ψ maps the t_i's to the corresponding grid points in $[0,r]$. Define $\tilde{\gamma}_j = P_r^j \circ \psi : [t_{(j-1)r}, t_{jr}] \to \mathbb{R}^n$. Then as ψ is affine, $\tilde{\gamma}_j$ is a polynomial of degree at most r. Note that $f = \tilde{\gamma}_j - \gamma : [t_{(j-1)r}, t_{jr}] \to \mathbb{R}^n$ is C^{r+2} and satisfies $f(t_{(j-1)r}) = f(t_{(j-1)r+1}) = \cdots = f(t_{jr}) = 0$. By (2)

$$f(t) = (t - t_{(j-1)r})(t - t_{(j-1)r+1}) \cdots (t - t_{jr}) h(t), \tag{3}$$

where $h : [t_{(j-1)r}, t_{jr}] \to \mathbb{R}^n$ by Lemma 1 is C^1. Still by proof of Lemma 1

$$h(t) = O(\frac{d^{r+1} f}{dt^{r+1}}) = O(\frac{d^{r+1}\gamma}{dt^{r+1}}) = O(1), \tag{4}$$

because $deg(\tilde{\gamma}_j) \leq r$ and $\frac{d^{r+1}\gamma}{dt^{r+1}}$ is $O(1)$. Thus by (3) and (4)

$$f(t) = O(\frac{1}{m^{r+1}}), \tag{5}$$

for $t \in [t_{(r-1)j}, t_{rj}]$. Differentiating function h (defined as a $r+1$-multiple integral of $f^{(r+1)}$ over the compact cube $[0,1]^{r+1}$; see proof of Lemma 1) yields

$$\dot{h}(t) = O(\frac{d^{r+2} f}{dt^{r+2}}) = O(\frac{d^{r+2}\gamma}{dt^{r+2}}) = O(1), \tag{6}$$

as $deg(\tilde{\gamma}_j) \leq r$. By (3) and (6) $f = O(\frac{1}{m^r})$ and hence for $t \in [t_{(j-1)r}, t_{jr}]$

$$\dot{\gamma}(t) - \dot{\tilde{\gamma}}_j(t) = \dot{f}(t) = O(\frac{1}{m^r}). \tag{7}$$

Let $V_{\dot{\gamma}}^{\perp}(t)$ be the orthogonal complement of the line spanned by $\dot{\gamma}(t)$. Since $\|\dot{\gamma}(t)\| = d(\gamma)$,

$$\dot{\tilde{\gamma}}_j(t) = \frac{< \dot{\tilde{\gamma}}_j(t), \dot{\gamma}(t) >}{d(\gamma)^2} \dot{\gamma}(t) + v(t), \tag{8}$$

where $v(t)$ is the orthogonal projection of $\dot{\tilde{\gamma}}_j(t)$ onto $V_{\dot{\gamma}}^{\perp}(t)$. Since $\dot{\tilde{\gamma}}_j(t) = \dot{f}(t) + \dot{\gamma}(t)$, we have $\dot{\tilde{\gamma}}_j(t) = (1 + \frac{<\dot{f}(t), \dot{\gamma}(t)>}{d(\gamma)^2}) \dot{\gamma}(t) + v(t)$. Furthermore, by (7), $v = O(\frac{1}{m^r})$. Since $< \dot{\gamma}(t), v(t) >= 0$, by the Binomial Theorem the norm $\|\dot{\tilde{\gamma}}_j(t)\| =$

$$\|\dot{\gamma}(t)\| \sqrt{1 + 2\frac{< \dot{f}(t), \dot{\gamma}(t) >}{d(\gamma)^2} + O(\frac{1}{m^{2r}})} = \|\dot{\gamma}(t)\| (1 + \frac{< \dot{f}(t), \dot{\gamma}(t) >}{d(\gamma)^2}) \tag{9}$$

up to the $O(\frac{1}{m^{2r}})$ term; note that by (7) $|2\frac{<\dot{f}(t),\dot{\gamma}(t)>}{d(\gamma)^2} + O(\frac{1}{m^{2r}})| < 1$ holds asymptotically. Integrating by parts, $\int_{t_{(j-1)r}}^{t_{jr}} (\|\dot{\tilde{\gamma}}_j(t)\| - \|\dot{\gamma}(t)\|)\,dt =$

$$\int_{t_{(j-1)r}}^{t_{jr}} \frac{< \dot{f}(t), \dot{\gamma}(t) >}{d(\gamma)} dt + O(\frac{1}{m^{2r+1}}) = -\int_{t_{(j-1)r}}^{t_{jr}} \frac{< f(t), \ddot{\gamma}(t) >}{d(\gamma)} dt + O(\frac{1}{m^{2r+1}}).$$

Since γ is compact and at least C^3 by (4) and $h = O(1)$ we have $< h(t), \ddot{\gamma}(t) >= O(1)$ and $< h(t), \gamma^{(3)}(t) >= O(1)$. Similarly, by (6) we have $< \dot{h}(t), \ddot{\gamma}(t) >= O(1)$. Hence, by (3) and Taylor's Theorem applied to $r(t) =< h(t), \ddot{\gamma}(t) >$ at $t = t_{(j-1)r}$, we get $< f(t), \ddot{\gamma}(t) >= (t - t_{(j-1)r})(t - t_{(j-1)r+1}) \ldots (t - t_{jr})(a + O(\frac{1}{m}))$, where a is constant in t and $O(1)$. Since sampling is uniform the integral $\int_{t_{(j-1)r}}^{t_{jr}} (t - t_{(j-1)r})(t - t_{(j-1)r+1}) \ldots (t - t_{jr}) dt$ vanishes when r is even. So $\frac{1}{d(\gamma)} \int_{t_{(j-1)r}}^{t_{jr}} < f(t), \ddot{\gamma}(t) > dt$ is either $O(\frac{1}{m^{r+2}})$ or $O(\frac{1}{m^{r+3}})$, according as r is odd or even. As $2r + 1 \geq r + 3$ (for $r \geq 2$) and $2r + 1 \geq r + 2$ (for $r \geq 1$),

$$\int_{t_{(j-1)r}}^{t_{jr}} (\|\dot{\tilde{\gamma}}_j(t)\| - \|\dot{\gamma}(t)\|)\,dt = \begin{cases} O(\frac{1}{m^{r+2}}) \text{ if } r \geq 1 \text{ is odd} \\ O(\frac{1}{m^{r+3}}) \text{ if } r \geq 2 \text{ is even.} \end{cases}$$

Take $\tilde{\gamma}$ to be a track-sum of the $\tilde{\gamma}_j$, i.e. $d(\tilde{\gamma}) = \Sigma_{j=0}^{\frac{m}{r}-1} d(\tilde{\gamma}_j) = d(\gamma) + O(\frac{1}{m^{r+p}})$, where p is 1 or 2 according as r is odd or even. This proves Theorem 1.

Notice that, by (5), perfectly uniform sampling permits estimates of γ with uniform $O(\frac{1}{m^{r+1}})$ error. Next we consider non-uniform samplings, for which piecewise-quadratic interpolation gives good length estimates.

4 ε-Uniform Sampling

Let $k = 4$, so that $\gamma : [0, 1] \to \mathbb{R}^n$ and its reparameterizations are at least C^4. Fix $0 < \varepsilon \leq 1$, and let the t_i's be sampled ε-uniformly. We are going to prove Theorem 2. Without loss of generality m is even. For each triple (q_i, q_{i+1}, q_{i+2}), where $0 \leq i \leq m - 2$, let $Q^i : [0, 2] \to \mathbb{R}^n$ be the quadratic curve (expressed in local parameter $s \in [0, 2]$) satisfying $Q^i(0) = q_i$, $Q^i(1) = q_{i+1}$, and $Q^i(2) = q_{i+2}$. Write $Q^i(s) = q_i + a_1 s + a_2 s^2$, where $s \in [0, 2]$. Then

$$a_0 = q_i, \quad a_1 = \frac{4q_{i+1} - 3q_i - q_{i+2}}{2} \quad \text{and} \quad a_2 = \frac{q_{i+2} - 2q_{i+1} + q_i}{2}. \tag{10}$$

By Taylor's Theorem $\gamma(t_q) = \gamma(t_i) + \dot{\gamma}(t_i)(t_q - t_i) + (1/2)\ddot{\gamma}(\xi_q))(t_q - t_i)^2$, for either $q = i + 1$ or $q = i + 2$ and some $t_i < \xi_q < t_q$. Combining the latter with $\gamma(t_i) = q_i$, $\gamma(t_{i+1}) = q_{i+1}$, $\gamma(t_{i+2}) = q_{i+2}$ and substituting into (10) yields

$$a_2 = (1/2)\dot{\gamma}(t_i)(t_{i+2} - 2t_{i+1} + t_i) + O(\frac{1}{m^2}). \tag{11}$$

Because sampling is ε-uniform the Mean Value Theorem gives

$$t_q - t_i = \phi'(\eta_q)\frac{1}{m} + O(\frac{1}{m^{1+\varepsilon}}), \tag{12}$$

for either $q = i+1$ or $q = i+2$ and some $t_i < \eta_q < t_q$. Thus by (11) and (12)

$$a_2 = \frac{t_{i+2} - 2t_{i+1} + t_i}{2}\dot{\gamma}(t_i) + O(\frac{1}{m^2}). \tag{13}$$

Furthermore

$$t_{i+2} - 2t_{i+1} + t_i = \phi(\frac{i+2}{m}) - \phi(\frac{i+1}{m}) - (\phi(\frac{i+1}{m}) - \phi(\frac{i}{m})) + O(\frac{1}{m^{1+\varepsilon}}) \tag{14}$$

because sampling is ε-uniform. By Taylor's Theorem the following holds

$$\phi(\frac{i+1}{m}) = \phi(\frac{i}{m}) + \dot{\phi}(\frac{i}{m})\frac{1}{m} \quad \text{and} \quad \phi(\frac{i+2}{m}) = \phi(\frac{i}{m}) + \dot{\phi}(\frac{i}{m})\frac{2}{m}, \tag{15}$$

up to a $O(\frac{1}{m^2})$ term. Substituting (15) into (14) and taking into account ε-uniformity renders

$$t_{i+2} - 2t_{i+1} + t_i = O(\frac{1}{m^{1+\varepsilon}}). \tag{16}$$

The latter combined with (13) and ε-uniform sampling yields

$$a_2 = O(\frac{1}{m^{1+\varepsilon}}) + O(\frac{1}{m^2}) = O(\frac{1}{m^{1+\varepsilon}}). \tag{17}$$

A similar argument results in

$$a_1 = \frac{4t_{i+1} - 3t_i - t_{i+2}}{2}\dot{\gamma}(t_i) + O(\frac{1}{m^2}) = O(\frac{1}{m}). \tag{18}$$

From (17) and (18),

$$\frac{dQ^i}{ds} = a_1 + 2sa_2 = O(\frac{1}{m}) \quad \text{and} \quad \frac{d^2Q^i}{ds^2} = 2a_2 = O(\frac{1}{m^{1+\varepsilon}}), \tag{19}$$

where $s \in [0,2]$. Let $\psi : [t_i, t_{i+2}] \to [0,2]$ be the quadratic $\psi(t) = b_0 + b_1 t + b_2 t^2$ satisfying $\psi(t_i) = 0$, $\psi(t_{i+1}) = 1$, and $\psi(t_{i+2}) = 2$ (although ψ depends on i we suppress this in the notation). Inspection reveals $b_1 = (t_{i+1} - t_i)^{-1} - b_2(t_{i+1} + t_i)$, and $b_2 = ((t_{i+1} - t_i) - (t_{i+2} - t_{i+1}))[(t_{i+1} - t_i)(t_{i+2} - t_{i+1})(t_{i+2} - t_i)]^{-1}$. Furthermore, as before, by ε-uniformity $(t_{i+1} - t_i) - (t_{i+2} - t_{i+1}) = O(\frac{1}{m^{1+\varepsilon}})$, and $m^3(t_{i+1} - t_i)(t_{i+2} - t_{i+1})(t_{i+2} - t_i) = O(1)$, where the right-hand side of the latter is bounded away from 0 (as ϕ is a diffeomorphism defined over a compact set $[0,1]$). Hence,

$$b_2 = O(m^{2-\varepsilon}) \qquad \text{and} \qquad \ddot{\psi}(t) = 2b_2 = O(m^{2-\varepsilon}). \tag{20}$$

As easily verified $(t_{i+1} - t_i)^{-1} = O(m)$. Hence, coupling $b_1 = (t_{i+1} - t_i)^{-1} - b_2(t_{i+1} + t_i)$ with (20) yields

$$\dot{\psi}(t) = b_1 + 2b_2 t = O(m) + b_2(2t - (t_{i+1} + t_i)) = O(m), \tag{21}$$

as sampling is ε-uniform and $2t - (t_{i+1} + t_i) = O(\frac{1}{m})$, for $t \in [t_i, t_{i+2}]$. In particular, ψ is a diffeomorphism for m large. Define $\tilde{\gamma}_i = Q^i \circ \psi : [t_i, t_{i+2}] \to$

$\mathbb{R}^n$. Then $\tilde{\gamma}_i$ is polynomial of degree at most 4. Its derivatives of order $2 \leq p \leq 4$, are $O(m^{(p-1)(1-\varepsilon)})$. Indeed, by (19), (20), (21), $deg(\psi) \leq 2$ and $deg(Q^i) \leq 2$

$$\ddot{\tilde{\gamma}}_i = Q^{i''}\dot{\psi}^2 + Q^{i'}\ddot{\psi} = O(\frac{1}{m^{1+\varepsilon}})O(m^2) + O(\frac{1}{m})O(m^{2-\varepsilon}) = O(m^{1-\varepsilon}), \quad (22)$$

$$\tilde{\gamma}_i^{(3)} = 3Q^{i''}\dot{\psi}\ddot{\psi} = O(\frac{1}{m^{1+\varepsilon}})O(m)O(m^{2-\varepsilon}) = O(m^{2-2\varepsilon}), \quad (23)$$

$$\tilde{\gamma}_i^{(4)} = 3Q^{i''}\ddot{\psi}^2 = O(\frac{1}{m^{1+\varepsilon}})O(m^{4-2\varepsilon}) = O(m^{3-3\varepsilon}). \quad (24)$$

Then $f = \tilde{\gamma}_i - \gamma : [t_i, t_{i+2}] \to \mathbb{R}^n$ is C^4 and satisfies $f(t_i) = f(t_{i+1}) = f(t_{i+2}) = 0$. By (22), (23), (24) and ε-uniformity we have

$$\frac{d^2 f}{dt^2} = O(m^{1-\varepsilon}), \quad \frac{d^3 f}{dt^3} = O(m^{2(1-\varepsilon)}), \quad \frac{d^4 f}{dt^4} = O(m^{3(1-\varepsilon)}). \quad (25)$$

Use Lemma 1 to write $f(t) = (t - t_i)(t - t_{i+1})(t - t_{i+2})h(t)$, where $h : [t_i, t_{i+2}] \to \mathbb{R}^n$ is C^1, respectively. Then again by Lemma 1 and (25) we have $h = O(\frac{d^3 f}{dt^3}) = O(m^{2(1-\varepsilon)})$. Furthermore (6) coupled with (25) renders $\dot{h} = O(m^{3(1-\varepsilon)})$. The latter combined with the ε-uniformity yields

$$\dot{f} = O(\frac{1}{m^{2\varepsilon}}) \quad \text{and} \quad f = O(\frac{1}{m^{1+2\varepsilon}}). \quad (26)$$

As in the proof of Theorem 1 define $V_{\dot{\gamma}}^{\perp}(t)$ to be the orthogonal complement to the space spanned by $\dot{\gamma}(t)$. Then expand $\dot{\tilde{\gamma}}_j(t)$ according to (8), where $v(t)$ is the orthogonal projection of $\dot{\tilde{\gamma}}_j(t)$ onto $V_{\dot{\gamma}}^{\perp}(t)$. Similarly to (9), by using (26) we arrive at $v = O(\frac{1}{m^{2\varepsilon}})$ and thus

$$\|\dot{\tilde{\gamma}}_j(t)\| = \|\dot{\gamma}(t)\|(1 + \frac{< \dot{f}(t), \dot{\gamma}(t) >}{d(\gamma)^2}) + O(\frac{1}{m^{4\varepsilon}}), \quad (27)$$

for which we use $\varepsilon \in (0, 1]$. Consequently, by (9), (26), and (27) the integral $\int_{t_i}^{t_{i+2}} (\|\dot{\tilde{\gamma}}_i(t)\| - \|\dot{\gamma}(t)\|)\, dt =$

$$\int_{t_i}^{t_{i+2}} \frac{< \dot{f}(t), \dot{\gamma}(t) >}{d(\gamma)} dt + O(\frac{1}{m^{1+4\varepsilon}}) = -\int_{t_i}^{t_{i+2}} \frac{< f(t), \ddot{\gamma}(t) >}{d(\gamma)} dt$$

up to $O(\frac{1}{m^{1+4\varepsilon}})$. Now $< f(t), \ddot{\gamma}(t) >= (t - t_i)(t - t_{i+1})(t - t_{i+2})r(t)$, where $r(t) =< h(t), \ddot{\gamma}(t) >$. Taylor's Theorem applied to r at t_i yields $r(t) = r(t_i) + (t - t_i)\dot{r}(\xi)$, for some $t_i < \xi < t_{i+2}$. Similarly to the argument used for uniform sampling, $r(t_i) = O(m^{2(1-\varepsilon)})$ and $\dot{r} = O(m^{3(1-\varepsilon)})$. Consequently, by (16) $\int_{t_i}^{t_{i+2}} (t - t_i)(t - t_{i+1})(t - t_{i+2}) dt = \frac{1}{12}(t_i - t_{i+2})^3 (t_{i+2} - 2t_{i+1} + t_i) = O(\frac{1}{m^{4+\varepsilon}})$ and hence the integral $\int_{t_i}^{t_{i+2}} \frac{< f(t), \ddot{\gamma}(t) >}{d(\gamma)}) dt =$

$$\frac{r(t_i)}{d(\gamma)} \int_{t_i}^{t_{i+2}} (t - t_i)(t - t_{i+1})(t - t_{i+2}) dt + O(\frac{1}{m^5}) O(m^{3(1-\varepsilon)})$$

is $O(\frac{1}{m^{2+3\varepsilon}})$. So again by ε-uniformity $\int_{t_i}^{t_{i+2}}(\|\dot{\tilde{\gamma}}_i(t)\|-\|\dot{\gamma}(t)\|)\,dt = O(\frac{1}{m^{2+3\varepsilon}})+O(\frac{1}{m^{1+4\varepsilon}}) = O(\frac{1}{m^{1+4\varepsilon}})$, and we finally arrive at $d(\tilde{\gamma}) = \Sigma_{j=0}^{\frac{m}{2}-1} d(Q^{2j}) = d(\gamma)+O(\frac{1}{m^{4\varepsilon}})$. This proves Theorem 2.

Notice that ε-uniform sampling, permits (by (26)) estimates of γ with uniform $O(\frac{1}{m^{1+2\varepsilon}})$ error. Moreover, by taking $\varepsilon = 1$ in Theorem 2 we obtain a stronger statement (as ϕ-perturbation of uniform sampling is still allowed; see Definition 1) than Theorem 1 when $r = 2$. Note also that Theorem 2 has nothing to say about the case $\varepsilon = 0$. This is dealt with in [15] using a different approach.

5 Experiments

Next we test the sharpness of the theoretical results in Theorems 1 and 2 with some numerical experiments. Our test curves are a semicircle and cubic $\gamma_s, \gamma_c : [0,1] \to \mathbb{R}^2$, given by $\gamma_s(t) = (\cos(\pi(1-t)), \sin(\pi(1-t)))$ and $\gamma_c(t) = (\pi t, (\frac{\pi t+1}{\pi+1})^3)$. Of course $d(\gamma_s) = \pi$, and numerical integration gives $d(\gamma_c) = 3.3452$. Experiments were performed with Mathematica.

5.1 Uniform Sampling

We first discuss convergence of length estimates for piecewise polynomial approximations and perfectly uniform sampling. Experiments were conducted for both test curves, with $r = 1,2,3,4$ for which Theorem 1 asserts errors that are $O(\frac{1}{m^2})$, $O(\frac{1}{m^4})$, $O(\frac{1}{m^4})$, and $O(\frac{1}{m^6})$, respectively. For each $r = 1,2,3,4$ the minimum and maximum number of interpolation points were $(min_1, max_1) = (min_2, max_2) = (7,101)$, $(min_3, max_3) = (7,100)$, and $(min_4, max_4) = (9,101)$. Let $\tilde{\gamma}_{r,m_r}$ represent a piecewise-r-degree polynomial interpolating m_r points. In each row of Tables 1 and 2 (for a fixed $1 \le r \le 4$) we list only some specific values obtained from the set of absolute errors $E_{m_r}(\gamma) = |d(\gamma) - d(\tilde{\gamma}_{r,m_r})|$ (here m_r indexes $\tilde{\gamma}_{r,m_r}$, where $min_r \le m_r \le max_r$ and $m_r = rn+1$), namely: $E_{min_r}^{max_r}(\gamma) = \max_{min_r \le m_r \le max_r} E_{m_r}(\gamma)$ and $E_{max_r}(\gamma)$. Moreover, for each r, in searching for the estimate of convergence rate $O(\frac{1}{m^\alpha})$ (where $m+1 = m_r$ is a number of interpolation points) a linear regression is carried out on pairs of points $(\log(m_r - 1), -\log(E_{m_r}(\gamma))$, where $min_r \le i_r \le max_r$ and $i_r = rn+1$. Here are the results. Both Tables 1 and 2 suggest that (in these cases at least) the statements in Theorem 1 are sharp. (The last two rows of Table 2 are somewhat irrelevant in that Lagrange interpolation returns, for $r \ge 3$, the same curve γ_c, up to machine precision.)

5.2 ε-Uniform Sampling

A full report on experiments with piecewise-quadratic Lagrange interpolation with ε-uniform sampling is given in [16]. We experimented with $\varepsilon_1 = 1$, $\varepsilon_{1/2} =$

Table 1. Results for length estimation $d(\gamma_s)$ of semicircle γ_s.

r	min_r	max_r	α (where $E_{m_r} \propto m^{-\alpha}$)	$E_{min_r}^{max_r}(\gamma_s)$	E_{max_r}
1	7	101	1.99943	0.0357641	1.29191×10^{-4}
2	7	101	3.98485	0.0036266	5.09874×10^{-8}
3	7	100	3.97964	0.0026509	3.98087×10^{-8}
4	9	101	5.98218	0.0001136	3.19167×10^{-11}

Table 2. Results for length estimation $d(\gamma_s)$ of cubic curve γ_c.

r	min_r	max_r	α (where $E_{m_r} \propto m^{-\alpha}$)	$E_{min_r}^{max_r}(\gamma_c)$	E_{max_r}
1	7	101	2.00006	0.0357641	5.18348×10^{-6}
2	7	201	4.09546	0.0036266	1.22657×10^{-12}
3	7	100	n/a [a]	5.90639×10^{-14} [a]	4.44089×10^{-16} [a]
4	9	101	n/a [a]	2.73115×10^{-13} [a]	4.44089×10^{-16} [a]

[a] not applicable (see above).

$1/2$, $\varepsilon_{1/3} = 1/3$, and for $l = 1, 2, 3$, with diffeomorphisms $\phi_l : [0,1] \to [0,1]$, given by $\phi_1(t) = t$, $\phi_2(t) = \frac{1}{\pi+1} t(\pi t + 1)$, and $\phi_3(t) = \frac{\exp(\pi t)-1}{\exp(\pi)-1}$. These functions are used to define first ε-uniform *random sampling*

$$t_i = \phi_l(\frac{i}{m}) + (Random[\,] - 0.5)\frac{1}{m^{1+\varepsilon}}, \tag{28}$$

where $Random[\,]$ takes the pseudorandom values from the interval $[0,1]$ and $0 \le i \le m$. In addition, we experimented with two other families of *skew-symmetric* ε-uniform samplings with ϕ_1 and $0 \le i \le m$:

$$(i)\ t_i = \frac{i}{m} + \frac{(-1)^{i+1}}{2m^{1+\varepsilon}} \qquad (ii)\ t_i = \begin{cases} \frac{i}{m} & \text{if } i \text{ even}, \\ \frac{i}{m} + \frac{1}{2m^{1+\varepsilon}} & \text{if } i = 4k+1, \\ \frac{i}{m} - \frac{1}{2m^{1+\varepsilon}} & \text{if } i = 4k+3. \end{cases} \tag{29}$$

In all cases $t_0 = 0, t_1 = 1$. Piecewise-quadratic interpolation was implemented for both kinds of sampling, with m even running from $m = 6$ up to $m = 100$ and to $m = 200$, respectively. These experiments with γ_s and γ_c showed faster convergence than proved in Theorem 2 for sampling (28). However, the statement of Theorem 2 appears to be sharp for the samplings (29): the observed rates of convergence nearly coincide with those asserted by the theorem: $\alpha_1 = 4$, $\alpha_{1/2} = 2$ and $\alpha_{1/3} = 4/3$. Note also that for 0-uniform sampling (29)(i), and for semicircle γ_s and cubic curve γ_c, a piecewise-quadratic Lagrange polynomial interpolant does not provide good estimates of $d(\gamma_s)$ and $d(\gamma_c)$, respectively (see Figure 2).

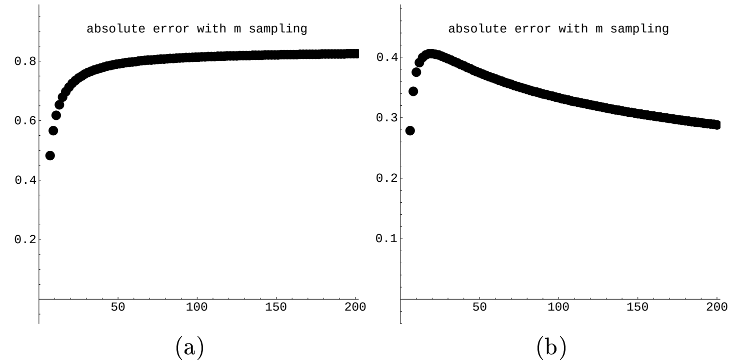

Fig. 2. Absolute errors plotted for 0-uniform skew sampling (29)(i) (where m_2 is even and $6 \leq m_2 \leq 200$): (**a**) $E_{m_2}(\gamma_s)$ against m_2. (**b**) $E_{m_2}(\gamma_c)$ against m_2.

6 Conclusion

The problem of estimating $d(\gamma)$ of a C^k curve seems rather straightforward when the parameter values $t_i \in [0,1]$ are given, for example when sampling is uniform. This paper examines a class of samplings for which the same simple methods give length estimates converging to the true value $d(\gamma)$, including investigation of convergence rates. Our results appear to be sharp for the class of samplings studied in this paper. Piecewise Lagrange interpolation does not work well for 0-uniform samplings (more elaborate methods for dealing with these are given in [15]) and so the class of ε-uniform samplings is of special interest where $0 < \varepsilon \leq 1$. In general, the relationship between convergence of length estimates and uniform convergence to the image of γ seems not quite straightforward. Because the methods used herein are relatively simple, they are also widely applicable. Unlike the situation in [15] there is no convexity requirement on γ, and there is no need to restrict attention to planar curves.

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